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Higher Poincaré lemma and integrability

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We prove the non-abelian Poincaré lemma in higher gauge theory in two different ways. That is, we show that every flat local connective structure is gauge trivial. The first method uses a result by Jacobowitz [J. Differ. Geom. **13**, 361 (1978)] which states solvability conditions for differential equations of a certain type. The second method extends a proof by Voronov [Proc. Am. Math. Soc. **140**, 2855 (2012)] and yields the explicit gauge parameters connecting a flat local connective structure to the trivial one. Finally, we show how higher flatness appears as a necessary integrability condition of a linear system which featured in recently developed twistor descriptions of higher gauge theories. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4929537]

I. INTRODUCTION AND RESULTS

Higher gauge theory[1–3] is an interesting generalization of ordinary gauge theory that describes consistently the parallel transport of extended objects. This requires the introduction of higher form potentials, and the usual no-go theorems concerning non-abelian higher form theories are circumvented by categorifying the mathematical structures underlying ordinary gauge theory.

The need to parallel transport extended objects arises, e.g., in string and M-theory, where point particles are replaced by one-, two-, and five-dimensional objects: the strings, the M2-, and M5-branes. In particular, there is a superconformal field theory in six dimensions which can be regarded as an effective description of stacks of multiple M5-branes.[4] Because the interactions of M5-branes are mediated by M2-branes ending on them in so-called self-dual strings, the theory should also capture the parallel transport of these strings. This fits the fact that in the abelian case corresponding to a single M5-brane, its field content comprises a 2-form potential. Altogether, it is therefore reasonable to expect that this theory—if it exists at the classical level—is a higher gauge theory.

Many approaches towards constructing this six-dimensional superconformal field theory have been followed. Within the framework of higher gauge theory, the twistor constructions of Refs. 5–7 seem particularly promising. Here, manifestly superconformal field equations are derived from a Penrose–Ward transform of holomorphic principal 2- and 3-bundles, which are holomorphic versions of non-abelian gerbes. There is in fact a one-to-one correspondence between gauge equivalence classes of solutions to the arising field equations and equivalence classes of the holomorphic principal 2- and 3-bundles. In proving this one-to-one correspondence, a higher Poincaré lemma enters, which says that flat connective structures are pure gauge. While it is unreasonable to assume that this statement is not true, we have not found it explicitly in the literature. In this paper, we provide two independent proofs of the higher Poincaré lemma, both for principal 2- and 3-bundles.

The difficulty in proving the higher Poincaré lemma is that one of the standard ways of showing the ordinary Poincaré lemma, the Frobenius theorem, cannot be readily extended beyond 1-form potentials. (Note that when speaking of the Poincaré lemma, we always refer to the statement that abelian or non-abelian flat connections are gauge equivalent to the trivial connection.) In particular,

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it does not seem to be clear what a higher generalization of the notion of foliation would be. We speculate about this in the Appendix, where we show how differential ideals are related to certain L_∞-structures on multivector fields, but the picture remains incomplete. Fortunately, the reformulation of the Frobenius theorem as an equation in differential forms has a generalization due to Jacobowitz.[8] This generalization is sufficient to establish a first proof of the higher Poincaré lemma for principal 2- and 3-bundles.

Another way of proving the Poincaré lemma has been recently followed by Voronov.[9] Here, the explicit gauge parameters connecting the flat connection to the trivial one are constructed from a Cauchy problem. We find nice generalizations of this proof to the case of principal 2- and 3-bundles. It should be noted that Voronov's proof holds for connections taking values in a Lie superalgebra or even in a $\mathbb{Z}$-graded Lie algebra, see also Ref. 10 in this context. As differential graded algebras can be regarded as duals to higher Lie or L_∞-algebras, Voronov's proof contains to some extent already a dual description of the Poincaré lemma for higher gauge theory based on Q-bundles and Q-groups. Our generalization of his proof, however, gives directly the picture in ordinary higher gauge theory.

Flat connections arise in twistor descriptions of gauge field equations as solutions to linear systems of the form $(\mathrm{d} + A)g = 0$, where g is a matrix group valued function and A is a matrix Lie-algebra valued one-form. This linear system directly implies that $A = \mathrm{d}gg^{-1}$ is pure gauge. Moreover, it can only have a solution if the curvature $F := \mathrm{d}A + \frac{1}{2}[A, A]$ vanishes. The Frobenius theorem or, equivalently, the Poincaré lemma then states that this condition is in fact sufficient for the existence of a solution. It is interesting to see if and how these statements generalize to the higher case. As we show, the higher analogue of having a matrix group for crossed modules of Lie groups is to have an underlying A_∞-algebra structure. If the products of this structure extend to the Lie groups, one can indeed write down a linear system containing a flat connective structure on a principal 2- or 3-bundle which implies that the connective structure is gauge equivalent to the trivial one and that the corresponding curvatures vanish. We expect that this observation has interesting applications in generalizing notions and structures from the theory of classical integrable systems to the higher setting.

This paper is organized as follows. In Section II, we review Jacobowitz's theorem and use it to give a first proof of the higher Poincaré lemma. In Section III, we show how Voronov's proof of the Poincaré lemma is extended to the higher situation. Finally, Section IV shows how higher flatness can be seen as a necessary integrability condition on a linear system. The Appendix contains some speculations relating L_∞-structures on multivector fields to differential ideals.

II. THE POINCARÉ LEMMA FOR HIGHER GAUGE THEORY

As the Poincaré lemma is a local statement, we shall be merely interested in the local description of higher gauge theories. That is, we consider local connective structures on principal n-bundles, which are encoded in certain differential forms on an open contractible patches of a smooth manifold. We ignore all issues related to patching these local objects to global ones.

The local description of higher gauge theory is readily derived, cf., e.g., Ref. 7. Consider the tensor product of the differential graded algebra of differential forms $\Omega^\bullet(U)$ on a patch U of a smooth manifold with a semistrict gauge Lie n-algebra in the form of an n-term strong homotopy Lie algebra. The result is another strong homotopy Lie algebra, whose Maurer–Cartan equations have solutions describing flat local connective structures on semistrict principal n-bundles over U. One can read off the definition of curvatures as well as the infinitesimal gauge transformations of the differential forms defining the connective structure. To derive the finite gauge transformations, however, one has to work a little harder.

We shall restrict our discussion to the case of principal 2- and 3-bundles with strict gauge 2- and 3-groups. It is hard to imagine that an analogous statement fails to hold in the semistrict case or for higher principal n-bundles, and a proof for these cases along similar lines to the ones below should exist. This, however, is not obvious. Moreover, one might want to use a different set up for such a proof, as, e.g., encoding higher gauge groups in simplicial manifolds.

A. A generalized Poincaré lemma

The usual Poincaré lemma states that the equation $\mathrm{d}\alpha = \beta$ involving some p- and $p+1$-forms α and β can be solved in an open, contractible region if and only if $\mathrm{d}\beta = 0$. In Ref. 8, Jacobowitz presented a generalization of this statement which we briefly review below. The precise definition of having local solutions is as follows.

Definition 2.1. We say that the equation $\mathrm{d}\omega = \Psi_{p+1}(x,\omega)$ *for a p-form* ω *is solvable in a region D, if for each* $x \in D$ *and for each* $\omega_0 \in \wedge^p T^*M|_x$*, there is an open neighborhood* $U_x \subset D$ *and an* $\omega \in \Omega^p(U_x)$ *such that* $\mathrm{d}\omega = \Psi_{p+1}(x,\omega)$ *and* $\omega|_x = \omega_0$.

The generalized Poincaré lemma reads then as follows.

Proposition 2.2. The equation $\mathrm{d}\omega = \Psi_{p+1}(x,\omega)$ *is solvable in a region D, if for all* $x \in D$ *there is a neighborhood* U_x *such that for all* $\omega_0 \in \Omega^p(U_x)$ *with* $\mathrm{d}\omega_0 = \Psi_{p+1}(\omega_0)$ *at x, we have* $\mathrm{d}\Psi_{p+1}(\omega_0) = 0$ *at x. This statement generalizes to systems of such equations with forms* ω *of varying degree.*

The proof found in Ref. 8 is a generalization of the usual proof of the Frobenius theorem.

Recall that the ordinary Frobenius theorem states that an involutive distribution $\mathscr{D}$ on a manifold M (i.e., a smoothly varying family of subspaces of the tangent bundle, on whose sections the Lie bracket of vector fields closes) corresponds to a regular foliation of M by submanifolds N. In modern language, the distribution is the annihilator of a differential ideal generated by 1-forms. Such a differential ideal comes with integral submanifolds. That is, for each point $p \in M$, we have an embedding $e : N_p \hookrightarrow M$ such that $p \in N_p$ and $e^*\alpha = 0$ for any form α in the differential ideal. These integral submanifolds correspond to the leaves of the foliation of M.

It does not seem to be completely clear how to generalize this picture to higher forms. The equation $\mathrm{d}\omega = \Psi_{p+1}(x,\omega)$ is certainly again encoded in a differential ideal which, however, is no longer generated exclusively by 1-forms. Such an ideal forms an exterior differential system, which admits integral submanifolds if and only if Cartan's test is passed, cf. Ref. 11. One issue with Cartan's test is that it does not work in the smooth, but only in the real analytic category. In the Appendix, we present some partial generalization of the notion of distribution, which amounts to a differential ideal. The conditions of Cartan's test, however, do not seem to have a clear interpretation in the context of generalized distributions.

B. Local flat connective structures on principal 2-bundles

A principal 2-bundle is essentially the non-abelian generalization of a gerbe, see Refs. 12–14. Connections on principal 2-bundles were discussed in detail in Ref. 1. Here, we will only need the local description over an open, contractible patch U of a smooth manifold M and the only non-trivial data will be the local connective structure over the patch U.

