SPECTRAL FUNCTIONS OF SUBORDINATE BROWNIAN MOTION ON CLOSED MANIFOLDS

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ABSTRACT. For a class of subordinators we investigate the spectrum of the infinitesimal generator of subordinate Brownian motion on a closed manifold. We consider three spectral functions of the generator: the zeta function, the heat trace and the spectral action. Each spectral function explicitly yields both probabilistic and geometric information, the latter through the classical heat invariants. All constructions are done with classical pseudodifferential operators and are fully analytically tractable.

1. INTRODUCTION

This paper investigates the correspondence between analytic, geometric and probabilistic objects. On a closed Riemannian manifold (i.e. compact and without boundary) we consider a subordinate Brownian motion. From the spectrum of its infinitesimal generator A we extract both geometric and probabilistic information using spectral functions that aggregate the spectrum suitably. Two of the best known examples of such functions are the heat trace $\mathrm{Trace}(e^{tA})$ and the number of eigenvalues $N(\lambda)$ below a certain level λ. The latter is associated with Weyl [48] who investigated the behaviour of $N(\lambda)$ for $\lambda \to \infty$ noting that this depends on the volume and dimension of the manifold.

In the tradition of Blumenthal and Getoor [12], Applebaum [3] computes the heat trace and Weyl asymptotics for the generator of the Cauchy process on certain compact Lie groups. He notes that the eigenvalue asymptotics differ markedly from the standard case for the Laplace operator. Bañuelos and Baudoin [6] give Weyl asymptotics for subordinate Brownian motion on general closed manifolds for a class of Laplace exponents (Bernstein functions) that are regularly varying at ∞. The authors derive the lowest-order heat trace asymptotics involving the volume of the manifold and prove their dependence on properties of the Bernstein function.

We extend this line of research by considering a subclass of the regularly varying Bernstein functions that lead to generators which are classical pseudodifferential operators. This allows the asymptotic computation of spectral functions to arbitrary order and reveals the connection between certain stochastic processes and invariance theory [26]. Our class of Bernstein functions is small enough to be analytically tractable yet large enough to be interesting in applications. For example, the class covers the relativistic α-stable processes.

To describe our contribution more precisely, let B_t be a Brownian motion on a closed manifold M of dimension n and let X_t be a subordinator, i.e. an increasing Lévy process on $[0, \infty)$ such that $X_0 = 0$ almost surely. The generator of the subordinate process B_{X_t} is given by $A = -f\left(-\frac{1}{2}\Delta\right)$ for f a Bernstein function and Δ the Laplace-Beltrami operator. One shows that A belongs to a suitable

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commutative subalgebra of the classical pseudodifferential operators on M that is generated by Δ. It is then straightforward to explicitly compute several spectral functions, namely the zeta function $\operatorname{Trace}((-A+c)^{-z})$ and the heat trace $\operatorname{Trace}(e^{tA})$ which also yields Weyl-type asymptotics for $-A$. To illustrate the limitations of this approach we also consider the spectral action $\operatorname{Trace}(\Phi((-A+c)/\lambda))$ for suitable Φ. Here, c is a positive constant depending on f. One of our key results is illustrated in the following diagram.

Stochastics		*Analysis & Geometry*
Process on M		Spectrum of the generator
B_{X_t}	$=\!=$	$\operatorname{Sp}(A)$
$\downarrow$		$\downarrow$
$\underbrace{\alpha = 2 \lim_{x\to 0^+} \frac{1}{x}\mathbb{E}\left(X_{T(x)^-}\right)}_{\text{Lévy's arcsine law with first passage time } T(x)=\inf\{t\geq 0 \mid X_t > x\}}$	$\longrightarrow$	$\underbrace{\operatorname{Trace}\left(e^{At}\right) \sim e^{ct}\sum_{j=0}^{\infty} c_j(\alpha, M) t^{(n-j)/\alpha} - e^{ct}\sum_{k=1}^{\infty} c'_k(\alpha, M) t^k \log t}_{\text{heat trace asymptotics for } t\to 0^+\text{, log terms only if } \alpha \text{ is rational}}$

Here, the c_j and c'_k depend on the classical heat invariants of M and on the coefficients in an asymptotic expansion of f in which the parameter α also plays a key role. This parameter can be interpreted probabilistically in terms of a first passage time. Logarithmic terms appear only if α is rational. As the classical heat invariants depend on geometric features of the manifold such as the volume and the scalar curvature, this shows that the heat trace of subordinate Brownian motion mixes geometric and probabilistic information.

The main advantages of our approach are: it is highly transparent and no local computations are needed; it immediately generalizes to quantum stochastic processes in noncommutative algebras; and it allows to distinguish geometry and probability, the former represented by the Laplace-Beltrami operator Δ and the latter being represented by the Laplace exponent f.

One could argue that a complicated calculation involving the eigenvalues of Δ yields the same results without resorting to pseudodifferential operators. However, two reasons justify the approach. First, the generator of the subordinate process is naturally a pseudodifferential operator and these operators are a convenient tool that allow the computation of the spectral functions. Moreover, we can appeal to the known connections between traces of powers of the Laplacian and the geometry of the manifold, cf. Proposition 4.3. This is also very much in the spirit of noncommutative geometry where one tries to infer geometric information from the spectrum of a strategically associated operator.

There is a related but orthogonal line of research that considers the heat trace asymptotics of subordinate Brownian motion on domains in Euclidean space with various boundary conditions [7, 8, 9, 41]. The authors treat the specific examples of the stable or relativistic stable processes using a combination of hard analysis and probabilistic methods. More general results for a class of Lévy processes and subordinate Brownian motion were recently obtained in [15]. A key issue in this direction is how to define the generator of processes when boundary conditions are present. A case at hand is applying Dirichlet boundary conditions when one considers subordinate processes: the two ways to define a generator would be as $f(\Delta_{\text{Dirichlet}})$ or $f(\Delta)_{\text{Dirichlet}}$ where the subscript indicates on which operator the boundary conditions are imposed, cf. [20] for a spectral comparison of these approaches. For manifolds with boundary the second type of generator may be probabilistically more natural, cf. also [15].

The use of pseudodifferential operators associated to Markov processes is extensively discussed in the seminal joint and individual work of Hoh, Jacob and Schilling building on [29], cf. also the comprehensive series [32, 33, 34] and the survey papers [17, 35]. The authors construct a global pseudodifferential calculus on Euclidean space that allows parametrices, cf. also [16]. The study of Markov processes on compact Lie groups using pseudodifferential operators is done in Applebaum [4]. The group properties of the manifold allow a global calculus. Neither calculus is, however, suitable for our purposes as we require classicality of the operators and work on general closed manifolds. Classical pseudodifferential operators were also applied in the context of derivative pricing in financial mathematics based on Normal Inverse Gaussian processes that are a special case of our class of Laplace exponents, cf. [18].

Our exposition is the companion paper of [25] which considers the analogous investigation on Euclidean space. At the expense of some minor overlaps we have, however, ensured that the present paper is self-contained.

This paper is organized as follows. The following section summarizes the key results. Section 3 introduces a commutative subalgebra of classical pseudodifferential operators including a parameter-dependent parametrix and complex powers. In Section 4 we use this calculus to compute certain spectral functions; this is also of independent interest. Proofs of the key results are collected in Section 5.

2. Key results

We briefly describe the probabilistic setup and summarize selected results. The reader is referred to [10] for details of Lévy processes, to [5, 30] for studies of stochastic processes on manifolds and for the potential theory of stable processes we refer to [13]. All proofs are deferred to Section 5.

Brownian motion. Let (M, g) be a compact manifold without boundary endowed with a Riemannian metric g. Let B_t be a canonical Brownian motion on M. The generator of the corresponding semigroup is $\frac{1}{2}\Delta$ with Δ the Laplace-Beltrami operator on M. It is defined as $\Delta = \text{div grad}$ and it can be locally described as

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \sqrt{g} g^{ij} \frac{\partial}{\partial x_i}$$

where $\sqrt{g}$ is the volume factor $\sqrt{\det(g_{ij})}$ and g^{ij} is the inverse of the metric tensor.

