ON THE QUENCHING OF A NONLOCAL PARABOLIC PROBLEM ARISING IN ELECTROSTATIC MEMS CONTROL

NIKOS I. KAVALLARIS, ANDREW A. LACEY, AND CHRISTOS V. NIKOLOPOULOS

ABSTRACT. We consider a nonlocal parabolic model for a micro-electro-mechanical system. Specifically, for a radially symmetric problem with monotonic initial data, it is shown that the solution quenches, so that touchdown occurs in the device, in a situation where there is no steady state. It is also shown that quenching occurs at a single point and a bound on the approach to touchdown is obtained. Numerical simulations illustrating the results are given.

1. INTRODUCTION

The main purpose of the current work is to investigate a singular mathematical behaviour, called *quenching*, of the solutions of the following non-local parabolic problem:

$$u_t - \Delta u = \frac{\lambda}{(1-u)^2 \left(1 + \alpha \int_\Omega \frac{1}{1-u}\, \mathrm{d}x\right)^2} \quad \text{in} \quad Q_T := \Omega \times (0,T), \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where λ, α are positive constants, $T > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and u_0 is a continuous function in $\overline{\Omega}$ such that $0 \le u_0 < 1$.

The motivation for studying problem (1.1)-(1.3) is that it arises as a mathematical model which describes operation of some electrostatic actuated micro-electro-mechanical systems (MEMS). The term "MEMS" more precisely refers to precision devices which combine mechanical processes with electrical circuits. In particular, electrostatic actuation is a popular application of MEMS. MEMS devices range in size from millimetres down to microns, and involve precision mechanical components that can be constructed using semiconductor manufacturing technologies. Various electrostatic actuated MEMS have been developed and used in a wide variety of devices applied as sensors and have fluid-mechanical, optical, radio frequency (RF), data-storage, and biotechnology applications. Examples of microdevices of this kind include microphones, temperature sensors, RF switches, resonators, accelerometers, micromirrors, micropumps, microvalves, data-storage devices etc., [3, 20, 23].

The principal part of such a electrostatic actuated MEMS device usually consists of an elastic plate suspended above a rigid ground plate. Typically the elastic plate (or membrane) is held fixed at two ends while the other two edges remain free to move. An alternative configuration could entail the plate or membrane being held fixed around its entire edge. When a potential difference V_d is applied between the membrane and the plate, the membrane deflects towards the ground plate. Under the realistic assumption that the width of the gap, between the membrane and the bottom plate, is small compared

Date: January 28, 2016.

1991 *Mathematics Subject Classification.* Primary 35K55, 35J60; Secondary 74H35, 74G55, 74K15.

Key words and phrases. Electrostatic MEMS, touchdown, quenching.

to the device length, and when the configuration of the two parallel plates is connected in series with a fixed voltage source and a fixed capacitor, then the deformation of the elastic membrane u is described by the dimensionless equation (1.1), see [19, 20]. In (1.1)

$$\lambda = \frac{V_d^2 L^2 \varepsilon_0}{2\mathcal{T} l^2},$$

$\mathcal{T}$ is the tension in the membrane, L the characteristic length (diameter) of the domain Ω, l the characteristic width of the gap between the membrane and the fixed ground plate (electrode), and ε_0 the permittivity of free space. The integral in (1.1) arises from the fact that the device is embedded in an electrical circuit with a capacitor of fixed capacitance. The parameter α denotes the ratio of this fixed capacitance to a reference capacitance of the device. Without loss of generality, we may assume that $\alpha = 1$. (The limiting case $\alpha = 0$ corresponds to the configuration where there is no capacitor in the circuit.) If the edges of the membrane are kept fixed then Dirichlet boundary conditions of the form (1.2) are imposed. It is usually supposed that the elastic membrane is initially at rest, so that $u(x, 0) \equiv 0$. However, in this work, we consider more general non-negative initial conditions $u(x, 0) = u_0(x) \geq 0$. For a more detailed derivation of (1.1) see [19, 20].

From many experiments it is clear that the applied voltage V_d controls the operation of the MEMS device. It is observed that when V_d exceeds a critical threshold V_{cr}, called the *pull-in voltage*, then the phenomenon of *touch-down* (or *pull-in instability* as it is also known in MEMS literature) occurs when the elastic membrane touches the rigid ground plate. For the mathematical problem (1.1)-(1.3), this means that there is some critical value λ_{cr} of the parameter λ above which singular behaviour should be anticipated. Focusing on the nonlinear term of problem (1.1), one can notice that such singular behaviour is possible only when u takes the value 1, a phenomenon known in the mathematical literature as *quenching*, see also Section 3. From the point of view of applications it is important to determine whether quenching occurs and, if it does, to clarify when, how and where it might happen.

When $\alpha = 0$ we obtain the following local (standard) parabolic problem,

$$\begin{aligned} & u_t - \Delta u = \frac{\lambda}{(1-u)^2} \quad \text{in} \quad Q_T\,, \\ & u = 0 \quad \text{on} \quad \partial\Omega \times (0, T)\,, \\ & u(x, 0) = u_0(x) \quad \text{for} \quad x \in \Omega\,, \end{aligned}$$

whose quenching behaviour has been extensively studied in the papers [6, 7, 11, 10, 17].

The quenching behaviour of the solutions of (1.1)-(1.3) has been also studied in [8, 9, 13] but questions regarding :
(i) occurrence of quenching for $\lambda > \lambda^*$ where λ^* is defined by (2.2);
(ii) determination of the quenching rate;
(iii) establishment of the form of the quenching set;
were left open. These questions are addressed in the current work for radially symmetric problems with the extra assumption that the initial data decreases with distance from the centre of the domain.

It is worth mentioning that the problem (1.1)-(1.3) shares some common features with a non-local problem which exhibits blow-up and models control of mass, and which was investigated in [12, 21]. However, the methods we use in this paper to examine finite-time quenching are rather different from the those of [12, 21].

The structure of the paper is as follows. In Section 2 a brief study of corresponding steady-state problem is presented; this will be used for proving the occurrence of finite-time quenching in Section 3. Section 4 is devoted to determination of the form of the quenching set and a bound on the quenching rate. Finally Section 5 presents some numerical results confirming some of the analytical results obtained in the preceding sections.

2. Steady-State Problem

In this section a brief study of the steady-state problem corresponding to (1.1)-(1.3) is provided. This problem has the form

$$\Delta w + \frac{\lambda}{(1-w)^2\left(1+\int_\Omega \frac{\mathrm{d}x}{1-w}\right)^2} = 0, \quad x \in \Omega, \qquad w = 0, \quad x \in \partial\Omega, \tag{2.1}$$

where we always have $0 \leq w < 1$ in $\overline{\Omega}$ for a (classical) solution of (2.1).

Let

$$\lambda^* := \sup\{\lambda > 0 : \text{ problem (2.1) admits a classical solution}\}, \tag{2.2}$$

then $\lambda^* < \infty$ for any dimension $N \geq 1$, see [8, 9]. For more on the structure of the solution set of (2.1) see [9].

For the purposes of the current work we will need the notion of a weak solution of problem (2.1). In particular we define the following form of weak solution for (2.1).

Definition 2.1. *A function $w \in H_0^1(\Omega)$ is called a* **weak finite-energy solution** *of* (2.1) *if there exists a sequence $\{w_j\}_{j=1}^{\infty} \in C^2(\Omega) \cap C_0(\Omega)$ satisfying as $j \to \infty$*

$$w_j \rightharpoonup w \quad \text{weakly in} \quad H_0^1(\Omega)\,, \tag{2.3}$$

$$w_j \to w \quad a.e., \tag{2.4}$$

$$\frac{1}{(1-w_j)^2} \to \frac{1}{(1-w)^2} \quad in \quad L^1(\Omega)\,, \tag{2.5}$$

$$\frac{1}{(1-w_j)} \to \frac{1}{(1-w)} \quad in \quad L^1(\Omega) \tag{2.6}$$

and

$$\Delta w_j + \frac{\lambda}{(1-w_j)^2\left(1+\int_\Omega \frac{\mathrm{d}x}{1-w_j}\right)^2} \to 0 \quad in \quad L^2(\Omega). \tag{2.7}$$

It follows that any weak finite-energy solution of (2.1) also satisfies

$$-\int_\Omega \nabla\phi \cdot \nabla w \,\mathrm{d}x + \lambda \frac{\int_\Omega \frac{\phi}{(1-w)^2}\,\mathrm{d}x}{\left(1+\int_\Omega \frac{\mathrm{d}x}{1-w}\right)^2} = 0 \quad \text{for all} \quad \phi \in H_0^1(\Omega),$$

i.e. it is a weak $H_0^1(\Omega)-$solution of (2.1) as well, see also [24].

