# Integral and differential equations for the moments of multistate models in health insurance

**Franck Adékambi[1] & Marcus C. Christiansen[2,3]**

[1] Department of Statistics, University of Johannesburg, South Africa

[2] Maxwell Institute for Mathematical Sciences, Edinburgh, UK

[3] Department of Actuarial Mathematics and Statistics, Heriot-Watt University, Edinburgh, UK

*The final publication is available at Taylor & Francis via*

*http://www.tandfonline.com/doi/full/10.1080/03461238.2014.971860*

**Abstract**

The moments of the random future liabilities of health insurance policies are key quantities for studying distributional properties of the future liabilities. Assuming that the randomness of the future health status of individual policyholders can be described by a Semi-Markovian multistate model, integral and differential equations are derived for moments of any order and for the moment generating function. Different representations are derived and discussed with a view to numerical solution methods.

*Keywords*: multistate life insurance; Semi-Markov model; conditional moments; Thiele's equation; numerical solution.

## 1. Introduction

Long-term health insurance such as permanent health insurance and long-term care insurance is commonly calculated similar to life insurance, using Markovian and Semi-Markovian multistate models. The stochastic process describing the health status of the policyholder is called the ‘random pattern of states’ of the policyholder. If durational effects (impact of the time since entering the current state on future transitions) are negligible, the random pattern of states can usually be defined as a Markovian process. Still, for many types of health insurance contracts there are significant durational effects, see Segerer (1993), Casasnovas et al. (2012), and Christiansen et al. (2015). In this case the more general Semi-Markovian approach is needed, i.e. we assume that the random pattern of states is a bivariate process consisting of (1) the random pattern of states and (2) the duration process (giving the time elapsed since the last transition) is Markovian.

For the risk management of health insurance portfolios, the discounted sum of future liabilities is a key quantity. For the calculation of premiums, reserves, risk margins and solvency requirements, apart from first-order moments further distributional properties are of interest that describe the fluctuations around expected values. The first moment provides a measure of the central tendency of the future payments. Moments centered at the mean of order two, three and four give a good indication of the shape of the distribution of the future payments, describing the dispersion around the average, the asymmetry and the flattening of the distribution. Moreover, the moments can potentially be used to build approximations of the distribution, e.g. using the normal power approximation.

This paper derives integral equations and differential equations for the non-central conditional moments of the discounted sum of future liabilities of a health insurance policy in a Semi-Markovian framework.

The integral/differential equations for the first-order moment are the well-known Thiele equations. While Thiele introduced his equations for the Markovian framework, Hoem (1972) and Helwich (2008) generalized Thiele’s ideas to the more general Semi-Markovian framework. In the Markovian framework, integral and differential equations are also available for all higher order moments. The variance was obtained as a double integral in the multistate Markov model by Hoem (1969), see also Amsler (1968) and Norberg (1991). Norberg (1992) used Martingale techniques to express the variance as a single integral. Higher order conditional moments of present values of payments related to a life insurance policy are presented in Norberg (1995). In the Semi-Markovian case, Helwich (2008) presented integral equations for loss variances.

To the knowledge of the authors, integral or differential equations for third- and higher-order moments have not been derived for the Semi-Markovian framework so far. Here we fill that gap.We will start with equations for the moment generating function of the discounted sum of future liabilities, from which we then derive the moments. Apart from being useful for the calculation of moments, the moment generating function is in itself an interesting quantity as it allows to calculate the full probability distribution via inversion formulas.

The paper is organized as follows. Sections 2 and 3 explain the Semi-Markovian modeling of the random pattern of states and the modeling of the insurance payments. In Section 4 we derive integral and differential equations for the moment generating function. Sections 5 to 7 present integral equations of type 1, partial differential equations, integral equations of type 2, and ordinary differential equations for the conditional moments. Section 8 discusses numerical and analytical solution methods and presents a numerical example.

## 2. Semi-Markovian model for the health status

Throughout this section we follow the presentation and notation of Christiansen (2012). Let the random pattern of states of an individual policyholder be given by a pure jump process $\left(\Omega, F, P, (X_t)_{t\geq 0}\right)$ with finite state space $S$ and right continuous paths with left-hand limits, representing the state of the policy at time $t \geq 0$. We further define the transition space $J := \{(i,j) \in S \times S \mid i \neq j\}$, the counting processes

$$N_{jk}(t) := \#\left\{\tau \in (0,t] \middle| X_\tau = k, X_{\tau-} = j\right\},\ (j,k) \in J$$

the time of the next jump after $t$

$$T(t) := \inf\left\{\tau > t \middle| X_\tau \neq X_{\tau-}\right\}$$

($\inf \varnothing := \infty$), the series of the jump times

$$S_0 := 0,\ S_n := T(S_{n-1}),\ n \in \mathbb{N},$$

and a process that gives for each time $t$ the time elapsed since entering the current state,

$$U_t := \max\{\tau \in [0,t] \mid X_u = X_t \ \textit{for all}\ u \in [t-\tau, t]\},$$

also denoted as duration process. Instead of using a jump process $(X_t)_{t\geq 0}$, some authors describe the random pattern of states by a chain of jumps. The two concepts are equivalent.

