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The solution of some discretionary stopping problems

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We present a methodology for obtaining explicit solutions to infinite time horizon optimal stopping problems involving general, one-dimensional, Itô diffusions, payoff functions that need not be smooth and state-dependent discounting. This is done within a framework based on dynamic programming techniques employing variational inequalities. The aim of this paper is to facilitate the solution of a wide variety of problems, particularly in finance or economics.

Keywords: stochastic control, optimal stopping, dynamic programming, finance.

1. Introduction

A fundamental problem in finance, economics or management science is concerned with determining the optimal time to execute an action that results in some payoff in a random environment. Examples of these types of problems include buying or selling an asset in a market, making a decision based on noisy economic data or operating a manufacturing facility in response to consumer demand. To address these types of problems, the theory of discretionary stopping has been widely employed in finance following Karlin (1962) and the development of so-called 'real options' theory, introduced by McDonald & Siegel (1986).

In order to address some problems of this type, this article presents a framework for obtaining explicit solutions to a wide variety of infinite time horizon optimal stopping problems. We assume that the stochastic system we study is driven by the Itô diffusion given by the stochastic differential equation (SDE)

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0 = x \in \mathcal{I}, \tag{1}$$

where the functions $b, \sigma : \mathcal{I} \to \mathbb{R}$ satisfy Assumptions 2.1 and 2.2 and $\mathcal{I}$ is an open interval with left endpoint $\alpha \geqslant -\infty$ and right endpoint $\beta \leqslant \infty$. Our objective is to select the $(\mathcal{F}_t)$-stopping-time, τ, that maximizes

$$\mathbb{E}_x[e^{-\Lambda_\tau} g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}],$$

where g is subject to the conditions in Assumption 2.4 and Λ is a state-dependent discounting factor defined by

$$\Lambda_t = \int_0^t r(X_s)\,\mathrm{d}s, \tag{2}$$

for some function, r, satisfying the conditions of Assumption 2.3.

The majority of financial and real options models in the current literature assume that the underlying asset's value dynamics are modelled by a geometric Brownian motion, the associated payoff function is affine and the discounting rate is constant. The objective of this paper is to significantly relax all of these assumptions and to provide a much more realistic modelling framework within which results of an explicit nature can be obtained.

Apart from offering economic modellers additional flexibility, developing the existing theory so that it can account for asset price dynamics driven by general Itô diffusions becomes essential once one recognizes that the value of assets that exist in equilibrium market conditions tend to fluctuate about some long-term mean level, rather than, on average, grow or fall exponentially, as modelled by a geometric Brownian motion. This observation, which is supported by empirical evidence (e.g. see Metcalf & Hassett, 1995; Sarkar, 2003), suggests that real asset dynamics should be modelled by mean-reverting diffusions rather than by a geometric Brownian motion.

Introducing state-dependent discounting enables a more realistic modelling framework for investment decisions in the presence of default risk. In practice, investment decision making involves the choice of a discounting rate that accounts for the time-value of money and the associated investment's depreciation rate as well as for the likelihood of the investment's default. In view of this observation, discounting should reflect the dependence of default likelihood of an investment project on the economic environment affecting the project, which, in an economic setting, might be related to the underlying asset's value or demand. In particular, the events of 2007–2008 highlighted the importance of including a state-dependent discount factor.

Considering general payoff functions, rather than affine ones, plainly provides significant additional modelling flexibility, which allows for the incorporation of tax effects on payoffs and enables utility based decision making, which, apart from the work of Henderson & Hobson (2002), and despite its fundamental importance, has hardly found its way into real options theory. Indeed, the accommodation of general utility functions into real option models is a major economic contribution of this paper.

However, the main benefit of accommodating general payoffs in the modelling framework is the ability to incorporate decisions to enter and exit a project that pays running payoffs, such as in Duckworth & Zervos (2000, 2001), Johnson & Zervos (2010) or Guo & Tomecek (2008), for example. The simplest manifestation of this decision problem is when a project is initiated at a cost, $G(X_t)$, and provides a running payoff, $H(X_t)$ and the objective is to select the $(\mathcal{F}_t)$-stopping-time, τ, that maximizes

$$\mathbb{E}_x\left[e^{-\Lambda_\tau}\left(-G(X_\tau)+\int_\tau^\infty e^{-\Lambda_s}H(X_s)\,\mathrm{d}s\right)\mathbf{1}_{\{\tau<\infty\}}\right]. \tag{3}$$

For example, X_t could represent the demand for electricity and H is the 'stack', a discontinuous function, representing the value of supplying the demand.

The theory of discretionary stopping has numerous applications and has attracted the interest of many researchers. important, older accounts of this theory include Dynkin (1963), Shiryaev (1978), El-Karoui (1979), Krylov (1980), Bensoussan & Lions (1982) and Salminen (1985). More recent contributions include Davis & Karatzas (1994), Beibel & Lerche (1997, 2000)), Guo & Shepp (2001), Alvarez (2001), Dayanik & Karatzas (2003), Dayanik (2008), Lerche & Urusov (2007), Lempa (2010), Christensen & Irle (2011) and Matomäki (2012). An extensive presentation of the theory can be found in Peskir & Shiryaev (2006), for example. The use of optimal stopping models has become widespread in finance and economics since the introduction of so-called 'real options' theory by McDonald & Siegel (1986), and has been described in Merton (1990), Dixit & Pindyck (1994), Trigeorgis (1996) and Shreve (2004). The approach taken in this paper is similar to that in Rüschendorf & Urusov (2008), Lamberton (2009), Johnson & Zervos (2010) and Lamberton & Zervos (2013).

The paper is organized as follows. Section 2 is concerned with a formulation of the optimal stopping problem and a set of assumptions for our problem to be well-posed while in Section 2.5 we discuss the practical implications of these assumptions. In Section 3, we present the methodology for identifying the boundaries for six 'elementary' problems and then, in Section 4, we demonstrate how these 'elementary' problems can be employed in solving stopping problems with non-standard payoffs. An Appendix provides the proof of a key result in solving the problem when a continuation region lies between two stopping regions.

2. Problem formulation and technical foundations

2.1 *Notation*

We denote by $\mathcal{I}$ a given open interval with left endpoint $\alpha \geqslant -\infty$ and right endpoint $\beta \leqslant \infty$, and by $\mathcal{B}(\mathcal{I})$ the Borel σ-algebra on $\mathcal{I}$. Given a point $c \in \mathcal{I}$, we adopt the convention $]c, c[=]c, c] = [c, c[= \emptyset$. Throughout the paper, we consider signed Radon measures, and we refer to them simply as 'measures'. Given such a measure, μ, on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ we denote the total variation of μ by $|\mu| = \mu^+ + \mu^-$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ.

A function $F : \mathcal{I} \to \mathbb{R}$ is the difference of two convex functions if and only if its left-hand side derivative, F'_-, exists and is of finite variation, and its second distributional derivative is a measure, which we denote by $F''(\mathrm{d}x)$. In this case, we have the Lebesgue decomposition

$$F''(\mathrm{d}x) = F''_{\mathrm{ac}}(x)\,\mathrm{d}x + F''_{\mathrm{s}}(\mathrm{d}x),$$

where $F''_{\mathrm{ac}}(x)\,\mathrm{d}x$ is absolutely continuous with respect to the Lebesgue measure and $F''_{\mathrm{s}}(\mathrm{d}x)$ is mutually singular with the Lebesgue measure.

2.2 *The underlying Itô diffusion*

We assume that the data of the one-dimensional Itô diffusion given by (1) in the introduction satisfy the following two assumptions.

ASSUMPTION 2.1 The functions $b, \sigma : \mathcal{I} \to \mathbb{R}$ are $\mathcal{B}(\mathcal{I})$-measurable,

$$\sigma^2(x) > 0 \quad \text{for all } x \in \mathcal{I},$$

$$\int_{\underline{\alpha}}^{\bar{\beta}} \frac{1 + |b(s)|}{\sigma^2(s)}\,\mathrm{d}s < \infty \quad \text{and} \quad \sup_{s \in [\underline{\alpha}, \bar{\beta}]} \sigma^2(s) < \infty \quad \text{for all } \alpha < \underline{\alpha} < \bar{\beta} < \beta.$$

With reference to Karatzas & Shreve (1991, Section 5.5.C), the conditions appearing in this assumption are sufficient for the SDE (1) to have a weak solution $\mathbb{S}_x$ that is unique in the sense of probability law up to a possible explosion time, for all initial conditions $x \in \mathcal{I}$.

ASSUMPTION 2.2 The solution of (1) is non-explosive.

This assumption means that the boundaries α and β are inaccessible to the diffusion starting in $\mathcal{I}$.

We denote by L^y the local-time process of X at level $y \in \mathcal{I}$. Given a measure μ on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ such that σ^{-2} is locally integrable with respect to $|\mu|$, we define the finite variation, continuous process A^μ by

$$A_t^\mu = \int_\alpha^\beta \frac{L_t^y}{\sigma^2(y)}\,\mu(\mathrm{d}y). \tag{4}$$

We also make the following assumption in relation to the discounting factor Λ, defined by (2).

