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G_2-structures and quantization of non-geometric M-theory backgrounds

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Abstract: We describe the quantization of a four-dimensional locally non-geometric M-theory background dual to a twisted three-torus by deriving a phase space star product for deformation quantization of quasi-Poisson brackets related to the nonassociative algebra of octonions. The construction is based on a choice of G_2-structure which defines a nonassociative deformation of the addition law on the seven-dimensional vector space of Fourier momenta. We demonstrate explicitly that this star product reduces to that of the three-dimensional parabolic constant R-flux model in the contraction of M-theory to string theory, and use it to derive quantum phase space uncertainty relations as well as triproducts for the nonassociative geometry of the four-dimensional configuration space. By extending the G_2-structure to a Spin(7)-structure, we propose a 3-algebra structure on the full eight-dimensional M2-brane phase space which reduces to the quasi-Poisson algebra after imposing a particular gauge constraint, and whose deformation quantisation simultaneously encompasses both the phase space star products and the configuration space triproducts. We demonstrate how these structures naturally fit in with previous occurences of 3-algebras in M-theory.

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Contents

1 Introduction and summary

Non-geometric backgrounds of string theory are of interest not only because of their potential phenomenological applications, but also because they make explicit use of string duality symmetries which allows them to probe stringy regimes beyond supergravity. Many of them can be obtained by duality transformations of geometric backgrounds in flux compactifications of ten-dimensional and eleven-dimensional supergravity (see e.g. [13, 25, 27] for reviews). They have been studied extensively for the NS-NS sector of ten-dimensional supergravity which involves non-geometric Q-fluxes and R-fluxes.

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One of the most interesting recent assertions concerning non-geometric strings is that they probe noncommutative and nonassociative deformations of closed string background geometries [12, 40]. This has been further confirmed through explicit string computations in left-right asymmetric worldsheet conformal field theory [1, 5, 9, 14, 22] and in double field theory [15], and also in a topological open membrane sigma-model [44] and in Matrix theory [19] which both suggest origins for non-geometric fluxes in M-theory. Quantization of these backgrounds through explicit constructions of phase space star products were provided in [4, 37, 44, 45], and subsequently applied to building nonassociative models of quantum mechanics [45] and field theories [8, 43]; the physical significance and viability of these nonassociative structures in quantum mechanics is clarified in [16, 17]. A nonassociative theory of gravity describing the low-energy effective dynamics of closed strings in locally non-geometric backgrounds is currently under construction [2, 6, 7, 11, 45].

Until very recently, however, there had been two pieces missing from this story: firstly, the M-theory version of this deformation of geometry and, secondly, the role played by the octonions which are the archetypical example of a nonassociative algebra. In [30], these two ingredients are treated simultaneously and shown to be related. Their approach is based on lifting the non-geometric string theory R-flux to M-theory within the context of SL(5) exceptional field theory, following [10], which extends the $\mathrm{SL}(4) = \mathrm{Spin}(3,3)$ double field theory of string theory with three-dimensional target spaces and is relevant for compactifications of eleven-dimensional supergravity to seven dimensions. They argue that the phase space of the four-dimensional locally non-geometric M-theory background, which is dual to a twisted three-torus, lacks a momentum mode and consequently is seven-dimensional. The corresponding classical quasi-Poisson brackets can then be mapped precisely onto the Lie 2-algebra generated by the imaginary octonions. In the contraction limit $g_s \to 0$ which reduces M-theory to IIA string theory, the quasi-Poisson brackets contract to those of the non-geometric string theory R-flux background obtained via T-duality from a geometric three-torus with H-flux. The goal of the present paper is to quantize these phase space quasi-Poisson brackets, and to use it to describe various physical and geometrical features of the non-geometric M-theory background.

For this, we derive a phase space star product which lifts that of the three-dimensional string theory R-flux background [44], in the sense that it reduces exactly to it in the appropriate contraction limit which shrinks the M-theory circle to a point; our derivation is based on extending and elucidating deformation quantization of the coordinate algebra related to the imaginary octonions that was recently considered in [36]. The contraction limit reduces the complicated combinations of trigonometric functions appearing in the resulting star product to the elementary algebraic functions of the string theory case. Our constructions exploit relevant facts from calibrated geometry, particularly the theory of G_2-structures and Spin(7)-structures, simplified to the case of flat space, that may in future developments enable an extension of these considerations to more general compactifications of M-theory on manifolds of G_2-holonomy. In contrast to the usual considerations of calibrated geometry, however, for deformation quantization our structure manifolds involve corresponding bivectors and trivectors, respectively, rather than the more conventional three-forms and four-forms. All of the relevant deformation quantities are underpinned by vector cross products, whose theory we review in the following.

In fact, in this paper we emphasise a common underlying mathematical feature of the star products which quantise non-geometric string theory and M-theory backgrounds: they all originate, via the Baker-Campbell-Hausdorff formula, from the theory of cross products on real vector spaces; non-trivial cross products only exist in dimensions three (where they are associative) and seven (where they are nonassociative). In the three-dimensional case, relevant for the quantisation of the string theory R-flux background, the vector cross product determines a 3-cocycle among Fourier momenta that appears as a phase factor in the associator for the star product, whereas in the seven-dimensional case, relevant for the quantisation of the M-theory R-flux background, the vector cross product determines a nonassociative deformation of the sum of Fourier momenta. In the generalisation to the full eight-dimensional M-theory phase space, wherein the physical seven-dimensional R-flux background arises as a certain gauge constraint, triple cross products determine an underlying 3-algebraic structure akin to those previously found in studies of multiple M2-branes (see e.g. [3] for a review). Higher associativity of the 3-bracket is governed by a 5-bracket, but it is not related in any simple way to a 5-vector. This parallels the situation with the lift of non-geometric string theory fluxes: unlike the NS-NS R-flux, the M-theory R-flux is not a multivector. This point of view should prove helpful in understanding generalisations of these considerations to both higher dimensions and to the treatment of missing momentum modes in M-theory backgrounds dual to non-toroidal string vacua.

Armed with the phase space star product, we can use it to describe various physical and geometrical features of the membrane phase space. In particular, we derive quantum uncertainty relations which explicitly exhibit novel minimal area cells in the M-theory phase space, as well as minimal volumes demonstrating a coarse-graining of both configuration space and phase space itself, in contrast to the string theory case [45]. We also derive configuration space triproducts, in the spirit of [2], which quantize the four-dimensional 3-Lie algebra A_4 and suggest an interpretation of the quantum geometry of the M-theory R-flux background as a foliation by fuzzy membrane worldvolume three-spheres; in the contraction limit $g_s \to 0$, these triproducts consistently reduce to those of the string theory configuration space which quantize the three-dimensional Nambu-Heisenberg 3-Lie algebra [2, 14]. In contrast to the string theory case, this curving of the configuration space by three-spheres also results in a novel associative but noncommutative deformation of the geometry of momentum space itself. The origin of these configuration space triproducts in the present case is most naturally understood in terms of quantisation of the 3-algebraic structure of the eight-dimensional membrane phase space: the G_2-structure, which determines the star product quantising the seven-dimensional phase space, extends to a Spin(7)-structure determining phase space triproducts that restrict to those on the four-dimensional configuration space.

The organisation of the remainder of this paper is as follows. In section 2 we briefly review relevant aspects of the parabolic non-geometric string theory R-flux model on a three-torus with constant fluxes and its deformation quantization; in particular, we point out that star product algebras of functions generally spoil the classical Malcev-Poisson algebraic structure that sometimes appears in discussions of nonassociativity in physics, see e.g. [17, 29]. As preparation for the M-theory lift of this model, in section 3 we review pertinent properties of the algebra of octonions and the associated linear algebra of vector cross products, and use them to derive a deformed summation operation on Fourier momenta that defines the pertinent star products. This technical formalism is then applied in section 4 to derive a star product quantising the phase space quasi-Poisson brackets proposed by [30], whose derivation we also review; we demonstrate in detail that it reduces appropriately to that of section 2 in the contraction limit that sends M-theory to IIA string theory, and further apply it to derive quantum uncertainty relations, as well as the nonassociative geometry of configuration space induced by the M-theory R-flux and the radius of the M-theory circle. Finally, after briefly reviewing how 3-algebra structures have arisen in other contexts in M-theory as motivation, in section 5 we extend the vector cross products to triple cross products and use them to postulate a novel 3-algebraic structure of the full eight-dimensional membrane phase space which reproduces the quasi-Poisson brackets of [30] upon imposing a suitable gauge fixing constraint; we describe a partial quantisation of this 3-algebra and show how it naturally encompasses both the phase space and the configuration space nonassociative geometry from section 4.

2 Quantization of string theory R-flux background

In this section we review and elaborate on features of the quantization of the parabolic phase space model for the constant string R-flux background in three dimensions.

2.1 Quasi-Poisson algebra for non-geometric string theory fluxes

In our situations of interest, a non-geometric string theory R-flux background originates as a double T-duality transformation of a supergravity background M of dimension D with geometric flux; this sends closed string winding number into momentum. The non-trivial windings, and hence the momentum modes in the R-flux background, are classified by the first homology group $H_1(M,\mathbb{Z})$. The R-flux is represented by a trivector or locally by a totally antisymmetric rank three tensor R^{ijk} in the framework of double field theory: it is given by taking suitable "covariant" derivatives $\hat{\partial}^i$ along the string winding directions of a globally well-defined bivector β^{jk}, which is related to an $O(D)\times O(D)$ rotation of the generalised vielbein containing the background metric and two-form B-field after T-duality.

In the parabolic flux model in three dimensions, the background M is a twisted three-torus which is a circle bundle over the two-torus T^2 whose degree $d\in\mathbb{Z}$ coincides with the cohomology class of the three-form H-flux $H=\mathrm{d}\,\mathrm{vol}_{T^3}$ in the original T-duality frame (consisting of a three-torus T^3). Since the first homology group is $H_1(M,\mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_d$, there are non-trivial windings along all three directions of M and the non-torsion winding numbers map to momentum modes of the R-flux model. In this way, the position and momentum coordinates $\boldsymbol{x}=(x^i)$ and $\boldsymbol{p}=(p_i)$ of closed strings propagating in the background of a constant R-flux $R^{ijk}=R\,\varepsilon^{ijk}$ (with $R=d$) define a quasi-Poisson structure on phase space T^*M with the classical brackets [40]

$$\{x^i,x^j\}=\tfrac{\ell_s^3}{\hbar^2}\,R^{ijk}\,p_k\,,\qquad \{x^i,p_j\}=\delta^i_j\qquad\text{and}\qquad \{p_i,p_j\}=0\,, \tag{2.1}$$

where ℓ_s is the string length and ε^{ijk}, $i,j,k = 1,2,3$, is the alternating symbol in three dimensions normalised as $\varepsilon^{123} = +1$; unless otherwise explicitly stated, in the following repeated indices are always understood to be summed over. It is convenient to rewrite (2.1) in a more condensed form as

$$\{x^I, x^J\} = \Theta^{IJ}(x) = \begin{pmatrix} \frac{\ell_s^3}{\hbar^2}\, R^{ijk}\, p_k & -\delta^i_j \\ \delta^i_j & 0 \end{pmatrix} \qquad \text{with} \quad x = (x^I) = (\boldsymbol{x}, \boldsymbol{p})\,, \tag{2.2}$$

which identifies the components of a bivector $\Theta = \frac{1}{2}\,\Theta^{IJ}(x)\,\frac{\partial}{\partial x^I} \wedge \frac{\partial}{\partial x^J}$. Strictly speaking, here the coordinates $\boldsymbol{x}$ live on a three-torus T^3, but as we are only interested in local considerations we take the decompactification limit and consider $\boldsymbol{x} \in \mathbb{R}^3$ throughout this paper. From the perspective of double field theory, in this frame the dual phase space coordinates $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}}$ have canonical Poisson brackets among themselves and vanishing brackets with $\boldsymbol{x}, \boldsymbol{p}$, and the totality of brackets among the double phase space coordinates $(\boldsymbol{x}, \boldsymbol{p}, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ can be rotated to any other T-duality frame via an $O(3,3)$ transformation [14, 15]; the same is true of the star product reviewed below [4]. For ease of notation, in this paper we restrict our attention to the R-flux frame and suppress the dependence on the dual coordinates $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$.

For any three functions f, g and h on phase space, the classical Jacobiator is defined as

$$\{f,g,h\} := \{f,\{g,h\}\} - \{\{f,g\},h\} - \{g,\{f,h\}\}\ . \tag{2.3}$$

By construction it is antisymmetric in all arguments, trilinear and satisfies the Leibniz rule. For the brackets (2.1) one finds

$$\{f,g,h\} = \frac{3\,\ell_s^3}{\hbar^2}\, R^{ijk}\, \partial_i f\, \partial_j g\, \partial_k h\,, \tag{2.4}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, which further obeys the fundamental identity of a 3-Lie algebra; in fact, when restricted to functions on configuration space, it defines the standard Nambu-Poisson bracket on $\mathbb{R}^3$. Hence the classical brackets of the constant R-flux background generate a nonassociative phase space algebra.

2.2 Phase space star product

To describe the quantization of closed strings in the R-flux background and their dynamics, as well as the ensuing nonassociative geometry of the non-geometric background, we define a star product by associating a (formal) differential operator $\hat{f}$ to a function f as

$$(f \star g)(x) = \hat{f} \triangleright g(x)\,, \tag{2.5}$$

where the symbol $\triangleright$ denotes the action of a differential operator on a function. In particular one has

$$x^I \star f = \hat{x}^I \triangleright f(x)\ . \tag{2.6}$$

The operators

$$\hat{x}^I = x^I + \tfrac{\mathrm{i}\,\hbar}{2}\, \Theta^{IJ}(x)\, \partial_J\,, \tag{2.7}$$

with $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial_{i+3} = \frac{\partial}{\partial p_i}$ for $i = 1, 2, 3$, close to an associative algebra of differential operators; in particular

$$[\hat{x}^i, \hat{x}^j] = \tfrac{\mathrm{i}\,\ell_s^3}{\hbar}\, R^{ijk}\,\big(\hat{p}_k + \mathrm{i}\,\hbar\,\partial_k\big) \ . \tag{2.8}$$

Taking (2.6) and (2.7) as a definition of the star product, one may easily calculate the quantum brackets

$$[x^I, x^J]_\star := x^I \star x^J - x^J \star x^I = \mathrm{i}\,\hbar\,\Theta^{IJ} \qquad \text{and} \qquad [x^i, x^j, x^k]_\star = -3\,\ell_s^3\, R^{ijk} \ , \tag{2.9}$$

which thereby provide a quantization of the classical brackets (2.1); in particular, the quantum 3-bracket represents a Nambu-Heisenberg algebra which quantizes the standard classical Nambu-Poisson bracket (2.4) on $\mathbb{R}^3$.

To define the star product $f \star g$ between two arbitrary functions on phase space, we introduce the notion of *Weyl star product* by requiring that, for any f, the differential operator $\hat{f}$ defined by (2.5) can be obtained by symmetric ordering of the operators $\hat{x}^I$. Let $\tilde{f}(k)$ denote the Fourier transform of $f(x)$, with $k = (k_I) = (\boldsymbol{k}, \boldsymbol{l})$ and $\boldsymbol{k} = (k_i), \boldsymbol{l} = (l^i) \in \mathbb{R}^3$. Then

$$\hat{f} = W(f) := \int \frac{\mathrm{d}^6 k}{(2\pi)^6}\ \tilde{f}(k)\ \mathrm{e}^{\,-\,\mathrm{i}\,k_I\,\hat{x}^I} \ . \tag{2.10}$$

For example, $W(x^I\,x^J) = \frac{1}{2}\,(\hat{x}^I\,\hat{x}^J + \hat{x}^J\,\hat{x}^I)$. Weyl star products satisfy

$$\big(x^{I_1} \cdots x^{I_n}\big) \star f = \frac{1}{n!} \sum_{\sigma \in S_n} x^{I_{\sigma(1)}} \star \big(x^{I_{\sigma(2)}} \star \cdots \star (x^{I_{\sigma(n)}} \star f) \cdots\big) \ , \tag{2.11}$$

where the sum runs over all permutations in the symmetric group S_n of degree n. It should be stressed that the correspondence $f \mapsto \hat{f}$ is not an algebra representation: since the star product that we consider here is not necessarily associative, in general $\widehat{f \star g} \neq \hat{f} \circ \hat{g}$.

To obtain an explicit form for the corresponding star product we first observe that since $[k_I\,x^I, k_J\,\Theta^{JL}\,\partial_L] = 0$ one can write

$$\mathrm{e}^{\,-\,\mathrm{i}\,k_I\,\hat{x}^I} = \mathrm{e}^{\,-\,\mathrm{i}\,k\cdot x}\ \mathrm{e}^{\,\frac{\hbar}{2}\,k_I\,\Theta^{IJ}(x)\,\partial_J} \ ,$$

with $\cdot$ the standard Euclidean inner product of vectors. By the relation $k_I k_L \Theta^{IJ} \partial_J \Theta^{LM} \partial_M = \frac{\ell_s^3}{\hbar^2}\, k_i\, k_l\, R^{lki}\,\partial_k = 0$ it follows that

$$\big(k_I\,\Theta^{IJ}\,\partial_J\big)^n = k_{I_1} \cdots k_{I_n}\ \Theta^{I_1 J_1} \cdots \Theta^{I_n J_n}\ \partial_{J_1} \cdots \partial_{J_n} \ .$$

One may also write

$$\big(\overleftarrow{\partial}_I\,\Theta^{IJ}\,\overrightarrow{\partial}_J\big)^n = \overleftarrow{\partial}_{I_1} \cdots \overleftarrow{\partial}_{I_n}\ \Theta^{I_1 J_1} \cdots \Theta^{I_n J_n}\ \overrightarrow{\partial}_{J_1} \cdots \overrightarrow{\partial}_{J_n} \ ,$$

where $\overleftarrow{\partial}_I$ and $\overrightarrow{\partial}_I$ stand for the action of the derivative $\frac{\partial}{\partial x^I}$ on the left and on the right correspondingly. Thus the Weyl star product representing quantization of the quasi-Poisson bracket (2.1) can be written in terms of a bidifferential operator as

$$(f \star g)(x) = \int \frac{\mathrm{d}^6 k}{(2\pi)^6}\ \tilde{f}(k)\ \mathrm{e}^{\,-\,\mathrm{i}\,k_I\,\hat{x}^I} \triangleright g(x) = f(x)\ \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,\overleftarrow{\partial}_I\,\Theta^{IJ}(x)\,\overrightarrow{\partial}_J}\ g(x) \ . \tag{2.12}$$

It is easy to see that (2.12) is Hermitean, $(f \star g)^* = g^* \star f^*$, and unital, $f \star 1 = f = 1 \star f$; it is moreover 2-cyclic and 3-cyclic under integration in the sense of [45]. This star product first appeared in [44] where it was derived using the Kontsevich formula for deformation quantization of twisted Poisson structures. Its realisation through an associative algebra of differential operators was first pointed out in [45]. The significance and utility of this star product in understanding non-geometric string theory is exemplified in [2, 4, 44, 45].

For later use, let us rewrite the star product $f \star g$ in integral form through the Fourier transforms $\tilde{f}$ and $\tilde{g}$ alone. The star product of plane waves is given by

$$\mathrm{e}^{\,\mathrm{i}\, k \cdot x} \star \mathrm{e}^{\,\mathrm{i}\, k' \cdot x} = \mathrm{e}^{\,\mathrm{i}\, \mathcal{B}(k,k') \cdot x}\,, \tag{2.13}$$

where

$$\mathcal{B}(k,k') \cdot x := (\boldsymbol{k} + \boldsymbol{k}') \cdot \boldsymbol{x} + (\boldsymbol{l} + \boldsymbol{l}') \cdot \boldsymbol{p} - \tfrac{\ell_s^3}{2\hbar}\, R\, \boldsymbol{p} \cdot (\boldsymbol{k} \times_\varepsilon \boldsymbol{k}') + \tfrac{\hbar}{2}\, \big(\boldsymbol{l} \cdot \boldsymbol{k}' - \boldsymbol{k} \cdot \boldsymbol{l}'\big)\,, \tag{2.14}$$

with $(\boldsymbol{k} \times_\varepsilon \boldsymbol{k}')_i = \varepsilon_{ijl}\, k_j\, k_l'$ the usual cross product of three-dimensional vectors. Then

$$(f \star g)(x) = \int \frac{\mathrm{d}^6 k}{(2\pi)^6}\, \frac{\mathrm{d}^6 k'}{(2\pi)^6}\, \tilde{f}(k)\, \tilde{g}(k')\, \mathrm{e}^{\,\mathrm{i}\, \mathcal{B}(k,k') \cdot x}\,. \tag{2.15}$$

In this form, the star product follows from application of the Baker-Campbell-Hausdorff formula to the brackets (2.9) [4, 44], whereas the nonassociativity of the star product is encoded in the additive associator

$$\mathcal{A}(k,k',k'') := \big(\mathcal{B}(\mathcal{B}(k,k'),k'')) - \mathcal{B}(k,\mathcal{B}(k',k''))\big) \cdot x = \tfrac{\ell_s^3}{2}\, R\, \boldsymbol{k} \cdot (\boldsymbol{k}' \times_\varepsilon \boldsymbol{k}'')\,, \tag{2.16}$$

which is antisymmetric in all arguments, and in fact defines a certain 3-cocycle [4, 44, 45].