We assume that the random pattern of states $(X_t)_{t\geq 0}$ is Semi-Markovian, i.e. the bivariate process $(X_t, U_t)_{t\geq 0}$ is a Markovian process, which means that for all $i \in S, u \geq 0$, and $t \geq t_n \geq \dots t_1 \geq 0,\ n \in \mathbb{N}$, we have

$$P\Big((X_t, U_t) = (i, u)\Big| X_{t_n}, U_{t_n}, \dots, X_{t_1}, U_{t_1}\Big) = P\Big((X_t, U_t) = (i, u)\Big| X_{t_n}, U_{t_n}\Big)$$

almost surely. In the following we always assume that the initial state $(X_0, U_0)$ is deterministic, i.e. we know the state of the policyholder when signing the contract. (Note that $U_0 = 0$ by definition). With this assumption and the Markov property for $(X_t, U_t)_{t\geq 0}$, the probability distribution of $(X_t, U_t)_{t \geq 0}$ is already uniquely defined by the transitions probability matrix

$$p(s,t,u,v) = \left(P\left(X_t = k, U_t \leq v \middle| X_s = j, U_s = u\right)\right)_{(j,k) \in S^2},$$

$0 \leq u \leq s \leq t < \infty$, $v \geq 0$. Alternatively, we can also uniquely define the probability distribution of $(X_t, U_t)_{t \geq 0}$ by specifying the probabilities

$$\begin{aligned}
&\overline{p}(s,t,u) = \left(\overline{p}_{jk}(s,t,u)\right)_{(j,k) \in S^2},\\
&\overline{p}_{jk}(s,t,u) := P\left(T(s) \leq t,\, X_{T(s)} = k \,\middle|\, X_s = j, U_s = u\right),\ j \neq k,\\
&\overline{p}_{jj}(s,t,u) := P\left(T(s) \leq t \,\middle|\, X_s = j, U_s = u\right).
\end{aligned}$$

A third way to uniquely define the probability distribution of $(X_t, U_t)_{t \geq 0}$ is to specify the cumulative transition intensity matrix

$$\begin{aligned}
&q(s,t) = \left(q_{jk}(s,t)\right)_{(j,k) \in S^2},\\
&q_{jk}(s,t) := \int_{(s,t]} \frac{\overline{p}_{jk}(s, d\tau, 0)}{1 - \overline{p}_{jj}(s, \tau-, 0)},\ 0 \leq s \leq t < \infty, j \neq k,\\
&q_{jj}(s,t) := -\sum_{k: k \neq j} q_{jk}(s,t).
\end{aligned}$$

If $q(s,t)$ is differentiable with respect to $t$, we can also define the transition intensity matrix

$$\mu_{jk}(t, t-s) := \frac{d}{dt} q(s,t) = \left( \frac{\frac{d}{dt}\overline{p}_{jk}(s,t,0)}{1 - \overline{p}_{jj}(s,t,0)} \right)_{(j,k) \in S \times S}.$$

The quantity $\mu_{jk}(t, t-s)$ gives the rate of transitions from state $j$ to state $k$ at time $t$ given that the current duration of stay in $j$ is $t - s$.

## 3. The health insurance contract

Payments between insurer and policyholder are of two types:

(a) The amount $b_{jk}(t,u)$ is payable if the policy jumps from state $j$ to state $k$ at time $t$ and the duration of stay in state $j$ was $u$.

(b) Annuity payments fall due during sojourns in a state and are defined by deterministic functions $B_j(s,t)$, $j \in S$. Given that the last transition occurred at time $s$, $B_j(s,t)$ is the total amount paid in $[s,t]$ during a sojourn in state $j$. We assume that the functions $B_j(s,\cdot)$ are right continuous and of bounded variation on compacts. In order to distinguish between payments from insurer to insured and vice versa, benefit payments get a positive sign and premium payments get a negative sign.

We assume that all contractual payments happen only on the time interval $[0,n]$, i.e. $n$ is the maximal duration of the contract.

By statute the insurer must at any time maintain a reserve in order to meet all future liabilities in respect of the contract. This reserve bears interest with some rate $\varphi(t)$. On the basis of this interest rate we define a discounting function,

$$v(s,t) := e^{-\int_s^t \varphi(r)dr}.$$

We can interpret $v(s,t)$ as the value at time $s$ of a unit payable at time $t \geq s$. Next, we study the present value of future payments between insurer and policyholder, that is, the discounted sum of all future benefit and premiums payments,

$$A(t) := \sum_{j\in S}\sum_{l=0}^{\infty} \int_{(t,n]} v(t,\tau) 1_{\{S_l \leq \tau < S_{l+1}\}} B_j(S_l, d\tau) + \sum_{(j,k)\in J} \int_{(t,n]} v(t,\tau) b_{jk}(\tau, U_\tau) dN_{jk}(\tau).$$

The quantity $A(t)$ is the random amount that an insurer would need at time $t$ in order to exactly meet all future obligations in respect of the contract. Since we assumed that there are no payments after time $n$, we have $A(t) = 0$ for $t > n$.

## 4. The conditional moment generating function

As mentioned in the introduction, our plan is to derive moments of $A(t)$ from the moment generating function. So let

$$M_j(t,\zeta,r) := E\left[e^{\zeta A(t)} \middle| X_t = j, U_t = r\right],$$

given that this conditional expectation exists. The following proposition characterizes $M_j(t,\zeta,r)$ by an integral equation system.