Principal 2-bundles come with a structure Lie 2-group. The most general Lie 2-groups are notoriously difficult to handle, and we therefore restrict our attention in this paper to strict such 2-groups. These are well-known to be equivalent to crossed modules of Lie groups, cf. Ref. 15.

Definition 2.3. A crossed module of Lie groups $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}, \rhd)$ *is a pair of Lie groups* G *and* H *together with a group homomorphism* $\mathsf{t} : \mathsf{H} \to \mathsf{G}$ *and an action by automorphism* $\rhd$ *of* G *on* H*. The group homomorphism and the action satisfy the following compatibility conditions for all* $g \in \mathsf{G}$ *and* $h, h_{1,2} \in \mathsf{H}$*:*

$$\mathsf{t}(g \rhd h) = g\mathsf{t}(h)g^{-1} \qquad \text{and} \qquad \mathsf{t}(h_1) \rhd h_2 = h_1 h_2 h_1^{-1}. \tag{1}$$

The first condition guarantees equivariance with respect to conjugation, while the second condition is the Peiffer identity.

Applying the tangent functor to a crossed module of Lie groups, we obtain the following.

Definition 2.4. A crossed module of Lie algebras $(\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g}, \rhd)$ *is a pair of Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *together with a Lie algebra homomorphism* $\mathsf{t}: \mathfrak{h} \to \mathfrak{g}$ *and an action by derivation* $\rhd$ *of* $\mathfrak{g}$ *on* $\mathfrak{h}$*. The compatibility conditions here read as*

$$\mathsf{t}(\gamma \rhd \chi) = [\gamma, \mathsf{t}(\chi)] \qquad \text{and} \qquad \mathsf{t}(\chi_1) \rhd \chi_2 = [\chi_1, \chi_2] \tag{2}$$

for all $\gamma \in \mathfrak{g}$ *and* $\chi, \chi_{1,2} \in \mathfrak{h}$.

The standard example of a crossed module of Lie groups is the automorphism 2-group $(\mathsf{G} \xrightarrow{\mathsf{t}} \mathsf{Aut}(\mathsf{G}), \rhd)$ of a Lie group G, where t is the embedding by the adjoint action and $\rhd$ is the automorphism action. Another example is the delooping $\mathsf{BU}(1) := (\mathsf{U}(1) \xrightarrow{\mathsf{t}} *, \rhd)$ of $\mathsf{U}(1)$, where $* = \{\mathbb{1}\}$ is the trivial group and t and $\rhd$ are trivial.

Instead of delving into the general definition of principal 2-bundles, we merely need the local description of their connective structures.

Definition 2.5. Given an open, contractible patch U of a smooth manifold M, a local connective structure over U of a principal 2-bundle with structure crossed module $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}, \rhd)$ *is given by a* $\mathsf{Lie}(\mathsf{G})$*-valued 1-form A together with a* $\mathsf{Lie}(\mathsf{H})$*-valued 2-form B over U. The corresponding curvatures read as*

$$\mathcal{F} := \mathrm{d}A + \tfrac{1}{2}[A, A] - \mathsf{t}(B) \qquad \text{and} \qquad H := \mathrm{d}B + A \rhd B. \tag{3}$$

An equivalence relation on local connective structures is given by gauge transformations, which are parameterized by a G*-valued function g together with a* $\mathsf{Lie}(\mathsf{H})$*-valued 1-form Λ as follows:*

$$\begin{aligned} A \mapsto \tilde{A} &:= g^{-1}Ag + g^{-1}\mathrm{d}g - \mathsf{t}(\Lambda), \\ B \mapsto \tilde{B} &:= g^{-1} \rhd B - \mathrm{d}\Lambda - \tilde{A} \rhd \Lambda - \tfrac{1}{2}[\Lambda, \Lambda], \\ \mathcal{F} \mapsto \tilde{\mathcal{F}} &:= g^{-1}\mathcal{F} g, \\ H \mapsto \tilde{H} &:= g^{-1} \rhd H - \tilde{\mathcal{F}} \rhd \Lambda. \end{aligned} \tag{4}$$

If a connective structure is to describe a consistent parallel transport of a 1-dimensional object along a surface, the curvature, also called "fake curvature," $\mathcal{F}$ has to vanish. Note that the equation $\mathcal{F} = 0$ is invariant under gauge transformations (4).

Definition 2.6. We call a local connective structure (A, B) flat, if $\mathcal{F} = 0$ and $H = 0$.

Note that a flat connective structure remains flat under gauge transformations (4).

We now have the following statement about flat connective structures.

Theorem 2.7. *For any flat local connective structure (A, B) on a patch U and any point $p \in U$, there is a neighborhood U_p of p such that (A, B) is pure gauge. That is, it can be written as*

$$\begin{aligned} A &= g^{-1}\mathrm{d}g - \mathsf{t}(\Lambda), \\ B &= -\mathrm{d}\Lambda - A \rhd \Lambda - \tfrac{1}{2}[\Lambda, \Lambda] \end{aligned} \tag{5}$$

for some G*-valued function g and* $\mathfrak{h}$*-valued 1-form Λ on $U_p \subset U$.*

Proof. For simplicity, we assume that G and H are matrix groups. The proof is, however, readily extended to the general case. We can rewrite Equation (5) as

$$\begin{aligned} \mathrm{d}g^{-1} &= -Ag^{-1} - \mathsf{t}(\Lambda)g^{-1} =: \Psi_1(g, \Lambda), \\ \mathrm{d}\Lambda &= -B - A \rhd \Lambda - \tfrac{1}{2}[\Lambda, \Lambda] =: \Psi_2(g, \Lambda). \end{aligned} \tag{6}$$

We regard (6) as a system of equations of the form $\mathrm{d}\omega = \Psi_{p+1}(\omega, x)$ with $\dim(\mathsf{G})$ 0-forms and $\dim(\mathsf{Lie}(\mathsf{H}))$ 1-forms. To apply Proposition 2.2, we merely have to show that $\mathrm{d}\Psi_1(g_0, \Lambda_0) = 0$ and $\mathrm{d}\Psi_2(g_0, \Lambda_0) = 0$ at any $x \in U$ if $\mathcal{F} = H = 0$ as well as $\mathrm{d}g_0^{-1} = \Psi_1(g_0, \Lambda_0)$ and $\mathrm{d}\Lambda_0 = \Psi_2(g_0, \Lambda_0)$ at x.

We compute

$$\begin{aligned} \mathrm{d}\Psi_1(g_0,\Lambda_0)|_x &= \big(-\mathrm{d}Ag_0^{-1} + A\wedge \mathrm{d}g_0^{-1} - \mathsf{t}(\mathrm{d}\Lambda_0)g_0^{-1} + \mathsf{t}(\Lambda_0)\wedge \mathrm{d}g_0^{-1}\big)\big|_x \\ &= \big(A\wedge Ag_0^{-1} - \mathsf{t}(B)g_0^{-1} + (A+\mathsf{t}(\Lambda_0))\wedge\Psi_1(g_0,\Lambda_0) - \mathsf{t}(\Psi_2(g_0,\Lambda_0))g_0^{-1}\big)\big|_x \\ &= 0 \end{aligned} \tag{7a}$$

and

$$\begin{aligned} \mathrm{d}\Psi_2(g_0,\Lambda_0)|_x &= (-\mathrm{d}B - \mathrm{d}A\rhd\Lambda_0 + A\rhd \mathrm{d}\Lambda_0 - [\mathrm{d}\Lambda_0,\Lambda_0])|_x \\ &= (A\rhd B + (A\wedge A - \mathsf{t}(B))\rhd\Lambda_0 + (A+\mathsf{t}(\Lambda_0))\rhd\Psi_2(g_0,\Lambda_0))|_x \\ &= 0. \end{aligned} \tag{7b}$$

Therefore by Proposition 2.2, Equation (6), and thus also (5), are solvable on U. According to Definition 2.1, this means that there is a solution in a neighborhood $U_p \subset U$ of each point $p \in U$. □

C. Local flat connective structures on principal 3-bundles

In this section, we extend the result of Sec. II B to local connective structures on principal 3-bundles. Principal 3-bundles are one step further in the categorification of principal bundles and form non-abelian generalizations of 2-gerbes. The full description of principal 3-bundles with connective structure is found in Ref. 6, see also Refs. 16 and 17 for partial earlier accounts.

Principal 3-bundles use Lie 3-groups as structure 3-groups, and we shall restrict ourselves to semistrict 3-groups for simplicity. Just as strict Lie 2-groups are categorically equivalent to crossed modules of Lie 2-groups, semistrict Lie 3-groups are equivalent to 2-crossed modules of Lie groups. We therefore start by recalling the latter notion.[18]

Definition 2.8. A 2-crossed module of Lie groups is a normal complex of Lie groups (i.e., a complex of Lie groups in which each image of t *is a normal subgroup of the next group)*

$$\mathsf{L} \xrightarrow{\ \mathsf{t}\ } \mathsf{H} \xrightarrow{\ \mathsf{t}\ } \mathsf{G}, \tag{8}$$

together with an action, $\rhd$, *of* G *on* H *and* L *by automorphism as well as a* G*-equivariant binary map* $\{\cdot,\cdot\}:\mathsf{H}\times\mathsf{H}\longrightarrow\mathsf{L}$ *satisfying the following conditions. For all* $h,h_{1,2,3}\in H$, $g\in\mathsf{G}$, *and* $\ell,\ell_{1,2}\in\mathsf{L}$, *we have*

(i) $\mathsf{t}(g\rhd\ell) = g\rhd\mathsf{t}(\ell)$ *and* $\mathsf{t}(g\rhd h) = g\mathsf{t}(h)g^{-1}$,
(ii) $\mathsf{t}(\{h_1,h_2\}) = \big(h_1h_2h_1^{-1}\big)\big(\mathsf{t}(h_1)\rhd h_2^{-1}\big)$,
(iii) $\{\mathsf{t}(\ell_1),\mathsf{t}(\ell_2)\} = \ell_1\ell_2\ell_1^{-1}\ell_2^{-1} := [\ell_1,\ell_2]$,
(iv) $\{h_1h_2,h_3\} = \{h_1,h_2h_3h_2^{-1}\}\,(\mathsf{t}(h_1)\rhd\{h_2,h_3\})$,
(v) $\{h_1,h_2h_3\} = \{h_1,h_2\}\{h_1,h_3\}\{\mathsf{t}(\{h_1,h_3\})^{-1},\mathsf{t}(h_1)\rhd h_2\}$,
(vi) $\{h,\mathsf{t}(\ell)\} = (\{\mathsf{t}(\ell),h\})^{-1}\,\ell\,\big(\mathsf{t}(h)\rhd\ell^{-1}\big)$.