2.3 Alternativity and Malcev-Poisson identity

A star product is *alternative* if the star associator of three functions

$$A_\star(f,g,h) := f \star (g \star h) - (f \star g) \star h$$

vanishes whenever any two of them are equal (equivalently $A_\star(f,g,h)$ is completely antisymmetric in its arguments, or 'alternating'). For such products the Jacobiator is proportional to the associator. Since the function (2.16) vanishes whenever any two of its arguments are equal, it follows that the star product (2.15) restricted to Schwartz functions is alternative. However, for generic smooth functions on phase space this property is violated; in fact, the simple example $A_\star\big(|\boldsymbol{x}|^2, |\boldsymbol{x}|^2, |\boldsymbol{x}|^2\big) = 2\,\mathrm{i}\, \frac{\ell_s^6}{\hbar^4}\, R^2\, \boldsymbol{p} \cdot \boldsymbol{x}$ shows that alternativity is even violated on the phase space coordinate algebra $\mathbb{C}[\boldsymbol{x}, \boldsymbol{p}]$.[1]

[1] To be more precise, alternativity generally holds only for functions with function-valued (rather than distribution-valued) Fourier transform, such as Lebesgue integrable functions. For some accidental cases, such as linear coordinate functions or powers of single coordinate generators, the star product is trivially alternative. The star products considered in this paper are best behaved on algebras of Schwartz functions, so we will often make this restriction. It is interesting to understand more precisely the class of functions on which the star product is alternative, but this is not relevant for the present paper; see [18] for a systematic and general analysis of the non-alternativity of a class of star products containing ours.

Another way of understanding this violation, which will be relevant in later sections, is via the observation of [36] that a necessary condition for the star product $f \star g$ to be alternative is that the corresponding classical bracket $\{f, g\}$ satisfies the Malcev-Poisson identity [54]. For any three functions f, g and h the Malcev-Poisson identity can be written as

$$\{f, g, \{f, h\}\} = \{\{f, g, h\}, f\} \ . \tag{2.17}$$

As a simple example, let us check both sides of (2.17) for the three functions $f = x^1$, $g = x^3\, p_1$ and $h = x^2$. Since $\{f, h\} = \{x^1, x^2\} = \frac{\ell_s^3}{\hbar^2}\, R\, p_3$ does not depend on $\boldsymbol{x}$, the left-hand side of (2.17) vanishes by (2.4). On the other hand, one has $\{f, g, h\} = -\frac{3\ell_s^3}{\hbar^2}\, R\, p_1$ and consequently for the right-hand side of (2.17) one finds $\{\{f, g, h\}, f\} = -\frac{3\ell_s^3}{\hbar^2}\, R\, \{p_1, x^1\} = \frac{3\ell_s^3}{\hbar^2}\, R$. It follows that the classical string R-flux coordinate algebra $\mathbb{C}[\boldsymbol{x}, \boldsymbol{p}]$ is *not* a Malcev algebra (beyond linear order in the phase space coordinates), and consequently no star product representing a quantization of (2.1) can be alternative.[2]

3 G_2-structures and deformation quantization

In this paper we are interested in the lift of the string theory phase space model of section 2 to M-theory. As the conjectural quasi-Poisson structure from [30], which we review in section 4, is intimately related to the nonassociative algebra of octonions, in this section we shall take a technical detour, recalling some of the algebraic and geometric features of octonions, together with their related linear algebra, in the form that we need in this paper. In particular we will derive, following [28, 36], a star product decribing quantization of the coordinate algebra based on the imaginary octonions, elucidating various aspects which will be important for later sections and which are interesting in their own right. The reader uninterested in these technical details may temporarily skip ahead to section 4.

3.1 Octonions

The algebra $\mathbb{O}$ of octonions is the best known example of a nonassociative but alternative algebra. Every octonion $X \in \mathbb{O}$ can be written in the form

$$X = k^0\, \mathbb{1} + k^A\, e_A \tag{3.1}$$

where $k^0, k^A \in \mathbb{R}$, $A = 1, \dots, 7$, while $\mathbb{1}$ is the identity element and the imaginary unit octonions e_A satisfy the multiplication law

$$e_A\, e_B = -\delta_{AB}\, \mathbb{1} + \eta_{ABC}\, e_C \ . \tag{3.2}$$

Here η_{ABC} is a completely antisymmetric tensor of rank three with nonvanishing values

$$\eta_{ABC} = +1 \qquad \text{for} \quad ABC = 123,\ 435,\ 471,\ 516,\ 572,\ 624,\ 673\ . \tag{3.3}$$

[2] The counterexamples always involve momentum dependence, and so alternativity can be restored in a suitable sense by restricting to configuration space $\boldsymbol{p} = \boldsymbol{0}$. We return to this point in section 4.5.

Introducing $f_i := e_{i+3}$ for $i = 1, 2, 3$, the algebra (3.2) can be rewritten as

$$\begin{aligned} e_i\, e_j &= -\delta_{ij}\, \mathbb{1} + \varepsilon_{ijk}\, e_k\,, \\ e_i\, f_j &= \delta_{ij}\, e_7 - \varepsilon_{ijk}\, f_k\,, \\ f_i\, f_j &= \delta_{ij}\, \mathbb{1} - \varepsilon_{ijk}\, e_k\,, \\ e_7\, e_i &= f_i \qquad \text{and} \qquad f_i\, e_7 = e_i\,, \end{aligned} \tag{3.4}$$

which emphasises a subalgebra $\mathbb{H}$ of quaternions generated by e_i; we will use this component form of the algebra $\mathbb{O}$ frequently in what follows.

The algebra $\mathbb{O}$ is neither commutative nor associative. The commutator algebra of the octonions is given by

$$[e_A, e_B] := e_A\, e_B - e_B\, e_A = 2\, \eta_{ABC}\, e_C\,, \tag{3.5}$$

which can be written in components as

$$\begin{aligned} [e_i, e_j] &= 2\, \varepsilon_{ijk}\, e_k \qquad \text{and} \qquad [e_7, e_i] = 2\, f_i\,, \\ [f_i, f_j] &= -2\, \varepsilon_{ijk}\, e_k \qquad \text{and} \qquad [e_7, f_i] = -2\, e_i\,, \\ [e_i, f_j] &= 2\, (\delta_{ij}\, e_7 - \varepsilon_{ijk}\, f_k)\ . \end{aligned} \tag{3.6}$$

The structure constants η_{ABC} satisfy the contraction identity

$$\eta_{ABC}\, \eta_{DEC} = \delta_{AD}\, \delta_{BE} - \delta_{AE}\, \delta_{BD} + \eta_{ABDE}\,, \tag{3.7}$$

where η_{ABCD} is a completely antisymmetric tensor of rank four with nonvanishing values

$$\eta_{ABCD} = +1 \qquad \text{for} \quad ABCD = 1267,\ 1346,\ 1425,\ 1537,\ 3247,\ 3256,\ 4567\ .$$

One may also represent the rank four tensor η_{ABCD} as the dual of the rank three tensor η_{ABC} through

$$\eta_{ABCD} = \tfrac{1}{6}\, \varepsilon_{ABCDEFG}\, \eta_{EFG}\,, \tag{3.8}$$

where $\varepsilon_{ABCDEFG}$ is the alternating symbol in seven dimensions normalized as $\varepsilon_{1234567} = +1$. Together they satisfy the contraction identity

$$\begin{aligned} \eta_{AEF}\, \eta_{ABCD} &= \delta_{EB}\, \eta_{FCD} - \delta_{FB}\, \eta_{ECD} + \delta_{EC}\, \eta_{BFD} - \delta_{FC}\, \eta_{BED} \\ &\quad + \delta_{ED}\, \eta_{BCF} - \delta_{FD}\, \eta_{BCE}\ . \end{aligned} \tag{3.9}$$

Taking into account (3.7), for the Jacobiator we get

$$[e_A, e_B, e_C] := [e_A, [e_B, e_C]] + [e_C, [e_A, e_B]] + [e_B, [e_C, e_A]] = -12\, \eta_{ABCD}\, e_D\,, \tag{3.10}$$

and the alternative property of the algebra $\mathbb{O}$ implies that the Jacobiator is proportional to the associator, i.e., $[X, Y, Z] = 6\, \big((X\, Y)\, Z - X\, (Y\, Z)\big)$ for any three octonions $X, Y, Z \in \mathbb{O}$.

JHEP02(2017)099

3.2 Cross products

An important related linear algebraic entity in this paper will be the notion of a *cross product* on a real inner product space [32] (see [33, 53] for nice introductions), generalising the well known cross product of vectors in three dimensions. They are intimately related to the four normed algebras over the field of real numbers $\mathbb{R}$ (namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$), and likewise cross products only exist for vector spaces of real dimensions 0, 1, 3 and 7. In dimensions 0 and 1 the cross product vanishes, in three dimensions it is the standard one $\times_\varepsilon$ (up to sign) which has appeared already in our discussion of the star product for the string theory R-flux background, while in seven dimensions it can be defined (uniquely up to orthogonal transformation) in a Cayley basis for vectors $\vec{k} = (k^A), \vec{k}' = (k'^A) \in \mathbb{R}^7$ by

$$(\vec{k} \times_\eta \vec{k}'\,)^A := \eta^{ABC}\, k^B\, k'^C \tag{3.11}$$

with the structure constants η^{ABC} introduced in (3.3). To help describe and interpret the underlying geometry of the seven-dimensional cross product, it is useful to note that it can be expressed in terms of the algebra of imaginary octonions by writing $X_{\vec{k}} := k^A\, e_A$ and observing that

$$X_{\vec{k} \times_\eta \vec{k}'} = \tfrac{1}{2}\, \big[X_{\vec{k}}, X_{\vec{k}'}\big]\ . \tag{3.12}$$

This bilinear product satisfies the defining properties of cross products [53]:

(C1) $\vec{k} \times_\eta \vec{k}' = -\vec{k}' \times_\eta \vec{k}$;

(C2) $\vec{k} \cdot (\vec{k}' \times_\eta \vec{k}''\,) = -\vec{k}' \cdot (\vec{k} \times_\eta \vec{k}''\,)$;

(C3) $|\vec{k} \times_\eta \vec{k}'\,|^2 = |\vec{k}|^2\, |\vec{k}'\,|^2 - (\vec{k} \cdot \vec{k}'\,)^2$, where $|\vec{k}| = \sqrt{\vec{k} \cdot \vec{k}}$ is the Euclidean vector norm.

As usual property **(C1)** is equivalent to the statement that the cross product $\vec{k} \times_\eta \vec{k}'$ is non-zero if and only if $\vec{k}$, $\vec{k}'$ are linearly independent vectors, property **(C2)** is equivalent to the statement that it is orthogonal to both $\vec{k}$ and $\vec{k}'$, while property **(C3)** states that its norm calculates the area of the triangle spanned by $\vec{k}$ and $\vec{k}'$ in $\mathbb{R}^7$. However, unlike the three-dimensional cross product $\times_\varepsilon$, due to (3.10) it does not obey the Jacobi identity: using (3.7) the Jacobiator is given by

$$\begin{aligned} \vec{J}_\eta(\vec{k}, \vec{k}', \vec{k}''\,) &:= (\vec{k} \times_\eta \vec{k}'\,) \times_\eta \vec{k}'' + (\vec{k}' \times_\eta \vec{k}''\,) \times_\eta \vec{k} + (\vec{k} \times_\eta \vec{k}''\,) \times_\eta \vec{k}' \\ &= 3\,\big(\,(\vec{k} \times_\eta \vec{k}'\,) \times_\eta \vec{k}'' + (\vec{k}' \cdot \vec{k}''\,)\,\vec{k} - (\vec{k} \cdot \vec{k}''\,)\,\vec{k}'\,\big)\,, \end{aligned} \tag{3.13}$$

which can be represented through the associator on the octonion algebra $\mathbb{O}$ as

$$X_{\vec{J}_\eta(\vec{k}, \vec{k}', \vec{k}''\,)} = \tfrac{1}{4}\, \big[X_{\vec{k}}, X_{\vec{k}'}, X_{\vec{k}''}\big] = \tfrac{3}{2}\, \big((X_{\vec{k}}\, X_{\vec{k}'})\, X_{\vec{k}''} - X_{\vec{k}}\, (X_{\vec{k}'}\, X_{\vec{k}''})\big)\ .$$

Hence properties **(C1)** and **(C2)** imply that the products $(\,\times_\eta\,,\,\cdot\,)$ make the vector space $V = \mathbb{R}^7$ into a pre-Courant algebra [56]. Only rotations in the 14-dimensional exceptional group $G_2 \subset \mathrm{SO}(7)$ preserve the cross product $\times_\eta$, where the action of G_2 can be described as the transitive action on the unit sphere $S^6 \subset V$ identified with the homogeneous space $S^6 \simeq G_2/\mathrm{SU}(3)$.[3] A G_2-*structure* on an oriented seven-dimensional vector space V is the choice of a cross product that can be written as (3.11) in a suitable oriented frame.

[3]This means that the Lie group G_2 is the stabilizer of a unit vector in V.

JHEP02(2017)099

3.3 Baker-Campbell-Hausdorff formula and vector star sums

Let us now work out the Baker-Campbell-Hausdorff formula for $\mathbb{O}$, which will be crucial for the derivations which follow. The alternative property

$$(X\,Y)\,Y = X\,(Y\,Y) \qquad \text{and} \qquad X\,(X\,Y) = (X\,X)\,Y\,,$$

for any pair of octonions $X, Y \in \mathbb{O}$, implies in particular that $(X\,X)\,X = X\,(X\,X)$. Hence the quantity $X^n := X\,(X\,(\cdots(X\,X)\cdots))$ is well-defined independently of the ordering of parantheses for all $n \geq 0$ (with $X^0 := \mathbb{1}$). This implies that power series in octonions are readily defined [39, 48], and in particular one can introduce the octonion exponential function $\mathrm{e}^{\,X} := \sum_{n\geq 0} \frac{1}{n!}\,X^n$. By setting $X = X_{\vec k} = k^A\,e_A$ with $\vec k = (k^A) \in \mathbb{R}^7$ and using the multiplication law (3.2), one can derive an octonionic version of de Moivre's theorem [39, 48]

$$\mathrm{e}^{\,X_{\vec k}} = \cos|\vec k|\ \mathbb{1} + \frac{\sin|\vec k|}{|\vec k|}\ X_{\vec k}\ . \tag{3.14}$$

We can take (3.14) to *define* the octonion exponential function $\mathrm{e}^{\,X_{\vec k}} \in \mathbb{O}$.

One can now repeat the derivation of [28], appendix C to obtain a closed form for the Baker-Campbell-Hausdorff formula for $\mathbb{O}$. Multiplying two octonion exponentials of the form (3.14) together using (3.2) we get

$$\begin{aligned}\mathrm{e}^{\,X_{\vec k}}\ \mathrm{e}^{\,X_{\vec k'}} &= \Big(\cos|\vec k|\cos|\vec k'| - \frac{\sin|\vec k|\sin|\vec k'|}{|\vec k|\,|\vec k'|}\ \vec k\cdot\vec k'\Big)\ \mathbb{1} \\ &\quad + \frac{\cos|\vec k'|\sin|\vec k|}{|\vec k|}\ X_{\vec k} + \frac{\cos|\vec k|\sin|\vec k'|}{|\vec k'|}\ X_{\vec k'} - \frac{\sin|\vec k|\sin|\vec k'|}{|\vec k|\,|\vec k'|}\ X_{\vec k\times_\eta\vec k'}\,,\end{aligned} \tag{3.15}$$

where $\vec k\times_\eta\vec k'$ is the seven-dimensional vector cross product (3.11) on V. On the other hand, the Baker-Campbell-Hausdorff expansion is defined by

$$\mathrm{e}^{\,X_{\vec k}}\ \mathrm{e}^{\,X_{\vec k'}} =: \mathrm{e}^{\,X_{\vec{\mathcal{B}}{}'_\eta(\vec k,\vec k'\,)}} = \cos\big|\vec{\mathcal{B}}{}'_\eta(\vec k,\vec k'\,)\big|\ \mathbb{1} + \frac{\sin\big|\vec{\mathcal{B}}{}'_\eta(\vec k,\vec k'\,)\big|}{\big|\vec{\mathcal{B}}{}'_\eta(\vec k,\vec k'\,)\big|}\ X_{\vec{\mathcal{B}}{}'_\eta(\vec k,\vec k'\,)}\ . \tag{3.16}$$

By comparing (3.15) and (3.16) we arrive at

$$\begin{aligned}\vec{\mathcal{B}}{}'_\eta(\vec k,\vec k'\,) &= \frac{\cos^{-1}\Big(\cos|\vec k|\cos|\vec k'| - \frac{\sin|\vec k|\sin|\vec k'|}{|\vec k|\,|\vec k'|}\ \vec k\cdot\vec k'\Big)}{\sin\cos^{-1}\Big(\cos|\vec k|\cos|\vec k'| - \frac{\sin|\vec k|\sin|\vec k'|}{|\vec k|\,|\vec k'|}\ \vec k\cdot\vec k'\Big)} \\ &\quad \times\Big(\frac{\cos|\vec k'|\sin|\vec k|}{|\vec k|}\ \vec k + \frac{\cos|\vec k|\sin|\vec k'|}{|\vec k'|}\ \vec k' - \frac{\sin|\vec k|\sin|\vec k'|}{|\vec k|\,|\vec k'|}\ \vec k\times_\eta\vec k'\Big)\ .\end{aligned} \tag{3.17}$$

To rewrite (3.17) in a more manageable form, we use the G_2-structure on V to define a binary operation $\circledast_\eta$ on the unit ball $B^7 \subset V$ consisting of vectors $\vec p$ with $|\vec p| \leq 1$. To any pair of vectors $\vec p, \vec p\,' \in B^7$, it assigns the vector

$$\vec p \circledast_\eta \vec p\,' = \epsilon_{\vec p,\vec p\,'}\,\Big(\sqrt{1-|\vec p\,'|^2}\ \vec p + \sqrt{1-|\vec p|^2}\ \vec p\,' - \vec p\times_\eta\vec p\,'\Big)\,, \tag{3.18}$$

where $\epsilon_{\vec{p}_1,\vec{p}_2} = \pm 1$ is the sign of $\sqrt{1-|\vec{p}_1|^2}\,\sqrt{1-|\vec{p}_2|^2} - \vec{p}_1 \cdot \vec{p}_2$ satisfying

$$\epsilon_{\vec{p}_1,\vec{p}_2}\,\epsilon_{\vec{p}_1 \circledast_\eta \vec{p}_2,\vec{p}_3} = \epsilon_{\vec{p}_1,\vec{p}_2 \circledast_\eta \vec{p}_3}\,\epsilon_{\vec{p}_2,\vec{p}_3}\,, \tag{3.19}$$

which follows by properties **(C1)** and **(C2)** of the cross product from section 3.2; these sign factors have a precise intrinsic origin that we shall describe in section 5.3. Using the properties **(C1)**–**(C3)** from section 3.2 we find

$$1 - |\vec{p} \circledast_\eta \vec{p}\,'\,|^2 = \Big(\sqrt{1-|\vec{p}\,|^2}\,\sqrt{1-|\vec{p}\,'\,|^2} - \vec{p}\cdot\vec{p}\,'\,\Big)^2 \geq 0\,, \tag{3.20}$$

and so the vector $\vec{p} \circledast_\eta \vec{p}\,'$ indeed also belongs to the unit ball $B^7 \subset V$. We call the binary operation (3.18) on B^7 the *vector star sum* of $\vec{p}$ and $\vec{p}\,'$. It admits an identity element given by the zero vector in V,

$$\vec{p} \circledast_\eta \vec{0} = \vec{p} = \vec{0} \circledast_\eta \vec{p}\,, \tag{3.21}$$

and the inverse of $\vec{p} \in B^7$ is $-\vec{p} \in B^7$,

$$\vec{p} \circledast_\eta (-\vec{p}\,) = \vec{0} = (-\vec{p}\,) \circledast_\eta \vec{p}\ .$$

It is noncommutative with commutator given by the G_2-structure as

$$\vec{p} \circledast_\eta \vec{p}\,' - \vec{p}\,' \circledast_\eta \vec{p} = -2\,\vec{p} \times_\eta \vec{p}\,'\,, \tag{3.22}$$

and using (3.19) we find that the corresponding associator is related to the Jacobiator (3.13) for the cross product (3.11) through

$$\vec{A}_\eta(\vec{p},\vec{p}\,',\vec{p}\,''\,) := (\vec{p} \circledast_\eta \vec{p}\,'\,) \circledast_\eta \vec{p}\,'' - \vec{p} \circledast_\eta (\vec{p}\,' \circledast_\eta \vec{p}\,''\,) = \tfrac{2}{3}\,\vec{J}_\eta(\vec{p},\vec{p}\,',\vec{p}\,''\,)\ . \tag{3.23}$$

It follows that the components of the associator (3.23) take the form

$$A_\eta(\vec{p},\vec{p}\,',\vec{p}\,''\,)^A = \eta^{ABCD}\,p^B\,p'^C\,p''^D\ .$$

It is non-vanishing but totally antisymmetric, and hence the seven-dimensional vector star sum (3.18) is nonassociative but alternative, making the ball $B^7 \subset V$ into a 2-group.

To extend the 2-group structure (3.18) over the entire vector space V we introduce the map

$$\vec{p} = \frac{\sin(\hbar\,|\vec{k}|)}{|\vec{k}|}\,\vec{k} \qquad \text{with} \quad k^A \in \mathbb{R}\ . \tag{3.24}$$

The inverse map is given by

$$\vec{k} = \frac{\sin^{-1}|\vec{p}\,|}{\hbar\,|\vec{p}\,|}\,\vec{p}\ .$$

Then for each pair of vectors $\vec{k},\vec{k}' \in V$, following [28], appendix C we can use the trigonometric identities

$$\sin\cos^{-1}s = \cos\sin^{-1}s = \sqrt{1-s^2} \tag{3.25}$$

for $-1 \leq s \leq 1$ to find that the deformed vector sum (3.17) can be written in terms of the vector star sum as

$$\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,) := \tfrac{1}{\hbar}\,\vec{\mathcal{B}}_\eta'(\hbar\,\vec{k},\hbar\,\vec{k}'\,) = \frac{\sin^{-1}|\vec{p} \circledast_\eta \vec{p}\,'\,|}{\hbar\,|\vec{p} \circledast_\eta \vec{p}\,'\,|}\,\vec{p} \circledast_\eta \vec{p}\,'\,\Big|_{\vec{p}=\vec{k}\,\sin(\hbar\,|\vec{k}|)/|\vec{k}|}\ . \tag{3.26}$$

From (3.26) one immediately infers the following properties:

(B1) $\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,) = -\vec{\mathcal{B}}_\eta(-\vec{k}',-\vec{k}\,)$;

(B2) $\vec{\mathcal{B}}_\eta(\vec{k},\vec{0}\,) = \vec{k} = \vec{\mathcal{B}}_\eta(\vec{0},\vec{k}\,)$;

(B3) Perturbative expansion: $\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,) = \vec{k}+\vec{k}' - 2\,\hbar\,\vec{k}\times_\eta\vec{k}' + O(\hbar^2)$;

(B4) The associator

$$\vec{\mathcal{A}}_\eta(\vec{k},\vec{k}',\vec{k}''\,) := \vec{\mathcal{B}}_\eta\big(\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,)\,,\,\vec{k}''\,\big) - \vec{\mathcal{B}}_\eta\big(\vec{k}\,,\,\vec{\mathcal{B}}_\eta(\vec{k}',\vec{k}''\,)\big)$$

is antisymmetric in all arguments.