ASSUMPTION 2.3 The function $r : \mathcal{I} \to]0, \infty[$ is $\mathcal{B}(\mathcal{I})$-measurable, there exists $r_0 > 0$ such that $r(x) \geqslant r_0$, for all $x \in \mathcal{I}$, and

$$\int_{\underline{\alpha}}^{\bar{\beta}} \frac{r(s)}{\sigma^2(s)}\, \mathrm{d}s < \infty \quad \text{for all } \alpha < \underline{\alpha} < \bar{\beta} < \beta.$$

2.3 *The solution of an associated ordinary differential equation (ODE)*

In the presence of Assumptions 2.1–2.3, the general solution of the second-order linear homogeneous ODE,

$$\tfrac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x) - r(x)f(x) = 0, \quad x \in \mathcal{I} \tag{5}$$

is given by

$$f(x) = A\phi(x) + B\psi(x), \tag{6}$$

for some constants $A, B \in \mathbb{R}$. The functions ϕ and ψ are C^1, their first derivatives are absolutely continuous functions,

$$0 < \phi(x) \quad \text{and} \quad \phi'(x) < 0 \quad \text{for all } x \in \mathcal{I}, \tag{7}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0 \quad \text{for all } x \in \mathcal{I} \tag{8}$$

and

$$\lim_{x \downarrow \alpha} \phi(x) = \lim_{x \uparrow \beta} \psi(x) = \infty. \tag{9}$$

In this context, ϕ and ψ are unique, modulo multiplicative constants and given any points $x_1 < x_2$ in $\mathcal{I}$ and weak solutions $\mathbb{S}_{x_1}$, $\mathbb{S}_{x_2}$ of the SDE (1), the functions ϕ and ψ satisfy

$$\phi(x_2) = \phi(x_1)\mathbb{E}_{x_2}[e^{-\Lambda_{\tau_{x_1}}}] \quad \text{and} \quad \psi(x_1) = \psi(x_2)\mathbb{E}_{x_1}[e^{-\Lambda_{\tau_{x_2}}}], \tag{10}$$

where τ_z denotes the first hitting time of $\{z\}$, defined by

$$\tau_z = \{t \geqslant 0 \mid X_t = z\}.$$

All of these claims are standard and can be found in various forms in references, such as Feller (1952), Breiman (1968), Itô & McKean (1974), Karlin & Taylor (1981), Rogers & Williams (1994) and Borodin & Salminen (2002).

The framework we adopt accommodates the commonly encountered Itô diffusions, including: the standard Brownian motion; the Ornstein–Uhlenbeck process; the geometric Brownian motion; the geometric Ornstein–Uhlenbeck process and the so-called Feller, square-root mean-reverting or Cox–Ingersoll–Ross process. When r is constant, the expressions for the general solutions (6) to the ODEs associated with all of these diffusions are all well known. In situations where ϕ and ψ are not known, it is possible to approximate them through simulation, for example, by employing (10).

Central to our analysis is the solution of the non-homogeneous ODE

$$\mathcal{L}R_\mu + \mu = 0, \tag{11}$$

where μ is a measure on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ and the measure-valued operator $\mathcal{L}$ is defined by

$$\mathcal{L}F(\mathrm{d}x) = \tfrac{1}{2}\sigma^2(x)F''(\mathrm{d}x) + b(x)F'_-(x)\,\mathrm{d}x - r(x)F(x)\,\mathrm{d}x \tag{12}$$

on the space of all functions $F : \mathcal{I} \to \mathbb{R}$ that are the difference of two convex functions. In addition, we recall the definition of (ϕ, ψ)-integrable measures (Johnson & Zervos, 2007, Definition 2.5).

Definition 2.1 A measure μ on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ is a (ϕ, ψ)-integrable measure if

$$\int_{]\alpha,\gamma[} \Psi(s)|\mu|(\mathrm{d}s) + \int_{[\gamma,\beta[} \Phi(s)|\mu|(\mathrm{d}s) < \infty \quad \text{for all } \gamma \in \mathcal{I},$$

where Φ and Ψ are defined by

$$\Phi(x) = \frac{2\phi(x)}{\sigma^2(x)\mathcal{W}(x)} \quad \text{and} \quad \Psi(x) = \frac{2\psi(x)}{\sigma^2(x)\mathcal{W}(x)}, \tag{13}$$

and here $\mathcal{W}$ is the Wronskian of ϕ, ψ, defined by

$$\mathcal{W}(x) := \phi(x)\psi'(x) - \phi'(x)\psi(x) > 0 \quad \text{for all } x \in \mathcal{I}.$$

Necessary and sufficient conditions for a measure μ on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ to be (ϕ, ψ)-integrable are (Lamberton & Zervos, 2013, Theorem 4.2)

$$\int_{\underline{\alpha}}^{\bar{\beta}} \frac{1}{\sigma^2(s)}|\mu|(\mathrm{d}s) < \infty \quad \text{and} \quad \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\,\mathrm{d}A_t^{|\mu|}\right] < \infty \tag{14}$$

for all $\alpha < \underline{\alpha} < \bar{\beta} < \beta$ and all $x \in \mathcal{I}$, where $A^{|\mu|}$ is defined as in (4).

2.4 *The objective of the optimization problem*

We adopt a weak formulation of the optimal stopping problem that we solve.

Definition 2.2 Given an initial condition $x \in \mathcal{I}$, a stopping strategy is a pair $(\mathbb{S}_x, \tau)$ such that $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W)$ is a weak solution to (1) and τ is an $(\mathcal{F}_t)$-stopping-time. We denote by $\mathcal{S}_x$ the set of all such stopping strategies.

With each stopping strategy, we associate the performance criterion

$$J(\mathbb{S}_x, \tau) = \mathbb{E}_x[e^{-\Lambda_\tau}g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}].$$

The objective of the optimal stopping problem is to maximize $J(\mathbb{S}_x, \tau)$ over all stopping strategies $(\mathbb{S}_x, \tau)$. Accordingly, we define the value function v by

$$v(x) = \sup_{(\mathbb{S}_x,\tau)\in\mathcal{S}_x} J(\mathbb{S}_x, \tau) \quad \text{for } x \in \mathcal{I}. \tag{15}$$

To ensure that our optimization problem is well-posed, we make the following assumption on the payoff, g.

ASSUMPTION 2.4 The function $g : \mathcal{I} \to \mathbb{R}$ is the difference of two convex functions, and the measure $\mathcal{L}g$ is (ϕ, ψ)-integrable. In addition,

$$\lim_{x \downarrow \alpha} \frac{|g(x)|}{\phi(x)} = \lim_{x \uparrow \beta} \frac{|g(x)|}{\psi(x)} = 0, \tag{16}$$

and the limits $\lim_{x\uparrow\beta} g(x)/\phi(x)$ and $\lim_{x\downarrow\alpha} g(x)/\psi(x)$ exist in $[-\infty, \infty]$.

Note that this assumption accommodates cases where $g(x) < 0$, for some $x \in \mathcal{I}$. Without loss of generality, our subsequent analysis could be developed by assuming that g is positive. However, within the context of finance and economics, it is important to be able to explicitly accommodate cases where the payoff is negative. For example, a widget might be sold at a price x, but the cost of production of the widget means the value to the producer of the widget might be negative.

2.5 *Implications of the problem formulation*

Under Assumptions 2.1–2.4 and setting $\mu(\mathrm{d}x) = -\mathcal{L}g(\mathrm{d}x)$, the following results have been established in Johnson & Zervos (2007, 2010) or in Lamberton & Zervos (2013).

The payoff function g can be expressed analytically as

$$\begin{aligned} g(x) &= -\left(\phi(x) \int_{]\alpha,x[} \Psi(s)\mathcal{L}g(\mathrm{d}s) + \psi(x) \int_{[x,\beta[} \Phi(s)\mathcal{L}g(\mathrm{d}s)\right) \\ &\equiv -\left(\phi(x) \int_{]\alpha,x]} \Psi(s)\mathcal{L}g(\mathrm{d}s) + \psi(x) \int_{]x,\beta[} \Phi(s)\mathcal{L}g(\mathrm{d}s)\right), \end{aligned} \tag{17}$$

and probabilistically as the $r(\cdot)$-potential of $A^{-\mathcal{L}g}$, specifically

$$g(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} \, \mathrm{d}A_t^{-\mathcal{L}g} \right],$$

where $A^{-\mathcal{L}g}$ is defined as in (4).

The payoff function g satisfies Dynkin's formula, i.e. given any $(\mathcal{F}_t)$-stopping times $\rho_1 < \rho_2 < \infty$,

$$\mathbb{E}_x[e^{-\Lambda_{\rho_2}} g(X_{\rho_2})] = \mathbb{E}_x[e^{-\Lambda_{\rho_1}} g(X_{\rho_1})] + \mathbb{E}_x \left[\int_{\rho_1}^{\rho_2} e^{-\Lambda_t} \, \mathrm{d}A_t^{\mathcal{L}g} \right]. \tag{18}$$

In addition, we have a transversality condition, namely, given an increasing sequence of $(\mathcal{F}_t)$-stopping times (ρ_n) such that $\lim_{n\to\infty} \rho_n = \infty$,

$$\lim_{n\to\infty} \mathbb{E}_x[e^{-\Lambda_{\rho_n}} |g(X_{\rho_n})| \mathbf{1}_{\{\rho_n < \infty\}}] = 0.$$

This condition implies that our value function should be finite.

Furthermore, using (17), we can calculate that

$$g'_+(x)\phi(x) - g(x)\phi'(x) = -\mathcal{W}(x)\int_{]x,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s), \tag{19}$$

$$g'_-(x)\phi(x) - g(x)\phi'(x) = -\mathcal{W}(x)\int_{[x,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s), \tag{20}$$

$$g'_+(x)\psi(x) - g(x)\psi'(x) = \mathcal{W}(x)\int_{]\alpha,x]}\Psi(s)\mathcal{L}g(\mathrm{d}s), \tag{21}$$

$$g'_-(x)\psi(x) - g(x)\psi'(x) = \mathcal{W}(x)\int_{]\alpha,x[}\Psi(s)\mathcal{L}g(\mathrm{d}s). \tag{22}$$

Also, given a C^1 function f, the calculation

$$\left(\frac{g}{f}\right)'_{\pm}(x) = \frac{g'_{\pm}(x)f(x) - g(x)f'(x)}{f^2(x)} \tag{23}$$

reveals that (19) and (20) are related to the slope of the function g/ϕ, while (21) and (22) relate to the slope of g/ψ.

