# SECOND-ORDER PARTIAL STANDING WAVE SOLUTION
# FOR A SLOPING BOTTOM

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This paper presents a second-order asymptotic solution in Lagrangian description for a nonlinear partial standing wave over a sloping bottom. The particle trajectories are obtained as a function of the nonlinear ordering parameters, wave steepness $\varepsilon$ and the bottom slope $\alpha$, to the second order. The analytical Lagrangian solution assumes irrotational flow and satisfies the boundary condition of constant pressure $p = 0$ at the free surface. This solution is applicable to progressive, standing and partial standing waves, shoaling from deep to shallow water. Mass transport and particle trajectory nonlinear partial standing waves on a sloping bottom are investigated using the closed form Lagrangian wave solution

*Keywords: Lagrangian solution; partial standing wave; sloping bottom; mass transport, particle trajectory; nonlinear waves.*

## 1. INTRODUCTION

The investigation of partial standing wave is a long-standing problem in ocean science and engineering (e.g. Rayleigh 1883; Longuet-Higgins 1953; Ünlüata and Mei 1970; Ng 2004b). Hughes and fowler (1995) described a partial standing waves on a horizontal bottom in front of an inclined coastal structure. Partial standing waves may also occur at the slope a coastal structure located in relatively deep water under moderate wave conditions. Brunoone and Tomasicchio (1997) analyze the the measured velocity distribution in a simple manner using the vertical distribution of the horizontal velocity variance. Kobayashi (2000) assumed the linear wave theory developed for a horizontal bottom is approximately valid locally even on the steep slope. He pointed out that the linear wave theory, however, is not accurate enough to resolve the cross-shore variation of relatively small reflected waves. Therefore, the non-linear wave theory is developed here to study the partial standing wave over a slope.

The motion of a fluid may be described by either observing the trajectory of a particle that is carried along with the flow through Lagrangian approach, or by the fluid velocity at a fixed position, through Eulerian approach. For an incompressible fluid, the Eulerian approach is clearly preferred because the governing equation is Laplace equation which is linear and it is also well known that the Eulerian description for a free surface is a differentiable single-valued function. On the other hand, the surface elevation is specified through the positions of the surface particles in the Lagrangian approach. Unlike an Eulerian surface, which is given as an implicit function, a Lagrangian form is expressed through a parametric representation of particle motion. Hence, the Lagrangian description is more appropriate for the free surface motion by overcoming some limitation of the classical Eulerian solutions (Naciri and Mei 1993, Ng 2004a~b, Buldakov et al. 2006, Chen and Hsu 2009, Hsu et al 2010).

In this study, we will construct a second order solution for partial standing surface waves propagating over a sloping bottom in a Lagrangian framework. The objective is to examine the effect of sloping bottom, wave steepness on surface waves, by using perturbation expansion of the particle trajectories in terms of wave steepness $\varepsilon$ and the bottom slope $\alpha$ to the second order. The asymptotic solutions for other physical quantities related to the wave motion are then obtained up to the second order. Finally, we examine the effect bottom slope and wave reflection on the particle trajectories in the presence of a sloping bottom which to our knowledge has not received much attention in the literature.

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## 2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

We consider a two dimensional partial standing wave in a water depth $d$ over a sloping bottom. The negative $x$-axis is outward to the sea from the still water level (SWL) at the shoreline, while the $y$-axis is taken positive vertically upward from the SWL, and the sea bottom is at $y = -d = \alpha x$, in which $\alpha$ denotes the bottom slope as shown in Fig. 1.
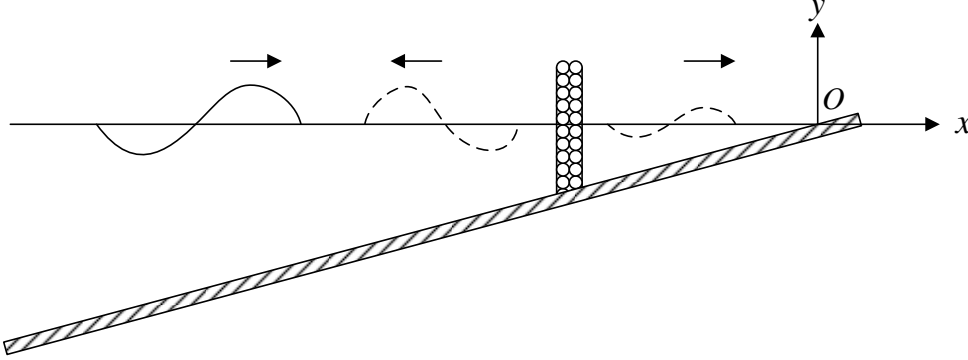


**Fig. 1.** Definition sketch for partial stand wave propagating on a uniformly sloping bottom.

The fluid motion in the Lagrangian description is described by keeping track of individual fluid particles. For two-dimensional flow, a fluid particle is identified by the horizontal and vertical parameters ( $x_0$ , $y_0$ ) known as Lagrangian labels. These labels are initial particle positions or undisturbed coordinated, which has been demonstrated by Lamb (1932), Yakubovich and Zenkovich (2001), or Chen et al. (2010). Then fluid motion is described by a set of trajectories $x(x_0, y_0, t)$ and $y(x_0, y_0, t)$, where $x$ and $y$ are the Cartesian coordinates. The dependent variables $x$ and $y$ denote the position of any particle at time $t$, and are functions of the independent variables $x_0$ , $y_0$ and $t$. In a system of Lagrangian description, the governing equations for two-dimensional irrotational free-surface flow are as follows:

The continuity equation is

$$\frac{\partial(x, y)}{\partial(x_0, y_0)} = x_{x_0} y_{y_0} - x_{y_0} y_{x_0} = 1. \tag{1}$$

