Solitons in one-dimensional Bose-Einstein condensate with higher-order interactions

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(Dated: August 30, 2017)

Abstract

We model a one-dimensional Bose-Einstein condensate with the one-dimensional Gross-Pitaevskii equation (1D GPE) incorporating higher-order interaction effects. Based on the F-expansion method, we analytically solve the 1D GPE, identifying the typical soliton solution under certain experimental settings within the general wave-like solution set, and demonstrating the applicability of the theoretical treatment that is employed.

PACS numbers: 05.45.-a; 03.75.Lm; 02.30.Jr

Keywords: nonlinear Schrödinger equation, higher-order nonlinearity, soliton

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I. INTRODUCTION

Solitons, as typical nonlinear phenomena in many branches of science, have been extensively investigated both theoretically and experimentally, in the past few decades. Ultracold quantum gases, where Bose-Einstein condensate (BEC)-related studies dominate in many respects, are ideal candidates for the study of nonlinear dynamics because of their easy control and flexible tunability.

The Gross-Pitaevskii equation (GPE), derived from the mean field approach, is proven to be the reliable model for the study of BEC-related phenomena in ultracold quantum gases. The one-dimensional case of the GPE (1D GPE) that models quasi-one-dimensional settings, such as a system with an elongated potential, has attracted much attention in recent years. The typical bright/dark soliton-type analytical solutions have been identified for regular cubic nonlinear interaction formulations in the 1D GPE, as demonstrated in many prior works [1–6]. Recently, however, higher-order nonlinear interactions have demonstrated their importance in relevant theoretical and experimental progress. Originating from the higher-order expansion of the two-body scattering phase shift at low momenta, the higher-order effects have emerged in some novel cold atom systems, such as BECs with embedded Rydberg molecules [7–10], and constitute particular ingredients in mean field formulations, such as the GPE [11]. In this study, for 1D systems modeled by the 1D GPE, by incorporating higher-order nonlinear interaction effects, we obtain the analytical solution to the 1D GPE based on the F-expansion method [12, 13] and identify possible solutions to the typical nonlinear features for both dark and bright solitons.

This paper is arranged as follows. The next section describes the 1D GPE model incorporating higher-order interactions and the F-expansion method. Section III demonstrates the detailed problem-solving steps for obtaining the analytical soliton solution of the GPE. Section IV gives some concluding remarks.

II. MODIFIED GPE WITH HIGHER-ORDER INTERACTION AND F-EXPANSION METHOD

A. Formulation of modified GPE

The GPE, which is derived from the mean field approach, typically considers only the s-wave two-body scattering length in the nonlinear interaction formulation. However, to incorporate higher-order effects in the two-body interaction dynamics, the contribution from higher partial waves must be considered. The inter-particle potential can be expressed in the following form [14]

$$V_{int}(\mathbf{r}-\mathbf{r}') = U_0[\delta(\mathbf{r}-\mathbf{r}') + \frac{g_2}{2}(\overleftarrow{\nabla}_r^2\delta(\mathbf{r}-\mathbf{r}') + \delta(\mathbf{r}-\mathbf{r}')\overrightarrow{\nabla}_r^2)] \tag{1}$$

where $U_0 = \frac{4\pi\hbar^2 a}{m}$ is the hard-core interaction. The higher-order scattering effects are incorporated in the parameter $g_2 = a^2/3 - ar_e/2$, where m, r_e, and a are the atomic mass, effective range, and s-wave scattering length, respectively. As the 1D setting is typical and some important experimental settings, such as elongated harmonic trapping are also quasi-one-dimensional, in this study, we consider the 1D case of the GPE. In addition to the higher-order two-body scattering effects that are to be incorporated in the modified GPE, the three-body effect is also incorporated as a competing nonlinear interaction candidate. The modified GPE in the dimensionless form can be expressed as:

$$i\frac{\partial}{\partial t}\psi = [-\frac{\partial^2}{\partial x^2} + kx^2 + (g_0|\psi|^2 + g_1|\psi|^4 + g_2\frac{\partial^2}{\partial x^2}|\psi|^2)]\psi \tag{2}$$