The methodology we employ to solve the stopping problem is based on the results in Lamberton & Zervos (2013, Section 6), where it is established that under Assumptions 2.1–2.4, the value function, v, associated with the optimal stopping problem and defined by (15), is of the form

$$v(x) = \begin{cases} A_j\phi(x) + B_j\psi(x) & \text{if } x \in \mathcal{C}_j \subseteq \mathcal{C}, \\ g(x) & \text{if } x \in \mathcal{D}. \end{cases} \tag{24}$$

Here $\mathcal{D}$ represents the stopping region, a closed set, while the continuation, or waiting, region is given by $\mathcal{C} = \mathcal{I} \setminus \mathcal{D}$ and $A_j, B_j \geqslant 0$ are specific to each component of the partition that makes up $\mathcal{C}$. In addition, v is the unique solution to the variational inequality

$$\max\{\mathcal{L}v(x),\ g(x) - v(x)\} = 0, \quad x \in \mathcal{I}, \tag{25}$$

in the sense of Definition 2.3, that satisfies the boundary condition

$$\lim_{x\downarrow\alpha}\frac{v(x)}{\phi(x)} = \lim_{x\downarrow\alpha}\frac{g(x)}{\phi(x)} = 0 \quad \text{and} \quad \lim_{x\uparrow\beta}\frac{v(x)}{\psi(x)} = \lim_{x\uparrow\beta}\frac{g(x)}{\psi(x)} = 0 \tag{26}$$

(see also (16)).

Definition 2.3 A function $v : \mathcal{I} \mapsto [0,\infty[$ is a solution of the variational inequality (25) if $v(x)$ is the difference of two convex functions, the measure $\mathcal{L}v$ is (ϕ,ψ)-integrable,

$$\text{the measure } \mathcal{L}v \text{ does not charge the set } \mathcal{C} = \{x \in \mathcal{I} \mid v(x) > g(x)\}, \tag{27}$$

$$-\mathcal{L}v \text{ is a positive measure on } (\mathcal{I}, \mathcal{B}(\mathcal{I})) \tag{28}$$

and

$$g(x) - v(x) \leqslant 0 \quad \text{for all } x \in \mathcal{I}. \tag{29}$$

3. The solution to six elementary stopping problems

3.1 *The cases*

This section of the paper considers six elementary cases distinguished by the behaviour of $\mathcal{L}g$, g/ϕ and g/ψ. These cases are not exhaustive but can be seen as the basic building blocks for addressing more complex situations (as in Example 4.3). In what follows, we denote by x_ϕ (respectively, x_ψ) the location at which a global maximum of the extension of g/ϕ (respectively, g/ψ) in $[\alpha, \beta]$, suggested by Assumption 2.4, occurs. The first two cases are the most basic ones, Cases III and IV are constructed by developing Cases I and II while Cases V and VI are, in turn, further developments of Cases III and IV. The function g is strictly positive at some point in all cases apart from in Case I.
Case I: $g(x) \leqslant 0$ for all $x \in \mathcal{I}$.
Given (16), this implies that

$$\lim_{x\downarrow\alpha} \frac{g(x)}{\phi(x)} = 0 \geqslant \frac{g(y)}{\phi(y)} \quad \text{and} \quad \lim_{x\uparrow\beta} \frac{g(x)}{\psi(x)} = 0 \geqslant \frac{g(y)}{\psi(y)} \quad \text{for all } y \in \mathcal{I}, \tag{30}$$

and suggest the definitions

$$x_\phi = \alpha \quad \text{and} \quad x_\psi = \beta.$$

The economic intuition behind this case is that given we have the choice of waiting forever, with a zero payoff, or taking a negative payoff in finite time, the optimal strategy would be to wait indefinitely.
Case II: $-\mathcal{L}g$ is a positive measure.
In view of (19–22) and (23), we can see that g/ϕ (respectively, g/ψ) is an increasing (respectively, decreasing) function, and given (16), these imply that g is everywhere positive. In this case, we define

$$x_\phi = \beta \quad \text{and} \quad x_\psi = \alpha.$$

Economically, (4), (18) and the inequality $\mathcal{L}g < 0$ imply that waiting destroys value. Therefore, in this case, there is no point in $\mathcal{I}$ where waiting enhances value, and the optimal strategy is to stop immediately. This case is symmetric to Case I.
Case III: There exists $x_r \in \mathcal{I}$ such that

$$\text{the restriction of } -\mathcal{L}g \text{ in } ([x_r, \beta[, \mathcal{B}([x_r, \beta[)) \text{ is a positive measure,} \tag{31}$$

$$\frac{g(x)}{\psi(x)} \leqslant \frac{g(x_r)}{\psi(x_r)} \quad \text{for all } x \in \,]\alpha, x_r[\tag{32}$$

and

$$g(x) > 0 \quad \text{for some } x \in [x_r, \beta[. \tag{33}$$

In view of (22) and (23),

$$\left(\frac{g}{\psi}\right)'_-(x) = \frac{\mathcal{W}(x)}{\psi^2(x)} \int_{]\alpha,x[} \Psi(s)\mathcal{L}g(\mathrm{d}s).$$

Therefore, a simple sufficient condition for (32) to be true is that the restriction of $\mathcal{L}g$ in $(]\alpha,x_r[,\mathcal{B}(]\alpha,x_r[))$ is a positive measure. Also, combining this identity with (16) and (31–33), we can see that there exists $x_\psi \in \mathcal{I}$ such that

$$\left(\frac{g}{\psi}\right)'_-(x) \quad \begin{cases} \geqslant 0 & \text{for all } x \in [x_r, x_\psi], \\ < 0 & \text{for all } x \in]x_r, \beta]. \end{cases}$$

These inequalities and (32) imply that g/ψ has a global maximum at x_ψ. In addition, since ψ is a strictly increasing function, (32) implies that $g(x) < g(x_\psi)$ for all $x \in]\alpha, x_\psi[$. Combining this observation and the fact that ϕ is strictly decreasing, (19–20), (23) and (31), we can see that

$$\frac{g(x)}{\phi(x)} \leqslant \frac{g(x_\psi)}{\phi(x_\psi)} \leqslant \lim_{x\uparrow\beta} \frac{g(x)}{\phi(x)} \quad \text{for all } x \in]\alpha, x_\psi[,$$

and we define $x_\phi = \beta$.

As we are going to see, the economic interpretation of this case is of a call option-type payoff.

Case IV: There exists $x_l \in \mathcal{I}$ such that

$$\text{the restriction of } -\mathcal{L}g \text{ in } (]\alpha,x_l],\mathcal{B}(]\alpha,x_l])) \text{ is a positive measure,}$$

$$\frac{g(x)}{\phi(x)} \leqslant \frac{g(x_l)}{\phi(x_l)} \quad \text{for all } x \in]x_l, \beta[$$

and

$$g(x) > 0 \quad \text{for some } x \in]\alpha, x_l].$$

This case is symmetric to Case III, we similarly deduce that g/ϕ has a global maximum at some $x_\phi \in]\alpha, x_l]$ and we define $x_\psi = \alpha$. The simplest example that falls under this case occurs when the restrictions of $-\mathcal{L}g$ in $(]\alpha,x_l],\mathcal{B}(]\alpha,x_l]))$ and $\mathcal{L}g$ in $(]x_l,\beta[,\mathcal{B}(]x_l,\beta[))$ are both positive measures and $g(x) > 0$ for some $x \in]\alpha, x_l]$. Economically, this case represents a put option-type payoff.

Case V: There exists $\mathcal{E}^b := [x_l, x_r] \subset \mathcal{I}$ such that

$$\text{the restriction of } -\mathcal{L}g \text{ in } (\mathcal{E}^b,\mathcal{B}(\mathcal{E}^b))) \text{ is a positive measure,}$$

$$\frac{g(x)}{\psi(x)} < \frac{g(x_l)}{\psi(x_l)} \quad \text{for all } x \in]\alpha, x_l[$$

and

$$\frac{g(x)}{\psi(x)} < \frac{g(x_r)}{\psi(x_r)} \quad \text{for all } x \in]x_r, \beta[,$$

while

$$\frac{g(x)}{\phi(x)} < \frac{g(x_l)}{\phi(x_l)} \quad \text{for all } x \in]\alpha, x_l[$$

and

$$\frac{g(x)}{\phi(x)} < \frac{g(x_r)}{\phi(x_r)} \quad \text{for all } x \in]x_r, \beta[.$$

Also, $g(x) > 0$ for some $x \in \mathcal{E}^b$.

With reference to the discussion of Case III, these conditions imply that there exists an $x_\psi \in [x_l, x_r]$ such that g/ψ is increasing for $x \in]x_l, x_\psi]$ and then strictly decreasing in $]x_\psi, x_r]$ so that the point x_ψ represents a global maximum. In addition, (19) and (20) mean that there exists a point $x_\phi \in [x_l, x_r]$ such that g/ϕ is strictly increasing for $x \in [x_l, x_\phi[$ and then decreases in $[x_\phi, x_r]$.

Since ψ is a strictly increasing function, we have that $g(y) < g(x_\psi)$ for all $y \in]\alpha, x_\psi[$, and, since ϕ is strictly decreasing, the implication is that

$$\frac{g(y)}{\phi(y)} < \frac{g(x_\psi)}{\phi(x_\psi)} \quad \text{for all } x \in]\alpha, x_\psi[,$$

and so $x_\psi \leqslant x_\phi$.

The simplest manifestation of this case would be when the restriction of $-\mathcal{L}g$ in $\mathcal{E}^b$ and the restriction of $\mathcal{L}g$ in $\mathcal{I} \setminus \mathcal{E}^b$ are positive measures and $g(x) > 0$ for some $x \in \mathcal{E}^b$. The case is a combination of Case III to the left of Case IV and economically it represents a butterfly option-type payoff.