Eq. (1) set the invariant condition on the volume of a Lagrangian particle, and the differentiation of Eq. (1) with respect to time is

$$x_{x_0 t} y_{y_0} - x_{y_0 t} y_{x_0} + x_{x_0} y_{y_0 t} - x_{y_0} y_{x_0 t} = \frac{\partial(x_t, y)}{\partial(x_0, y_0)} + \frac{\partial(x, y_t)}{\partial(x_0, y_0)} = 0, \tag{2}$$

and irrotational flow condition are

$$x_{x_0 t} x_{y_0} - x_{y_0 t} x_{x_0} + y_{x_0 t} y_{y_0} - y_{y_0 t} y_{x_0} = \frac{\partial(x_t, x)}{\partial(x_0, y_0)} + \frac{\partial(y_t, y)}{\partial(x_0, y_0)} = 0, \tag{3}$$

$$\frac{\partial \phi}{\partial x_0} = x_t x_{x_0} + y_t y_{x_0} \quad , \quad \frac{\partial \phi}{\partial y_0} = x_t x_{y_0} + y_t y_{y_0}. \tag{4}$$

The Eq. (4) defines the corresponding Lagrangian velocity potential by Chen (2010), the fundamental physical relationships defining the equations above have been derived previously (Lamb 1932; Miche 1944; Pierson 1962; Yakubovich and Zenkovich 2001, Chen et al. 2010).
The Bernoulli equation for irrotational flow in Lagrangian description is

$$\frac{P}{\rho} = -\frac{\partial \phi}{\partial t} - g y + \frac{1}{2}\left(x_t^2 + y_t^2\right), \tag{5}$$

Where subscripts $x_0$, $y_0$, and $t$ denote partial differentiation with respect to the specified variable, $P(x_0, y_0, t)$ is water pressure, $\phi(x_0, y_0, t)$ a velocity potential function in Lagrangian system.

The wave motion has to satisfy a number of boundary conditions at the bottom and on the free water surface:

(a) On an immovable and impermeable sloping plane with an inclination to the horizon, the no-flux bottom boundary condition gives

$$y_t - \alpha x_t = 0, \quad y = y_0 = -d \tag{6}$$

(b). The dynamic boundary condition of zero pressure at the free surface is

$$P = 0, \quad y_0 = 0 \tag{7}$$

(c). A stationary mass transport condition is required as waves propagate toward the beach. A horizontal hydrostatic pressure gradient, to balance the radiation stress of the progressive wave, will drive a return flow and a hence boundary condition should be imposed. This condition is necessary for the uniqueness of the solution and requires that at any cross-section of the x–y plane, the mass transport should vanish:

$$\int_{-d}^{0} U_L\, dy - U(\alpha) \int_{-d}^{0} U_L^c\, dy = 0, \quad U(\alpha) = \begin{cases} 0, \alpha \neq 0 \\ 1, \alpha = 0 \end{cases} \quad 0 \geq y \geq -d, \tag{8}$$

the superscript $c$ denotes the physical quantity at offshore boundary $x \to -\infty$. Because of the nonlinear effect, waves over constant depth induce a net flux of water. Thus, a constant depth mass transport term $U_L^c$ is introduced in (8) which is adjusted by a unit function $U(\alpha)$ to ensure that it can be reduced to the constant depth condition when the bottom slope is equal to zero.

## 3. ASYMPTOTIC SOLUTIONS

To solve Eqs. (1)~(8), it is assumed that relevant physical quantities can be expanded as a double power series in terms of the bottom slope $\alpha$ and nonlinear parameter $\varepsilon$. Thus, the particle displacements $x$ and $y$, the potential function $\phi$, wave pressure $P$, wave number $k$ and Lagrangian wave frequency $\sigma$ can be obtained as.

$$x = x_0 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \alpha^n \left[ f_{m,n}^+(x_0, y_0, t_1) + f_{m,n}^{*+}(x_0, y_0, t_0) + f_{m,n}^-(x_0, y_0, t_1) + f_{m,n}^{*-}(x_0, y_0, t_0) \right]$$

$$= x_0 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \alpha^n \left[ A_{m,n}^+(x_0, y_0)F_{m,n}^+(S^+) + A_{m,n}^{*+}(x_0, y_0)t_0 + A_{m,n}^-(x_0, y_0)F_{m,n}^-(S^-) + A_{m,n}^{*-}(x_0, y_0)t_0 \right] \tag{9}$$

$$y = y_0 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \alpha^n \left[ g_{m,n}^+(x_0, y_0, t_1) + g_{m,n}^{*+}(x_0, y_0, t_0) + g_{m,n}^-(x_0, y_0, t_1) + g_{m,n}^{*-}(x_0, y_0, t_0) \right]$$

$$= y_0 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \alpha^n \left\lfloor B_{m,n}^+(x_0, y_0)G_{m,n}^+(S^+) + B_{m,n}^{*+}(x_0, y_0)t_0 + B_{m,n}^-(x_0, y_0)G_{m,n}^-(S^-) + B_{m,n}^{*-}(x_0, y_0)t_0 \right\rfloor \tag{10}$$

$$\phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \alpha^n \left[ \begin{array}{l} \Phi_{m,n}^+(x_0, y_0, t_1) + \phi_{m,n}^{*+}(x_0, y_0, t_0) + \Phi_{m,n}^-(x_0, y_0, t_1) \\ + \phi_{m,n}^{*-}(x_0, y_0, t_0) + \int M_{m,n,0}(x_0, t_0)dx_0 \end{array} \right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \alpha^n \left[ \begin{array}{l} \phi_{m,n}^+(x_0, y_0)F_{m,n}^+(S^+) + \phi_{m,n}^{*+}(x_0, y_0, t_0) + \phi_{m,n}^- F_{m,n}^-(S^-) \\ + \phi_{m,n}^{*-}(x_0, y_0, t_0) + \int M_{m,n,0}(x_0, t_0)dx_0 \end{array} \right], \tag{11}$$

$$P = -\rho g y_0 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \alpha^n P_{m,n}(x_0, y_0, t_1), \tag{12}$$