**Proposition 4.1**

*The conditional moment generating functions* $M_j(t,\zeta,r)$ *satisfy the integral equation system*

$$M_j(t,\zeta,r)=\left(1-\overline{p}_{jj}(t,n,r)\right)e^{\zeta\left(\int\limits_{(t,n]} v(t,s)B_j(t-r,ds)\right)}$$
$$+\sum_{k:k\neq j}\int\limits_{(t,n]} e^{\zeta\left(\int\limits_{(t,\tau]} v(t,s)B_j(t-r,ds)\,+\,v(t,\tau)b_{jk}(\tau,\tau-t+r)\right)} M_k(\tau,\zeta v(t,\tau),\,0)\,\overline{p}_{jk}(t,d\tau,r)$$

*for all* $j\in S, 0\le r\le t\le n$ *and for all* $\zeta$ *for which the conditional moment generating function exists.*

**Proof**

Since $T(t)$ is almost surely greater than $t$, we can rewrite $M_j(t,\zeta,r)$ to

$$M_j(t,\zeta,r)=E\left[e^{\zeta A(t)}\middle|X_t=j,U_t=r\right]$$
$$=\sum_{k\neq j}\int\limits_{(t,\infty]} E\left[e^{\zeta A(t)}\middle|X_t=j,U_t=r,X_{T(t)}=k,T(t)=\tau\right]\overline{p}_{jk}(t,d\tau,r).$$

Conditional on $\left\{X_t=j,U_t=r,X_{T(t)}=k,T(t)=\tau\right\}$ the discounted future payments $A(t)$ equal

$$A(t)=\int\limits_{(t,\tau]} v(t,s)B_j(t-r,ds)+\ v(t,\tau)b_{jk}(\tau,\tau-t+r)\ +v(t,\tau)\,A(\tau).$$

The first two of the three addends on the right hand side are deterministic, so we have

$$E\left[e^{\zeta A(t)}\middle|X_t=j,U_t=r,X_{T(t)}=k,T(t)=\tau\right]$$
$$=e^{\zeta\left(\int\limits_{(t,\tau]} v(t,s)B_j(t-r,ds)+\ v(t,\tau)b_{jk}(\tau,\tau-t+r)\right)}\ E\left[e^{\zeta v(t,\tau)A(\tau)}\middle|X_t=j,U_t=r,X_{T(t)}=k,T(t)=\tau\right].$$

The event $\left\{X_t=j,U_t=r,X_{T(t)}=k,T(t)=\tau\right\}$ can be equivalently expressed by $\left\{U_s=s-r,X_s=j,s\in[t-r,\tau),U_\tau=0,X_\tau=k\right\}$. Thus, using that $(X_t,\,U_t)_{t\ge 0}$ is a Markovian process, we get

$$E\left[e^{\zeta v(t,\tau)A(\tau)}\middle|X_t=j,U_t=r,X_{T(t)}=k,T(t)=\tau\right]=E\left[e^{\zeta v(t,\tau)A(\tau)}\,\middle|\,X_\tau=k,U_\tau=0\right]$$
$$=M_k\left(\tau,\zeta v(t,\tau),0\right)$$

almost everywhere. All in all, we obtain the equation

$$M_j(t,\zeta,r)=\sum_{k:k\neq j}\int\limits_{(t,\infty]} e^{\zeta\left(\int\limits_{(t,\tau]} v(t,s)B_j(t-r,ds)\ +\ v(t,\tau)b_{jk}(\tau,\tau-t+r)\right)} M_k(\tau,\zeta v(t,\tau),\,0)\,\overline{p}_{jk}(t,d\tau,r).$$

Since all payments are restricted to the time interval $[0,n]$, we have $A(\tau)=0$ and, thus, $M_j(\tau,\zeta,r)=1$ for all $\tau>n$. Furthermore, for $\tau>n$ the exponential function has the simpler form

$$e^{\zeta\left(\int\limits_{(t,\tau]} v(t,s)B_j(t-r,\,ds)\;+\;v(t,\tau)b_{jk}(\tau,\tau-t+r)\right)}=e^{\zeta\left(\int\limits_{(t,n]} v(t,s)B_j(t-r,ds)\right)}.$$

By splitting the integration range $(t,\infty)$ into $(n,\infty]$ and $(t,n]$ and using the above facts for $\tau>n$, we get

$$\begin{aligned}M_j(t,\zeta,r)&=e^{\zeta\left(\int\limits_{(t,n]} v(t,s)B_j(t-r,ds)\right)}\sum_{k:k\neq j}\int\limits_{(n,\infty]}\overline{p}_{jk}(t,d\tau,r)\\&+\sum_{k:k\neq j}\int\limits_{(t,n]} e^{\zeta\left(\int\limits_{(t,\tau]} v(\mathrm{t},s)B_j(t-r,ds)\;+\;v(t,\tau)b_{jk}(\tau,\tau-t+r)\right)}M_k(\tau,\zeta v(t,\tau),\,0)\,\overline{p}_{jk}(t,d\tau,r).\end{aligned}$$

Finally, using the fact that

$$\sum_{k:k\neq j}\int\limits_{(n,\infty]}\overline{p}_{jk}(t,d\tau,r)=1-\overline{p}_{jj}(t,n,r)\ ,$$

we arrive at the statement of the proposition.

□

A sufficient condition for the existence of the conditional moment generating function $M_j(t,\zeta,r)$ is boundedness of $A(t)$. Indeed, for insurance contracts from real life we can always find a finite bound for $A(t)$.

**Remark 4.2**

*According to Helwich (2008), for $j\neq k$ we have*

$$\overline{p}_{jk}(t,d\tau,r)=\left(1-\overline{p}_{jj}(t,\tau-,r)\right)\,q_{jk}(t-r,d\tau),$$

*and if the transition intensity matrix exists, we can also write*

$$\begin{aligned}\overline{p}_{jk}(t,d\tau,r)&=\left(1-\overline{p}_{jj}(t,\tau,r)\right)\mu_{jk}(\tau,\,\tau-t+r)\,d\tau\\&=e^{-\sum_{l:l\neq j}\int_t^\tau \mu_{jl}(u,\,u-t+r)du}\;\mu_{jk}(\tau,\,\tau-t+r)\,d\tau.\end{aligned}$$