where g_0 and g_1 are the two-body and three-body nonlinear interaction strength coefficients (leading order) respectively, with $g_0 = \frac{4\pi\hbar^2 a}{m}$ based on the standard Gross-Pitaevskii equation formulation, and g_1 is of order a^2 because of three-body interaction. The term with the interaction coefficient g_2 is the contribution from higher-order wave scattering effects. Here k is the harmonic trapping strength. For the elongated potential, it is the longitudinal trapping strength, which is much smaller than the axial trapping strength. The tight confinement in the axial direction of the elongated trap freezes the transverse motion of atoms, which makes it valid that we investigate the nonlinear properties with the one dimensional system modeled by Eq. (2). We will investigate the typical nonlinear properties of the system modeled by Eq. (2).

B. *F*-expansion method

The F-expansion method can be utilized to solve nonlinear partial differential equations of the following form,

$$G(u, u_t, u_x, u_{xx}, \ldots) = 0 \tag{3}$$

where $u(x,t)$ is the function to be determined. G is a polynomial of $u(x,t)$ and its partial derivatives of various orders. With $F(\xi)$ being a function of $\xi = p(t)x + q(t)$, the key idea of the F-expansion method is to express $u(x,t)$ as a polynomial of $F(\xi)$

$$\frac{d^2}{d\xi^2}F(\xi) = c_0(2F^3(\xi) + \frac{3}{2}\lambda F^2(\xi) + \mu F(\xi) + \frac{1}{2}\eta) \tag{4}$$

where λ, μ, and η are constants to be determined in the following problem-solving steps. By multiplying both sides of Eq. (4) by $dF(\xi)/d\xi$ and integrating with ξ once, we obtain the expression for $dF(\xi)/d\xi$

$$\frac{dF(\xi)}{d\xi} = \pm\sqrt{c_0(F^4(\xi) + \lambda F^3(\xi) + \mu F^2(\xi) + \eta F(\xi) + \varsigma)} \tag{5}$$

We can express $u(x,t)$ as

$$u(x,t) = \sum_{i=0}^{m} h_i(t)F^i(\xi), \quad h_m(t) \neq 0 \tag{6}$$

The next step involves substituting Eq. (6) into the original nonlinear partial differential equation (Eq. 3) and by using Eqs. (4) and (5), we can fix m by balancing between the nonlinear term and highest differential term. Then, we express G in Eq. (3), as a polynomial of $F(\xi)$ and another polynomial of $F(\xi)$ multiplied by $dF(\xi)/d\xi$. We solve Eq. (3) by setting the coefficients of all the terms, ($F^i(\xi)$ and $F^j(\xi)dF(\xi)/d\xi$), of the reformulated G to zero. This will give a set of over determined ordinary differential equations (ODEs) of $h_i(t)$, which will put $u(x,t)$ in an explicit form if the ODEs can be solved consistently.

III. SOLITON-TYPE SOLUTION OF GPE WITH HIGHER-ORDER INTERACTION

A. Problem reformulation with coupled phase-modulus transformation

To eliminate the integrable constraints, we adopt the coupled phase-modulus transformation through the parameter function $\sigma(t)$ as follows,

$$x' = \sigma(t')x, \tag{7a}$$

$$t' = t, \tag{7b}$$

$$\psi(x,t) = \sigma^{1/2}(t')\exp[i\frac{\sigma_{t'}(t')}{\sigma(t')}x^2)]\varphi(x',t') \tag{8}$$

By substituting transformations (8) and (7) into Eq. (2), and changing the notation from (x',t') to (x,t), the original modified GPE is transformed to the following form,

$$i\varphi_t + \sigma^2(t)\varphi_{xx} + \left[\frac{k(t)}{\sigma^2(t)} + \frac{1}{4}\left(\frac{\sigma_t(t)}{\sigma(t)}\right)^2 - \frac{1}{4}\left(\frac{\sigma_t(t)}{\sigma(t)}\right)_t\right]x^2\varphi$$
$$+g_0\sigma(t)|\varphi|\varphi + g_1\sigma^2(t)|\varphi|^2\varphi + g_2\sigma(t)(|\varphi|^2)_{xx}\varphi = 0 \tag{9a}$$