Set

$$\widehat{\lambda} := \sup\left\{\lambda > 0 : \text{ problem (2.1) admits a weak finite-energy solution}\right\}.$$

We now restrict our discussion to radially symmetric problems, so that we may take $\Omega = B_1 = B_1(0) = \{x \in \mathbb{R}^N : ||x||_2 < 1\}$ with solutions which are decreasing in $r = ||x||_2$. The relation between λ^* and $\widehat{\lambda}$ is then provided by the following:

Proposition 2.2. *For radially symmetric problems, with radially decreasing solutions, the suprema of the spectra of the classical and weak problems are identical:* $\lambda^* = \widehat{\lambda}$.

Proof. Since any classical solution of (2.1) is also a weak finite-energy solution, $\lambda^* \leq \widehat{\lambda}$.

On the other hand, we can take λ_1 arbitrarily close to $\widehat{\lambda}$ so that there is a weak finite-energy solution w_1 for $\lambda = \lambda_1$. Since w_1 is decreasing with $0 \leq w \leq 1$, either $w < 1$ for $0 < r \leq 1$ or there is some $s > 0$ such that $w = 1$ for $0 \leq r < s$. In the latter case $\int_\Omega (1-w)^{-1}\,\mathrm{d}x$ becomes infinite so that w is then a weak finite-energy solution of $\Delta w = 0$ satisfying $0 \leq w \leq 1$, as well as the boundary condition $w = 0$, giving $w \equiv 0$. We must then have $w_1(r) < 1$ for $r > 0$ and it follows that w_1 is regular for $r > 0$.

For $N = 1$, simple integration now gives that the solution is classical. For $N \geq 2$, following now [9] (see also [14]), the (classical) problem can be solved in $r > 0$ to find that there is precisely one limiting value of λ, say λ_*, for which $w(0) = 1$ and w is then a weak finite-energy solution but not classical. Depending upon the value of N, $\lambda_* < \lambda^*$ or $\lambda_* = \lambda^*$. In either case, $\lambda^* = \widehat{\lambda}$. □

3. Finite-Time Quenching

Local-in-time existence of a solution to problem (1.1)-(1.3) is established in Corollary 2.4 of [9] by constructing a proper lower-upper pair or solutions. Moreover the solution u exists as long as $u < 1$ and it ceases to exist once u reaches 1. In particular we have:

Definition 3.1. *The solution $u(x,t)$ of problem* (1.1)-(1.3) *quenches at some point $x^* \in \Omega$ in finite time $0 < T_q < \infty$ if there exist sequences $\{x_n\}_{n=1}^\infty \in \Omega$ and $\{t_n\}_{n=1}^\infty \in (0,\infty)$ with $x_n \to x^*$ and $t_n \to T_q$ as $n \to \infty$ such that $u(x_n,t_n) \to 1-$ as $n \to \infty$. In the case where $T_q = \infty$ we say that $u(x,t)$ quenches in infinite time at x^*.*

The set

$$\mathcal{Q} = \Big\{ x^* \in \overline{\Omega} \;\Big|\; \text{there exists a sequence } (x_k,t_k)_{k\in\mathbb{N}} \subset \Omega \times (0,T_q) \text{ such that } x_k \to x^*, t_k \to T_q \text{ and } u(x_k,t_k) \to 1 \text{ as } k \to \infty \Big\}$$

is called the quenching set.

Some finite-time quenching results for large values of the parameter λ have been proved in [8, 13], while the authors in [9] proved a quenching result for initial data u_0 quite close to 1.

In the current section we are working towards the improvement of the preceding quenching results under some circumstances. Before we proceed with the proof of our quenching results we will present some auxiliary lemmata.

First of all we point out that problem (1.1)-(1.3) admits an energy functional of the form

$$E[u](t) \equiv E(t) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{\lambda}{\left(1 + \int_\Omega (1-u)^{-1}\,\mathrm{d}x\right)} > 0, \tag{3.1}$$

which decreases with respect to time for a solution of problem (1.1)-(1.3). More precisely,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_\Omega u_t^2(x,t)\,\mathrm{d}x < 0 \quad \text{for} \quad t > 0\,, \tag{3.2}$$

or, equivalently,

$$E(t)=\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x+\frac{\lambda}{\left(1+\int_\Omega (1-u)^{-1}\,\mathrm{d}x\right)}\le E(0)=E_0<\infty. \tag{3.3}$$

A key estimate for proving our quenching result is given by the following:

Lemma 3.2. *Let u be a global-in-time solution of problem* (1.1)-(1.3). *Then there is a sequence $\{t_j\}_{j=1}^\infty \uparrow \infty$ as $j\to\infty$ such that*

$$\lambda\int_\Omega u_j(1-u_j)^{-2}\,\mathrm{d}x\le C_1H^2(u_j), \tag{3.4}$$

for some positive constant C_1, where $u_j=u(\cdot,t_j)$ and

$$H(u_j):=1+\int_\Omega (1-u_j)^{-1}\,\mathrm{d}x>0. \tag{3.5}$$

Proof. Suppose that the problem (1.1)-(1.3) has a global-in-time solution $u(x,t)=u(x,t;\lambda)$. Then, multiplying equation (1.1) by u and integrating over Ω, we derive

$$\begin{aligned}\int_\Omega u\,u_t\,\mathrm{d}x &= \int_\Omega u\left[\Delta u+\frac{\lambda(1-u)^{-2}}{\left(1+\int_\Omega (1-u)^{-1}\,\mathrm{d}x\right)^2}\right]\mathrm{d}x\\ &= -\int_\Omega |\nabla u|^2\,\mathrm{d}x+\frac{\lambda\int_\Omega u(1-u)^{-2}\,\mathrm{d}x}{\left(1+\int_\Omega (1-u)^{-1}\,\mathrm{d}x\right)^2}\\ &= -2E(t)+\frac{2\lambda}{1+\int_\Omega (1-u)^{-1}\,\mathrm{d}x}+\frac{\lambda\int_\Omega u(1-u)^{-2}\,\mathrm{d}x}{\left(1+\int_\Omega (1-u)^{-1}\,\mathrm{d}x\right)^2},\end{aligned} \tag{3.6}$$

using also integration by parts and relation (3.1).

By virtue of Hölder's inequality and (3.3), (3.6) implies

$$\begin{aligned}\lambda\int_\Omega u(1-u)^{-2}\,\mathrm{d}x &= 2E(t)H^2(u)-2\lambda H(u)+H^2(u)\int_\Omega u\,u_t\,\mathrm{d}x\\ &\le 2E_0H^2(u)+||u(\cdot,t)||_2\,||u_t(\cdot,t)||_2\,H^2(u)\\ &\le 2E_0H^2(u)+|\Omega|^{1/2}\,||u_t(\cdot,t)||_2\,H^2(u).\end{aligned} \tag{3.7}$$

On the other hand, the energy dissipation formula (3.2) reads

$$0\le\int_\tau^t\int_\Omega u_t^2(x,s)\,\mathrm{d}x\,\mathrm{d}s=E(\tau)-E(t)$$

and thus from (3.3) we deduce that

$$\int_\tau^\infty\int_\Omega u_t^2(x,s)\,\mathrm{d}x\,\mathrm{d}s\le C<\infty, \tag{3.8}$$

where the constant C is independent of τ.