One can explicitly compute the products $\big(\mathrm{e}^{\,X_{\vec{k}}}\,\mathrm{e}^{\,X_{\vec{k}'}}\,\big)\,\mathrm{e}^{\,X_{\vec{k}''}}$ and $\mathrm{e}^{\,X_{\vec{k}}}\,\big(\mathrm{e}^{\,X_{\vec{k}'}}\,\mathrm{e}^{\,X_{\vec{k}''}}\,\big)$ of octonion exponentials using (3.14) and (3.15), and after a little calculation using the identities (3.20) and (3.25) one finds for the associator

$$\vec{\mathcal{A}}_\eta(\vec{k},\vec{k}',\vec{k}''\,) = \frac{\sin^{-1}\big|(\vec{p}\circledast_\eta\vec{p}'\,)\circledast_\eta\vec{p}''\,\big|}{\hbar\,\big|(\vec{p}\circledast_\eta\vec{p}'\,)\circledast_\eta\vec{p}''\,\big|}\ \vec{A}_\eta(\vec{p},\vec{p}',\vec{p}''\,)\Big|_{\vec{p}=\vec{k}\,\sin(\hbar\,|\vec{k}|)/|\vec{k}|} \ . \tag{3.27}$$

3.4 Quasi-Poisson algebra and Malcev-Poisson identity

Consider the algebra of classical brackets on the coordinate algebra $\mathbb{C}[\vec{\xi}\,]$ which is isomorphic to the algebra (3.5),

$$\{\xi_A,\xi_B\}_\eta = 2\,\eta_{ABC}\,\xi_C \ , \tag{3.28}$$

where $\vec{\xi}=(\xi_A)$ with $\xi_A\in\mathbb{R}$, $A=1,\dots,7$. This bracket is bilinear, antisymmetric and satisfies the Leibniz rule by definition. Introducing $\sigma^i := \xi_{i+3}$ for $i=1,2,3$ and $\sigma^4 := \xi_7$, one may rewrite (3.28) in components as

$$\begin{aligned} \{\xi_i,\xi_j\}_\eta &= 2\,\varepsilon_{ijk}\,\xi_k \qquad \text{and} \qquad \{\sigma^4,\xi_i\}_\eta = 2\,\sigma^i \ , \\ \{\sigma^i,\sigma^j\}_\eta &= -2\,\varepsilon^{ijk}\,\xi_k \qquad \text{and} \qquad \{\sigma^4,\sigma^i\}_\eta = -2\,\xi_i \ , \\ \{\sigma^i,\xi_j\}_\eta &= -2\,(\delta^i_j\,\sigma^4 - \varepsilon^i{}_{jk}\,\sigma^k) \ . \end{aligned} \tag{3.29}$$

Using (3.10) the non-vanishing Jacobiators can be written as

$$\begin{aligned} \{\xi_i,\xi_j,\sigma^k\}_\eta &= -12\,(\varepsilon_{ij}{}^k\,\sigma^4 + \delta^k_j\,\sigma_i - \delta^k_i\,\sigma_j) \ , \\ \{\xi_i,\sigma^j,\sigma^k\}_\eta &= 12\,(\delta^j_i\,\xi_k - \delta^k_i\,\xi_j) \ , \\ \{\sigma^i,\sigma^j,\sigma^k\}_\eta &= 12\,\varepsilon^{ijk}\,\sigma^4 \ , \\ \{\xi_i,\xi_j,\sigma^4\}_\eta &= 12\,\varepsilon_{ijk}\,\sigma^k \ , \\ \{\xi_i,\sigma^j,\sigma^4\}_\eta &= 12\,\varepsilon_i{}^{jk}\,\xi_k \ , \\ \{\sigma^i,\sigma^j,\sigma^4\}_\eta &= -12\,\varepsilon^{ijk}\,\sigma^k \ . \end{aligned} \tag{3.30}$$

The Malcev-Poisson identity (2.17) is satisfied for monomials. However, we can show in an analogous way as in section 2.3 that it is violated in general on $\mathbb{C}[\vec{\xi}\,]$. For this, consider $f=\xi_1$, $g=\xi_3\,\sigma^1$ and $h=\xi_2$. Using (3.29) and (3.30) one finds that the left-hand side of (2.17) is given by $\{f,g,\{f,h\}_\eta\}_\eta = -24\,\xi_3\,\sigma^3$, while for the right-hand side one has $\{\{f,g,h\}_\eta,f\}_\eta = 24\,(\xi_2\,\sigma^2 - \xi_3\,\sigma^3)$. We conclude that the Malcev-Poisson identity for the classical brackets (3.28) is violated. This is in contrast to the well-known fact that the pre-Lie algebra (3.5) of imaginary octonions defines a Malcev algebra, due to the identity (3.9) and the multiplication law (3.2) on the finite-dimensional algebra $\mathbb{O}$. Hence the Malcev identity (2.17) holds for octonions, while it is violated in general for the quasi-Poisson structure (3.28) on the infinite-dimensional polynomial algebra $\mathbb{C}[\vec{\xi}\,]$; this is also implied by the general results of [18].

3.5 G_2-symmetric star product

Let us now work out the quantization of the classical brackets (3.28). Consider the quasi-Poisson bivector

$$\Theta_\eta := \eta_{ABC}\,\xi_C\,\psi^A\wedge\psi^B \qquad \text{with} \quad \psi^A = \partial^A = \tfrac{\partial}{\partial\xi_A} \tag{3.31}$$

defining the brackets (3.28). It can be regarded as a pre-homological potential on $T^*\Pi V$ with coordinates (ψ^A,ξ_B) and canonical Poisson bracket; then the corresponding derived brackets are $[[\xi_A,\xi_B]]_{\Theta_\eta} = \{\xi_A,\xi_B\}_\eta$ giving $T^*\Pi V$ the structure of a symplectic nearly Lie 2-algebra [49]. We can extend this structure to the entire algebra of functions by defining a star product through

$$(f\star_\eta g)(\vec{\xi}\,) = \int \frac{\mathrm{d}^7\vec{k}}{(2\pi)^7}\,\frac{\mathrm{d}^7\vec{k}'}{(2\pi)^7}\ \tilde{f}(\vec{k}\,)\,\tilde{g}(\vec{k}'\,)\ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,)\cdot\vec{\xi}}\,, \tag{3.32}$$

where again $\tilde{f}$ stands for the Fourier transform of the function f and $\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,)$ is the deformed vector sum (3.26). By definition it is the Weyl star product.

Due to the properties **(B1)** and **(B2)** from section 3.3 of the deformed vector addition $\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,)$, this star product is Hermitean, $(f\star_\eta g)^* = g^*\star_\eta f^*$, and unital, $f\star_\eta 1 = f = 1\star_\eta f$. It can be regarded as a quantization of the dual of the pre-Lie algebra (3.5) underlying the octonion algebra $\mathbb{O}$; in particular, by property **(B3)** it provides a quantization of the quasi-Poisson bracket (3.28): defining $[f,g]_{\star_\eta} = f\star_\eta g - g\star_\eta f$, we have

$$\lim_{\hbar\to 0}\frac{[f,g]_{\star_\eta}}{\mathrm{i}\,\hbar} = 2\,\xi_A\,\eta_{ABC}\,\partial^B f\,\partial^C g = \{f,g\}_\eta\ . \tag{3.33}$$

Property **(B4)** implies that the star product (3.32) is alternative on monomials and Schwartz functions, but not generally because of the violation of the Malcev-Poisson identity discussed in section 3.4.

Let us calculate $\xi_A\star_\eta f$ explicitly using (3.32). We have

$$\xi_A\star_\eta f = -\int\frac{\mathrm{d}^7\vec{k}'}{(2\pi)^7}\ \xi_D\ \frac{\partial\mathcal{B}_\eta(\vec{k},\vec{k}'\,)^D}{\partial k^A}\bigg|_{\vec{k}=\vec{0}}\ \tilde{f}(\vec{k}'\,)\ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\eta(\vec{0},\vec{k}'\,)\cdot\vec{\xi}}$$

and after some algebra one finds

$$\frac{\partial\mathcal{B}_\eta(\vec{k},\vec{k}'\,)^D}{\partial k^A}\bigg|_{\vec{k}=\vec{0}} = -\hbar\,\eta_{ADE}\,k'^{\,E} + \delta_{AD}\,\hbar\,|\vec{k}'\,|\cot(\hbar\,|\vec{k}'\,|) + \frac{k'_A\,k'_D}{|\vec{k}'\,|^2}\,\Big(\hbar\,|\vec{k}'\,|\cot(\hbar\,|\vec{k}'\,|)-1\Big)\ .$$

Taking into account property **(B2)** from section 3.3 and integrating over $\vec{k}'$, we arrive at

$$\xi_A \star_\eta f = \Big(\xi_A + \mathrm{i}\,\hbar\,\eta_{ABC}\,\xi_C\,\partial^B \\ + \big(\xi_A\,\triangle_{\vec{\xi}} - (\vec{\xi}\cdot\nabla_{\vec{\xi}}\,)\,\partial_A\big)\,\triangle_{\vec{\xi}}^{-1}\,\big(\hbar\,\triangle_{\vec{\xi}}^{1/2}\,\coth(\hbar\,\triangle_{\vec{\xi}}^{1/2}\,) - 1\big)\Big) \triangleright f(\vec{\xi}\,) \tag{3.34}$$

where $\triangle_{\vec{\xi}} = \nabla_{\vec{\xi}}^2 = \partial_A\,\partial^A$ is the flat space Laplacian in seven dimensions. In particular for the Jacobiator one finds

$$[\xi_A, \xi_B, \xi_C]_{\star_\eta} = 12\,\hbar^2\,\eta_{ABCD}\,\xi_D\,, \tag{3.35}$$

which thereby provides a quantization of the classical 3-brackets (3.30).

3.6 SL(3)-symmetric star product

Setting $e_A = 0$ for $A = 4,5,6,7$ (equivalently $f_i = e_7 = 0$) reduces the nonassociative algebra of octonions $\mathbb{O}$ to the associative algebra of quaternions $\mathbb{H}$, whose imaginary units e_i generate the $\mathfrak{su}(2)$ Lie algebra $[e_i, e_j] = 2\,\varepsilon_{ijk}\,e_k$. To see this reduction at the level of our vector products, consider the splitting of the seven-dimensional vector space V according to the components of section 3.4 with $\vec{k} = (\boldsymbol{l}, \boldsymbol{k}, k_4)$, where $\boldsymbol{l} = (l^i), \boldsymbol{k} = (k_i) \in \mathbb{R}^3$. With respect to this decomposition, by using (3.6) the seven-dimensional cross product can be written in terms of the three-dimensional cross product as

$$\vec{k} \times_\eta \vec{k}' = \big(\boldsymbol{l} \times_\varepsilon \boldsymbol{l}' - \boldsymbol{k} \times_\varepsilon \boldsymbol{k}' + k_4'\,\boldsymbol{k} - k_4\,\boldsymbol{k}'\,,\, \boldsymbol{k} \times_\varepsilon \boldsymbol{l}' - \boldsymbol{l} \times_\varepsilon \boldsymbol{k}' + k_4\,\boldsymbol{l}' - k_4'\,\boldsymbol{l}\,,\, \boldsymbol{l}\cdot\boldsymbol{k}' - \boldsymbol{k}\cdot\boldsymbol{l}'\,\big)\,. \tag{3.36}$$

The symmetry group G_2 preserving $\times_\eta$ contains a closed SL(3) subgroup acting on these components as $\boldsymbol{l} \mapsto g\,\boldsymbol{l}$, $\boldsymbol{k} \mapsto (g^{-1})^\top\,\boldsymbol{k}$ and $k_4 \mapsto k_4$ for $g \in \mathrm{SL}(3)$. In particular, reduction to the three-dimensional subspace spanned by e_i gives

$$(\boldsymbol{l}, \boldsymbol{0}, 0) \times_\eta (\boldsymbol{l}', \boldsymbol{0}, 0) = (\boldsymbol{l} \times_\varepsilon \boldsymbol{l}', \boldsymbol{0}, 0)\,, \tag{3.37}$$

and so yields the expected three-dimensional cross product. This reduction is implemented in all of our previous formulas by simply replacing $\times_\eta$ with $\times_\varepsilon$ throughout. In particular, the corresponding Jacobiator $\boldsymbol{J}_\varepsilon$ from (3.13) now vanishes by a well-known identity for the cross product in three dimensions; as a consequence, the pair $(\times_\varepsilon\,,\,\cdot\,)$ defines a Courant algebra structure on the vector space $\mathbb{R}^3$ [56]. The cross product $\times_\varepsilon$ is preserved by the full rotation group $\mathrm{SO}(3) \subset G_2$ in this case, acting transitively on the unit sphere $S^2 \simeq \mathrm{SO}(3)/\mathrm{SO}(2)$ in $\mathbb{R}^3$.

Similarly, the reduction of the seven-dimensional vector star sum (3.18) on this three-dimensional subspace reproduces the three-dimensional vector star sum $\circledast_\varepsilon$ from [38],

$$(\boldsymbol{q}, \boldsymbol{0}, 0) \circledast_\eta (\boldsymbol{q}', \boldsymbol{0}, 0) = (\boldsymbol{q} \circledast_\varepsilon \boldsymbol{q}', \boldsymbol{0}, 0)\,, \tag{3.38}$$

which by (3.23) is now associative; as a consequence, it makes the unit ball $B^3 \subset \mathbb{R}^3$ into a non-abelian group. The reduction of the deformed vector sum (3.26) reproduces the three-dimensional vector sum $\boldsymbol{\mathcal{B}}_\varepsilon(\boldsymbol{l}, \boldsymbol{l}'\,)$ from [28, 38],

$$\vec{\mathcal{B}}_\eta\big((\boldsymbol{l}, \boldsymbol{0}, 0)\,,\,(\boldsymbol{l}', \boldsymbol{0}, 0)\big) = \big(\boldsymbol{\mathcal{B}}_\varepsilon(\boldsymbol{l}, \boldsymbol{l}'\,), \boldsymbol{0}, 0\big)\,, \tag{3.39}$$

JHEP02(2017)099

with vanishing associator (3.27). From (3.39) it follows that, for functions f, g on this three-dimensional subspace, the corresponding star product $(f \star_\varepsilon g)(\boldsymbol{\xi}, \mathbf{0}, 0)$ from (3.32) reproduces the associative star product of [28, 38] for the quantization of the dual of the Lie algebra $\mathfrak{su}(2)$. In the general case (3.36), the evident similarity with the terms in the vector sum (2.14) will be crucial for what follows.

4 Quantization of M-theory R-flux background

In this section we use the constructions of section 3 to derive a suitable star product which quantizes the four-dimensional locally non-geometric M-theory background which is dual to a twisted torus [30]. We demonstrate explicitly that it is the lift of the star product which quantizes the string theory R-flux background of section 2, by showing that it reduces to the star product of section 2.2 in the weak string coupling limit which reduces M-theory to IIA string theory. We apply this construction to the description of the quantum mechanics of M2-branes in the non-geometric background, as well as of the noncommutative and nonassociative geometry these membranes probe.

4.1 Quasi-Poisson algebra for non-geometric M-theory fluxes

Let us start by reviewing the derivation of the classical quasi-Poisson algebra for the four-dimensional non-geometric M-theory background from [30], beginning again with some general considerations. String theory on a background M is dual to M-theory on the total space of an oriented circle bundle

$$\begin{array}{ccc} S^1 & \hookrightarrow & \widetilde{M} \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

over M, where the radius $\lambda \in \mathbb{R}$ of the circle fibre translates into the string coupling constant g_s. T-duality transformations become U-duality transformations sending membrane wrapping numbers to momentum modes, which are classified by the second homology group $H_2(\widetilde{M}, \mathbb{Z})$. The homology groups of $\widetilde{M}$ are generally related to those of M through the Gysin exact sequence

$$\cdots \longrightarrow H_k(\widetilde{M}, \mathbb{Z}) \xrightarrow{\pi_*} H_k(M, \mathbb{Z}) \xrightarrow{\cap e} H_{k-2}(M, \mathbb{Z}) \xrightarrow{\pi^!} H_{k-1}(\widetilde{M}, \mathbb{Z}) \longrightarrow \cdots \tag{4.1}$$

where π_* and $\pi^!$ are the usual pushforward and Gysin pullback on homology, and $\cap e$ is the cap product with the Euler class $e \in H^2(M, \mathbb{Z})$ of the fibration. For instance, in the case of a trivial fibration $\widetilde{M} = M \times S^1$, wherein $e = 0$, the Gysin sequence collapses to a collection of short exact sequences, and in particular by the Künneth theorem there is a splitting $H_2(\widetilde{M}, \mathbb{Z}) \simeq H_1(M, \mathbb{Z}) \oplus H_2(M, \mathbb{Z})$. More generally, for an automorphism g of M we can define a twisted lift to an M-bundle $\widetilde{M}_g$ over S^1 whose total space is the quotient of $M \times \mathbb{R}$ by the $\mathbb{Z}$-action[4]

$$(\boldsymbol{x}, t) \longmapsto \big(g^n(\boldsymbol{x}), t + 2\pi\, n\, \lambda\big)\,,$$

where $\boldsymbol{x} \in M$, $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. The Gysin sequence (4.1) shows that $H_1(M, \mathbb{Z})$ generally classifies "vertical" wrapping modes around the S^1-fibre which are dual to momenta along M, whereas $H_2(M, \mathbb{Z})$ classifies "horizontal" wrapping modes dual to momenta along the S^1-fibre.

In the situations we are interested in from section 2.1, the lift of the non-geometric string theory R-flux can be described locally within the framework of SL(5) exceptional field theory as a quantity $R^{\mu,\nu\rho\alpha\beta}$: it is derived by taking suitable "covariant" derivatives $\hat{\partial}^{\mu\nu}$ along the membrane wrapping directions of a trivector $\Omega^{\rho\alpha\beta}$, which is related to an SO(5) rotation of the generalised vielbein containing the background metric and three-form C-field after U-duality. Its first index is a vector index while the remaining indices define a completely antisymmetric rank four tensor. Because of the possibly non-trivial Euler class $e \in H^2(M, \mathbb{Z})$, it is proposed in [30] that, generally, the phase space $T^*\widetilde{M}$ of the locally non-geometric background in M-theory is constrained to a codimension one subspace defined by the momentum slice

$$R^{\mu,\nu\rho\alpha\beta}\, p_\mu = 0\,, \tag{4.2}$$

reflecting the absence of momentum modes in the dual R-flux background.

This proposal was checked explicitly in [30] for the parabolic toroidal flux model in three dimensions from section 2.1, wherein M is a twisted three-torus. In the M-theory lift to the four-manifold $\widetilde{M} = M \times S^1$, with local coordinates $(x^\mu) = (\boldsymbol{x}, x^4)$ where $\boldsymbol{x} \in M$ and $x^4 \in S^1$, by Poincaré duality the second homology group is $H_2(M, \mathbb{Z}) = H^1(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, which does not contain the requisite non-trivial two-cycle that would allow for non-trivial wrapping modes dual to momenta along the x^4-direction, i.e., $p_4 = 0$. As a consequence the phase space of the M-theory lift of the R-flux background is only seven-dimensional and lacks a momentum space direction. The only non-vanishing components of the M-theory R-flux in this case are $R^{4,\mu\nu\alpha\beta} = R\, \varepsilon^{\mu\nu\alpha\beta}$, $\mu, \nu, \dots = 1, 2, 3, 4$, where $\varepsilon^{\mu\nu\alpha\beta}$ is the alternating symbol in four dimensions normalised as $\varepsilon^{1234} = +1$. Note that this reduction only occurs in the presence of non-trivial flux: when $d = 0$ there is no torsion in the homology or cohomology of the torus $M = T^3$ and Poincaré-Hodge duality implies $H_2(M, \mathbb{Z}) = H_1(M, \mathbb{Z}) = H^1(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, so that all two-cycles are homologically non-trivial.

