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Deformation theory of nearly Kähler manifolds

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ABSTRACT

Nearly Kähler manifolds are the Riemannian 6-manifolds admitting real Killing spinors. Equivalently, the Riemannian cone over a nearly Kähler manifold has holonomy contained in G_2 . In this paper we study the deformation theory of nearly Kähler manifolds, showing that it is obstructed in general. More precisely, we show that the infinitesimal deformations of the homogeneous nearly Kähler structure on the flag manifold are all obstructed to second order.

1. Introduction

A *Killing spinor* on a Riemannian spin manifold (M^n, g) is a spinor ψ such that

$$\nabla_X \psi = \alpha X \cdot \psi,$$

for some $\alpha \in \mathbb{C}$ and every vector field X . Here \cdot denotes Clifford multiplication. Killing spinors appeared in work of Friedrich [9] on the first eigenvalue of the Dirac operator. It was shown in [9] that every manifold with a Killing spinor is Einstein: $\text{Ric}(g) = 4(n-1)\alpha^2 g$. In particular, one of three cases must hold: (i) $\alpha \neq 0$ is purely imaginary and M is non-compact; (ii) if $\alpha = 0$ then ψ is a parallel spinor and therefore (M, g) has restricted holonomy contained in $SU(\frac{n}{2})$, $Sp(\frac{n}{4})$, G_2 or $Spin(7)$; (iii) $\alpha \neq 0$ is real: ψ is called a *real Killing spinor* and M (if complete) is compact with finite fundamental group. In the real case one can always assume that $\alpha = \pm \frac{1}{2}$ by scaling the metric.

There is in fact a relation between parallel spinors and real Killing spinors: by work of Bär [2], the cone over a manifold with a real Killing spinor has a parallel spinor and conversely simply connected manifolds with a real Killing spinor are the cross-sections of Riemannian cones with holonomy contained in $SU(\frac{n}{2})$, $Sp(\frac{n}{4})$, G_2 or $Spin(7)$, depending on the dimension n and the number of linearly independent real Killing spinors.

Nearly Kähler manifolds are the 6-dimensional Riemannian manifolds admitting real Killing spinors. The cone over a nearly Kähler manifold has holonomy contained in G_2 .

REMARK. The name nearly Kähler was introduced by Gray [11] to denote a special class of almost Hermitian manifolds in every even dimension. What we call here nearly Kähler manifolds are often referred to as *strict* nearly Kähler manifolds of dimension 6. The terminology *Gray manifolds* has also been used, cf. [16, Definition 4.1].

Despite the spinorial point of view will play a role in this paper, we prefer to relate the holonomy reduction of the cone $C(M)$ over a nearly Kähler manifold to the existence of a closed and co-closed *stable* 3-form rather than to the existence of a parallel spinor. From this point of view a nearly Kähler structure is an $SU(3)$ structure with special torsion: a pair of differential forms (ω, Ω) , where ω is a non-degenerate 2-form and Ω is a complex volume form

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satisfying appropriate algebraic compatibility conditions and the first order PDE system

$$d\omega = 3 \operatorname{Re} \Omega, \quad d \operatorname{Im} \Omega = -2\omega^2.$$

There are currently only six known examples of simply connected nearly Kähler manifolds. Four of these are homogeneous and were known since 1968 [23]: the round 6–sphere endowed with the non-integrable almost complex structure induced by octonionic multiplication on $\mathbb{R}^7 \simeq \operatorname{Im} \mathbb{O}$ and the 3–symmetric spaces $\mathbb{C}\mathbb{P}^3 = Sp(2)/U(1) \times Sp(1)$, $S^3 \times S^3 = SU(2)^3/SU(2)$ and $\mathbb{F}_3 = SU(3)/T^2$. Recently two inhomogeneous nearly Kähler structures on S^6 and $S^3 \times S^3$ were found in [8]. Finite quotients of the homogeneous nearly Kähler manifolds have also been studied [7]. This scarcity of examples should be contrasted with the infinitely many known examples of manifolds with real Killing spinors in other dimensions: Sasaki–Einstein, 3–Sasaki and nearly parallel G_2 manifolds (the cross-sections of Calabi–Yau, hyperkähler and $Spin(7)$ cones, respectively).

In this paper we study the deformation theory of nearly Kähler manifolds. In [16] Moroianu–Nagy–Simmelmann studied infinitesimal deformations of nearly Kähler structures, identifying them with an eigenspace of the Laplacian acting on coclosed primitive $(1, 1)$ –forms. The question of whether nearly Kähler 6–manifolds have smooth, unobstructed deformations was however left open. Because of the scarcity of examples it would be very interesting to understand whether it is possible to obtain new nearly Kähler manifolds by deforming known ones. Understanding whether nearly Kähler deformations are in general obstructed is also important for applications to the theory of G_2 *conifolds* (asymptotically conical and conically singular G_2 manifolds) developed by Karigiannis–Lotay [13]. Finally, possible constructions of new examples of nearly Kähler manifolds based on singular perturbation methods require as a preliminary step the study of the deformation theory of nearly Kähler manifolds (and its extension to certain singular nearly Kähler spaces).

It is instructive to recall known results about the deformation theory of manifolds with real Killing spinors in other dimensions. Continuous families of Sasaki–Einstein structures are certainly known, e.g. the regular Sasaki–Einstein 5–manifolds obtained from del Pezzo surfaces of degree $d \leq 4$ via the Calabi ansatz have non-trivial moduli. However in general Sasaki–Einstein manifolds have obstructed deformations (*cf.* [21] for the relation between integrability of infinitesimal deformations and K –stability in the more general context of constant scalar curvature Sasaki metrics). By a result of Pedersen–Poon [20] 3–Sasaki manifolds are rigid. In [1] Alexandrov–Simmelmann study infinitesimal deformations of nearly parallel G_2 structures. As in the nearly Kähler case these are identified with a certain subspace of an eigenspace of the Laplacian acting on 3–forms. It is unclear whether infinitesimal deformations of nearly parallel G_2 manifolds are unobstructed in general.

Given what is known about deformations of Sasaki–Einstein manifolds, it is not surprising that nearly Kähler 6–manifolds have obstructed deformations in general, as we show in this paper. In [14, Theorem 6.12] Koiso showed that infinitesimal deformations of Einstein metrics are in general obstructed. He exhibited Einstein symmetric spaces with non-trivial infinitesimal Einstein deformations which cannot be integrated to second order. We will follow a similar strategy. In [17] Moroianu–Simmelmann calculated the space of infinitesimal deformations of the homogeneous nearly Kähler manifolds. They found that $\mathbb{C}\mathbb{P}^3$ and $S^3 \times S^3$ are rigid while the flag manifold \mathbb{F}_3 has an 8–dimensional space of infinitesimal deformations [17, Corollary 6.1]. (The case of the round 6–sphere is special, since there are more than two Killing spinors in this case. The space of nearly Kähler structures compatible with the round metric is an $\mathbb{R}\mathbb{P}^7$ –bundle over S^6 . Since the round metric does not admit any Einstein deformation, there are no infinitesimal nearly Kähler deformations other than the ones coming from this family.)

In this paper we address the question of deciding whether the homogeneous nearly Kähler structure on the flag manifold admits genuine nearly Kähler deformations.

THEOREM. *The infinitesimal nearly Kähler deformations of the flag manifold \mathbb{F}_3 are all obstructed.*

The proof of the Theorem (and the paper itself) is divided into two distinct parts. In a first step we will obtain a deeper understanding of the deformation theory of nearly Kähler structures in general beyond the infinitesimal level considered in [16]. The main tool is a certain Dirac-type operator on nearly Kähler manifolds, which appears as a certain combination of differential, codifferential and type decomposition acting on differential forms. The use of Dirac operators as tools in the deformation theory of manifolds with special geometric structures is not new. Nördstrom [18, Chapter 3] used Dirac operators to streamline certain steps in the deformation theory of manifolds with special holonomy. More precisely, mapping properties of Dirac operators are used to establish slice theorems for the action of the diffeomorphism group. This approach has turned out to be particularly useful in the deformation theory of *non-compact* manifolds with special holonomy, in particular asymptotically cylindrical manifolds [19] and conifolds [13].

Besides the application to the deformation theory of nearly Kähler structures, we will show that Dirac-type operators can also be used to obtain interesting results about Hodge theory on nearly Kähler manifolds. In particular, we will give an elementary proof of a result of Verbitsky [22, Theorem 6.2 and Remark 6.4] on the type decomposition of harmonic forms on nearly Kähler manifolds.

A second important ingredient in our treatment of the deformation theory of nearly Kähler manifolds is Hitchin's interpretation of nearly Kähler structures as (constrained) critical points of a Hamiltonian function on the infinite dimensional symplectic vector space $\Omega_{exact}^3 \times \Omega_{exact}^4$. This description will allow us to interpret the nearly Kähler equations as the vanishing of a smooth map $\Phi: \Omega_{exact}^3 \times \Omega_{exact}^4 \times \Omega^1 \rightarrow \Omega_{exact}^3 \times \Omega_{exact}^4$, rather than as equations on the space of $SU(3)$ structures. The main advantage of this approach is to introduce additional free parameters that can be used to reduce the cokernel of the linearisation $D\Phi$ and pin down exactly where possible obstructions to integrate infinitesimal deformations could lie.

The general deformation theory of nearly Kähler structures is then applied to the specific case of the flag manifold \mathbb{F}_3 . The framework introduced in the first part of the paper will make possible the explicit calculation of the non-vanishing obstructions to integrate the infinitesimal deformations of the homogeneous nearly Kähler structure on \mathbb{F}_3 to second order.

As we have already mentioned, Alexandrov–Simmelmann [1] studied infinitesimal deformations of nearly parallel G_2 manifolds. In particular, by [1, §8] the normal homogeneous nearly parallel G_2 manifolds are all rigid except for the Aloff–Wallach manifold $SU(3)/U(1)$, which admits an 8-dimensional space of infinitesimal deformations isomorphic to \mathfrak{su}_3 . It is likely that the methods of this paper could also be used to analyse the integrability of these infinitesimal deformations.

The paper is organised as follows. In Section 2 we collect various preliminary results about 6-manifolds endowed with an $SU(3)$ structure. The notion of a *stable* form introduced by Hitchin in [12] will be central in our exposition. These results are known and we collect them here simply for the convenience of the reader. In Section 3 we study algebraic compatibilities on nearly Kähler manifolds between differential and codifferential and the decomposition of the space of differential forms into types corresponding to irreducible representations of $SU(3)$. We will then introduce the Dirac-type operator mentioned above and study its mapping properties. As a first application, we will derive results about the Hodge theory of nearly Kähler manifolds. Section 4 discusses the deformation problem of nearly Kähler manifolds. The Dirac-type operator is used to define a slice for the action of the diffeomorphism group while Hitchin's interpretation of nearly Kähler structures as (constrained) critical points allows to re-write the nearly Kähler equations as the vanishing of a certain non-linear map $\Phi: \Omega_{exact}^3 \times \Omega_{exact}^4 \times \Omega^1 \rightarrow \Omega_{exact}^3 \times \Omega_{exact}^4$. We study the linearisation $D\Phi$ at a nearly Kähler structure and identify its cokernel and

therefore possible obstructions to integrate infinitesimal deformations to genuine nearly Kähler deformations. Deformations of the homogeneous nearly Kähler structure on the flag manifold are finally studied in Section 5. We introduce the notion of second order deformations, recall the result of Moroianu–Semmelmann [17, Corollary 6.1] on the existence of an 8–dimensional space of infinitesimal nearly Kähler deformations and, via explicit calculations and representation theoretic considerations, we show that these are all obstructed to second order.

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2. $SU(3)$ structures on 6–manifolds

In this preliminary section we collect various known facts about $SU(3)$ structures on 6–manifolds. These results are well-known and we collect them here simply for the convenience of the reader.

2.1. Stable forms

Following Hitchin [12], the notion of a stable form will be central in our exposition.

DEFINITION 1. A differential form $\phi \in \Lambda^p(\mathbb{R}^n)^*$ is *stable* if its $GL(n, \mathbb{R})$ –orbit in $\Lambda^p(\mathbb{R}^n)^*$ is open.

In dimension 6, there are only three possibilities for stable forms [12, §2]:

- (1) a stable 2–form ω (a non-degenerate 2–form) with open orbit isomorphic to $GL(6, \mathbb{R})/Sp(6, \mathbb{R})$;
- (2) a stable 4–form σ , with stabiliser $Sp(6, \mathbb{R})$;
- (3) a stable 3–form ρ , with stabiliser $SL(3, \mathbb{C})$.

Note that in all three cases the stabiliser is in fact contained in $SL(6, \mathbb{R})$ and therefore to each stable form one can associate a volume form dv . For example one can define $dv(\omega) = \frac{1}{n!}\omega^n$ for a stable 2–form ω in dimension $2n$. Using the homogeneous behaviour of the map dv , for every stable p –form ϕ Hitchin defines its *dual* $\hat{\phi}$, a $(n-p)$ –form such that $dv(\phi)$ is proportional to $\phi \wedge \hat{\phi}$. In dimension 6 we have:

- (1) for a stable 2–form ω , $\hat{\omega} = \frac{1}{2}\omega^2$;
- (2) for a stable 4–form σ , $\hat{\sigma}$ is the unique non-degenerate 2–form such that $\sigma = \frac{1}{2}\hat{\sigma}^2$;
- (3) for a stable 3–form ρ , $\hat{\rho}$ is the unique 3–form such that $\rho + i\hat{\rho}$ is a nowhere vanishing complex volume form.

In particular, in dimension 6 the real part of a complex volume form Ω uniquely determines Ω itself. Moreover, since its stabiliser is $SL(3, \mathbb{C})$, a stable 3–form $\text{Re } \Omega$ defines an almost complex structure J : a 1–form α is of type $(1, 0)$ if and only if $\alpha \wedge \Omega = 0$.

DEFINITION 2. An $SU(3)$ –structure on a 6–manifold M is a pair of smooth differential forms $(\omega, \text{Re } \Omega)$ such that

- (i) ω is a stable 2–form with Hitchin’s dual $\frac{1}{2}\omega^2$;
- (ii) $\text{Re } \Omega$ is a stable 3–form with Hitchin’s dual $\text{Im } \Omega$;

(iii) the following algebraic constraints are satisfied:

$$\omega \wedge \operatorname{Re} \Omega = 0, \quad \frac{1}{6} \omega^3 = \frac{1}{4} \operatorname{Re} \Omega \wedge \operatorname{Im} \Omega. \quad (2.1)$$

The two constraints guarantee that the stabiliser of the pair $(\omega, \operatorname{Re} \Omega)$ is exactly $SU(3) = Sp(6, \mathbb{R}) \cap SL(3, \mathbb{C})$. We could of course define an $SU(3)$ -structure as the choice of a pair of stable differential forms $\sigma \in \Omega^4(M)$ and $\operatorname{Re} \Omega \in \Omega^3(M)$ satisfying (2.1) with $\omega = \hat{\sigma}$.