Case VI: There exists $\mathcal{E}^s :=]x_l, x_r[\subset \mathcal{I}$ such that

$$\text{the restriction of } \mathcal{L}g \text{ in } (\mathcal{E}^s, \mathcal{B}(\mathcal{E}^s)) \text{ is a positive measure with } \mathcal{L}g(\mathcal{E}^s) > 0, \tag{34}$$

while,

$$\text{the restriction of } -\mathcal{L}g \text{ in } (\mathcal{I} \setminus \mathcal{E}^s, \mathcal{B}(\mathcal{I} \setminus \mathcal{E}^s)) \text{ is a positive measure.} \tag{35}$$

In addition, the limits $\lim_{x\downarrow\alpha} g(x)/\psi(x), \lim_{x\uparrow\beta} g(x)/\phi(x)$ exist,

$$\lim_{x\downarrow\alpha} \frac{g(x)}{\psi(x)} > \frac{g(y)}{\psi(y)} \quad \text{and} \quad \lim_{x\uparrow\beta} \frac{g(x)}{\phi(x)} > \frac{g(y)}{\phi(y)} \quad \text{for all } y \in \mathcal{I}, \tag{36}$$

$$\text{while } g(x) > 0 \quad \text{for some } x \in \mathcal{I}. \tag{37}$$

Together, (36) and (37) imply that $\lim_{x\downarrow\alpha} g(x) > 0$ and $\lim_{x\uparrow\beta} g(x) > 0$ and as in Case II, we define $x_\phi = \beta$ and $x_\psi = \alpha$. However, unlike Case II we cannot stop for some $x \in \mathcal{E}^s$ by (28) given (34). This suggests that the stopping problem related to this case is one of the first exit time of the diffusion from an interval, rather than one of locating the first hitting time of a point, which characterize Cases I–V. The economic interpretation is of a straddle-type payoff.

REMARK 3.1 We have ordered the six cases we consider by their increasing complexity. We could have ordered them by the locations of x_ϕ and x_ψ:

Location of x_ϕ	Location of x_ψ	Case
$x_\phi = \alpha$	$x_\psi = \beta$	Case I
$\alpha < x_\phi < x_\psi = \beta$	$x_\psi = \beta$	Case IV
$\alpha < x_\psi < x_\phi < \beta$	$\alpha < x_\psi < x_\phi < \beta$	Case V
$x_\phi = \beta$	$\alpha < x_\psi < x_\phi = \beta$	Case III
$x_\phi = \beta$	$x_\psi = \alpha$	If: $-\mathcal{L}g$ is positive, then Case II; otherwise Case VI.

3.2 *Analytic considerations for solving Cases* III *and* VI

Case III is associated with call option-type payoffs and, in this case, we would expect that there is a single boundary point, x^*, so that $\mathcal{C} =]\alpha, x^*[$ and $\mathcal{D} = [x^*, \beta[$. In this context, the calculation

$$v(x) = \mathbb{E}_x[e^{-\Lambda_{\tau_{x^*}}} g(x^*)] = \frac{g(x^*)}{\psi(x^*)}\psi(x) \quad \text{for all } x < x^*, \tag{38}$$

implies that

$$A = 0 \quad \text{and} \quad B = \frac{g(x^*)}{\psi(x^*)}. \tag{39}$$

Furthermore, the observation that

$$v(x) \geqslant \mathbb{E}_x[e^{-\Lambda_{\tau_z}}]g(z) = \frac{g(z)}{\psi(z)}\psi(x) \quad \text{for all } x < x^* \text{ and } z \geqslant x,$$

combined with (38) suggests that $x^* = x_\psi$. The fact that g/ψ has a global maximum at x_ψ (see the discussion after the statement of Case III in the previous subsection), gives rise to the system of inequalities

$$\left(\frac{g}{\psi}\right)'_+(x^*) \leqslant 0 \leqslant \left(\frac{g}{\psi}\right)'_-(x^*), \tag{40}$$

which are equivalent to

$$\int_{]\alpha,x^*]} \Psi(s)\mathcal{L}g(\mathrm{d}s) \leqslant 0 \leqslant \int_{]\alpha,x^*[} \Psi(s)\mathcal{L}g(\mathrm{d}s).$$

It is worth noting that (39) and (40) are equivalent to

$$B\psi(x^*) = g(x^*),$$
$$g'_+(x^*) \leqslant B\psi'(x^*) \leqslant g'_-(x^*),$$

which reduce to the system of equations

$$B\psi(x^*) = g(x^*) \quad \text{and} \quad B\psi'(x^*) = g'(x^*),$$

associated with the so-called 'smooth-pasting' of optimal stopping if g is C^1, in particular at the point x^*.

We associate Case VI with straddle option-type payoffs and the continuation and stopping regions are given by

$$\mathcal{C} =]a, b[\quad \text{and} \quad \mathcal{D} =]\alpha, a] \cup [b, \beta[$$

for some $a \in]\alpha, x_l[, b \in]x_r, \beta[$. The intuition that we developed in the discussion of Case III, above, suggests the system of inequalities

$$A\phi(a) + B\psi(a) = g(a), \quad A\phi(b) + B\psi(b) = g(b), \tag{41}$$

$$A\phi'(a) + B\psi'(a) \leqslant g'_-(a), \quad A\phi'(b) + B\psi'(b) \leqslant g'_-(b), \tag{42}$$

$$A\phi'(a) + B\psi'(a) \geqslant g'_+(a), \quad A\phi'(b) + B\psi'(b) \geqslant g'_+(b). \tag{43}$$

It is worth noting that these are equivalent to the identities

$$\frac{g(a)}{A\phi(a)+B\psi(a)}=\frac{g(b)}{A\phi(b)+B\psi(b)}=1$$

and the requirement that the function

$$\frac{g(x)}{A\phi(x)+B\psi(x)} \quad \text{for } x\in\mathfrak{I}$$

has maxima at a and b. Indeed this equivalent characterization has been central to the approach to solving optimal stopping problems developed in Beibel & Lerche (1997, 2000) and Christensen & Irle (2011).

In order to identify a and b, and hence the values for A and B, observe that by using (19–22), (41–43) can be rearranged into the following set of inequalities:

$$-\int_{]a,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s)\leqslant B\leqslant-\int_{[a,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s), \tag{44}$$

$$-\int_{]b,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s)\leqslant B\leqslant-\int_{[b,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s), \tag{45}$$

$$-\int_{]\alpha,a]}\Psi(s)\mathcal{L}g(\mathrm{d}s)\geqslant A\geqslant-\int_{]\alpha,a[}\Psi(s)\mathcal{L}g(\mathrm{d}s), \tag{46}$$

$$-\int_{]\alpha,b]}\Psi(s)\mathcal{L}g(\mathrm{d}s)\geqslant A\geqslant-\int_{]\alpha,b[}\Psi(s)\mathcal{L}g(\mathrm{d}s). \tag{47}$$

These are equivalent to the following system of inequalities:

$$q^{\mathrm{o}}_{\phi}(a,b)\geqslant 0 \quad\text{and}\quad q^{\mathrm{c}}_{\phi}(a,b)\leqslant 0, \tag{48}$$

$$q^{\mathrm{o}}_{\psi}(a,b)\geqslant 0 \quad\text{and}\quad q^{\mathrm{c}}_{\psi}(a,b)\leqslant 0, \tag{49}$$

where

$$q^{\mathrm{c}}_{\phi}(y,z):=\int_{[y,z]\cap\mathfrak{I}}\Phi(s)\mathcal{L}g(\mathrm{d}s), \tag{50}$$

$$q^{\mathrm{o}}_{\phi}(y,z):=\int_{]y,z[}\Phi(s)\mathcal{L}g(\mathrm{d}s), \tag{51}$$

$$q^{\mathrm{o}}_{\psi}(y,z):=\int_{]y,z[}\Psi(s)\mathcal{L}g(\mathrm{d}s) \tag{52}$$

and

$$q^{\mathrm{c}}_{\psi}(y,z):=\int_{[y,z]\cap\mathfrak{I}}\Psi(s)\mathcal{L}g(\mathrm{d}s), \tag{53}$$

for $\alpha\leqslant y<z\leqslant\beta$.

We prove the following result in the Appendix.

LEMMA A.3 Suppose that the problem data is such that the conditions of Case VI, (34–37), hold. Then there exist $a \in \mathcal{I} \setminus \mathcal{E}^s$ and $b \in \mathcal{I} \setminus \mathcal{E}^s$ satisfying the system of equations (48) and (49).

REMARK 3.2 Note that the condition (34) in the statement of Case VI can be relaxed to

$$\int_{\mathcal{E}^s} \Phi(s)\mathcal{L}g(\mathrm{d}s) > 0 \quad \text{and} \quad \int_{\mathcal{E}^s} \Phi(s)\mathcal{L}g(\mathrm{d}s) > 0$$

if the inequalities (A.7) and (A.8) in Lemma A.2 can be shown to be true. If the relaxed condition holds but either of (A.7) or (A.8) fail, the implication is that there is a subset of the stopping region in $\mathcal{E}^s$. We give an example of applying this observation in Section 4.2.

3.3 *The solution of the problems*

We now solve the various control problems described in Cases I–VI by constructing explicit solutions of the variational inequalities (25) of the form (24) that satisfies the requirements of (26) and Definition 2.3.

THEOREM 3.1 Suppose that Assumptions 2.1–2.4 hold. We have the following solutions to the discretionary stopping problem we have formulated as Cases I–VI.
Case I. Given any initial condition $x \in \mathcal{I}$, the value function v is given by $v(x) = 0$ and $\mathcal{C} = \mathcal{I}$. In this case, the stopping strategy $(\mathbb{S}_x^*, \infty) \in \mathcal{S}_x$ is optimal.
Case II. Given any initial condition $x \in \mathcal{I}$, then the value function v is given by $v(x) = g(x)$, $\mathcal{D} = \mathcal{I}$ and the stopping strategy $(\mathbb{S}_x^*, 0) \in \mathcal{S}_x$ is optimal.
Case III. Given any initial condition $x \in \mathcal{I}$, the value function v is given by

$$v(x) = \begin{cases} B\psi(x) & \text{if } x \in \mathcal{C} =]\alpha, x_\psi[, \\ g(x) & \text{if } x \in \mathcal{D} = [x_\psi, \beta[, \end{cases} \tag{54}$$

with $B = g(x_\psi)/\psi(x_\psi) \geqslant 0$. Furthermore, given any initial condition $x \in \mathcal{I}$, the stopping strategy $(\mathbb{S}_x^*, \tau^*) \in \mathcal{S}_x$, where $\mathbb{S}_x^*$ is a weak solution to (1) and

$$\tau^* = \inf\{t \geqslant 0 \mid X_t \in \mathcal{D}\}$$

is optimal.
Case IV. Given any initial condition $x \in \mathcal{I}$, the value function v is given by

$$v(x) = \begin{cases} g(x) & \text{if } x \in \mathcal{D} =]\alpha, x_\phi], \\ A\phi(x) & \text{if } x \in \mathcal{C} =]x_\phi, \beta[, \end{cases} \tag{55}$$

with $A = g(x_\phi)/\phi(x_\phi) \geqslant 0$. Furthermore, given any initial condition $x \in \mathcal{I}$, the stopping strategy $(\mathbb{S}_x^*, \tau^*) \in \mathcal{S}_x$, where $\mathbb{S}_x^*$ is a weak solution to (1) and

$$\tau^* = \inf\{t \geqslant 0 \mid X_t \in \mathcal{D}\}$$

is optimal.