Since $A(t)=0$ for all $t>n$ and, thus, $M_j(t,\zeta,r)=1$ for all $t>n$, we can solve the integral equation according to Proposition 4.1 backwards in time, starting from the terminal condition

$$M_j(n+,\zeta,r):=\lim_{\varepsilon\searrow 0} M_j(n+\varepsilon,\zeta,r)=1, \quad 0\le r\le n, \zeta\in\mathbb{R}.$$

However, solving the integral equation is not trivial even if we just aim for a numerical solution. As the function $M_k(\tau,\zeta v(t,\tau),0)$ on the right hand side of the integral equation in Proposition 4.1 does not depend on $r$, we can simplify the three-dimensional problem to a two-dimensional problem: First, calculate the functions $M_k(t,\zeta,0)$ by solving the integral equation just on the two-dimensional subspace with $r=0$, and, second, calculate $M_j(t,\zeta,r)$ on the basis of the functions $M_k(t,\zeta,0)$ by simple integration. Note that in the second step the integral equation becomes an explicit formula.

If we add extra assumptions on the differentiability of the payments functions and the transition intensities, we can also derive a partial differential equation.

**Proposition 4.3**

*Let the derivatives* $b_j(t,t-s):=\frac{\partial}{\partial t}B_j(s,t)$ *exist and let the functions* $b_j(t,r)$, $b_{jk}(t,r)$, $\mu_{jk}(t,r)$ *be continuously differentiable in their second argument. Then the functions* $M_j(t,\zeta,r)$ *satisfy the partial differential equation system*

$$\begin{aligned}\frac{\partial}{\partial t}M_j(t,\zeta,r) &= \zeta\varphi(t)\frac{\partial}{\partial\zeta}M_j(t,\zeta,r)-\frac{\partial}{\partial r}M_j(t,\zeta,r)-\zeta b_j(t,r)M_j(t,\zeta,r)\\ &\quad -\sum_{k:k\neq j}\left(e^{\zeta b_{jk}(t,r)}M_k(t,\zeta,0)-M_j(t,\zeta,r)\right)\mu_{jk}(t,r),\\ M_j(n,\zeta,r) &= 1\end{aligned}$$

*for all* $j\in S$, $0\le r\le t\le n$ *and for all* $\zeta$ *for which the conditional moment generating function exists.*

**Proof**

From Proposition 4.1 and Remark 4.2 we get

$$M_j(t,\zeta,r) = e^{-\sum\limits_{l:l\neq j}\int\limits_t^n \mu_{jl}(u,\,u-t+r)du}\, e^{\zeta\left(\int\limits_t^n v(t,s)b_j(s,s-t+r)ds\right)}$$
$$+\sum_{k:k\neq j}\int\limits_t^n e^{\zeta\left(\int\limits_t^\tau v(t,s)b_j(s,s-t+r)ds\,+\,v(t,\tau)b_{jk}(\tau,\tau-t+r)\right)} M_k(\tau,\zeta v(t,\tau),\,0)$$
$$\times e^{-\sum_{l:l\neq j}\int_t^\tau \mu_{jl}(u,\,u-t+r)du}\,\mu_{jk}(\tau,\,\tau-t+r)\,d\tau.$$

Differentiating on both sides with respect to $t$ yields

$$\frac{\partial}{\partial t}M_j(t,\zeta,r) = -\sum_{k:k\neq j} e^{\zeta b_{jk}(t,r)} M_k(t,\zeta,0)\,\mu_{jk}(t,r)$$
$$-\zeta b_j(t,r)M_j(t,\zeta,r)$$
$$+M_j(t,\zeta,r)\sum_{l:l\neq j}\mu_{jl}(t,r)$$
$$+\left.\frac{d}{dx}\right|_{x=t} e^{-\sum\limits_{l:l\neq j}\int\limits_t^n \mu_{jl}(u,\,u-t+r)du}\, e^{\zeta\left(\int\limits_t^n v(x,s)b_j(s,s-t+r)ds\right)}$$
$$+\left.\frac{d}{dx}\right|_{x=t}\sum_{k:k\neq j}\int\limits_t^n e^{\zeta\left(\int\limits_t^\tau v(x,s)b_j(s,s-t+r)ds\,+\,v(x,\tau)b_{jk}(\tau,\tau-t+r)\right)} M_k(\tau,\zeta v(x,\tau),\,0)$$
$$\times e^{\sum\limits_{l:l\neq j}\int\limits_t^\tau \mu_{jl}(u,\,u-t+r)du}\,\mu_{jk}(\tau,\,\tau-t+r)\,d\tau$$
$$+\left.\frac{d}{dx}\right|_{x=t} e^{-\sum\limits_{l:l\neq j}\int\limits_t^n \mu_{jl}(u,\,u-t+r)du}\, e^{\zeta\left(\int\limits_t^n v(t,s)b_j(s,s-x+r)ds\right)}$$
$$+\left.\frac{d}{dx}\right|_{x=t}\sum_{k:k\neq j}\int\limits_t^n e^{\zeta\left(\int\limits_t^\tau v(t,s)b_j(s,s-x+r)ds\,+\,v(t,\tau)b_{jk}(\tau,\tau-x+r)\right)} M_k(\tau,\zeta v(t,\tau),\,0)$$
$$\times e^{\sum\limits_{l:l\neq j}\int\limits_t^\tau \mu_{jl}(u,\,u-x+r)du}\,\mu_{jk}(\tau,\,\tau-x+r)\,d\tau.$$