The map $\{\cdot,\cdot\}$ is called the Peiffer lifting and measures the failure of $(\mathsf{H}\xrightarrow{t}\mathsf{G},\rhd)$ to be a crossed module. Sometimes we use $\mathsf{L}\rightarrow\mathsf{H}\rightarrow\mathsf{G}$ to denote 2-crossed modules. Lie 2-crossed modules are generalizations of Lie crossed modules. In particular, we can obtain Lie crossed modules from Lie 2-crossed modules by taking L to be the trivial Lie group. Moreover, the Lie 2-crossed module $(\mathsf{L}\xrightarrow{\mathsf{t}}\mathsf{H},\rhd)$ together with the induced action

$$h\rhd\ell := \ell\{\mathsf{t}(\ell)^{-1},h\} \tag{9}$$

for all $h\in\mathsf{H}$ and $\ell\in\mathsf{L}$ also forms a Lie crossed module.

Applying the tangent functor to normal sequence (8), we obtain the axioms for 2-crossed modules of Lie algebras.

Definition 2.9. Let $(\mathfrak{l},\mathfrak{h},\mathfrak{g})$ *be a triple of Lie algebras. A 2-crossed module of Lie algebras (or a differential Lie 2-crossed module) is a complex of Lie algebras (i.e., a complex in which the image of* *each term is an ideal of the next)*

$$\mathfrak{l} \xrightarrow{\mathsf{t}} \mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g}, \tag{10}$$

together with actions $\triangleright$ *of* $\mathfrak{g}$ *on* $\mathfrak{l}$ *and* $\mathfrak{h}$ *by derivation as well as a* $\mathfrak{g}$*-equivariant bilinear map,* $\{\cdot,\cdot\} : \mathfrak{h}\times\mathfrak{h} \longrightarrow \mathfrak{l}$ *satisfying the following conditions:*

(i) $\mathsf{t}(\gamma \triangleright \lambda) = \gamma \triangleright \mathsf{t}(\lambda)$ *and* $\mathsf{t}(\gamma \triangleright \chi) = [\gamma, \mathsf{t}(\chi)]$,
(ii) $\mathsf{t}(\{\chi_1,\chi_2\}) = [\chi_1,\chi_2] - \mathsf{t}(\chi_1) \triangleright \chi_2$,
(iii) $\{\mathsf{t}(\lambda_1), \mathsf{t}(\lambda_2)\} = [\lambda_1,\lambda_2]$,
(iv) $\{[\chi_1,\chi_2],\chi_3\} = \mathsf{t}(\chi_1) \triangleright \{\chi_2,\chi_3\} + \{\chi_1,[\chi_2,\chi_3]\} - \mathsf{t}(\chi_2) \triangleright \{\chi_1,\chi_3\} - \{\chi_2,[\chi_1,\chi_3]\}$,
(v) $\{\chi_1,[\chi_2,\chi_2]\} = \{\mathsf{t}(\{\chi_1,\chi_2\}),\chi_3\} - \{\mathsf{t}(\{\chi_1,\chi_3\}),\chi_2\}$,
(vi) $-\{\mathsf{t}(\lambda),\chi\} = \{\chi,\mathsf{t}(\lambda)\} + \mathsf{t}(\chi) \triangleright \lambda$

for every $\gamma \in \mathfrak{g}$, $\chi, \chi_{1,2,3} \in \mathfrak{h}$, *and* $\lambda, \lambda_{1,2} \in \mathfrak{l}$.

Note that a Lie 2-crossed module of Lie groups can be partially linearized to obtain more general actions, as, e.g., the action of G onto $\mathfrak{h}$. More details on 2-crossed modules can be found in Refs. 6 and 16.

The local description of a connective structure on a principal 3-bundle is now readily given, cf. Ref. 6.

Definition 2.10. Let U *be a contractible patch of a smooth manifold* M*. A local connective structure over* U *of a principal 3-bundle with structure 2-crossed module* $(\mathsf{L}\to\mathsf{H}\to\mathsf{G},\triangleright,\{\cdot,\cdot\})$ *can be expressed as a triple of Lie algebra valued forms* (A,B,C)*, where* $A \in \Omega^1(U,\mathsf{Lie}(\mathsf{G}))$, $B \in \Omega^2(U,\mathsf{Lie}(\mathsf{H}))$*, and* $C \in \Omega^3(U,\mathsf{Lie}(\mathsf{L}))$*. Corresponding curvatures are defined according to*

$$\mathcal{F} := \mathrm{d}A + \frac{1}{2}[A,A] - \mathsf{t}(B),\ \ \mathcal{H} := \mathrm{d}B + A \triangleright B - \mathsf{t}(C),\ \ G := \mathrm{d}C + A \triangleright C + \{B,B\}. \tag{11}$$

Gauge transformations act on the Lie algebra valued forms according to

$$\begin{aligned}
&A \mapsto \tilde{A} := g^{-1}Ag + g^{-1}\mathrm{d}g - \mathsf{t}(\Lambda),\\
&B \mapsto \tilde{B} := g^{-1} \triangleright B - (\mathrm{d} + \tilde{A}\,\triangleright)\Lambda - \tfrac{1}{2}\mathsf{t}(\Lambda) \triangleright \Lambda - \mathsf{t}(\Sigma),\\
&C \mapsto \tilde{C} := g^{-1} \triangleright C - \big((\mathrm{d} + \tilde{A}\,\triangleright) + \mathsf{t}(\Lambda)\,\triangleright\big)\Sigma + \{\tilde{B} + \tfrac{1}{2}(\mathrm{d} + \tilde{A}\,\triangleright)\Lambda + \tfrac{1}{2}[\Lambda,\Lambda],\ \Lambda\} +\\
&\qquad\qquad + \{\Lambda, \tilde{B} - \tfrac{1}{2}(\mathrm{d} + \tilde{A}\,\triangleright)\Lambda - \tfrac{1}{2}[\Lambda,\Lambda]\},\\
&\mathcal{F} \mapsto \tilde{\mathcal{F}} := g^{-1}\mathcal{F}g,\\
&\mathcal{H} \mapsto \tilde{\mathcal{H}} := g^{-1} \triangleright \mathcal{H} - \tilde{\mathcal{F}} \triangleright \Lambda,\\
&G \mapsto \tilde{G} := g^{-1} \triangleright G - \big(\tilde{\mathcal{F}} \triangleright (\Sigma - \tfrac{1}{2}\{\Lambda,\Lambda\})\big) + \{\Lambda, \tilde{\mathcal{H}}\} - \{\tilde{\mathcal{H}},\Lambda\} - \{\Lambda, \tilde{\mathcal{F}} \triangleright \Lambda\},
\end{aligned} \tag{12}$$

where g *is a* G*-valued function and* Λ *and* Σ *are* $\mathsf{Lie}(\mathsf{H})$ *and* $\mathsf{Lie}(\mathsf{L})$*-valued* 1*-and* 2*-forms, respectively.*

For consistency of the parallel transport described by this local connective structure, it is necessary that both the 2- and 3-form fake curvatures $\mathcal{F}$ and $\mathcal{H}$ vanish.

Definition 2.11. A local connective structure (A,B,C) *is said to be flat, if all curvatures vanish:* $\mathcal{F} = 0$, $\mathcal{H} = 0$*, and* $G = 0$.

Again, note that as in the case of principal 2-bundles, flat connective structures on principal 3-bundles remain flat under gauge transformations (12).

The Poincaré lemma here reads as follows.