Remark 2.1. Following the probabilistic convention, the Laplace operator just defined has negative spectrum whereas the geometric literature sometimes uses the convention that $\Delta = -\text{div grad}$ leading to positive spectrum.

Subordinator. A *subordinator* is an increasing Lévy process X_t with values in $[0, \infty)$ and $X_0 = 0$ a.s. that is independent of B_t. The generating function of the subordinator can be written as

$$\mathbb{E}\left(e^{-\lambda X_t}\right) = e^{-tf(\lambda)},$$

with *Laplace exponent* f which is a Bernstein function.

Definition 2.2 (([42, Definition 3.1])). A function $f : (0, \infty) \to \mathbb{R}$ is a *Bernstein function* if f is smooth, $f(\lambda) \geq 0$ and $(-1)^{k-1} f^{(k)}(\lambda) \geq 0$ for $k \in \mathbb{N}$.

The Bochner subordination principle yields a simple description of the generator of the subordinate Brownian motion in terms of the Bernstein function.

Proposition 2.3 (([6]))**.** *The infinitesimal generator of* B_{X_t} *is given by* $-f\left(-\frac{1}{2}\Delta\right)$.

Any Bernstein function can be given in Lévy-Khintchin form as

$$f(\lambda) = a + b\lambda + \int_0^\infty \left(1 - e^{-\lambda t}\right) \mu(dt), \tag{1}$$

for constants $a, b \geq 0$ and μ a measure (the Lévy measure) on $(0, \infty)$ satisfying $\int_0^\infty \min\{t, 1\}\, \mu(dt) < \infty$. The function f is uniquely determined by the *Lévy characteristic triplet* (a, b, μ). We restrict ourselves to Bernstein functions with characteristic triplet $(0, 0, \mu)$ whose Lévy measure possesses a locally integrable density m with respect to Lebesgue measure, the *Lévy density*. Reasonable assumptions on this density turn $-f\left(-\frac{1}{2}\Delta\right)$ into a classical pseudodifferential operator, cf. [25].

To state the assumptions we recall the definition of asymptotic expansions.

Definition 2.4. Suppose that $m : (0, \infty) \to \mathbb{R}$ is a function. We say that $m(t) \sim \sum_{k=0}^\infty p_k t^{a_k}$ as $t \to 0^+$ if $p_k \in \mathbb{R}, a_k \uparrow \infty$ and

$$\lim_{t \to 0^+} t^{-a_N} \left(m(t) - \sum_{k=0}^N p_k t^{a_k} \right) = 0$$

for every $N \geq 0$. Analogously for $t \to \infty$ when we require $a_k \downarrow -\infty$.

The assumptions on the Lévy density then read as follows.

Hypothesis 2.5. Let $f(\lambda) = \int_0^\infty \left(1 - e^{-\lambda t}\right) m(t) dt$ be a Bernstein function with locally integrable density $m : (0, \infty) \to \mathbb{R}$ such that

(i) it has an asymptotic expansion $m(t) \sim t^{-1-\alpha/2} \sum_{k=0}^\infty p_k t^k$ as $t \to 0^+$ for some $\alpha \in (0, 2)$,
(ii) m is of rapid decay at ∞, i.e. $m(t)t^\beta$ is bounded a.e. for $t > 1$ for all $\beta \in \mathbb{R}$,
(iii) $\overline{m}(0, \infty) < 0$ where $\overline{m}(0, \infty) = \int_0^\infty \left(m(t) - p_0 t^{-1-\alpha/2}\right) dt$.

Assumption (i) yields an asymptotic expansion of f for large λ and the second assumption makes f smooth at the origin. Overall the assumptions ensure that $f\left(-\frac{1}{2}\Delta\right) - \overline{m}(0, \infty)$ is a classical pseudodifferential operator with positive spectrum, cf. Proposition 5.1.

The α-stable processes with $f(\lambda) = \lambda^{\alpha/2}$ and density $m(t) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} t^{-1-\alpha/2}$ are not in this class of Bernstein functions. However, the class contains the following examples from [42] where $\alpha \in (0, 2)$, $a > 0$ and Γ denotes the Gamma function.

Bernstein function f	Lévy density m
$(\lambda + 1)^{\alpha/2} - 1$	$\frac{\alpha/2}{\Gamma(1-\alpha/2)} e^{-t} t^{-\alpha/2-1}$
$\lambda/(\lambda + a)^{\alpha/2}$	$\frac{\sin(\pi\alpha/2)\Gamma(1-\alpha/2)}{\pi} e^{-at} t^{\alpha/2-2}(at + 1 - \alpha/2)$
$\lambda(1 - e^{-2\sqrt{\lambda+a}})/\sqrt{\lambda + a}$	$\frac{e^{-1/t-at}\left(2+t\left(e^{1/t}-1\right)(1+2at)\right)}{2\sqrt{\pi}t^{5/2}}$
$\Gamma\left(\frac{\lambda+a}{2a}\right)/\Gamma(\lambda/2a)$	$\frac{a^{3/2}e^{2at}}{2\sqrt{\pi}(e^{2at}-1)^{3/2}}$
$\Gamma(\alpha\lambda/2 + 1)/\Gamma(\alpha\lambda/2 + 1 - \alpha/2)$	$\frac{e^{-2t/\alpha}}{\Gamma(1-\alpha/2)(1-e^{-2t/\alpha})^{1+\alpha/2}}$.

Remark 2.6. The literature typically uses "local" and "global" scaling classes that characterize the behaviour of the Bernstein function as $\lambda \to \infty$, cf. for example [14] and [40]. The Bernstein functions satisfying Hypothesis 2.5 are regularly varying at ∞ and belong to the intersection of the local upper and lower scaling classes used in the cited papers with parameters arbitrarily close to α. The global scaling conditions need not be satisfied as can be seen in the case of the relativistic stable processes.

Probabilistic interpretation. We touch upon the relationship between the asymptotic expansion of the Lévy density on the one hand and probabilistic properties of the subordinator and the subordinate Brownian motion on the other. The first connection is phrased in terms of a first-passage time and follows from Lévy's arcsine law.

Proposition 2.7 (([25, Theorem 2]))**.** *Let X_t be the subordinator corresponding to the Bernstein function f satisfying Hypothesis 2.5. For any $x > 0$ define the first passage time strictly above x by $T(x) = \inf\{t \geq 0 | X_t > x\}$. Then*

$$\alpha = 2 \lim_{x \to 0^+} \tfrac{1}{x} \mathbb{E}\left(X_{T(x)-}\right).$$

The lowest-order coefficient is thus given as

$$p_0 = \frac{1}{t} \frac{1}{\Gamma(-\alpha/2)} \lim_{\lambda \to \infty} \lambda^{-\alpha/2} \log \mathbb{E}\left(e^{-\lambda X_t}\right),$$

for $t > 0$ with similar expressions for the higher-order coefficients.

Furthermore, the parameter α governs the transition density of the subordinate process on M, cf. also [14, 40] and references therein for a general class of processes on $\mathbb{R}^n$.

Proposition 2.8 (([27, Theorem 1]))**.** *The transition density p of the process B_{X_t}, i.e., the heat kernel for $-f\left(-\frac{1}{2}\Delta\right)$, satisfies the estimates*

$$p(t, x, y) \leq \frac{Ct}{(d(x, y) + t^{1/\alpha})^{n+\alpha}}$$

for a constant $C > 0$ where $x, y \in M$ and $d(x, y)$ is distance between x and y.

More intuitively we can view the term $p_0 t^{-1-\alpha/2}$ in the asymptotic expansion of the Lévy density as coming from a tempered stable process with Lévy density $p_0 e^{-\lambda t} t^{-1-\alpha/2}$ for some $\lambda > 0$. As in the comments after Proposition 4.2 of [23] one can interpret α and p_0 as follows.

- The parameter α determines the local behaviour of the process: small values of α mean that the process B_{X_t} exhibits quieter periods interrupted by big jumps. If α is close to 2, then the process is similar to a Brownian motion.
- The parameter p_0 determines the frequency of jumps: the larger this parameter is, the more often B_{X_t} shows jumps larger than a given size.