On $\mathbb{R}^6 \simeq \mathbb{C}^3$ with holomorphic coordinates (z_1, z_2, z_3) we define the standard parallel $SU(3)$ -structure $(\omega_0, \operatorname{Re} \Omega_0)$ by

$$\omega_0 = \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3), \quad \Omega_0 = dz_1 \wedge dz_2 \wedge dz_3.$$

An $SU(3)$ structure on a 6-manifold M in the sense of Definition 2 defines a reduction of the structure group of the tangent bundle of M to $SU(3)$ by considering the sub-bundle of the frame-bundle of M defined by $\{u: \mathbb{R}^6 \xrightarrow{\sim} T_x M \mid u^*(\omega_x, \operatorname{Re} \Omega_x) = (\omega_0, \operatorname{Re} \Omega_0)\}$.

REMARK 1. Since $SU(3) \subset SU(4) \simeq Spin(6)$ is precisely the stabiliser of a non-zero vector in \mathbb{C}^4 , we could also define an $SU(3)$ structure as the choice of a spin structure on M together with a non-vanishing spinor.

Note that every $SU(3)$ -structure induces a Riemannian metric g because $SU(3) \subset SO(6)$. Hence from now on we will identify without further notice TM and T^*M using the metric g (and \mathbb{R}^6 with $(\mathbb{R}^6)^*$ using the flat metric g_0). We will instead distinguish between differential df and gradient ∇f of a function f .

REMARK 2. The identification of tangent and cotangent bundle using the metric leads us to define the action of the almost complex structure J on 1-forms by $(J\gamma)(X) = -\gamma(JX)$ for every 1-form γ and tangent vector X .

2.2. Decomposition of the space of differential forms

The decomposition into irreducible representations of the $SU(3)$ -representation $\Lambda^* \mathbb{R}^6$ is well-known. This is usually stated after complexification in terms of the (p, q) -type decomposition induced by the standard complex structure J_0 and in terms of primitive forms. We will stick with real representations and find more convenient to use the uniform notation Λ_ℓ^k for an irreducible component of $\Lambda^k \mathbb{R}^6$ of dimension ℓ .

LEMMA 2.1. We have the following orthogonal decompositions into irreducible $SU(3)$ representations:

$$\Lambda^2 \mathbb{R}^6 = \Lambda_1^2 \oplus \Lambda_6^2 \oplus \Lambda_8^2,$$

where $\Lambda_1^2 = \mathbb{R} \omega$, $\Lambda_6^2 = \{X \lrcorner \operatorname{Re} \Omega \mid X \in \mathbb{R}^6\}$ and Λ_8^2 is the space of primitive forms of type $(1, 1)$.

$$\Lambda^3 \mathbb{R}^6 = \Lambda_6^3 \oplus \Lambda_{1 \oplus 1}^3 \oplus \Lambda_{12}^3,$$

where $\Lambda_6^3 = \{X \wedge \omega \mid X \in \mathbb{R}^6\}$, $\Lambda_{1 \oplus 1}^3 = \mathbb{R} \operatorname{Re} \Omega \oplus \mathbb{R} \operatorname{Im} \Omega$ and Λ_{12}^3 is the space of primitive forms of type $(1, 2) + (2, 1)$, i.e. $\Lambda_{12}^3 = \{S_* \operatorname{Re} \Omega \mid S \in \operatorname{Sym}^2(\mathbb{R}^6), SJ + JS = 0\}$.

REMARK. Here an endomorphism $S \in \operatorname{End}(\mathbb{R}^6)$ acts on a differential p -form ϕ by $S_* \phi(X_1, \dots, X_p) = -\sum_{j=1}^p \phi(X_1, \dots, SX_j, \dots, X_p)$.

For a 6–manifold M endowed with an $SU(3)$ –structure (ω, Ω) we denote by $\Omega_\ell^k(M)$ the space of smooth sections of the bundle over M with fibre Λ_ℓ^k .

By definition Λ_6^2 is isomorphic to Λ^1 via the metric and the contraction with $\text{Re } \Omega$. The adjoint of this map with respect to the flat metric g_0 will be denoted by $\alpha: \Lambda_6^2 \rightarrow \Lambda^1$. Note that for a form $\eta \in \Lambda_6^2$ we have

$$\eta = \frac{1}{2}\alpha(\eta) \lrcorner \text{Re } \Omega. \quad (2.2)$$

The following identities follow from [16, Equations (12), (17), (18) and (19)].

LEMMA 2.2. *In the decomposition of Lemma 2.1 the Hodge–* operator is given by:*

- (i) $*\omega = \frac{1}{2}\omega^2$;
- (ii) $*(X \lrcorner \text{Re } \Omega) = -JX \wedge \text{Re } \Omega = X \wedge \text{Im } \Omega$;
- (iii) $*(\eta_0 \wedge \omega) = -\eta_0$ for all $\eta_0 \in \Omega_3^2$;
- (iv) $*(X \wedge \omega) = \frac{1}{2}X \lrcorner \omega^2 = JX \wedge \omega$;
- (v) $*\text{Re } \Omega = \text{Im } \Omega$ and $*\text{Im } \Omega = -\text{Re } \Omega$;
- (vi) $*(S_* \text{Re } \Omega) = -S_* \text{Im } \Omega = (JS)_* \text{Re } \Omega$;

We use Lemma 2.2 to deduce useful identities and characterisations of the different types of forms.

LEMMA 2.3. *If $\eta = \eta_0 + \lambda\omega + X \lrcorner \text{Re } \Omega \in \Omega^2$ with $\eta_0 \in \Omega_3^2$, then the following holds:*

- (i) $*(\eta \wedge \omega) = -\eta_0 + 2\lambda\omega + X \lrcorner \text{Re } \Omega$;
- (ii) $*(\eta \wedge \eta \wedge \omega) = -|\eta_0|^2 + 6\lambda^2 + 2|X|^2$;
- (iii) $*(\eta \wedge \text{Re } \Omega) = 2JX$ and $*(\eta \wedge \text{Im } \Omega) = -2X$;
- (iv) $*(\eta \wedge \omega^2) = 6\lambda$.

In particular, $\eta \in \Omega_3^2$ iff $\eta \wedge \omega^2 = 0 = \eta \wedge \text{Re } \Omega$ iff $\eta \wedge \omega = -\eta$.*

Proof. (i) follows immediately from Lemma 2.2 (i)–(iii). In turn, (i) implies (ii) by the identity

$$\eta \wedge \eta \wedge \omega = \eta \wedge *^2(\eta \wedge \omega),$$

the fact that the decomposition of Lemma 2.1 is orthogonal and $|\omega|^2 = 3$, $\langle X \lrcorner \text{Re } \Omega, X \lrcorner \text{Re } \Omega \rangle = 2|X|^2$. The identities (iii) follow immediately from [16, Equations (3) and (4)] and (iv) from $2*\omega = \omega^2$ and $|\omega|^2 = 3$. \square

We have similar identities on 3–forms, with analogous proof.

LEMMA 2.4. *If $\sigma = X \wedge \omega + \lambda \text{Re } \Omega + \mu \text{Im } \Omega + S_* \text{Re } \Omega \in \Omega^3$, then the following holds:*

- (i) $*(\sigma \wedge \omega) = 2JX$;
- (ii) $*(\sigma \wedge \text{Re } \Omega) = -4\mu$;
- (iii) $*(\sigma \wedge \text{Im } \Omega) = 4\lambda$.

In particular, $\rho \in \Omega_{12}^3$ iff $\rho \wedge \omega = 0 = \rho \wedge \Omega$.

Finally, it will be useful to have an explicit formula for the linearisation of Hitchin’s duality map for stable forms in terms of the Hodge–* and the decomposition of forms into types.

PROPOSITION 2.5. *Given an $SU(3)$ structure $(\omega, \operatorname{Re} \Omega)$ on M^6 let $\sigma \in \Omega^4(M)$ and $\rho \in \Omega^3(M)$ be forms with small enough C^0 -norm so that $\frac{1}{2}\omega^2 + \sigma$ and $\operatorname{Re} \Omega + \rho$ are still stable forms. Decomposing into types we write $\sigma = \sigma_1 + \sigma_6 + \sigma_8$ and $\rho = \rho_6 + \rho_{1\oplus 1} + \rho_{12}$.*

(i) *The image $\hat{\sigma}$ of σ under the linearisation of Hitchin's duality map at $\frac{1}{2}\omega^2$ is*

$$\hat{\sigma} = \frac{1}{2} * \sigma_1 + * \sigma_6 - * \sigma_8.$$

(ii) *The image $\hat{\rho}$ of ρ under the linearisation of Hitchin's duality map at $\operatorname{Re} \Omega$ is*

$$\hat{\rho} = *(\rho_6 + \rho_{1\oplus 1}) - * \rho_{12}.$$

Proof. In order to prove the first statement, observe that $\hat{\sigma}$ is the unique 2-form such that $\sigma = \omega \wedge \hat{\sigma}$. Apply $*$ to this identity and use Lemma 2.3.(i). The formula for $\hat{\rho}$ follows from [16, Lemma 3.3] and the last three identities in Lemma 2.2. \square

2.3. Nearly Kähler manifolds

Given a subgroup G of $SO(n)$, we define a G -structure on a Riemannian manifold (M^n, g) as a sub-bundle \mathcal{P} of the orthogonal frame bundle of M with structure group G . The *intrinsic torsion* of \mathcal{P} is a measure of how much \mathcal{P} is far from being parallel with respect to the Levi-Civita connection ∇ of (M, g) . More precisely, restricting ∇ to \mathcal{P} yields a $\mathfrak{so}(n)$ -valued 1-form θ on \mathcal{P} . Choose a complement \mathfrak{m} of the Lie algebra of G in $\mathfrak{so}(n)$. Projection of θ onto \mathfrak{m} yields a 1-form T on M with values in the bundle $\mathcal{P} \times_G \mathfrak{m}$. This is the (intrinsic) torsion of the G -structure \mathcal{P} .

For an $SU(3)$ -structure on a 6-manifold M one can check that $\nabla(\omega, \Omega) = T_*(\omega, \Omega)$ where T acts on differential forms via the representation of $\mathfrak{m} \subset \mathfrak{so}(6)$ on $\Lambda^*(\mathbb{R}^6)$. It turns out that T itself is uniquely recovered by knowledge of the anti-symmetric part of $T_*(\omega, \Omega)$, i.e. the knowledge of $d\omega$ and $d\Omega$.

PROPOSITION 2.6 Chiossi-Salamon [6, §1]; cf. also [4, Theorem 2.9]. *Let (ω, Ω) be an $SU(3)$ structure. Then there exists functions $w_1, \hat{w}_1 \in \Omega^0$, $w_2, \hat{w}_2 \in \Omega^2_8$, $w_3 \in \Omega^3_{12}$ and vector fields w_4, w_5 on M such that*

$$\begin{aligned} d\omega &= 3w_1 \operatorname{Re} \Omega + 3\hat{w}_1 \operatorname{Im} \Omega + w_3 + w_4 \wedge \omega, \\ d \operatorname{Re} \Omega &= 2\hat{w}_1 \omega^2 + w_5 \wedge \operatorname{Re} \Omega + w_2 \wedge \omega, \\ d \operatorname{Im} \Omega &= -2w_1 \omega^2 - Jw_5 \wedge \operatorname{Re} \Omega + \hat{w}_2 \wedge \omega. \end{aligned}$$

DEFINITION 3. An $SU(3)$ structure on a 6-manifold M is called a *nearly Kähler* structure if $\hat{w}_1, w_2, \hat{w}_2, w_3, w_4, w_5$ all vanish and $w_1 = 1$. In other words a nearly Kähler structure satisfies

$$d\omega = 3 \operatorname{Re} \Omega, \quad d \operatorname{Im} \Omega = -2\omega^2. \quad (2.3)$$

As remarked in the Introduction, (2.3) are equivalent to the requirement that

$$\varphi = r^2 dr \wedge \omega + r^3 \operatorname{Re} \Omega$$

is a closed and coclosed “conical” G_2 structure on the cone $C(M) = \mathbb{R}^+ \times M$. Thus the cone $C(M)$ endowed with the metric $g_C = dr^2 + r^2 g$ induced by φ has holonomy contained in G_2 and in particular is Ricci-flat. As a consequence, nearly Kähler manifolds are Einstein with positive Einstein constant normalised so that $\operatorname{Scal} = 30$. In particular every complete nearly Kähler manifold is compact with finite fundamental group.

In Remark 1 we observed that every 6-manifold with an $SU(3)$ structure is spin and endowed with a unit spinor ψ . The nearly Kähler equations (2.3) have an equivalent interpretation as a

first order differential equation on ψ [2, Theorem 2]. Indeed, every G_2 manifold admits a parallel spinor. Restricting the parallel spinor on the cone $C(M)$ to the cross-section M induces a *real Killing spinor* ψ on every nearly Kähler manifold, i.e. (possibly after changing orientation) a unit spinor ψ such that

$$\nabla_X \psi = \frac{1}{2} X \cdot \psi \quad (2.4)$$

for every vector field X . Clifford multiplication by the volume form Vol (i.e. the complex structure on the spinor bundle) yields a second Killing spinor $\text{Vol} \cdot \psi$ satisfying (recall that $X \cdot \text{Vol} \cdot \psi = -\text{Vol} \cdot X \cdot \psi$)

$$\nabla_X (\text{Vol} \cdot \psi) = -\frac{1}{2} X \cdot (\text{Vol} \cdot \psi).$$

In the rest of the paper (M, ω, Ω) will denote a complete (hence compact) nearly Kähler 6-manifold and ψ will denote the real Killing spinor on M satisfying (2.4).

3. Hodge theory on nearly Kähler manifolds

The main goal of this section is to introduce a Dirac-type operator D on a nearly Kähler manifold M and study its mapping properties. D differs from the standard Dirac operator by a lower order term. This operator arises as a certain composition of differential, codifferential and type decomposition on differential forms and will play a central role in the study of the deformation theory of nearly Kähler manifolds. Furthermore, we find that the mapping properties of the operator D have interesting consequences on the Hodge theory of nearly Kähler manifolds. In particular, we will give an elementary proof of results of Verbitsky [22] on the type decomposition of harmonic forms on nearly Kähler manifolds.

3.1. Differential and co-differential on nearly Kähler manifolds

Before giving the definition of the Dirac operator D we need to study how the exterior differential $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ and its adjoint behave with respect to the decomposition of $\Omega^*(M)$ introduced in Lemma 2.1. We need an additional piece of notation and few simple observations.

Let $\Lambda: \Omega^k(M) \rightarrow \Omega^{k-2}(M)$ be the point-wise adjoint of wedging with ω and note that $\Lambda(X \wedge \omega) = 2X$ for every vector field X . Recall also that $\alpha: \Omega^3(M) \rightarrow \Omega^1(M)$ is the point-wise adjoint of the map $X \mapsto X \lrcorner \text{Re} \Omega$. Furthermore we have the following identities:

$$*(X \wedge \sigma) = (-1)^p X \lrcorner * \sigma \quad (3.1)$$

for every p -form σ and $X \in \mathcal{X}(M)$;

$$(X \lrcorner \text{Re} \Omega) \wedge \omega = -JX \wedge \text{Re} \Omega, \quad (3.2)$$

which follows immediately by contracting $\text{Re} \Omega \wedge \omega = 0$ by X .

$$(X \lrcorner \text{Re} \Omega) \wedge \text{Re} \Omega = X \wedge \omega^2, \quad (X \lrcorner \text{Re} \Omega) \wedge \text{Im} \Omega = JX \wedge \omega^2 \quad (3.3)$$

which are [16, Equations (3) and (4)].

$$*Y = \frac{1}{2} JY \wedge \omega^2 \quad (3.4)$$

which is a consequence of (3.1). As a consequence,

$$d^*Y = -*(dJY \wedge \frac{1}{2}\omega^2). \quad (3.5)$$

PROPOSITION 3.1. *For every $f \in C^\infty(M)$, $\eta \in \Omega^2_8(M)$ and $X \in \mathcal{X}(M)$ we have*

- (i) $d(f\omega) = df \wedge \omega + 3f \text{Re} \Omega$;

- (ii) $d^*(f\omega) = -Jdf$;
- (iii) $d\eta = -\frac{1}{2}Jd^*\eta \wedge \omega + \rho$ for some $\rho \in \Omega_{12}^3$;
- (iv) $d(X \lrcorner \text{Re } \Omega) = -(\frac{1}{2}\alpha(dJX) + 3X) \wedge \omega - \frac{1}{2}(d^*X) \text{Re } \Omega - \frac{1}{2}d^*(JX) \text{Im } \Omega + \rho'$ for some $\rho' \in \Omega_{12}^3$;
- (v) $d^*(X \lrcorner \text{Re } \Omega) = J\alpha(dJX)$.