We then utilize the following ansatz for Eq. (9),

$$\varphi(x,t) = v(x,t)e^{i\theta(x,t)} \tag{10}$$

When substituting Eq. (10) into Eq. (9), we obtain the equations for $v(x,t)$ and $\theta(x,t)$ as follows,

$$v^2\theta_t + \sigma^2(t)(vv_{xx} + v^2\theta_x^2) + \alpha(t)x^2v^2 + \beta_1(t)v^4 + \beta_2(t)v^6$$
$$+\beta_3(t)\sigma^2(t)(v^2)_{xx}v^2 = 0 \tag{11a}$$

$$v_t + \sigma^2(t)(2v_x\theta_x + v\theta_{xx}) = 0 \tag{11b}$$

where $\alpha(t) = k(t)/\sigma^2(t) + \frac{1}{4}(\sigma_t(t)/\sigma(t))^2 - \frac{1}{4}(\sigma_t(t)/4\sigma(t))_t, \beta_n(t) = -g_{n-1}\sigma^n(t)(n = 1,2)$, and $\beta_3 = -g_2\sigma(t)$. Equation (11) is in the form for which the F-expansion method can be applied. This is shown explicitly in the following subsection.

B. Soliton-type solution

In order to obtain a possible soliton-type solution for Eq. (11), we adopt the following ansatz,

$$v(x,t) = h(t)F(\xi) \tag{12}$$

$$\theta(x,t) = \Phi(t)x^2 + \Gamma(t)x + \Omega(t) \tag{13}$$

where $\xi = p(t)x + q(t)$ and the F-expansion formulation takes the following form

$$\left(\frac{dF(\xi)}{d\xi}\right)^2 = a_4F^4 + a_2F^2 + a_0 \quad \text{or} \tag{14a}$$

$$F\frac{d^2F}{d\xi^2} = 4a_4F^4 + 2a_2F^2 \tag{14b}$$

where $a_n(n = 4,3,2,1)$ are determined in later steps. By substituting ansatz (12) and (13) into Eq. (11) and using Eqs. (14a) and (14b), we obtain two polynomials of $x^iF^j(\frac{dF(\xi)}{d\xi})^k, i,j,k$ are integers. By setting all the coefficients of the polynomials to zero, we have the following set of ODEs of t

$Eq.(11a)$

$$x^2F^2: \quad \Phi'(t) + 4\sigma^2(t)\Phi^2(t) + \alpha(t) = 0 \tag{15a}$$

$$xF^2: \quad \Gamma'(t) + 4\sigma^2(t)\Phi(t)\Gamma(t) = 0 \tag{15b}$$

$$F^6: \quad g_1(\sigma h^2)^2 + 6g_2(\sigma h^2)(\sigma p)^2a_4 = 0 \tag{15c}$$

$$F^4: \quad 4(\sigma p)^2a_4 - g_0(\sigma h^2)^2 - 4g_2(\sigma h^2)(\sigma p)^2a_2 = 0 \tag{15d}$$

$$F^2: \quad \Omega'(t) + (\sigma p)^2a_2 - g_2(\sigma h^2)(\sigma p)^2a_0 = 0 \tag{15e}$$

$Eq.(11b)$

$$xF': \quad p'(t) + 4\sigma^2(t)\Phi(t)p(t) = 0 \tag{15f}$$

$$F': \quad q'(t) + 2\sigma^2(t)\Gamma(t)p(t) = 0 \tag{15g}$$

$$F: \quad h'(t) + 2\sigma^2(t)\Phi(t)h(t) = 0 \tag{15h}$$

From Eqs. (15c,d,e) we can obtain the following important results,

$$\sigma(t)p(t) = C_1 \tag{16a}$$

$$\sigma(t)h^2(t) = C_2 \tag{16b}$$

where C_1 and C_2 are constants to be determined by the initial experimental setting. Again, from Eqs. (15c,d,e), we have

$$a_4 = -\frac{g_1 C_2}{6 g_2 C_1} \tag{17a}$$

$$a_2 = -\frac{g_1}{6 g_2^2 C_1} - \frac{g_0}{4 g_2 C_1^2} \tag{17b}$$

$$a_0 = -\frac{g_1}{6 g_2^3 C_1 C_2} - \frac{g_0}{4 g_2^2 C_1^2 C_2} + \frac{\Omega'(t)}{g_2 C_1^2 C_2} \tag{17c}$$