The zeta function. The first key result illuminates the pole structure of the zeta function. We phrase this in terms of the shifted positive operator $f(-\frac{1}{2}\Delta) - \overline{m}(0, \infty)$.

Theorem 2.9. *Let f be a Bernstein function satisfying Hypothesis 2.5 and set $\overline{m}(0, \infty) = \int_0^\infty \left(m(t) - p_0 t^{-1-\alpha/2}\right) dt$. Then the zeta function*

$$\zeta(z) = \mathrm{Trace}\left(\left[f\left(-\tfrac{1}{2}\Delta\right) - \overline{m}(0, \infty)\right]^{-z}\right)$$

is meromorphic on $\mathbb{C}$ with at most simple poles at the points $z_j = (n - j)/\alpha$ for $j = 0, 1, 2, \ldots$ More precisely, the singularity structure of the function $\Gamma(z)\zeta(z)$ can *be expressed as*

$$\Gamma(z)\zeta(z) \sim \sum_{j=0}^{\infty} \frac{c_j}{z - \frac{n-j}{\alpha}} + \sum_{k=1}^{\infty} \frac{c'_k}{(z+k)^2}$$

in the sense that for large N, the left hand side minus the sums for $j,k \le N$ in the right hand side is holomorphic for $\mathrm{Re}\, z > \max\{(n-N-1)/\alpha, -N-1\}$. The complex coefficients c_j, c'_k depend on the p_k and the classical heat invariants of M. The double poles only appear if α is rational.

The heat trace expansion. Our second main result is gives the asymptotic expansion of the heat trace of $-f(-\frac{1}{2}\Delta)$.

Theorem 2.10. *Let f be a Bernstein function satisfying Hypothesis 2.5. Then the heat trace expansion of $-f\left(-\frac{1}{2}\Delta\right)$ as $t \to 0^+$ is given by*

$$\mathrm{Trace}\left(e^{-tf\left(-\frac{1}{2}\Delta\right)}\right) \sim e^{-\overline{m}(0,\infty)t}\left(\sum_{j=0}^{\infty} c_j t^{-(n-j)/\alpha} - \sum_{k=1}^{\infty} c'_k t^k \log t\right)$$

with coefficients c_j, c'_k from Theorem 2.9. The logarithmic terms only appear if α is rational. Here, the constant $\overline{m}(0,\infty)$ is given by $\overline{m}(0,\infty) = \int_0^\infty \left(m(t) - p_0 t^{-1-\alpha/2}\right) dt$.

The coefficients can in principle be computed to arbitrary order. In dimension $n \ge 3$ the geometric content of the expansion becomes visible in c_0 and c_2. In lowest orders we find

$$\mathrm{Trace}\left(e^{-tf\left(-\frac{1}{2}\Delta\right)}\right) \sim e^{-\overline{m}(0,\infty)t}\left[c_0 t^{-n/\alpha} + c_2 t^{-(n-2)/\alpha} + \cdots\right]$$

with coefficients given by

$$c_0 = \frac{\Gamma(n/\alpha)}{\alpha(2\pi)^n} a_0^{-n/\alpha} \Omega_n \mathrm{vol}(M)$$

$$c_2 = \frac{\Gamma\left(\frac{n-2}{\alpha}\right)}{\alpha(2\pi)^n}\left[a_0^{-(n-2)/\alpha} \frac{(\frac{n}{2}-1)\Omega_n}{6} \int_M s\sqrt{g}\, d^n x - \frac{n-2}{\alpha} a_0^{-(n-2)/\alpha - 1} a_1 \Omega_n \mathrm{vol}(M)\right],$$

where

$$a_0 = -p_0 \Gamma(-\alpha/2)(\tfrac{1}{2})^{\alpha/2}, \quad a_1 = -p_1 \Gamma(-\alpha/2+1)(\tfrac{1}{2})^{\alpha/2-1}.$$

Here $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$ the volume of the unit sphere in $\mathbb{R}^n$, s the scalar curvature of M and $\sqrt{g}$ the Riemannian density. The coefficients mix geometric and probabilistic information. The powers of $\frac{1}{2}$ are due to the generator of Brownian motion being $\frac{1}{2}$ times the Laplace operator. Using the identity $\Gamma(z+1) = z\Gamma(z)$, the coefficient c_0 becomes $c_0 = a_0^{-n/\alpha} \frac{\Gamma(n/\alpha+1)\mathrm{vol}(M)}{\Gamma(n/2+1)(4\pi)^{n/2}}$ in which form it appears in [6] for the subordinator $f(\lambda) = \lambda^{\alpha/2}$ with $a_0 = 1$.

Weyl asymptotics. An immediate consequence concerns the asymptotics of the eigenvalues of the positive operator $f\left(-\frac{1}{2}\Delta\right)$, cf. [12] for a probabilistic discussion of this.

Corollary 2.11. *Denote by $N(\lambda)$ the number of eigenvalues of $f\left(-\frac{1}{2}\Delta\right)$ less than λ. Under Hypothesis 2.5 we have*

$$N(\lambda) \sim a_0^{-n/\alpha} \frac{\mathrm{vol}(M)}{(4\pi)^{n/2}\Gamma(1+n/2)} \lambda^{n/\alpha} \tag{2}$$

as $\lambda \to \infty$ in the sense of Definition 2.4 with $a_0 = -p_0\Gamma(-\alpha/2)(\frac{1}{2})^{\alpha/2}$.

The analogous result for the subordinator $f(\lambda) = \lambda^{\alpha/2}$ is a special case in [3, 6] and reads

$$N(\lambda) \sim \frac{\mathrm{vol}(M)}{(4\pi)^{n/2}\Gamma(1+n/2)}\lambda^{n/\alpha}.$$

The authors consider the generator Δ instead of $\frac{1}{2}\Delta$ so that there are no powers of $\frac{1}{2}$.

The probabilistic and geometric information represented by the c_j and c'_k, i.e. the short-time asymptotics of the Lévy density and the heat invariants of the manifold appears to be the key information that can be extracted from the spectrum of $-f(-\frac{1}{2}\Delta)$ in a pseudodifferential operator approach, cf. the discussion in Remark 4.8.

3. A global pseudodifferential operator calculus

We introduce a commutative subalgebra of the algebra of classical pseudodifferential operators on a closed manifold. This allows us to construct complex powers (and thence our spectral functions) globally without resorting to symbol considerations in local coordinates. The subalgebra is motivated by the abstract pseudodifferential operators introduced in [22].

As most of this material is well known we do not give full proofs of all results but rather indicate where our approach leads to simplifications.

3.1. **A subalgebra of pseudodifferential operators.** We briefly recall selected aspects of classical pseudodifferential operators on closed manifolds, cf. §4.3 of [45] for details. For an open set $X \subseteq \mathbb{R}^n$ denote by $C^\infty(X)$ the smooth functions $u : X \to \mathbb{C}$ and by $C_0^\infty(X)$ the smooth functions of compact support. For a closed manifold M let $H^s(M)$ be the usual Sobolev spaces for $s \in \mathbb{R}$.

Any pseudodifferential operator A on M can be represented in local coordinates as an oscillatory integral. Formally for $X \subset \mathbb{R}^n$ open, we require for any $u \in C_0^\infty(X)$ that

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot y}\sigma^A(x,\xi)\hat{u}(\xi)d\xi$$

where $x \cdot y$ denotes the standard inner product in $\mathbb{R}^n$ and $\hat{u} = \int e^{-ix\cdot y}u(\xi)d\xi$ is the Fourier transform of u. We call $\sigma^A \in C^\infty(X \times \mathbb{R}^n)$ the *symbol* of A and demand that for any multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ and any compact set $K \subset X$ there is a $z \in \mathbb{C}$ and a constant C_{α,β_K} such that

$$|\partial_\xi^\alpha \partial_x^\beta \sigma^A(x,\xi)| \le C_{\alpha,\beta,K}\langle\xi\rangle^{\mathrm{Re}\, z-|\alpha|-\beta|}$$

for all $x \in K$ and $\xi \in \mathbb{R}^n$. Here, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, $|\alpha| = \alpha_1 + \cdots \alpha_n$ and $\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_1}^{\alpha_1}$. We call z the *order* of the symbol (and the corresponding operator) and denote the space of such symbols by $S^z(X \times \mathbb{R}^n)$. Recall that *classical* pseudodifferential operators have asymptotic expansions of the symbol that are homogeneous in the covariable ξ. This means we can write

$$\sigma^A(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\sigma_{z-j}^A(x,\xi)$$

in the sense that $\sigma^A(x,\xi) - \sum_{j=0}^N \chi(\xi)\sigma_{z-j}^A(x,\xi) \in S^{z-N}(X \times \mathbb{R}^n)$ and the $\sigma_{z-j}^A(x,\xi)$ are homogeneous of degree $z-j$ in ξ, i.e. $\sigma_{z-j}^A(x,\tau\xi) = \tau^{z-j}\sigma_{z-j}^A(x,\xi)$ for any $\tau > 0$. Here χ is a bump function supported away from the origin and equal to 1 for $|\xi| \ge 1$. The symbol of the highest order is called the *principal symbol* of A.