Proof. (i) follows immediately from (2.3).

Since ω^2 is closed, $d^*(f\omega) = - * d(\frac{1}{2}f\omega^2) = - *(df \wedge \frac{1}{2}\omega^2)$ and (3.4) completes the proof of (ii).

In order to prove (iii), differentiate $\eta \wedge \Omega = 0$ and use the fact that $\eta \wedge d\Omega = -2i\eta \wedge \omega^2 = 0$. Lemma 2.4 then implies that $d\eta$ has zero component in the complex line spanned by Ω . Similarly, differentiating the equality $\eta \wedge \omega + *\eta = 0$ and using Lemma 2.4.(i) yields $\Delta d\eta = -Jd^*\eta$, cf. [16, Lemma 4.3].

The identity (iv) is proved in a similar way. Consider $d(X \lrcorner \text{Re } \Omega) \wedge \omega$: by (3.2) and (3.3)

$$d(X \lrcorner \text{Re } \Omega) \wedge \omega = -3X \wedge \omega^2 - dJX \wedge \text{Re } \Omega.$$

Lemma 2.3.(iii) and (3.4) now imply that the Ω_6^3 -component of $d(X \lrcorner \text{Re } \Omega)$ is

$$-(\frac{1}{2}\alpha(dJX) + 3X) \wedge \omega$$

as claimed. On the other hand, by (3.3) and the fact that ω^2 is closed

$$d(X \lrcorner \text{Re } \Omega) \wedge \text{Re } \Omega = dX \wedge \omega^2, \quad d(X \lrcorner \text{Re } \Omega) \wedge \text{Im } \Omega = dJX \wedge \omega^2.$$

(iv) now follows from (3.5) and Lemma 2.3.(iv).

In order to prove (v) observe that $X \lrcorner \text{Re } \Omega = JX \lrcorner \text{Im } \Omega$ and therefore by (3.1),

$$d^*(X \lrcorner \text{Re } \Omega) = *d(JX \wedge \text{Re } \Omega) = *(dJX \wedge \text{Re } \Omega).$$

Lemma 2.3.(iii) concludes the proof. □

We actually need a bit more: the Ω_{12}^3 component of $d(X \lrcorner \text{Re } \Omega)$ is equal to the Ω_{12}^3 component of $d^*(X \wedge \text{Re } \Omega)$. In particular, if $d(X \lrcorner \text{Re } \Omega) \in \Omega_{12}^3$ then $d(X \lrcorner \text{Re } \Omega) = 0$.

LEMMA 3.2. *For every vector field $X \in \mathcal{X}(M)$ we have*

$$d(X \lrcorner \text{Re } \Omega) = (J\alpha(dX) + 2X) \wedge \omega - d^*X \text{Re } \Omega - d^*(JX) \text{Im } \Omega + d^*(X \wedge \text{Re } \Omega).$$

Proof. If $\rho = \mathcal{L}_X \text{Re } \Omega$ then the linearisation of Hitchin's duality map $\hat{\rho}$ must be

$$\hat{\rho} = \mathcal{L}_X \text{Im } \Omega = d(X \lrcorner \text{Im } \Omega) - 4JX \wedge \omega = -d(JX \lrcorner \text{Re } \Omega) - 4JX \wedge \omega.$$

On the other hand, by Proposition 3.1.(iv)

$$\rho = d(X \lrcorner \text{Re } \Omega) = -(\frac{1}{2}\alpha(dJX) + 3X) \wedge \omega - \frac{1}{2}d^*X \text{Re } \Omega - \frac{1}{2}d^*(JX) \text{Im } \Omega + \rho_0$$

and Proposition 2.5 now implies

$$\begin{aligned} \hat{\rho} &= -(\frac{1}{2}J\alpha(dJX) + 3JX) \wedge \omega + \frac{1}{2}d^*(JX) \text{Re } \Omega - \frac{1}{2}d^*X \text{Im } \Omega - *\rho_0 \\ &= -(J\alpha(dJX) + 6JX) \wedge \omega + d^*(JX) \text{Re } \Omega - d^*X \text{Im } \Omega - *d(X \lrcorner \text{Re } \Omega). \end{aligned}$$

Comparing the two expressions for $\hat{\rho}$ we conclude that

$$d(JX \lrcorner \text{Re } \Omega) = (J\alpha(dJX) + 2JX) \wedge \omega - d^*(JX) \text{Re } \Omega + d^*X \text{Im } \Omega + *d(X \lrcorner \text{Re } \Omega).$$

Up to changing X into JX , the Lemma is now proved since by (3.1)

$$d^*(JX \wedge \text{Re } \Omega) = *d(X \lrcorner \text{Re } \Omega).$$

REMARK 3. Equating the two different ways of writing $\hat{\rho}$ also yields the identity

$$\alpha(dX) = J\alpha(dJX) + 4JX \quad (3.6)$$

of [17, Lemma 3.2]. Note also that integrating by parts the identity of Lemma 3.2 with a compactly supported X yields

$$\alpha(d^*\rho) = -J\alpha(*d\rho) \quad (3.7)$$

for every $\rho \in \Omega_{1,2}^3(M)$. In particular, if $\rho \in \Omega_{1,2}^3$ is coclosed then $d\rho \in \Omega_3^4$. Indeed, since $\rho \wedge \omega = 0 = \rho \wedge \operatorname{Re} \Omega$ by Lemma 2.4 we also know that $d\rho$ has not component in Ω_1^4 .

3.2. The Dirac operator on nearly Kähler manifolds

Every 6-manifold M with an $SU(3)$ structure is spin and it is endowed with a unit spinor ψ . As an $SU(3)$ representation the real spinor bundle $\mathcal{S}(M)$ is isomorphic to $\Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2$, where the isomorphism is

$$(f, g, X) \mapsto f\psi + g \operatorname{Vol} \cdot \psi + X \cdot \psi.$$

For comparison with the Dirac-type operator we will define in the next subsection, we want now to derive a formula for the Dirac operator \mathcal{D} on a nearly Kähler manifold in terms of this isomorphism.

Recall that on a nearly Kähler manifold we can assume the defining unit spinor ψ satisfies (2.4). In particular, $\mathcal{D}\psi = -3\psi$ and $\mathcal{D}(\operatorname{Vol} \cdot \psi) = 3(\operatorname{Vol} \cdot \psi)$. Thus

$$\mathcal{D}(f\psi + g \operatorname{Vol} \cdot \psi) = -3f\psi + 3g \operatorname{Vol} \cdot \psi + (\nabla f - J\nabla g) \cdot \psi,$$

since the almost complex structure J satisfies

$$JX \cdot \psi = \operatorname{Vol} \cdot X \cdot \psi = -X \cdot \operatorname{Vol} \cdot \psi. \quad (3.8)$$

On the other hand,

$$\mathcal{D}(X \cdot \psi) = \sum_{i=1}^6 e_i \cdot \nabla_{e_i} X \cdot \psi - X \cdot \psi - X \cdot \mathcal{D}\psi = dX \cdot \psi + (d^*X)\psi + 2X \cdot \psi.$$

Now decompose dX into types: $dX = \frac{1}{3}d^*(JX)\omega + \frac{1}{2}\alpha(dX) \lrcorner \operatorname{Re} \Omega + \pi_8(dX)$. We have to determine the action of 2-forms on ψ .

LEMMA 3.3. For any 2-form $\eta = \lambda\omega + Y \lrcorner \operatorname{Re} \Omega + \eta_0$ with $\eta_0 \in \Omega_8^2$ we have

$$\eta \cdot \psi = 3\lambda \operatorname{Vol} \cdot \psi + 2JY \cdot \psi.$$

Proof. Forms of type Ω_8^2 are exactly those that act trivially on ψ_0 . On the other hand, writing $\omega = \sum_{i=1}^3 e_i \wedge Je_i$ for an $SU(3)$ -adapted orthonormal co-frame $\{e_1, Je_1, e_2, Je_2, e_3, Je_3\}$,

$$\omega \cdot \psi = \sum_{i=1}^3 (e_i \wedge Je_i) \cdot \psi = \sum_{i=1}^3 e_i \cdot Je_i \cdot \psi = - \sum_{i=1}^3 e_i \cdot e_i \cdot \operatorname{Vol} \cdot \psi = 3 \operatorname{Vol} \cdot \psi.$$

Here we used [3, Equation (1.3)], the fact that e_i is orthogonal to Je_i and (3.8).

In order to calculate the action of Ω_6^2 , observe that the intrinsic torsion of a nearly Kähler structure is $-\operatorname{Im} \Omega$ [17, p. 3] and that this acts as 4 on ψ and annihilates spinors of the form $X \cdot \psi$, cf. for example [5, Lemma 2]. Thus, using [3, Equations (1.3) and (1.4)],

$$(Y \lrcorner \operatorname{Re} \Omega) \cdot \psi = (JY \lrcorner \operatorname{Im} \Omega) \cdot \psi = -\frac{1}{2}(JY \cdot \operatorname{Im} \Omega + \operatorname{Im} \Omega \cdot JY) \cdot \psi = 2JY \cdot \psi.$$

PROPOSITION 3.4. For every $f, g \in C^\infty(M)$ and $X \in \mathcal{X}(M)$ we have

$$\begin{aligned} \mathcal{D}(f\psi + g \operatorname{Vol} \cdot \psi + X \cdot \psi) = & (d^*X - 3f)\psi + (d^*JX + 3g) \operatorname{Vol} \cdot \psi \\ & + (\nabla f - J\nabla g - \alpha(dJX) - 2X) \cdot \psi. \end{aligned}$$

Proof. Use Lemma 3.3 and (3.6) to rewrite $J\alpha(dX)$ as $-\alpha(dJX) - 4X$. \square

3.3. A Dirac-type operator

Consider now the first order operator

$$D: \Omega^1 \oplus \Omega^0 \oplus \Omega^0 \rightarrow \Omega^1 \oplus \Omega^0 \oplus \Omega^0$$

defined by

$$D(X, f, g) = \left(\frac{1}{2}\alpha(dJX) + 3X + df - Jdg, d^*X + 6f, d^*(JX) - 6g \right). \quad (3.9)$$

PROPOSITION 3.5. D is a self-adjoint elliptic operator.

Proof. Proposition 3.4 shows that, after choosing appropriate isomorphisms between $\mathcal{S}(M)$ and $\Omega^0 \oplus \Omega^0 \oplus \Omega^1$ we can identify D with \mathcal{D} up to a self-adjoint zeroth order term. \square

REMARK. For every $s \in \mathbb{R}$ one can define a Dirac operator D_s associated with the connection $\nabla^s = \nabla + \frac{s}{2}T$, where ∇ is the Levi-Civita connection and $T = -\operatorname{Im} \Omega$ is the intrinsic torsion (when $s = 1$ this is the so-called *canonical Hermitian connection*). None of these Dirac operators coincides with the operator D defined in (3.9).

Our interest in the operator D arises from the following identities. Consider the operator

$$D^+: \Omega_{1 \oplus 6}^2 \oplus \Omega_1^4 \longrightarrow \Omega_{1 \oplus 1 \oplus 6}^3 \quad (3.10)$$

defined by

$$(f\omega + X \lrcorner \operatorname{Re} \Omega, \frac{1}{2}g\omega^2) \longmapsto \pi_{1 \oplus 1 \oplus 6} \left(d(f\omega - X \lrcorner \operatorname{Re} \Omega) + \frac{1}{2}d^*(g\omega^2) \right).$$

Since by Proposition 3.1 the image of $(f\omega + X \lrcorner \operatorname{Re} \Omega, \frac{1}{2}g\omega^2)$ is

$$\left(\frac{1}{2}\alpha(dJX) + df - Jdg + 3X \right) \wedge \omega + \frac{1}{2}(d^*X + 6f) \operatorname{Re} \Omega + \frac{1}{2}(d^*JX - 6g) \operatorname{Im} \Omega,$$

D^+ coincides with D after choosing appropriate identifications of $\Omega_{1 \oplus 6}^2 \oplus \Omega_1^4$ and $\Omega_{1 \oplus 1 \oplus 6}^3$ with $\Omega^0 \oplus \Omega^0 \oplus \Omega^1$. Similarly, the operator

$$D^-: \Omega_{1 \oplus 1 \oplus 6}^3 \longrightarrow \Omega_{1 \oplus 6}^4 \oplus \Omega_1^2 \quad (3.11)$$

defined by

$$\sigma = JX \wedge \omega - g \operatorname{Re} \Omega + f \operatorname{Im} \Omega \longmapsto (\pi_{1 \oplus 6} d\sigma, \pi_1 d^* \sigma)$$

coincides with D after identifying $\Omega_{1 \oplus 6}^4 \oplus \Omega_1^2$ with $\Omega^0 \oplus \Omega^0 \oplus \Omega^1$ by

$$(f, g, X) \longmapsto \left((X \lrcorner \operatorname{Re} \Omega) \wedge \omega - \frac{1}{3}f\omega^2, \frac{2}{3}g\omega \right).$$

Much of what follows relies on the mapping properties of the operator D (and therefore of D^\pm).

PROPOSITION 3.6. Let (M, g, ω, Ω) be a complete nearly Kähler 6-manifold not isometric to the round 6-sphere. Then the kernel (and cokernel, since D is self-adjoint) of D consists of Killing fields that preserve the whole $SU(3)$ structure.

Proof. Suppose that (X, f, g) lies in the kernel of D . Then

- (a) $\frac{1}{2}\alpha(dJX) + df - Jdg + 3X = 0$;
- (b) $d^*X + 6f = 0$;
- (c) $d^*(JX) - 6g = 0$.

We apply $d^* \circ J$ to the identity (a): using (ii) and (v) in Proposition 3.1 and (c) we obtain $d^*dg + 18g = 0$ and therefore $g = 0$. Hence

- (a') $\frac{1}{2}\alpha(dJX) + df + 3X = 0$;
- (b') $d^*X + 6f = 0$;
- (c') $d^*(JX) = 0$.

Now, (a') and (b') together with (3.5) imply

$$dJX = -\frac{1}{3}(d^*X)\omega + \frac{1}{2}\alpha(dJX)\lrcorner\text{Re}\Omega + \pi_8(dJX) = 2f\omega - (df + 3X)\lrcorner\text{Re}\Omega + \pi_8(dJX).$$

Using Proposition 3.1 we differentiate this identity and take the inner product with $\text{Re}\Omega$:

$$0 = \frac{1}{4}\langle d^2(JX), \text{Re}\Omega \rangle = 6f + \frac{1}{2}d^*(df + 3X) = \frac{1}{2}(d^*df - 6f).$$

By Obata's Theorem $f = 0$ under the assumptions of the Proposition.