From Eq. (15e), $\Omega(t)$ is the freely varying parametric function. For $a_4 > 0$ and $a_2 < 0$, with a proper setting, choosing $\Omega'(t) = \frac{(2g_1C_1+3g_0g_2)^2}{96g_1g_2C_1} + \frac{g_1C_1}{6g_2^2} + \frac{g_0}{4g_2}$, we have $a_0 = \frac{a_2^2}{4a_4}$, making Eq. (14a) have the following form,

$$\frac{dF(\xi)}{d\xi} = \pm(a_4)^{1/2}(F^2(\xi) + \frac{a_2}{2a_4}) \tag{18}$$

Equation (18) is integrable and possesses the following solution,

$$F(\xi) = \sqrt{\frac{|a_2|}{2a_4}} \tanh((\frac{a_2}{2})^{1/2}\xi + C_0) \tag{19}$$

From ansatz (8) and (12), and using Eq. (19), we have

$$|\psi(\xi)| = D_1 C_2^{1/2} \sqrt{\frac{|a_2|}{2a_4}} \tanh((\frac{a_2}{2})^{1/2}\xi + C_0) = D_1 K \tanh(K D_0 x + q_0(t)) \tag{20}$$

where

$$K = (\frac{1}{2g_2} + \frac{3g_0}{4C_1 g_1})^{1/2} \tag{21a}$$

$$D_0 = |\frac{g_1 C_1}{6 g_2}|^{1/2} \tag{21b}$$

$$q_0(t) = \frac{K D_0 q(t)}{C_1} + C_0 \tag{21c}$$

D_1 is the normalization constant and C_0 is the constant determined by the initial condition. We can see that solution (20) is of the dark soliton type. Equation (2) possesses the analytical solution of the dark soliton type.

In Eq. (15e), when $\Omega'(t) + (\sigma p)^2 a_2 = 0$, we have $a_0 = 0$. For this case,

$$\frac{dF(\xi)}{d\xi} = \pm(a_4)^{1/2}\sqrt{F^4(\xi) + \frac{a_2}{a_4}F^2(\xi)} \tag{22}$$

which is integrable. For $a_2 < 0$

$$F(\xi) = \sqrt{\frac{a_2}{|a_4|}}\text{sech}((a_2)^{1/2}\xi + C_0')$$
$$|\psi(\xi)| = D_2 C_2^{1/2}\sqrt{\frac{a_2}{|a_4|}}\text{sech}((a_2)^{1/2}\xi + C_0') = \sqrt{2}D_2 K\text{sech}(\sqrt{2}KD_0 x + q_0'(t)) \qquad (23)$$

where D_2 is the normalization constant, $q_0'(t) = \sqrt{2}\frac{KD_0 q(t)}{C_1} + C_0'$,and C_0' is a constant determined by the initial condition. It is not difficult to notice that solution (23) is of the bright soliton type. Thus, the system described by the 1D GPE (Eq. (2)) supports both the dark and bright soliton-type solutions. In the limits that the higher-order effects are to be neglected, g_2 and g_1 are very small. In Eq. (21a), $\frac{1}{2g_2}$ is very large, but C_1 is set to a value such that K is of normal size ($K \ll |\frac{1}{2g_2}|$). From $|\frac{1}{2g_2}| = |\frac{3g_0}{4C_1 g_1}| - K$ or $|\frac{3g_0}{4C_1 g_1}| + K$, we know that $|\frac{3g_0}{4C_1 g_1}| \simeq |\frac{1}{2g_2}|$, or $D_0 = |\frac{g_1 C_1}{6g_2}|^{1/2} \simeq |\frac{g_0}{4}|^{1/2}$. From the normalization conditions of formula (20) and (23), we can see that C_1 should be chosen in such a way that $K = \frac{|2g_0|^{1/2}}{4}$. The dark soliton type and bright soliton type solutions reduce to