The main conjecture of [30] is that the classical brackets of this seven-dimensional phase space are given by the quasi-Poisson brackets of section 3.4 after a suitable choice of affine structure on the vector space $\mathbb{R}^7$, i.e., a choice of linear functions. For this, let us introduce the 7×7 matrix

$$\varLambda = \big(\varLambda^{AB}\big) = \frac{1}{2\hbar} \begin{pmatrix} 0 & \sqrt{\lambda\, \ell_s^3\, R}\ \mathbb{1}_3 & 0 \\ 0 & 0 & \sqrt{\lambda^3\, \ell_s^3\, R} \\ -\lambda\, \hbar\ \mathbb{1}_3 & 0 & 0 \end{pmatrix} \tag{4.3}$$

[4]The precise sort of automorphism should be specified by the intended application; for example, g could be an automorphism preserving some background form field.

with $\mathbb{1}_3$ the 3×3 identity matrix. The matrix Λ is non-degenerate as long as all parameters are non-zero, but it is not orthogonal. Using it we define new coordinates

$$\vec{x} = \big(x^A\big) = \big(\boldsymbol{x}, x^4, \boldsymbol{p}\big) := \Lambda\,\vec{\xi} = \tfrac{1}{2\hbar}\,\Big(\sqrt{\lambda\,\ell_s^3\,R}\;\boldsymbol{\sigma}\,,\;\sqrt{\lambda^3\,\ell_s^3\,R}\;\sigma^4\,,\;-\lambda\,\hbar\;\boldsymbol{\xi}\Big)\;. \tag{4.4}$$

From the classical brackets (3.28) one obtains the quasi-Poisson algebra

$$\{x^A, x^B\}_\lambda = 2\,\lambda^{ABC}\,x^C \qquad \text{with} \quad \lambda^{ABC} := \Lambda^{AA'}\,\Lambda^{BB'}\,\eta_{A'B'C'}\,\Lambda^{-1}_{C'C}\,, \tag{4.5}$$

which can be written in components as

$$\begin{aligned}
\{x^i, x^j\}_\lambda &= \tfrac{\ell_s^3}{\hbar^2}\,R^{4,ijk4}\,p_k \qquad \text{and} \qquad \{x^4, x^i\}_\lambda = \tfrac{\lambda\,\ell_s^3}{\hbar^2}\,R^{4,1234}\,p^i\,, \\
\{x^i, p_j\}_\lambda &= \delta^i_j\,x^4 + \lambda\,\varepsilon^i{}_{jk}\,x^k \quad\;\; \text{and} \qquad \{x^4, p_i\}_\lambda = \lambda^2\,x_i\,, \\
\{p_i, p_j\}_\lambda &= -\lambda\,\varepsilon_{ijk}\,p^k\,,
\end{aligned} \tag{4.6}$$

where we recall that the M-theory radius λ incorporates the string coupling constant g_s. The corresponding Jacobiators are

$$\{x^A, x^B, x^C\}_\lambda = -12\,\lambda^{ABCD}\,x^D \qquad \text{with} \quad \lambda^{ABCD} := \Lambda^{AA'}\,\Lambda^{BB'}\,\Lambda^{CC'}\,\eta_{A'B'C'D'}\,\Lambda^{-1}_{D'D}\,,$$

with the components

$$\begin{aligned}
\{x^i, x^j, x^k\}_\lambda &= \tfrac{3\,\ell_s^3}{\hbar^2}\,R^{4,ijk4}\,x^4\,, \\
\{x^i, x^j, x^4\}_\lambda &= -\tfrac{3\,\lambda^2\,\ell_s^3}{\hbar^2}\,R^{4,ijk4}\,x_k\,, \\
\{p_i, x^j, x^k\}_\lambda &= \tfrac{3\,\lambda\,\ell_s^3}{\hbar^2}\,R^{4,1234}\,\big(\delta^j_i\,p^k - \delta^k_i\,p^j\big)\,, \\
\{p_i, x^j, x^4\}_\lambda &= \tfrac{3\,\lambda^2\,\ell_s^3}{\hbar^2}\,R^{4,ijk4}\,p_k\,, \\
\{p_i, p_j, x^k\}_\lambda &= -3\,\lambda^2\,\varepsilon_{ij}{}^k\,x^4 - 3\,\lambda\,\big(\delta^k_j\,x_i - \delta^k_i\,x_j\big)\,, \\
\{p_i, p_j, x^4\}_\lambda &= 3\,\lambda^3\,\varepsilon_{ijk}\,x^k\,, \\
\{p_i, p_j, p_k\}_\lambda &= 0\;.
\end{aligned} \tag{4.7}$$

The crucial observation of [30] is that in the contraction limit $\lambda = 0$ which shrinks the M-theory circle to a point, i.e., the weak string coupling limit $g_s \to 0$ which reduces M-theory to IIA string theory, the classical brackets (4.6) and (4.7) of the M-theory R-flux background reduce to the quasi-Poisson structure (2.1) and (2.4) of the string theory R-flux background; in this limit the circle fibre coordinate x^4 is central in the algebra defined by (4.6) and so may be set to any non-zero constant value, which we conveniently take to be $x^4 = 1$. In the following we will extend this observation to the quantum level. As in section 3.4 the classical coordinate algebra here is not a Malcev algebra, which is another way of understanding the violation of the Malcev-Poisson identity from section 2.3 in the contraction limit $\lambda = 0$.

JHEP02(2017)099

4.2 Phase space star product

We will now quantize the brackets (4.6). For this, we use the G_2-symmetric star product (3.32) to define a star product of functions on the seven-dimensional M-theory phase space by the prescription

$$(f \star_\lambda g)(\vec{x}\,) = (f_\Lambda \star_\eta g_\Lambda)(\vec{\xi}\,)\big|_{\vec{\xi}=\Lambda^{-1}\vec{x}} \tag{4.8}$$

where $f_\Lambda(\vec{\xi}\,) := f(\Lambda\,\vec{\xi}\,)$. Using the deformed vector addition $\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,)$ from (3.26), we can write (4.8) as

$$(f \star_\lambda g)(\vec{x}\,) = \int \frac{\mathrm{d}^7\vec{k}}{(2\pi)^7}\,\frac{\mathrm{d}^7\vec{k}'}{(2\pi)^7}\;\tilde{f}(\vec{k}\,)\,\tilde{g}(\vec{k}'\,)\;\mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\eta(\Lambda\,\vec{k},\Lambda\,\vec{k}'\,)\cdot\Lambda^{-1}\,\vec{x}}\;. \tag{4.9}$$

The star product (4.9) may also be written in terms of a (formal) bidifferential operator as

$$(f \star_\lambda g)(\vec{x}\,) = f(\vec{x}\,)\;\mathrm{e}^{\,\mathrm{i}\,\vec{x}\cdot(\Lambda^{-1}\,\vec{\mathcal{B}}_\eta(-\mathrm{i}\,\Lambda\,\overleftarrow{\partial},-\mathrm{i}\,\Lambda\,\overrightarrow{\partial}\,)+\mathrm{i}\,\overleftarrow{\partial}+\mathrm{i}\,\overrightarrow{\partial}\,)}\;g(\vec{x}\,)\,,$$

which identifies it as a cochain twist deformation [45]. For the same reasons as (3.32) the star product (4.9) is unital, Hermitean and Weyl, and it is alternative on monomials and Schwartz functions.

To show that (4.9) provides a quantization of the brackets (4.6), we calculate $x^A \star_\lambda f$ by making the change of affine structure (4.4) in (3.34) to get

$$x^A \star_\lambda f = \hat{x}^A \triangleright f$$

where

$$\hat{x}^A = x^A + \mathrm{i}\,\hbar\,\lambda^{ABC}\,x^C\,\partial_B + \hbar^2\,\big(x^A\,\tilde{\triangle}_{\vec{x}} - (\vec{x}\cdot\tilde{\nabla}_{\vec{x}}\,)\,\tilde{\partial}^A\big)\,\chi\big(\hbar^2\,\tilde{\triangle}_{\vec{x}}\big)\,, \tag{4.10}$$

with

$$\tilde{\nabla}_{\vec{x}} = \big(\tilde{\partial}^A\big) := \big(\Lambda^{BA}\,\tfrac{\partial}{\partial x^B}\big) = \tfrac{1}{2\hbar}\,\Big(\sqrt{\lambda\,\ell_s^3\,R}\;\nabla_{\boldsymbol{x}}\,,\;\sqrt{\lambda^3\,\ell_s^3\,R}\;\tfrac{\partial}{\partial x^4}\,,\;-\lambda\,\hbar\;\nabla_{\boldsymbol{p}}\Big) \tag{4.11}$$

and

$$\tilde{\triangle}_{\vec{x}} = \tilde{\nabla}_{\vec{x}}^2 = \tfrac{\lambda}{4\hbar^2}\,\Big(\ell_s^3\,R\,\triangle_{\boldsymbol{x}} + \lambda^2\,\ell_s^3\,R\,\tfrac{\partial^2}{\partial x_4^2} + \lambda\,\hbar^2\,\triangle_{\boldsymbol{p}}\Big)\;. \tag{4.12}$$

We have also introduced the (formal) differential operator

$$\chi\big(\tilde{\triangle}_{\vec{x}}\big) := \tilde{\triangle}_{\vec{x}}^{-1}\,\Big(\tilde{\triangle}_{\vec{x}}^{1/2}\,\coth\tilde{\triangle}_{\vec{x}}^{1/2} - 1\Big)\;.$$

We thus find for the algebra of star commutators and Jacobiators

$$[x^A,x^B]_{\star_\lambda} = 2\,\mathrm{i}\,\hbar\,\lambda^{ABC}\,x^C \qquad \text{and} \qquad [x^A,x^B,x^C]_{\star_\lambda} = 12\,\hbar^2\,\lambda^{ABCD}\,x^D\;. \tag{4.13}$$

Written in components, these quantum brackets coincide with those of [30], eq. (3.30).[5]

[5] Our definition of the Jacobiator differs from that of [30] by a factor of -3. We have also corrected the expression for the 3-bracket $[x^i,x^j,x^4]_{\star_\lambda}$ which is missing a factor λ^2 in [30], eq. (3.30).

We will now show that this quantization is the correct M-theory lift of the quantization of the string theory R-flux background from section 2.2, in the sense that the star product (4.9) reduces to (2.15) in the contraction limit $\lambda = 0$; this calculation will also unpackage the formula (4.9) somewhat. For this, we need to show that the quantity $\mathcal{B}_\eta(\Lambda\,\vec{k},\Lambda\,\vec{k}'\,)\cdot\Lambda^{-1}\,\vec{x}$ reduces to (2.14) in the $\lambda \to 0$ limit. We do this by carefully computing the contractions of the various vector products comprising (3.26) and (3.18).

First, let us introduce

$$\begin{aligned}\vec{p}_\Lambda &:= \frac{\sin\big(\hbar\,|\Lambda\,\vec{k}|\big)}{|\Lambda\,\vec{k}|}\;\Lambda\,\vec{k} \\ &= \frac{\sin\Big(\frac12\sqrt{\lambda\,(\lambda\,\hbar^2\,\boldsymbol{l}^2+\ell_s^3\,R\,\boldsymbol{k}^2+\lambda^2\,\ell_s^3\,R\,k_4^2)}\Big)}{\sqrt{\lambda\,(\lambda\,\hbar^2\,\boldsymbol{l}^2+\ell_s^3\,R\,\boldsymbol{k}^2+\lambda^2\,\ell_s^3\,R\,k_4^2)}}\;\Big(\sqrt{\lambda\,\ell_s^3\,R}\;\boldsymbol{k}\,,\,\sqrt{\lambda^3\,\ell_s^3\,R}\;k_4\,,\,-\lambda\,\hbar\,\boldsymbol{l}\Big)\end{aligned} \tag{4.14}$$

in the conventions of section 3.6. Evidently

$$\lim_{\lambda\to0}\,\frac{\sin\big(\hbar\,|\Lambda\,\vec{k}|\big)}{|\Lambda\,\vec{k}|} = \hbar\,,$$

so that from (4.14) we find the limit

$$\lim_{\lambda\to0}\,\vec{p}_\Lambda = \vec{0}\ . \tag{4.15}$$

From the identity (3.20) we thus find

$$\lim_{\lambda\to0}\,\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big| = 0 \tag{4.16}$$

and

$$\lim_{\lambda\to0}\,\frac{\sin^{-1}\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|}{\hbar\,\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|} = \frac{1}{\hbar}\ .$$

These limits imply that

$$\lim_{\lambda\to0}\,\frac{\sin^{-1}\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|}{\hbar\,\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|}\,\Big(\sqrt{1-|\vec{p}_\Lambda|^2}\;\vec{p}_\Lambda^{\,\prime}+\sqrt{1-|\vec{p}_\Lambda^{\,\prime}|^2}\;\vec{p}_\Lambda\Big)\cdot\Lambda^{-1}\,\vec{x} = \big(\vec{k}+\vec{k}'\,\big)\cdot\vec{x}\ .$$

Next, using (3.36) one easily finds

$$\begin{aligned}2\big(\Lambda\,\vec{k}\times_\eta\Lambda\,\vec{k}'\,\big)\cdot\Lambda^{-1}\,\vec{x} &= \lambda\,\boldsymbol{x}\cdot(\boldsymbol{k}\times_\varepsilon\boldsymbol{l}'-\boldsymbol{l}\times_\varepsilon\boldsymbol{k}'\,)+\lambda\,\boldsymbol{x}\cdot(k_4'\,\boldsymbol{l}-k_4\,\boldsymbol{l}'\,)+x^4\,(\boldsymbol{k}\cdot\boldsymbol{l}'-\boldsymbol{l}\cdot\boldsymbol{k}'\,) \\ &\quad+\tfrac{\ell_s^3\,R}{\hbar^2}\,\boldsymbol{p}\cdot(\boldsymbol{k}\times_\varepsilon\boldsymbol{k}')-\lambda\,\boldsymbol{p}\cdot(\boldsymbol{l}\times_\varepsilon\boldsymbol{l}')+\tfrac{\lambda\,\ell_s^3\,R}{\hbar^2}\,\boldsymbol{p}\cdot(k_4\,\boldsymbol{k}'-k_4'\,\boldsymbol{k})\end{aligned} \tag{4.17}$$

and from (4.17) we compute

$$\begin{aligned}\lim_{\lambda\to0}\,\frac{\sin^{-1}\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|}{\hbar\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|}\,\big(\vec{p}_\Lambda\times_\eta\vec{p}_\Lambda^{\,\prime}\big)\cdot\Lambda^{-1}\,\vec{x} &= \lim_{\lambda\to0}\,\bigg(\frac{\sin^{-1}\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|}{\hbar\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|}\,\frac{\sin\big(\hbar|\Lambda\vec{k}|\big)}{|\Lambda\,\vec{k}|}\,\frac{\sin\big(\hbar|\Lambda\vec{k}'\,|\big)}{|\Lambda\,\vec{k}'\,|} \\ &\qquad\qquad\times\,\big(\Lambda\,\vec{k}\times_\eta\Lambda\,\vec{k}'\,\big)\cdot\Lambda^{-1}\,\vec{x}\bigg) \\ &= \tfrac{1}{2\hbar}\,\big(\ell_s^3\,R\,\boldsymbol{p}\cdot(\boldsymbol{k}\times_\varepsilon\boldsymbol{k}'\,)+\hbar^2\,x^4\,(\boldsymbol{k}\cdot\boldsymbol{l}'-\boldsymbol{l}\cdot\boldsymbol{k}'\,)\big)\ .\end{aligned}$$

Putting everything together we conclude that

$$\lim_{\lambda\to 0} \vec{\mathcal{B}}_\eta(\Lambda\,\vec k,\Lambda\,\vec k'\,)\cdot\Lambda^{-1}\,\vec x = (\boldsymbol{k}+\boldsymbol{k}'\,)\cdot\boldsymbol{x} + (k_4+k_4'\,)\,x^4 + (\boldsymbol{l}+\boldsymbol{l}'\,)\cdot\boldsymbol{p} \\ - \tfrac{1}{2\hbar}\,\big(\ell_s^3\,R\,\boldsymbol{p}\cdot(\boldsymbol{k}\times_\varepsilon\boldsymbol{k}'\,) + \hbar^2\,x^4\,(\boldsymbol{k}\cdot\boldsymbol{l}'-\boldsymbol{l}\cdot\boldsymbol{k}'\,)\big) \ . \qquad (4.18)$$

Up to the occurance of the circle fibre coordinate x^4, this expression coincides exactly with (2.14). In the dimensional reduction of M-theory to IIA string theory we restrict the algebra of functions to those which are constant along the x^4-direction; they reduce the Fourier space integrations in (4.9) to the six-dimensional hyperplanes $k_4 = k_4' = 0$. From (4.10) we see that $x^4 \star_\lambda f = x^4\, f + O(\lambda)$, and hence the coordinate x^4 is central in the star product algebra of functions in the limit $\lambda \to 0$; as before we may therefore set it to any non-zero constant value, which we take to be $x^4 = 1$. In this way the $\lambda \to 0$ limit of the star product (4.9) reduces exactly to (2.15),

$$\lim_{\lambda\to 0}\,(f\star_\lambda g)(\vec x\,) = (f\star g)(x) \ .$$

With similar techniques, one shows that the dimensional reduction of the M-theory associator from (3.27) coincides precisely with the string theory associator (2.16):

$$\lim_{\lambda\to 0}\,\vec{\mathcal{A}}_\eta(\Lambda\,\vec k,\Lambda\,\vec k',\Lambda\,\vec k''\,)\cdot\Lambda^{-1}\,\vec x = \mathcal{A}(k,k',k''\,) \ . \qquad (4.19)$$

4.3 Closure and cyclicity

We are not quite done with our phase space quantization of the non-geometric M-theory background because the star product (4.9), in contrast to its string theory dual counterpart (2.15) at $\lambda = 0$, has the undesirable feature that it is neither 2-cyclic nor 3-cyclic in the sense of [45]; these properties are essential for a sensible nonassociative phase space formulation of the quantisation of non-geometric strings [45], for matching with the expectations from worldsheet conformal field theory in non-geometric string backgrounds [2, 44], and in the construction of physically viable actions for a nonassociative theory of gravity underlying the low-energy limit of non-geometric string theory [8]. It is natural to ask for analogous features involving M2-branes and a putative nonassociative theory of gravity underlying the low-energy limit of non-geometric M-theory.

Although for the classical brackets (4.5) of Schwartz functions one has $\int \mathrm{d}^7\vec x\ \{f,g\}_\lambda = \int \mathrm{d}^7\vec x\ \partial_A(2\,\lambda^{ABC}\,x^C\,f\,\partial_B g) = 0$, this is no longer true for the quantum brackets $[f,g]_{\star_\lambda}$. The issue is that the star product (4.9) is not closed with respect to Lebesgue measure on $\mathbb{R}^7$, i.e., $\int \mathrm{d}^7\vec x\ f\star_\lambda g \neq \int \mathrm{d}^7\vec x\ f\,g$, and no modification of the measure can restore closure. In particular, using (4.10) a simple integration by parts shows that

$$\int \mathrm{d}^7\vec x\ \big(x^A\star_\lambda f - x^A\,f\big) = 6\,\hbar^2\int \mathrm{d}^7\vec x\ \chi\big(\hbar^2\,\tilde\triangle_{\vec x}\,\big)\,\tilde\partial^A \triangleright f \ . \qquad (4.20)$$

To overcome this problem we seek a gauge equivalent star product

$$f\bullet_\lambda g = \mathcal{D}^{-1}\big(\mathcal{D}f\star_\lambda\mathcal{D}g\big) \qquad \text{with} \quad \mathcal{D} = 1+O(\lambda) \ . \qquad (4.21)$$

The construction of the invertible differential operator $\mathcal{D}$ implementing this gauge transformation is analogous to the procedure used in [38]. Order by order calculations, see e.g. [36], show that $\mathcal{D}$ contains only even order derivatives, and in fact it is a functional $\mathcal{D} = \mathcal{D}(\hbar^2\,\tilde{\triangle}_{\vec{x}})$. Since $\mathcal{D} \triangleright x^A = x^A$, we have

$$x^A \bullet_\lambda f = \mathcal{D}^{-1}\big(\mathcal{D}x^A \star_\lambda \mathcal{D}f\big) = \mathcal{D}^{-1}\,\hat{x}^A\,\mathcal{D} \triangleright f \ . \tag{4.22}$$

Using now $[\mathcal{D}^{-1}, x^C] = -2\,\hbar^2\,\mathcal{D}^{-2}\,\mathcal{D}'\,\tilde{\partial}^C$, where $\mathcal{D}'$ stands for the (formal) derivative of $\mathcal{D}$ with respect to its argument, together with the explicit form (4.10) we find

$$x^A \bullet_\lambda f = x^A \star_\lambda f - 2\,\hbar^2\,\mathcal{D}^{-1}\,\mathcal{D}'\,\tilde{\partial}^A \triangleright f \ . \tag{4.23}$$

The requirement $\int \mathrm{d}^7\vec{x}\ x^A \bullet_\lambda f = \int \mathrm{d}^7\vec{x}\ x^A\,f$, with the help of (4.20) and (4.23), gives the elementary Cauchy problem

$$\mathcal{D}^{-1}\,\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t} = 3\,\frac{\sqrt{t}\coth\sqrt{t}-1}{t} \qquad \text{with} \quad \mathcal{D}(0) = 1 \ ,$$

whose solution finally yields

$$\mathcal{D} = \Big(\big(\hbar\,\tilde{\triangle}_{\vec{x}}^{1/2}\big)^{-1}\sinh\big(\hbar\,\tilde{\triangle}_{\vec{x}}^{1/2}\big)\Big)^6 \ .$$

We may therefore write the star product (4.21) as

$$\begin{aligned}(f \bullet_\lambda g)(\vec{x}\,) = \int \frac{\mathrm{d}^7\vec{k}}{(2\pi)^7}\,\frac{\mathrm{d}^7\vec{k}'}{(2\pi)^7}\ \tilde{f}(\vec{k}\,)\,\tilde{g}(\vec{k}'\,)\ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\eta(\varLambda\,\vec{k},\varLambda\,\vec{k}'\,)\,\cdot\,\varLambda^{-1}\,\vec{x}} \\ \times\left(\frac{\sin\big(\hbar\,|\varLambda\,\vec{k}|\big)\sin\big(\hbar\,|\varLambda\,\vec{k}'\,|\big)}{\hbar\,|\varLambda\,\vec{k}|\,|\varLambda\,\vec{k}'\,|}\ \frac{|\vec{\mathcal{B}}_\eta(\varLambda\,\vec{k},\varLambda\,\vec{k}'\,)|}{\sin\big(\hbar\,|\vec{\mathcal{B}}_\eta(\varLambda\,\vec{k},\varLambda\,\vec{k}'\,)|\big)}\right)^6 \ .\end{aligned} \tag{4.24}$$

This star product still provides a quantization of the brackets (4.6),

$$\lim_{\substack{\hbar,\ell_s\to 0\\ \ell_s^3/\hbar^2=\text{constant}}}\ \frac{[f,g]_{\bullet_\lambda}}{\mathrm{i}\,\hbar} = \{f,g\}_\lambda \ .$$

Using the limits computed in section 4.2, the extra factors in (4.24) are simply unity in the contraction limit $\lambda \to 0$, and so the star product (4.24) still dimensionally reduces to (2.15),

$$\lim_{\lambda\to 0}\,(f \bullet_\lambda g)(\vec{x}\,) = (f \star g)(x) \ .$$

It is Hermitean, $(f \bullet_\lambda g)^* = g^* \bullet_\lambda f^*$, and unital, $f \bullet_\lambda 1 = f = 1 \bullet_\lambda f$, but it is no longer a Weyl star product, i.e., it does not satisfy (2.11); in particular, the star products of plane waves $\mathrm{e}^{\,\mathrm{i}\,\vec{k}\cdot\vec{x}} \bullet_\lambda \mathrm{e}^{\,\mathrm{i}\,\vec{k}'\cdot\vec{x}}$ are no longer given simply by the Baker-Campbell-Hausdorff formula. However, it is now closed,

$$\int \mathrm{d}^7\vec{x}\ f \bullet_\lambda g = \int \mathrm{d}^7\vec{x}\ f\,g \ ,$$

JHEP02(2017)099

which identifies it as the Kontsevich star product; in particular, the desired 2-cyclicity property follows: $\int \mathrm{d}^7\vec{x}\ [f,g]_{\bullet_\lambda} = 0$. The closure condition can be regarded as the absence of noncommutativity (and nonassociativity) among free fields.