$$k = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon^m \alpha^n k_{m,n}(x_0, y_0) \tag{13}$$

$$\sigma = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon^m \alpha^n \sigma_{m,n}(x_0, y_0), \tag{14}$$

where $S^+$ is the phase function of the incident wave $S^+ = \int k dx_0 - \sigma t$, $S^-$ is the phase function of the reflection wave $S^- = \int k dx_0 + \sigma t$, $x(x_0, y_0, t)$ (e.g. Longuet-Higgins 1953; Mei 1985; Ng 2004b) $x(x_0, y_0, t)$ and $y(x_0, y_0, t)$ are the particle displacements and the Lagrangian variable $(x_0, y_0)$ are any two characteristic parameters, $\varepsilon$ is the nonlinear ordering parameter characterizing the wave steepness, $M_{m,n,0}$ is the return flow, $t_1$ is $\sigma t$ and $t_0$ is $\sigma_{0,0} t$ for simplicity. The superscript "+" denotes the physical quantity of incident wave, and the superscript "-" denotes the physical quantity of reflection wave. Additional, the subscript "*" denotes the no-periodic function. $\sigma = 2\pi / T$ is the angular frequency of the particle motion or the Lagrangian angular frequency, where $T$ is the period of particle motion. For a relatively gentle bottom slope $\alpha$, it may be assumed that the $q$-th differentiation of $A_{m,n}^+$, $A_{m,n}^{*+}$, $A_{m,n}^-$, $A_{m,n}^{*-}$, $B_{m,n}^+$, $B_{m,n}^{*+}$, $B_{m,n}^-$, $B_{m,n}^{*-}$, $\phi_{m,n}^+$, $\phi_{m,n}^-$, $M_{m,n,0}$ and $k_{m,n}$ with respect to $x_0$ are in the order of $\alpha^q$:

$$\begin{aligned}
(\frac{d^q k_{m,n}}{dx_0^q}, \frac{\partial^q M_{m,n,0}}{\partial x_0^q}, \frac{d^q A_{m,n}^+}{dx_0^q}, \frac{d^q A_{m,n}^{*+}}{dx_0^q}, \frac{d^q B_{m,n}^+}{dx_0^q}, \frac{d^q B_{m,n}^{*+}}{dx_0^q}, \frac{d^q \phi_{m,n}^+}{dx_0^q} \\
\frac{d^q A_{m,n}^-}{dx_0^q}, \frac{d^q A_{m,n}^{*-}}{dx_0^q}, \frac{d^q B_{m,n}^-}{dx_0^q}, \frac{d^q B_{m,n}^{*-}}{dx_0^q}, \frac{d^q \phi_{m,n}^-}{dx_0^q}) = O(\alpha^q), q, n \in 0, 1, 2 \cdots N
\end{aligned} \tag{15}$$

Substituting Eqs. (9)~(14) into Eqs. (1)~(8), and collecting the terms of the like order in $\varepsilon$ and $\alpha$, we obtain the necessary equations to each order of approximation. Then different orders of $\varepsilon(m)$ and $\alpha(n)$ may be separated, yielding a set of partial differential equations for each index *(m,n)*. It is assumed that the $\alpha = O(\varepsilon)$ which is the same as Chen et al. (2005, 2006). Following these procedures, analytical solutions can then be obtained.

### 3.1 $\varepsilon^1 \alpha^0$ -order approximation

The solution is not affected by the sloping bottom and is given by:

$$\begin{cases}
f_{1,0}^+ = -B^+(ch)\sin S^+, f_{1,0}^- = -B^-(ch)\sin S^-, \\
g_{1,0}^+ = B^+(sh)\cos S, g_{1,0}^- = B^-(sh)\cos S^-, \\
\Phi_{1,0}^+ = B^+ \frac{\sigma_{0,0}}{k_{0,0}}(ch)\sin S^+, \Phi_{1,0}^- = -B^- \frac{\sigma_{0,0}}{k_{0,0}}(ch)\sin S^-, \\
\frac{P_{1,1}}{\rho} = -g \frac{\sinh k_{0,0} y_0}{\cosh^2 k_{0,0} d}[(B^+ \cos S^+ + B^- \cos S^-)], \\
\sigma_{0,0}^2 = gk_{0,0} \tanh k_{0,0} d, ch = \frac{\cosh k_{0,0}(y_0 + d)}{\cosh k_{0,0} d}, \\
sh = \frac{\sinh k_{0,0}(y_0 + d)}{\cosh k_{0,0} d}.
\end{cases}$$

### 3.2 $\varepsilon^1 \alpha^1$ -order approximation

To the next order in $O(\varepsilon^1 \alpha^1)$, A general solution for $A_{1,1}^+$ , $A_{1,1}^-$ , $B_{1,1}^+$ and $B_{1,1}^-$ , which satisfies both continuity equation and irrotational flow condition, can be obtained.

Based on the above solutions, let us briefly discuss the effect of bottom slope on the free surface displacement $y(x_0, y_0 = 0, t)$. First, the correction to the free surface displacement at $O(\varepsilon^1 \alpha^1)$ is $90^\circ$ out

of phase with respect to the leading order solution ($\varepsilon^1\alpha^0$). Second, the wave amplitude is enhanced and the phase is modified due to the effect of the slope. The solutions of $\varepsilon^1\alpha^1$ are determined as