Theorem 2.12. *For any flat local connective structure* (A,B,C) *on a patch* U *and any point* $p \in U$*, there is a neighborhood* $U_p \subset U$ *of* p *such that* (A,B,C) *is pure gauge. That is, it can be written as*

$$\begin{aligned}
A &= g^{-1}\mathrm{d}g - \mathsf{t}(\Lambda),\\
B &= -(\mathrm{d} + A \rhd)\Lambda - \tfrac{1}{2}\mathsf{t}(\Lambda) \rhd \Lambda - \mathsf{t}(\Sigma),\\
C &= -\big((\mathrm{d} + A \rhd) + \mathsf{t}(\Lambda) \rhd\big)\Sigma + \{B + \tfrac{1}{2}(\mathrm{d} + A \rhd)\Lambda + \tfrac{1}{2}[\Lambda,\Lambda],\ \Lambda\} +\\
&\qquad + \{\Lambda, B - \tfrac{1}{2}(\mathrm{d} + A \rhd)\Lambda - \tfrac{1}{2}[\Lambda,\Lambda]\}
\end{aligned} \tag{13}$$

for some G*-valued function* g*,* $\mathsf{Lie}(\mathsf{H})$*-valued 1-form* Λ*, and* $\mathsf{Lie}(\mathsf{L})$*-valued 2-form* Σ *on* U_p*.*

Proof. The proof is fully analogous to that of Theorem 2.7, but considerably more involved. We therefore only outline the computations. First, we rewrite (13) as follows:

$$\begin{aligned}
\mathrm{d}g^{-1} &= -Ag^{-1} - \mathsf{t}(\Lambda)g^{-1} =: \Psi_1(g,\Lambda,\Sigma),\\
\mathrm{d}\Lambda &= -B - A \rhd \Lambda - \tfrac{1}{2}\mathsf{t}(\Lambda) \rhd \Lambda - \mathsf{t}(\Sigma) =: \Psi_2(g,\Lambda,\Sigma),\\
\mathrm{d}\Sigma &= -C - A \rhd \Sigma - \mathsf{t}(\Lambda) \rhd \Sigma + \{B + \tfrac{1}{2}(\mathrm{d} + A \rhd)\Lambda + \tfrac{1}{2}[\Lambda,\Lambda],\ \Lambda\} +\\
&\qquad + \{\Lambda, B - \tfrac{1}{2}(\mathrm{d} + A \rhd)\Lambda - \tfrac{1}{2}[\Lambda,\Lambda]\} =: \Psi_3(g,\Lambda,\Sigma).
\end{aligned} \tag{14}$$

Proposition 2.2 guarantees that (14) are solvable on U, if $\mathrm{d}\Psi_{1,2,3}(g_0,\Lambda_0,\Sigma_0)$ vanish at any $x \in U$ if $\mathcal{F} = \mathcal{H} = G = 0$ as well as

$$\mathrm{d}g_0^{-1}|_x = \Psi_1(g_0,\Lambda_0,\Sigma_0)|_x, \quad \mathrm{d}\Lambda_0|_x = \Psi_2(g_0,\Lambda_0,\Sigma_0)|_x, \quad \mathrm{d}\Sigma_0|_x = \Psi_3(g_0,\Lambda_0,\Sigma_0)|_x. \tag{15}$$

We now have to rewrite $\mathrm{d}\Psi_{1,2,3}(g_0,\Lambda_0,\Sigma_0)$ in terms of quantities which we know at x. The exterior derivative will hit either a potential n-form or a gauge parameter. The exterior derivatives of the gauge parameters are given in (14) and the exterior derivatives of the potential n-forms can be rewritten using the flatness equations $\mathcal{F} = \mathcal{H} = G = 0$.

Putting everything together, we find after a lengthy calculation that given (15), $\mathrm{d}\Psi_{1,2,3}(g_0,\Lambda_0,\Sigma_0)$ indeed vanish for flat local connective structures. Again, solvability of (14) over U implies that for all $p \in U$ there exists a neighborhood U_p over which (14) have a solution. □

III. CONSTRUCTIVE PROOF OF THE POINCARÉ LEMMA

We come now to a constructive proof of the Poincaré lemma which yields the explicit gauge transformation trivializing a flat local connective structure. Our proof will be a direct generalization of that of Ref. 9, where the author constructs a solution to a Cauchy problem, relating pullbacks of flat connections along homotopic maps by gauge transformation. On a contractible patch of a smooth manifold, flat connections are therefore gauge equivalent to pullbacks along constant maps. This implies that flat connections are locally pure gauge.

A. Poincaré lemma on principal 2-bundles

Let U be an open, contractible patch of a smooth manifold M. Over $U \times [0,1]$, let $(\hat{A},\hat{B})$ be a local connective structure with underlying crossed module of Lie groups $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}, \rhd)$. Let $(\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g}, \rhd)$ denote the corresponding crossed module of Lie algebras. To simplify our notation, we assume that G and H are matrix groups. We decompose the differential forms $\hat{A}$ and $\hat{B}$ according to

$$\hat{A} = \hat{A}_x + \mathrm{d}t\,\hat{A}_t \qquad \text{and} \qquad \hat{B} = \hat{B}_x + \mathrm{d}t\,\hat{B}_t, \tag{16}$$

where $\frac{\partial}{\partial t} \lrcorner \hat{A}_x = 0$ and $\frac{\partial}{\partial t} \lrcorner \hat{B}_x = 0$. Similarly, we decompose the exterior derivative

$$\mathrm{d}\omega = \mathrm{d}_x\omega + \mathrm{d}t\,\frac{\partial}{\partial t}\omega = \mathrm{d}_x\omega + \mathrm{d}t\,\dot{\omega}. \tag{17}$$

We are interested in solutions $g \in C^\infty(U \times [0,1], \mathsf{G})$ and $\Lambda \in \Omega^1(U \times [0,1], \mathfrak{h})$ to the following Cauchy problem, which arises by considering gauge transformations of the components $\hat{A}_t$ and $\hat{B}_t$ to 0, cf. (4),

$$\dot{g} = -\hat{A}_t\, g + g\mathsf{t}(\Lambda_t) \qquad \text{and} \qquad \dot{\Lambda}_x = g^{-1} \rhd \hat{B}_t + \mathrm{d}_x\Lambda_t + \left(g^{-1}A_x g + g^{-1}\mathrm{d}_x g\right) \rhd \Lambda_t, \tag{18a}$$

with initial conditions

$$g(x,0) = \mathbb{1}_{\mathsf{G}} \qquad \text{and} \qquad \Lambda(x,0) = 0 \qquad \text{for} \qquad x \in U. \tag{18b}$$

Proposition 3.1. Let (g,Λ) be a solution to the Cauchy problem (18). Then

$$-g_1^{-1}\mathrm{d}g_1 + \int_0^1 \mathrm{d}t\, \frac{\partial}{\partial t} \lrcorner (g^{-1}\hat{\mathcal{F}} g) = g_1^{-1}\hat{A}_x|_{t=1} g_1 - \hat{A}_x|_{t=0} - \mathsf{t}(\Lambda_x)|_{t=1}, \tag{19}$$

where $g_1 \coloneqq g(x,1)$ and $\hat{\mathcal{F}}$ is the fake curvature of the local connective structure $(\hat{A},\hat{B})$.

Proof. First, using (18a), we readily compute

$$\frac{\partial}{\partial t}\left(g^{-1}\mathrm{d}_x g\right) = -g^{-1}(\mathrm{d}_x\hat{A}_t)g + \mathsf{t}\left(g^{-1}\mathrm{d}_x g \rhd \Lambda_t\right) + \mathrm{d}_x\mathsf{t}(\Lambda_t) \tag{20}$$

and

$$\frac{\partial}{\partial t}\left(g^{-1}\hat{A}_x g\right) = g^{-1}\left(\dot{\hat{A}}_x + [\hat{A}_t,\hat{A}_x]\right) g + \mathsf{t}(g^{-1}\hat{A}_x g \rhd \Lambda_t). \tag{21}$$

Moreover,

$$g_1^{-1}\mathrm{d}g_1 = (g^{-1}\mathrm{d}_x g)\big|_{t=1} \qquad \text{and} \qquad \mathrm{d}_x g|_{t=0} = 0. \tag{22}$$

We would now like to rewrite (20) and (21) in terms of the fake curvature of $(\hat{A},\hat{B})$. Note that

$$\begin{aligned}\int_0^1 \mathrm{d}t\, \frac{\partial}{\partial t}\lrcorner(g^{-1}\hat{\mathcal{F}}g) &= \int_0^1 \mathrm{d}t\; g^{-1}\Big(-\mathrm{d}_x\hat{A}_t + \dot{\hat{A}}_x + [\hat{A}_t,\hat{A}_x] - \mathsf{t}(\hat{B}_t)\Big)g \\ &= \int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\left(g^{-1}\mathrm{d}_x g\right) + \int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\left(g^{-1}\hat{A}_x g\right) + \\ &\quad - \int_0^1 \mathrm{d}t\;\mathsf{t}\Big(g^{-1} \rhd \hat{B}_t + \mathrm{d}_x\Lambda_t + \left(g^{-1}A_x g + g^{-1}\mathrm{d}_x g\right) \rhd \Lambda_t\Big).\end{aligned} \tag{23}$$

Using (18a) and (22), we can further simplify this to

$$\int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\lrcorner(g^{-1}\hat{\mathcal{F}}g) = g_1^{-1}\mathrm{d}g_1 + \int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}(g^{-1}\hat{A}_x g) - \int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\mathsf{t}(\Lambda_x), \tag{24}$$

which is obviously equivalent to (19). □

Next, we prove an analogous statement involving the 3-form curvature $\hat{H}$ of $(\hat{A},\hat{B})$.

Proposition 3.2. Let (g,Λ) be a solution to Cauchy problem (18). Then

$$\begin{aligned}&\mathrm{d}_x\Lambda_1 + g_1^{-1}\mathrm{d}g_1 \rhd \Lambda_x|_{t=1} - (\Lambda_x \wedge \Lambda_x)|_{t=1} + \left(g_1^{-1}\hat{A}_x|_{t=1}g_1\right) \rhd \Lambda_x|_{t=1} = \\ &\quad - \int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\lrcorner\left(g^{-1} \rhd \hat{H} - (g^{-1}\hat{\mathcal{F}}g) \rhd \Lambda\right) + g_1^{-1} \rhd \hat{B}_x|_{t=1} - \hat{B}_x|_{t=0},\end{aligned} \tag{25}$$

where $g_1 \coloneqq g(x,1)$, $\Lambda_1 = \Lambda(x,1)$, and $\hat{\mathcal{F}}$ and $\hat{H}$ are the fake and 3-form curvatures of the local connective structure $(\hat{A},\hat{B})$.