Because the fourth and fifth added on the right hand side equal

$$\zeta\varphi(t)\frac{d}{d\zeta}e^{-\sum\limits_{l:l\neq j}\int\limits_t^n \mu_{jl}(u,\,u-t+r)du}\, e^{\zeta\left(\int\limits_t^n v(t,s)b_j(s,s-t+r)ds\right)}$$
$$+\zeta\varphi(t)\frac{d}{d\zeta}\sum_{k:k\neq j}\int\limits_t^n e^{\zeta\left(\int\limits_t^\tau v(t,s)b_j(s,s-t+r)ds\,+\,v(t,\tau)b_{jk}(\tau,\tau-t+r)\right)} M_k(\tau,\zeta v(t,\tau),\,0)\, e^{\sum\limits_{l:l\neq j}\int\limits_t^\tau \mu_{jl}(u,\,u-t+r)du}\,\mu_{jk}(\tau,\,\tau-t+r)\,d\tau$$
$$=\zeta\varphi(t)\frac{\partial}{\partial\zeta}M_j(t,\zeta,r)$$

and since the sixth and seventh term on the right hand side equal

$$-\frac{\partial}{\partial r}e^{-\sum_{l:l\neq j}\int_t^n \mu_{jl}(u,u-t+r)du}e^{\zeta\left(\int_t^n v(t,s)b_j(s,s-t+r)ds\right)}$$

$$-\frac{\partial}{\partial r}\sum_{k:k\neq j}\int_{(t,\infty)} e^{\zeta\left(\int_t^\tau v(t,s)b_j(s,s-t+r)ds\,+\,v(t,\tau)b_{jk}(\tau,\tau-t+r)\right)}M_k\left(\tau,\zeta v(t,\tau),0\right)e^{\sum_{l:l\neq j}\int_t^\tau \mu_{jl}(u,u-t+r)du}\mu_{jk}\left(\tau,\tau-t+r\right)d\tau$$

$$=-\frac{\partial}{\partial r}M_j\left(t,\zeta,r\right),$$

we obtain the statement of the proposition. The terminal condition follows from the fact that $A(t)=0, t>n$ and, thus, $M_j\left(n+,\zeta,r\right)=M_j\left(n,\zeta,r\right)=1$.

□

## 5. Integral equations of type 1 for the conditional moments

On the basis of the integral equations for the conditional moment generating function of the random variable $A(t)$, we can derive integral equations for the conditional moments of $A(t)$, defined as

$$V_j^m\left(t,\,r\right):=E\left[A(t)^m\,|X_t=j,U_t=r\right].$$

**Proposition 5.1**

*Given that the conditional moment generating function exists in a neighborhood of $\zeta=0$, the conditional moment functions $V_j^m\left(t,\,r\right)$ satisfy the integral equation system*

$$V_j^m\left(t,r\right)=\left(1-\overline{p}_{jj}\left(t,n,r\right)\right)\left(\int_{(t,n]} v(t,s)\,B_j\left(t-r,ds\right)\right)^m$$

$$+\sum_{k:k\neq j}\sum_{l=0}^{m}\binom{m}{l}\int_{(t,n]}\left(\int_{(t,\tau]} v(t,\,s)B_j\left(t-r,ds\right)+v(t,\,\tau)b_{jk}\left(\tau,\tau-t+r\right)\right)^{m-l}$$
$$\times\, v^l\left(t,\,\tau\right)V_k^l\left(\tau,0\right)\overline{p}_{jk}\left(t,d\tau,r\right).$$

*for all* $j\in S, 0\le r\le t\le n$, $m\in\mathbb{N}$.

**Proof**

The conditional moment $V_j^m\left(t,\,r\right)$ of $m$-th order can be obtained by evaluating the $m$-th order derivative of the conditional moment generating function at $\zeta=0$. By applying Proposition 4.1, we get

$$\frac{\partial^m}{\partial \zeta^m} M_j(t,\zeta,r) = \sum_{k:k\neq j} \sum_{l=0}^{m} \int_{(t,n]} \binom{m}{l} \left( \int_{(t,\tau]} v(t,s) B_j(t-r,ds) + v(t,\tau) b_{jk}(\tau,\tau-t+r) \right)^{m-l} v(t,\tau)^l$$

$$\times \frac{\partial^l}{\partial \zeta^l} M_k(\tau,\ \zeta v(t,\tau),\ 0)\ e^{\zeta\left( \int_{(t,\tau]} v(t,s) B_j(t-r,ds) + v(t,\tau) b_{jk}(\tau,\tau-t+r) \right)} \overline{p}_{jk}(t,d\tau,r).$$

Evaluating this equation at $\zeta = 0$ and using

$$\frac{\partial^l}{\partial \zeta^l} M_k(\tau,0,0) = E\left[ A(\tau)^l \,|\, X_\tau = k, U_\tau = 0 \right] = V_k^l(\tau,\ 0)$$

leads us to the statement of the proposition.

□

**Remark 5.2**

*We show here that Proposition 5.1 is a generalization of Thiele's integral equation of type 1. With the notation* $V_j^1(t,\ r) = V_j(t,\ r)$, *from Proposition 5.1 we get for* $m = 1$ *the integral equation system*

$$\begin{aligned} V_j(t,r) &= \left(1 - \overline{p}_{jj}(t,n,r)\right) \int_{(t,n]} v(t,s) B_j(t-r,ds) \\ &+ \sum_{k:k\neq j} \int_{(t,n]} v(t,\tau) V_k(\tau,\ 0)\, \overline{p}_{jk}(t,d\tau,r) \\ &+ \sum_{k:k\neq j} \int_{(t,n]} \left( \int_{(t,\tau]} v(t,s) B_j(t-r,ds) + v(t,\ \tau) b_{jk}(\tau,\tau-t+r) \right) \overline{p}_{jk}(t,d\tau,r),\ j \in S. \end{aligned}$$