$$
\begin{cases}
f_{1,1}^+ = \dfrac{\alpha B^+}{\cosh k_{0,0}d}\{[\dfrac{k_{0,0}^2(y_0+d)^2}{D\sinh 2k_{0,0}d} - k_{0,0}(y_0+d) + \dfrac{1}{D^2\tanh k_{0,0}d}]\cosh k_{0,0}(y_0+d) \\
\quad +[\dfrac{k_{0,0}(y_0+d)}{D^2\tanh k_{0,0}d} + \dfrac{2k_{0,0}(y_0+d)}{D\sinh 2k_{0,0}d} - 1]\sinh k_{0,0}(y_0+d)\}\cos S^+, \\[6pt]
f_{1,1}^- = \dfrac{\alpha B^-}{\cosh k_{0,0}d}\{[\dfrac{k_{0,0}^2(y_0+d)^2}{D\sinh 2k_{0,0}d} - k_{0,0}(y_0+d) + \dfrac{1}{D^2\tanh k_{0,0}d}]\cosh k_{0,0}(y_0+d) \\
\quad +[\dfrac{k_{0,0}(y_0+d)}{D^2\tanh k_{0,0}d} + \dfrac{2k_{0,0}(y_0+d)}{D\sinh 2k_{0,0}d} - 1]\sinh k_{0,0}(y_0+d)\}\cos S^-, \\[6pt]
g_{1,1}^+ = \dfrac{\alpha B^+}{\cosh k_{0,0}d}\{[\dfrac{k_{0,0}^2(y_0+d)^2}{D\sinh 2k_{0,0}d} - k_{0,0}(y_0+d) + \dfrac{1}{D^2\tanh k_{0,0}d}]\sinh k_{0,0}(y_0+d) \\
\quad +[\dfrac{k_{0,0}(y_0+d)}{D^2\tanh k_{0,0}d} + \dfrac{2k_{0,0}(y_0+d)}{D\sinh 2k_{0,0}d} - 1]\cosh k_{0,0}(y_0+d)\}\sin S^+, \\[6pt]
g_{1,1}^- = \dfrac{\alpha B^-}{\cosh k_{0,0}d}\{[\dfrac{k_{0,0}^2(y_0+d)^2}{D\sinh 2k_{0,0}d} - k_{0,0}(y_0+d) + \dfrac{1}{D^2\tanh k_{0,0}d}]\sinh k_{0,0}(y_0+d) \\
\quad +[\dfrac{k_{0,0}(y_0+d)}{D^2\tanh k_{0,0}d} + \dfrac{2k_{0,0}(y_0+d)}{D\sinh 2k_{0,0}d} - 1]\cosh k_{0,0}(y_0+d)\}\sin S^-, \\[6pt]
\phi_{1,1}^+ = -\dfrac{\sigma_{0,0}}{k_{0,0}}\dfrac{\alpha B^+}{\cosh k_{0,0}d}\{[\dfrac{k_{0,0}^2(y_0+d)^2}{D\sinh 2k_{0,0}d} - k_{0,0}(y_0+d)]\cosh k_{0,0}(y_0+d) \\
\quad +\dfrac{k_{0,0}(y_0+d)}{D^2\tanh k_{0,0}d}\sinh k_{0,0}(y_0+d)\}\sin S^+, \\[6pt]
\phi_{1,1}^- = \dfrac{\sigma_{0,0}}{k_{0,0}}\dfrac{\alpha B^-}{\cosh k_{0,0}d}\{[\dfrac{k_{0,0}^2(y_0+d)^2}{D\sinh 2k_{0,0}d} - k_{0,0}(y_0+d)]\cosh k_{0,0}(y_0+d) \\
\quad +\dfrac{k_{0,0}(y_0+d)}{D^2\tanh k_{0,0}d}\sinh k_{0,0}(y_0+d)\}\sin S^-, \\[6pt]
\dfrac{P_{1,1}}{\rho} = -[\sigma_0\phi_{1,1}^+ + gB^+]\sin S^+ - [\sigma_0\phi_{1,1}^- - gB^-]\sin S^- \\[6pt]
k_{0,1} = \sigma_{0,1} = 0.
\end{cases}
\tag{16}
$$

$$
B^+ = \frac{a^+}{\tanh k_{0,0}d}, \quad a^+ = \frac{a_0^+}{\sqrt{k_{0,0}d\,\text{sech}^2\,k_{0,0}d + \tanh k_{0,0}d}} = a_0^+ K_s
\tag{17}
$$

$$
B^- = \frac{a^-}{\tanh k_{0,0}d}, \quad a^- = \frac{a_0^-}{\sqrt{k_{0,0}d\,\text{sech}^2\,k_{0,0}d + \tanh k_{0,0}d}} = Ra_0^+ K_s
\tag{18}
$$

where, $R$ is the reflection coefficient; $a_0^+$ is the amplitude of the incident waves in deep water and $a^+$ is that on the sloping bottom; $a_0^+$ and $a^+$ are related as follows,

$$
a^+ = a_0^+/\sqrt{D\tanh k_0 d} = a_0^+ K_s .
\tag{19}
$$

where parameter $K_s$ is the conventional shoaling coefficient that are same as Mei (1985). The solution includes the sloping bottom effect by the sloping bottom at this order and wave reflection.

### 3.3 $\varepsilon^2\alpha^0$ -order approximation

In the same manner as for $O(\varepsilon^1\alpha^1)$ is solved. Although laborious, the procedure to obtain the solutions at this order is lengthy but straightforward manipulations, the solutions can be given by