Proof. In this case we have

$$\mathrm{d}_x\Lambda_x|_{t=0} = 0 \qquad \text{and} \qquad \mathrm{d}_x\Lambda_1 = \mathrm{d}_x\Lambda_x|_{t=1}. \tag{26}$$

Moreover, by direct differentiation and using (18a), we obtain

$$\frac{\partial}{\partial t}\left(d_x\Lambda_x\right) = \mathrm{d}_x\left(g^{-1} \rhd \hat{B}_t + \mathrm{d}_x\Lambda_t + (g^{-1}\hat{A}_x g + g^{-1}\mathrm{d}_x g) \rhd \Lambda_t\right), \tag{27}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left(g^{-1}\mathrm{d}_x g \rhd \Lambda_x\right) &= \left(-g^{-1}(\mathrm{d}_x\hat{A}_t)g + \mathsf{t}(g^{-1}\mathrm{d}_x g \rhd \Lambda_t) + \mathrm{d}_x\mathsf{t}(\Lambda_t)\right) \rhd \Lambda_x + \\ &\quad + g^{-1}\mathrm{d}_x g \rhd \left(g^{-1} \rhd \hat{B}_t + \mathrm{d}_x\Lambda_t + \left(g^{-1}\hat{A}_x g + g^{-1}\mathrm{d}_x g\right) \rhd \Lambda_t\right).\end{aligned} \tag{28}$$

Thus, considering the expressions of the fake and the 3-curvatures of a local connective structure $(\hat{A},\hat{B})$ yields

$$
\begin{aligned}
\int_0^1 \mathrm{d}t\, \frac{\partial}{\partial t}\lrcorner\Big(-g^{-1}\hat{\mathcal{F}}g\rhd\Lambda+g^{-1}\rhd\hat{H}\Big) = \\
&\int_0^1 \mathrm{d}t\, \Big(g^{-1}\mathrm{d}_x\hat{A}_t g\rhd\Lambda_x - g^{-1}\Big(\dot{\hat{A}}_x+[\hat{A}_t,\hat{A}_x]\Big)g\rhd\Lambda_x\Big)+\\
&+\int_0^1 \mathrm{d}t\, \big(-g^{-1}(\mathrm{d}_x\hat{A}_x+\hat{A}_x\wedge\hat{A}_x)g\rhd\Lambda_t+g^{-1}\mathsf{t}(\hat{B}_x)g\rhd\Lambda_t\big)+\\
&+\int_0^1 \mathrm{d}t\, \Big(g^{-1}\mathsf{t}(\hat{B}_t)g\rhd\Lambda_x+g^{-1}\rhd\Big(\dot{\hat{B}}_x+\hat{A}_t\rhd\hat{B}_x-\mathrm{d}_x\hat{B}_t-\hat{A}_x\rhd\hat{B}_t\Big)\Big).
\end{aligned}
\tag{29}
$$

But by direct differentiation and using (18a), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big((g^{-1}\hat{A}_x g)\rhd\Lambda_x\Big) = &\Big(g^{-1}\Big(\dot{\hat{A}}_x+[\hat{A}_t,\hat{A}_x]\Big)g\Big)\rhd\Lambda_x+\mathsf{t}(g^{-1}\hat{A}_x g\rhd\Lambda_t)\rhd\Lambda_x+\\
&+(g^{-1}\hat{A}_x g)\rhd\big(g^{-1}\rhd\hat{B}_t+\mathrm{d}_x\Lambda_t+g^{-1}\hat{A}_x g\rhd\Lambda_t+g^{-1}\mathrm{d}_x g\rhd\Lambda_t\big)
\end{aligned}
\tag{30}
$$

and

$$
\frac{\partial}{\partial t}\big(g^{-1}\rhd\hat{B}_x\big) = \big(g^{-1}\hat{A}_t-\mathsf{t}(\Lambda_t)g^{-1}\big)\rhd\hat{B}_x+g^{-1}\rhd\dot{\hat{B}}_x. \tag{31}
$$

Now applying (18a), after combining (27), (28), (30), and (31), gives

$$
\begin{aligned}
\int_0^1 \mathrm{d}t\, \frac{\partial}{\partial t}\lrcorner\Big(-g^{-1}\hat{\mathcal{F}}g\rhd\Lambda+g^{-1}\rhd\hat{H}\Big) = \\
&\int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\big(-d_x\Lambda_x\big)+\int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\big(-g^{-1}\mathrm{d}_x g\rhd\Lambda_x\big)+\\
&+\int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\big(-(g^{-1}\hat{A}_x g)\rhd\Lambda_x\big)+\int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\big(g^{-1}\rhd\hat{B}_x\big)+\\
&+\int_0^1 \mathrm{d}t\,\big(\mathsf{t}(\dot{\Lambda}_x)\rhd\Lambda_x\big).
\end{aligned}
\tag{32}
$$

After simplification of (32) using (18b) and (26), we finally arrive at

$$
\begin{aligned}
\mathrm{d}_x\Lambda_1+g_1^{-1}\mathrm{d}g_1\rhd\Lambda_x|_{t=1}-(\Lambda_x\wedge\Lambda_x)|_{t=1}+\big(g_1^{-1}\hat{A}_x\big|_{t=1}g_1\big)\rhd\Lambda_x|_{t=1} = \\
-\int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\lrcorner\Big(-g^{-1}\hat{\mathcal{F}}g\rhd\Lambda+g^{-1}\rhd\hat{H}\Big)+g_1^{-1}\rhd\hat{B}_x|_{t=1}-\hat{B}_x|_{t=0}.
\end{aligned}
\tag{33}
$$

□

We can now follow[9] further and consider homotopic maps $h_{0,1}(x): U \rightrightarrows V$ between local patches U and V of some smooth manifolds. Let $h(x,t): U\times[0,1]\to V$ with $h(x,0)=h_0(x)$ and $h(x,1)=h_1(x)$ be a homotopy satisfying $\frac{\partial}{\partial t}h(x,t)|_{t=0,1}=0$. Because the pullback is compatible with the wedge product and the exterior derivative, Propositions 3.1 and 3.2 yield the following corollary.

Corollary 3.3. The pullbacks of a local connective structure (A,B) on the patch V of some manifold along homotopic maps $h_{0,1}: U\rightrightarrows V$ are related as follows:

$$
\begin{aligned}
-g_1^{-1}\mathrm{d}g_1+\mathsf{t}(\Lambda_{1,x})+\int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\lrcorner(g^{-1}h^*(\mathcal{F})g) &= g_1^{-1}h_1^*(A_x)g_1-h_0^*A_x,\\
\mathrm{d}_x\Lambda_1+(g_1^{-1}h_1^*(A_x)g_1)\rhd\Lambda_{1,x}+(g_1^{-1}\mathrm{d}g_1)\rhd\Lambda_{1,x}-(\Lambda_{1,x}\wedge\Lambda_{1,x}) &= \\
-\int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\lrcorner\big(g^{-1}\rhd h^*H-g^{-1}h^*(\mathcal{F})g\rhd\Lambda\big)&+g_1^{-1}\rhd h_1^*B_x-h_0^*B_x,
\end{aligned}
\tag{34}
$$

where h denotes a homotopy between h_0 and h_1 with $\frac{\partial}{\partial t}h(x,t)|_{t=0,1} = 0$, (g,Λ) is a solution of Cauchy problem (18) and $g_1 = g(x,1)$, $\Lambda_1 = \Lambda(x,1)$. In particular, the pullbacks for flat connective structures are gauge equivalent.

This corollary can now be used to prove the Poincaré lemma. Consider an open contractible patch U of a smooth manifold and regard it as a subset of some vector space $\mathbb{R}^d$ containing the origin 0_U. We are interested in the homotopy $h(x,t) : U \times [0,1] \to U$ with $h(x,t) = xtk(t)$ between U and the point $0_U \in U$, where $k(t)$ is a smooth function such that $k'(t)|_{t=0,1} = 0$, $k(0) = 0$ and $k(1) = 1$. Note that the pullback of the connective structure on U along h_0 vanishes, which implies the following theorem.

Theorem 3.4 (Higher Poincaré lemma). *Flat local connective structures are gauge equivalent to the trivial connective structure.*

B. Poincaré lemma on principal 3-bundles

An interesting aspect of our proof in Sec. III A was that it was not necessary to extend the interval $[0,1]$ used in the case of ordinary principal 2-bundles to $[0,1]^2$. The latter arises if one wants to define the general transport 2-functor from the path 2-groupoid to the delooping of the strict Lie 2-group corresponding to the crossed module $\mathsf{H} \to \mathsf{G}$, cf. Ref. 19.

Therefore, and since all the terms in the formulas contained in our proof have clear meanings, one can, in principle, readily generalize our proof to the case of local connective structures on principal 3-bundles. Let us here concisely summarize the steps.

We start from a local connective structure $(\hat{A},\hat{B},\hat{C})$ on $U \times [0,1]$, where U is a contractible patch of some smooth manifold. Let $\mathsf{L} \to \mathsf{H} \to \mathsf{G}$ be the relevant 2-crossed module and $\mathfrak{l} \to \mathfrak{h} \to \mathfrak{g}$ the corresponding linearization. The Cauchy problem is again given by equations stating that the components of the connective structures along $\mathrm{d}t$ can be gauged away. Here, we have

$$\begin{aligned} \dot{g} &= -\hat{A}_t g + g\mathsf{t}(\Lambda_t), \\ \dot{\Lambda}_x &= g^{-1} \triangleright \hat{B}_t + \mathrm{d}_x\Lambda_t + (g^{-1}A_x g + g^{-1}\mathrm{d}_x g) \triangleright \Lambda_t - \mathsf{t}(\Sigma_t), \\ \dot{\Sigma}_x &= g^{-1} \triangleright \hat{C}_t + \ldots, \end{aligned} \tag{35}$$

where $\ldots$ stands for terms easily read off from Equation (12). As their explicit forms are not illuminating, we suppress them here. This will then lead to statements analogous to Propositions 3.1 and 3.2, which are of the form

$$\int_0^1 \mathrm{d}t \frac{\partial}{\partial t} \lrcorner (\tilde{\hat{K}}) = \tilde{\hat{P}}|_{t=1} - \hat{P}|_{t=0}. \tag{36}$$

Here, $\hat{P}$ is the potential n-form for $n = 1,2,3$ and $\hat{K}$ is the corresponding curvature $n+1$-form. Furthermore, $\tilde{\hat{P}}$ and $\tilde{\hat{K}}$ denote gauge transformed objects.