Now (3.8) yields the existence of a sequence $\{t_j\}_{j=1}^\infty\uparrow\infty$ such that

$$||u_t(\cdot,t_j)||_2^2=\int_\Omega u_t^2(x,t_j)\,\mathrm{d}x\to 0 \quad \text{as} \quad t_j\to\infty, \tag{3.9}$$

and thus by virtue of (3.7)

$$\lambda \int_\Omega u_j(1-u_j)^{-2}\,\mathrm{d}x \le C_1 H^2(u_j), \tag{3.10}$$

for some $C_1 > 0$. □

The next step is to provide another key estimate for $H(u)$ which will allow us not only to prove finite-time quenching but also to characterize the form of the quenching set. However, such an estimate according to our method can be only obtained for the radial symmetric case, i.e. when $\Omega = B_1$. Then problem (1.1)-(1.3) in N dimensions is written as

$$u_t - \Delta_r u = F(r,t), \quad (r,t) \in (0,1)\times(0,T), \tag{3.11}$$

$$u_r(0,t) = u(1,t) = 0, \quad t \in (0,T), \tag{3.12}$$

$$0 \le u(r,0) = u_0(r) < 1, \quad 0 < r < 1, \tag{3.13}$$

where $\Delta_r u := u_{rr} + (N-1)r^{-1}u_r$ for $N \ge 1$ and

$$F(r,t) = \lambda k(t)(1-u(r,t))^{-2}, \tag{3.14}$$

for

$$k(t) = \left(1 + N\omega_N \int_0^1 r^{N-1}\,(1-u(r,t))^{-1}\,\mathrm{d}r\right)^{-2},$$

where $\omega_N = |B_1| = \pi^{N/2}/\Gamma(N/2)$ stands for the volume of the N-dimensional unit sphere $B_1(0)$ in $\mathbb{R}^N$ and Γ is the gamma function.

Condition $u_r(0,t) = 0$, for $N \ge 1$ is imposed to guarantee the regularity of the solution u. If we consider radial decreasing initial data $u_0(r)$, i.e., $u_0'(r) \le 0$, then it is a standard result that the monotonicity property is inherited by u so that $u_r(r,t) \le 0$ for $r > 0$ and $t > 0$.

For the sake of simplicity we obtain the desired estimate for $H(1-v)$ where v is defined as $v := 1-u$. Then $v \to 0+$ if $u \to 1-$. Moreover v satisfies

$$v_t - v_{rr} - (N-1)r^{-1}v_r = -fv^{-2}, \quad (r,t) \in (0,1)\times(0,T), \tag{3.15}$$

$$v_r(0,t) = 0, \; v(1,t) = 1, \quad t \in (0,T), \tag{3.16}$$

$$0 < v(r,0) = v_0(r) \le 1, \quad 0 < r < 1, \tag{3.17}$$

where

$$f = f(t) = \frac{\lambda}{(1 + N\omega_N \int_0^1 r^{N-1} v^{-1}\,\mathrm{d}r)^2}. \tag{3.18}$$

Then we have the following:

Lemma 3.3. *Consider symmetric and radial increasing initial data $v_0(r)$. Then for any $k > 2/3$ there exists a positive constant $C(k)$ such that*

$$1 - u(r,t) \ge C(k) r^k \quad \textit{for} \quad (r,t) \in (0,1)\times(0,T_{max}) \tag{3.19}$$

where T_{max} is the maximum existence time of solution u.

Furthermore, there exists C_2 uniform in λ and independent of time t such that

$$H(u) = H(1-v) \le C_2 \quad \textit{for any} \quad 0 < t < T_{max}. \tag{3.20}$$

Proof. Fixing some a, $1 < a < 2$, there are some $t_1 > 0$ and $\epsilon_1 > 0$ such that
$$v_r > \epsilon_1 r v^{-a} \ \text{ at } t = t_1 \text{ for } 0 < r < 1. \tag{3.21}$$
We define
$$z = r^{N-1} v_r \tag{3.22}$$
and it is then easy to check, [5], by differentiating (3.15), that
$$z_t - z_{rr} + (N-1)r^{-1} z_r = 2r^{N-1} f v^{-3} v_r\,. \tag{3.23}$$
We define
$$J = z - \epsilon r^N v^{-a} \tag{3.24}$$
where
$$0 < \epsilon < \epsilon_1\,.$$
Then
$$J_t = z_t + a\epsilon r^N v^{-a-1} v_t\,, \tag{3.25}$$
$$J_r = z_r + a\epsilon r^N v^{-a-1} v_r - N\epsilon r^{N-1} v^{-a}, \tag{3.26}$$
and
$$J_{rr} = z_{rr} + a\epsilon r^N v^{-a-1} v_{rr} + 2Na\epsilon r^{N-1} v^{-a-1} v_r - a(a+1)\epsilon r^N v^{-a-2} v_r^2 - N(N-1)\epsilon r^{N-2} v^{-a}. \tag{3.27}$$
We define a function $G(\epsilon)$ by
$$G(\epsilon) = \frac{\epsilon^{\frac{2}{a+1}}}{(\epsilon^{\frac{1}{a+1}} + \frac{N\omega_N}{Na+N-2}(a+1)^{\frac{a}{a+1}} 2^{\frac{1}{a+1}})^2}. \tag{3.28}$$
Our choice of ϵ is then, more precisely, given by
$$0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}\,, \tag{3.29}$$
where $\epsilon_2 > 0$ is chosen to satisfy
$$\epsilon_2 < \sup\left\{\epsilon : \epsilon \le \min\left\{\frac{1}{N}, \left(\frac{2-a}{2a}\right)\right\} \lambda G(\epsilon)\right\}\,; \tag{3.30}$$
a small ϵ_2 satisfying (3.30) can be found since $G(\epsilon)$ is of order $\epsilon^{\frac{2}{a+1}} \gg \epsilon$ for ϵ small (recall that $a > 1$). Then
$$J > 0 \ \text{ for } 0 < r \le 1 \text{ at } t = t_1. \tag{3.31}$$
As long as $J > 0$,
$$z > \epsilon r^N v^{-a} \quad \Rightarrow \quad v_r > \epsilon r v^{-a} \quad \Rightarrow \quad v > \left(\frac{(a+1)\epsilon}{2}\right)^{\frac{1}{a+1}} r^{\frac{2}{a+1}} \tag{3.32}$$
which will lead to (3.19).

Then
$$\int_0^1 r^{N-1} v^{-1}\, \mathrm{d}r < \left(\frac{2}{(a+1)\epsilon}\right)^{\frac{1}{a+1}} \int_0^1 r^{\frac{Na+N-2}{a+1}-1}\, \mathrm{d}r = \left(\frac{2}{(a+1)\epsilon}\right)^{\frac{1}{a+1}} \left(\frac{a+1}{Na+N-2}\right)$$
so
$$f(t) = \frac{\lambda}{(1 + N\omega_N \int_0^1 r^{N-1} v^{-1}\, \mathrm{d}r)^2} > \lambda G(\epsilon)\,. \tag{3.33}$$
In particular, $f(t) > \lambda G(\epsilon)$ in a neighbourhood of $t = t_1$.

We suppose for a contradiction that

$$\text{there is some } t_2 \in (t_1, T_{max}) \text{ such that } f(t_2) = \lambda G(\epsilon) \text{ with } f(t) > \lambda G(\epsilon) \text{ for } t_1 \le t < t_2. \tag{3.34}$$

Now

$$J = 0 \ \text{ on } r = 0. \tag{3.35}$$

On the boundary $r = 1$ we have

$$J = v_r - \epsilon \ \text{ and then}$$

$$J_r = v_{rr} + (N-1)v_r + a\epsilon v_r - N\epsilon = (f - (N-1)v_r) + (N-1)v_r + a\epsilon v_r - N\epsilon = f + a\epsilon v_r - N\epsilon$$

$$\text{so both} \quad J_r - a\epsilon J = f + a\epsilon^2 - N\epsilon > f - N\epsilon \ \text{ on } r = 1 \tag{3.36}$$

$$\text{and, since } v_r \ge 0 \text{ on } r = 1, \quad J_r \ge f - N\epsilon \ \text{ on } r = 1. \tag{3.37}$$

Provided that

$$\epsilon < f/N\,, \tag{3.38}$$

either (3.36) or (3.37) gives a positive boundary condition on $r = 1$.