Case V. Given any initial condition $x \in \mathcal{I}$, then $\mathcal{C} =]\alpha, x_\psi[\cup]x_\phi, \beta[$ and the value function v is given by

$$v(x) = \begin{cases} B\psi(x) & \text{if } x \in \mathcal{C}_1 =]\alpha, x_\psi[, \\ g(x) & \text{if } x \in \mathcal{D} = [x_\psi, x_\phi], \\ A\phi(x) & \text{if } x \in \mathcal{C}_2 =]x_\phi, \beta[, \end{cases} \tag{56}$$

with $B = g(x_\psi)/\psi(x_\psi) \geqslant 0$ and $A = g(x_\phi)/\phi(x_\phi) \geqslant 0$. Furthermore, given any initial condition $x \in \mathcal{I}$, the stopping strategy $(\mathbb{S}_x^*, \tau^*) \in \mathcal{S}_x$, where $\mathbb{S}_x^*$ is a weak solution to (1) and

$$\tau^* = \inf\{t \geqslant 0 \mid X_t \in \mathcal{D}\}$$

is optimal.

Case VI. If (34–37) hold, then there exists a unique pair $a \in]\alpha, x_l] \subset \mathcal{I} \setminus \mathcal{E}^s$ and $b \in [x_r, \beta[\subset \mathcal{I} \setminus \mathcal{E}^s$ such that (48) and (49) are true. In these circumstances, given any initial condition $x \in \mathcal{I}$, then $\mathcal{C} =]a, b[$ and the value function v is given by

$$v(x) = \begin{cases} g(x) & \text{if } x \in \mathcal{D}_1 =]\alpha, a], \\ A\phi(x) + B\psi(x) & \text{if } x \in \mathcal{C} =]a, b[, \\ g(x) & \text{if } x \in \mathcal{D}_2 = [b, \beta[, \end{cases} \tag{57}$$

with

$$A = \frac{g(b)\psi(a) - g(a)\psi(b)}{\phi(b)\psi(a) - \phi(a)\psi(b)}, \tag{58}$$

$$B = \frac{g(a)\phi(b) - g(b)\phi(a)}{\phi(b)\psi(a) - \phi(a)\psi(b)}. \tag{59}$$

Furthermore, given any initial condition $x \in \mathcal{I}$, the stopping strategy $(\mathbb{S}_x^*, \tau^*) \in \mathcal{S}_x$, where $\mathbb{S}_x^*$ is a weak solution to (1) and

$$\tau^* = \inf\{t \geqslant 0 \mid X_t \in \mathcal{D}_1 \cup \mathcal{D}_2\}$$

is optimal.

Proof of Case I. In view of (30), the function $v \equiv 0$ plainly satisfies the variational inequality (25). □

Proof of Case II. Since $\mathcal{C} = \emptyset$, (27) is true while (28) is satisfied because $\mathcal{L}g$ is negative for all $x \in \mathcal{I}$. Since $v(x) = g(x)$ for all $x \in \mathcal{I}$, (26) and (29) are satisfied. □

Proof of Case III. Firstly, (26) holds by (16) and since $\lim_{x \downarrow \alpha}(\psi(x)/\phi(x)) = 0$ by (9). Because $x_\psi \in [x_r, \beta[$ and $\mathcal{L}g < 0$ in this interval, (27) and (28) are true. Since x_ψ represents a strictly positive global maximum of g/ψ (see the discussion following the statement of Case III), we have that

$$g(x) \leqslant \frac{g(x_\psi)}{\psi(x_\psi)}\psi(x) = B\psi(x) \quad \text{for all } x \in]\alpha, x_\psi[$$

and (54) satisfies (29). □

Proof of Case IV. Firstly, (26) holds by (16) and since $\lim_{x\uparrow\beta}(\phi(x)/\psi(x))=0$ by (9). Because $x_\phi \in]\alpha, x_l[$ and $\mathcal{L}g<0$ in this interval, (27) and (28) are true. Since x_ϕ represents a strictly positive global maximum of g/ϕ we have that

$$g(x) \leqslant \frac{g(x_\phi)}{\psi(x_\phi)}\phi(x) = A\phi(x) \quad \text{for all } x \in]x_\phi, \beta, [$$

and (29) is satisfied. □

Proof of Case V. We can regard Case V as being composed of sub-problems (moving from α to β) Case III, Case II and then Case IV. The proof of this case is constructed, on the Bellman principle, by identifying the optimal solution to the sub-problems and 'pasting' these together at points in the stopping region.

We note that since $x_\phi, x_\psi \in \mathcal{E}^b$, and $\mathcal{L}g<0$ in this interval, (27) and (28) are true. As in Cases III and IV, $A, B>0$ and (56) satisfies (29) while (26) holds by (16) and (9). □

Proof of Case VI. We begin by noting that Lemma A.3 proves the existence of a unique pair $a \in]\alpha, x_l] \subset \mathcal{I} \setminus \mathcal{E}^s$ and $b \in [x_r, \beta[\subset \mathcal{I} \setminus \mathcal{E}^s$ such that (48) and (49) are true. To see that $A, B>0$ observe that, from (47–44)

$$A = -\int_\alpha^a \Psi(s)\mathcal{L}g(\mathrm{d}s) > 0,$$

$$B = -\int_b^\beta \Phi(s)\mathcal{L}g(\mathrm{d}s) > 0,$$

with the inequalities being a consequence of (35).

To see that (57) satisfies (29), recall that (48) and (49) imply that the points a, b define maximal turning points of the function

$$\frac{g(x)}{A\phi(x)+B\psi(x)} \quad \text{for } x \in \mathcal{I}$$

(see, for example, Beibel & Lerche, 2000; Lempa, 2010) and so

$$A\phi(x)+B\psi(x) \geqslant g(x) \quad \text{for all } x \in \mathcal{C}.$$

Also, for v given by (57), (26) holds by (16), (27) is true and, similarly while (28) is true since $a, b \in \mathcal{I} \setminus \mathcal{E}^s$. □

4. Three examples based on a geometric Brownian Motion

Now we present some concrete examples. In each case, X is a geometric Brownian motion such that

$$\mathrm{d}X_t = bX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t,$$

for constants b, σ and r is a constant and it is well known that in these cases

$$\phi(x) = x^m \quad \text{and} \quad \psi(x) = x^n,$$

where $m < 0 < n$ are given by

$$\left(\frac{1}{2} - \frac{b}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

4.1 *Some payoffs involving ϕ and ψ*

Consider a payoff function of the form

$$g(x) = \begin{cases} \psi(x) & \text{if } x < 1, \\ 1 & \text{if } x = 1, \\ \phi(x) & \text{if } x > 1. \end{cases}$$

This payoff satisfies Assumption 2.4 and the conditions of Case II and the value function is given by

$$v(x) = \begin{cases} \psi(x) & \text{if } x < 1, \\ 1 & \text{if } x = 1, \\ \phi(x) & \text{if } x > 1. \end{cases}$$

Now consider the payoff

$$g(x) = \min\{\psi(x), c\}$$

for $c \in \mathcal{I}$. This payoff also satisfies Assumption 2.4 and the conditions of Case II. The strategy will not be changed as c increases, and so by taking the limit as $c \uparrow \infty$ we can argue that the case $g(x) = \psi(x)$ conforms to Case II, despite the fact that the payoff does not satisfy (16). Similar arguments can be applied to the payoff

$$g(x) = \min\{\phi(x), c\},$$

inferring that $g(x) = \phi(x)$ also conforms to Case II.

Finally in this section consider the payoff

$$g(x) = \begin{cases} \min\{\phi(x), c\} & \text{if } x < 1, \\ 1 & \text{if } x = 1, \\ \min\{\psi(x), c\} & \text{if } x > 1. \end{cases} \quad \text{for } c \in]1, \beta[.$$

This payoff satisfies a the conditions of Case VI with

$$\mathcal{E}^s =]\phi^{-1}(c), \psi^{-1}(c)[$$

and the value function associated with the problem is given by

$$v(x) = \begin{cases} c & \text{if } x \in \mathcal{D}_1 =]\alpha, a], \\ A\phi(x) + B\psi(x) & \text{if } x \in \mathcal{C} =]a, b[, \\ c & \text{if } x \in \mathcal{D}_2 = [b, \beta[, \end{cases}$$

with $a=\phi^{-1}(c)$ and $b=\psi^{-1}(c)$ and

$$A=\frac{c(\psi(b)-\psi(a))}{c\psi(b)-\phi(b)\psi(a)}>0,$$
$$B=\frac{c(\phi(a)-\phi(b))}{c\phi(a)-\phi(b)\psi(a)}>0.$$

4.2 *Relaxing condition* (34)

Consider the case when $b=0$ and $\sigma=0.2$ and there is a constant discount rate of $r=0.01$. In this case, we have that

$$\phi(x)=x^{1/2-\sqrt{3/4}} \quad \text{and} \quad \psi(x)=x^{1/2+\sqrt{3/4}}.$$

If we have a relatively straightforward payoff function given by

$$g(x)=\max(5-x,1)$$

with

$$\mathcal{L}g(x)=\begin{cases}0.01(x-5), & x\in\,]\alpha,4[,\\ 0.32, & x=4.0,\\ -0.01, & x\in\,]4,\beta[,\end{cases}$$

we cannot define $\mathcal{E}^s:=]x_l,x_r[\subset\mathcal{I}$ such that (34) holds. However, we can relax (34) as identified in Remark 3.2.