It remains to show that a vector field X such that $dJX = -3X\lrcorner\text{Re}\Omega + \pi_8(dJX)$ and $d^*(JX) = 0 = d^*X$ must preserve the $SU(3)$ structure. Since $d^*X = 0$ and thus $dJX \wedge \omega^2 = 0$, by Lemma 2.3.(i) we have

$$18\|X\|_{L^2}^2 - \|\pi_8(dJX)\|_{L^2}^2 = \int_M dJX \wedge dJX \wedge \omega = 3 \int_M JX \wedge dJX \wedge \text{Re}\Omega = 18\|X\|_{L^2}^2.$$

Thus $0 = dJX + 3X\lrcorner\text{Re}\Omega = \mathcal{L}_X\omega$. Moreover, by Proposition 3.1.(iv) $d(X\lrcorner\text{Re}\Omega) \in \Omega_{12}^3$ and therefore the formula of Lemma 3.2 implies that $\mathcal{L}_X\text{Re}\Omega = d(X\lrcorner\text{Re}\Omega) = 0$. \square

REMARK. When M is the round 6-sphere with its standard nearly Kähler structure the kernel of D consists of elements of the form $(X - \nabla f, f, 0)$ where X is a Killing field such that $\mathcal{L}_X\omega = 0 = \mathcal{L}_X\text{Re}\Omega$ and f satisfies $d^*df = 6f$.

3.4. Hodge decomposition on nearly Kähler manifolds

As immediate corollaries of Proposition 3.6 we obtain useful decompositions of the space of 3-forms and 4-forms. First we fix some notation.

DEFINITION 4. We denote by \mathcal{K} the space of Killing fields of M .

By [15, Corollary 3.2] if M is not isometric to the round 6-sphere then every $X \in \mathcal{K}$ preserves the whole $SU(3)$ structure. Hence by Proposition 3.6 when M is not isometric to the round 6-sphere the kernel and cokernel of D , and therefore of D^\pm in (3.10) and (3.11), are isomorphic to \mathcal{K} .

PROPOSITION 3.7. *Let (M, ω, Ω) be a nearly Kähler manifold not isometric to the round 6-sphere. Then the following holds.*

- (i) $\Omega^3 = \{X \wedge \omega \mid X \in \mathcal{K}\} \oplus d\Omega_{1\oplus 6}^2 \oplus d^*\Omega_1^4 \oplus \Omega_{12}^3$.
- (ii) We have an L^2 -orthogonal decomposition $\Omega_{\text{exact}}^3 = d\Omega_{1\oplus 6}^2 \oplus \Omega_{12, \text{exact}}^3$.
- (iii) $\Omega^4 = \{(X\lrcorner\text{Re}\Omega) \wedge \omega \mid X \in \mathcal{K}\} \oplus d\Omega_{1\oplus 1\oplus 6}^3 \oplus \Omega_8^4$. More precisely, for every 4-form σ there exist unique $X \in \mathcal{K}$, $Y \in \mathcal{K}^{\perp L^2} \subset \Omega^1$, $f \in \Omega^0$ and $\sigma_0 \in \Omega_8^4$ such that

$$\sigma = (X\lrcorner\text{Re}\Omega) \wedge \omega + d(JY \wedge \omega + f \text{Im}\Omega) + \sigma_0.$$

- (iv) We have an L^2 -orthogonal decomposition $\Omega_{\text{exact}}^4 = d\Omega_{1\oplus 1\oplus 6}^3 \oplus \Omega_{8, \text{exact}}^4$.

Proof. The identification of D^+ with D and Proposition 3.6 immediately imply (i). In order to deduce (ii) from (i) observe that $2d^*(X \wedge \omega) = - * \mathcal{L}_X \omega^2 = 0$ for all $X \in \mathcal{K}$. In particular, $\{X \wedge \omega \mid X \in \mathcal{K}\} \oplus d^* \Omega_1^4$ is L^2 -orthogonal to exact forms (and point-wise orthogonal to Ω_{12}^3). As for the L^2 -orthogonality statement, observe that by Remark 3 if $\rho_0 \in \Omega_{12}^3$ is closed then $d^* \rho_0 \in \Omega_8^2$.

Similarly, the decomposition (iii) follows from the identification of D^- with D and Proposition 3.6. Thus every 4-form σ can be written as $\sigma = (X \lrcorner \text{Re } \Omega) \wedge \omega + d(JY \wedge \omega - g \text{Re } \Omega + f \text{Im } \Omega) + \sigma_0$ with $X \in \mathcal{K}$, $Y \in \mathcal{K}^{\perp L^2}$ and $\sigma_0 \in \Omega_8^4$. These constraints uniquely specify (X, Y, f, g, σ_0) up to prescribing $d^*(JY \wedge \omega - g \text{Re } \Omega + f \text{Im } \Omega) \wedge \omega^2$. Now, going back to the first step in the proof of Proposition 3.6 one can show that for every (Y', f') with $Y' \in \mathcal{K}^{\perp L^2}$ there exists a unique choice of g' , namely $g' = \frac{1}{3} d^*(JY')$, such that every solution (Y, f, g) to $D(Y, f, g) = (Y', f', g')$ satisfies $g = 0$.

Finally, (iii) and the fact that $d^*((X \lrcorner \text{Re } \Omega) \wedge \omega) = - * \mathcal{L}_X \text{Re } \Omega = 0$ for every $X \in \mathcal{K}$ imply (iv). As for the L^2 -orthogonality statement, observe that by Proposition 3.1.(iii) if $\sigma_0 \in \Omega_8^4$ is closed then $d^* \sigma_0 \in \Omega_{12}^3$. \square

Proposition 3.7 plays a crucial role in our treatment of the deformation theory of nearly Kähler manifolds. As an additional interesting application, we recover results of Verbitsky [22, Theorem 6.2 and Remark 6.4] on the cohomology of nearly Kähler 6-manifolds. Note that the full information about the cohomology of nearly Kähler manifolds is contained in degree 2 and 3 since $\pi_1(M)$ is finite.

THEOREM 3.8. *Let (M, ω, Ω) be a complete nearly Kähler manifold. Then every harmonic 2-form on M lies in Ω_8^2 and every harmonic 3-form lies in Ω_{12}^3 .*

Proof. If M is diffeomorphic to S^6 the result is vacuous. We can therefore assume that (M, g) is not isometric to the round 6-sphere.

Let η be a closed and coclosed 2-form. By Proposition 3.7.(iii) we can write

$$\eta = X \lrcorner \text{Re } \Omega + d^* \rho + \eta_0,$$

with $X \in \mathcal{K}$, $\rho \in \Omega_{1 \oplus 1 \oplus 6}^3$ and $\eta_0 \in \Omega_8^2$.

Now, on one side $d^* \eta = 0$ is equivalent to $d^* \eta_0 = -d^*(X \lrcorner \text{Re } \Omega) = 6JX$. However,

$$\langle d^* \eta_0, JX \rangle_{L^2} = \langle \eta_0, dJX \rangle_{L^2} = -3 \langle \eta_0, X \lrcorner \text{Re } \Omega \rangle_{L^2} = 0$$

for every Killing field X . Thus $X = 0 = d^* \eta_0$ and therefore $d\eta_0 \in \Omega_{12}^3$ by Proposition 3.1.(iii). Then

$$0 = \langle \eta, d^* \rho \rangle_{L^2} = \|d^* \rho\|_{L^2}^2 + \langle \rho, d\eta_0 \rangle_{L^2} = \|d^* \rho\|_{L^2}^2.$$

Similarly, let ρ be a closed and coclosed 3-form and write

$$\rho = X \wedge \omega + d\eta + d^*(f\omega^2) + \rho_0$$

for $X \in \mathcal{K}$, $\eta \in \Omega_{1 \oplus 6}^2$, $f \in C^\infty$ and $\rho_0 \in \Omega_{12}^3$.

We already observed that $X \wedge \omega$ is L^2 -orthogonal to exact forms whenever X is a Killing field. Furthermore, since in this case $d^* JX = 0$ we also have

$$\langle X \wedge \omega, d^*(f\omega^2) \rangle_{L^2} = \langle d(X \wedge \omega), f\omega^2 \rangle_{L^2} = \langle dX \wedge \omega - 3X \wedge \text{Re } \Omega, f\omega^2 \rangle_{L^2} = 0.$$

Thus integration by parts using the fact that ρ is closed and coclosed yields

$$0 = \langle \rho, d^*(f\omega^2) \rangle_{L^2} = \|d^*(f\omega^2)\|_{L^2}^2 + \langle d\rho_0, f\omega^2 \rangle_{L^2}, \quad 0 = \langle \rho, d\eta \rangle_{L^2} = \|d\eta\|_{L^2}^2 + \langle d^* \rho_0, \eta \rangle_{L^2}. \quad (3.12)$$

Since $\rho_0 \wedge \omega = 0$, differentiation yields $d\rho_0 \wedge \omega = 3\rho_0 \wedge \operatorname{Re} \Omega = 0$ and therefore the first identity in (3.12) implies

$$\rho = X \wedge \omega + d\eta + \rho_0.$$

In particular $d\rho_0 = -d(X \wedge \omega)$. Consider the Ω_6^4 -component of this identity. We have

$$-d(X \wedge \omega) = -dX \wedge \omega + 3X \wedge \operatorname{Re} \Omega = -dX \wedge \omega + 3(JX \lrcorner \operatorname{Re} \Omega) \wedge \omega$$

by (3.2). Moreover, by (3.6) and the fact that $\frac{1}{2}\alpha(dJX) = -3X$ since $X \in \mathcal{K}$ we have

$$-\pi_6(dX \wedge \omega) = -\left(\frac{1}{2}\alpha(dX) \lrcorner \operatorname{Re} \Omega\right) \wedge \omega = (JX \lrcorner \operatorname{Re} \Omega) \wedge \omega.$$

Thus $\pi_6(d\rho_0) = 4(JX \lrcorner \operatorname{Re} \Omega) \wedge \omega$. However, by Lemma 3.2

$$\langle d\rho_0, (JX \lrcorner \operatorname{Re} \Omega) \wedge \omega \rangle_{L^2} = \langle d\rho_0, X \wedge \operatorname{Re} \Omega \rangle_{L^2} = \langle \rho_0, d(X \lrcorner \operatorname{Re} \Omega) \rangle_{L^2} = 0$$

since $X \in \mathcal{K}$. We conclude that $X = 0$ and $\rho = d\eta + \rho_0$.

In particular, $d\rho_0 = 0$. Now, on one side $*(d^*\rho_0 \wedge \omega^2) = -2*(d*\rho_0 \wedge \omega) = 0$ since $*\rho_0 \in \Omega_{12}^3$. On the other side, $d\rho_0 = 0$ and (3.7) imply that $d*\rho_0$ has no Ω_6^4 component. We conclude that $d^*\rho_0 \in \Omega_8^2$ and therefore $\langle d^*\rho_0, \eta \rangle_{L^2} = 0$. In view of the second identity in (3.12) the Theorem is proved. \square

REMARK. Theorem 3.8 has an interesting application, suggested to us by Gonalo Oliveira, to gauge theory on nearly Kähler manifolds. On every nearly Kähler manifold one can define the notion of a *pseudo-Hermitian-Yang-Mills* connection: a connection A on a principal G -bundle whose curvature F_A is a primitive $(1, 1)$ -form with values in the adjoint bundle. The interest in pseudo-Hermitian-Yang-Mills connections on a nearly Kähler manifold M arises from the fact that they correspond to scale-invariant G_2 -instantons on the Riemannian cone over M . In particular, pseudo-Hermitian-Yang-Mills connections on the round S^6 model isolated singularities of G_2 instantons on smooth G_2 manifolds. Now, Theorem 3.8 implies that every line bundle L over a nearly Kähler manifold admits a pseudo-Hermitian-Yang-Mills connection. Indeed, the connection with curvature given by the harmonic representative of $-2\pi i c_1(L)$ is pseudo-Hermitian-Yang-Mills.

4. Deformations of nearly Kähler manifolds

Let (M, ω, Ω) be a nearly Kähler manifold not isometric to the round 6-sphere. We are going to study the problem of deforming (ω, Ω) to a nearby nearly Kähler structure (ω', Ω') .

This will be done in two steps. First we use Proposition 3.7 to define a slice for the action of the diffeomorphism group on the space of $SU(3)$ structures close to (ω, Ω) . This choice of slice will allow us to give a streamlined proof of the identification of the space of infinitesimal nearly Kähler deformations with an eigenspace of the Laplacian acting on coclosed primitive $(1, 1)$ forms, a result due to Moroianu–Nagy–Simmelmann [16].

In order to proceed beyond the linearised level, we find convenient to exploit some observations due to Hitchin [12] to enlarge the spaces under considerations: we will reinterpret the nearly Kähler equations (2.3) as the vanishing of a smooth map on the space of pairs of an exact stable 4-form and an exact stable 3-form, without requiring a priori any compatibility condition. Studying the mapping properties of the linearisation of this map will identify possible obstructions to integrate infinitesimal nearly Kähler deformations to genuine nearly Kähler structures.

4.1. Deformations of $SU(3)$ structures

Let M be a 6-manifold. We denote by \mathcal{C} the space of $SU(3)$ structures on M , i.e. the set of all $(\operatorname{Re} \Omega, \operatorname{Im} \Omega, \frac{1}{2}\omega^2, \omega) \in \Omega^3(M) \times \Omega^3(M) \times \Omega^4(M) \times \Omega^2(M)$ such that $\operatorname{Re} \Omega$ is a stable 3-form with dual $\operatorname{Im} \Omega$, $\frac{1}{2}\omega^2$ is a stable 4-form with dual ω and the constraints $\omega \wedge \operatorname{Re} \Omega = 0 = 4\omega^3 - 3\operatorname{Re} \Omega \wedge \operatorname{Im} \Omega$ are satisfied. We will label an $SU(3)$ structure by $(\operatorname{Re} \Omega, \frac{1}{2}\omega^2)$.

In fact, for the purposes of doing analysis it will be more appropriate to introduce Hölder spaces and consider the subspace $\mathcal{C}^{k,\alpha}$ of $SU(3)$ structures such that ω, Ω are of class $C^{k,\alpha}$ for some $k \geq 1$ and $\alpha \in (0, 1)$. Thus $\mathcal{C}^{k,\alpha}$ is continuously embedded as a submanifold of the space of differential forms of class $C^{k,\alpha}$. For ease of notation we will drop the index k,α when this will not be essential.

Given a point $\mathfrak{c} = (\operatorname{Re} \Omega, \frac{1}{2}\omega^2) \in \mathcal{C}$ we now define the tangent space $T_{\mathfrak{c}}\mathcal{C}$ and an exponential map $\exp: \mathcal{U} \rightarrow \mathcal{C}$ from a sufficiently small neighbourhood \mathcal{U} of the origin in $T_{\mathfrak{c}}\mathcal{C}$.