We denote the algebra of classical pseudodifferential operators of order $z \in \mathbb{C}$ on M by $\Psi DO_{cl}^z(M)$. Any $A \in \Psi DO_{cl}^z(M)$ gives rise to a continuous linear map $H^s(M) \to H^{s-\mathrm{Re}\, z}(M)$.

We say that a classical pseudodifferential operator A of order z is *elliptic* if its principal symbol vanishes nowhere: $\sigma_z^A(x,\xi) \neq 0$ for $x \in X$ and $\xi \neq 0$. It is standard (cf. §5.5 of [45]) that an elliptic pseudodifferential operator $A \in \Psi DO_{cl}^z(M)$ has a *parametrix* $B \in \Psi DO_{cl}^{-z}(M)$, i.e. an inverse up to order $-\infty$ such that $AB - I$ and $BA - I$ belong to $\Psi DO^{-\infty}(M)$.

To make our presentation accurate we introduce the operator

$$\Delta_1 = -\Delta + \epsilon\Pi$$

where Π is the projection onto the kernel of Δ and $\epsilon > 0$ is smaller than the lowest non-zero eigenvalue of $-\Delta$. Note that Δ_1 has positive spectrum so it is invertible. Moreover, Δ and Δ_1 differ by an operator of order $-\infty$ on the scale of Sobolev spaces and have the same symbol

$$\sigma^{\Delta_1}(x,\xi) = ||\xi||_g^2 = \sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j.$$

The principal symbol of Δ_1^μ is given by $\sigma_{2\mu}^{\Delta_1^\mu}(x,\xi) = ||\xi||_g^{2\mu}$ by the construction in Chapter 11.2 of [45], equation (11.11).

We mimic the notion of classicality on the level of operators by formally replacing the covariable with Δ_1.

Definition 3.1. For $\mu \in \mathbb{C}$ we define $\Psi^{2\mu}$ to be the set of linear operators $A : C^\infty(M) \to C^\infty(M)$ such that

$$A \sim a_0\Delta_1^\mu + a_1\Delta_1^{\mu-1} + a_2\Delta_1^{\mu-2} + \cdots$$

with $a_k \in \mathbb{C}$ for $k = 0, 1, 2, \ldots$ where the asymptotic expansion is in the sense that for any $N = 1, 2, \ldots$ we have

$$A - \sum_{k=0}^N a_k\Delta_1^{\mu-k} \in \Psi DO_{cl}^{-N}(M).$$

We let $\Psi^* = \cup_{\mu\in\mathbb{C}}\Psi^\mu$ be the union of all such operators.

Lemma 3.2. *The space Ψ^* is a commutative algebra. Also, $\Psi^{2\mu} \subset \Psi DO_{cl}^{2\mu}(M)$.*

Proof. The algebraic property follows from addition and multiplication of asymptotic sums. Any such A is a classical pseudodifferential operator as the $\Delta_1^{\mu-k}$ are classical pseudodifferential operators of order $2(\mu-k)$ for $k = 0, 1, 2, \ldots$, cf. [44]. □

3.2. **A parameter-dependent parametrix.** We now construct a parametrix of $A - \lambda$ for $A \in \Psi^\mu$ with $\mu > 0$ and λ in a sector in the complex plane. This is analogous to the parameter-dependent parametrices in a global calculus of classical pseudodifferential operators on $\mathbb{R}^n$ from [39]. We make the following assumption.

Hypothesis 3.3. Let $A \in \Psi^{2\mu}$ for some $\mu > 0$ with asymptotic expansion

$$A \sim a_0\Delta_1^\mu + a_1\Delta_1^{\mu-1} + a_2\Delta_1^{\mu-2} + \cdots,$$

where $a_0 > 0$ and $a_k \in \mathbb{C}$ for $k \geq 1$. Let Λ be a sector in the left half of the complex plane with apex at the origin: $\Lambda = \{re^{i\phi} \in \mathbb{C} | \pi - \rho < \phi < \pi + \rho\}$ for some $\rho \in (0, \pi/2)$. Assume that

(i) the principal symbol $\sigma_{2\mu}^A(x,\xi)$ is positive for nonzero ξ: $\sigma_{2\mu}^A(x,\xi) - \lambda \neq 0$ for any $\lambda \in (-\infty, 0]$ and $\xi \neq 0$; and
(ii) the spectrum of A is contained in the interval $(0, \infty)$; in particular, A is invertible with bounded inverse.

Remark 3.4. Conditions (i) and (ii) are simply the conditions for constructing the complex powers A^z, cf. §1 of [44] or Chapter 10.1 of [45]. The first assumption is automatically satisfied as $\sigma_{2\mu}^A(x,\xi) = a_0||\xi||_g^{2\mu}$ and is given for completeness only.

Let Λ_δ be a the set $\Lambda \cup \{|\lambda| \leq \delta\}$ for $\delta > 0$ so small that the spectrum of A does not intersect the disk $\{\lambda \in \mathbb{C} | |\lambda| < \delta\}$. For the shifted Laplacian $\Delta_1 = -\Delta + \epsilon\Pi$ we choose $\delta < \epsilon$.

Proposition 3.5. *Assume Hypothesis 3.3. Then there is a family of operators $B(\lambda) \in \Psi DO_{cl}^{-2\mu}(M)$ depending on $\lambda \in \Lambda_\delta$ such that the following holds.*

(i) $B(\lambda)$ is a parametrix for $A - \lambda$, i.e. $(A - \lambda I)B(\lambda) - I$ and $B(\lambda)(A - \lambda I) - I$ belong to $\Psi DO^{-\infty}(M)$ uniformly in $\lambda \in \Lambda_\delta$.

(ii) There is an asymptotic expansion $B(\lambda) \sim b_{-2\mu}(\lambda) + b_{-2\mu-2}(\lambda) + b_{-2\mu-4}(\lambda) + \cdots$ where each $b_{-2\mu-2k}(\lambda) \in \Psi DO_{cl}^{-2\mu-2k}(M)$ can be explicitly expressed in terms of Δ_1.

(iii) We have $|\lambda|^2((A - \lambda)^{-1} - B(\lambda)) \in \Psi DO^{-\infty}(M)$ uniformly in $\lambda \in \Lambda_\delta$.

In lowest orders the expansion of the parametrix reads

$$\left.\begin{aligned} b_{-2\mu}(\lambda) &= (a_0\Delta_1^\mu - \lambda)^{-1}, \\ b_{-2\mu-2}(\lambda) &= -a_1\Delta_1^{\mu-1}(a_0\Delta_1^\mu - \lambda)^{-2}, \\ b_{-2\mu-4}(\lambda) &= a_1^2\Delta_1^{2\mu-2}(a_0\Delta_1^\mu - \lambda)^{-3} - a_2\Delta_1^{\mu-2}(a_0\Delta_1^\mu - \lambda)^{-2}. \end{aligned}\right\}$$

Proof. This is analogous to the arguments in Section 3.2 of [39].