On this basis, we can define $x_l=3.95$ and $x_r=4.05$ and calculate that

$$\int_{\mathcal{E}^s}\Psi(s)\mathcal{L}g(\mathrm{d}s)\approx 3.82, \quad \int_{\mathcal{E}^s}\Phi(s)\mathcal{L}g(\mathrm{d}s)\approx 0.346.$$

If we define $u^*=x_l$, we deduce that $v^*=166.97$, while if we set $v_*=x_r$, we deduce that $u_*=0.06$. We now need to check that these choices satisfy (A.7) and (A.8) of Lemma A.2 and calculate

$$q^c_\phi(u_*,v_*)=-23.54<0 \quad \text{while,}\ q^o_\phi(u^*,v^*)=0.157>0.$$

These results establish the existence and uniqueness of the points (a,b) appearing in (57).

It is relatively easy to approximate (a,b) numerically, deducing $a=1.34$, $b=88.6$ so that

$$v(x)=\begin{cases}5-x & \text{if } x\in\mathcal{D}_1=]\alpha,1.34],\\ 4.0731x^{1/2-\sqrt{3/4}}+0.0004612x^{1/2+\sqrt{3/4}} & \text{if } x\in\mathcal{C}=]1.34,88.6[,\\ 1 & \text{if } x\in\mathcal{D}_2=[88.6,\beta[.\end{cases}$$

4.3 *Two staircase payoffs*

Our third example involves two functions that do not satisfy Assumption 2.4, but never the less demonstrate the usefulness of considering complex stopping problems in terms of Cases I–VI. Consider the case when $b=0$ and $\sigma=0.2$ and there is a constant discount rate of $r=0.01$ and two ‘staircase’ type payoffs, as discussed in Bronstein *et al.* (2005),

$$g_1(x) = \begin{cases} 0 & \text{if } x < 2, \\ 1 & \text{if } 2 \leqslant x < 4, \\ 4 & \text{if } 4 \leqslant x < 6, \\ 9 & \text{if } 6 \leqslant x < 8, \\ 16 & \text{if } 8 \leqslant x < 10, \\ 25 & \text{if } 10 \leqslant x, \end{cases} \qquad g_2(x) = \begin{cases} 0 & \text{if } x < 2, \\ 2 & \text{if } 2 \leqslant x < 4, \\ 4 & \text{if } 4 \leqslant x < 6, \\ 6 & \text{if } 6 \leqslant x < 8, \\ 8 & \text{if } 8 \leqslant x < 10, \\ 10 & \text{if } 10 \leqslant x. \end{cases}$$

Since these functions are not continuous, they do not satisfy the conditions of Assumption 2.4 apart from (16). However, any points at which the payoff function is discontinuous will be part of the continuation region. On this basis, it is possible to construct a value function associated with these 'staircase' payoff functions that conform to Definition 2.3 by considering the sign of $\mathcal{L}g$ and stationary points of the functions g/ϕ and g/ϕ.

There are turning points of $g_{(1,2)}/\psi$ at $2, 4, 6, 8$ and 10, and we note that

$$0.3880 = \frac{g_1}{\psi}(2) < \frac{g_1}{\psi}(4) < \frac{g_1}{\psi}(6) < \frac{g_1}{\psi}(8) < \frac{g_1}{\psi}(10) = 1.1.0763,$$

while

$$0.7759 = \frac{g_2}{\psi}(2) > \frac{g_2}{\psi}(4) > \frac{g_2}{\psi}(6) > \frac{g_2}{\psi}(8) > \frac{g_2}{\psi}(10) = 0.4305.$$

These sequences mean that the solutions to the two problems are very different. With g_1 there is a global maximum turning point at $x = 10$, and we have Case III with

$$v_1(x) = \begin{cases} 1.1.0763 x^{1/2+\sqrt{3/4}} & \text{if } x \in \mathcal{C} =]\alpha, 10[, \\ 25 & \text{if } x \in \mathcal{D} = [10, \beta[, \end{cases}$$

and (26), (28) and (29) are satisfied.

Now, with g_2 we have the situation of a series of sub-intervals, as described in the proof of Case V. We have Case III in the interval $]0, 2]$, Case VI for the intervals $[2, 4], [4, 6], [6, 8], [8, 10]$ and Case II for $[10, \beta[$. For the four versions of Case VI, we employ the relaxation in Remark 3.2 and define each jump location as $j = 4, 6, 8, 10$. The right-hand boundary of the four intervals must be continuous fit at j, while, employing (41–43), the left-hand boundary will satisfy smooth fit (see also Bronstein *et al.*, 2005, Lemma 4) if

$$\frac{\psi'(j-2)}{\phi'(j-2)} \leqslant \frac{j\psi(j-2) - (j-2)\psi(j)}{j\phi(j-2) - (j-2)\phi(j)}.$$

If this condition is not satisfied, we will also have only continuous fit at the left-hand boundary, $(j-2)$. In the case under consideration, it can be deduced that $\mathcal{D} = \{\{2\}, \{4\}, \{6\}, \{8\}, \{10\}\}$ and it is easy to establish that

$$v_2(x) = \begin{cases} 0.7759x^{1/2+\sqrt{3/4}} & \text{if } x \in]\alpha, 2[, \\ 2 & \text{if } x = 2, \\ 0.8263x^{1/2-\sqrt{3/4}} + 0.5272x^{1/2+\sqrt{3/4}} & \text{if } x \in]2, 4[, \\ 4 & \text{if } x = 4, \\ 1.8162x^{1/2-\sqrt{3/4}} + 0.4375x^{1/2+\sqrt{3/4}} & \text{if } x \in]4, 6[, \\ 6 & \text{if } x = 6, \\ 2.9443x^{1/2-\sqrt{3/4}} + 0.3868x^{1/2+\sqrt{3/4}} & \text{if } x \in]6, 8[, \\ 8 & \text{if } x = 8, \\ 4.1899x^{1/2-\sqrt{3/4}} + 0.3528x^{1/2+\sqrt{3/4}} & \text{if } x \in]8, 10[, \\ 10 & \text{if } x \geqslant 10, \end{cases}$$

and (26), (28) and (29) are satisfied.

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Appendix. The existence and uniqueness of a, b appearing in (48) and (49)

We first establish the following preliminary result.

LEMMA A.1 If the problem data are such that the conditions of Case VI hold, then

$$q_\phi^c(\alpha,\beta) \equiv q_\phi^o(\alpha,\beta) = \lim_{u\downarrow\alpha} \int_{]u,\beta[} \Phi(s)\mathcal{L}g(\mathrm{d}s) < 0 \tag{A.1}$$

and

$$q_\psi^c(\alpha,\beta) \equiv q_\psi^o(\alpha,\beta) = \lim_{v\uparrow\beta} \int_{]\alpha,v[} \Psi(s)\mathcal{L}g(\mathrm{d}s) < 0. \tag{A.2}$$

Furthermore, if $q_\phi^o(\alpha,\beta) > -\infty$, then

$$q_\phi^c(\alpha,\beta) \equiv q_\phi^o(\alpha,\beta) = -\lim_{x\downarrow\alpha} \frac{g(x)}{\psi(x)} \in]-\infty, 0[, \tag{A.3}$$

while, if $q_\psi^o(\alpha,\beta) > -\infty$, then

$$q_\psi^c(\alpha,\beta) \equiv q_\psi^o(\alpha,\beta) = -\lim_{x\uparrow\beta} \frac{g(x)}{\phi(x)} \in]-\infty, 0[. \tag{A.4}$$

Proof. First we note that all the limits in (A.1) and (A.2) exist thanks to (35) and the monotone convergence theorem. Also we note that (36) and (37) imply that $g(x) > 0$ for all x sufficiently close to α. Combining this observation with (16), we can see that $g(x)/\phi(x)$ is decreasing to zero as x decreases to α. Therefore, (19), (20) and (23) imply that (A.1) is true. Similarly, we can see that (A.2) is also true.

The (ϕ,ψ)-integrability of $\mathcal{L}g$ (see Definition 2.1) implies that the inequality $q_\phi^o(\alpha,\beta) > -\infty$ is equivalent to the inequality

$$\int_{]\alpha,x[} \Phi(s)\mathcal{L}g(\mathrm{d}s) > -\infty \quad \text{for all } x > \alpha.$$

In this case, (35) implies that

$$-\infty < \int_{]\alpha,x[} \Phi(s)\mathcal{L}g(\mathrm{d}s) \leqslant 0 \quad \text{for all } x \in]\alpha, x_l].$$

It follows, from the definition (13) of Φ, Ψ, the fact that ϕ/ψ is a decreasing function, (35) and the dominated convergence theorem, that

$$0 = \lim_{x\downarrow\alpha} \int_{]\alpha,x[} \Phi(s)\mathcal{L}g(\mathrm{d}s) = \lim_{x\downarrow\alpha} \int_{]\alpha,x[} \frac{\phi(s)}{\psi(s)}\Psi(s)\mathcal{L}g(\mathrm{d}s)$$
$$\leqslant \lim_{x\downarrow\alpha} \frac{\phi(x)}{\psi(x)} \int_{]\alpha,x[} \Psi(s)\mathcal{L}g(\mathrm{d}s) \leqslant 0.$$

Combining the final limit with (17), we obtain (A.3).