LEMMA 4.1. $T_{\mathfrak{c}}\mathcal{C}$ is the set of all $(\rho, \hat{\rho}, \sigma, \hat{\sigma}) \in \Omega^3(M) \times \Omega^3(M) \times \Omega^4(M) \times \Omega^2(M)$ such that

$$\begin{aligned} \rho &= -X \wedge \omega + 3\lambda \operatorname{Re} \Omega + 3\mu \operatorname{Im} \Omega + \rho_0, \\ \hat{\rho} &= -JX \wedge \omega - 3\mu \operatorname{Re} \Omega + 3\lambda \operatorname{Im} \Omega - *\rho_0, \\ \sigma &= \hat{\sigma} \wedge \omega, \quad \hat{\sigma} = X \lrcorner \operatorname{Re} \Omega + 2\lambda\omega + \eta_0, \end{aligned}$$

for some $X \in \mathcal{X}(M)$, $\lambda, \mu \in \Omega^0(M)$, $\rho_0 \in \Omega_{12}^3(M)$, $\eta_0 \in \Omega_8^2(M)$.

Proof. The Lemma is a straightforward consequence of the linearisation of the constraints (2.1) and Proposition 2.5. \square

Note that linear $SU(3)$ structures on \mathbb{R}^6 are parametrised by $GL(6, \mathbb{R})/SU(3)$. Following [10], let $GL(TM)$ denote the space of bundle isomorphisms of TM and $SU(TM)$ the subspace of those bundle isomorphisms that preserve $(\operatorname{Re} \Omega, \frac{1}{2}\omega^2)$ at each point. Then $\mathcal{C} \simeq GL(TM)/SU(TM)$ under the map that associate to each $g \in GL(TM)$ the $SU(3)$ structure $g_*(\operatorname{Re} \Omega, \frac{1}{2}\omega^2)$.

From this perspective $T_{\mathfrak{c}}\mathcal{C}$ is given by elements of the form $A_*(\operatorname{Re} \Omega, \frac{1}{2}\omega^2)$ for a section A of $\mathcal{P}_{SU}(M) \times_{SU(3)} \mathfrak{m}$. Here $\mathcal{P}_{SU}(M)$ is the principal $SU(3)$ bundle on M defined by the $SU(3)$ structure \mathfrak{c} , \mathfrak{m} is a complement of $\mathfrak{su}(3)$ in $\mathfrak{gl}(6, \mathbb{R})$ and $\mathcal{P}_{SU}(M) \times_{SU(3)} \mathfrak{m}$ is the bundle associated with the representation \mathfrak{m} of $SU(3)$. For a sufficiently small neighbourhood \mathcal{U} of the origin in $T_{\mathfrak{c}}\mathcal{C}$, these identifications and the exponential map $\mathfrak{m} \rightarrow GL(6, \mathbb{R})$ induce an exponential map

$$\exp: \mathcal{U} \longrightarrow \mathcal{C}, \quad \exp(A_*\mathfrak{c}) = \sum_{n=0}^{\infty} \frac{1}{n!} (A_*)^n \mathfrak{c} = \mathfrak{c} + A_*\mathfrak{c} + \frac{1}{2}A_*A_*\mathfrak{c} + \dots \quad (4.1)$$

which is a homeomorphism onto its image.

4.2. Slice to the action of the diffeomorphism group

The first step in studying the deformation theory of nearly Kähler structures is to find a slice for the action of the diffeomorphism group.

Let $\mathfrak{c} = (\operatorname{Re} \Omega, \frac{1}{2}\omega^2)$ be a nearly Kähler structure such that the induced metric is not isometric to the round metric on S^6 . Denote by $\mathcal{O}_{\mathfrak{c}}$ the orbit of \mathfrak{c} under the action of the group $\operatorname{Diff}_0^{k+1,\alpha}(M)$ of $C^{k+1,\alpha}$ -diffeomorphisms of M isotopic to the identity. The tangent space $T_{\mathfrak{c}}\mathcal{O}_{\mathfrak{c}}$ is the space of Lie derivatives $\mathcal{L}_X(\operatorname{Re} \Omega, \frac{1}{2}\omega^2)$ for $X \in C^{k,\alpha}(TM)$.

We now use Proposition 3.7 to define a complement \mathcal{W} of $T_{\mathfrak{c}}\mathcal{O}_{\mathfrak{c}}$ in $T_{\mathfrak{c}}\mathcal{C}$. Given $(\rho, \hat{\rho}, \sigma, \hat{\sigma}) \in T_{\mathfrak{c}}\mathcal{C}$, write

$$\sigma = -(X \lrcorner \operatorname{Re} \Omega) \wedge \omega + d(JY \wedge \omega - f \operatorname{Im} \Omega) + \sigma_0$$

for unique $X \in \mathcal{K}$, $Y \in \mathcal{K}^{\perp L^2}$, $f \in \Omega^0$ and $\sigma_0 \in \Omega_{\mathfrak{g}}^2$. The term $d(JY \wedge \omega) = \mathcal{L}_Y(\frac{1}{2}\omega^2)$ is a Lie derivative. Thus up to an element in $T_{\mathfrak{c}}\mathcal{O}_{\mathfrak{c}}$ we can assume that

$$\sigma = (-X + \nabla f) \lrcorner \operatorname{Re} \Omega \wedge \omega + 2f\omega^2 + \sigma_0. \quad (4.2a)$$

Then Lemma 4.1 forces

$$\rho = (X + df) \wedge \omega + 3f \operatorname{Re} \Omega + 3g \operatorname{Im} \Omega + \rho_0. \quad (4.2b)$$

for some $\rho_0 \in \Omega_{12}^3$. $\hat{\rho}$, $\hat{\sigma}$ are then determined by ρ and σ via Proposition 2.5.

Hence Proposition 3.7 yields a splitting $T_{\mathfrak{c}}\mathcal{C} = T_{\mathfrak{c}}\mathcal{O}_{\mathfrak{c}} \oplus \mathcal{W}$ where $\mathcal{W} \simeq \mathcal{K} \times \Omega^0(M) \times \Omega^0(M) \times \Omega_{\mathfrak{g}}^2(M) \times \Omega_{12}^3(M)$ is the space of all $(\rho, \hat{\rho}, \sigma, \hat{\sigma}) \in T_{\mathfrak{c}}\mathcal{C}$ with ρ and σ of the form (4.2).

PROPOSITION 4.2 [18, Theorems 3.1.4 and 3.1.7]. *There exists an open neighbourhood $\mathcal{U} \subset \mathcal{W}$ of the origin such that $\mathcal{S} = \exp(\mathcal{U})$ is a slice for the action of $\operatorname{Diff}_0^{k+1, \alpha}(M)$ on a sufficiently small neighbourhood of $\mathfrak{c} = (\operatorname{Re} \Omega, \frac{1}{2}\omega^2) \in \mathcal{C}$.*

4.3. Infinitesimal deformations

With this choice of slice we can easily determine infinitesimal deformations of the nearly Kähler structure \mathfrak{c} , thus recovering the result of Morianu–Nagy–Semmelmann [16, Theorem 4.2] as a simple consequence of Proposition 3.7.

THEOREM 4.3. *Let (M, ω, Ω) be a nearly Kähler manifold non-isometric to the round 6–sphere. Then infinitesimal deformations of the nearly Kähler structure modulo diffeomorphisms are in one to one correspondence with pairs $(\rho_0, \sigma_0) \in \Omega_{12, \text{exact}}^3 \oplus \Omega_{\mathfrak{g}, \text{exact}}^4$ satisfying*

$$-d * \sigma_0 - 3\rho_0 = 0, \quad -d * \rho_0 + 4\sigma_0 = 0. \quad (4.3)$$

Proof. Linearising the nearly Kähler equations (2.3) we see that an infinitesimal deformations $(\rho, \hat{\rho}, \sigma, \hat{\sigma}) \in T_{\mathfrak{c}}\mathcal{C}$ of \mathfrak{c} as an $SU(3)$ structure is an infinitesimal nearly Kähler deformation if

$$d\hat{\sigma} - 3\rho = 0 = d\hat{\rho} + 4\sigma. \quad (4.4)$$

In particular, $(\rho, \sigma) \in \Omega_{\text{exact}}^3 \oplus \Omega_{\text{exact}}^4$. Thus if we further assume that $(\rho, \hat{\rho}, \sigma, \hat{\sigma}) \in \mathcal{W}$ then by Proposition 3.7.(iv) we have

$$\sigma = -d(f \operatorname{Im} \Omega) + \sigma_0$$

with $\sigma_0 \in \Omega_{\mathfrak{g}, \text{exact}}^4$. Lemma 4.1 then implies that

$$\rho = d(f\omega) + g \operatorname{Im} \Omega + \rho_0$$

for some function g and $\rho_0 \in \Omega_{12, \text{exact}}^3$ and $\hat{\sigma} = 2f\omega - \nabla f \lrcorner \operatorname{Re} \Omega - *\sigma_0$.

Since ρ is exact (in particular, closed) and $d\rho_0 \wedge \omega = 0$ for all $\rho_0 \in \Omega_{12}^3$ we have

$$0 = d\rho \wedge \omega = -2g\omega^3$$

and therefore $g = 0$.

Consider now the equation $d\hat{\sigma} - 3\rho = 0$: by Proposition 3.1.(ii) and (iv)

$$0 = \frac{1}{4} \langle d\hat{\sigma} - 3\rho, \operatorname{Re} \Omega \rangle = \frac{1}{2} (d^*df - 6f)$$

and therefore $f = 0$ by Obata's Theorem. \square

As noted in [16, Theorem 4.2], a solution (ρ_0, σ_0) of (4.3) is uniquely determined by the coclosed primitive $(1, 1)$ -form $*\sigma_0 \in \Omega_{\mathfrak{g}}^2$. This satisfies $\Delta(*\sigma_0) = 12(*\sigma_0)$. The space of

infinitesimal nearly Kähler deformations is therefore identified with the intersection of the space of coclosed primitive $(1, 1)$ -forms with the eigenspace of eigenvalue 12 of the Laplacian acting on 2-forms.

4.4. The deformation problem

To proceed further we would like to reformulate the nearly Kähler equations (2.3) as the vanishing of a smooth map Φ on the slice \mathcal{S} . In fact, it will be convenient to enlarge the space \mathcal{C} of $SU(3)$ structures dropping the requirement that the constraints (2.1) are satisfied. This approach to the deformation theory of nearly Kähler manifolds is very natural from the point of view introduced in Hitchin's seminal paper [12].

Hitchin's first observation is that if $\operatorname{Re} \Omega$ is a stable 3-form with dual $\operatorname{Im} \Omega$ and $\frac{1}{2}\omega^2$ is a stable 4-form with dual ω satisfying the nearly Kähler equations (2.3) then $(\operatorname{Re} \Omega, \frac{1}{2}\omega^2)$ defines an $SU(3)$ structure, *i.e.* the compatibility constraints (2.1) are automatically satisfied. Indeed, observe that

$$3\omega \wedge \operatorname{Re} \Omega = \omega \wedge d\omega = d\left(\frac{1}{2}\omega^2\right) = 0.$$

Since $\operatorname{Im} \Omega$ is the dual of $\operatorname{Re} \Omega$ then $\omega \wedge \operatorname{Im} \Omega = 0$ also. Differentiating this identity we obtain

$$0 = d\omega \wedge \operatorname{Im} \Omega + \omega \wedge d\operatorname{Im} \Omega = 3\operatorname{Re} \Omega \wedge \operatorname{Im} \Omega - 2\omega^3.$$

Avoiding to impose the constraints (2.1) from the start let us gain a vector field and a function as additional free parameters.

The appearance of a second vector field as an additional parameter follows from the action of the diffeomorphism group. In order to explain this last point, following Hitchin [12] we interpret nearly Kähler structures as critical points of a diffeomorphism-invariant functional on an open set in the infinite dimensional symplectic vector space $\Omega_{\text{exact}}^3 \oplus \Omega_{\text{exact}}^4$.

Let $\tilde{\mathcal{C}}$ be the space of stable forms $(\operatorname{Re} \Omega, \frac{1}{2}\omega^2) \in \Omega_{\text{exact}}^3 \times \Omega_{\text{exact}}^4$. We denote by $\operatorname{Im} \Omega$ and ω the respective duals. For each element in $\tilde{\mathcal{C}}$ Hitchin defines volume functionals $V_3(\operatorname{Re} \Omega)$ and $V_4(\frac{1}{2}\omega^2)$ such that for any $(\rho, \sigma) \in \Omega_{\text{exact}}^3 \oplus \Omega_{\text{exact}}^4$

$$\delta V_3(\rho) = - \int \operatorname{Im} \Omega \wedge \rho, \quad \delta V_4(\sigma) = \int \omega \wedge \sigma.$$

Furthermore, consider the natural pairing P on $\Omega_{\text{exact}}^3 \times \Omega_{\text{exact}}^4$ defined by

$$P(\rho, \sigma) = \int \alpha \wedge \sigma = - \int \rho \wedge \beta$$

for $d\alpha = \rho$ and $d\beta = \sigma$. Now define a functional $\mathcal{E}: \tilde{\mathcal{C}} \rightarrow \mathbb{R}$ by

$$\mathcal{E}(\operatorname{Re} \Omega, \frac{1}{2}\omega^2) = 3V_3(\operatorname{Re} \Omega) + 4V_4(\frac{1}{2}\omega^2) - 12P(\operatorname{Re} \Omega, \frac{1}{2}\omega^2).$$

Then for every $(\rho, \sigma) \in \Omega_{\text{exact}}^3 \times \Omega_{\text{exact}}^4$ we have

$$\delta \mathcal{E}(\rho, \sigma) = -3 \int (\operatorname{Im} \Omega + 4\beta) \wedge \rho + 4 \int (\omega - 3\alpha) \wedge \sigma$$

where $d\alpha = \operatorname{Re} \Omega$ and $d\beta = \frac{1}{2}\omega^2$. Critical points of \mathcal{E} are therefore pairs $(\operatorname{Re} \Omega, \frac{1}{2}\omega^2) \in \tilde{\mathcal{C}}$ satisfying $d\operatorname{Im} \Omega + 2\omega^2 = 0 = d\omega - 3\operatorname{Re} \Omega = 0$, *i.e.* nearly Kähler structures.

Since \mathcal{E} is diffeomorphism invariant it is clear that $\delta \mathcal{E}$ vanishes in the direction of Lie derivatives at any point in $\tilde{\mathcal{C}}$ and this explain the freedom to throw in the equations an extra vector field.

In the following Proposition we take stock of these observations to rewrite the nearly Kähler equations as the vanishing of a smooth map.

PROPOSITION 4.4. *Suppose that $(\operatorname{Re}\Omega, \frac{1}{2}\omega^2) \in \widetilde{\mathcal{C}}$ satisfies*

$$d\omega - 3\operatorname{Re}\Omega = 0, \quad d\operatorname{Im}\Omega + 2\omega^2 = d * d(Z \lrcorner \operatorname{Re}\Omega)$$

for some vector field Z . Here the Hodge- $*$ is associated with a fixed background metric.

Then $(\operatorname{Re}\Omega, \frac{1}{2}\omega^2)$ defines a nearly Kähler structure.

Proof. It is enough to show that $d(Z \lrcorner \operatorname{Re}\Omega) = 0$.

First of all observe as above that $d\omega - 3\operatorname{Re}\Omega = 0 = d\omega \wedge \omega$ imply that $\omega \wedge \operatorname{Re}\Omega = 0$ and therefore $(Z \lrcorner \operatorname{Re}\Omega) \wedge \omega^2 = 0$.