Now

$$\begin{aligned} J_t - J_{rr} + (N-1)r^{-1}J_r &= 2r^{N-1}fv^{-3}v_r + (a\epsilon r^N v^{-a-1})(2(N-1)r^{-1}v_r - fv^{-2}) - 2Na\epsilon r^{N-1}v^{-a-1}v_r \\ &\quad + a(a+1)\epsilon r^N v^{-a-2}v_r^2 + N(N-1)\epsilon r^{N-2}v^{-a} - N(N-1)\epsilon r^{N-2}v^{-a} \\ &> (2r^{N-1}fv^{-3} + 2(N-1)a\epsilon r^{N-1}v^{-a-1} - 2Na\epsilon r^{N-1}v^{-a-1})v_r - a\epsilon f r^N v^{-a-3} \\ &= 2(fv^{-3} - a\epsilon v^{-a-1})z - a\epsilon f r^N v^{-a-3} \\ &= 2(fv^{-3} - a\epsilon v^{-a-1})J + 2(fv^{-3} - a\epsilon v^{-a-1})\epsilon r^N v^{-a} - a\epsilon f r^N v^{-a-3} \\ &= 2(fv^{-3} - a\epsilon v^{-a-1})J + \epsilon(2-a)f r^N v^{-a-3} - 2a\epsilon^2 r^N v^{-2a-1}\,. \end{aligned}$$

Thus

$$J_t - J_{rr} + (N-1)r^{-1}J_r > 2(fv^{-3} - a\epsilon v^{-a-1})J\,, \tag{3.39}$$

as long as

$$\epsilon < \frac{(2-a)}{2a} f\,. \tag{3.40}$$

Following the standard arguments for the maximum principle, we now show that $J > 0$ for $0 < r \le 1$, $t_1 \le t \le t_2$.

In $0 < r \le 1$, $t_1 \le t \le t_2$, because $v > 0$, the coefficient of J in (3.39) is bounded. We can then define a new variable $\tilde{J} = \mathrm{e}^{-D_1 t}J$ which then satisfies boundary condition (3.35), boundary inequality (3.36) and

$$\tilde{J}_t - \tilde{J}_{rr} + (N-1)r^{-1}\tilde{J}_r > -D_2\tilde{J}\,, \tag{3.41}$$

where D_1 and D_2 are positive constants. Should $\tilde{J}$ be non-positive somewhere (with $r > 0$), it must take a non-positive minimum at some (r_3, t_3) with $0 < r_3 \le 1$ and $t_1 < t_3 \le t_2$.

For $r_3 = 1$, (3.37) gives $\tilde{J}_r > 0$ on $r = 1$, leading to a contradiction, so the supposed minimum must have $0 < r_3 < 1$, where $\tilde{J}_t \le 0$, $\tilde{J}_r = 0$ and $\tilde{J}_{rr} \ge 0$. With $\tilde{J} \le 0$ and $D_2 > 0$, (3.41) gives another contradiction. Hence both $\tilde{J}$ and J remain positive in $r > 0$ for $t_1 \le t \le t_2$.

This now gives that (3.33) holds at $t = t_2$, contradicting the assumption (3.34). Thus, as long as the solution exists, $f(t) > \lambda G(\epsilon)$ for $t \ge t_1$.

It follows that $J > 0$, (3.19) holds and

$$\int_0^1 r^{N-1} v^{-1}\,\mathrm{d}r < \frac{1}{Na+N-2}(a+1)^{\frac{a}{a+1}}\left(\frac{2}{\epsilon}\right)^{\frac{1}{a+1}} \tag{3.42}$$

for all $t \geq t_1$, if (3.15)-(3.17) have a global solution, and up to and including the quenching time, if the solution quenches. Thanks to the definition of $H(u)$ (3.42) implies the desired estimate. □

Remark 3.4. *An estimate similar to* (3.19) *has been also obtained in* [8] *but only for the one-dimensional case. Furthermore, here we also prove that the exponent* 2/3 *is optimal for the validity of* (3.19)*, see Section 4.*

3.1. **Quenching for $\lambda > \lambda^*$.** We now can combine Lemma 3.2 and Lemma 3.3 to derive the following quenching result:

Theorem 3.5. *Consider symmetric and radial decreasing initial data $u_0(r)$. Then for any $\lambda > \lambda^*$ the solution of problem* (3.11)-(3.13) *quenches in finite time $T_q < \infty$.*

Proof. Let $\lambda > \lambda^*$ and assume that problem (3.11)-(3.13) has a global-in-time solution. Then (3.4) in conjunction with (3.20) yields

$$\lambda N\omega_N \int_0^1 r^{N-1} u_j (1-u_j)^{-2}\,\mathrm{d}r \leq C_3\,, \quad \text{for any} \quad t > 0, \tag{3.43}$$

where the constant C_3 is independent of j.

From this and (3.20) we have

$$\begin{aligned} N\omega_N \int_0^1 \frac{r^{N-1}\,\mathrm{d}r}{(1-u_j)^2} &= N\omega_N \int_0^1 \frac{r^{N-1}\,\mathrm{d}r}{(1-u_j)} + N\omega_N \int_0^1 \frac{r^{N-1}u_j\,\mathrm{d}r}{(1-u_j)^2} \\ &\leq (C_2-1) + C_3/\lambda := C_4 \end{aligned} \tag{3.44}$$

where C_4 is independent of j.

From the energy dissipation formula (3.3) we also have

$$||\nabla u_j||^2_{L^2(B_1)} \leq C_5 < \infty, \tag{3.45}$$

with constant C_5 being independent of j as well. Passing to a subsequence, if necessary, relation (3.45) implies the existence of a function w such that

$$u_j \rightharpoonup w \quad \text{in} \quad H_0^1(B_1), \tag{3.46}$$

$$u_j \to w \quad \text{a.e.} \quad \text{in} \quad B_1. \tag{3.47}$$

For $N \geq 2$ by virtue of (3.19) we directly derive that $1/(1-u_j)^2$ is uniformly integrable and since $1/(1-u_j)^2 \to 1/(1-w)^2$, a.e. in B_1, due to (3.47), we deduce

$$\frac{1}{(1-u_j)^2} \to \frac{1}{(1-w)^2} \quad \text{as} \quad j \to \infty \quad \text{in} \quad L^1(B_1), \tag{3.48}$$

applying the dominated convergence theorem. Similarly we also derive

$$H(u_j) \to H(w) \quad \text{as} \quad j \to \infty \quad \text{in} \quad L^1(B_1). \tag{3.49}$$

Note that the weak formulation of (3.11) along the sequence $\{t_j\}$ is given by

$$\int_{B_1} \frac{\partial u_j}{\partial t}\,\phi\,\mathrm{d}x = -\int_{B_1} \nabla u_j \cdot \nabla\phi\,\mathrm{d}x + \lambda H^{-1}(u_j)\int_{B_1} \phi(1-u_j)^{-2}\,\mathrm{d}x \quad \text{as} \quad j \to \infty \tag{3.50}$$

for any $\phi \in H_0^1(B_1)$.

Passing to the limit as $j \to \infty$ in (3.50), and in conjunction with (3.9), (3.46), (3.48) and (3.49), we derive

$$\Delta u_j + \frac{\lambda}{(1-u_j)^2\Big(1+\int_\Omega \frac{\mathrm{d}x}{1-u_j}\Big)^2} \to 0 \quad \text{in} \quad L^2(\Omega),$$

which implies that w is an weak finite-energy solution of problem (2.1) corresponding to $\lambda > \lambda^*$, contradicting the result of Proposition 2.2.