We can derive (A.4) following similar reasoning. □

LEMMA A.2 Suppose that the problem data are such that the conditions of Case VI hold. Suppose also that there exist points

$$\alpha \leqslant u_* \leqslant u^* \leqslant x_l$$

and

$$x_r \leqslant v_* \leqslant v^* \leqslant \beta$$

such that

$$q^c_\psi(u_*, v_*) \leqslant 0 \leqslant q^o_\psi(u_*, v_*), \tag{A.5}$$

$$q^c_\psi(u^*, v^*) \leqslant 0 \leqslant q^o_\psi(u^*, v^*), \tag{A.6}$$

$$q^c_\phi(u_*, v_*) < 0 \tag{A.7}$$

and

$$q^o_\phi(u^*, v^*) > 0 \tag{A.8}$$

all hold. Then there exist $a \in]\alpha, u^*] \subset \mathcal{I} \setminus \mathcal{E}^s$ and $b \in [v_*, \beta[\subset \mathcal{I} \setminus \mathcal{E}^s$ satisfying the system of inequalities (48) and (49).

Proof. With each $u \in [u_*, u^*]$, we associate the set

$$\mathcal{S}_\psi(u) = \{v \in [v_*, v^*] \mid q^c_\psi(u, v) \leqslant 0\},$$

which is non-empty thanks to the inequalities

$$q^c_\psi(u, v^*) \leqslant q^c_\psi(u^*, v^*) \leqslant 0$$

that follow from (35) and (A.6). If we define

$$l(u) = \inf \mathcal{S}_\psi(u) \quad \text{for } u \in [u_*, u^*], \tag{A.9}$$

then we can see that (A.5) and (A.6) imply that

$$l(u_*) = v_* \quad \text{and} \quad l(u^*) \leqslant v^*. \tag{A.10}$$

Furthermore, l is increasing because the function $u \mapsto q^c_\psi(u, v)$ (respectively, $v \mapsto q^c_\psi(u, v)$) is increasing in $]\alpha, x_l]$ (respectively, decreasing in $[x_r, \beta[$) thanks to (35). In particular, the definition of l in (A.9) implies that

$$l(u_1) = l(u_2) \quad \text{for all } u_1 < u_2 \text{ such that } \mathcal{L}g([u_1, u_2[) = 0. \tag{A.11}$$

Given any $u \in [u_*, u^*]$ and any decreasing sequence (v_n) in $\mathcal{S}_\psi(u)$ such that

$$\lim_{n\to\infty} v_n = l(u),$$

we use (35) and the dominated convergence theorem to calculate

$$q^c_\psi(u, l(u)) = \lim_{n\to\infty} q^c_\psi(u, v_n) \leqslant 0. \tag{A.12}$$

If $l(u) = v_*$, then (35) and (A.5) imply that

$$q^o_\psi(u, l(u)) = q^o_\psi(u, v_*) \geqslant q^o_\psi(u_*, v_*) \geqslant 0. \tag{A.13}$$

On the other hand, if $l(u) > v_*$, then the definition of l and the monotone convergence theorem imply that

$$q^o_\psi(u, l(u)) \geqslant \int_{[u,l(u)[\cap\mathcal{I}} \Psi(s)\mathcal{L}g(\mathrm{d}s) = \lim_{v\uparrow l(u)} q^c_\psi(u, v) \geqslant 0. \tag{A.14}$$

For future reference, we also note that,

$$\text{if } l(u^*) = v^* = \beta, \quad \text{then } l(u_1) < \beta \text{ for all } u_1 < u^* \text{ such that } -\mathcal{L}g([u_1, u^*[) > 0. \tag{A.15}$$

To see this claim, we argue by contradiction. To this end, we consider any $u_1 < u^*$ such that $-\mathcal{L}g([u_1, u^*[) > 0$ and we assume $l(u_1) = \beta$. In this context, (35) implies that

$$q^c_\psi(u_1, l(u_1)) = q^c_\psi(u_1, \beta) < q^c_\psi(u^*, \beta) \leqslant 0.$$

In view of this inequality and the monotone convergence theorem, we can see that

$$0 > q^c_\psi(u_1, \beta) = \lim_{v\uparrow\beta} q^c_\psi(u_1, v).$$

It follows that there exists $v_1 < \beta$ such that $q^c_\psi(u_1, v) \leqslant 0$ for all $v \in [v_1, \beta[$, which combined with the definition of l, implies that $l(u_1) \leqslant v_1 < \beta$ and the contradiction has been established.

We now consider the set

$$\mathcal{S}_\phi = \{u \in [u_*, u^*] \mid q^c_\phi(u, l(u)) \leqslant 0\},$$

which is non-empty thanks to (A.7) and (A.9), and we define

$$a = \sup \mathcal{S}_\phi > \alpha. \tag{A.16}$$

If $u_* = \alpha$, then the inequality here is a consequence of the inequalities

$$\lim_{u\downarrow\alpha} q^c_\phi(u, l(u)) \leqslant \lim_{u\downarrow\alpha} q^c_\phi(u, v_*) = q^c_\phi(u_*, v_*) < 0,$$

which follows from (35), the monotone convergence theorem and (A.7).

We next define

$$b = l(a) < \beta.$$

If we establish this inequality as well as the inequalities

$$q^c_\phi(a, l(a)) \leqslant 0 \leqslant q^o_\phi(a, l(a)),$$

then the proof of existence will be complete thanks to (A.12–A.15). To this end, observe that since $a > \alpha$ (see (A.16)), we can use (35), the fact that l is increasing and the dominated convergence theorem to calculate

$$\begin{aligned} q^c_\phi(a, l(a)) &\leqslant \int_{[a, \lim_{\varepsilon\downarrow 0} l(a-\varepsilon)[} \Phi(s)\mathcal{L}g(\mathrm{d}s) \\ &= \lim_{\varepsilon\downarrow 0} q^c_\phi(a-\varepsilon, l(a-\varepsilon)) \leqslant 0. \end{aligned} \tag{A.17}$$

In particular, this inequality and (A.7) imply that $l(a) < \beta$. In the case that $a = u^*$, we can also see that

$$q^o_\phi(u^*, l(u^*)) \geqslant q^o_\phi(u^*, v^*) > 0 \tag{A.18}$$

thanks to (35), (A.8) and (A.10).

To proceed further, we note that the inequality $a < u^*$ can only be true if $\mathcal{L}g([a, u^*[) < 0$ because otherwise the inequalities

$$q^c_\phi(a, l(a)) = q^c_\phi(u^*, l(a)) \geqslant q^c_\phi(u^*, v^*) \geqslant q^o_\phi(u^*, v^*) > 0$$

would contradict (A.17). Therefore, if $a < u^*$, then the inequality $l(a) < \beta$ holds true thanks to (A.15). Finally, if $a < u^*$, then (35), the fact that l is increasing, the inequality $l(a) < \beta$, the dominated convergence theorem and the definition of a imply that

$$\begin{aligned} q^o_\phi(a, l(a)) &\geqslant q^c_\phi(a, \lim_{\varepsilon\downarrow 0} l(a+\varepsilon)) \\ &= \int_{]a, \lim_{\varepsilon\downarrow 0} l(a+\varepsilon)]} \Phi(s)\mathcal{L}g(\mathrm{d}s) \\ &= \lim_{\varepsilon\downarrow 0} q^c_\phi(a+\varepsilon, l(a+\varepsilon)) \geqslant 0 \end{aligned}$$

and the proof is complete. □

REMARK A.1 While a and b may not be unique, the definitions (A.16) and (A.9) mean that they are defined uniquely.

REMARK A.2 Many practical problems in finance and economics will need to be modelled by diffusions for which there are no analytic expressions for ϕ and ψ. If this is the case, then the conditions of

Lemma A.2 can be checked by noting that, as a consequence of (19–23) combined with (50–53),

$$q_\psi^c(u,v) = \frac{\psi^2}{\mathcal{W}}(v)\left(\frac{g}{\psi}\right)'_+(v) - \frac{\psi^2}{\mathcal{W}}(u)\left(\frac{g}{\psi}\right)'_-(u),$$

$$q_\psi^o(u,v) = \frac{\psi^2}{\mathcal{W}}(v)\left(\frac{g}{\psi}\right)'_-(v) - \frac{\psi^2}{\mathcal{W}}(u)\left(\frac{g}{\psi}\right)'_+(u),$$

$$q_\phi^c(u,v) = \frac{\phi^2}{\mathcal{W}}(v)\left(\frac{g}{\phi}\right)'_+(v) - \frac{\phi^2}{\mathcal{W}}(u)\left(\frac{g}{\phi}\right)'_-(u)$$

and

$$q_\phi^o(u,v) = \frac{\phi^2}{\mathcal{W}}(v)\left(\frac{g}{\phi}\right)'_-(v) - \frac{\phi^2}{\mathcal{W}}(u)\left(\frac{g}{\phi}\right)'_+(u).$$

The functions ϕ, ψ can be estimated over the interval of interest by applying Monte–Carlo simulation to (10). The accuracy of these estimates can then be measured by observing that we should have $\mathcal{L}\phi = \mathcal{L}\psi = 0$.

We are now in a position to give the main result in this Appendix.

LEMMA A.3 Suppose that the problem data are such that the conditions of Case VI, (34–37), hold. Then there exist $a \in \mathcal{I}\setminus\mathcal{E}^s$ and $b \in \mathcal{I}\setminus\mathcal{E}^s$ satisfying the system of equations (48) and (49).