Moreover, since $d\operatorname{Re}\Omega = 0$ by Proposition 2.6 we also have $\int (Y \lrcorner \operatorname{Re}\Omega) \wedge d\operatorname{Im}\Omega = 0$ for every vector field Y . Indeed, we can conformally rescale ω so that both constraints (2.1) are satisfied. Denote by $\widetilde{\omega}$ this rescaled form. Then $(\widetilde{\omega}, \Omega)$ is an $SU(3)$ structure. In particular, Proposition 2.6 can now be applied: $d\operatorname{Im}\Omega$ has no component of type Ω_6^4 with respect to $(\widetilde{\omega}, \Omega)$. Finally, note that $\int (Y \lrcorner \operatorname{Re}\Omega) \wedge d\operatorname{Im}\Omega = \langle (Y \lrcorner \operatorname{Re}\Omega) \wedge \widetilde{\omega}, d\operatorname{Im}\Omega \rangle_{L^2}$, where the L^2 inner product is computed using the metric induced by $(\widetilde{\omega}, \Omega)$.

Integrating by parts we therefore obtain

$$\|d(Z \lrcorner \operatorname{Re}\Omega)\|_{L^2}^2 = \int (Z \lrcorner \operatorname{Re}\Omega) \wedge (d\operatorname{Im}\Omega + 2\omega^2) = 0.$$

Fix a nearly Kähler structure $(\operatorname{Re}\Omega, \frac{1}{2}\omega^2)$ and assume that the induced metric is not isometric to the round metric on S^6 . For every $\rho \in \Omega_{exact}^3$ and $\sigma \in \Omega_{exact}^4$ sufficiently small in $C^{k,\alpha}$ the forms $\operatorname{Re}\Omega' = \operatorname{Re}\Omega + \rho$ and $\frac{1}{2}\omega^2 + \sigma$ are still stable forms. We therefore have a “linear” exponential map $\widetilde{\exp}$ from a neighbourhood of the origin in $\Omega_{exact}^3 \times \Omega_{exact}^4$ into a sufficiently small neighbourhood of $(\operatorname{Re}\Omega, \frac{1}{2}\omega^2)$ in $\widetilde{\mathcal{C}}$.

Proposition 3.7 could be used to define a slice to the action of the diffeomorphism group also in this case (but we won’t really need this beyond the tangent space level): consider the image under $\widetilde{\exp}$ of a small neighbourhood \mathcal{U} of the origin in the subspace $\widetilde{\mathcal{W}}$ of $\Omega_{exact}^3 \times \Omega_{exact}^4$ consisting of pairs (ρ, σ) where $\sigma = d(g\operatorname{Im}\Omega) + \sigma_0$ with $\sigma_0 \in \Omega_{8,exact}^4$.

Now define a map

$$\begin{aligned} \Phi: \mathcal{U} \times C^{k+1,\alpha}(TM) &\longrightarrow C^{k-1,\alpha}(\Lambda^3 T^*M \oplus \Lambda^4 T^*M), \\ \Phi(\rho, \sigma, Z) &= (d\omega' - 3\operatorname{Re}\Omega', d\operatorname{Im}\Omega' + 2\omega' \wedge \omega' + d * d(Z \lrcorner \operatorname{Re}\Omega')). \end{aligned} \quad (4.5)$$

Here $\operatorname{Im}\Omega'$ is the dual of $\operatorname{Re}\Omega' = \operatorname{Re}\Omega + \rho$, ω' is the dual of $\frac{1}{2}\omega^2 + \sigma$ and the Hodge- $*$ is computed with respect to the metric induced by the nearly Kähler structure $(\operatorname{Re}\Omega, \frac{1}{2}\omega^2)$. By Proposition 4.4 nearly Kähler structures close to $(\operatorname{Re}\Omega, \frac{1}{2}\omega^2)$ are parametrised modulo diffeomorphisms by the zero locus of Φ .

4.5. Obstructions

The linearisation $D\Phi$ of (4.5) at $(\operatorname{Re}\Omega, \frac{1}{2}\omega^2, 0)$ is defined by

$$d\hat{\sigma} - 3\rho, \quad d\hat{\rho} + 4\sigma + d * d(Z \lrcorner \operatorname{Re}\Omega), \quad (4.6)$$

where $(\rho, \sigma) \in \widetilde{\mathcal{W}}$ and $\hat{\rho}, \hat{\sigma}$ are their images under the linearisation of Hitchin’s duality map in Proposition 2.5. Using Proposition 3.7 we can write:

$$\rho = d(f\omega + X \lrcorner \operatorname{Re}\Omega) + \rho_0 \quad (4.7a)$$

with $\rho_0 \in \Omega_{12,exact}^3$ and $X \in \mathcal{K}^{\perp L^2}$ and

$$\hat{\rho} = *d(f\omega) - 4JX \wedge \omega - *\rho_0 + d(X \lrcorner \operatorname{Im}\Omega) = Jdf \wedge \omega + 3f \operatorname{Im}\Omega - 4JX \wedge \omega - *\rho_0 + d(X \lrcorner \operatorname{Im}\Omega) \quad (4.7b)$$

(here we used the fact that $d(X \lrcorner \operatorname{Re} \Omega) = \mathcal{L}_X \operatorname{Re} \Omega$ to calculate the corresponding component $\mathcal{L}_X \operatorname{Im} \Omega$ of $\hat{\rho}$);

$$\sigma = -d(g \operatorname{Im} \Omega) + \sigma_0, \quad (4.7c)$$

$$\hat{\sigma} = - * d(g \operatorname{Im} \Omega) - 2g\omega - * \sigma_0 = -\nabla g \lrcorner \operatorname{Re} \Omega + 2g\omega - * \sigma_0. \quad (4.7d)$$

In order to describe the zero locus of Φ using the Implicit Function Theorem we need to study the mapping properties of $D\Phi$, i.e. given $(\alpha, \beta) \in \Omega_{exact}^3(M) \times \Omega_{exact}^4(M)$ we need to study the equation

$$D\Phi(\rho, \sigma, Z) = (\alpha, \beta).$$

PROPOSITION 4.5. *A pair $(\alpha, \beta) \in \Omega_{exact}^3 \times \Omega_{exact}^4$ lies in the image of $D\Phi$ if and only if*

$$\langle d^* \alpha + 3 * \beta, \xi \rangle_{L^2} = 0$$

for every co-closed primitive $(1, 1)$ form ξ such that $\Delta \xi = 12\xi$.

Proof. We have to solve

$$d\hat{\sigma} - 3\rho = \alpha, \quad d\hat{\rho} + 4\sigma + d * d(Z \lrcorner \operatorname{Re} \Omega) = \beta \quad (4.8)$$

for $\rho, \alpha \in \Omega_{exact}^3$, $\sigma, \beta \in \Omega_{exact}^4$ and a vector field Z .

Using (4.7) we calculate

$$\begin{aligned} d\hat{\sigma} - 3\rho &= d\left(- (3X + \nabla g) \lrcorner \operatorname{Re} \Omega + (2g - 3f)\omega\right) - (d * \sigma_0 + 3\rho_0), \\ d\hat{\rho} + 4\sigma &= d\left((-4JX + J\nabla f) \wedge \omega + (3f - 4g) \operatorname{Im} \Omega\right) + (-d * \rho_0 + 4\sigma_0). \end{aligned}$$

Now set $u = 2g - 3f$, $v = -3f + 4g$ and $U = -3X - \nabla g$. Note that the map $(f, g) \mapsto (u, v)$ is invertible and observe that $-4X + \nabla f = \frac{4}{3}(U - \nabla u) + \nabla v$. Thus

$$\begin{aligned} d\hat{\sigma} - 3\rho &= d\left(U \lrcorner \operatorname{Re} \Omega + u\omega\right) - (d * \sigma_0 + 3\rho_0), \\ d\hat{\rho} + 4\sigma &= d\left(\frac{1}{3}J(4U - 4\nabla u + 3\nabla v) \wedge \omega - v \operatorname{Im} \Omega\right) + (-d * \rho_0 + 4\sigma_0). \end{aligned}$$

Moreover, by Lemma 3.2

$$*d(Z \lrcorner \operatorname{Re} \Omega) = d\left(JZ \lrcorner \operatorname{Re} \Omega + \frac{1}{3}(d^* JZ)\omega\right) - J\left(\alpha(dJZ) + 2Z - \frac{1}{3}J\nabla d^*(JZ)\right) \wedge \omega - (d^* Z) \operatorname{Im} \Omega.$$

Use Proposition 3.7 to write $\alpha = d\eta + \alpha_0$ and $\beta = d\zeta + \beta_0$ with $\eta \in \Omega_{1 \oplus 6}^2$, $\alpha_0 \in \Omega_{8, exact}^2$, $\zeta \in \Omega_{1 \oplus 1 \oplus 6}^3$ with $\zeta \wedge \operatorname{Im} \Omega = 0$ and $\beta_0 \in \Omega_{12, exact}^3$.

We first look for a solution to (4.8) assuming $\alpha_0 = 0 = \beta_0$.

Observe that U and u are uniquely determined by η up to an element of \mathcal{K} . Thus we reduced to study the mapping properties of the operator

$$D': (Z, v) \mapsto \left(\alpha(dJZ) + 2Z - \frac{1}{3}J\nabla d^*(JZ) - \nabla v, d^* Z + v\right)$$

of $\Omega^1 \times \Omega^0$ into itself. By changing v into $-\frac{1}{2}v$ and setting $w = \frac{1}{6}d^*(JZ)$ the equation $D'(Z, v) = (2X_0, \mu_0)$ can be rewritten as

$$\frac{1}{2}\alpha(dJZ) + Z + dv - Jdw = X_0, \quad d^* Z - 2v = \mu_0, \quad d^* JZ - 6w = 0.$$

By Proposition 3.4 D' can therefore be identified with a self-adjoint operator which coincides with \mathcal{D} up to a zeroth order term. The same arguments as in the proof of Proposition 3.6 show that D' has trivial kernel (and cokernel). Indeed, suppose that

$$\frac{1}{2}\alpha(dJZ) + Z + dv - Jdw = 0, \quad d^* Z - 2v = 0, \quad d^* JZ - 6w = 0.$$

Applying $d^* \circ J$ to the first equation and using the third we find $0 = d^* dw + d^*(JZ) = d^* dw + 6w$ and therefore $w = 0$. Thus the first two equations now imply

$$dJZ = -\frac{2}{3}v\omega - (Z + \nabla v) \lrcorner \operatorname{Re} \Omega + \pi_8(dJZ).$$

In particular, (using $|\operatorname{Re} \Omega|^2 = 4$)

$$0 = \frac{1}{2} \langle d(dJZ), \operatorname{Re} \Omega \rangle = -4v + d^*Z + d^*dv = d^*dv - 2v$$

and therefore $v = 0$ also. Finally, since $d^*Z = 0$, by Lemma 2.3.(i) we have

$$2\|Z\|_{L^2}^2 - \|\pi_8(dJZ)\|_{L^2}^2 = \int_M dJZ \wedge dJZ \wedge \omega = 3 \int_M JZ \wedge dJZ \wedge \operatorname{Re} \Omega = 6\|Z\|_{L^2}^2$$

and therefore $Z = 0$.

We have therefore reduced to solve the equation

$$-d^* \sigma_0 - 3\rho_0 = \alpha_0, \quad -d^* \rho_0 + 4\sigma_0 = \beta_0,$$

for $\alpha_0 \in \Omega_{12,exact}^3$ and $\beta_0 \in \Omega_{8,exact}^4$.

Now, let us rewrite this system of equations as a second order PDE for the coclosed primitive $(1, 1)$ -form $\hat{\sigma}_0 = - * \sigma_0$: taking d^* of the first equation and using $*$ of the second one we find

$$d^* d\hat{\sigma}_0 - 12\hat{\sigma}_0 = d^* \alpha_0 + 3 * \beta_0.$$

Conversely, given a solution $\hat{\sigma}_0$ of this equation, set $3\rho_0 = d\hat{\sigma}_0 - \alpha_0 \in \Omega_{12,exact}^3$ to get a solution of the first order system.

It is now clear that a solution of (4.8) exists if and only if $d^* \alpha_0 + 3 * \beta_0$ is L^2 -orthogonal to the space \mathcal{O} of primitive coclosed $(1, 1)$ -forms which are eigenforms for the Laplacian with eigenvalue 12. Furthermore, observe that for every $\eta \in \Omega_{1\oplus 6}^2$ and $\zeta \in \Omega_{1\oplus 1\oplus 6}^3$

$$\langle d^* d\eta, \xi \rangle_{L^2} = 12 \langle \eta, \xi \rangle_{L^2} = 0, \quad \langle *d\zeta, \xi \rangle_{L^2} = -\langle \zeta, *d\xi \rangle_{L^2} = 0$$

for every $\xi \in \mathcal{O}$ since $d\xi \in \Omega_{12}^3$. We therefore conclude that (4.8) is solvable if and only if $d^* \alpha + 3 * \beta$ is L^2 -orthogonal to \mathcal{O} . \square

Proposition 4.5 strongly suggests that the deformation theory of nearly Kähler manifolds is in general obstructed. In the next section we study a concrete example that shows that this is indeed the case.

5. Deformations of the flag manifold

In this final section we study in details an example. In [17] Moroianu–Nagy–Simmelmann study infinitesimal deformations of the homogeneous nearly Kähler manifolds (with the exclusion of S^6 with its standard nearly Kähler structure that has to be considered separately, cf. [16, Theorem 5.1]). They found that $S^3 \times S^3$ and $\mathbb{C}\mathbb{P}^3$ are rigid while the flag manifold $\mathbb{F}_3 = SU(3)/T^2$ admits an 8-dimensional space of infinitesimal nearly Kähler deformations. It has been an open problem to decide whether any of these deformations can be integrated to produce new examples of complete nearly Kähler manifolds. Here we apply the framework developed in the previous section to show that all the infinitesimal deformations of the flag manifolds are obstructed.

5.1. Second order deformations

Inspired by work of Koiso [14], we find convenient to introduce the notion of second order deformations and show that infinitesimal deformations of the flag manifold are obstructed to second order.

DEFINITION 5. Given a nearly Kähler structure $(\text{Re } \Omega_0, \frac{1}{2}\omega_0^2)$ and an infinitesimal nearly Kähler deformation (ρ_1, σ_1) a *second order deformation* in the direction of (ρ_1, σ_1) is a 4-tuple $(\rho_2, \rho'_2, \sigma_2, \sigma'_2) \in \Omega^3(M) \times \Omega^3(M) \times \Omega^4(M) \times \Omega^2(M)$ such that

$$\begin{aligned} \text{Re } \Omega &= \text{Re } \Omega_0 + \epsilon \rho_1 + \frac{1}{2}\epsilon^2 \rho_2, & \text{Im } \Omega &= \text{Im } \Omega_0 + \epsilon \hat{\rho}_1 + \frac{1}{2}\epsilon^2 \rho'_2, \\ \frac{1}{2}\omega^2 &= \frac{1}{2}\omega_0^2 + \epsilon \sigma_1 + \frac{1}{2}\epsilon^2 \sigma_2, & \omega &= \omega_0 + \epsilon \hat{\sigma}_1 + \frac{1}{2}\epsilon^2 \sigma'_2, \end{aligned}$$

is a nearly Kähler structure up to terms of order $o(\epsilon^2)$. We say that the infinitesimal deformation (ρ_1, σ_1) is *obstructed to second order* if there exists no second order deformation in its direction.