1. We first construct the parametrix in the form of an explicit asymptotic expansion

$$B(\lambda) \sim b_{-2\mu}(\lambda) + b_{-2\mu-2}(\lambda) + b_{-2\mu-4}(\lambda) + \cdots$$

with operators $b_{-2\mu-2k}(\lambda) \in \Psi DO_{cl}^{-2\mu-2k}(M)$. Without loss of generality $a_0 = 1$. We determine the $b_{-2\mu-2k}(\lambda)$ by the formal ansatz

$$\left[(\Delta_1^\mu - \lambda) + a_1\Delta_1^{\mu-1} + \cdots\right][b_{-2\mu}(\lambda) + b_{-2\mu-2}(\lambda) + \cdots] = 1.$$

Collecting according to the orders of operators yields

$$\begin{aligned} b_{-2\mu}(\lambda) &= (\Delta_1^\mu - \lambda)^{-1} \\ b_{-2\mu-2k}(\lambda) &= -(\Delta_1^\mu - \lambda)^{-1}\left[a_1\Delta_1^{\mu-1}b_{-2\mu-2(k-1)}(\lambda) + \cdots + a_k\Delta_1^{\mu-k}b_{-2\mu}(\lambda)\right] \end{aligned}$$

for $k \geq 1$. In closed form this can be expressed as

$$b_{-2\mu-2k}(\lambda) = \sum_{l=1}^{k} b_{kl}(\Delta_1^\mu - \lambda)^{-(l+1)}, \tag{3}$$

with $b_{kl} \in \Psi DO_{cl}^{2l\mu-2k}(M)$ given by

$$b_{kl} = (-1)^l \left(\sum_{|j|=k} a_{j_1}\cdots a_{j_l}\right)\Delta_1^{l\mu-k}. \tag{4}$$

Here $j = (j_1, \ldots, j_n) \in \mathbb{N}^l$ is a multi-index of length l and $|j| = j_1 + \cdots + j_n$.

2. To see that $B(\lambda)$ is a classical pseudodifferential operator note that from spectral considerations the operator $\Delta_1^\mu - \lambda$ is invertible for $\lambda \in \Lambda_\delta$. It is also an elliptic pseudodifferential operator. This means that it has a parametrix $B'(\lambda)$ with $(\Delta_1^\mu - \lambda)B'(\lambda) = I + R_1(\lambda)$ and $B'(\lambda)(\Delta_1^\mu - \lambda) = I + R_2(\lambda)$ with $R_1, R_2 \in \Psi DO^{-\infty}(M)$. By the identity

$$(\Delta_1^\mu - \lambda)^{-1} - B'(\lambda) = B'(\lambda)R_1(\lambda) + R_2(\lambda)(\Delta_1^\mu - \lambda)^{-1}R_1(\lambda) \tag{5}$$

the inverse $(\Delta_1^\mu - \lambda)^{-1}$ and the parametrix $B'(\lambda)$ differ by an operator of order $-\infty$ so that $(\Delta_1^\mu - \lambda)^{-1} \in \Psi DO_{cl}^{-2\mu}(M)$. This also implies that $b_{-2\mu-2k} \in \Psi DO_{cl}^{-2\mu-2k}(M)$. Let $B(\lambda)$ be the operator obtained by asymptotically summing the $b_{-2\mu-2k}$, cf. Proposition 3.4 of [45].

3. From the definition of the $b_{-2\mu-2k}$ we find

$$|\lambda| b_{-2\mu}(\lambda) \in \Psi DO_{cl}^{-2\mu}(M), \quad |\lambda|^2 b_{-2\mu-2k}(\lambda) \in \Psi DO_{cl}^{-2\mu-2k}(M) \tag{6}$$

for $k = 1, 2, \ldots$ uniformly in $\lambda \in \Lambda_\delta$.

4. To see that $B(\lambda)$ is a parametrix of $A - \lambda$ set $B_N(\lambda) = \sum_{k=0}^{N-1} b_{-2\mu-2k}(\lambda)$. Then

$$(A - \lambda)B(\lambda) = (A - \lambda)B_N(\lambda) + (A - \lambda)(B(\lambda) - B_N(\lambda)).$$

By construction, we have $|\lambda|(A-\lambda)(B(\lambda)-B_N(\lambda)) \in \Psi DO_{cl}^{-2\mu-2N}(M)$ and $|\lambda|((A-\lambda)B_N(\lambda)-I) \in \Psi DO_{cl}^{-2\mu-2N}(M)$ uniformly in $\lambda \in \Lambda_\delta$. As N was arbitrary, we find that $B(\lambda)$ is a right parametrix and by a similar argument also a left parametrix.

5. The assertion in (iii) follows from (6) using (5) for the difference $(A - \lambda)^{-1} - B(\lambda)$. If we set $R_1 = (A - \lambda)B(\lambda) - I$ and $R_2 = B(\lambda)(A - \lambda) - I$, then by step 4 we have $|\lambda|^2 R_i \in \Psi DO^{-\infty}(M)$ so that the claim follows. □

3.3. **Complex powers.** We construct the complex powers of certain operators in Ψ^* and show that the complex powers also belong to this class.

For Re $z < 0$ the complex powers of A are defined by a Dunford integral

$$A_z = \frac{i}{2\pi} \int_{\partial\Lambda_\delta} \lambda^z (A - \lambda)^{-1} d\lambda, \tag{7}$$

where $\partial\Lambda_\delta$ is a parametrization of the boundary of Λ_δ. The power $\lambda^z = e^{z \log \lambda}$ is given by the main branch of the logarithm. The integral converges for Re $(z) < 0$ to a bounded operator in $L^2(M)$ since $||(A - \lambda)^{-1}|| \leq C/|\lambda|$ for some $C > 0$ from spectral considerations.

One defines $A^z = A^k A_{z-k}$ for arbitrary $z \in \mathbb{C}$ by choosing $k \in \mathbb{N}$ sufficiently large so that Re $z - k < 0$. Lemma 3 of [44] shows that the complex powers have the group property and that this definition is independent of k.

We can now show that A^z also belongs to $\Psi^{\mu z}$, i.e. has a suitable asymptotic expansion.

Theorem 3.6. *Assume Hypothesis 3.3. Then the operator A^z belongs to $\Psi^{2\mu z}$ for any $z \in \mathbb{C}$ and there are operators $b_{2\mu z-2k} \in \Psi^{2\mu z-2k}$ for $k = 0, 1, 2, \ldots$ such that*

$$A^z \sim b_{2\mu z} + b_{2\mu z-2} + b_{2\mu z-4} + \cdots .$$

For Re $z < 0$ *these operators are given as Dunford integrals*

$$b_{2\mu z-2k} = \frac{i}{2\pi} \int_{\Lambda_\delta} \lambda^z b_{-2\mu-2k}(\lambda) d\lambda$$

with $b_{-2\mu-2k}(\lambda)$ from Proposition 3.5.

We mention for completeness that the map $z \mapsto A^z$ is a holomorphic family of operators between Sobolev spaces, cf. Theorem 3 of [44]. This will allow us to define the zeta function $\zeta_A(z) = \text{Trace}(A^{-z})$ as a meromorphic function.

The lowest-order terms in the asymptotic expansion of A^z are given as

$$\left.\begin{aligned} b_{\mu z}(\Delta_1) &= a_0^z \Delta_1^{\mu z} \\ b_{\mu z-2}(\Delta_1) &= z a_0^{z-1} a_1 \Delta_1^{\mu z-1} \\ b_{\mu z-4}(\Delta_1) &= \left[\tfrac{z(z-1)}{2} a_0^{z-2} a_1^2 + z a_0^{z-1} a_2\right] \Delta_1^{\mu z-2}, \end{aligned}\right\} \tag{8}$$

where we assumed Re $z < 0$.

Proof. We show the claim analogously to Section 3.2 of [39]. Without loss of generality $a_0 = 1$. We need only consider the case Re $z < 0$. As usual, the idea is to replace the resolvent in (7) by a parameter-dependent parametrix. We write

$$\frac{i}{2\pi}\int_{\Lambda_\delta}\lambda^z(A-\lambda)^{-1}d\lambda = \frac{i}{2\pi}\int_{\Lambda_\delta}\lambda^z B(\lambda)d\lambda + \frac{i}{2\pi}\int_{\Lambda_\delta}\lambda^z\left[(A-\lambda)^{-1} - B(\lambda)\right]d\lambda \quad (9)$$

with $B(\lambda)$ be as in Proposition 3.5. By Proposition 3.5 (iii) the second summand defines an operator of order $-\infty$ on the scale of Sobolev spaces.