On the other hand, for $N = 1$ by using (3.9), (3.11), (3.14), (3.44) and (3.45) we deduce that $(u_j)_x$ is bounded in $W^{1,1}(-1,1)$ and thus, by virtue of Sobolev's inequality,

$$(u_j)_x \quad \text{is bounded in} \quad L^\infty(-1,1) \tag{3.51}$$

Furthermore

$$\big[(1-u_j)^{-1}\big] \quad \text{is bounded in} \quad W^{1,1}(-1,1)$$

since

$$\big[(1-u_j)^{-1}\big]_x = \frac{(u_j)_x}{(1-u_j)^2} \quad \text{is bounded in} \quad L^1(-1,1),$$

due to (3.44) and (3.51), and

$$(1-u_j)^{-1} \quad \text{is bounded in} \quad L^1(-1,1),$$

by virtue of (3.20).

Therefore Sobolev's inequality guarantees that

$$\big[(1-u_j)^{-1}\big] \quad \text{is bounded in} \quad L^\infty(-1,1)$$

and thus

$$\big[(1-u_j)^{-2}\big] \quad \text{is bounded in} \quad L^\infty(-1,1).$$

Now by virtue of (3.47) we derive that

$$(1-u_j)^{-1} \to (1-w)^{-1} \quad \text{and} \quad (1-u_j)^{-2} \to (1-w)^{-2} \qquad \text{as} \quad j \to \infty \quad \text{in} \quad L^\infty(-1,1).$$

Consequently,

$$\Delta w + \frac{\lambda}{(1-w)^2\Big(1+\int_\Omega \frac{\mathrm{d}x}{1-w}\Big)^2} = 0 \quad \text{in} \quad L^2(\Omega),$$

where the non-local term is bounded and hence elliptic regularity arguments entail that w classical steady solution, again contradicting Proposition 2.2. □

Remark 3.6. *Theorem 3.5 improves the results of Theorems 4.1 in* [8] *and Theorem 5.2, 5.3 in* [13]. *Indeed, the earlier results have provided finite-time quenching only for large values of the parameter λ, without giving a threshold for λ above which quenching occurs.*

3.2. **Quenching for large initial data.** To prove quenching for big initial data, i.e. for $0 < u_0(x) < 1$ close to 1, we employ the widely used classical technique of Kaplan, [15]. The estimate is provided by Lemma 3.3, permitting us to treat the non-local problem (3.11)-(3.13) as a local one. In particular we have:

Theorem 3.7. *For any $\lambda > 0$ there exist symmetric initial data u_0 satisfying the assumptions of Theorem 3.5 that are close to 1 such that the solution u of (3.11)-(3.13) quenches in finite time $T_q < \infty$.*

Proof. Set $\lambda_1 = \lambda_1(B_1) > 0$ the principal eigenvalue of the following problem

$$-\Delta\phi = \lambda\phi, \quad x \in B_1, \quad \phi(x) = 0, \quad x \in \partial B_1\,,$$

with associated positive eigenfunction $\phi_1(x)$ normalized so that

$$\int_{B_1} \phi_1(x)\,\mathrm{d}x = 1.$$

Let us assume that problem (3.11)-(3.13) has a global-in-time solution, i.e. $T_{max} = \infty$ so that $0 < u(x,t) < 1$ for any $(x,t) \in B_1 \times (0,+\infty)$.

Multiplying (3.11) by ϕ_1, integrating over B_1 and using Green's second identity, we obtain, via Lemma 3.3,

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -\lambda_1 A(t) + \frac{\lambda\int_{B_1}\phi_1(1-u)^{-2}\,\mathrm{d}x}{H^2(u)} \\ &\geq -\lambda_1 A(t) + \frac{\lambda\int_{B_1}\phi_1(1-u)^{-2}\,\mathrm{d}x}{C_2^2} \end{aligned} \tag{3.52}$$

where $A(t) = \int_{B_1} u\,\phi_1\,\mathrm{d}x$. Applying Jensen's inequality to (3.52),

$$\frac{\mathrm{d}A}{\mathrm{d}t} \geq -\lambda_1 A(t) + \frac{\lambda}{C_2^2}(1-A(t))^{-2} \quad \text{for any} \quad t > 0\,. \tag{3.53}$$

Choosing $\gamma \in (0,1)$ so that

$$\Psi(s) := \frac{\lambda}{C_2^2}(1-s)^{-2} - \lambda_1 s > 0 \qquad \text{for all } s \in [\gamma, 1),$$

then by choosing u_0 close enough to 1 such that $A(0) \geq \gamma$, relation (3.52) yields

$$\frac{\mathrm{d}A}{\mathrm{d}t} \geq \Psi(A(t)) > 0 \quad \text{for any} \quad t > 0,$$

which then leads to

$$t \leq \int_{A(0)}^{A(t)} \frac{\mathrm{d}s}{\Psi(s)} \leq \int_{A(0)}^{1} \frac{\mathrm{d}s}{\Psi(s)} < \infty,$$

contradicting the assumption $T_{max} = \infty$. This completes the proof of the theorem. □

4. Local Behaviour at Quenching

In this section we obtain some limited results about the manner of finite-time quenching, shown to take place in the previous section.

4.1. **Single-point quenching.** Our main result is that for the class of problems under consideration, namely radially symmetric with monotonic decreasing initial data, quenching, when it occurs, takes place at a single point, the origin:

Theorem 4.1. *If we consider initial data as in Theorem 3.5 so that the solution of problem* (3.11)-(3.13) *quenches in finite time $T_q < \infty$, the quenching occurs only at the origin $r = 0$.*

Proof. The proof follows immediately from from (3.19). □

Due to the non-locality, obtaining the sharp profile of the standard (local) problem (cf. [4]) for (3.11)-(3.13) might be hard. However it can be rigorously shown that the exponent 2/3 in (3.19) is optimal, at least in the sense that (3.19) cannot be true for any exponent $k < 2/3$.

The optimality of the exponent 2/3 is a consequence of the following result.

Proposition 4.2. *Let T_q be the quenching time of the solution u of* (3.11)-(3.13) *then*

$$\lim_{t\to T_q} ||(1-u)^{-1}||_m = \infty \quad \text{for any} \quad m > \frac{3N}{2} > 1. \tag{4.1}$$

Proof. First note $\theta = (1-u)^{-1}$ satisfies $\theta_t - \Delta\theta \leq f(t)\theta^4 \leq \lambda\theta^4$ with $\theta = 1$ on $\partial\Omega$. Next fix any $\Lambda > 0$ and assume that $||\theta(t_0)||_m \leq \Lambda$ for some $m > 3N/2 > 1$ and $t_0 \in (0, T_q)$.

By virtue of [21, Theorem 15.2, Example 51.27], see also [1, Theorem 1] and [22, Theorem 1], we have that problem

$$\begin{aligned} &z_t - \Delta z = \lambda(1+z)^4 \quad \text{in} \quad \Omega \times (t_0, T_q) \\ &z = 0 \quad \text{on} \quad \partial\Omega \times (t_0, T_q) \\ &z(x, t_0) = \theta(x, t_0) \quad x \in \Omega \end{aligned}$$

is well posed and in particular there exists $\tau > 0$ such that

$$||z(t_0 + s)||_\infty \leq K s^{-N/2m}, \quad s \in (0, \tau], \tag{4.2}$$

where K, τ depend only on $\Lambda, m, \Omega, \lambda$. By comparison θ exists and satisfies $\theta \leq z + 1$ on $[t_0, t_0 + \tau]$. Since $\lim_{t\to T_q} ||\theta(t)||_\infty = \infty$ it follows that $||\theta(t)||_m > \Lambda$ for all $t \in (\max(0, T_q - \tau), T_q)$ and thus (4.1). □

Remark 4.3. *It should be noted in the critical case $m = m_c = 3N/2$ that the time τ in* (4.2) *depends on $\theta(t_0)$ and not just on $\|\theta(t_0)\|_m$ (see* [1] *and* [21, Remarks 15.4 and 16.2(iv)]*). Therefore, in this case, it is no longer certain that $T_q < \infty$ implies the finite-time blow-up of the norm $\|\theta(t_0)\|_m$.*