Under the gauge transformation (4.21) the star associator and Jacobiator transform covariantly:

$$A_{\bullet_\lambda}(f,g,h) = \mathcal{D}^{-1} A_{\star_\lambda}(\mathcal{D}f, \mathcal{D}g, \mathcal{D}h) \qquad \text{and} \qquad [f,g,h]_{\bullet_\lambda} = \mathcal{D}^{-1}[\mathcal{D}f, \mathcal{D}g, \mathcal{D}h]_{\star_\lambda} \ .$$

The star product (4.9) is alternative on the space of Schwartz functions, and since the differential operator $\mathcal{D}$ preserves this subspace it follows that the star product (4.24) is also alternative on Schwartz functions, i.e., it satisfies

$$A_{\bullet_\lambda}(f,g,h) = \tfrac{1}{6}\, [f,g,h]_{\bullet_\lambda} \ . \tag{4.25}$$

By 2-cyclicity the integrated star Jacobiator of Schwartz functions vanishes, and together with (4.25) we arrive at the desired 3-cyclicity property

$$\int \mathrm{d}^7\vec{x}\ (f \bullet_\lambda g) \bullet_\lambda h = \int \mathrm{d}^7\vec{x}\ f \bullet_\lambda (g \bullet_\lambda h) \ .$$

This property can be regarded as the absence on-shell of nonassociativity (but not noncommutativity) among cubic interactions of fields.

4.4 Uncertainty relations

The closure and cyclicity properties of the gauge equivalent star product $\bullet_\lambda$ enable a consistent formulation of nonassociative phase space quantum mechanics, along the lines given in [45] (see also [46], section 4.5 for a review). In particular, this framework provides a concrete and rigorous derivation of the novel uncertainty principles which are heuristically expected to arise from the commutation relations (4.13) that quantize the brackets (4.6) and (4.7) capturing the nonassociative geometry of the M-theory R-flux background. It avoids the problems arising from the fact that our nonassociative algebras are not alternative (which has been the property usually required in previous treatments of nonassociativity in quantum mechanics).

In this approach, observables f are real-valued functions on the seven-dimensional phase space that are multiplied together with the star product (4.24); dynamics in the quantum theory with classical Hamiltonian H is then implemented via the time evolution equations

$$\frac{\partial f}{\partial t} = \frac{\mathrm{i}}{\hbar}\, [\mathsf{H}, f]_{\bullet_\lambda} \ .$$

States are characterized by normalized phase space wave functions ψ_a and statistical probabilities μ_a. Expectation values are computed via the phase space integral

$$\langle f \rangle = \sum_a \mu_a \int \mathrm{d}^7\vec{x}\ \psi_a^* \bullet_\lambda (f \bullet_\lambda \psi_a) \,,$$

which using closure and cyclicity can be expressed in terms of a normalized real-valued state function $S = \sum_a \mu_a\, \psi_a \bullet_\lambda \psi_a^*$ as $\langle f \rangle = \int \mathrm{d}^7\vec{x}\ f\, S$.

From the non-vanishing Jacobiators (4.7) in the present case we expect in fact to obtain a coarse graining of the M-theory phase space, rather than just the configuration space as it happens in the reduction to the string theory R-flux background [45]. This can be quantified by computing the expectation values of oriented area and volume uncertainty operators following the formalism of [45]. In this prescription, we can define the area operator corresponding to directions x^A, x^B as

$$\mathsf{A}^{AB} = \mathfrak{Im}\big([\tilde{x}^A, \tilde{x}^B]_{\bullet_\lambda}\big) = -\,\mathrm{i}\,\big(\tilde{x}^A \bullet_\lambda \tilde{x}^B - \tilde{x}^B \bullet_\lambda \tilde{x}^A\big)\,, \tag{4.26}$$

while the volume operator in directions x^A, x^B, x^C is

$$\mathsf{V}^{ABC} = \tfrac{1}{3}\,\mathfrak{Re}\big(\tilde{x}^A \bullet_\lambda [\tilde{x}^B, \tilde{x}^C]_{\bullet_\lambda} + \tilde{x}^C \bullet_\lambda [\tilde{x}^A, \tilde{x}^B]_{\bullet_\lambda} + \tilde{x}^B \bullet_\lambda [\tilde{x}^C, \tilde{x}^A]_{\bullet_\lambda}\big)\,, \tag{4.27}$$

where $\tilde{x}^A := x^A - \langle x^A\rangle$ are the operator displacements appropriate to the description of quantum uncertainties. Explicit computations using the fact that the star product $\bullet_\lambda$ is alternative on monomials give the operators (4.26) and (4.27) as

$$\mathsf{A}^{AB} = 2\,\hbar\,\lambda^{ABC}\,\tilde{x}^C \qquad \text{and} \qquad \mathsf{V}^{ABC} = \tfrac{1}{6}\,[\tilde{x}^A, \tilde{x}^B, \tilde{x}^C]_{\bullet_\lambda} = 2\,\hbar^2\,\lambda^{ABCD}\,\tilde{x}^D\,. \tag{4.28}$$

Let us now write the expectation values of the operators (4.28) in components. For the fundamental area measurement uncertainties (or minimal areas) we obtain

$$\begin{aligned}
\langle \mathsf{A}^{ij}\rangle &= \tfrac{\ell_s^3}{\hbar}\,\big|R^{4,ijk4}\,\langle p_k\rangle\big| && \text{and} && \langle \mathsf{A}^{4i}\rangle = \tfrac{\lambda\,\ell_s^3}{\hbar}\,\big|R^{4,1234}\,\langle p^i\rangle\big|\,,\\
\langle \mathsf{A}^{x^i,p_j}\rangle &= \hbar\,\big|\delta^i_j\,\langle x^4\rangle + \lambda\,\varepsilon^i{}_{jk}\,\langle x^k\rangle\big| && \text{and} && \langle \mathsf{A}^{x^4,p_i}\rangle = \lambda^2\,\hbar\,\langle x_i\rangle\,,\\
\langle \mathsf{A}^{p_i,p_j}\rangle &= \lambda\,\hbar\,\big|\varepsilon_{ijk}\,\langle p^k\rangle\big|\,.
\end{aligned} \tag{4.29}$$

The first expression $\langle \mathsf{A}^{ij}\rangle$ demonstrates, as in the string theory case [45], an area uncertainty on M proportional to the magnitude of the transverse momentum, while the second expression $\langle \mathsf{A}^{4i}\rangle$ gives area uncertainties along the M-theory circle proportional to the momentum transverse to the fibre direction. The third expression $\langle \mathsf{A}^{x^i,p_j}\rangle$ describes phase space cells of position and momentum in the same direction with area $\hbar\,|\langle x^4\rangle|$, together with new cells proportional to the transverse directions; in the contraction limit $\lambda = 0$ it reduces to the standard minimal area (for $x^4 = 1$) governed by the Heisenberg uncertainty principle. For $\lambda \neq 0$ the area uncertainties $\langle \mathsf{A}^{x^\mu,p_i}\rangle$ suggest that there are new limitations to the simultaneous measurements of transverse position and momentum in the seven-dimensional M-theory phase space, induced by a non-zero string coupling constant g_s, although we will see below that this interpretation is somewhat subtle. The final expression $\langle \mathsf{A}^{p_i,p_j}\rangle$ is also a new area uncertainty particular to M-theory, yielding cells in momentum space with area proportional to $\lambda\,\hbar$ (and to the magnitude of the transverse momentum), a point to which we return in section 4.6.

For the fundamental volume measurement uncertainties (or minimal volumes) we obtain

$$\begin{aligned}
\langle \mathsf{V}^{ijk}\rangle &= \tfrac{\ell_s^3}{2}\,\big|R^{4,ijk4}\,\langle x^4\rangle\big| && \text{and} && \langle \mathsf{V}^{ij4}\rangle = \tfrac{\lambda^2\,\ell_s^3}{2}\,\big|R^{4,ijk4}\,\langle x_k\rangle\big|\,,\\
\langle \mathsf{V}^{p_i,x^j,x^k}\rangle &= \tfrac{\lambda\,\ell_s^3}{2}\,\big|R^{4,1234}\,\big(\delta^j_i\,\langle p^k\rangle - \delta^k_i\,\langle p^j\rangle\big)\big| && \text{and} && \langle \mathsf{V}^{p_i,x^j,x^4}\rangle = \tfrac{\lambda^2\,\ell_s^3}{2}\,R^{4,ijk4}\,\langle p_k\rangle\,,\\
\langle \mathsf{V}^{p_i,p_j,x^k}\rangle &= \tfrac{\lambda\,\hbar^2}{2}\,\big|\lambda\,\varepsilon_{ij}{}^k\,\langle x^4\rangle + \delta^k_j\,\langle x_i\rangle - \delta^k_i\,\langle x_j\rangle\big| && \text{and} && \langle \mathsf{V}^{p_i,p_j,x^4}\rangle = \tfrac{\lambda^3\,\hbar^2}{2}\,\big|\varepsilon_{ijk}\,\langle x^k\rangle\big|\,.
\end{aligned} \tag{4.30}$$

JHEP02(2017)099

They demonstrate volume uncertainties in position coordinates x^μ, x^ν, x^α proportional to the magnitude of the transverse coordinate direction; in particular, there is a volume uncertainty on M proportional to the magnitude of the circle fibre coordinate x^4, which reduces for $x^4 = 1$ to the expected minimal volume in non-geometric string theory [45]. A geometric interpretation of these position volume uncertainties will be provided in section 4.5. There are also phase space cubes for position and momentum in the same direction as well as in transverse directions, reflecting the fact that the corresponding nonassociating triples of M-theory phase space coordinates cannot be measured simultaneously to arbitrary precision; these new volume uncertainties vanish in the contraction limit $\lambda = 0$. In the string theory limit the volume uncertainties can be interpreted as the non-existence of D-particles in the R-flux background due to the Freed-Witten anomaly in the T-dual H-flux frame [15, 30]; it would interesting to understand the corresponding meaning in the presence of non-geometric M-theory R-fluxes, which involves the full seven-dimensional M-theory phase space. However, there are no minimal volumes in momentum space, as we discuss further in section 4.6.

The present situation is much more complicated in the case of the actual quantum uncertainty principles imposing limitations to position and momentum measurements; they encode positivity of operators in nonassociative phase space quantum mechanics [45]. To calculate the uncertainty relations amongst phase space coordinates, we use the Cauchy-Schwarz inequality derived in [45] to obtain the uncertainty relations

$$\Delta x^A \, \Delta x^B \geq \tfrac{1}{2} \left| \left\langle [x^A, x^B]_{\circ_\lambda} \right\rangle \right| , \tag{4.31}$$

where

$$[x^A, x^B]_{\circ_\lambda} \bullet_\lambda \psi := x^A \bullet_\lambda (x^B \bullet_\lambda \psi) - x^B \bullet_\lambda (x^A \bullet_\lambda \psi) \tag{4.32}$$

for any phase space wave function ψ.

We first observe that from (4.10) one obtains the commutator

$$[\hat{x}^A, \hat{x}^B] = 2\,\mathrm{i}\,\hbar\,\lambda^{ABC}\,\hat{x}^C - 4\,\hbar^2\,\lambda^{ABDE}\,x^E\,\partial_D \ . \tag{4.33}$$

It is easy check that in the limit $\lambda \to 0$ the relations (4.33) reproduce the algebra of differential operators (2.8) for the string theory R-flux background. From the contraction identity (3.7) we can rewrite (4.33) as

$$[\hat{x}^A, \hat{x}^B] = 2\,\mathrm{i}\,\hbar\,\lambda^{ABC}\,\hat{x}^C + 4\,\hbar^2\,\lambda^{ABC}\,\lambda^{CDE}\,x^E\,\partial_D + x^B\,\bar{\partial}^A - x^A\,\bar{\partial}^B \ ,$$

where

$$\big(\bar{\partial}^A\big) := 4\,\hbar^2\,\big(\,\Lambda^{BA}\,\tilde{\partial}_B\,\big) = \big(\lambda\,\ell_s^3\,R\,\nabla_{\boldsymbol{x}}\,,\ \lambda^3\,\ell_s^3\,R\,\tfrac{\partial}{\partial x^4}\,,\ \lambda^2\,\hbar^2\,\nabla_{\boldsymbol{p}}\big) \ .$$

Next we calculate

$$\begin{aligned} x^A \star_\lambda (x^B \star_\lambda \psi) - x^B \star_\lambda (x^A \star_\lambda \psi) &= [\hat{x}^A, \hat{x}^B] \triangleright \psi \\ &= 2\,\mathrm{i}\,\hbar\,\lambda^{ABC}\,\big(x^C \star_\lambda \psi - 2\,\mathrm{i}\,\hbar\,\lambda^{CDE}\,x^E\,\partial_D\psi\big) \\ &\quad + (x^B\,\bar{\partial}^A - x^A\,\bar{\partial}^B) \triangleright \psi \\ &= 2\,\mathrm{i}\,\hbar\,\lambda^{ABC}\,\psi \star_\lambda x^C + (x^B\,\bar{\partial}^A - x^A\,\bar{\partial}^B) \triangleright \psi \ . \end{aligned} \tag{4.34}$$

To translate the expression (4.34) into the definition (4.32) via the closed star product $\bullet_\lambda$, we use the gauge transformation (4.22) to obtain

$$\begin{aligned}
[x^A, x^B]_{\circ_\lambda} \bullet_\lambda \psi &= \mathcal{D}^{-1}\big(x^A \star_\lambda (x^B \star_\lambda \mathcal{D}\psi) - x^B \star_\lambda (x^A \star_\lambda \mathcal{D}\psi)\big) \\
&= \mathcal{D}^{-1}\big(2\,\mathrm{i}\,\hbar\,\lambda^{ABC}\,\mathcal{D}\psi \star_\lambda x^C + (x^B\,\bar\partial^A - x^A\,\bar\partial^B) \triangleright \mathcal{D}\psi\big) \\
&= 2\,\mathrm{i}\,\hbar\,\lambda^{ABC}\,\psi \bullet_\lambda x^C + (x^B\,\bar\partial^A - x^A\,\bar\partial^B) \triangleright \psi\,,
\end{aligned} \tag{4.35}$$

where in the last equality we used $\mathcal{D} = \mathcal{D}(\hbar^2\,\tilde\triangle_{\vec x})$.

The explicit computation of the uncertainty relations (4.31) is complicated by the second term in the last line of (4.35). The differential operator $x^A\,\bar\partial^B - x^B\,\bar\partial^A$ is of order $O(\lambda)$, and it can be regarded as a generator of "twisted" rotations in the phase space plane spanned by the vectors x^A and x^B; in the limit $\lambda \to 0$, the result (4.35) reproduces exactly the corresponding calculation from ([45], eq. (5.33)) where this problem does not arise. If we restrict to states which are rotationally invariant in this sense, so that the corresponding wave functions ψ obey $(x^B\,\bar\partial^A - x^A\,\bar\partial^B) \triangleright \psi = 0$, and which obey the "symmetry" condition of [45], then the corresponding uncertainty relations (4.31) for phase space coordinate measurements reads as

$$\Delta x^A\,\Delta x^B \geq \hbar\,\big|\lambda^{ABC}\,\langle x^C\rangle\big|\,,$$

with similar interpretations as those of the area measurement uncertainties derived in (4.29). However, the uncertainty relations (4.31) seem too complicated to suggest a universal lower bound which does not depend on the choice of state.

4.5 Configuration space triproducts and nonassociative geometry

Thus far all of our considerations have applied to phase space, and it is now natural to look at polarizations which suitably reduce the physical degrees of freedom as is necessary in quantization. From the perspective of left-right asymmetric worldsheet conformal field theory, closed strings probe the nonassociative deformation of the R-flux background through phase factors that turn up in off-shell correlation functions of tachyon vertex operators, which can be encoded in a triproduct of functions on configuration space M [14]; this triproduct originally appeared in [24, 55] (see also [12]) as a candidate deformation quantization of the canonical Nambu-Poisson bracket on $\mathbb{R}^3$. This geometric structure was generalised to curved spaces with non-constant fluxes within the framework of double field theory in [15], and in [2] it was shown that these triproducts descend precisely from polarisation of the phase space star product along the leaf of zero momentum $\boldsymbol{p} = \mathbf{0}$ in phase space T^*M. Although at present we do not have available a quantum theory that would provide an M2-brane analog of the computation of conformal field theory correlation functions for closed strings propagating in constant non-geometric R-flux compactifications, we can imitate this latter reduction of the phase space star product in our case and derive triproducts which geometrically describe the quantization of the four-dimensional M-theory configuration space.

For this, we consider functions which depend only on configuration space coordinates, that we denote by $\vec{x}_0 = (\boldsymbol{x}, x^4, \boldsymbol{0}) \in \mathbb{R}^4$, and define the product

$$\big(f \,\triangle^{(2)}_\lambda\, g\big)(\vec{x}_0) := (f \star_\lambda g)(\boldsymbol{x}, x^4, \boldsymbol{p})\big|_{\boldsymbol{p}=\boldsymbol{0}} = \int \frac{\mathrm{d}^4\vec{k}}{(2\pi)^4}\, \frac{\mathrm{d}^4\vec{k}'}{(2\pi)^4}\, \tilde{f}(\vec{k}\,)\, \tilde{g}(\vec{k}'\,)\ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\eta(\Lambda\,\vec{k},\Lambda\,\vec{k}'\,)\cdot \Lambda^{-1}\,\vec{x}_0} .$$

From (3.36) we see that the cross product of four-dimensional vectors $\vec{k} = (\boldsymbol{0}, \boldsymbol{k}, k_4) \in \mathbb{R}^4$ gives

$$\Lambda\,\vec{k} \times_\eta \Lambda\,\vec{k}' = \tfrac{\lambda\,\ell_s^3\,R}{4\hbar^2}\,\big(-\boldsymbol{k}\times_\varepsilon \boldsymbol{k}' + \lambda\,k_4'\,\boldsymbol{k} - \lambda\,k_4\,\boldsymbol{k}'\,,\,\boldsymbol{0}\,,\,0\big)\,,$$

and it is therefore orthogonal to $\Lambda^{-1}\,\vec{x}_0$, i.e., $\big(\Lambda\,\vec{k}\times_\eta \Lambda\,\vec{k}'\,\big)\cdot\Lambda^{-1}\,\vec{x}_0 = 0$. Hence the source of noncommutativity and nonassociativity vanishes in this polarization, and using the variables (4.14) we can write the product succinctly as

$$\big(f \,\triangle^{(2)}_\lambda\, g\big)(\vec{x}_0) = \int \frac{\mathrm{d}^4\vec{k}}{(2\pi)^4}\, \frac{\mathrm{d}^4\vec{k}'}{(2\pi)^4}\, \tilde{f}(\vec{k}\,)\, \tilde{g}(\vec{k}'\,)\ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{T}}^{(2)}_\Lambda(\vec{k},\vec{k}'\,)\cdot\Lambda^{-1}\,\vec{x}_0}\,, \tag{4.36}$$

where we introduced the deformed vector sum

$$\vec{\mathcal{T}}^{(2)}_\Lambda(\vec{k},\vec{k}'\,) = \frac{\sin^{-1}\big|\vec{p}_\Lambda \circledast_\eta \vec{p}_\Lambda^{\,\prime}\big|}{\hbar\,\big|\vec{p}_\Lambda \circledast_\eta \vec{p}_\Lambda^{\,\prime}\big|}\,\Big(\sqrt{1-|\vec{p}_\Lambda^{\,\prime}|^2}\ \vec{p}_\Lambda + \sqrt{1-|\vec{p}_\Lambda|^2}\ \vec{p}_\Lambda^{\,\prime}\Big)\,.$$

It has a perturbative expansion given by

$$\Lambda^{-1}\,\vec{\mathcal{T}}^{(2)}_\Lambda(\vec{k},\vec{k}'\,) = \vec{k} + \vec{k}' + O\big(\sqrt{\lambda}\,\big)\,.$$

The product (4.36) inherits properties of the phase space star product $\star_\lambda$; in particular, since $\vec{\mathcal{T}}^{(2)}_\Lambda(\vec{k},\vec{0}\,) = \vec{k} = \vec{\mathcal{T}}^{(2)}_\Lambda(\vec{0},\vec{k}\,)$, it is unital:

$$f \,\triangle^{(2)}_\lambda\, 1 = f = 1 \,\triangle^{(2)}_\lambda\, f\,.$$

It is commutative and associative, as expected from the area uncertainties $\langle \mathsf{A}^{\mu\nu}\rangle$ of (4.29) in this polarisation; from the limits (4.15) and (4.16) it follows that it reduces at $\lambda = 0$ to the ordinary pointwise product of fields on $\mathbb{R}^3$, as anticipated from the corresponding string theory result [2, 14]. However, the product (4.36) is *not* generally the pointwise product of functions on $\mathbb{R}^4$, $f \,\triangle^{(2)}_\lambda\, g \neq f\,g$; in particular

$$x^\mu \,\triangle^{(2)}_\lambda\, f = x^\mu\,f + \hbar^2\,\big(x^\mu\,\tilde{\triangle}_{\vec{x}_0} - (\vec{x}_0\cdot\tilde{\nabla}_{\vec{x}_0})\,\tilde{\partial}^\mu\big)\,\chi\big(\hbar^2\,\tilde{\triangle}_{\vec{x}_0}\big)\triangleright f\,,$$

so that off-shell membrane amplitudes in this case experience a commutative and associative deformation. Moreover, $\int \mathrm{d}^4\vec{x}_0\ f \,\triangle^{(2)}_\lambda\, g \neq \int \mathrm{d}^4\vec{x}_0\ f\,g$, but this can be rectified by defining instead a product $\blacktriangle^{(2)}_\lambda$ based on the closed star product (4.24); then $\int \mathrm{d}^4\vec{x}_0\ f\,\blacktriangle^{(2)}_\lambda g = \int \mathrm{d}^4\vec{x}_0\ f\,g$.