To construct the asymptotic expansion recall that $B(\lambda)$ can be expressed as the asymptotic sum $\sum_{k=0}^{\infty} b_{-2\mu-2k}(\lambda)$. Thus, the first integral in (9) has an asymptotic expansion

$$\frac{i}{2\pi}\int_{\Lambda_\delta}\lambda^z B(\lambda)d\lambda \sim \sum_{k=0}^{\infty} b_{2\mu z-2k},$$

where each summand can be rewritten using (3) and (4) as

$$b_{2\mu z-2k} = \frac{i}{2\pi}\int_{\Lambda_\delta}\lambda^z b_{-2\mu-2k}(\lambda)d\lambda = \sum_{l=1}^{k} b_{kl}\frac{i}{2\pi}\int_{\Lambda_\delta}\lambda^z\left(\Delta_1^\mu - \lambda\right)^{-(l+1)}d\lambda.$$

Each integral can be computed explicitly. For $k = 0$ we have $b_{-2\mu}(\lambda) = (\Delta_1^\mu - \lambda)^{-1}$ so that

$$b_{2\mu z}(\Delta_1) = \Delta_1^{\mu z}.$$

For $b_{2\mu z-2k}$ with $k = 1, 2, \ldots$ we observe that from spectral considerations the functions

$$\Lambda_\delta \to \Psi^0 : \lambda \mapsto |\lambda|^k(\Delta_1^\mu - \lambda)^{-k}$$
$$\Lambda_\delta \to \Psi^{-2\mu k} : \lambda \mapsto (\Delta_1^\mu - \lambda)^{-k}$$

are bounded uniformly in λ in operator norm so that we can integrate by parts

$$\frac{i}{2\pi}\int_{\Lambda_\delta}\lambda^z\left(\Delta_1^\mu - \lambda\right)^{-(l+1)}d\lambda = \frac{z(z-1)\cdots(z-(l-1))}{l!}(\Delta_1^\mu)^{z-l}$$

belonging to $\Psi^{2\mu(z-l)}$. We had $b_{kl} \in \Psi^{2\mu l-2k}$ so that overall $b_{2\mu z-2k} \in \Psi^{2\mu z-2k}$. □

4. Spectral functions of the global pseudodifferential operators

We now compute three spectral functions of certain $A \in \Psi^*$.

4.1. The zeta function.

The above construction of the complex powers allows us to investigate the zeta function Trace(A^{-z}). It is meromorphic and the residues at the poles are given in terms of the *Wodzicki residue* [37, 49] which we denote by RES. For a classical pseudodifferential operator A the Wodzicki residue can be computed explicitly in terms of the symbol. Let σ_{-n}^A be the homogeneous function of degree $-n$ in the symbol expansion. Then

$$\mathrm{RES}(A) = \int_{S^*(M)} \sigma_{-n}^A ds,$$

with integration over the unit sphere in the cotangent bundle $T^*(M)$.

Theorem 4.1. *Let M be a closed manifold of dimension n. With $A \in \Psi^{2\mu}$ satisfying Hypothesis 3.3 the zeta function*

$$\zeta_A(z) = \mathrm{Trace}(A^{-z})$$

is analytic for Re $z > n/2\mu$. *It can be extended to a meromorphic function on the whole complex plane with at most simple poles. The poles are located in the set*

$$\mathcal{P} = \left\{ \tfrac{n-j}{2\mu} \,\middle|\, j = 0, 1, 2, \ldots \right\}$$

with residues given by

$$\operatorname*{res}_{z=(n-j)/2\mu} \zeta_A(z) = \frac{1}{2\mu(2\pi)^n} \mathrm{RES}\left(A^{-(n-j)/2\mu}\right)$$

for $j \neq n$. The zeta function $\zeta_A(z)$ has a removable singularity at $z = 0$.

Proof. The arguments of the proof of Theorem 13.1 of [45] apply here. The residue calculation appeared in [37]. □

Corollary 4.2. *The singularity structure of the function $\Gamma(z)\zeta_A(z)$ can be expressed as*

$$\Gamma(z)\zeta_A(z) \sim \sum_{j=0}^{\infty} \frac{c_j}{z - \frac{n-j}{2\mu}} + \sum_{k=1}^{\infty} \frac{c'_k}{(z+k)^2},$$

in the sense that for large N, the left hand side minus the sums for $j, k \leq N$ in the right hand side is holomorphic for $\mathrm{Re}\, z > \max\{(n-N-1)/2\mu, -N-1\}$.

The double poles appear if the poles of the zeta function and of the Gamma function (located at $0, -1, -2, \ldots$) coincide. This can only happen if the order of A is rational. In Section 4.2 we illustrate the computation of the lowest-order coefficients.

It is well known that the residues of the zeta function of the Laplace operator yield the classical heat invariants, cf. [26] for a general discussion.

Proposition 4.3. *Let (M, g) be a closed Riemannian manifold of dimension n with Riemannian metric g. Define Ω_n to be the volume of the unit sphere in $\mathbb{R}^n$ and $\mathrm{vol}(M)$ to be the volume of the manifold M under the canonical Riemannian density $\sqrt{g}$. Then*

$$\mathrm{RES}\left(\Delta_1^{-n/2}\right) = \Omega_n \mathrm{vol}(M)$$
$$\mathrm{RES}\left(\Delta_1^{-(n-1)/2}\right) = 0$$

If the dimension satisfies $n \geq 3$, then

$$\mathrm{RES}\left(\Delta_1^{-(n-2)/2}\right) = \frac{(\frac{n}{2}-1)\Omega_n}{6} \int_M s\sqrt{g}\, d^n x$$

with s the scalar curvature of M.

Proof. The claim for $\mathrm{RES}\left(\Delta_1^{-n/2}\right)$ is Proposition 7.7 of [28], however, with non-normalized Riemannian density on M; the result originally appeared in [44].

The second assertion on the vanishing of the Wodzicki residue is a consequence of the fact that the odd heat invariants of the Laplace operator vanish by Lemma 1.8.2 (d) of [26]. The heat invariants correspond precisely to the residues of the zeta function.

Finally, the third assertion is the Kastler-Kalau-Walze theorem [36, 38] with normalization as in Theorem 7.8 of [28]. It also follows from the correspondence of heat invariants and residues of the zeta function [1]. □

4.2. **The heat operator and the heat trace.** A consequence of the pole structure of the zeta function is the short-time asymptotic expansion of the heat trace.

Theorem 4.4. *With $A \in \Psi^{2\mu}$ satisfying Hypothesis 3.3 the asymptotics of the heat trace $\mathrm{Trace}\left(e^{-tA}\right)$ as $t \to 0^+$ are given as*

$$\mathrm{Trace}\left(e^{-tA}\right) \sim \sum_{j=0}^{\infty} c_j t^{-(n-j)/2\mu} - \sum_{k=0}^{\infty} c'_k t^k \log t \tag{10}$$

with coefficients c_j, c'_k as in Corollary 4.2.