Corollary 4.4. *Relation* (3.19) *is not valid for any $k < 2/3$.*

Proof. First note that since $T_q = T_{max} < \infty$ it is easily seen by the proof of Lemma 3.3 that (3.19) is valid up to T_{max}. Assume now that there is $k_0 < 2/3$ such that

$$1 - u(r,t) \geq C(k_0) r^{k_0} \quad \text{for} \quad (r,t) \in (0,1) \times (0, T_{max}]$$

then

$$\lim_{t\to T_{max}} ||(1-u)^{-1}||_m = \int_0^1 \frac{r^{N-1}}{(1-u(T_{max}))^m}\, dr \leq C^{-1}(k_0) \int_0^1 r^{N-1-mk_0}\, dr < \infty$$

for $m > 3N/2$ close to $3N/2$, contradicting Proposition 4.2. □

4.2. **Lower bound of quenching rate.** We recall that considering radial decreasing initial data u_0 then u inherits this property and hence

$$M(t) := \max_{x\in \bar{B}_1} u(x,t) = u(0,t).$$

The next result provides a lower estimate of the quenching rate:

Theorem 4.5. *The lower bound of the quenching rate of problem* (3.11)-(3.13) *is given by*

$$M(t) \geq 1 - \widehat{C}(T_q - t)^{1/3} \quad for \quad 0 < t < T_q\,, \tag{4.3}$$

where $\widehat{C}$ is a positive constant independent of time t.

Proof. It can be easily checked that the function $M(t)$ is Lipschitz continuous and hence, by Rademacher's theorem, is almost everywhere differentiable, see [5, 16]. Furthermore, since u is decreasing in r, $\Delta_r u(0,t) \leq 0$ for all $t \in (0, T_q)$. Therefore, for any t where $\mathrm{d}M/\mathrm{d}t$ exists, we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} \leq \lambda \frac{(1-M(t))^{-2}}{\left(1+\int_{B_1}\frac{1}{1-u}\,\mathrm{d}x\right)^2} \leq \lambda \frac{(1-M(t))^{-2}}{(1+N\omega_N)^2} \quad \text{for a.e.} \quad t \in (0, T_q),$$

which yields

$$\int_{M(t)}^{1} (1-s)^2\,\mathrm{d}s \leq \lambda C(T_q - t),$$

for $C = 1/(1+N\omega_N)^2$, giving the desired estimate

$$M(t) \geq 1 - \widehat{C}(T_q - t)^{1/3} \quad \text{for} \quad 0 < t < T_q\,,$$

where $\widehat{C} = (3\lambda C)^{1/3}$. □

5. Numerical Results

We now carry out a brief numerical study of problem (1.1)-(1.3) for the one-dimensional case. Here the form of the problem is taken as

$$u_t = u_{xx} + \frac{\lambda}{(1-u)^2\left(1+\int_0^1 \frac{1}{1-u}\,\mathrm{d}x\right)^2}, \quad 0 < x < 1, \quad t > 0, \tag{5.1}$$

$$u(0,t) = 0, \quad u(1,t) = 0,$$

$$u(x,0) = u_0(x).$$

A moving mesh adaptive method, based on the techniques suggested in [4], is used. This captures the behaviour of the solution near a singularity. More specifically we take initially a partition of $M+1$ points in $[0,1]$, $\xi_0 = 0$, $\xi_0 + \delta\xi = \xi_1, \cdots, \xi_M = 1$. For the solution $u = u(x,t)$, we introduce a computational coordinate ξ in the interval $[0,1]$ and we consider the mesh points X_i to be the images of the points ξ_i (uniform mesh) under the map $x(\xi,t)$ so that $X_i(t) = x(i\delta\xi, t)$. Given this transformation, we have, for the approximation of the solution $u_i(t) \simeq u(X_i(t), t)$, that $\dfrac{\mathrm{d}u(X_i(t),t)}{\mathrm{d}t} = u_t(X_i,t) + u_x\dot{X}_i$ or $u_t = \dfrac{\mathrm{d}u}{\mathrm{d}t} - u_x x_t$.

The way that the map, $x(\xi,t)$, is determined is controlled by the monitor function $\mathcal{M}(u)$ which, in a sense, follows the evolution of the singularity. This function is determined by the scale invariants of the problem, [2]. In our case, for the semilinear parabolic equation of the form $v_t = v_{xx} - \lambda/(v^2(1+\int_0^1 v^{-2}\,\mathrm{d}x)^2)$ for $v = 1-u$, an appropriate monitor function should be $\mathcal{M}(v) = |v|^{-2}$ in this case.

At the same time we need also a rescaling of time of the form $\dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{\mathrm{d}u}{\mathrm{d}\tau}\dfrac{\mathrm{d}\tau}{\mathrm{d}t}$ for $\dfrac{\mathrm{d}t}{\mathrm{d}\tau} = g(u)$, where $g(u)$ is a function determining the way that the time scale changes as the solution approaches the singularity, and is given by $g(u) = 1/\|\mathcal{M}(u)\|_\infty$ (again see [2]).

In addition the evolution of $X_i(t)$ is given by a moving mesh PDE which has the form $-x_{\tau\xi\xi} = \varepsilon^{-1} g(u)\left(\mathcal{M}(u)x_\xi\right)_\xi$. Here ε is a small parameter accounting for the relaxation time scale.

Thus finally we obtain a system of ODE's for X_i and u_i and the ODE system takes the form

$$\begin{aligned} \frac{\mathrm{d}t}{\mathrm{d}\tau} &= g(u), \\ u_\tau - x_\tau u_x &= g(u)\left(u_{xx} + \frac{\lambda}{(1-u)^2\left(1+\int_0^1 \frac{1}{1-u}\,\mathrm{d}x\right)^2}\right), \\ -x_{\tau\xi\xi} &= \frac{g(u)}{\varepsilon}\left(\mathcal{M}(u)x_\xi\right)_\xi . \end{aligned}$$

We may apply now a discretization in space and we have

$$\begin{aligned} u_x(X_i,\tau) &\simeq \delta_x u_i(\tau) := \frac{u_{i+1}(\tau) - u_{i-1}(\tau)}{X_{i+1}(\tau) - X_{i-1}(\tau)}, \\ u_{xx}(X_i,\tau) &\simeq \delta_x^2 u_i(\tau) := \left(\frac{u_{i+1}(\tau) - u_i(\tau)}{X_{i+1}(\tau) - X_i(\tau)} - \frac{u_i(\tau) - u_{i-1}(\tau)}{X_i(\tau) - X_{i-1}(\tau)}\right)\frac{2}{X_{i+1}(\tau) - X_{i-1}(\tau)}, \\ x_{\xi\xi}(\xi_i,\tau) &\simeq \delta_\xi^2 x_i(\tau) := \frac{X_{i+1}(\tau) - 2X_i(\tau) + X_{i-1}(\tau)}{\delta\xi^2}, \\ \left(\mathcal{M}(u)x_\xi\right)_\xi &\simeq \delta_\xi(\mathcal{M}\delta_\xi x) := \left(\frac{\mathcal{M}_{i+1} - \mathcal{M}_i}{2}\frac{x_{i+1} - x_i}{\delta\xi} - \frac{\mathcal{M}_i - \mathcal{M}_{i-1}}{2}\frac{x_i - x_{i-1}}{\delta\xi}\right)\frac{1}{\delta\xi}. \end{aligned}$$

Therefore the resulting ODE system to be solved, for

$$\begin{aligned} y &= \left(t(\tau), u_1(\tau), u_2(\tau), \ldots u_M(\tau), X_1(\tau), X_2(\tau), \ldots X_M(\tau)\right), \\ &= \left(t(\tau), \mathbf{u}, \mathbf{X}\right), \qquad \mathbf{u}, \mathbf{X} \in \mathbb{R}^M, \end{aligned}$$

will have the form

$$A(\tau, y)\frac{\mathrm{d}y}{\mathrm{d}\tau} = b(\tau, y),$$

where the matrix $A \in \mathbb{R}^{2n+1}$ has the block form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & -\delta_x u \\ 0 & 0 & -\delta_\xi^2 \end{bmatrix}, \quad y = \begin{bmatrix} t(\tau) \\ \mathbf{u} \\ \mathbf{X} \end{bmatrix}, \quad b = g(u)\begin{bmatrix} 1 \\ \delta_x^2\mathbf{u} + \lambda\frac{1}{(1-\mathbf{u})^2(1+\mathcal{I}(u))^2} \\ \delta_\xi(\mathcal{M}\delta x_\xi) \end{bmatrix}.$$

where $\mathcal{I}(u)$ is an approximation of the integral $\int_0^1 \frac{1}{1-u}\,\mathrm{d}x$, using, for example, Simpson's rule.