These equations describe the relation between pullbacks of a local connective structure along homotopic maps. In particular, they imply that the pullbacks of flat local connective structures along homotopic maps are gauge equivalent. Considering again the homotopy $h(x,t) : U \times [0,1] \to U$ with $h(x,t) = xtk(t)$ implies that flat local connective structures are pure gauge.

IV. THE POINCARÉ LEMMA AND INTEGRABILITY

In the context of integrable systems, we often encounter linear systems of the form

$$\nabla g := (\mathrm{d} + A)g = 0, \tag{37}$$

where g is a G-valued function for some matrix Lie group G, d is a differential, and A is a $\mathsf{Lie}(\mathsf{G})$-valued 1-form. For example, in the Penrose–Ward transform,[20] d is a relative exterior derivative along a fibration and A is a relative differential 1-form. Acting with ∇ on (37), we obtain $\nabla(\nabla g) = 0$, which is equivalent to $Fg := (\nabla)^2 g = 0$. Note that the product in $Fg = 0$ is just an ordinary matrix product. Multiplying by g^{-1} from the left, we see that the existence of a solution g

requires that the curvature F of ∇ vanishes. Moreover, re-arranging Equation (37) directly yields the relation $A = g\mathrm{d}g^{-1}$, implying $F = 0$.

In this section, we demonstrate how these statements translate to linear systems involving connective structures on principal 2-bundles.

A. Underlying 2-term A_∞- and L_∞-algebras

To write down Equation (37), it is crucial to have a matrix Lie group such that the expressions $\mathrm{d}g$ and Ag make sense. Analogously, we consider a crossed module of matrix Lie groups $\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}$, which yields matrix products between elements of $\mathfrak{g} := \mathsf{Lie}(\mathsf{G})$ and G as well as $\mathfrak{h} := \mathsf{Lie}(\mathsf{H})$ and H. The Lie brackets are recovered by antisymmetrization of the matrix product. For our construction, we need a further product which turns into the action $\triangleright \colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ upon antisymmetrization. As we will explain now, the right context to look for such a product is an associative 2-term A_∞-algebra.

Recall that an associative 2-term A_∞-algebra is a graded vector space $\mathcal{A} := \mathcal{A}_{-1} \oplus \mathcal{A}_0 := \mathfrak{h} \oplus \mathfrak{g}$ together with "products" $m_1 : \mathcal{A} \to \mathcal{A}$ and $m_2 : \mathcal{A}^{\otimes 2} \to \mathcal{A}$ of degrees 1 and 0, respectively, such that

$$m_1 \circ m_1 = 0, \quad m_1 \circ m_2 = m_2 \circ (m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_1), \quad m_2 \circ (\mathbb{1} \otimes m_2 - m_2 \otimes \mathbb{1}) = 0. \tag{38}$$

The first equation says that m_1 is a differential, the second equation states the compatibility between this differential and the product m_2, and the third equation implies that the product m_2 is associative. We use the usual sign convention for the maps m_i,

$$(m_i \otimes m_j)(a_1 \otimes a_2) = (-1)^{\tilde{m}_j \tilde{a}_1} m_i(a_1) \otimes m_j(a_2), \tag{39}$$

where $\tilde{a}_1$ denotes the total parity of $a_1 \in \mathcal{A}^{\otimes i}$ and $\tilde{m}_j := 2 - j$.

If we antisymmetrize the products m_i to antisymmetric products μ_i, we obtain a 2-term L_∞-algebra, cf. Refs. 21–23. Associative 2-term L_∞-algebras, in turn, are equivalent to crossed modules of Lie algebras. More explicitly, the map t is identified with $m_1 = \mu_1$, the commutator on $\mathfrak{g} := \mathcal{A}_0$ is given by $\mu_2 : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the action of $\mathfrak{g}$ onto $\mathfrak{h} := \mathcal{A}_{-1}$ is given by $\mu_2 : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$, and the commutator on $\mathfrak{h}$ is given by $\mu_2 \circ (\mu_1 \otimes \mathbb{1})$. Altogether, we conclude that the higher analogue of demanding a matrix Lie algebra structure instead of merely a Lie algebra structure implies to ask for an A_∞-algebra underlying the L_∞-algebra corresponding to the crossed module of Lie algebras. Finally, we demand that the A_∞-product can be continued to a product between the A_∞-algebra and the crossed module of Lie groups $\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}$, such that we have products

$$m_2 : \mathcal{A}_0 \times \mathsf{G} \to \mathcal{A}_0, \quad m_2 : \mathcal{A}_0 \times \mathsf{H} \to \mathcal{A}_{-1} \qquad \text{and} \qquad m_2 : \mathcal{A}_{-1} \times \mathsf{G} \to \mathcal{A}_{-1}. \tag{40}$$

We now arrived at a complete higher analogue of having a matrix Lie group.

As a non-trivial example for such a structure, consider the crossed module of Lie groups $\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G} = \mathsf{GL}(n, \mathbb{C}) \xrightarrow{\mathrm{id}} \mathsf{GL}(n, \mathbb{C})$. The action $\triangleright$ is just the adjoint action, and we define

$$m_2(a,b) := \begin{cases} ab & a,b \in \mathsf{G} \cup \mathsf{Lie}(\mathsf{G}), \\ ab & a \in \mathsf{G} \cup \mathsf{Lie}(\mathsf{G}),\ \ b \in \mathsf{H} \cup \mathsf{Lie}(\mathsf{H}), \\ ab^{-1} & a \in \mathsf{Lie}(\mathsf{H}),\ \ b \in \mathsf{G}, \\ -ab & a \in \mathsf{Lie}(\mathsf{H}),\ \ b \in \mathsf{Lie}(\mathsf{G}). \end{cases} \tag{41}$$

It is not clear to us how to construct such an A_∞-algebra for an arbitrary crossed module of Lie algebras $\mathfrak{h} \to \mathfrak{g}$, but we strongly suspect that there is such a construction. Even if such a construction did not exist, we could impose a restriction to crossed modules admitting such a construction. This set is not empty, as the above example shows. Note that the A_∞-algebra resulting from the construction $\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0$ will contain the crossed module as a 2-term L_∞-subalgebra and therefore might be larger than $\mathfrak{g} \oplus \mathfrak{h}$.

To deal with connections and their curvatures, we have to allow for differential forms on some contractible region U taking values in the subspace $\mathfrak{h} \oplus \mathfrak{g}$ of the A_∞-algebra $\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0$. Recall that $\Omega^\bullet(U)$ is a differential graded algebra, and there is a natural tensor product between differential graded algebras and A_∞-algebras. This product yields an A_∞-algebra $\tilde{\mathcal{A}} := \Omega^\bullet(U) \otimes \mathcal{A}$ where the total degree of an element is the sum of the degree in $\mathcal{A}$ and its form degree. The products are given by

$$\tilde{m}_1(a) := \mathrm{d}a + (-1)^p m_1(a) \qquad \text{and} \qquad \tilde{m}_2 = m_2 \tag{42}$$

for $a \in \Omega^p(U) \otimes \mathcal{A}$. As a shortcut, we shall write $a * b := \tilde{m}_2(a,b)$. To rewrite these products in terms of the maps t and $\rhd$ of the crossed module which are independent of the form degree, we choose the convention of moving all form degrees to the left. Whenever two odd elements are moved past each other, a sign has to be inserted. For example, we have

$$\begin{aligned} A * B + B * A &:= \tilde{m}_2(A,B) + \tilde{m}_2(B,A) \\ &= \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\kappa (m_2(A_\mu, \tfrac{1}{2}B_{\nu\kappa}) - m_2(\tfrac{1}{2}B_{\nu\kappa}, A_\mu)) \\ &= \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\kappa (A_\mu \rhd \tfrac{1}{2}B_{\nu\kappa}) \\ &=: A \rhd B, \end{aligned} \tag{43}$$

where we used some coordinates (x^μ) on U to illustrate the issue.

B. Higher flatness as an integrability condition

The above constructions now suggest a higher generalization of the covariant derivative $\mathrm{d} + A$ to the operator

$$\nabla := \tilde{m}_1 + A * - B*, \tag{44}$$

where we inserted a sign for convenience. This operator will act on formal sums consisting of differential forms with values in G, $\mathfrak{g}$, and $\mathfrak{h}$. The detailed action is given in the following lemma.