Next we define a triproduct for three functions f, g and h of $\vec{x}_0 = (\boldsymbol{x}, x^4) \in \mathbb{R}^4$ by a similar rule:

$$\big(f \,\triangle^{(3)}_\lambda\, g \,\triangle^{(3)}_\lambda\, h\big)(\vec{x}_0) := \big((f \star_\lambda g)\star_\lambda h\big)(\boldsymbol{x}, x^4, \boldsymbol{p})\big|_{\boldsymbol{p}=\boldsymbol{0}}\,.$$

JHEP02(2017)099

As before, the Fourier integrations truncate to four-dimensional subspaces and we can write

$$\big(f \,\triangle^{(3)}_\lambda\, g \,\triangle^{(3)}_\lambda\, h\big)(\vec x_0) = \int \frac{\mathrm{d}^4\vec k}{(2\pi)^4}\,\frac{\mathrm{d}^4\vec k'}{(2\pi)^4}\,\frac{\mathrm{d}^4\vec k''}{(2\pi)^4}\ \tilde f(\vec k\,)\,\tilde g(\vec k'\,)\,\tilde h(\vec k''\,)\ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\eta(\vec{\mathcal{B}}_\eta(\varLambda\,\vec k,\varLambda\,\vec k'\,),\varLambda\,\vec k''\,)\,\cdot\,\varLambda^{-1}\,\vec x_0}\ .$$

As in the calculation which led to (3.27), we can compute the deformed vector addition from products of octonion exponentials

$$\mathrm{e}^{\,X_{\vec{\mathcal{B}}_\eta(\vec{\mathcal{B}}_\eta(\varLambda\,\vec k,\varLambda\,\vec k'\,),\varLambda\,\vec k''\,)}} = \big(\,\mathrm{e}^{\,X_{\varLambda\,\vec k}}\ \mathrm{e}^{\,X_{\varLambda\,\vec k'}}\,\big)\ \mathrm{e}^{\,X_{\varLambda\,\vec k''}}$$

using (3.14) and (3.15), together with the identities (3.20) and (3.25). The final result is a bit complicated in general, but is again most concisely expressed in terms of the variables (4.14). Exploiting again the property that the vector cross product $\vec p_\varLambda^{\,\prime}\times_\eta\vec p_\varLambda$ lives in the orthogonal complement $\mathbb{R}^3$ to the four-dimensional subspace $\mathbb{R}^4$ in $\mathbb{R}^7$ containing $\vec p_\varLambda$, so that $\big(\vec p_\varLambda^{\,\prime}\times_\eta\vec p_\varLambda\big)\cdot\varLambda^{-1}\,\vec x_0=0$, after a bit of calculation one finds that the triproduct can be written as

$$\big(f \,\triangle^{(3)}_\lambda\, g \,\triangle^{(3)}_\lambda\, h\big)(\vec x_0) = \int \frac{\mathrm{d}^4\vec k}{(2\pi)^4}\,\frac{\mathrm{d}^4\vec k'}{(2\pi)^4}\,\frac{\mathrm{d}^4\vec k''}{(2\pi)^4}\ \tilde f(\vec k\,)\,\tilde g(\vec k'\,)\,\tilde h(\vec k''\,)\ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{T}}^{(3)}_\varLambda(\vec k,\vec k',\vec k''\,)\,\cdot\,\varLambda^{-1}\,\vec x_0}\ , \tag{4.37}$$

where we defined the deformed vector sum

$$\begin{aligned}\vec{\mathcal{T}}^{(3)}_\varLambda(\vec k,\vec k',\vec k''\,) = \frac{\sin^{-1}\big|(\vec p_\varLambda\circledast_\eta\vec p_\varLambda^{\,\prime})\circledast_\eta\vec p_\varLambda^{\,\prime\prime}\big|}{\hbar\,\big|(\vec p_\varLambda\circledast_\eta\vec p_\varLambda^{\,\prime})\circledast_\eta\vec p_\varLambda^{\,\prime\prime}\big|}\,\Big(\vec A_\eta(\vec p_\varLambda,\vec p_\varLambda^{\,\prime},\vec p_\varLambda^{\,\prime\prime}\,) + \epsilon_{\vec p_\varLambda^{\,\prime},\vec p_\varLambda^{\,\prime\prime}}\,\sqrt{1-\big|\vec p_\varLambda^{\,\prime}\circledast_\eta\vec p_\varLambda^{\,\prime\prime}\big|^2}\ \vec p_\varLambda \\ + \epsilon_{\vec p_\varLambda,\vec p_\varLambda^{\,\prime\prime}}\,\sqrt{1-\big|\vec p_\varLambda\circledast_\eta\vec p_\varLambda^{\,\prime\prime}\big|^2}\ \vec p_\varLambda^{\,\prime} + \epsilon_{\vec p_\varLambda,\vec p_\varLambda^{\,\prime}}\,\sqrt{1-\big|\vec p_\varLambda\circledast_\eta\vec p_\varLambda^{\,\prime}\big|^2}\ \vec p_\varLambda^{\,\prime\prime}\Big)\end{aligned} \tag{4.38}$$

which contains the associator (3.27). It has a perturbative expansion given by

$$\begin{aligned}\varLambda^{-1}\,\vec{\mathcal{T}}^{(3)}_\varLambda(\vec k,\vec k',\vec k''\,) = \vec k+\vec k'+\vec k''+\tfrac{\hbar^2}{2}\,\big(2\varLambda^{-1}\,\vec A_\eta(\varLambda\,\vec k,\varLambda\,\vec k',\varLambda\,\vec k''\,)\\ +|\varLambda\,\vec k'+\varLambda\,\vec k''\,|^2\ \vec k+|\varLambda\,\vec k+\varLambda\,\vec k''\,|^2\ \vec k'+|\varLambda\,\vec k+\varLambda\,\vec k'\,|^2\ \vec k''\,\big)+O(\lambda)\ .\end{aligned}$$

Using the triproduct (4.37), we then define a completely antisymmetric quantum 3-bracket in the usual way by

$$[f_1,f_2,f_3]_{\triangle^{(3)}_\lambda} := \sum_{\sigma\in S_3}\,(-1)^{|\sigma|}\ f_{\sigma(1)}\,\triangle^{(3)}_\lambda\,f_{\sigma(2)}\,\triangle^{(3)}_\lambda\,f_{\sigma(3)}\ .$$

It reproduces the 3-brackets from (4.13) amongst linear functions that encodes the nonassociative geometry of configuration space,

$$[x^\mu,x^\nu,x^\alpha]_{\triangle^{(3)}_\lambda} = -12\,\hbar^2\,\lambda^{\mu\nu\alpha\beta}\,x^\beta$$

for $\mu,\nu,\alpha,\beta=1,2,3,4$. For $\lambda=1$, these are just the brackets (up to rescaling) of the 3-Lie algebra A_4,

$$[x^\mu,x^\nu,x^\alpha]_{\triangle^{(3)}_1} = 3\,\ell_s^3\,R\,\varepsilon^{\mu\nu\alpha\beta}\,x^\beta\ , \tag{4.39}$$

familiar from studies of multiple M2-branes in M-theory where it describes the polarisation of open membranes ending on an M5-brane into fuzzy three-spheres [3]; indeed, the brackets (4.39) quantize the standard Nambu-Poisson structure on the three-sphere $S^3 \subset \mathbb{R}^4$ of radius $\sqrt{3\,\ell_s^3\,R/\hbar^2}$. In the present case we are in a sector that involves only membranes of M-theory and excludes M5-branes, but we can nevertheless interpret the nonassociative geometry modelled on (4.39): it represents a (discrete) foliation of the M-theory configuration space $\mathbb{R}^4$ by fuzzy membrane worldvolume three-spheres,[6] and in this sense our triproduct (4.37) gives a candidate deformation quantization of the standard Nambu-Poisson structure on S^3. We will say more about this perspective in section 5.

From the limits (4.15) and (4.16), together with (4.19), we see that the triproduct (4.37) reproduces that of the string theory configuration space $\mathbb{R}^3$ in the contraction limit $\lambda \to 0$ [2, 14]; in particular

$$\lim_{\lambda\to 0}\,[x^i,x^j,x^k]_{\triangle^{(3)}_\lambda} = -3\,\ell_s^3\,R\,\varepsilon^{ijk}\ . \tag{4.40}$$

Thus while the string theory triproduct represents a deformation quantization of the Nambu-Heisenberg 3-Lie algebra, its lift to M-theory represents a deformation quantization of the 3-Lie algebra A_4. Since the associator $\vec{A}_\eta\big(\vec{p}_\Lambda,\vec{p}^{\,\prime}_\Lambda,\vec{p}^{\,\prime\prime}_\Lambda\big)$ is $\vec{0}$ whenever any of its arguments is the zero vector, using (3.21) we have $\vec{\mathcal{T}}^{(3)}_\Lambda(\vec{k},\vec{k}^{\,\prime},\vec{0}\,) = \vec{\mathcal{T}}^{(3)}_\Lambda(\vec{k},\vec{0},\vec{k}^{\,\prime}\,) = \vec{\mathcal{T}}^{(3)}_\Lambda(\vec{0},\vec{k},\vec{k}^{\,\prime}\,) = \vec{\mathcal{T}}^{(2)}_\Lambda(\vec{k},\vec{k}^{\,\prime}\,)$ and so we obtain the unital property

$$f\ \triangle^{(3)}_\lambda\ g\ \triangle^{(3)}_\lambda\ 1 = f\ \triangle^{(3)}_\lambda\ 1\ \triangle^{(3)}_\lambda\ g = 1\ \triangle^{(3)}_\lambda\ f\ \triangle^{(3)}_\lambda\ g = f\ \triangle^{(2)}_\lambda\ g\ ,$$

as expected from the $\lambda \to 0$ limit [2, 14]. However, in contrast to the string theory triproduct, here the M-theory triproduct does *not* trivialise on-shell, i.e., $\int \mathrm{d}^4\vec{x}_0\ f\ \triangle^{(3)}_\lambda\ g\ \triangle^{(3)}_\lambda\ h \neq \int \mathrm{d}^4\vec{x}_0\ f\,g\,h$. This is again related to the fact that the precursor phase space star product $\star_\lambda$ is not closed, and presumably one can find a suitable gauge equivalent triproduct $\blacktriangle^{(3)}_\lambda$ analogous to the closed star product $\bullet_\lambda$ that we derived in section 4.3. In section 5 we will give a more intrinsic definition of this triproduct in terms of an underlying Spin(7)-symmetric 3-algebra on the membrane phase space.

We close the present discussion by sketching two generalisations of these constructions in light of the results of [2, 14, 15]. Firstly, one can generalize these derivations to work out explicit n-triproducts for any $n \geq 4$, which in the string theory case would represent the off-shell contributions to n-point correlation functions of tachyon vertex operators in the R-flux background [14]. For functions $f_1,\dots,f_n$ of $\vec{x}_0 = (\boldsymbol{x},x^4) \in \mathbb{R}^4$ we set

$$\begin{aligned}\big(f_1\ \triangle^{(n)}_\lambda\ \cdots\ \triangle^{(n)}_\lambda\ f_n\big)(\vec{x}_0) &:= \Big(\big(\cdots\big((f_1\star_\lambda f_2)\star_\lambda f_3\big)\star_\lambda\cdots\big)\star_\lambda f_n\Big)(\boldsymbol{x},x^4,\boldsymbol{p})\Big|_{\boldsymbol{p}=\boldsymbol{0}}\\ &= \int\ \prod_{a=1}^n\frac{\mathrm{d}^4\vec{k}_a}{(2\pi)^4}\ \tilde{f}_a(\vec{k}_a)\ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\eta(\vec{\mathcal{B}}_\eta(\dots,(\vec{\mathcal{B}}_\eta(\Lambda\,\vec{k}_1,\Lambda\,\vec{k}_2),\Lambda\,\vec{k}_3),\dots\,),\Lambda\,\vec{k}_n)\,\cdot\,\Lambda^{-1}\,\vec{x}_0}\end{aligned}$$

and as before the nested compositions of vector additions can be computed from products of corresponding octonion exponentials $\big(\cdots\big(\big(\mathrm{e}^{\,X_{\Lambda\,\vec{k}_1}}\ \mathrm{e}^{\,X_{\Lambda\,\vec{k}_2}}\big)\ \mathrm{e}^{\,X_{\Lambda\,\vec{k}_3}}\big)\cdots\big)\ \mathrm{e}^{\,X_{\Lambda\,\vec{k}_n}}$. The calculation simplifies again by dropping all vector cross products $\Lambda\,\vec{k}_a\times_\eta\Lambda\,\vec{k}_b$ (which do not contribute to the inner product with $\Lambda^{-1}\,\vec{x}_0$), by correspondingly dropping many higher

[6] See e.g. [23] for an analogous description of a noncommutative deformation of $\mathbb{R}^3$ in terms of discrete foliations by fuzzy two-spheres.

iterations of associator terms using the contraction identity (3.9), and by making repeated use of the identities from section 3.3. Here we only quote the final result:

$$\big(f_1 \, \triangle^{(n)}_\lambda \, \cdots \, \triangle^{(n)}_\lambda \, f_n\big)(\vec{x}_0) = \int \, \prod_{a=1}^n \, \frac{\mathrm{d}^4\vec{k}_a}{(2\pi)^4} \, \tilde{f}_a(\vec{k}_a) \, \mathrm{e}^{\,\mathrm{i}\, \vec{\mathcal{T}}^{(n)}_\Lambda(\vec{k}_1,\dots,\vec{k}_n)\cdot \Lambda^{-1}\,\vec{x}_0} \, ,$$

where

$$\begin{aligned}
\vec{\mathcal{T}}^{(n)}_\Lambda(\vec{k}_1,\dots,\vec{k}_n) &= \frac{\sin^{-1}\big|\big(\cdots\big((\vec{p}_{1\Lambda}\circledast_\eta\vec{p}_{2\Lambda})\circledast_\eta\vec{p}_{3\Lambda}\big)\circledast_\eta\cdots\big)\circledast_\eta\vec{p}_{n\Lambda}\big|}{\hbar\,\big|\big(\cdots\big((\vec{p}_{1\Lambda}\circledast_\eta\vec{p}_{2\Lambda})\circledast_\eta\vec{p}_{3\Lambda}\big)\circledast_\eta\cdots\big)\circledast_\eta\vec{p}_{n\Lambda}\big|} \\
&\times\Bigg(\sum_{a=1}^n\sqrt{1-\big|\big(\cdots\big((\vec{p}_{1\Lambda}\circledast_\eta\vec{p}_{2\Lambda})\circledast_\eta\cdots\big)\circledast_\eta\widehat{\vec{p}_{a\Lambda}}\big)\circledast_\eta\cdots\big)\circledast_\eta\vec{p}_{n\Lambda}\big|^2}\;\epsilon_a\,\vec{p}_{a\Lambda} \\
&\times\sum_{a<b<c}\sqrt{1-\big|\big(\cdots\big(\vec{p}_{1\Lambda}\circledast_\eta\cdots\big)\circledast_\eta\widehat{\vec{p}_{a\Lambda}}\,\big)\circledast_\eta\cdots\big)\circledast_\eta\widehat{\vec{p}_{b\Lambda}}\big)\circledast_\eta\cdots\big)\circledast_\eta\widehat{\vec{p}_{c\Lambda}}\big)\circledast_\eta\cdots\big)\circledast_\eta\vec{p}_{n\Lambda}\big|^2} \\
&\times\epsilon_{abc}\;\vec{A}_\eta\big(\vec{p}_{a\Lambda},\vec{p}_{b\Lambda},\vec{p}_{c\Lambda}\big)\Bigg)
\end{aligned}$$

and $\widehat{\vec{p}_{a\Lambda}}$ denotes omission of $\vec{p}_{a\Lambda}$ for $a=1,\dots,n$; here we abbreviated signs of square roots analogous to those in (4.38) by ϵ_a and ϵ_{abc}. Again we see that

$$\vec{\mathcal{T}}^{(n)}_\Lambda\big(\vec{k}_1,\dots,(\vec{k}_a{=}\vec{0}\,),\dots,\vec{k}_n\big) = \vec{\mathcal{T}}^{(n-1)}_\Lambda\big(\vec{k}_1,\dots,\widehat{\vec{k}_a},\dots,\vec{k}_n\big)\, ,$$

which implies that the n-triproducts obey the expected unital property

$$f_1 \, \triangle^{(n)}_\lambda \, \cdots \, \triangle^{(n)}_\lambda \, (f_a{=}1) \, \triangle^{(n)}_\lambda \, \cdots \, \triangle^{(n)}_\lambda \, f_n = f_1 \, \triangle^{(n-1)}_\lambda \, \cdots \, \triangle^{(n-1)}_\lambda \, \widehat{f_a} \, \triangle^{(n-1)}_\lambda \, \cdots \, \triangle^{(n-1)}_\lambda \, f_n$$

for $a=1,\dots,n$. As previously, these n-triproducts reduce to those of the string theory R-flux background in the limit $\lambda\to 0$ [2, 14].

Secondly, one can consider more general foliations of the M-theory phase space by leaves of constant momentum $\boldsymbol{p}=\bar{\boldsymbol{p}}$. This would modify the product (4.36) by introducing phase factors

$$\begin{aligned}
&\exp\bigg(\frac{\sin^{-1}\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|}{\hbar\,\big|\vec{p}_\Lambda\circledast_\eta\vec{p}_\Lambda^{\,\prime}\big|}\,\frac{\mathrm{i}\,\ell_s^3}{2\hbar^2}\,R\,\bar{\boldsymbol{p}}\cdot\big(\boldsymbol{p}_\Lambda\times_\varepsilon\boldsymbol{p}_\Lambda'+\lambda\,p_{\Lambda 4}\,\boldsymbol{p}_\Lambda'-\lambda\,p_{\Lambda 4}'\,\boldsymbol{p}_\Lambda\big)\bigg) \\
&\qquad = \exp\bigg(\frac{\mathrm{i}\,\ell_s^3}{2\hbar}\,R\,\bar{\boldsymbol{p}}\cdot\big(\boldsymbol{k}\times_\varepsilon\boldsymbol{k}'+\lambda\,k_4\,\boldsymbol{k}'-\lambda\,k_4'\,\boldsymbol{k}\big)+O\big(\sqrt{\lambda}\,\big)\bigg)
\end{aligned}$$

into the integrand, exactly as for the Moyal-Weyl type deformation of the string theory R-flux background which is obtained at $\lambda=0$ [2] (see (2.14)). This turns the product $\triangle^{(2)}_\lambda$ into a noncommutative (but still associative) star product. One can likewise include such phase factors into the calculations of higher n-triproducts $\triangle^{(n)}_\lambda$ to obtain suitable noncommutative deformations. In the string theory setting, the physical meaning of these non-zero constant momentum deformations is explained in [2], and it would be interesting to understand their interpretation in the M-theory lift.

JHEP02(2017)099

4.6 Noncommutative geometry of momentum space

Polarisation along leaves of constant momentum is of course not the only possibility; see [2] for a general discussion of polarised phase space geometry in our context. A natural alternative polarisation is to set $x^\mu = 0$ and restrict to functions on momentum space $\boldsymbol{p} \in \mathbb{R}^3$. This is particularly interesting in the M-theory R-flux background: in the string theory case momentum space itself undergoes no deformation, whereas here we see from the brackets (4.6) and (4.7) that momentum space experiences a noncommutative, but associative, deformation by the M-theory radius λ alone. As pointed out already in section 3.6, in this polarisation the nonassociative star product (4.9) reproduces the associative star product on $\mathbb{R}^3$ for quantisation of the dual of the Lie algebra $\mathfrak{su}(2)$. Unlike the star product of configuration space functions, the star product $\star_\lambda$ restricts to the three-dimensional momentum space, so the projections employed in section 4.5 are not necessary and one can work directly with the star product restricted to $\mathbb{R}^3$; for functions f and g of $\boldsymbol{p} \in \mathbb{R}^3$ it reads as

$$(f \star_\lambda g)(\boldsymbol{p}) = \int \frac{\mathrm{d}^3\boldsymbol{l}}{(2\pi)^3}\, \frac{\mathrm{d}^3\boldsymbol{l}'}{(2\pi)^3}\, \tilde{f}(\boldsymbol{l})\, \tilde{g}(\boldsymbol{l}'\,)\, \mathrm{e}^{-\frac{2\,\mathrm{i}}{\lambda}\, \boldsymbol{\mathcal{B}}_\varepsilon(-\frac{\lambda}{2}\,\boldsymbol{l}, -\frac{\lambda}{2}\,\boldsymbol{l}'\,)\cdot \boldsymbol{p}} \ . \tag{4.41}$$

Thus the M-theory momentum space itself experiences an associative, noncommutative deformation of its geometry, independently of the R-flux. We will say more about this purely membrane deformation in section 5.