For the solution of the above system a standard ODE solver, such as the matlab function "ode15i", can be used.

We first plot, in Figure 1, the solution for $\lambda > \lambda^* = 8.533$, [18], to give quenching. The initial data are taken to be zero in this case; we use $M = 141$. In can be observed that the solutions flattens for a time (this will be while it lies close to the steady state corresponding to $\lambda = \lambda^*$).

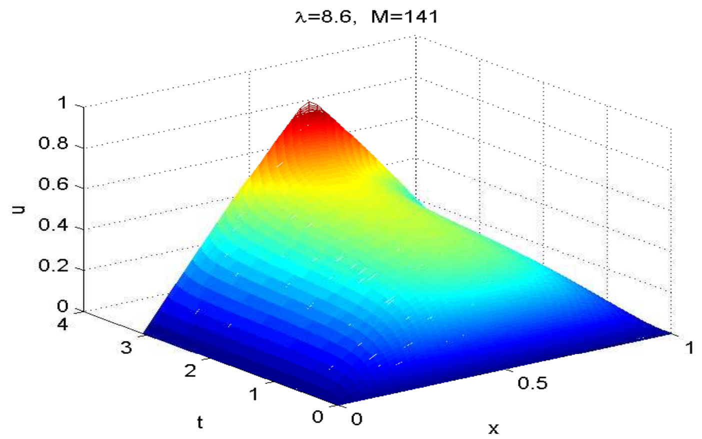

FIGURE 1. *The numerical solution of problem (5.1) with* $u_0 \equiv 0$*, for* $\lambda = 8.6$ *and taking* $M = 141$*.*

Using now $\lambda = 10$, we can see the evolution of the solution profile against space for various times in Figure 2. Again we take $u_0 \equiv 0$ and use $M = 141$.

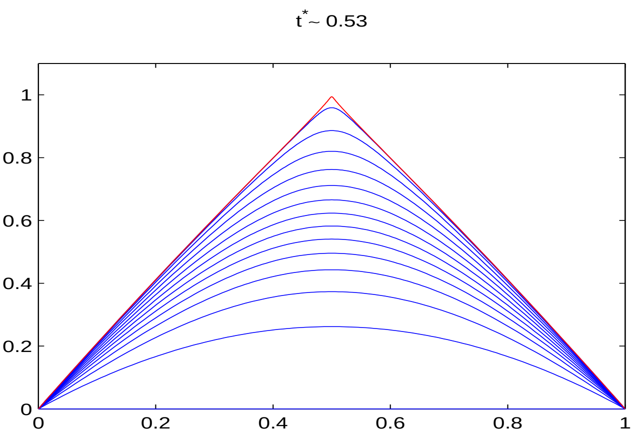

FIGURE 2. *Profile of the numerical solution of problem (5.1) against* x *for* $\lambda = 10$ *and* $u_0 \equiv 0$*, taking* $M = 141$*.*

Regarding the behaviour of the solution near quenching with respect to time, our numerical simulations indicate that we get an approximate $t^{1/3}$ dependence. More precisely near the quenching time $T = T_q$, $\ln\left(1 - u(\frac{1}{2}, t)\right) \propto \ln(T - t)$ with constant of proportionality $\frac{1}{3}$. This is demonstrated in Figure 3. For the local spatial dependence, a plot of $\ln u(x, T)$ against $\ln(x - \frac{1}{2})$, in Figure 4, shows that $u(x, T)$ behaves approximately like $\sim C(x - \frac{1}{2})^{\frac{2}{3}}$

near quenching. These numerical results agree both with the bound of Theorem 4.5 and the asymptotic results on quenching of [4, 8, 10, 11]. (Note that a more accurate local asymptotic form of the quenching profile is expected to be, [4], $u \sim C(x-\frac{1}{2})^{\frac{2}{3}}|\ln(x-\frac{1}{2}|)|^{-\frac{1}{3}}$. Note also that the local quenching behaviour of our non-local problem is expected to be like that of the standard problem, (1.4), because of the boundedness of the integral in the non-local term, Lemma 3.3.)

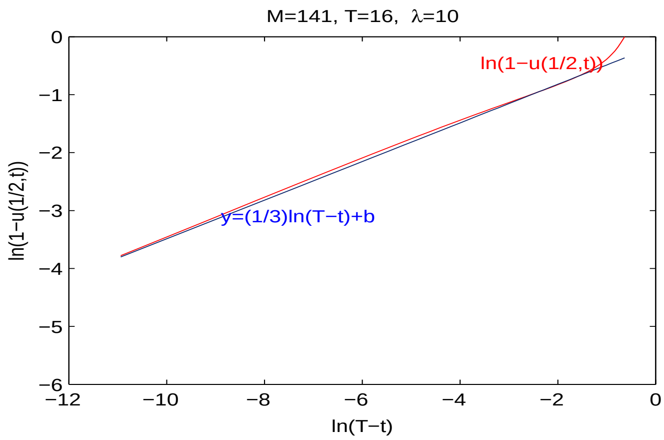

FIGURE 3. *Plot of* $y = \ln\big(1-u(\frac{1}{2},t)\big)$ *(red curve) against* $\ln(T-t)$ *for* $\lambda = 10$. *The straight line (blue) has slope* $\frac{1}{3}$ *and indicates good agreement between* $1-u(\frac{1}{2},t)$ *and const.*$\times(T-t)^{\frac{1}{3}}$.

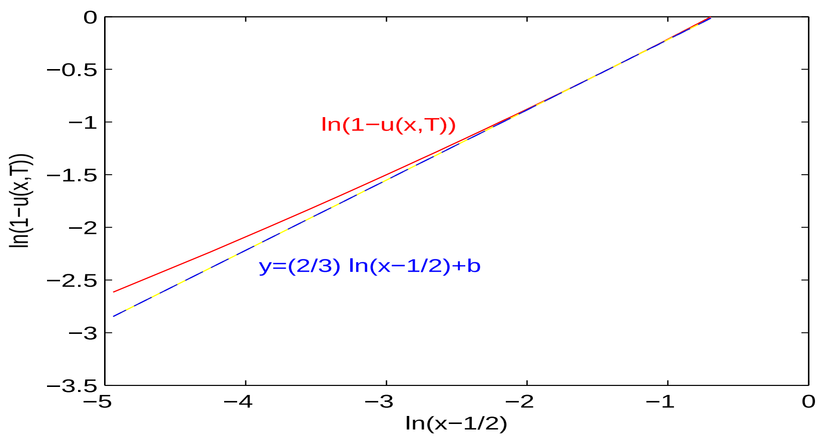

FIGURE 4. *Plot of* $\ln(1-u(x,T))$ *(solid curve) against* $\ln(x-\frac{1}{2})$, *for* $\lambda = 10$. *The blue dashed line has slope* $2/3$.

We look briefly at the radial symmetric problem in two dimensions,

$$\begin{aligned} u_t &= u_{rr} + \frac{1}{r}u_r + \frac{\lambda}{(1-u)^2\left(1+4\pi\int_0^1 \frac{r}{1-u}\,\mathrm{d}r\right)^2}, \quad 0<r<1, \quad t>0\,, \\ u_r(0,t) &= 0, \quad u(1,t)=0\,, \\ u(r,0) &= u_0(r)\,, \end{aligned} \tag{5.2}$$

taking, for a change, $\alpha = 2$.