Lemma 4.1. For $g \in \Omega^0(U) \otimes \mathsf{G}$, $X \in \Omega^p(U) \otimes \mathfrak{g}$, and $Y \in \Omega^q(U) \otimes \mathfrak{h}$, we have the following two equations:

$$\begin{aligned} \nabla(g + X + Y) &= \mathrm{d}g + \mathrm{d}X + \mathrm{d}Y + (-1)^q \mathsf{t}(Y) + Ag + AX + A * Y - B * g - B * X, \\ \nabla^2(g + X + Y) &= \mathcal{F}g + \mathcal{F}X + \mathcal{F} * Y - H * g - H * X. \end{aligned}$$

Proof. The first equation follows directly. To compute the second equation, recall that $\tilde{m}_1$ satisfies by definition a Leibniz rule $\tilde{m}_1(a * b) := \tilde{m}_1(a) * b + (-1)^{\tilde{a}} a * m_1(b)$. We then have

$$\begin{aligned} \nabla^2(g + X + Y) =&\ \nabla\big(\tilde{m}_1(g + X + Y) + A * (g + X + Y) - B * (g + X + Y)\big) \\ =&\ \tilde{m}_1\big(A * (g + X + Y) - B * (g + X + Y)\big) + \\ &+ A * \tilde{m}_1(g + X + Y) - B * \tilde{m}_1(g + X + Y) + \\ &+ A * A * (g + X + Y) - A * B * (g + X + Y) + \\ &- B * A(g + X) - B * A * Y - B * B * (g + X) \\ =&\ \tilde{m}_1(A) * (g + X + Y) - \tilde{m}_1(B) * (g + X + Y) + \\ &+ A * A * (g + X + Y) - A * B * (g + X) - B * A(g + X) \\ =&\ \mathcal{F}(g + X) + \mathcal{F} * Y - H * (g + X), \end{aligned} \tag{45}$$

as claimed. □

We have now everything at our disposal to consider the higher analogue of linear system (37) in the context of local connective structures on principal 2-bundles.

Theorem 4.2. *The equation*

$$\nabla(g - \Lambda * g) = 0 \quad \textit{for} \quad g \in \Omega^0(U) \otimes \mathsf{G},\ \Lambda \in \Omega^1(U) \otimes \mathfrak{h} \tag{46}$$

implies that the local connective structure (A, B) is pure gauge and that the curvature $\nabla^2 = (\mathcal{F} - H)$ vanishes.

Proof. Using Lemma 4.1 with $X = 0$ and $Y = -\Lambda * g$, we obtain

$$\nabla(g - \Lambda * g) = \mathrm{d}g - (\mathrm{d}\Lambda) * g - \Lambda * (\mathrm{d}g) + \mathsf{t}(\Lambda)g + Ag - A * \Lambda * g - B * g = 0. \tag{47}$$

We can split this equation by form degree into

$$\begin{aligned} 0 &= \mathrm{d}g + Ag + \mathsf{t}(\Lambda)g, \\ B * g &= (-\mathrm{d}\Lambda - A \rhd \Lambda) * g - \Lambda * (\mathrm{d}g + Ag). \end{aligned} \tag{48}$$

The first equation states that A is pure gauge. Note that $\Lambda * \mathsf{t}(\Lambda) = -\frac{1}{2}[\Lambda, \Lambda]$, which is due to

$$\begin{aligned} \Lambda * \mathsf{t}(\Lambda) &= \tfrac{1}{2}\big(\Lambda * \mathsf{t}(\Lambda) + \Lambda * \mathsf{t}(\Lambda) - \mathsf{t}(\Lambda * \Lambda)\big) \\ &= \tfrac{1}{2}\big(\Lambda * \mathsf{t}(\Lambda) - \mathsf{t}(\Lambda) * \Lambda\big) = -\tfrac{1}{2}\mathsf{t}(\Lambda) \rhd \Lambda \\ &= -\tfrac{1}{2}[\Lambda, \Lambda], \end{aligned} \tag{49}$$

where we use the fact that t is a derivation with respect to m_2 and the Peiffer identity. Using this identity together with $Y * g * g^{-1} = Y$, we can reformulate the second equation in (48) as

$$B = -\mathrm{d}\Lambda - A \rhd \Lambda - \tfrac{1}{2}[\Lambda, \Lambda], \tag{50}$$

and the total local connective structure (A, B) is pure gauge. A local connective structure which is pure gauge is clearly flat. Equivalently, Lemma 4.1 implies that $0 = \nabla^2(g - \Lambda * g) = \mathcal{F} g - \mathcal{F} * \Lambda * g + H * g$ and therefore leads to the same conclusion. □

Altogether, we saw how the usual solution and integrability condition for linear system (37) can be translated to categorified case (46) by means of an associative 2-term A_∞-algebra.

Finally, let us comment on the case of principal 3-bundles. Again, the extension of the discussion in Sec. IV A to the case of local connective structures on principal 3-bundles is more or less a mere technicality. One starts from an associative 3-term A_∞-algebra whose products extend to a 2-crossed module of matrix Lie groups. The covariant derivative is extended by adding a 3-form potential and the generalizations of linear system (46) are rather straightforward. The same holds for the derivation of the analogous statements to Theorem 4.2.

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APPENDIX: HIGHER DISTRIBUTIONS LEADING TO DIFFERENTIAL IDEALS

In this appendix, we briefly present a relation between certain higher distributions and differential ideals, generalizing the correspondence between ordinary involutive distributions and differential ideals generated by 1-forms. This is a first step towards a generalized Frobenius theorem.

Recall that a distribution $\mathscr{D}$ is a smoothly varying family of subspaces $\mathscr{D}_x$ of the fibers T_xM of the tangent bundle of some manifold M. It is involutive if the Lie algebra of vector fields closes on sections of $\mathscr{D}$. That is, for any point $p \in M$, there is a neighborhood U_p and vector fields $X_1, \ldots, X_r \in \mathfrak{X}(U_p)$ such that the X_i are linearly independent and at each point $x \in U_p$, $\mathscr{D}_x$ is spanned by the X_i. Extending these vector fields to a local basis $X_1, \ldots, X_d$ of TM, we have

$$[X_i, X_j] = f_{ij}^k X_k \qquad \text{with} \qquad f_{\bar{i}\bar{j}}^{\underline{k}} = 0, \tag{A1}$$

where the f_{ij}^k are functions on U_p and overlined and underlined indices $\bar{i}$ and $\underline{i}$ denote indices $i \leq r$ and $i > r$, respectively.

Recall that by the Frobenius theorem, such an involutive distribution induces a regular foliation of the manifold M.

The Lie algebra of vector fields in (A1) has a dual Chevalley–Eilenberg algebra, which is encoded in the relations

$$\mathrm{d}\theta^k = -\tfrac{1}{2} f_{ij}^k \theta^i \wedge \theta^j, \tag{A2}$$

where the 1-forms θ^i locally span T^*M and satisfy $\theta^i(X_j) = \delta^i_j$. Note that because of $f^{\underline{k}}_{\bar{i}\bar{j}} = 0$, the 1-forms $\theta^{\underline{i}}$ form a differential ideal.

This yields the modern formulation of the Frobenius theorem, which states that for a differential ideal on a manifold M which is generated by 1-forms, there are submanifolds $e : N_p \hookrightarrow M$ for each point $p \in M$ such that $p \in N_p$ and $e^*\alpha = 0$ for any α in the differential ideal.

Let us now generalize the correspondence between certain distribution and differential ideals. We start by recalling some basic facts on multivector fields.

Consider a patch U of a d-dimensional manifold M together with the set of multivector fields $\mathfrak{X}^\bullet(U) := \Gamma(TU) \oplus \Gamma(\wedge^2 TU) \oplus \cdots \oplus \Gamma(\wedge^d TU)$. On $\mathfrak{X}^\bullet(U)$, there is a natural generalization of the Lie bracket, which fulfills the Leibniz rule with respect to the $\wedge$-product.

Definition A.1. The Schouten–Nijenhuis bracket is the bilinear extension to $\mathfrak{X}^\bullet(U)$ of

$$\begin{aligned}&[V_1 \wedge \cdots \wedge V_m, W_1 \wedge \cdots \wedge W_n]_S := \\ &\qquad \sum_{i,j=1}^{m,n} (-1)^{i+j} [V_i, W_j] \wedge V_1 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge V_m \wedge W_1 \wedge \cdots \wedge \hat{W}_j \wedge \cdots \wedge W_n,\end{aligned} \tag{A3}$$

where V_i, $W_j \in \mathfrak{X}^1(U)$ and $\hat{}$ indicates an omission.

Note that the Schouten–Nijenhuis bracket turns the complex $\mathfrak{X}^\bullet(U)$ into a graded Lie algebra L_0. This graded Lie algebra has a dual Chevalley–Eilenberg algebra description in terms of forms in $\Omega^\bullet(U)$. Given a local basis $\theta^i, \xi^a, \ldots$ of linearly independent 1-forms, 2-forms, ..., spanning $T^*_x U$, $\wedge^2 T^*_x U, \ldots$ at every $x \in U$, we have

$$\mathrm{d}\theta^i = -\tfrac{1}{2} f^i_{jk} \theta^j \wedge \theta^k,\ \mathrm{d}\xi^a = -d^a_{ib} \theta^i \wedge \xi^b, \ldots, \tag{A4}$$

where the f^i_{jk} are the structure constants of the Lie algebra of vector fields and the additional structure constants d^a_{ib} are functions on U determined by the f^i_{jk}. As the θ^i, $\xi^a, \ldots$ form a complete basis, we can also write these relations as

$$\begin{aligned}\mathrm{d}\theta^i &= -\tfrac{1}{2} \tilde{f}^i_{jk} \theta^j \wedge \theta^k + \tilde{t}^i_a \xi^a, \\ \mathrm{d}\xi^a &= -\tilde{d}^a_{ib} \theta^i \wedge \xi^b - \tfrac{1}{3!} \tilde{c}^a_{ijk} \theta^i \wedge \theta^j \wedge \theta^k, \\ &\ldots\end{aligned} \tag{A5}$$

where $\xi^a = m^a_{ij} \theta^i \wedge \theta^j$ and

$$f^i_{jk} = \tilde{f}^i_{jk} + \tilde{t}^i_a m^a_{jk},\ d^a_{ib} m^b_{jk} = \tilde{d}^a_{ib} m^b_{jk} + \tilde{c}^a_{ijk}, \ldots. \tag{A6}$$

Equation (A5) describes the Chevalley–Eilenberg algebra of a strong homotopy Lie algebra, see Refs. 24 and 25 for a definition and more details.