Second order deformations coincide with the second derivative of a curve of nearly Kähler structures.

PROPOSITION 5.1. Given a nearly Kähler structure $(\text{Re } \Omega_0, \frac{1}{2}\omega_0^2)$ and an infinitesimal nearly Kähler deformation (ρ_1, σ_1) suppose that there exists a curve $(\text{Re } \Omega_\epsilon, \frac{1}{2}\omega_\epsilon^2)$ of nearly Kähler structures such that

$$(\text{Re } \Omega_\epsilon, \frac{1}{2}\omega_\epsilon^2)|_{\epsilon=0} = (\text{Re } \Omega_0, \frac{1}{2}\omega_0^2), \quad \frac{d}{d\epsilon}(\text{Re } \Omega_\epsilon, \frac{1}{2}\omega_\epsilon^2)|_{\epsilon=0} = (\rho_1, \sigma_1).$$

Then $\frac{d^2}{d\epsilon^2}(\text{Re } \Omega_\epsilon, \frac{1}{2}\omega_\epsilon^2)|_{\epsilon=0}$ defines a second order deformation in the direction of (ρ_1, σ_1) .

In particular, an infinitesimal nearly Kähler deformation obstructed to second order cannot be integrated to a curve of nearly Kähler structures.

The previous section suggests the following approach to find second (and higher) order deformations of a nearly Kähler structure $(\text{Re } \Omega_0, \frac{1}{2}\omega_0^2)$. Namely, we look for (formal) power series defining stable exact forms

$$\begin{aligned} \text{Re } \Omega_\epsilon &= \text{Re } \Omega_0 + \epsilon \rho_1 + \frac{1}{2}\epsilon^2 \rho_2 + \dots, \\ \frac{1}{2}\omega_\epsilon^2 &= \frac{1}{2}\omega_0^2 + \epsilon \sigma_1 + \frac{1}{2}\epsilon^2 \sigma_2 + \dots \end{aligned}$$

with $\rho_i \in \Omega_{exact}^3$ and $\sigma_i \in \Omega_{exact}^4$, and a vector field

$$Z_\epsilon = \epsilon Z_1 + \frac{1}{2}\epsilon^2 Z_2 + \dots,$$

satisfying the equation in Proposition 4.4, i.e.

$$d\omega_\epsilon - 3 \text{Re } \Omega_\epsilon = 0, \quad d \text{Im } \Omega_\epsilon + 4 \left(\frac{1}{2}\omega_\epsilon^2 \right) + d * d(Z_\epsilon \lrcorner \text{Re } \Omega_\epsilon) = 0, \quad (5.1)$$

where $\text{Im } \Omega_\epsilon$ and ω_ϵ are the duals of $\text{Re } \Omega_\epsilon$ and $\frac{1}{2}\omega_\epsilon^2$ respectively.

Given an infinitesimal nearly Kähler deformation (ρ_1, σ_1) , set $Z_1 = 0$ and look for $(\rho_2, \sigma_2) \in \Omega_{exact}^3 \times \Omega_{exact}^4$ such that (5.1) are satisfied up to terms of order $O(\epsilon^3)$. Explicitly, we can write the duals $\text{Im } \Omega_\epsilon$ and ω_ϵ of $\text{Re } \Omega_\epsilon$ and $\frac{1}{2}\omega_\epsilon^2$, respectively, as

$$\text{Im } \Omega_\epsilon = \text{Im } \Omega_0 + \epsilon \hat{\rho}_1 + \frac{1}{2}\epsilon^2 (\hat{\rho}_2 - Q_3(\rho_1)) + O(\epsilon^3), \quad \omega_\epsilon = \omega_0 + \epsilon \hat{\sigma}_1 + \frac{1}{2}\epsilon^2 (\hat{\sigma}_2 - Q_4(\sigma_1)) + O(\epsilon^3),$$

where $\hat{\cdot}$ denotes the linearisation of Hitchin's duality map for stable forms in Proposition 2.5 and $Q_3(\rho_1), Q_4(\sigma_1)$ are quadratic expressions yielding the quadratic term of Hitchin's duality map. We therefore look for a solution (ρ_2, σ_2, Z_2) of

$$d\hat{\sigma}_2 - 3\rho_2 = dQ_4(\sigma_1), \quad d\hat{\rho}_2 + 4\sigma_2 + d * d(Z_2 \lrcorner \text{Re } \Omega_0) = dQ_3(\rho_1). \quad (5.2)$$

By the previous section we know that there are obstructions to solve these equations. Assume nonetheless that a solution exists. Then we show that (ρ_2, σ_2) defines a second order deformation in the sense of Definition 5.

LEMMA 5.2. Assume a solution (ρ_2, σ_2, Z_2) to (5.2) exists. Then $d(Z_2 \lrcorner \text{Re } \Omega_0) = 0$ and

$$\left(\rho_2, \hat{\rho}_2 - Q_3(\rho_1), \sigma_2, \hat{\sigma}_2 - Q_4(\sigma_1) \right)$$

defines a second order deformation in the direction of (ρ_1, σ_1) .

Proof. We need to show that $d(Z_2 \lrcorner \text{Re } \Omega_0) = 0$ and that the compatibility constraints (2.1) are satisfied up to order $O(\epsilon^3)$. The proof is similar to the one of Proposition 4.4.

First observe that since ω_ϵ is the dual of $\frac{1}{2}\omega_\epsilon^2$, as the notation indicates $\frac{1}{2}\omega_\epsilon^2$ is proportional to the square of ω_ϵ . In particular,

$$\hat{\sigma}_1 \wedge \hat{\sigma}_1 + \omega_0 \wedge \hat{\sigma}_2 - \omega_0 \wedge Q_4(\sigma_1) = \sigma_2 \in \Omega_{exact}^4.$$

Hence, using $d\omega_0 = 3 \text{Re } \Omega_0$, $d\hat{\sigma}_1 = 3\rho_1$ and the first equation in (5.2) we obtain

$$\begin{aligned} 3\left(\hat{\sigma}_1 \wedge \rho_1 + \omega_0 \wedge \rho_2 + \hat{\sigma}_2 \wedge \text{Re } \Omega_0 - Q_4(\sigma_1) \wedge \text{Re } \Omega_0\right) &= d\left(\hat{\sigma}_1 \wedge \hat{\sigma}_1 + \omega_0 \wedge \hat{\sigma}_2 - \omega_0 \wedge Q_4(\sigma_1)\right) \\ &= d\sigma_2 = 0, \end{aligned}$$

i.e. $\omega_\epsilon \wedge \text{Re } \Omega_\epsilon = O(\epsilon^3)$.

In particular, for every vector field Y we have $(Y \lrcorner \text{Re } \Omega_\epsilon) \wedge \frac{1}{2}\omega_\epsilon^2 = O(\epsilon^3)$, i.e.

$$8\sigma_1 \wedge (Z \lrcorner \rho_1) + 4\sigma_2 \wedge (Y \lrcorner \text{Re } \Omega_0) + 2\omega_0^2 \wedge (Y \lrcorner \rho_2) = 0.$$

Moreover, by Proposition 2.6 the fact that $d \text{Re } \Omega_\epsilon = O(\epsilon^3)$ implies that $\int_M d \text{Im } \Omega_\epsilon \wedge (Y \lrcorner \text{Re } \Omega_\epsilon) = O(\epsilon^3)$ for every vector field Y , i.e.

$$\int_M d \text{Im } \Omega_0 \wedge (Y \lrcorner \rho_2) + 2d\hat{\rho}_1 \wedge (Y \lrcorner \rho_1) + d(\hat{\rho}_2 - Q_3(\rho_1)) \wedge (Y \lrcorner \text{Re } \Omega_0) = 0.$$

Using $d \text{Im } \Omega_0 = -2\omega_0^2$ and $d\hat{\rho}_1 = -4\sigma_1$ we therefore have

$$\int_M d(\hat{\rho}_2 - Q_3(\rho_1)) \wedge (Y \lrcorner \text{Re } \Omega_0) = \int_M 8\sigma_1 \wedge (Y \lrcorner \rho_1) + 2\omega_0^2 \wedge (Y \lrcorner \rho_2).$$

We can now show that $d(Z_2 \lrcorner \text{Re } \Omega_0) = 0$. Indeed,

$$\begin{aligned} \|d(Z_2 \lrcorner \text{Re } \Omega_0)\|_{L^2}^2 &= - \int_M (Z \lrcorner \text{Re } \Omega_0) \wedge \left(d(\hat{\rho}_2 - Q_3(\rho_1)) + 4\sigma_2 \right) \\ &= - \int_M 8\sigma_1 \wedge (Z \lrcorner \rho_1) + 4\sigma_2 \wedge (Y \lrcorner \text{Re } \Omega_0) + 2\omega_0^2 \wedge (Y \lrcorner \rho_2) = 0 \end{aligned}$$

Finally, since $\text{Im } \Omega_\epsilon$ is the dual of $\text{Re } \Omega_\epsilon$ and $\omega_\epsilon \wedge \text{Re } \Omega_\epsilon = O(\epsilon^3)$ then $\omega_\epsilon \wedge \text{Im } \Omega_\epsilon = O(\epsilon^3)$ also, i.e.

$$2\hat{\sigma}_1 \wedge \hat{\rho}_1 + \omega_0 \wedge \left(\hat{\rho}_2 - Q_3(\rho_1) \right) + \left(\hat{\sigma}_2 - Q_4(\sigma_1) \right) \wedge \text{Im } \Omega_0 = 0.$$

Taking the differential of this expression we find

$$6\rho_1 \wedge \hat{\rho}_1 - 8\hat{\sigma}_1 \wedge \sigma_1 + 3 \text{Re } \Omega_0 \wedge \left(\hat{\rho}_2 - Q_3(\rho_1) \right) - 4\omega_0 \wedge \sigma_2 + 3\rho_2 \wedge \text{Im } \Omega_0 - 2\left(\hat{\sigma}_2 - Q_4(\sigma_1) \right) \wedge \omega_0^2 = 0,$$

where we used the fact that $d\hat{\rho}_2 + 4\sigma_2 = dQ_3(\rho_1)$. Up to a constant factor the RHS is exactly the order ϵ^2 coefficient of $3 \text{Re } \Omega_\epsilon \wedge \text{Im } \Omega_\epsilon - 2\omega_\epsilon^3$ and therefore the proof is complete. \square

Conversely, every second order deformation $(\rho_2, \rho'_2, \sigma_2, \sigma'_2)$ in the sense of Definition 5 satisfies $d\sigma'_2 - 3\rho_2 = 0$ and $d\rho'_2 + 4\sigma_2 = 0$ and the constraints (2.1) force $\rho'_2 = \hat{\rho}_2 - Q_3(\rho_1)$ and $\sigma'_2 = \hat{\sigma}_2 - Q_4(\sigma_1)$. Thus we have a complete one-to-one correspondence between second order deformations in the sense of Definition 5 and solutions to (5.2).

We therefore reduced to study the solvability of the equation

$$d\hat{\sigma} - 3\rho = dQ_4(\sigma_1), \quad d\hat{\rho} + 4\sigma + d * d(Z \lrcorner \text{Re } \Omega_0) = dQ_3(\rho_1).$$

By Proposition 5.3 we know that there are obstructions to solve these equations: a solution exists if and only if

$$\langle d^*dQ_4(\sigma_1) + 3 * dQ_3(\rho_1), \hat{\sigma} \rangle_{L^2} = 12 \langle Q_4(\sigma_1), \hat{\sigma} \rangle_{L^2} - 3 \langle Q_3(\rho_1), *d\hat{\sigma} \rangle_{L^2} = 0 \quad (5.3)$$

for every coclosed primitive $(1, 1)$ form $\hat{\sigma}$ such that $\Delta\hat{\sigma} = 12\hat{\sigma}$.

5.2. Infinitesimal deformations of the flag manifold

We now recall the result of Moroianu–Semmelmann [17, Corollary 6.1] on infinitesimal nearly Kähler deformations of the flag manifold \mathbb{F}_3 . We will then show that these infinitesimal deformations are all obstructed to second order.

Introduce left-invariant 1-forms e_1, \dots, e_6 on \mathbb{F}_3 dual to the vector fields

$$E_{12} - E_{21}, \quad i(E_{12} + E_{21}), \quad E_{13} - E_{31}, \quad i(E_{13} + E_{31}), \quad E_{23} - E_{32}, \quad i(E_{23} + E_{32}),$$

where E_{ij} is the 3×3 matrix with 1 in position ij and all other entries zero.

The complex 1-forms

$$\theta_1 = e_2 - ie_1, \quad \theta_2 = e_4 + ie_3, \quad \theta_3 = e_6 - ie_5$$

span the space of $(1, 0)$ forms on \mathbb{F}_3 with respect to the almost complex structure J_0 induced by the homogeneous nearly Kähler structure. The nearly Kähler structure (ω_0, Ω_0) is given by

$$\omega_0 = \frac{i}{2}(\theta_1 \wedge \bar{\theta}_1 + \theta_2 \wedge \bar{\theta}_2 + \theta_3 \wedge \bar{\theta}_3), \quad \Omega_0 = \theta_1 \wedge \theta_2 \wedge \theta_3.$$

Given an element $\xi \in \mathfrak{su}_3$ define functions $v_1, v_2, v_3, x_1, \dots, x_6$ on \mathbb{F}_3 by

$$g^{-1}\xi g = \begin{pmatrix} \frac{i}{2}v_1 & x_1 + ix_2 & x_3 + ix_4 \\ -x_1 + ix_2 & \frac{i}{2}v_2 & x_5 + ix_6 \\ -x_3 + ix_4 & -x_5 + ix_6 & \frac{i}{2}v_3 \end{pmatrix} = i \begin{pmatrix} \frac{1}{2}v_1 & \bar{z}_1 & z_2 \\ z_1 & \frac{1}{2}v_2 & \bar{z}_3 \\ \bar{z}_2 & z_3 & \frac{1}{2}v_3 \end{pmatrix},$$

where

$$z_1 = x_2 + ix_1, \quad z_2 = x_4 - ix_3, \quad z_3 = x_6 + ix_5.$$

Here $g \in SU(3)$ but the functions v_1, \dots, v_3 descends to functions on the flag manifold $\mathbb{F}_3 = SU(3)/T^2$.

Moroianu–Semmelmann show that for each element $\xi \in \mathfrak{su}_3$ the 2-form

$$\hat{\sigma}_\xi = \frac{i}{2}(v_3 \theta_1 \wedge \bar{\theta}_1 + v_2 \theta_2 \wedge \bar{\theta}_2 + v_1 \theta_3 \wedge \bar{\theta}_3)$$

is a primitive coclosed $(1, 1)$ form with $\Delta\hat{\sigma}_\xi = 12\hat{\sigma}_\xi$.