$$
\begin{cases}
f_{2,0}^+ = -\dfrac{3}{8}k_{0,0}(B^+)^2\dfrac{ch2}{\sinh^2 k_{0,0}d}\sin 2S^+ + \dfrac{1}{4}k_{0,0}(B^+)^2\dfrac{\sin 2S^+}{\cosh^2 k_{0,0}d}\\[2mm]
\qquad +\dfrac{1}{8}k_{0,0}B^+B^-\dfrac{ch2}{\cosh^2 k_{0,0}d}\sin(2\int k dx_0),\\[2mm]
f_{2,0}^- = -\dfrac{3}{8}k_{0,0}(B^-)^2\dfrac{ch2}{\sinh^2 k_{0,0}d}\sin 2S^- + \dfrac{1}{4}k_{0,0}(B^-)^2\dfrac{\sin 2S^-}{\cosh^2 k_{0,0}d}\\[2mm]
\qquad +\dfrac{1}{2}k_{0,0}B^+B^-\dfrac{\sin(2\int k dx_0)}{\cosh^2 k_{0,0}d} + \dfrac{1}{8}k_{0,0}B_{1,0}B_{R1,0}\dfrac{ch2}{\cosh^2 k_{0,0}d}\sin(2\int k dx_0),\\[2mm]
f_{2,0}^{*+} = \dfrac{1}{2}k_{0,0}(B^+)^2(ch2)t_0 - \dfrac{k_{0,0}[(a_0^+)^2-(a_0^-)^2]}{2k_{0,0}d\tanh k_{0,0}d}t_0 + U(\alpha)\dfrac{k_{0,0}[(a_0^+)^2-(a_0^-)^2]}{2k_{0,0}d\tanh k_0 d_0}t_0,\\[2mm]
f_{2,0}^{*-} = -\dfrac{1}{2}k_{0,0}(B^-)^2(ch2)t_0,
\end{cases}
$$

$$
\begin{cases}
g_{2,0}^+ = \dfrac{3}{8}k_{0,0}(B^+)^2\dfrac{(sh2)}{\sinh^2 k_{0,0}d}\cos 2S^+ + \dfrac{1}{4}k_{0,0}(B^+)^2 sh2\\[2mm]
\qquad -\dfrac{1}{8}k_{0,0}B^+B^-\dfrac{sh2}{\cosh^2 k_{0,0}d}\cos(2\int k dx_0) + \dfrac{1}{2}\dfrac{(a_0^+)^2 k_0}{\sinh 2k_0 d_0} - \dfrac{1}{2}\dfrac{(a^+)^2 k_{0,0}}{\sinh 2kd},\\[2mm]
g_{2,0}^- = \dfrac{3}{8}k_{0,0}(B^-)^2\dfrac{sh2}{\sinh^2 k_{0,0}d}\cos 2S^- + \dfrac{1}{4}k_{0,0}(B^-)^2 sh2 + \dfrac{1}{2}k_{0,0}B^+B^-(sh2)\cos(2\sigma t)\\[2mm]
\qquad -\dfrac{1}{8}k_{0,0}B^+B^-\dfrac{sh2}{\cosh^2 k_{0,0}d}\cos(2\int k dx_0) + \dfrac{1}{2}\dfrac{(a_0^-)^2 k_0}{\sinh 2k_0 d_0} - \dfrac{1}{2}\dfrac{(a^-)^2 k_{0,0}}{\sinh 2kd},\\[2mm]
\phi_{2,0}^+ = \dfrac{3}{8}\sigma_{0,0}(B^+)^2\dfrac{ch2}{\sinh^2 k_{0,0}d}\sin 2S^+ - \dfrac{1}{2}\sigma_{0,0}(B^+)^2\dfrac{\sin 2S^+}{\cosh^2 k_{0,0}d}\\[2mm]
\qquad +\int\{\dfrac{k_{0,0}[(a_0^+)^2-(a_0^-)^2]}{2k_{0,0}d\tanh k_{0,0}d}\sigma_{0,0} + U(\alpha)\dfrac{k_{0,0}[(a_0^+)^2-(a_0^-)^2]}{2k_{0,0}d\tanh k_0 d_0}\sigma_{0,0}\}dx_0,\\[2mm]
\phi_{2,0}^- = -\dfrac{3}{8}\sigma_{0,0}(B^-)^2\dfrac{ch2}{\sinh^2 k_{0,0}d}\sin 2S^- + \dfrac{1}{2}\sigma_{0,0}(B^-)^2\dfrac{\sin 2S^-}{\cosh^2 k_{0,0}d}\\[2mm]
\qquad -\sigma_{0,0}B^+B^-(ch2)\sin(2\sigma t) + \dfrac{1}{4}\sigma_{0,0}B^+B^-(1+3\tanh^2 k_{0,0}d)\sin(2\sigma t),\\[2mm]
\phi_{2,0}^{*+} = -\dfrac{1}{4}(a_0^+)^2\sigma_{0,0}^2\dfrac{1}{\sinh^2 k_0 d_0}t;\quad \phi_{2,0}^{*-} = -\dfrac{1}{4}(a_0^-)^2\sigma_{0,0}^2\dfrac{1}{\sinh^2 k_0 d_0}t,\\[2mm]
ch2 = \dfrac{\cosh 2k_{0,0}(y_0+d)}{\cosh^2 k_{0,0}d},\; sh2 = \dfrac{\sinh 2k_{0,0}(y_0+d)}{\cosh^2 k_{0,0}d}.
\end{cases}
\tag{20}
$$

In Eq. (20), $k_0$ is the wave number in deep water. The horizontal Lagrangian particle trajectory, $x$, in the second order approximation includes periodic components $f_{2,0}^+$ and $f_{2,0}^-$ which are similar to that of the second-order Lagrangian solution at a constant depth, non-periodic function $f_{2,0}^{*+}$ and $f_{2,0}^{*-}$ that increases linearly in time and represents the mass transport and the return flow term. This implies that on average a fluid particle moves forward and does not form a closed orbit as it occurs in the first-order approximation. The velocity of Lagrangian particle is given by *dx/dt*, so differentiating non-periodic function $(f_{2,0}^{*+}+f_{2,0}^{*-})$ with respect to time, we can obtain the mass transport velocity $U_L$ of particle as

$$U_L = \frac{1}{2}[(B^+)^2 - (B^-)^2]\sigma_{0,0}k_{0,0}ch2 - \frac{gk_{0,0}^2(a^+)^2}{2k_{0,0}d\sigma_{0,0}} + U(\alpha)\frac{gk_{0,0}k_0(a_0^+)^2}{2k_{0,0}d\sigma_{0,0}},$$ (21)

When the reflection coefficient is zero, the first term in Eq. (21) becomes the wave drift velocity solution of a progressive wave over the whole water column. This is the second-order drift current solution obtained previously by Longuet-Higgins (1953) for constant water depth. When the reflection coefficient is unity, namely, $R = 1$ in case of complete reflection, $U_L$ becomes zero, the wave has no mass transport, which implies that particles moves up and down via space and time symmetrically. The last two terms in Eq. (28) are for the return flow. These terms have been obtained previously by Chen et al. (2012).