*Using Fubini's Theorem and rearranging terms, we obtain*

$$\begin{aligned} V_j(t,r) &= \int_{(t,n]} v(t,\tau)\left(1 - \overline{p}_{jj}(t,\tau,r)\right) B_j(t-r,d\tau) \\ &+ \sum_{k:k\neq j} \int_{(t,n]} v(t,\tau)\left(V_k(\tau,0) + b_{jk}(\tau,\tau-t+r)\right) \overline{p}_{jk}(t,d\tau,r),\ \ j \in S, \end{aligned}$$

*which is exactly Thiele's integral equation of type 1, cf. Helwich (2008) and Christiansen (2012).*

**Remark 5.3**

*Under the assumptions of Proposition 5.1, the* $m$*-th conditional central moment of* $A(t)$ *is given by*

$$C_j^m(t,r) := E\Big[\big(A(t)-V_j(t,r)\big)^m \,|X_t = j, U_t = r\Big] = \sum_{l=0}^{m}\binom{m}{l} V_j^l(t,r)\,\big(-V_j(t,r)\big)^{m-l}.$$

*After calculating the conditional moments $V_j^l(t,\, r)$ as solutions of the integral equation system in Proposition 5.1, this formula allows us to get the central moments.*

## 6. Partial differential equations for the conditional moments

Here we derive partial differential equations for the conditional moments $V_j^l(t,r)$ from the partial differential equation system for the conditional moment generating function according to Proposition 4.3.

**Proposition 6.1**

*Assume that the derivatives $b_j(t,t-s) := \frac{\partial}{\partial t} B_j(s,t)$ exist and assume that the functions $b_j(t,r)$, $b_{jk}(t,r)$, $\boldsymbol{\mu}_{jk}(t,\, r)$ are continuously differentiable in their second argument. Given that the conditional moment generating function exists in a neighborhood of $\zeta = 0$, the conditional moment functions $V_j^m(t,\, r)$ satisfy the partial differential equation system*

$$\begin{aligned}
\frac{\partial}{\partial t} V_j^m(t,r) &= -\frac{\partial}{\partial r} V_j^m(t,r) + m\varphi(t) V_j^m(t,r) - m\, b_j(t,r) V_j^{m-1}(t,r) \\
&\quad - \sum_{k:k\neq j}\left(\sum_{l=0}^{m}\left(\binom{m}{l}\big(b_{jk}(t,r)\big)^l V_k^{m-l}(t,0)\right) - V_j^m(t,r)\right)\boldsymbol{\mu}_{jk}(t,r), \\
V_j^m(n,r) &= 0
\end{aligned}$$

*for all $j \in S, 0 \le r \le t \le n$, $m \in \mathbb{N}$.*

**Proof**

By applying Proposition 4.3 and differentiating $m$-times with respect to $\zeta$, we get

$$
\begin{aligned}
&\frac{\partial^m}{\partial\zeta^m}\frac{\partial}{\partial t}M_j(t,\zeta,r)\\
&=\zeta\varphi(t)\frac{\partial^{m+1}}{\partial\zeta^{m+1}}M_j(t,\zeta,r)+m\varphi(t)\frac{\partial^m}{\partial\zeta^m}M_j(t,\zeta,r)-\frac{\partial^m}{\partial\zeta^m}\frac{\partial}{\partial r}M_j(t,\zeta,r)\\
&\quad-\zeta b_j(t,r)\frac{\partial^m}{\partial\zeta^m}M_j(t,\zeta,\, r)-m\,b_j(t,r)\frac{\partial^{m-1}}{\partial\zeta^{m-1}}M_j(t,\zeta,\, r)\\
&\quad-\sum_{k:k\neq j}\left(\sum_{l=0}^{m}\left(\binom{m}{l}e^{\zeta b_{jk}(t,r)}\left(b_{jk}(t,r)\right)^l\frac{\partial^{m-l}}{\partial\zeta^{m-l}}M_k(t,\zeta,\,0)\right)-\frac{\partial^m}{\partial\zeta^m}M_j(t,\zeta,\,r)\right)\mu_{jk}(t,\,r).
\end{aligned}
$$

Setting $\zeta=0$ and using the fact that the $m$-th order derivative of the conditional moment generating function at $\zeta=0$ satisfies the equation

$$
\left.\frac{\partial^m}{\partial\zeta^m}\right|_{\zeta=0}M_j(t,\zeta,r)=E\left[\left(A(t)\right)^m e^{\zeta A(t)}\Big|_{\zeta=0}\,|X_t=j,U_t=r\right]=V_j^m(t,r)\,,
$$

we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}V_j^m(t,r)&=m\varphi(t)V_j^m(t,r)-\frac{\partial}{\partial r}V_j^m(t,r)-m\,b_j(t,r)V_j^{m-1}(t,r)\\
&\quad-\sum_{k:k\neq j}\left(\sum_{l=0}^{m}\left(\binom{m}{l}\left(b_{jk}(t,r)\right)^l V_k^{m-l}(t,0)\right)-V_j^m(t,r)\right)\mu_{jk}(t,r)\,,
\end{aligned}
$$

which is the partial differential equation system as stated in the proposition. The terminal condition follows from the calculation

$$
V_j^m(n,r)=\frac{\partial^m}{\partial\zeta^m}M_j(n,0,r)=\frac{\partial^m}{\partial\zeta^m}1=0\,,\quad m\in\mathbb{N}.
$$

□

**Remark 6.2**

*Proposition 6.1 is a generalization of Thiele's differential equation. With the notation* $V_j^1(t,\,r)=V_j(t,\,r)$ *we get from Proposition 6.1*

$$\frac{\partial}{\partial t}V_j(t,r) = -\frac{\partial}{\partial r}V_j(t,r) + \varphi(t)V_j(t,r) - b_j(t,r)$$
$$- \sum_{k:k\neq j}\left(b_{jk}(t,r) + V_k(t,0) - V_j(t,r)\right)\mu_{jk}(t,r),$$
$$V_j(n,r) = 0,$$