Define ρ_ξ by $d\hat{\sigma}_\xi = 3\rho_\xi$. Since $v_1 + v_2 + v_3 = 0$ and $d(\theta_1 \wedge \bar{\theta}_1) = d(\theta_2 \wedge \bar{\theta}_2) = d(\theta_3 \wedge \bar{\theta}_3)$ by [17, Equation (38)], we have

$$d\hat{\sigma}_\xi = \frac{i}{2}(dv_3 \wedge \theta_1 \wedge \bar{\theta}_1 + dv_2 \wedge \theta_2 \wedge \bar{\theta}_2 + dv_1 \wedge \theta_3 \wedge \bar{\theta}_3).$$

Moreover, using [17, Equation (40)] and the complex notation we have introduced, we compute

$$dv_1 = \text{Im}(z_1\theta_1 - z_2\theta_2), \quad dv_2 = \text{Im}(z_3\theta_3 - z_1\theta_1), \quad dv_3 = \text{Im}(z_2\theta_2 - z_3\theta_3).$$

Hence

$$\rho_\xi = -\frac{1}{6} \text{Re}(\theta_1 \wedge \theta_2 \wedge \alpha_3 + \theta_2 \wedge \theta_3 \wedge \alpha_1 + \theta_3 \wedge \theta_1 \wedge \alpha_2), \quad \alpha_i = z_j\bar{\theta}_k + z_k\bar{\theta}_j, \quad (5.4)$$

where for each $i = 1, 2, 3$ the two indices j, k are chosen so that $\epsilon_{ijk} = 1$.

Set $\sigma_\xi = \hat{\sigma}_\xi \wedge \omega_0$. By Theorem 4.3 the pair (ρ_ξ, σ_ξ) is an infinitesimal nearly Kähler deformation.

5.3. The quadratic term

In view of (5.3), we have now to calculate the quadratic terms $Q_4(\sigma_\xi)$ and $Q_3(\rho_\xi)$.

The 2-form $Q_4(\sigma_\xi)$ is implicitly defined as follows. Given ξ and chosen $\epsilon > 0$ sufficiently small, consider the stable 4-form $\frac{1}{2}\omega_\epsilon = \frac{1}{2}\omega_0^2 + \epsilon\sigma_\xi$ and its dual

$$\omega_\epsilon = \omega_0 + \epsilon\hat{\sigma}_\xi - \frac{\epsilon^2}{2}Q_4(\hat{\sigma}_\xi) + O(\epsilon^3).$$

As the notation suggests $\frac{1}{2}\omega_\epsilon^2$ is proportional to the square of ω_ϵ . In particular,

$$\hat{\sigma}_\xi \wedge \hat{\sigma}_\xi - \omega_0 \wedge Q_4(\sigma_\xi) = 0. \quad (5.5)$$

From this formula we could calculate explicitly $Q_4(\sigma_\xi)$ but we will see below that this is not necessary.

Similarly, $Q_3(\rho_\xi)$ is defined so that

$$\text{Im } \Omega_0 - \epsilon * \rho_\xi - \frac{\epsilon^2}{2}Q_3(\rho_\xi) + O(\epsilon^3)$$

is the dual of the stable 3-form $\text{Re } \Omega_0 + \epsilon\rho_\xi$. While it is possible to compute $Q_3(\rho_\xi)$ explicitly following the algorithm to construct the dual of a stable 3-form [12, Equation (2) and §8.2] we find it quicker to produce by hand a complex volume form Ω_ϵ such that

$$\text{Re } \Omega_\epsilon = \text{Re } \Omega_0 + \epsilon\rho_\xi + O(\epsilon^3).$$

Then the quadratic term of $\text{Im } \Omega_\epsilon$ will yield an explicit expression for $Q_3(\rho_\xi)$.

Consider the complex 1-forms

$$\theta_i^\epsilon = \theta_i - \frac{\epsilon}{6}(z_k\bar{\theta}_j + z_j\bar{\theta}_k) + \frac{\epsilon^2}{36}\bar{z}_i(\bar{z}_i\bar{\theta}_i - \bar{z}_j\bar{\theta}_j - \bar{z}_k\bar{\theta}_k),$$

where as before for each $i = 1, 2, 3$ the two indices j, k are chosen so that $\epsilon_{ijk} = 1$. For ϵ sufficiently small $\theta_1^\epsilon, \theta_2^\epsilon, \theta_3^\epsilon$ span a 3-dimensional subspace of the space of complex 1-forms and therefore define an almost complex structure. We now set $\Omega_\epsilon = \theta_1^\epsilon \wedge \theta_2^\epsilon \wedge \theta_3^\epsilon$.

Expanding $\Omega_\epsilon = \Omega_0 + \epsilon\Omega_1 + \epsilon^2\Omega_2 + O(\epsilon^3)$, one can check that $\rho_\xi = \text{Re } \Omega_1$ and $\text{Re } \Omega_2 = 0$. Thus $Q_3(\rho_\xi) = -2\text{Im } \Omega_2$. Via a straightforward calculation we conclude that

$$9Q_3(\rho_\xi) = \text{Im}(\bar{\theta}_1 \wedge \bar{\theta}_2 \wedge \beta_3 + \bar{\theta}_2 \wedge \bar{\theta}_3 \wedge \beta_1 + \bar{\theta}_3 \wedge \bar{\theta}_1 \wedge \beta_2), \quad \beta_i = z_i(z_i\theta_i - z_j\theta_j - z_k\theta_k), \quad (5.6)$$

with the usual convention for indices.

5.4. Representation theoretic considerations

We now use basic representation theoretic observations to reduce to a minimum the number of calculations necessary to prove that infinitesimal deformations of the flag manifold are obstructed to second order.

By (5.3) for every $\xi \in \mathfrak{su}_3$ the infinitesimal nearly Kähler deformation (ρ_ξ, σ_ξ) is integrable to second order if and only if

$$\Phi(\xi, \xi') := \langle 12Q_4(\sigma_\xi), \hat{\sigma}_{\xi'} \rangle_{L^2} - 3\langle Q_3(\rho_\xi), *d\hat{\sigma}_{\xi'} \rangle_{L^2} = 0$$

for every $\xi' \in \mathfrak{su}_3$.

By invariance of the nearly Kähler structure with respect to the natural action of $SU(3)$ on \mathbb{F}_3 , Φ defines an Ad-invariant map $\Phi: \mathfrak{su}_3 \times \mathfrak{su}_3 \rightarrow \mathbb{R}$ which is quadratic in the first argument and linear in the second. It must therefore correspond to an element of $\text{Hom}_{SU(3)}(\mathfrak{su}_3, \text{Sym}^2(\mathfrak{su}_3))$. This is a one dimensional space. A generator $\xi \mapsto L_\xi$ can be defined as follows. Identify \mathfrak{su}_3 with $\Lambda_3^2(\mathbb{R}^6)^*$. Given $\xi, \xi' \in \mathfrak{su}_3$ we have

$$\xi \wedge \xi' = -\frac{1}{6}\langle \xi, \xi' \rangle \omega_0 + \omega_0 \wedge L_\xi(\xi')$$

for a unique primitive (1, 1) form $L_\xi(\xi')$. Here ω_0 is the standard $SU(3)$ -invariant Kähler form on $\mathbb{R}^6 \simeq \mathbb{C}^3$. We therefore conclude that $\Phi(\xi, \xi') = C\Phi_0(\xi^2, \xi')$ for some constant $C \in \mathbb{R}$. Here $\xi^2 \in \text{Sym}^2(\mathfrak{su}_3)$ and

$$\Phi_0(\xi^2, \xi') = \langle L_{\xi'}, \langle \xi, \cdot \rangle \xi \rangle.$$

Furthermore, we will now argue that in order to determine whether $C \neq 0$ it is enough to calculate the numbers $\Phi(\xi, \xi)$ for $\xi \in \mathfrak{su}_3$. To this end, it is enough to show that $\Phi_0(\xi^2, \xi)$ is a non-zero multiple of the unique cubic invariant polynomial on \mathfrak{su}_3 :

$$\Phi_0(\xi^2, \xi) = \langle L_\xi, \langle \xi, \cdot \rangle \xi \rangle = \sum_{i=1}^8 \langle L_\xi(e_i), \langle \xi, e_i \rangle \xi \rangle = \langle L_\xi(\xi), \xi \rangle = \xi \wedge *L_\xi(\xi) = -\xi \wedge \xi \wedge \xi$$

by Lemma 2.2.(iii). Here e_1, \dots, e_8 is an orthonormal basis of \mathfrak{su}_3 .

5.5. Obstructed deformations

We have now all the ingredients to prove the main theorem of this section.

THEOREM 5.3. *The infinitesimal deformations of the homogeneous nearly Kähler structure on the flag manifold \mathbb{F}_3 are all obstructed.*

Proof. By Proposition 5.1 it is enough to show that the infinitesimal deformations are obstructed to second order in the sense of Definition 5.

Consider the map $\Phi: \mathfrak{su}_3 \times \mathfrak{su}_3 \rightarrow \mathbb{R}$ introduced in the previous section and in particular its values on vectors of the form (ξ, ξ) with $\xi \in \mathfrak{su}_3$. We know that $\Phi(\xi, \xi) = iC \det(\xi)$ for some $C \in \mathbb{R}$. We want to show that $C \neq 0$.

Now, $\Phi(\xi, \xi) = \int_{\mathbb{F}_3} f_\xi([g]) dv$, where $[g] \in \mathbb{F}_3 = SU(3)/T^2$ and f_ξ is the function on \mathbb{F}_3 defined by

$$f_\xi([g]) = \langle 12Q_4(\sigma_\xi), \hat{\sigma}_\xi \rangle - 3\langle Q_3(\rho_\xi), *d\hat{\sigma}_\xi \rangle.$$

By the definition of $\hat{\sigma}_\xi$ and $3\rho_\xi = d\hat{\sigma}_\xi$ in terms of the functions v_1, \dots, v_6 , it is clear that $f_\xi([h^{-1}g]) = f_{h\xi h^{-1}}([g])$. We therefore conclude that $f_\xi([g]) = P(g^{-1}\xi g)$, where P is a polynomial of degree 3 on \mathfrak{su}_3 .

In fact we can use our formulas for $Q_4(\sigma_\xi)$ and $Q_3(\rho_\xi)$ to explicitly calculate P . Indeed, using $d\hat{\sigma}_\xi = 3\rho_\xi$ we have

$$f_\xi([g]) dv = 12 Q_4(\sigma_\xi) \wedge * \hat{\sigma}_\xi + 9 Q_3(\rho_\xi) \wedge \rho_\xi.$$

Now use (5.5) and the fact that $\hat{\sigma}_\xi \in \Omega_8^2$ to write

$$12 Q_4(\sigma_\xi) \wedge * \hat{\sigma}_\xi = -12 Q_4(\sigma_\xi) \wedge \omega_0 \wedge \hat{\sigma}_\xi = -12 \hat{\sigma}_\xi \wedge \hat{\sigma}_\xi \wedge \hat{\sigma}_\xi = -12 v_1 v_2 v_3 \omega_0^3.$$

On the other hand, using (5.4) and (5.6) we calculate

$$\begin{aligned} 9 Q_3(\rho_\xi) \wedge \rho_\xi &= -\frac{1}{6} \operatorname{Im} \left(\sum_{i=1}^3 \bar{\theta}_j \wedge \bar{\theta}_k \wedge \beta_i \right) \wedge \operatorname{Re} \left(\sum_{i=1}^3 \theta_j \wedge \theta_k \wedge \alpha_i \right) = \\ &= -\frac{1}{12} \operatorname{Im} \left((\bar{\theta}_1 \wedge \bar{\theta}_2 \wedge \beta_3 + \bar{\theta}_2 \wedge \bar{\theta}_3 \wedge \beta_1 + \bar{\theta}_3 \wedge \bar{\theta}_1 \wedge \beta_2) \wedge (\theta_1 \wedge \theta_2 \wedge \alpha_3 + \theta_2 \wedge \theta_3 \wedge \alpha_1 + \theta_3 \wedge \theta_1 \wedge \alpha_2) \right) \\ &= \frac{1}{2} \operatorname{Im} (z_1 z_2 z_3 \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \bar{\theta}_1 \wedge \bar{\theta}_2 \wedge \bar{\theta}_3) = -\operatorname{Re}(z_1 z_2 z_3) \operatorname{Re} \Omega_0 \wedge \operatorname{Im} \Omega_0, \end{aligned}$$

since $\Omega_0 = \theta_1 \wedge \theta_2 \wedge \theta_3$ and $\Omega_0 \wedge \bar{\Omega}_0 = -2i \operatorname{Re} \Omega_0 \wedge \operatorname{Im} \Omega_0$.

We conclude that

$$f_\xi([g]) = P(g^{-1}\xi g) = -72 v_1 v_2 v_3 - 4 \operatorname{Re}(z_1 z_2 z_3).$$

In order to calculate the mean value of P on \mathbb{F}_3 we appeal to the Peter–Weyl Theorem.

First of all observe that we can lift f_ξ to $SU(3)$ as a T^2 -invariant function and calculate $\Phi(\xi, \xi)$ up to a positive factor by considering the mean value of f_ξ on $SU(3)$. Indeed, by [17, Lemma 5.4] the nearly Kähler metric on \mathbb{F}_3 is induced by $-\frac{1}{12}B$, where B is the Killing form of $SU(3)$.

The Peter-Weyl Theorem says that for any compact Lie group G

$$L^2(G) = \bigoplus_{\gamma \in \hat{G}} V_\gamma \otimes V_\gamma^*,$$

where \hat{G} is the set of (non-isomorphic) irreducible G -representations. The Peter-Weyl isomorphism is explicit: to a pair $v \otimes \alpha \in V_\gamma \otimes V_\gamma^*$ we associate the function $f(g) = \alpha(\gamma(g^{-1})v)$. Moreover, each summand of the Peter-Weyl decomposition is an eigenspace for the Laplacian Δ_G with eigenvalue that can be determined from the highest weight γ .

Now, the decomposition of $Sym^3(\mathfrak{su}_3)$ into irreducible representations of $SU(3)$ contains a unique copy of the trivial representation, corresponding to the unique cubic Ad-invariant polynomial $i \det$ on \mathfrak{su}_3 . Thus we write $Sym^3(\mathfrak{su}_3) = V_0 \oplus \cdots \oplus V_n$ with $V_0 \simeq \mathbb{R}$ and V_i , $i > 0$, non trivial representations γ_i (it is not important what these actually are nor that each of these appears without multiplicity). We must have $P(g^{-1}\xi g) = \sum_{i=0}^n \alpha_i (\gamma_i(g^{-1})(\xi^3)_i)$ for some $\alpha_i \in V_i^*$. Here for every $\xi \in \mathfrak{su}_3$ we write $\xi^3 = (\xi^3)_0 + \cdots + (\xi^3)_n$ according to the decomposition of $Sym^3(\mathfrak{su}_3)$. In this way we obtain the decomposition of f_ξ into eigenspaces for the Laplacian on $SU(3)$. In particular, we can compute the mean value of f_ξ simply by calculating the inner product of the cubic polynomial P with the unique invariant cubic polynomial $i \det$.

Now, $i \det(g^{-1}\xi g) = \frac{1}{8}v_1v_2v_3 - \frac{1}{2}(v_3|z_1|^2 + v_2|z_2|^2 + v_1|z_3|^2) + 2\operatorname{Re}(z_1z_2z_3)$. Since the monomials $v_1v_2v_3$ and $\operatorname{Re}(z_1z_2z_3)$ appear with coefficients of the same sign both in P and $i \det$ and $\operatorname{Span}(e_1, \dots, e_6)$ is orthogonal to the sub-algebra of diagonal matrices (so that $v_1v_2v_3$ and $\operatorname{Re}(z_1z_2z_3)$ are orthogonal polynomials), we conclude that P has a non-zero inner product with $i \det$ and the proof is complete. \square

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