In Fig. 2, the dimensionless mass transport velocity is given for different reflection coefficient with incident wave steepness $k_0 H_0 = 0.06\pi$ and bottom slope $\alpha = 1/10$. The mass transport velocity decreases when the magnitude of reflection increases. In addition, the mass transport velocity increases when the water depth decreases.

The vertical trajectory $y$ in this order includes a second harmonic component, a Lagrangian mean level that is a function of $y_0$ and independent of time and a mean sea level change. From Eq. 29d and Eq. 29e the present theory derives a second order Lagrangian mean level of partial standing wave $\overline{y}(x_0, y_0, t)$ for all particles at different values of vertical marked level $y_0$.

$$\overline{y}(x_0, y_0, t) = \frac{1}{4}k_{0,0}[(B^+)^2 + (B^-)^2]ch2 - \frac{1}{4}k_{0,0}B^+B^-\frac{ch2}{\cosh^2 k_{0,0}d}\cos(2\int kdx_0)$$
$$+ \frac{1}{2}\frac{(1+R)(a_0^+)^2 k_{0,0}}{\sinh 2k_{0,0}d_0} - \frac{1}{2}\frac{(1+R)(a^+)^2 k_{0,0}}{\sinh 2k_{0,0}d}$$ (29)

This second-order vertical mean level of particle decays with water depth. The mean sea level change was first predicted by Longuet-Higgins and Stewart (1964) as the consequence of radiation stresses. If we consider the case of waves originating from deep-water depth without wave reflection R=0, the wave set-down, $(a_0^+)^2 k_0 / 2\sinh 2k_0 d_0 - (a^+)^2 k_{0,0} / 2\sinh 2k_{0,0}d$, is exactly the one that has been obtained by Longuet-Higgins and Stewart (1962). If we consider the case of constant depth with complete reflection $R = 1$, $-\frac{1}{4}k_{0,0}B^+B^-\frac{ch2}{\cosh^2 k_{0,0}d}\cos(2\int kdx_0)$, is same as that derived by Ng(2004b) which holds true only at the free surface of pure standing waves.
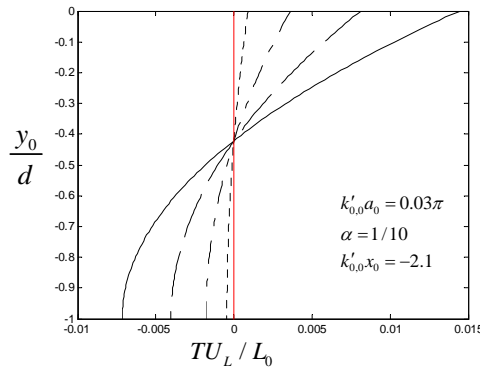


$\dfrac{y_0}{d}$

$TU_L / L_0$

$k'_{0,0}a_0 = 0.03\pi$
$\alpha = 1/10$
$k'_{0,0}x_0 = -2.1$

**Fig. 2. Dimensionless mass transport velocity** $TU_L / L_0$ **versus dimensionless water depth** $y_0 / d$ **in five different wave reflection R. (Solid line:** $R = 0$**, dash line:** $R = 0.25$**, dash-dotted line:** $R = 0.5$**, dot line:** $R = 0.75$**, red solid line:** $R = 1$**).**

## 4. RESULTS AND DISCUSSIONS

The Lagrangian solution for water-particle displacement developed in this study can be employed to demonstrate the validity for water particle motion. The parametric functions for the water particle at any position in Lagrangian coordinates $(x, y)$ have been obtained as follows.

$$x(x_0, y_0, t) = x_0 + \varepsilon^1\alpha^o(f_{1,0}^+ + f_{1,0}^-) + \varepsilon^1\alpha^1(f_{1,1}^+ + f_{1,1}^-) + \varepsilon^2\alpha^o(f_{2,0}^+ + f_{2,0}^{*+} + f_{2,0}^- + f_{2,0}^{*-})$$ (31)

$$y(x_0, y_0, t) = y_0 + \varepsilon^1\alpha^o(g_{1,0}^+ + g_{1,0}^-) + \varepsilon^1\alpha^1(g_{1,1}^+ + g_{1,1}^-) + \varepsilon^2\alpha^o(g_{2,0}^+ + g_{2,0}^-)$$ (32)

Figure 2(a) shows that the variation of surface elevation of a standing wave over the sloping bottom with complete reflection by a virtual vertical wall at the position $k_{0,0}x = -2.1$ at section (1). T Fig. 2(b) to (f) shows the second order trajectories of at five sections (1) to (5) indicated in figure 2(a) over a sloping bottom, with anti-node in Fig. 3b and Fig. 3f and the node in Fig. 3d. The particle trajectory moves almost vertically but not exactly under the anti-node due to affected by bottom slopes in Fig 3f. Nevertheless, the particle trajectory moves vertically at the wall in Fig 3b. As shown by Fig. 3d, the particle trajectories under the node change from a concave at the surface to a straight line parallel to the bottom at the as discussed in Zou et al. (2003), Zou & Hay (2003) and Chen et al. (2012). When surface particles are not at the node or anti-node, at Fig. 3e where the wave make transition from anti-node to node, the particle trajectory follows almost straight line inclining upwards with the higher ; while at Fig. 3f, where the wave make transition from node to anti-node, the particle trajectory follows almost straight line inclining downwards.