$$|\psi(\xi)|_{dark} = \frac{\sqrt{2}}{4}D_1\sqrt{|g_0|}\tanh(\frac{\sqrt{2}}{8}|g_0|x + q_0(t)) \qquad (24)$$
$$|\psi(\xi)|_{bright} = \frac{\sqrt{2}}{4}D_2\sqrt{|g_0|}\text{sech}(\frac{1}{4}|g_0|x + q_0'(t)) \qquad (25)$$

which are just soliton type solutions of the regular one-dimensional cubic nonlinear Schrödinger equation in the $g_2 \to 0$ and $g_1 \to 0$ limits of Eq. (2), with $g_0 < 0$ and $g_0 > 0$ corresponding to the bright soliton type and dark soliton type cases, respectively. Different from the three-dimensional case regarding incorporating higher-order interaction effects reported in prior work [15], where the higher-order interaction coefficient, g_2 plays an important role ensuring the stability of the system, the one-dimensional scenario setting supports stable bright and dark solitons. The parameters of these solitons, such as amplitude and width, depend principally on the leading lower order interaction coefficient g_0. Higher-order interaction coefficients g_1 and g_2 are several orders of magnitude smaller than g_0 and usually will not alter the qualitative features that arise from the dominant leading coefficient, g_0. From formula, (21a, b), we can get

$$D_0 = \frac{1}{2}[\frac{|g_0|}{1 - 2g_2 K^2}]^{1/2} \qquad (26)$$

where $K$is of order $\sqrt{|g_0|}$. For $g_2 \ll K^{-2}$, we can see $D_0 \simeq \frac{1}{2}|g_0|^{1/2}$. Since $g_2 = a^2/3 - ar_e/2$ and s-wave scattering length can be adjusted across a broad range via the Feshbach resonance experimental technique, we can visually capture the soliton shape difference arising from higher-order interaction effects for g_2 reaching $0.1K^{-2}$ through Fig 1 and Fig 2. However, in most cases $g_2 \ll K^{-2}$ holds and we cannot see a difference in the pictorial curves of soliton solutions (19) and (23), when compared to the case without consideration of the higher-order interaction (with $g_1 = 0$, $g_2 = 0$).