We can understand this noncommutative deformation by again restricting configuration space coordinates to the three-sphere $(\boldsymbol{x}, x^4) \in S^3 \subset \mathbb{R}^4$ of radius $\frac{1}{\lambda}$, similarly to the fuzzy membrane foliations we described in section 4.5, although the present discussion also formally applies to a vanishing R-flux. Then on the upper hemisphere $x^4 \geq 0$ the quantization of the brackets (4.6) yields the star commutators

$$[p_i, p_j]_{\star_\lambda} = -\,\mathrm{i}\,\hbar\,\lambda\,\varepsilon_{ijk}\,p^k \qquad \text{and} \qquad [x^i, p_j]_{\star_\lambda} = \mathrm{i}\,\hbar\,\lambda\,\sqrt{\tfrac{1}{\lambda^2} - |\boldsymbol{x}|^2}\ \delta^i_j + \mathrm{i}\,\hbar\,\lambda\,\varepsilon^i{}_{jk}\,x^k \ .$$

These commutation relations reflect the fact that the configuration space is curved: they show that the momentum coordinates $\boldsymbol{p}$ are realised as right-invariant derivations on configuration space $S^3 \subset \mathbb{R}^4$, and as a result generate the brackets of the Lie algebra $\mathfrak{su}(2)$. This noncommutativity has bounded position coordinates, which is consistent with the minimal momentum space areas $\langle \mathsf{A}^{p_i,p_j} \rangle$ computed in (4.29). At weak string coupling $\lambda = 0$, the three-sphere decompactifies and one recovers the canonical quantum phase space algebra of flat space $\mathbb{R}^3$. The star product (4.41) enables order by order computations of M-theory corrections to closed string amplitudes in this sense. The noncommutative geometry here parallels that of three-dimensional quantum gravity [26], where however the roles of position and momentum coordinates are interchanged.

5 Spin(7)-structures and M-theory 3-algebra

In this final section we shall describe some preliminary steps towards extending the quantum geometry of the R-flux compactification described in section 4 to the full eight-dimensional M-theory phase space. It involves replacing the notion of G_2-structure with that of a Spin(7)-structure and the quasi-Poisson algebra with a suitable 3-algebra, as anticipated on general grounds in lifts of structures from string theory to M-theory. We show, in particular, that quantisation of this 3-algebra naturally encompasses the triproducts from section 4.5 and the deformed geometry of the membrane momentum space from section 4.6.

5.1 Triple cross products

The constructions of this section will revolve around the linear algebraic notion of a *triple cross product* on an eight-dimensional real inner product space W [32, 33, 53]. For three vectors $K, K', K'' \in W = \mathbb{R}^8$, their triple cross product $K \times_\phi K' \times_\phi K'' \in W$ is defined by

$$(K \times_\phi K' \times_\phi K'')^{\hat{A}} := \phi^{\hat{A}\hat{B}\hat{C}\hat{D}}\, K^{\hat{B}}\, K'^{\hat{C}}\, K''^{\hat{D}}\,, \tag{5.1}$$

where $\hat{A}, \hat{B}, \dots = 0, 1, \dots, 7$ and $\phi_{\hat{A}\hat{B}\hat{C}\hat{D}}$ is a completely antisymmetric tensor of rank four with the nonvanishing values

$$\begin{aligned}\phi_{\hat{A}\hat{B}\hat{C}\hat{D}} = +1 \qquad \text{for} \quad \hat{A}\hat{B}\hat{C}\hat{D} = \ & 0123,\ 0147,\ 0165,\ 0246,\ 5027,\ 3045,\ 3076, \\ & 4567,\ 2365,\ 2374,\ 1537,\ 2145,\ 2176,\ 3164\ .\end{aligned}$$

It can be written more succinctly in terms of the structure constants of the octonion algebra $\mathbb{O}$ as

$$\phi_{0ABC} = \eta_{ABC} \qquad \text{and} \qquad \phi_{ABCD} = \eta_{ABCD}\ . \tag{5.2}$$

Following [34, 35], the tensor $\phi_{\hat{A}\hat{B}\hat{C}\hat{D}}$ satisfies the self-duality relation

$$\varepsilon_{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\hat{G}\hat{H}}\, \phi_{\hat{E}\hat{F}\hat{G}\hat{H}} = \phi_{\hat{A}\hat{B}\hat{C}\hat{D}}\,,$$

where $\varepsilon_{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\hat{G}\hat{H}}$ is the alternating symbol in eight dimensions normalized as $\varepsilon_{01234567} = +1$. It also obeys the contraction identity

$$\begin{aligned}\phi_{\hat{A}\hat{B}\hat{C}\hat{D}}\, \phi_{\hat{A}'\hat{B}'\hat{C}'\hat{D}} = \ & \delta_{\hat{A}\hat{A}'}\, \delta_{\hat{B}\hat{B}'}\, \delta_{\hat{C}\hat{C}'} + \delta_{\hat{A}\hat{B}'}\, \delta_{\hat{B}\hat{C}'}\, \delta_{\hat{C}\hat{A}'} + \delta_{\hat{A}\hat{C}'}\, \delta_{\hat{B}\hat{A}'}\, \delta_{\hat{C}\hat{B}'} \\ & - \delta_{\hat{A}\hat{A}'}\, \delta_{\hat{B}\hat{C}'}\, \delta_{\hat{C}\hat{B}'} - \delta_{\hat{A}\hat{B}'}\, \delta_{\hat{B}\hat{A}'}\, \delta_{\hat{C}\hat{C}'} - \delta_{\hat{A}\hat{C}'}\, \delta_{\hat{B}\hat{B}'}\, \delta_{\hat{C}\hat{A}'} \\ & - \delta_{\hat{A}\hat{A}'}\, \phi_{\hat{B}\hat{C}\hat{B}'\hat{C}'} - \delta_{\hat{B}\hat{A}'}\, \phi_{\hat{C}\hat{A}\hat{B}'\hat{C}'} - \delta_{\hat{C}\hat{A}'}\, \phi_{\hat{A}\hat{B}\hat{B}'\hat{C}'} \\ & - \delta_{\hat{A}\hat{B}'}\, \phi_{\hat{B}\hat{C}\hat{C}'\hat{A}'} - \delta_{\hat{B}\hat{B}'}\, \phi_{\hat{C}\hat{A}\hat{C}'\hat{A}'} - \delta_{\hat{C}\hat{B}'}\, \phi_{\hat{A}\hat{B}\hat{C}'\hat{A}'} \\ & - \delta_{\hat{A}\hat{C}'}\, \phi_{\hat{B}\hat{C}\hat{A}'\hat{B}'} - \delta_{\hat{B}\hat{C}'}\, \phi_{\hat{C}\hat{A}\hat{A}'\hat{B}'} - \delta_{\hat{C}\hat{C}'}\, \phi_{\hat{A}\hat{B}\hat{A}'\hat{B}'}\ .\end{aligned} \tag{5.3}$$

Despite its appearance, the triple cross product is *not* a simple iteration of the cross product $\times_\eta$, but it can also be expressed in terms of the algebra of octonions. For this, we choose a split $W = \mathbb{R} \oplus V$ with $K = (k_0, \vec{k}\,) \in W$, and consider the octonion $X_K := k_0\, \mathbb{1} + k^A\, e_A$ together with its conjugate $\bar{X}_K := k_0\, \mathbb{1} - k^A\, e_A$. As the commutator of two octonions is purely imaginary, the cross product from (3.12) can in fact be expressed in terms of eight-dimensional vectors as

$$X_{\vec{k} \times_\eta \vec{k}'} = \tfrac{1}{2}\, \big[X_K, X_{K'}\big]\ . \tag{5.4}$$

The natural extension of (5.4) to three vectors $K, K', K'' \in W$ is the antisymmetrization of $\big(X_K\, \bar X_{K'}\big)\, X_{K''}$, which can be simplified by repeated use of properties of the octonion algebra to give

$$X_{K\times_\phi K'\times_\phi K''} = \tfrac12 \big((X_K\, \bar X_{K'})\, X_{K''} - (X_{K''}\, \bar X_{K'})\, X_K \big) \ . \tag{5.5}$$

This trilinear product satisfies the defining properties of triple cross products [53]:

(TC1) $K_1 \times_\phi K_2 \times_\phi K_3 = (-1)^{|\sigma|}\, K_{\sigma(1)} \times_\phi K_{\sigma(2)} \times_\phi K_{\sigma(3)}$ for all permutations $\sigma \in S_3$;

(TC2) $K_1 \cdot (K_2 \times_\phi K_3 \times_\phi K_4) = -K_2 \cdot (K_1 \times_\phi K_3 \times_\phi K_4)$;

(TC3) $|K_1 \times_\phi K_2 \times_\phi K_3| = |K_1 \wedge K_2 \wedge K_3|$.

From the definitions above it follows that

$$\begin{aligned} K \times_\phi K' \times_\phi K'' = \big(&\vec k \cdot (\vec k\,' \times_\eta \vec k\,'') \,, \\ &\tfrac{1}{12}\, \vec J_\eta(\vec k, \vec k\,', \vec k\,'') - k_0\, (\vec k\,' \times_\eta \vec k\,'') - k_0'\, (\vec k\,'' \times_\eta \vec k\,) - k_0''\, (\vec k \times_\eta \vec k\,') \big) \ . \end{aligned} \tag{5.6}$$

Only rotations in the 21-dimensional spin group $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ preserve the triple cross product, where the action of Spin(7) can be described as the transitive action on the unit sphere $S^7 \subset W$ identified with the homogeneous space $S^7 \simeq \mathrm{Spin}(7)/G_2$; the Lie group Spin(7) is isomorphic to the double cover of SO(7), with the two copies of SO(7) corresponding to the upper and lower hemispheres of S^7 and G_2 the unique lift to a subgroup of Spin(7). A Spin(7)-*structure* on an oriented eight-dimensional vector space W is the choice of a triple cross product that can be written as (5.1) in a suitable oriented frame.

An important feature of a Spin(7)-structure is that it can be used to generate G_2-structures. For this, let $\hat k \in W$ be a fixed unit vector and let $V_{\hat k}$ be the orthogonal complement to the real line spanned by $\hat k$ in W. Then $W = \mathbb{R} \oplus V_{\hat k}$, and using (5.2) the seven-dimensional subspace $V_{\hat k}$ carries a cross product ([53], Theorem 6.15)

$$\vec k \times_{\hat k} \vec k\,' := \hat k \times_\phi \vec k \times_\phi \vec k\,' \,, \tag{5.7}$$

for $\vec k, \vec k\,' \in V_{\hat k}$.

5.2 Phase space 3-algebra

Let us set $\lambda = 1$, and consider the symmetries underlying the quasi-Poisson bracket relations (4.6) and (4.7). For this, we rewrite the bivector (3.31) in component form as

$$\Theta_\eta = \tfrac12\, \varepsilon_{ijk}\, \xi_k\, \tfrac{\partial}{\partial \xi_i} \wedge \tfrac{\partial}{\partial \xi_j} + \xi_i\, \big(\tfrac{\partial}{\partial \sigma^4} \wedge \tfrac{\partial}{\partial \sigma^i} + \varepsilon_{ijk}\, \tfrac{\partial}{\partial \sigma^j} \wedge \tfrac{\partial}{\partial \sigma^k} \big) - \tfrac{\partial}{\partial \xi_i} \wedge \big(\sigma^4\, \tfrac{\partial}{\partial \sigma^i} + \varepsilon_{ijk}\, \sigma^j\, \tfrac{\partial}{\partial \sigma^k} \big) \ .$$

From our discussion of G_2-structures from section 3.2, it follows that this bivector is invariant under the subgroup $G_2 \subset \mathrm{SO}(7)$ which is generated by antisymmetric 7×7 matrices $S = (s_{AB})$ satisfying $\eta_{ABC}\, s_{BC} = 0$ for $A = 1, \dots, 7$. Applying the affine transformation (4.3) (with $\lambda = 1$) generically breaks this symmetry to an $\mathrm{SO}(4) \times \mathrm{SO}(3)$ subgroup of SO(7). As G_2 contains no nine-dimensional subgroups (the maximal compact subgroup $\mathrm{SU}(3) \subset G_2$ is eight-dimensional), the residual symmetry group is $G_2 \cap \big(\mathrm{SO}(4) \times \mathrm{SO}(3)\big)$

JHEP02(2017)099

(see e.g. [41] for a description of the corresponding regular subalgebra $G_2[\alpha]$ of the Lie algebra of G_2). This exhibits the non-invariance of the quasi-Poisson algebra under SL(4) and SL(3) observed in [30]; however, as also noted by [30], the SL(3) symmetry is restored in the contraction limit by the discussion of section 3.6, and indeed the string theory quasi-Poisson algebra and its quantization from section 2 are SL(3)-invariant, being based on the three-dimensional cross product.

This symmetry breaking may be attributed to the specific choice of frame wherein the R-flux has non-vanishing components $R^{4,\mu\nu\alpha\beta}$ and the momentum constraint (4.2) is solved by $p_4 = 0$. In [30] it is suggested that this constraint could be implemented in a covariant fashion on the eight-dimensional phase space by the introduction of some sort of "Nambu-Dirac bracket". Here we will offer a slightly different, but related, explicit proposal for such a construction based on the Spin(7)-structures introduced in section 5.1. The impetus behind this proposal is that, in the lift from string theory to M-theory, 2-brackets should be replaced by suitable 3-brackets, as has been observed previously on many occasions (see e.g. [3, 31] for reviews); it is naturally implied by the Lie 2-algebra structure discussed at the beginning of section 3.5. This is prominent in the lift via T-duality of the SO(3)-invariant D1-D3-brane system in IIB string theory to the SO(4)-symmetric M2-M5-brane system wherein the underlying Lie algebra $\mathfrak{su}(2)$, representing the polarisation of D1-branes into fuzzy two-spheres S^2_F, is replaced by the 3-Lie algebra A_4, representing the polarization of M2-branes into fuzzy three-spheres S^3_F (see e.g. [23, 50–52] for reviews in the present context); we have already adapted a similar point of view in our considerations of sections 4.5 and 4.6. Based on this, the BLG model uses a 3-algebra for the underlying gauge symmetry to construct the $\mathcal{N} = 2$ worldvolume theory on the M-theory membrane, which is related to $\mathcal{N} = 6$ Chern-Simons theories, while the Moyal-Weyl type deformation of the coordinate algebra of D3-branes in a flat two-form B-field background of 10-dimensional supergravity lifts to Nambu-Heisenberg 3-Lie algebra type deformations of the coordinate algebra of M5-branes in a flat three-form C-field background of 11-dimensional supergravity [21]. If we moreover adopt the point of view of [44] that closed strings in R-flux compactifications should be regarded as boundary degrees of freedom of open membranes whose topological sector is described by an action that induces a phase space quasi-Poisson structure on the boundary, then this too has a corresponding lift to M-theory: in that case the action for an open topological 4-brane induces a 3-bracket structure on the boundary [47], regarded as the worldvolume of M2-branes in the M-theory R-flux background.

Taking this perspective further, we will adapt the point of view of [20]: in some systems with gauge symmetry, 3-brackets $\{\{f, g, h\}\}$ of fields can be defined without gauge-fixing, in contrast to quasi-Poisson brackets which depend on a gauge choice, such that for any gauge-fixing condition $G = 0$ the quasi-Poisson bracket $\{f, g\}_G$ in that gauge is simply given by $\{f, g\}_G = \{\{f, g, G\}\}$. This is analogous to the procedure of reducing a 3-Lie algebra to an ordinary Lie algebra by fixing one slot of the 3-bracket (see e.g. [23, 55]), and it can be used to dimensionally reduce the BLG theory of M2-branes to the maximally supersymmetric Yang-Mills theory of D2-branes [42]. It is also reminescent of the relation (5.7) between cross products and triple cross products, which motivates an application of this construction to the full eight-dimensional M-theory phase space subjected to the constraint (4.2). Writing $\varXi = (\xi_0, \vec{\xi}\,) \in \mathbb{R}^8$, we extend the G_2-symmetric bivector Θ_η to the Spin(7)-symmetric trivector

$$\Phi := \tfrac{1}{3}\,\phi_{\hat A\hat B\hat C\hat D}\,\varXi^{\hat D}\,\tfrac{\partial}{\partial \varXi^{\hat A}}\wedge\tfrac{\partial}{\partial \varXi^{\hat B}}\wedge\tfrac{\partial}{\partial \varXi^{\hat C}} = \tfrac{\partial}{\partial \xi_0}\wedge\Theta_\eta + \eta_{ABCD}\,\xi_D\,\tfrac{\partial}{\partial \xi_A}\wedge\tfrac{\partial}{\partial \xi_B}\wedge\tfrac{\partial}{\partial \xi_C} \tag{5.8}$$

which generates a 3-algebra structure on coordinate functions $\mathbb{C}[\varXi]$ with 3-brackets

$$\big\{\!\!\big\{\varXi_{\hat A},\varXi_{\hat B},\varXi_{\hat C}\big\}\!\!\big\}_\phi = 2\,\phi_{\hat A\hat B\hat C\hat D}\,\varXi_{\hat D}\ .$$

The trivector (5.8) is a Nambu-Poisson tensor if these 3-brackets satisfy the fundamental identity [55], thus defining a 3-Lie algebra structure on $\mathbb{C}[\varXi]$, so the Jacobiator (2.3) is now replaced by the 5-bracket

$$\begin{aligned}\{f_1,f_2,g,h,k\}_\phi := {}&\{\!\{f_1,f_2,\{\!\{g,h,k\}\!\}_\phi\}\!\}_\phi - \{\!\{\{\!\{f_1,f_2,g\}\!\}_\phi,h,k\}\!\}_\phi\\ &- \{\!\{g,\{\!\{f_1,f_2,h\}\!\}_\phi,k\}\!\}_\phi - \{\!\{g,h,\{\!\{f_1,f_2,k\}\!\}_\phi\}\!\}_\phi\,,\end{aligned}$$

which is natural from the index structure of the M-theory R-flux. One can compute it explicitly on linear functions $\varXi_{\hat A}$ by using the contraction identity (5.3) to get

$$\begin{aligned}\{\varXi_{\hat A},\varXi_{\hat B},\varXi_{\hat C},\varXi_{\hat D},\varXi_{\hat E}\}_\phi = {}& 12\,\big(\,\delta_{\hat A\hat C}\,\phi_{\hat D\hat E\hat B\hat F} + \delta_{\hat A\hat D}\,\phi_{\hat E\hat C\hat B\hat F} + \delta_{\hat A\hat E}\,\phi_{\hat C\hat D\hat B\hat F}\\ &- \delta_{\hat B\hat C}\,\phi_{\hat D\hat E\hat A\hat F} - \delta_{\hat B\hat D}\,\phi_{\hat E\hat C\hat A\hat F} - \delta_{\hat B\hat E}\,\phi_{\hat C\hat D\hat A\hat F}\,\big)\,\varXi^{\hat F}\\ &- 12\,\big(\,\varXi_{\hat A}\,\phi_{\hat B\hat C\hat D\hat E} - \varXi_{\hat B}\,\phi_{\hat A\hat C\hat D\hat E}\,\big)\ .\end{aligned}$$

To write the 3-brackets of phase space coordinates, we use (4.4) to define an affine transformation of the vector space $\mathbb{R}^8$ given by $X = (\vec{x},p_4) = (x^\mu,p_\mu) = \big(\varLambda\,\vec{\xi},-\frac{\lambda}{2}\,\xi_0\big)$, which we have chosen to preserve SO(4)-symmetry of momentum space; this breaks (at $\lambda=1$) the symmetry of the trivector (5.8) to a subgroup $\mathrm{Spin}(7)\cap\big(\mathrm{SO}(4)\times\mathrm{SO}(4)\big)$ of SO(8). Then the 3-brackets are given by

$$\begin{aligned}\big\{\!\!\big\{x^A,x^B,x^C\big\}\!\!\big\}_\phi &= -\lambda^{ABCD}\,x^D - \tfrac{2}{\lambda}\,\varLambda^{AA'}\,\varLambda^{BB'}\,\varLambda^{CC'}\,\eta_{A'B'C'}\,p_4\,,\\ \big\{\!\!\big\{p_4,x^A,x^B\big\}\!\!\big\}_\phi &= -\tfrac{\lambda}{2}\,\lambda^{ABC}\,x^C\ .\end{aligned}$$

Altogether, the phase space 3-algebra of the non-geometric M-theory R-flux background is summarised by the 3-brackets

$$\begin{aligned}\{\!\{x^i,x^j,x^k\}\!\}_\phi &= \tfrac{\ell_s^3}{2\hbar^2}\,R^{4,ijk4}\,x^4 \qquad\text{and}\qquad \{\!\{x^i,x^j,x^4\}\!\}_\phi = -\tfrac{\lambda^2\,\ell_s^3}{2\hbar^2}\,R^{4,ijk4}\,x_k\ ,\\ \{\!\{p^i,x^j,x^k\}\!\}_\phi &= \tfrac{\lambda^2\,\ell_s^3}{2\hbar^2}\,R^{4,ijk4}\,p_4 + \tfrac{\lambda\,\ell_s^3}{2\hbar^2}\,R^{4,ijk4}\,p_k\,,\\ \{\!\{p_i,x^j,x^4\}\!\}_\phi &= \tfrac{\lambda^2\,\ell_s^3}{2\hbar^2}\,R^{4,1234}\,\delta_i^j\,p_4 + \tfrac{\lambda^2\,\ell_s^3}{2\hbar^2}\,R^{4,ijk4}\,p_k\,,\\ \{\!\{p_i,p_j,x^k\}\!\}_\phi &= -\tfrac{\lambda^2}{2}\,\varepsilon_{ij}{}^k\,x^4 - \tfrac{\lambda}{2}\,\big(\delta_j^k\,x_i - \delta_i^k\,x_j\big)\,,\\ \{\!\{p_i,p_j,x^4\}\!\}_\phi &= \tfrac{\lambda^3}{2}\,\varepsilon_{ijk}\,x^k \qquad\text{and}\qquad \{\!\{p_i,p_j,p_k\}\!\}_\phi = 2\,\lambda\,\varepsilon_{ijk}\,p_4\,,\\ \{\!\{p_4,x^i,x^j\}\!\}_\phi &= -\tfrac{\lambda\,\ell_s^3}{2\hbar^2}\,R^{4,ijk4}\,p_k \qquad\text{and}\qquad \{\!\{p_4,x^i,x^4\}\!\}_\phi = \tfrac{\lambda^2\,\ell_s^3}{2\hbar^2}\,R^{4,1234}\,p^i\,,\\ \{\!\{p_4,p_i,x^j\}\!\}_\phi &= \tfrac{\lambda}{2}\,\delta_i^j\,x^4 + \tfrac{\lambda^2}{2}\,\varepsilon_i{}^{jk}\,x_k\,,\\ \{\!\{p_4,p_i,x^4\}\!\}_\phi &= \tfrac{\lambda^3}{2}\,x_i \qquad\text{and}\qquad \{\!\{p_4,p_i,p_j\}\!\}_\phi = \tfrac{\lambda^2}{2}\,\varepsilon_{ijk}\,p^k\ .\end{aligned} \tag{5.9}$$

JHEP02(2017)099

For any constraint

$$G(X)=0$$

on the eight-dimensional phase space, one can now define quasi-Poisson brackets through

$$\{f,g\}_G := \{\{f,g,G\}\}_\phi \ .$$

In particular, for the constraint function $G(X)=-\frac{2}{\lambda}\,p_4$ these brackets reproduce the quasi-Poisson brackets $\{f,g\}_\lambda$ of the seven-dimensional phase space from section 4.1; the more general choice $G(X)=R^{\mu,\nu\rho\alpha\beta}\,p_\mu$ constrains the 3-algebra of the eight-dimensional phase space to the codimension one hyperplane defined by (4.2) (which in the gauge $p_4=0$ is orthogonal to the p_4-direction). Moreover, setting $p_4=0$ in the remaining 3-brackets from (5.9) reduces them to the seven-dimensional phase space Jacobiators from section 4.1; we shall see this feature at the quantum level later on. Finally, we observe that in the limit $\lambda\to 0$ reducing M-theory to IIA string theory, the only non-vanishing 3-brackets from (5.9) are $\{\{x^i,x^j,x^k\}\}_\phi$, which at $x^4=1$ reproduce the string theory Jacobiator (2.3), and $\{\{p_4,x^I,x^J\}\}_\phi$, which at $x^4=1$ reproduce the bivector (2.2). This framework thereby naturally explains the SO(4)-invariance (at $\lambda=1$) of the M-theory brackets (4.6) and (4.7) described in [30]: the trivector (5.8) can be alternatively modelled on the space $S_-(\mathbb{R}^4)$ of negative chirality spinors over $\mathbb{R}^4$ (which is a quaternionic line bundle over $\mathbb{R}^4$) with respect to the splitting $W=\mathbb{R}^4\oplus\mathbb{R}^4$ (see e.g. [34] for details).