The trajectories of a progressive wave over a sloping bottom without reflection is shown in Fig. 4. The particles do not move in closed orbital motion and each particle advances a larger movement in the horizontal direction at the free surface. Near the bottom, the trajectory becomes more like an ellipse because the vertical velocity of the particle decreases exponentially with the vertical position of particle, in contrast to the trajectories near the mean water level. Fig. 4b shows that the particle orbit near the surface has an upward convex point in the shallow water at section 1.

Fig. 5 shows that the particle orbit of partial standing wave over a sloping bottom a wave reflection R=0.5. Under the node, due to the increase in wave reflection, the vertical excursion of particle decreases while the horizontal excursion increases (cf. Fig. 5d. However, the particle under the anti-node gives the opposite result. In Fig. 2-5 also show that the mass transport velocity indicated by the non-closed orbital motion decrease with increasing reflection R which is consistent with the results in figure 2.
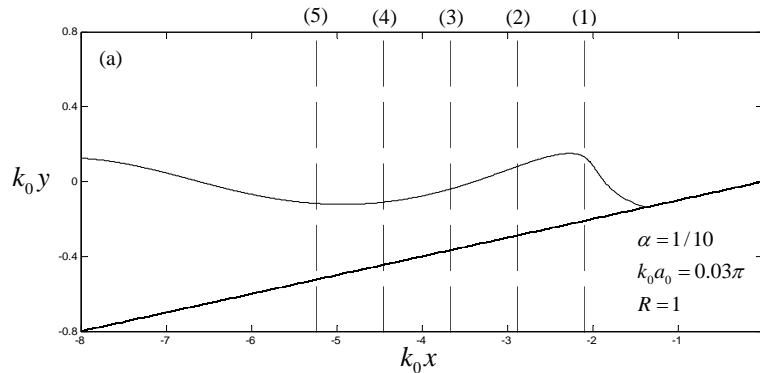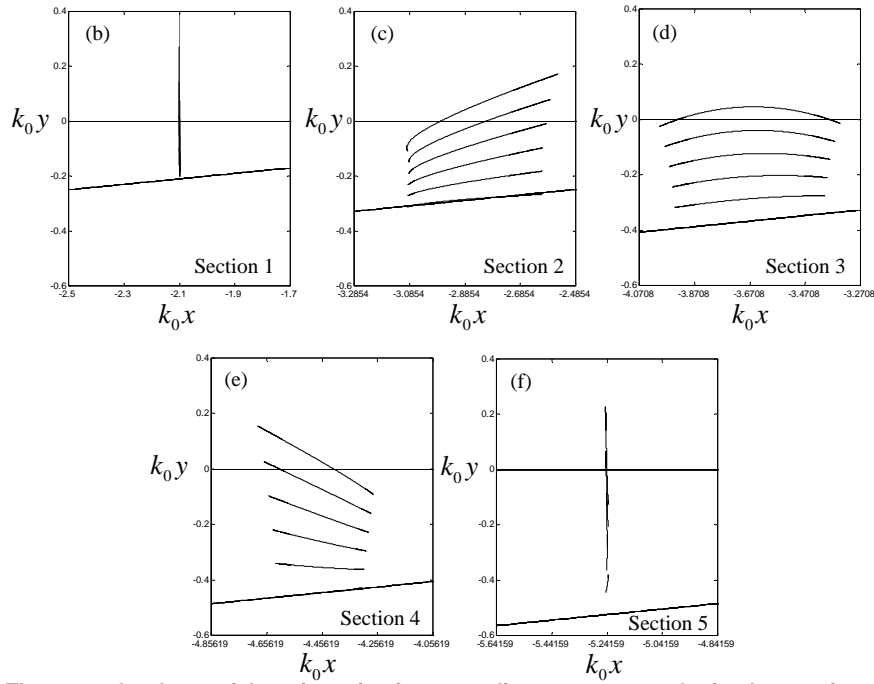
**Fig. 3 a-f. The second order particle trajectories for a standing wave over a sloping bottom for complete wave reflection R=1.**
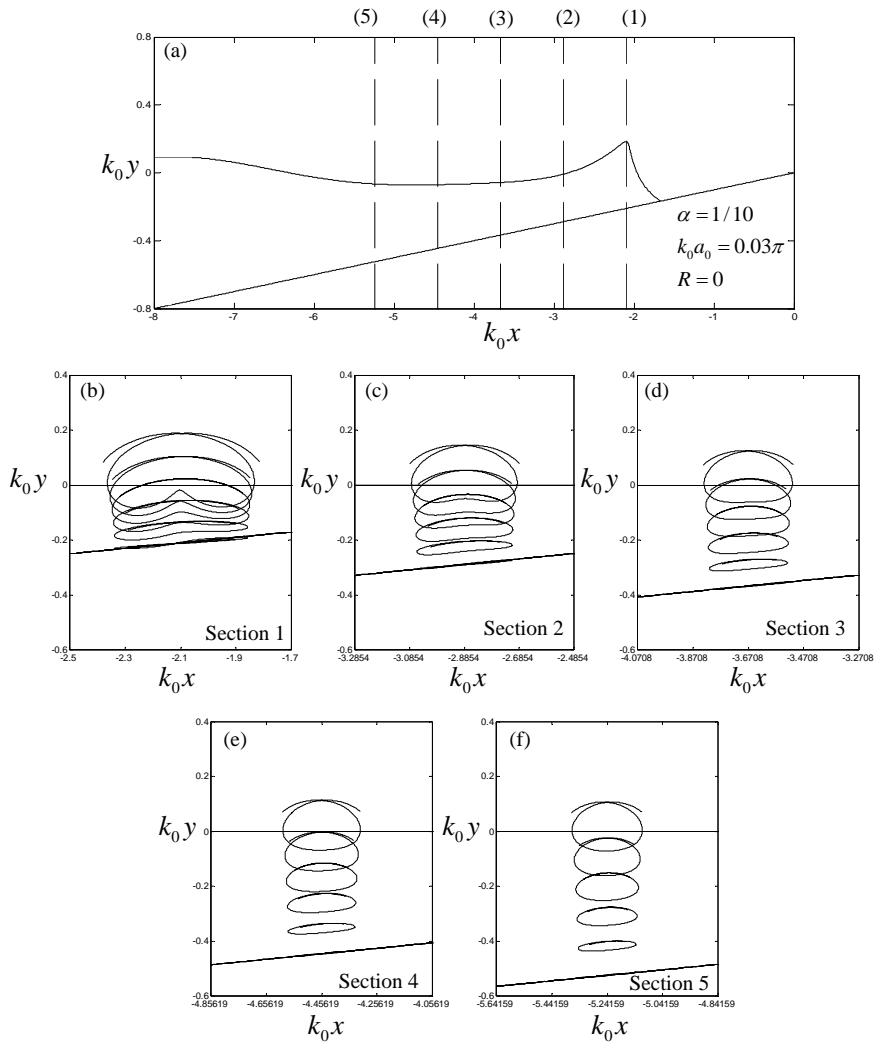