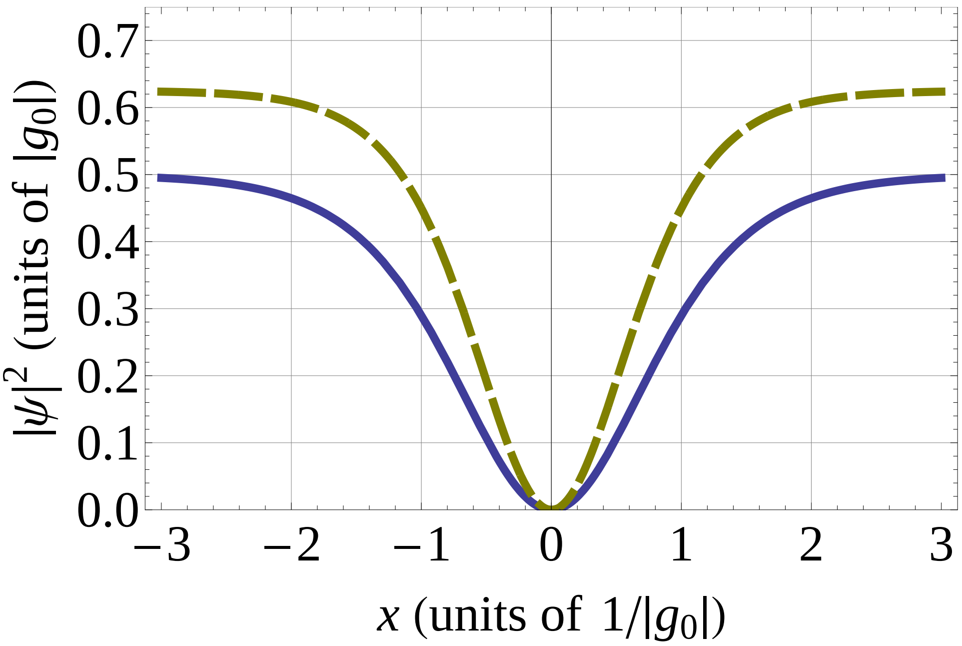

FIG. 1: Dark soliton solution $|\psi(x,t)|$ with $g_1 = 0$ and $g_2 = 0$ vs. x at time when $q_0(t) = 0$ (solid line) compared with dark soliton solution $|\psi(x,t)|$ with same setting except $g_2 = 0.1K^{-2}$ (dashed line).

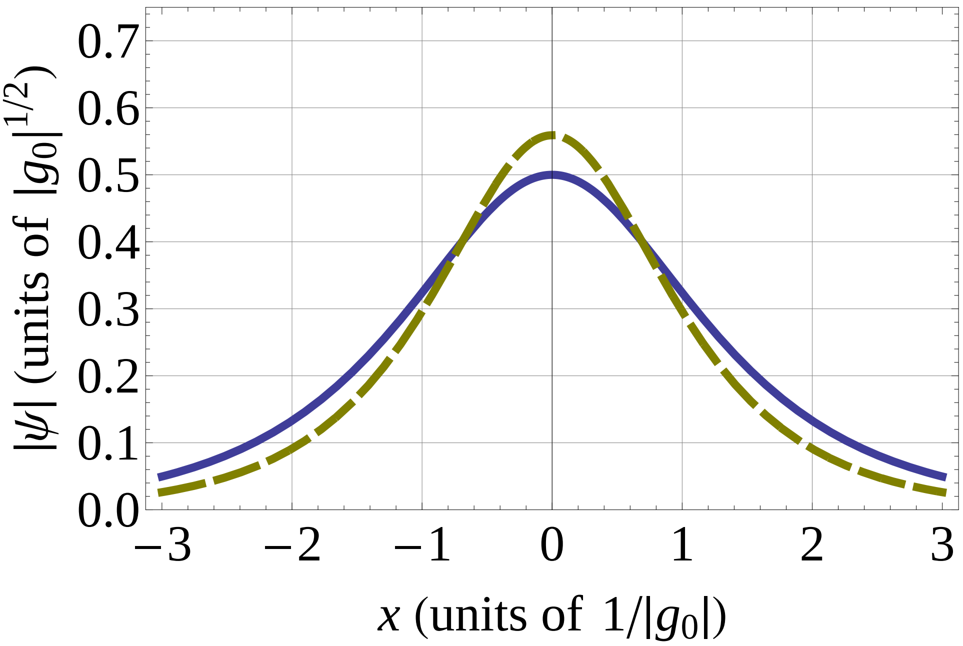

FIG. 2: Bright soliton solution $|\psi(x,t)|$ with $g_1 = 0$ and $g_2 = 0$ vs. x at time when $q_0(t) = 0$ (solid line) compared with bright soliton solution $|\psi(x,t)|$ with same setting except $g_2 = 0.1K^{-2}$ (dashed line).

IV. CONCLUSION

We studied the 1D GPE with the higher-order nonlinear interaction formulation, which incorporates the higher-order two-body effect and leading-order three-body scattering effects. Through the F-expansion method, combined with the coupled modulus-phase transformation, we analytically solve the GPE, identifying both the dark and bright soliton-type solutions without introducing any additional integrability constraints. The analytical results derived here can be used to guide experimental investigations in scenarios such as optical fibers and BECs, which exhibit higher-order nonlinear effects.

Acknowledgments

This work was supported by the National Natural Science Foundation (NSF) of China under Grant Nos. 11791240178, 11674338, 11547024.

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