*which is exactly Thiele's differential equation, cf. Helwich (2008) and Christiansen (2012).*

## 7. Integral equations of type 2 for the conditional moments

As Helwich (2008) points out, there exist two versions of Thiele's integral equation, denoted as type 1 and type 2. While the equations of type 1 still involve the probabilities $\left(1-\overline{p}_{jj}(t,\tau,r)\right)$, the equations of type 2 only need the transition intensities. In Section 5 we generalized Thiele's integral equation of type 1 to higher order moments. Here we show the generalization of Thiele's integral equation of type 2. For reasons explained later on, we will work here with the transformed prospective reserves

$$W_j^m(t,s) := V_j^m(t,t-s)$$

for $0 \le s \le t$, which can be easily reversed via $V_j^m(t,r) = W_j^m(t,t-r)$ for $0 \le r \le t$. Recall that the tuple $(t,s)$ describes the contract time and the absolute time of the last jump, whereas the transformed tuple $(t,t-s)$ describes the contract time and the relative duration since the last jump.

**Proposition 7.1**

*Given that the conditional moment generating function exists in a neighborhood of $\zeta = 0$, the functions $W_j^m(t,r)$ satisfy the integral equation system*

$$W_j^m(t,s) = -\sum_{t<u\le n}\sum_{h=0}^{m-1}\binom{m}{h}\left(-\Delta B_j(s,u)\right)^{m-h}W_j^h(u,s)$$
$$-m\int_t^n \varphi(u)W_j^m(u,s)\,du + \sum_{k:k\neq j}\int_t^n W_j^m(\tau,s)\,\mu_{jk}(\tau,\tau-s)\,d\tau$$
$$+\sum_{k:k\neq j}\sum_{l=0}^{m}\binom{m}{l}\int_t^n\left(b_{jk}(\tau,\tau-s)\right)^{m-l}W_k^l(\tau,\tau)\,\mu_{jk}(\tau,\tau-s)\,d\tau$$
$$+m\int_{(t,n]} W_j^{m-1}(u-,s)\,B_j^{cont}(s,du)$$

*for all* $j \in S, 0 \le r \le t \le n$, $m \in \mathbb{N}$, *where* $\Delta B_j(s,u) := B_j(s,u) - B_j(s,u-)$ *and* $B_j^{cont}(s,\cdot)$ *denotes the continuous part of the finite variation function* $B_j(s,\cdot)$, *i.e.*

$$B_j^{cont}(s,t) := B_j(s,t) - \sum_{s \le u \le t} \Delta B_j(s,u).$$

**Proof**

From Proposition 5.1 we know that

$$\begin{aligned} W_j^m(t,s) &= \left(1 - \overline{p}_{jj}(t,n,t-s)\right)\left(\int_{(t,n]} v(t,\tau) B_j(s,d\tau)\right)^m \\ &\quad + \sum_{k:k\neq j}\sum_{l=0}^{m}\binom{m}{l}\int_{(t,n]}\left(\int_{(t,\tau]} v(t,u) B_j(s,du) + v(t,\tau) b_{jk}(\tau,\tau-s)\right)^{m-l} \\ &\qquad \times v^l(t,\tau) W_k^l(\tau,\tau)\, \overline{p}_{jk}(t,d\tau,t-s), \end{aligned} \tag{7.1}$$

which – by using Remark 4.2 – can be rewritten to

$$\begin{aligned} W_j^m(t,s) &= (v(s,t))^{-m} e^{\sum_{l:l\neq j}\int_s^t \mu_{jl}(u,u-s)du}\left( e^{-\sum_{l:l\neq j}\int_s^n \mu_{jl}(u,u-s)du}\left(\int_{(t,n]} v(s,\tau) B_j(s,d\tau)\right)^m \right. \\ &\quad + \sum_{k:k\neq j}\sum_{l=0}^{m}\binom{m}{l}\int_{(t,n]}\left(\int_{(s,\tau]} v(s,u) B_j(s,du) - \int_{(s,t]} v(s,u) B_j(s,du) + v(s,\tau) b_{jk}(\tau,\tau-s)\right)^{m-l} \\ &\qquad \left. \times v^l(s,\tau) W_k^l(\tau,\tau) e^{-\sum_{l:l\neq j}\int_s^\tau \mu_{jl}(u,u-s)du} \mu_{jk}(\tau,\tau-s)\, d\tau \right) \\ &= (v(s,t))^{-m} e^{\sum_{l:l\neq j}\int_s^t \mu_{jl}(u,u-s)du}\left( e^{-\sum_{l:l\neq j}\int_s^n \mu_{jl}(u,u-s)du}\left(\int_{(t,n]} v(s,\tau) B_j(s,d\tau)\right)^m \right. \\ &\quad + \sum_{k:k\neq j}\sum_{l=0}^{m}\sum_{h=0}^{m-l}\binom{m}{l}\binom{m-l}{h}\left(-\int_{(s,t]} v(s,u) B_j(s,du)\right)^h \\ &\qquad \times \int_{(t,n]}\left(\int_{(s,\tau]} v(s,u) B_j(s,du) + v(s,\tau) b_{jk}(\tau,\tau-s)\right)^{m-l-h} \\ &\qquad \left. \times v^l(s,\tau) W_k^l(\tau,\tau) e^{-\sum_{l:l\neq j}\int_s^\tau \mu_{jl}(u,u-s)du} \mu_{jk}(\tau,\tau-s)\, d\tau \right). \end{aligned}$$