Proof. The existence of e^{-tA} as a trace-class operator is clear. The asymptotics are obtained as a Mellin transform of the zeta function, cf. the detailed account in Chapter 3.3.3 of [43]. □

Example 4.5. Consider a closed Riemannian manifold of dimension $n \geq 3$. Choose a $A \in \Psi^{2\mu}$ satisfying Hypothesis 3.3 and let $\mathcal{P}$ be the set of poles of ζ_A. From (8) we infer

$$\zeta_A(z) = \phi_0(z)\text{Trace}\left(\Delta_1^{-\mu z}\right)+\phi_1(z)\text{Trace}\left(\Delta_1^{-\mu z-1}\right)+\phi_2(z)\text{Trace}\left(\Delta_1^{-\mu z-2}\right)+\text{mero}$$

with coefficient functions

$$\begin{aligned}\phi_0(z) &= a_0^{-z}\\ \phi_1(z) &= z a_0^{-(z+1)} a_1\\ \phi_2(z) &= \tfrac{z}{2} a_0^{-(z+2)}\left[(z+1)a_1^2 - 2a_0 a_2\right].\end{aligned}$$

Here mero stands for a meromorphic function that is analytic for Re $z > (n-3)/2\mu$. We consider the three right-most points of $\mathcal{P}$ separately:

$z_0 = \frac{n}{2\mu}$: only Trace $\left(\Delta_1^{-\mu z}\right)$ can have a pole with residue $\frac{1}{2\mu(2\pi)^n}\text{RES}\left(\Delta^{-n/2}\right)$. Hence

$$\begin{aligned}c_0 &= \operatorname*{res}_{z=z_0} \Gamma(z)\zeta(z)\\ &= \frac{\Gamma(n/2\mu)}{2\mu(2\pi)^n} a_0^{-n/2\mu}\text{RES}\left(\Delta_1^{-n/2}\right)\\ &= \frac{\Gamma(n/2\mu)}{2\mu(2\pi)^n} a_0^{-n/2\mu}\Omega_n \text{vol}(M) \qquad (11)\end{aligned}$$

$z_1 = \frac{n-1}{2\mu}$: at this point only Trace $(\Delta^{-\mu z})$ can have a pole. However, due to RES $\left(\Delta^{-(n-1)/2}\right) = 0$ by Proposition 4.3 this trace is regular at z_1 so that $c_1 = 0$.

$z_2 = \frac{n-2}{2\mu}$: arguing as before we find

$$\begin{aligned}c_2 &= \operatorname*{res}_{z=z_2} \Gamma(z)\zeta(z)\\ &= \frac{\Gamma(z_2)}{2\mu(2\pi)^n}\left(\phi_0(z_2)\text{RES}\left(\Delta^{-(n-2)/2}\right) + \phi_1(z_2)\text{RES}\left(\Delta^{-n/2}\right)\right)\\ &= \frac{\Gamma(z_2)}{2\mu(2\pi)^n}\left(\phi_0(z_2)\frac{(\frac{n}{2}-1)\Omega_n}{6}\int_M s\sqrt{g}\, d^n x + \phi_1(z_2)\Omega_n \text{vol}(M)\right).\end{aligned}$$

We thus obtain the heat kernel expansion

$$\text{Trace}\left(e^{-tA}\right) \sim c_0 t^{-n/2\mu} + c_2 t^{-(n-2)/2\mu} + \cdots.$$

Upon setting $A = \Delta_1$, i.e. $\mu = 1$, $a_0 = 1, a_1 = a_2 = \ldots = 0$ we recover the usual heat trace asymptotics of the Laplace operator.

As a consequence of this result we find Weyl-type asymptotics.

Corollary 4.6. *Denote by $N(\lambda)$ the number of number of eigenvalues of A less than or equal to λ. Under the assumptions of Theorem 4.4 we have*

$$N(\lambda) \sim a_0^{-n/2\mu}\frac{\text{vol}(M)}{(4\pi)^{n/2}\Gamma(1+n/2)}\lambda^{n/2\mu}$$

as $\lambda \to \infty$.

Proof. There are several methods to derive the eigenvalue asymptotics from the zeta function or the heat trace, cf. the discussion following Theorem 6.1.1. of [2]. One can for example use Karamata's Tauberian theorem: if the function Trace(e^{-tA}) has the asymptotics $t^{-n/2\mu}(c_0+c_1t+\cdots)$ as $t \to 0^+$, then $N(\lambda) \sim c_0/\Gamma(n/2\mu+1)\lambda^{n/2\mu}$, cf. the proof of Theorem 3.1 in [47]. The claim follows using (11) and standard properties of the Gamma function. □

4.3. **The spectral action.** To complete the picture (and to indicate the limitations of our approach) we now investigate a third spectral function, viz. the *spectral action principle*. It was introduced in order to apply tools from noncommutative geometry to quantum field theory, cf. [19]. It is also of high importance in noncommutative geometry itself.

Broadly speaking, the spectral action is defined by $\mathrm{Trace}\left(\Phi\left(D/\lambda\right)\right)$ where D is the Dirac operator, $\lambda \in (0,\infty)$ is the *cut-off parameter* and Φ is any positive function such that $\Phi(D/\lambda)$ is trace-class. This includes the functions $\Phi(x) = e^{-x}$ corresponding to the heat trace and $\Phi(x) = x^{-z}$ corresponding to the zeta function. The terminology cut-off parameter is based on the case when Φ is a cut-off function with support in $[0,1]$ so that $\mathrm{Trace}\left(\Phi\left(D/\lambda\right)\right)$ merely counts the number of eigenvalues of D in $[0,\lambda]$. The spectral action is very hard to compute in general, so one develops an asymptotic expansion for large λ.

In our context we consider the spectral action $\mathrm{Trace}(\Phi(A/\lambda))$ which represents a general way to aggregate the eigenvalues of A. The simplest but instructive case is when Φ is represented as a Laplace transform.

Proposition 4.7. *Let $A \in \Psi^{2\mu}$ satisfy Hypothesis 3.3. Suppose that Φ is given as a Laplace transform $\Phi(x) = \int_0^\infty e^{-tx}\hat{\Phi}(t)dt$ for a function $\hat{\Phi}$ which is of rapid decay at 0 and ∞ (i.e. the function Φ is smooth at the origin). Then*

$$\mathrm{Trace}\left(\Phi(A/\lambda)\right) \sim \sum_{j=0}^{\infty} c_j \Phi_j \lambda^{(n-j)/2\mu} - \sum_{k=1}^{\infty} c'_k \Phi'_k \lambda^{-k} \log\lambda \tag{12}$$

as $\lambda \to \infty$ with coefficients c_j, c'_k from Corollary 4.2. Here

$$\Phi_j = \begin{cases} \frac{1}{\Gamma((n-j)/2\mu)} \int_0^\infty u^{(n-j)/2\mu-1}\Phi(u)du & \text{for } \frac{n-j}{2\mu} > 0 \\ \int_0^\infty u^{-(n-j)/2\mu}\hat{\Phi}(u)du & \text{for } \frac{n-j}{2\mu} < 0 \end{cases}$$

$$\Phi'_k = \frac{(-1)^{k+1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\log u + C}{u} \Phi^{(k)}(-u)du$$

with $\mathrm{Re}\, c > 0$ *and C is Euler's constant. If $l = -\frac{n-j}{2\mu}$ is a natural number, then $\Phi_j = (-1)^l \Phi^{(l)}(0)$, i.e. the spectral action for rational μ depends on the Taylor series of Φ at 0.*

Proof. We start with the heat trace expansion (10). As in the proof of Theorem 1.145 of [21] we argue as follows:

$$\begin{aligned} \mathrm{Trace}(\Phi(A)) &= \mathrm{Trace}\left(\int_0^\infty e^{-tA}\hat{\Phi}(t)dt\right) \\ &= \int_0^\infty \mathrm{Trace}(e^{-tA})\hat{\Phi}(t)dt \\ &\sim \sum_{j=0}^{\infty} c_j \int_0^\infty \hat{\Phi}(t)t^{-(n-j)/2\mu}dt + \sum_{k=1}^{\infty} c'_k \int_0^\infty \hat{\Phi}(t)t^k \log t dt. \end{aligned}$$

If $(n-j)/2\mu > 0$, then by standard properties of the Laplace transform we find

$$\int_0^\infty \hat{\Phi}(t)t^{-(n-j)/2\mu}dt = \frac{1}{\Gamma\left((n-j)/2\mu\right)} \int_0^\infty u^{(n-j)/2\mu-1}\Phi(u)du.$$

Denote the Laplace transform of a function f by $\mathcal{L}[f]$. Then one has by Parseval's formula

$$\begin{aligned}\int_0^\infty \hat{\Phi}(t)t^k \log t dt &= \lim_{s\to 0}\int_0^\infty e^{-st}\hat{\Phi}(t)t^k \log t dt\\ &= \lim_{s\to 0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathcal{L}[\log t](u)\mathcal{L}\left[t^k\hat{\Phi}(t)\right](s-u)du\\ &= \lim_{s\to 0}\frac{(-1)^{k+1}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\log u+\gamma}{u}\Phi^{(k)}(s-u)du\end{aligned}$$

with $\mathrm{Re}\, c > 0$ and γ being Euler's constant. □

Remark 4.8. We first note that the Weyl asymptotics of Corollary 4.6 formally follow from (12) by setting Φ equal to the indicator function on $[0,1]$ and considering only the highest order in λ. However, due to Gibb's phenomenon we cannot represent the indicator function as a Laplace transform. To overcome this, different approaches have been developed that allow more general Φ such as step functions, cf. [24, 31]. In a broader perspective, even if we had techniques for general Φ, Proposition 4.6 suggests that we only recover the coefficients of the heat trace expansion (or the singularities of the zeta function). This indicates the limits of the pseudodifferential operator approach to extract information from the spectrum of an infinitesimal generator of a stochastic process.