We plot the solution for $\lambda = 71$ in Figure 5. Figure 6 shows the profile of the solution for various times. We again take $u_0 \equiv 0$ and use $M = 141$. We see that the behaviour is very similar to the one-dimensional problem. Temporary flattening can again be observed.

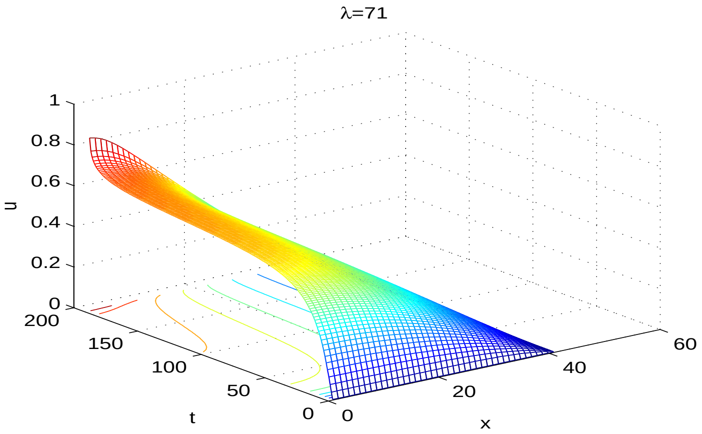

FIGURE 5. *The numerical solution of problem (5.2), for $\lambda = 71$ with $u_0 \equiv 0$ using $M = 141$.*

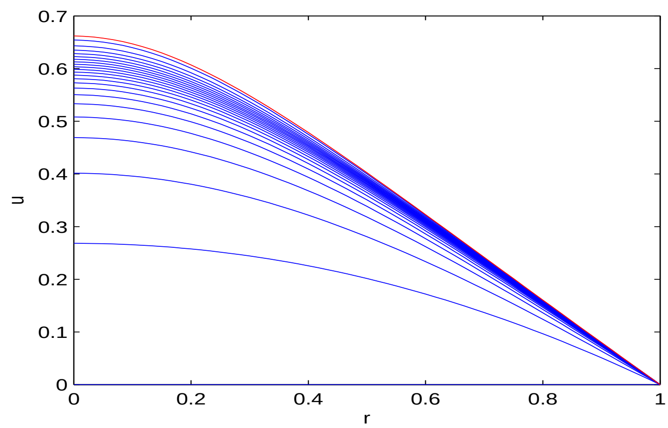

FIGURE 6. *Profiles for various times of the numerical solution of problem (5.2), for $\lambda = 71$ with $u_0 \equiv 0$ using $M = 141$.*

6. Discussion

Our results clearly easily extend to other problems of a similar form. For example, the analysis holds with $F(r,t)$, the right-hand side in (3.11), again given by $F(r,t) = \lambda k(t)(1-u(r,t))^{-2}$ now with $k(t) = \mathcal{F}(\int_\Omega (1-u)^{-1}\,\mathrm{d}x)$ and $\mathcal{F}(s)$ positive and satisfying $\mathcal{F}(s) \gg s^{-(a+1)}$ as $s \to \infty$ for some $a \in (1,2)$. Other non-linear functions of u multiplying $k(t)$ are also possible.

Local behaviour which has been established for the "standard" problem should be expected to carry over to these non-local problems on account of the boundedness of $\int_\Omega (1-u)^{-1}\,\mathrm{d}x$.

We also expect the results to hold for non-monotone initial data in the unit ball and for asymmetric problems in more general domains Ω.

Acknowledgement. The authors would like to thank the anonymous referee for their stimulating comments. In particular, their suggestions and recommendations regarding Theorem 3.5, Proposition 4.2 and Corollary 4.4 significantly improved the paper.

References

[1] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data, J. Anal. Math. **68** (1996), 277–304.
[2] C. J. Budd, J. F. Williams, How to adaptively resolve evolutionary singularities in differential equations with symmetry, J. Eng. Math. **6**6 (2010) 217–236.
[3] P. Esposito, N. Ghoussoub, Y. Guo, Mathematical analysis of partial differential equations modeling electrostatic MEMS, Courant Lecture Notes in Mathematics, 20. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2010.
[4] S. Filippas, J.S. Guo, Quenching profiles for one-dimensional semilinear heat equations, Quart. Appl. Math. **51** (1993) 713–729.
[5] A. Friedman, B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. **34** (1985) 425–447.
[6] G. Flores, G. Mercado, J.A. Pelesko, N. Smyth, Analysis of the dynamics and touchdown in a model of electrostatic MEMS, SIAM J. Appl. Math. **67** (2007) 434–446.
[7] N. Ghoussoub, Y. Guo, On the partial differential equations of electrostatic MEMS devices II: dynamic case, Nonlinear Diff. Eqns. Appl. **15** (2008) 115–145.
[8] J.-S. Guo, B. Hu, C.-J. Wang, A nonlocal quenching problem arising in micro-electro mechanical systems, Quart. Appl. Math. **67** (2009) 725–734.
[9] J.-S. Guo, N.I. Kavallaris, On a nonlocal parabolic problem arising in electromechanical MEMS control, Disc. Cont. Dynam. Systems **32** (2012) 1723–1746.
[10] Y. Guo, On the partial differential equations of electrostatic MEMS devices III: refined touchdown behavior, J. Diff. Eqns. **244** (2008) 2277–2309.
[11] Y. Guo, Z. Pan, M.J. Ward, Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties, SIAM J.Appl. Math. **166** (2006) 309–338.
[12] B. Hu & H-M. Yin, *Semilinear parabolic equations with prescribed energy*, Rend. Circ. Mat. Palermo **44(3)** (1995) 479–505.
[13] K-M. Hui, The existence and dynamic properties of a parabolic nonlocal MEMS equation, Nonlinear Analysis: Theory, Methods & Applications **74** (2011) 298–316.
[14] D.D. Joseph, T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rat. Mech. Anal. **49** (1973) 241–269.
[15] S. Kaplan, On the growth of solutions of quasilinear parabolic equations, Comm. Pure Appl. Maths. **16** (1963) 327–330.
[16] N.I. Kavallaris, T. Nadzieja, On the blow-up of the non-local thermistor problem, Proc. Edin. Math. Soc. **50** (2007) 389–409.
[17] N.I. Kavallaris, T. Miyasita, T. Suzuki, Touchdown and related problems in electrostatic MEMS device equation, Nonlinear Diff. Eqns. Appl. **15** (2008) 363–385.

[18] N. I. Kavallaris, A. A. Lacey, C. V. Nikolopoulos, D. E. Tzanetis, A hyperbolic nonlocal problem modelling MEMS technology, Rocky Mountain J. Math. **41** (2011) 505–534.
[19] J.A. Pelesko, A.A. Triolo, Nonlocal problems in MEMS device control, J. Eng. Math. **41** (2001) 345–366.
[20] J.A. Pelesko, D.H. Bernstein, Modeling MEMS and NEMS, Chapman Hall and CRC Press, 2002.
[21] P. Quittner and Ph. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2007.
[22] F.B.Weissler, Local existence and nonexistence for semilinear parabolic equations in L^p, Indiana Univ. Math. J. **29** (1980), 79–102.
[23] M. Younis, MEMS Linear and Nonlinear Statics and Dynamics, Springer, New York, 2011.
[24] G.-F. Zheng, Some dichotomy results for the quenching problem, Acta Math. Sinica **28** (2012) 1491–1506.

Department of Mathematics, University of Chester, Thornton Science Park, Pool Lane, Ince, Chester CH2 4NU, UK
E-mail address: n.kavallaris@chester.ac.uk

Maxwell Institute for Mathematical Sciences & School of Mathematical and Computer Sciences, Heriot-Watt University, Riccarton, Edinburgh, EH14 4AS, UK
E-mail address: A.A.Lacey@hw.ac.uk

Department of Mathematics, University of Aegean, Gr-83200 Karlovassi, Samos, Greece
E-mail address: cnikolo@aegean.gr