**Fig. 4 a-f. The second order particle trajectories for a progressive wave over a sloping bottom without wave reflection R=0.**
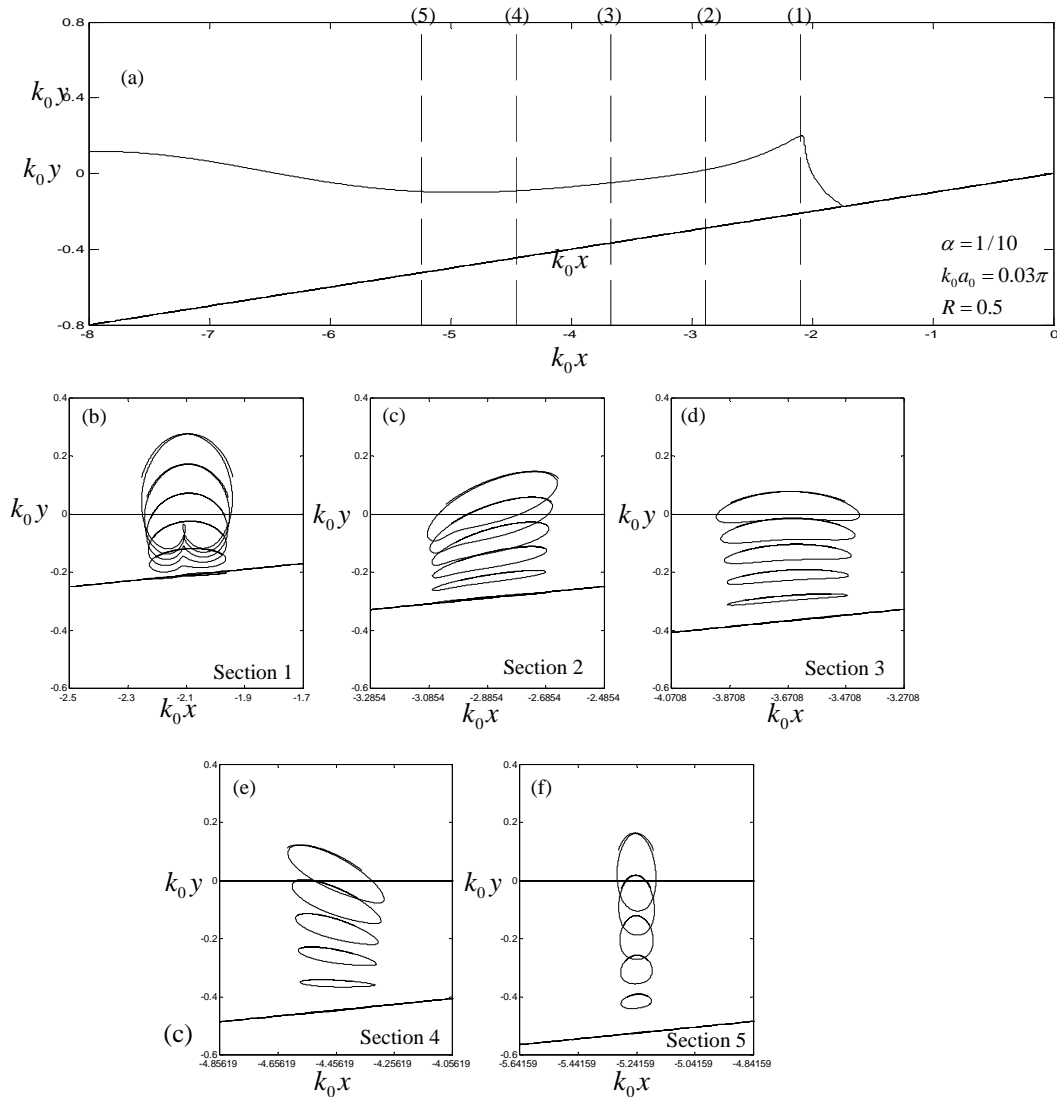


**Fig. 5 a-f. The second order particle trajectories for a progressive wave over a sloping bottom for wave reflection R= 0.5.**

## 5. CONCLUSIONS

We constructed a new second-order Lagrangian asymptotic solution for partial standing wave over a uniform sloping bottom for any given wave reflection R. The solution, developed in explicit form, includes parametric functions for water-particle motion and the wave velocity in Lagrangian description. When wave is shoaling from deep to breaking point toward the shore, the wave particle trajectory evolves in the cross shore direction over a sloping bottom. This generalized solution reduces to that of progressive wave when wave reflection $R = 0$, and standing wave when wave reflection $R = 1$. The solution also provides information for the process of successive deformation of a wave profile and water particle trajectory. The solution to the nonlinear boundary-value problem is presented after including a mean return current which is needed to maintain zero mass flux in a bounded domain.

## ACKNOWLEDGEMENTS

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