Proposition A.2. The tilded structure constants in (A5) define a strong homotopy Lie algebra on the graded vector space of multivector fields $\mathfrak{X}^\bullet(U)$.

In particular, in terms of a basis $X_i \in \mathfrak{X}^1(U)$, $Y_a \in \mathfrak{X}^2(U), \ldots$ dual to that of $\Omega^\bullet(U)$ used above, we have the following higher brackets:

$$\begin{aligned}\mu_1(Y_a) &= \tilde{t}^i_a X_i, & \mu_2(X_i, X_j) &= \tilde{f}^k_{ij} X_k, \\ \mu_2(X_i, Y_a) &= \tilde{d}^b_{ia} Y_b, & \mu_3(X_i, X_j, X_k) &= \tilde{c}^a_{ijk} Y_a, \\ &\ldots\end{aligned} \tag{A7}$$

The two underlying Chevalley–Eilenberg complexes of the Lie algebra L_0 given by the Schouten–Nijenhuis bracket and any L_∞-algebra on $\mathfrak{X}^\bullet(U)$ given by a rewriting as in (A5) are essentially identical. Therefore, there is an L_∞-algebra isomorphism between these, which motivates the following definition.

Definition A.3. An L_∞-algebra associated to the Lie algebra L_0 is an L_∞-algebra-structure on $\mathfrak{X}^\bullet(U)$ with higher brackets as in (A7) obtained by a rewriting of the underlying Chevalley–Eilenberg algebra of L_0 as in (A5).

Finally, note that we can truncate the structures introduced above from $\mathfrak{X}^\bullet(U)$ to multivector fields of a maximal degree n. In particular, we can evidently truncate the Schouten–Nijenhuis bracket to the complex

$$\mathfrak{X}_{(n)}(U) = TU \longleftarrow \wedge^2 TU \longleftarrow \wedge^3 TU \longleftarrow \cdots \longleftarrow \wedge^n TU \tag{A8}$$

by setting

$$[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q] := 0 \tag{A9}$$

for $X_i, Y_i \in \mathfrak{X}^1(U)$ and $p+q > n+1$. The associated L_∞-algebras come then with higher brackets satisfying

$$\mu_k(X_1, \ldots, X_k) := 0 \tag{A10}$$

for homogeneously graded $X_i \in \mathfrak{X}^{|X_i|} \subset \mathfrak{X}_{(n)}(U)$ and $k > n+1$ or $|X_1| + \cdots |X_k| > n+1$.

We now come to a generalization of the notion of distribution based on multivector fields.

Definition A.4. An $n-$distribution on a d-dimensional manifold M with $n \leq d$ is a sequence of distributions $\mathscr{D} = (\mathscr{D}_1, \ldots, \mathscr{D}_n)$ such that $\mathscr{D}_i$ is a distribution in $\wedge^i TM$.

The notion of a pre-involutive distribution is now defined as follows.

Definition A.5. An n-distribution $\mathscr{D}$ on a manifold M is called pre-involutive, if there is an L_∞-algebra associated to L_0, which closes on $\mathscr{D}$.

In the case $n = 1$, the above two definitions trivially reduce to those of an ordinary distribution and an ordinary involutive distribution.

In the following, let again $X_i \in \mathfrak{X}^1(U)$, $Y_a \in \mathfrak{X}^2(U)$, …form a local basis spanning TU, $\wedge^2 TU, \ldots$ and let $X_i, i \leq r_1, Y_a, a \leq r_2, \ldots$ span a pre-involutive n-distribution $\mathscr{D} = (\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_n)$. We shall again underline indices larger than r_i and overline indices that are less or equal to r_i. Using this notation, we can characterize the structure constants of L_∞-algebras on pre-involutive n-distributions in more detail.

Lemma A.6. The closure of an L_∞-algebra associated to L_0 on a pre-involutive n-distribution is equivalent to its structure constants $s^\alpha_{\beta_1\cdots\beta_k} = (\tilde{t}^i_a, \tilde{f}^k_{ij}, \tilde{d}^b_{ia}, \tilde{c}^a_{ijk}, \ldots)$ satisfying

$$s^{\underline{\alpha}}_{\overline{\beta}_1\cdots\overline{\beta}_k} = 0. \tag{A11}$$

Let us now switch to the dual picture and consider the Chevalley–Eilenberg description of the above n-term L_∞-algebra. That is, we have a local basis of forms $\theta^i \in \Omega^1(U)$, $\xi^a \in \Omega^2(U)$, …with $\mathrm{i}_{X_i}\theta^j = \delta^j_i$, $\mathrm{i}_{Y_a}\xi^b = \delta^b_a$, etc. Closure of an associated L_∞-algebra on a pre-involutive n-distribution amounts here to the following.

Theorem A.7. *The forms $\theta^{\underline{i}}, \xi^{\underline{a}}, \ldots$ spanning the annihilators of the distributions contained in a pre-involutive n-distribution generate a differential ideal.*

Proof. The Chevalley–Eilenberg description of the L_∞-algebra associated to L_0 is of the form

$$\mathrm{d}\omega^\alpha = \sum_k s^\alpha_{\beta_1\cdots\beta_k}\omega^{\beta_1} \wedge \ldots \wedge \omega^{\beta_k} \tag{A12}$$

for general forms $\omega^\alpha \in \Omega^1(U) \oplus \cdots \oplus \Omega^n(U)$. With Lemma (A.6), we conclude that

$$-\mathrm{d}\omega^{\underline{\alpha}} = \sum_k s^{\underline{\alpha}}_{\underline{\beta}_1\beta_2\cdots\beta_k} \omega^{\underline{\beta}_1} \wedge \omega^{\beta_2} \wedge \ldots \wedge \omega^{\beta_k}, \tag{A13}$$

which states that the $\omega^{\underline{\alpha}}$ generate a differential ideal. □

Note that in the case $n = 1$, this is just the familiar statement that the annihilator of an integrable distribution spans a differential ideal.

[1] J. C. Baez and U. Schreiber, e-print arXiv:hep-th/0412325 [hep-th].
[2] H. Sati, U. Schreiber, and J. Stasheff, in *Quantum Field Theory*, edited by B. Fauser, J. Tolksdorf, and E. Zeidler (Birkhauser, 2009), p. 303; e-print arXiv:0801.3480 [math.DG].
[3] J. C. Baez and J. Huerta, Gen. Relativ. Gravitation **43**, 2335 (2011); e-print arXiv:1003.4485 [hep-th].
[4] E. Witten, in *Proceedings of Strings 95* (University of Southern California, 1995); e-print arXiv:hep-th/9507121.
[5] C. Saemann and M. Wolf, Commun. Math. Phys. **328**, 527 (2014); e-print arXiv:1205.3108 [hep-th].
[6] C. Saemann and M. Wolf, Lett. Math. Phys. **104**, 1147 (2014); e-print arXiv:1305.4870 [hep-th].
[7] B. Jurco, C. Saemann, and M. Wolf, J. High Energy Phys. **1504**, 087 (2015); e-print arXiv:1403.7185 [hep-th].
[8] H. Jacobowitz, J. Differ. Geom. **13**, 361 (1978).
[9] T. Voronov, Proc. Am. Math. Soc. **140**, 2855 (2012); e-print arXiv:0905.0287 [math.DG].
[10] K. Igusa, e-print arXiv:0912.0249 [math.AT].
[11] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, *Mathematical Sciences Research Institute Publications* (Springer-Verlag, New York, 1991), Vol. 18, p. viii+475.
[12] L. Breen and W. Messing, Adv. Math. **198**, 732 (2005); e-print arXiv:math.AG/0106083.
[13] P. Aschieri, L. Cantini, and B. Jurčo, Commun. Math. Phys. **254**, 367 (2005); e-print arXiv:hep-th/0312154.
[14] T. Bartels, "Higher gauge theory I: 2-Bundles," Ph.D. thesis, University of California, Riverside, CA, 2006; e-print arXiv: math.CT/0410328 [math.CT].
[15] J. C. Baez and A. D. Lauda, Theory Appl. Categories **12**, 423 (2004); e-print arXiv:math.QA/0307200 [math].
[16] J. F. Martins and R. Picken, Differ. Geom. Appl. **29**, 179 (2011); e-print arXiv:0907.2566 [math.CT].
[17] B. Jurčo, Int. J. Geom. Methods Mod. Phys. **08**, 49 (2011); e-print arXiv:0911.1552 [math.DG].
[18] D. Conduché, J. Pure Appl. Algebra **34**, 155 (1984).
[19] U. Schreiber and K. Waldorf, Homol., Homotopy Appl. **13**, 143 (2011); e-print arXiv:0802.0663 [math.DG].
[20] R. S. Ward, Phys. Lett. A **61**, 81 (1977).
[21] J. D. Stasheff, Trans. Am. Math. Soc. **108**, 275 (1963).
[22] J. D. Stasheff, Trans. Am. Math. Soc. **108**, 293 (1963).
[23] J. Stasheff, *Quantum Groups (Leningrad, 1990)*, Lecture Notes in Mathematics Vol. 1510 (Springer, Berlin, 1992), p. 120137.
[24] T. Lada and J. Stasheff, Int. J. Theor. Phys. **32**, 1087 (1993); e-print arXiv:hep-th/9209099.
[25] M. Markl, S. Shnider, and J. Stasheff, *Mathematical Surveys and Monographs* (American Mathematical Society, 2002).