5. Proof of the key results

The generator of the subordinate Brownian motion is a pseudodifferential operator in the commutative algebra Ψ^* obtained from applying a Bernstein function to the operator $\frac{1}{2}\Delta$. We state the claim for the positive operator $f(-\frac{1}{2}\Delta)$.

Proposition 5.1. *Let f be a Bernstein function satisfying Hypothesis 2.5 whose density m of the Lévy measure has the asymptotic expansion*

$$m(t) \sim t^{-1-\alpha/2}\left(p_0 + p_1 t + p_2 t^2 + \cdots\right)$$

as $t \to 0^+$ for some $\alpha \in (0,2)$ and real coefficients p_k with $p_0 > 0$. Set $\overline{m}(0,\infty) = \int_0^\infty \left(m(t) - p_0 t^{-1-\alpha/2}\right)dt$. Then $f\left(-\frac{1}{2}\Delta\right) - \overline{m}(0,\infty) \in \Psi^\alpha$ and

$$f\left(-\tfrac{1}{2}\Delta\right) - \overline{m}(0,\infty) \sim a_0\Delta_1^{\alpha/2} + a_1\Delta_1^{\alpha/2-1} + a_2\Delta_1^{\alpha/2-2} + \cdots, \tag{13}$$

in the sense of Definition 3.1 with

$$a_k = -p_k\Gamma(-\alpha/2+k)\left(\tfrac{1}{2}\right)^{\alpha/2-k}. \tag{14}$$

Moreover, $f\left(-\frac{1}{2}\Delta\right) - \overline{m}(0,\infty)$ satisfies Hypothesis 3.3 so that it has complex powers in Ψ^*.

Note that the leading term on the right hand side of (13) can be viewed as the generator of a stable process. This drives both the heat kernel estimates in Proposition 2.8 and the Weyl estimates in Corollary 2.11.

To prove the proposition we need a version of Watson's Lemma.

Lemma 5.2 **(([11, Chapter 4.1])).** *Let $m : (0,\infty) \to \mathbb{R}$ be a function. We assume that m is locally integrable, there is an $a > 0$ such that $|m(t)| \leq e^{at}$ for all $t \geq 1$ and it has an asymptotic expansion $m(t) \sim \sum_{k=0}^\infty p_k t^{\alpha_k}$ as $t \to 0^+$ where the a_k are real numbers such that $\alpha_0 > -1$ and α_k increases monotonically as $k \to \infty$. We then have the asymptotic expansion*

$$\int_0^\infty e^{-xt}m(t)dt \sim \sum_{k=0}^\infty p_k\Gamma(\alpha_k+1)x^{-1-\alpha_k}$$

as $x \to \infty$ where Γ denotes the Gamma-function.

of Proposition 5.1. We first note that $A = f(-\frac{1}{2}\Delta) - \overline{m}(0,\infty)$ is a classical pseudodifferential operator by Theorem 1 of [46]. This applies since f has symbol-like properties by Proposition 1 (v) of [25] in the sense that f is smooth on $[0,\infty)$ and for every $l \in \mathbb{N}$ there is a constant $C_l \geq 0$ such that $|f^{(l)}(\lambda)| \leq C_l \lambda^{\alpha/2-l}$ as $\lambda \to \infty$.

It remains to prove the validity of the asymptotic expansion (13) in the sense of Definition 3.1. Set $\overline{m}(t) = m(t) - p_0 t^{-1-\alpha/2}$. It belongs to $L^1(0,\infty)$ and has the asymptotic expansion

$$\overline{m}(t) \sim t^{-1-\alpha/2}\left(p_1 t + p_2 t^2 + \cdots\right)$$

as $t \to 0^+$. We decompose $f(\lambda) = \int_0^\infty (1-e^{-\lambda t}) m(t) dt$ as

$$\begin{aligned} f(\lambda) &= \int_0^\infty (1-e^{-\lambda t})\overline{m}(t)dt + p_0 \int_0^\infty (1-e^{-\lambda t}) t^{-1-\alpha/2} dt \\ &= \overline{m}(0,\infty) - \int_0^\infty e^{-\lambda t}\overline{m}(t)dt + p_0 \Gamma(-\alpha/2)\lambda^{\alpha/2} \end{aligned} \tag{15}$$

Applying Lemma 5.2 to the integral in (15) yields the asymptotic expansion

$$f(\lambda) - \overline{m}(0,\infty) \sim a_0\lambda^{\alpha/2} + a_1\lambda^{\alpha/2-1} + a_2\lambda^{\alpha/2-2} + \cdots$$

in the sense of Definition 2.4.

Now let $\lambda_k \geq 0$ be the eigenvalues of $-\Delta$ in increasing order. By Definition 2.4 we have for any $N \in \mathbb{N}$ that

$$\lim_{k\to\infty} \lambda_k^{\alpha/2+N}\left(f(\lambda_k/2) - \overline{m}(0,\infty) - \sum_{j=0}^{N} a_j \lambda_k^{\alpha/2-j}\right) = 0,$$

so that the left hand side is bounded as a function of λ. This means in terms of operators that

$$\Delta_1^{\alpha/2+N}\left(f(\tfrac{1}{2}\Delta) - \overline{m}(0,\infty) - \sum_{j=0}^{N} a_j \Delta_1^{\alpha/2-j}\right)$$

has bounded eigenvalues. Since it is a classical pseudodifferential operator, it must have order 0 or less. This means

$$f(-\tfrac{1}{2}\Delta) - \overline{m}(0,\infty) - \sum_{j=0}^{N} a_j \Delta_1^{\alpha/2-j} \in \Psi DO_{cl}^{-\alpha/2-N}(M)$$

so that $f(-\frac{1}{2}\Delta) - \overline{m}(0,\infty) \sim \sum_{j=0}^\infty a_j \Delta_1^{\alpha/2-j}$ in the sense of Definition 3.1.

Finally, Hypothesis 3.3 is also satisfied: the principal symbol of $f(-\frac{1}{2}\Delta) - \overline{m}(0,\infty)$ is given by $a_0||\xi||_g^{2\mu}$ which only takes positive values for $\xi \neq 0$ and the spectrum of the operator is contained in the interval $(0,\infty)$. □

It is now easy to prove the key theorems. We set $A = f\left(\frac{1}{2}\Delta\right) - \overline{m}(0,\infty)$ as in Proposition 5.1 with coefficients a_k from (14). Theorem 2.9 follows from Theorem 4.1 and Theorem 2.10 is a consequence of Theorem 4.4 with explicit computations from Example 4.5.

Finally, Corollary 2.11 follows from Corollary 4.6 which gives the eigenvalue asymptotics of the shifted operator $f(-\frac{1}{2}\Delta) - \overline{m}(0,\infty)$ in terms of $\lambda^{n/\alpha}$. This translates to eigenvalue asymptotics of $f(-\frac{1}{2}\Delta)$ in terms of $(\lambda + \overline{m}(0,\infty))^{n/\alpha}$. However, $(\lambda + \overline{m}(0,\infty))^{n/\alpha}/\lambda^{n/\alpha} \to 1$ as $\lambda \to \infty$ whence the result.

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