Proof. In view of Lemma A.2, it suffices to find points $u_* \leqslant u^*$ in $]\alpha, x_l]$ and $v_* \leqslant v^*$ in $[x_r, \beta[$ satisfying (A.5–A.8). To this end, we consider four possible cases.
Case (a): If

$$q_\psi^o(\alpha, x_r) \equiv \int_{]\alpha,x_r[} \Psi(s)\mathcal{L}g(\mathrm{d}s) < 0 \quad \text{and} \quad q_\psi^o(x_l, \beta) \equiv \int_{]x_l,\beta[} \Psi(s)\mathcal{L}g(\mathrm{d}s) < 0, \tag{A.19}$$

then we define

$$u^* = x_l \quad \text{and} \quad v_* = x_r.$$

Combining the inequality

$$\int_{]x_l,x_r[} \Psi(s)\mathcal{L}g(\mathrm{d}s) > 0,$$

which follows from (34), with (35), (A.2) and the second inequality in (A.19) we can see that there exist unique $u_* \in]\alpha, x_l]$ and $v^* \in [x_r, \beta[$ such that

$$\int_{[u_*,x_r[} \Psi(s)\mathcal{L}g(\mathrm{d}s) \leqslant 0 \leqslant q_\psi^o(u_*, x_r)$$

and

$$\int_{]x_l,v^*]} \Psi(s)\mathcal{L}g(\mathrm{d}s) \leqslant 0 \leqslant q_\psi^o(x_l, v^*).$$

The inequalities (A.5) and (A.6) with $u_*, u^* = x_l, v_* = x_r$ and v^* hold true because $\mathcal{L}g(\{x_l\}) \leqslant 0$ and $\mathcal{L}g(\{x_r\}) \leqslant 0$.

To see the inequalities in (A.7) and (A.8), we recall that ϕ (respectively, ψ) is strictly decreasing (respectively, increasing), the definition (13) of Φ, Ψ and we note that (34) and (35) imply that

$$
\begin{aligned}
0 &\geqslant \frac{\phi(x_l)}{\psi(x_l)} \int_{[u_*,x_r[} \Psi(s)\mathcal{L}g(\mathrm{d}s) \\
&= \int_{[u_*,x_l]} \frac{\psi(s)\phi(x_l)}{\psi(x_l)\phi(s)} \Phi(s)\mathcal{L}g(\mathrm{d}s) + \int_{]x_l,x_r[} \frac{\psi(s)\phi(x_l)}{\psi(x_l)\phi(s)} \Phi(s)\mathcal{L}g(\mathrm{d}s) \\
&> \int_{[u_*,x_l]} \Phi(s)\mathcal{L}g(\mathrm{d}s) + \int_{]x_l,x_r[} \Phi(s)\mathcal{L}g(\mathrm{d}s) \\
&\geqslant q^c_\phi(u_*, v_*)
\end{aligned}
$$

and (A.7) is satisfied. Similarly, we can see that

$$
\begin{aligned}
0 &\leqslant \frac{\phi(x_l)}{\psi(x_l)} q^o_\psi(u^*, v^*) \\
&= \frac{\phi(x_l)}{\psi(x_l)} \int_{]x_l,v^*[} \Psi(s)\mathcal{L}g(\mathrm{d}s) \\
&= \int_{]x_l,x_{r_i}[} \frac{\psi(s)\phi(x_l)}{\psi(x_l)\phi(s)} \Phi(s)\mathcal{L}g(\mathrm{d}s) + \int_{[x_r,v^*[} \frac{\psi(s)\phi(x_l)}{\psi(x_l)\phi(s)} \Phi(s)\mathcal{L}g(\mathrm{d}s) \\
&< \int_{]x_l,x_{r_i}[} \Phi(s)\mathcal{L}g(\mathrm{d}s) + \int_{[x_r,v^*[} \Phi(s)\mathcal{L}g(\mathrm{d}s) \\
&= q^o_\phi(u^*, v^*)
\end{aligned}
$$

and (A.8) is satisfied.
Case (b): If

$$
q^o_\psi(\alpha, x_r) \equiv \int_{]\alpha,x_r[} \Psi(s)\mathcal{L}g(\mathrm{d}s) < 0 \quad \text{and} \quad q^o_\psi(x_l, \beta) \equiv \int_{]x_l,\beta[} \Psi(s)\mathcal{L}g(\mathrm{d}s) \geqslant 0, \tag{A.20}
$$

then we define

$$
v_* = x_r \quad \text{and} \quad v^* = \beta.
$$

We can show that there exists a point $u_* \in]\alpha, x_l]$ such that the inequalities (A.5) and (A.7) hold true in exactly the same way as in Case (a). Combining the second inequality in (A.20), the inequality

$$
q^o_\psi(x_l, x_r) = \int_{]x_l,x_r[} \Phi(s)\mathcal{L}g(\mathrm{d}s) > 0,
$$

which follows from (34), with (35) and (A.2) we can also see that there exists $u^* \in]u_*, x_l]$ such that

$$
q^c_\psi(u^*, \beta) \leqslant 0 \leqslant q^o_\psi(u^*, \beta).
$$

To show that the points u^* and $v^* = \beta$ are such that the inequality (A.8) is true, we first note that, without loss of generality, we may assume that the integral in (A.2) is not equal to $-\infty$. In this context, we combine (A.4) in Lemma A.1 with (17) to calculate that

$$\begin{aligned}\lim_{x\uparrow\beta}\frac{g(x)}{\phi(x)} &= -\int_{]\alpha,\beta[}\Psi(s)\mathcal{L}g(\mathrm{d}s)\\ &= -\int_{]\alpha,u^*[}\Psi(s)\mathcal{L}g(\mathrm{d}s) + \int_{[u^*,\beta[}\Psi(s)\mathcal{L}g(\mathrm{d}s)\\ &= -\int_{]\alpha,u^*[}\Psi(s)\mathcal{L}g(\mathrm{d}s) + q^c_\psi(u^*,v^*)\end{aligned}$$

implying that

$$\int_{]\alpha,u^*[}\Psi(s)\mathcal{L}g(\mathrm{d}s) = q^c_\psi(u^*,v^*) - \lim_{x\uparrow\beta}\frac{g(x)}{\phi(x)}.$$

Combining this result with (17), we obtain

$$\begin{aligned}-\frac{g(u^*)}{\phi(u^*)} &= \int_{]\alpha.u^*[}\Psi(s)\mathcal{L}g(\mathrm{d}s) + \frac{\psi(u^*)}{\phi(u^*)}\int_{[u^*,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s),\\ &= q^c_\psi(u^*,v^*) - \lim_{x\uparrow\beta}\frac{g(x)}{\phi(x)} + \frac{\psi(u^*)}{\phi(u^*)}\int_{[u^*,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s),\end{aligned}$$

yielding

$$\frac{\psi(u^*)}{\phi(u^*)}\int_{[u^*,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s) = \lim_{x\uparrow\beta}\frac{g(x)}{\phi(x)} - \frac{g(u^*)}{\phi(u^*)} - q^c_\psi(u^*,v^*) > 0,$$

with the inequality being a consequence of the second inequality in (36) and (A.6). Using (35), this result implies that

$$0 < \int_{[u^*,\beta[}\Phi(s)\mathcal{L}g(\mathrm{d}s) \leqslant q^o_\phi(u^*,\beta)$$

and the inequality (A.8) is true.
Case (c): If

$$q^o_\psi(\alpha,x_r) \equiv \int_{]\alpha,x_r[}\Psi(s)\mathcal{L}g(\mathrm{d}s) \geqslant 0 \quad \text{and} \quad q^o_\psi(x_l,\beta) \equiv \int_{]x_l,\beta[}\Psi(s)\mathcal{L}g(\mathrm{d}s) < 0, \tag{A.21}$$

then we define

$$u_* = \alpha \quad \text{and} \quad u^* = x_l.$$

We can show that there exists a point $v^* \in [x_r,\beta[$ such that the inequalities (A.6) and (A.8) hold true in exactly the same way as in Case (a). Combining the first inequality in (A.21), the inequality

$$q^o_\psi(x_l,x_r) = \int_{]x_l,x_r[}\Phi(s)\mathcal{L}g(\mathrm{d}s) > 0,$$

which follows from (34), with (35) and (A.2) we can also see that there exists $v_* \in]x_r, v^*[$ such that

$$q^c_\psi(\alpha,v_*) \leqslant 0 \leqslant q^o_\psi(\alpha,v_*).$$

To show that the points $u_* = \alpha$ and v_* are such that the inequality (A.7) is true, we first note that, without loss of generality, we may assume that the integral in (A.2) is not equal to $-\infty$. In this context, we combine (A.4) in Lemma A.1 with (17) to calculate that

$$-\frac{g(v_*)}{\psi(v_*)} = \frac{\phi(v_*)}{\psi(v_*)} \int_{]\alpha, v_*]} \Psi(s) \mathcal{L} g(\mathrm{d}s) + \int_{]v_*, \beta[} \Phi(s) \mathcal{L} g(\mathrm{d}s)$$

and so

$$\begin{aligned}
q_\phi^c(u_*, v_*) &= \int_{]\alpha, v_*]} \Phi(s) \mathcal{L} g(\mathrm{d}s) \\
&= \int_{]\alpha, \beta[} \Phi(s) \mathcal{L} g(\mathrm{d}s) - \int_{]v_*, \beta[} \Phi(s) \mathcal{L} g(\mathrm{d}s) \\
&= \frac{g(v_*)}{\psi(v_*)} + \frac{\phi(v_*)}{\psi(v_*)} \int_{]\alpha, v_*]} \Psi(s) \mathcal{L} g(\mathrm{d}s) - \frac{g}{\psi}(\alpha) \\
&\leqslant \frac{g(v_*)}{\psi(v_*)} - \frac{g}{\psi}(\alpha) \\
&< 0,
\end{aligned}$$

given that, by (A.5),

$$\int_{]\alpha, v_*]} \Psi(s) \mathcal{L} g(\mathrm{d}s) \leqslant 0$$

and employing the second inequality in (36).

Case (d): If

$$q_\psi^o(\alpha, x_r) \equiv \int_{]\alpha, x_r[} \Psi(s) \mathcal{L} g(\mathrm{d}s) \geqslant 0 \quad \text{and} \quad q_\psi^o(x_l, \beta) \equiv \int_{]x_l, \beta[} \Psi(s) \mathcal{L} g(\mathrm{d}s) \geqslant 0,$$

then we define

$$u_* = \alpha \quad \text{and} \quad v^* = \beta.$$

We can show the inequalities in (A.5) and (A.7) in exactly the same was as in Case (c) and the inequalities in (A.6) and (A.8) in exactly the same was as in Case (b). □