These consistency checks support our proposal for the M-theory phase space 3-algebra. It suggests that in the absence of R-flux, $R^{\mu,\nu\rho\alpha\beta}=0$, the brackets (5.9) describe the free phase space 3-algebra structure of M2-branes with the non-vanishing 3-brackets

$$\begin{aligned}
\{\{p_i,p_j,x^k\}\} &= -\tfrac{\lambda^2}{2}\,\varepsilon_{ij}{}^k\,x^4-\tfrac{\lambda}{2}\,\big(\delta^k_j\,x_i-\delta^k_i\,x_j\big) &&\text{and}\quad \{\{p_i,p_j,x^4\}\}=\tfrac{\lambda^3}{2}\,\varepsilon_{ijk}\,x^k\ ,\\
\{\{p_i,p_j,p_k\}\} &= 2\,\lambda\,\varepsilon_{ijk}\,p_4 &&\text{and}\quad \{\{p_4,p_i,x^j\}\}=\tfrac{\lambda}{2}\,\delta^j_i\,x^4+\tfrac{\lambda^2}{2}\,\varepsilon_i{}^{jk}\,x_k\ ,\\
\{\{p_4,p_i,x^4\}\} &= \tfrac{\lambda^3}{2}\,x_i &&\text{and}\quad \{\{p_4,p_i,p_j\}\}=\tfrac{\lambda^2}{2}\,\varepsilon_{ijk}\,p^k\ ,
\end{aligned}\tag{5.10}$$

which as expected all vanish in the weak string coupling limit $\lambda\to 0$. Quantization of this 3-algebra then provides a higher version of the noncommutative geometry of the M-theory momentum space discussed in section 4.6. In the following we will provide further evidence for these assertions.

5.3 Vector trisums

At this stage the next natural step is to quantise the 3-algebra (5.9). Although at present we do not know how to do this in generality, we can show how the star product of section 4.2 and the configuration space triproducts of section 4.5 naturally arise from the 3-algebraic structure of the full membrane phase space; this is based on a natural ternary extension of the vector star sum described in section 3.3. To motivate its construction, let us first provide an alternative eight-dimensional characterisation of the binary operation (3.18) on the unit ball $B^7\subset V\subset W$. For this, we note that for arbitrary vectors $P,P'\in W$ one has [53]

$$X_P\,X_{P'}=\big(p_0\,p_0'-\vec p\cdot\vec p\,'\big)\,\mathbb{1}+p_0\,X_{\vec p\,'}+p_0'\,X_{\vec p}+X_{\vec p\times_\eta\vec p\,'}\ .\tag{5.11}$$

By restricting to vectors P from the unit sphere $S^7 \subset W$, i.e., $|P| = 1$, we can easily translate this identity to the vector star sum (3.18): we fix a hemisphere $p_0 = \pm\sqrt{1-|\vec p\,|^2}$, such that for vectors $\vec p, \vec p\,'$ in the ball $B^7 = S^7/\mathbb{Z}_2 \simeq \mathrm{SO}(7)/G_2$ we reproduce the seven-dimensional vector star sum through

$$X_{\vec p \circledast_\eta \vec p\,'} = \mathfrak{Im}\big(X_{P'}\, X_P\big) = \tfrac{1}{2}\,\big(X_{P'}\, X_P - \bar X_P\, \bar X_{P'}\big)\,, \tag{5.12}$$

where the sign factor $\epsilon_{\vec p, \vec p\,'}$ from (3.18), which is the sign of the real part of (5.11), ensures that the result of the vector star sum remains in the same hemisphere. This interpretation of the vector star sum is useful for deriving various properties. For example, since the algebra of octonions $\mathbb{O}$ is a normed algebra, for $|P| = 1$ we have $|X_P| = 1$ and

$$\big|X_{P'}\, X_P\big|^2 = \big|\mathfrak{Re}(X_{P'}\, X_P)\big|^2 + \big|\mathfrak{Im}(X_{P'}\, X_P)\big|^2 = 1\ .$$

From (5.11) and (5.12) we then immediately infer the identity (3.20).

By comparing the representation (5.4) of the vector cross product with the definition (5.12) of the vector star sum, we can apply the same reasoning to the triple cross product represented through (5.5). We define a ternary operation on $B^7 = S^7/\mathbb{Z}_2 \subset W$ by

$$X_{\vec p \circledast_\phi \vec p\,' \circledast_\phi \vec p\,''} := \mathfrak{Im}\big((X_P\, X_{P'})\, X_{P''}\big)\,, \tag{5.13}$$

and call it a *vector trisum*. To obtain an explicit expression for it we observe that

$$(X_P\, X_{P'})\, X_{P''} = \tfrac{1}{2}\,\big((X_P\, X_{P'})\, X_{P''} + (X_{P''}\, X_{P'})\, X_P\big) + X_{P \times_\phi \bar P' \times_\phi P''}\,,$$

which leads to

$$\begin{aligned} \vec p \circledast_\phi \vec p\,' \circledast_\phi \vec p\,'' = \epsilon_{\vec p, \vec p\,', \vec p\,''}\, \Big(& \epsilon_{\vec p\,', \vec p\,''}\, \sqrt{1-\big|\vec p\,' \circledast_\eta \vec p\,''\big|^2}\ \vec p + \epsilon_{\vec p, -\vec p\,''}\, \sqrt{1-\big|\vec p \circledast_\eta (-\vec p\,'')\big|^2}\ \vec p\,' \\ & + \epsilon_{\vec p, \vec p\,'}\, \sqrt{1-\big|\vec p \circledast_\eta \vec p\,'\big|^2}\ \vec p\,'' + \vec A_\eta(\vec p, \vec p\,', \vec p\,'') \\ & + \sqrt{1-|\vec p\,|^2}\, (\vec p\,' \times_\eta \vec p\,'') + \sqrt{1-|\vec p\,'|^2}\, (\vec p\,'' \times_\eta \vec p\,) + \sqrt{1-|\vec p\,''|^2}\, (\vec p \times_\eta \vec p\,')\Big). \end{aligned} \tag{5.14}$$

Here $\epsilon_{\vec p, \vec p\,', \vec p\,''} = \pm 1$ is the sign of $\mathfrak{Re}\big((X_P\, X_{P'})\, X_{P''}\big)$ satisfying

$$\epsilon_{\vec p, \vec p\,'} = \epsilon_{\vec p, \vec p\,', \vec 0} = \epsilon_{\vec p, \vec 0, \vec p\,'} = \epsilon_{\vec 0, \vec p, \vec p\,'}\,,$$

which as previously ensures that the result of the vector trisum remains in the same hemisphere of S^7. As before, since $|X_P| = 1$ we have

$$\big|(X_P\, X_{P'})\, X_{P''}\big|^2 = \big|\mathfrak{Re}\big((X_P\, X_{P'})\, X_{P''}\big)\big|^2 + \big|\mathfrak{Im}\big((X_P\, X_{P'})\, X_{P''}\big)\big|^2 = 1\ .$$

This implies that

$$\begin{aligned} 1 - \big|\vec p \circledast_\phi \vec p\,' \circledast_\phi \vec p\,''\big|^2 &= \big|\mathfrak{Re}(X_P\, X_{P'})\, X_{P''}\big|^2 \\ &= \Big(\sqrt{1-|\vec p\,|^2}\, \sqrt{1-|\vec p\,'|^2}\, \sqrt{1-|\vec p\,''\,|^2} - \sqrt{1-|\vec p\,''|^2}\ \vec p\cdot \vec p \\ &\qquad - \sqrt{1-|\vec p\,|^2}\ \vec p\,'\cdot \vec p\,'' - \sqrt{1-|\vec p\,'|^2}\ \vec p\cdot \vec p\,'' - \vec p\,''\cdot(\vec p \times_\eta \vec p\,')\Big)^2 \ \geq\ 0\,, \end{aligned}$$

which shows that the vector $\vec p \circledast_\phi \vec p\,' \circledast_\phi \vec p\,''$ indeed also belongs to the unit ball $B^7 \subset V$.

JHEP02(2017)099

The relation between the vector trisum $\vec{p} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p}\,''$ and the vector star sum $\vec{p} \circledast_\eta \vec{p}\,'$ can be described as follows. From the definitions (5.12) and (5.13) it easily follows that

$$\vec{p} \circledast_\eta \vec{p}\,' = \vec{p} \circledast_\phi \vec{0} \circledast_\phi \vec{p}\,' = \vec{0} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p} = \vec{p}\,' \circledast_\phi \vec{p} \circledast_\phi \vec{0} \ . \tag{5.15}$$

By (5.7), for $\hat{p} = (1, \vec{0}\,) \in W$ there is an analogous relation

$$\vec{p} \times_\eta \vec{p}\,' = \vec{p} \times_\phi \hat{p} \times_\phi \vec{p}\,' = \hat{p} \times_\phi \vec{p}\,' \times_\phi \vec{p} = \vec{p}\,' \times_\phi \vec{p} \times_\phi \hat{p} \ .$$

From (5.6) it follows that the antisymmetrization of the vector trisum $\vec{p} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p}\,''$ reproduces the imaginary part of the triple cross product $\mathfrak{Im}(X_{P \times_\phi P' \times_\phi P''})$; this is analogous to the relation (3.22) between the vector star sum and the cross product. However, in contrast to (5.4), for the antisymmetrization of the product of three octonions one sees

$$\mathfrak{Im}\big(X_{P \times_\phi P' \times_\phi P''}\big) \neq \mathfrak{Im}\big(X_{\vec{p} \times_\phi \vec{p}\,' \times_\phi \vec{p}\,''}\big) \ ,$$

because the antisymmetrization of the vector trisum $\vec{p} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p}\,''$ contains terms involving $p_0 = \pm\sqrt{1 - |\vec{p}\,|^2}$.

To extend the vector trisum (5.14) over the entire vector space $V \subset W$, we again apply the map (3.24). Then for $\vec{k}, \vec{k}\,', \vec{k}\,'' \in V$, the corresponding mapping of the vector trisum is given by

$$\vec{\mathcal{B}}_\phi(\vec{k}, \vec{k}\,', \vec{k}\,''\,) := \frac{\sin^{-1} |\vec{p} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p}\,''\,|}{\hbar\, |\vec{p} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p}\,''\,|} \ \vec{p} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p}\,'' \Big|_{\vec{p} = \vec{k} \sin(\hbar\, |\vec{k}|)/|\vec{k}|} \ . \tag{5.16}$$

From (5.16) one immediately infers the following properties:

(TB1) $\vec{\mathcal{B}}_\phi(\vec{k}_1, \vec{k}_2, \vec{k}_3) = (-1)^{|\sigma|}\, \vec{\mathcal{B}}_\phi(-\vec{k}_{\sigma(1)}, -\vec{k}_{\sigma(2)}, -\vec{k}_{\sigma(3)})$ for all permutations $\sigma \in S_3$;

(TB2) $\vec{\mathcal{B}}_\eta(\vec{k}, \vec{k}\,'\,) = \vec{\mathcal{B}}_\phi(\vec{k}, \vec{0}, \vec{k}\,'\,) = \vec{\mathcal{B}}_\phi(\vec{0}, \vec{k}\,', \vec{k}\,) = \vec{\mathcal{B}}_\phi(\vec{k}\,', \vec{k}, \vec{0}\,)$;

(TB3) Perturbative expansion:

$$\begin{aligned}\vec{\mathcal{B}}_\phi(\vec{k}, \vec{k}\,', \vec{k}\,''\,) &= \vec{k} + \vec{k}\,' + \vec{k}\,'' + \hbar\,\big(\vec{k} \times_\eta \vec{k}\,' + \vec{k}\,'' \times_\eta \vec{k} + \vec{k}\,' \times_\eta \vec{k}\,''\,\big) \\ &\quad + \tfrac{\hbar^2}{2}\,\big(2\,\vec{A}_\eta(\vec{k}, \vec{k}\,', \vec{k}\,''\,) - \big|\vec{k}\,' + \vec{k}\,''\,\big|^2\, \vec{k} - \big|\vec{k} + \vec{k}\,''\,\big|^2\, \vec{k}\,' - \big|\vec{k}\,' + \vec{k}\,\big|^2\, \vec{k}\,''\,\big) \\ &\quad + O\big(\hbar^3\big) \ ;\end{aligned}$$

(TB4) The higher associator

$$\begin{aligned}\vec{\mathcal{A}}_\phi(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4, \vec{k}_5) &:= \vec{\mathcal{B}}_\phi\big(\vec{k}_1\,,\, \vec{k}_2\,,\, \vec{\mathcal{B}}_\phi(\vec{k}_3, \vec{k}_4, \vec{k}_5)\big) - \vec{\mathcal{B}}_\phi\big(\vec{\mathcal{B}}_\phi(\vec{k}_1, \vec{k}_2, \vec{k}_3)\,,\, \vec{k}_4\,,\, \vec{k}_5\big) \\ &\quad - \vec{\mathcal{B}}_\phi\big(\vec{k}_3\,,\, \vec{\mathcal{B}}_\phi(\vec{k}_1, \vec{k}_2, \vec{k}_4)\,,\, \vec{k}_5\big) - \vec{\mathcal{B}}_\phi\big(\vec{k}_3\,,\, \vec{k}_4\,,\, \vec{\mathcal{B}}_\phi(\vec{k}_1, \vec{k}_2, \vec{k}_5)\big)\end{aligned}$$

is antisymmetric in all arguments.

5.4 Phase space triproducts

We now define a triproduct on the M-theory phase space, analogously to the star product of section 4.2, by setting

$$(f \diamond_\lambda g \diamond_\lambda h)(\vec{x}\,) = \int \frac{\mathrm{d}^7\vec{k}}{(2\pi)^7}\, \frac{\mathrm{d}^7\vec{k}'}{(2\pi)^7}\, \frac{\mathrm{d}^7\vec{k}''}{(2\pi)^7}\, \tilde{f}(\vec{k}\,)\, \tilde{g}(\vec{k}'\,)\, \tilde{h}(\vec{k}''\,)\, \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\phi(\varLambda\vec{k},\varLambda\vec{k}',\varLambda\vec{k}''\,)\cdot\varLambda^{-1}\vec{x}}\,, \tag{5.17}$$

where as before $\tilde{f}$ stands for the Fourier transform of the function f and $\vec{\mathcal{B}}_\phi(\vec{k},\vec{k}',\vec{k}''\,)$ is the deformed vector sum (5.16). Property **(TB2)** from section 5.3 implies that the triproduct $f \diamond_\lambda g \diamond_\lambda h$ is related to the star product $f \star_\lambda g$, in precisely the same way that the triple cross product is related to the cross product, through the unital property

$$f \star_\lambda g = f \diamond_\lambda 1 \diamond_\lambda g = 1 \diamond_\lambda g \diamond_\lambda f = g \diamond_\lambda f \diamond_\lambda 1\ . \tag{5.18}$$

If we define a quantum 3-bracket in the usual way by

$$[\![f_1,f_2,f_3]\!]_{\diamond_\lambda} := \sum_{\sigma\in S_3} (-1)^{|\sigma|}\ f_{\sigma(1)} \diamond_\lambda f_{\sigma(2)} \diamond_\lambda f_{\sigma(3)}\,, \tag{5.19}$$

then by property **(TB3)** it has a perturbative expansion given to lowest orders by

$$[\![f,g,h]\!]_{\diamond_\lambda} = -\,\mathrm{i}\,\hbar\,\big(f\,\{g,h\}_\lambda + g\,\{h,f\}_\lambda + h\,\{f,g\}_\lambda\big) - 3\,\hbar^2\,\{f,g,h\}_\lambda + O(\hbar,\lambda)\ . \tag{5.20}$$

The fact that the perturbative expansion (5.20) contains terms without derivatives can be easily understood from the property (5.18), which together with (5.19) implies

$$[\![f,g,1]\!]_{\diamond_\lambda} = -3\,[f,g]_{\star_\lambda}\ . \tag{5.21}$$

Although this feature may seem unusual from the conventional perspective of deformation quantization, it is exactly the quantum version of the gauge fixing of 3-brackets to 2-brackets on the reduced M-theory phase space that we discussed in section 5.2.

All this generalises the properties of the triproducts described in section 4.5; indeed it is easy to see that the phase space triproduct $\diamond_\lambda$ reduces at $\boldsymbol{p}=\boldsymbol{0}$ to the configuration space triproduct $\triangle_\lambda^{(3)}$ by comparing (5.14) and (5.16) with (4.38). In particular, it is straightforward to show that the 3-bracket (5.19) reproduces the string theory 3-bracket on configuration space in the collapsing limit of the M-theory circle, as in (4.40): from the calculations of section 4.2 we find

$$\begin{aligned}\lim_{\lambda\to0}\ \vec{\mathcal{B}}_\phi(\varLambda\vec{k},\varLambda\vec{k}'\,,\varLambda\vec{k}''\,)\cdot\varLambda^{-1}\,\vec{x} = {}& (\boldsymbol{k}+\boldsymbol{k}'\,+\boldsymbol{k}''\,)\cdot\boldsymbol{x} + (k_4+k_4'\,+k_4''\,)\,x^4 + (\boldsymbol{l}+\boldsymbol{l}'\,+\boldsymbol{l}''\,)\cdot\boldsymbol{p}\\ &-\tfrac{\ell_s^3\,R}{2\hbar}\,\boldsymbol{p}\cdot(\boldsymbol{k}'\times_\varepsilon\boldsymbol{k}''\,+\boldsymbol{k}\times_\varepsilon\boldsymbol{k}''\,+\boldsymbol{k}\times_\varepsilon\boldsymbol{k}'\,)\\ &+\tfrac{\hbar}{2}\,x^4\,(\boldsymbol{k}'\cdot\boldsymbol{l}''-\boldsymbol{l}'\cdot\boldsymbol{k}''\,-\boldsymbol{k}''\cdot\boldsymbol{l}+\boldsymbol{l}''\cdot\boldsymbol{k}\,+\boldsymbol{k}\cdot\boldsymbol{l}'-\boldsymbol{l}\cdot\boldsymbol{k}'\,)\\ &-\tfrac{\ell_s^3\,R}{2}\,x^4\,\boldsymbol{k}\cdot(\boldsymbol{k}'\times_\varepsilon\boldsymbol{k}''\,)\,,\end{aligned}$$

so that in this limit the 3-bracket (5.19) for the configuration space coordinates $\boldsymbol{x}$ upon setting $\boldsymbol{p}=\boldsymbol{0}$ yields

$$\lim_{\lambda\to0}\,[\![x^i,x^j,x^k]\!]_{\diamond_\lambda}\Big|_{\boldsymbol{p}=\boldsymbol{0}} = -3\,\ell_s^3\,R\,\varepsilon^{ijk}\ .$$

JHEP02(2017)099

The phase space triproduct (5.17) also naturally quantises the 3-algebraic structure (5.10) of the membrane momentum space. Restricting (5.14) to vectors $\vec{p} = (\boldsymbol{q}, \boldsymbol{0}, 0)$ yields a non-vanishing vector trisum of vectors $\boldsymbol{q} \in \mathbb{R}^3$ with vanishing associator $\boldsymbol{J}_\varepsilon$ and cross products $\times_\eta$ replaced with $\times_\varepsilon$. The restriction of the triproduct (5.17) to functions of $\boldsymbol{p}$ alone thereby produces a non-trivial momentum space triproduct which quantises the 3-algebra (5.10).

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JHEP02(2017)099

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