By applying Ito's formula with respect to the time variable $t$ and using

$$\frac{d}{dt}(v(s,t))^{-m} = m\varphi(t)(v(s,t))^{-m}$$

we get that

$$W_j^m(dt,s) = W_j^m(t,s) - W_j^m(t-,s) + m\varphi(t)W_j^m(t,s)\,dt + \left(\sum_{l:l\neq j}\mu_{jl}(t,t-s)\right)W_j^m(t,s)\,dt$$
$$-\sum_{k:k\neq j}\sum_{l=0}^{m}\binom{m}{l}\left(b_{jk}(t,t-s)\right)^{m-l}W_k^l(\tau,\tau)\,\mu_{jk}(t,t-s)\,dt$$
$$-m\,W_j^{m-1}(t-,s)\,B_j^{cont}(s,dt),$$

since

$$d\left(\int_{(t,n]} v(s,\tau)\,B_j(s,d\tau)\right)^m = m\left(\int_{[t,n]} v(s,\tau)\,B_j(s,d\tau)\right)^{m-1} v(s,t)\,B_j^{cont}(s,dt)$$

and

$$d\left(\sum_{l=0}^{m}\sum_{h=0}^{m-l}\binom{m}{l}\binom{m-l}{h}\left(-\int_{(s,t]} v(s,u)B_j(s,du)\right)^h\right)$$
$$= m\sum_{l=0}^{m-1}\sum_{h=0}^{m-1-l}\binom{m-1}{l}\binom{m-1-l}{h}\left(-\int_{(s,t)} v(s,u)B_j(s,du)\right)^h v(s,t)\,B_j^{cont}(s,dt)\,.$$

Integrating over the interval $(t,n]$ and using $W_j^m(n,s) = V_j^m(n,\,t-s) = 0$ leads to

$$W_j^m(t,s) = \sum_{t<u\leq n}\left(W_j^m(u,s) - W_j^m(u-,s)\right)$$
$$-m\int_t^n \varphi(u)W_j^m(u,s)\,du + \sum_{k:k\neq j}\int_t^n W_j^m(\tau,s)\,\mu_{jk}(\tau,\tau-s)\,d\tau$$
$$+\sum_{k:k\neq j}\sum_{l=0}^{m}\binom{m}{l}\int_t^n\left(b_{jk}(\tau,\tau-s)\right)^{m-l}W_k^l(\tau,\tau)\,\mu_{jk}(\tau,\tau-s)\,d\tau$$
$$+m\int_{(t,n]} W_j^{m-1}(u-,s)\,B_j^{cont}(s,du).$$

By applying formula (7.1), Remark 4.2, and the Binomial Theorem, the term $W_j^m(u-,\,s)$ can be rewritten to

$$
\begin{aligned}
W_j^m(u-,s) &= \left(1-\bar p_{jj}(u,n,u-s)\right)\left(\int_{[u,n]} v(u,\tau)\,B_j(s,d\tau)\right)^m \\
&\quad + \sum_{k:k\neq j}\sum_{l=0}^{m}\binom{m}{l}\int_u^n\left(\int_{[u,\tau]} v(u,r)B_j(s,dr) + v(u,\tau)b_{jk}(\tau,\tau-s)\right)^{m-l} \\
&\qquad\qquad \times v^l(u,\tau)W_k^l(\tau,\tau)\left(1-\bar p_{jj}(u,\tau,u-s)\right)\mu_{jk}(\tau,\tau-s)\,d\tau \\
&= \sum_{h=0}^{m}\binom{m}{h}\left(1-\bar p_{jj}(u,n,u-s)\right)\left(\int_{(u,n]} v(u,\tau)\,B_j(s,d\tau)\right)^{m-h}\left(-\Delta B_j(s,u)\right)^h \\
&\quad + \sum_{k:k\neq j}\sum_{l=0}^{m}\sum_{h=0}^{m-l}\binom{m}{l}\binom{m-l}{h}\int_u^n\left(\int_{(u,\tau]} v(u,r)B_j(s,dr) + v(u,\tau)b_{jk}(\tau,\tau-s)\right)^{m-l-h} \\
&\qquad\qquad \times \left(-\Delta B_j(s,u)\right)^h v^l(u,\tau)W_k^l(\tau,\tau)\left(1-\bar p_{jj}(u,\tau,u-s)\right)\mu_{jk}(\tau,\tau-s)\,d\tau.
\end{aligned}
$$

Changing the order of summation and using

$$
\binom{m}{l}\binom{m-l}{h} = \binom{m}{h}\binom{m-h}{l},
$$

we can show that

$$
\begin{aligned}
W_j^m(u-,\,s) &= \sum_{h=0}^{m}\binom{m}{h}\left(-\Delta B_j(s,\,u)\right)^h\Bigg(\left(1-\bar p_{jj}(u,n,u-s)\right)\left(\int_{(u,n]} v(u,\tau)\,B_j(s,d\tau)\right)^{m-h} \\
&\quad + \sum_{k:k\neq j}\sum_{l=0}^{m-h}\binom{m-h}{l}\int_u^n\left(\int_{(u,\tau]} v(u,\,r)B_j(s,dr) + v(u,\tau)b_{jk}(\tau,\tau-s)\right)^{(m-h)-l} \\
&\qquad\qquad \times v^l(u,\,\tau)W_k^l(\tau,\tau)\left(1-\bar p_{jj}(u,\tau,u-s)\right)\mu_{jk}(\tau,\tau-s)\,d\tau\Bigg) \\
&= \sum_{h=0}^{m}\binom{m}{h}\left(-\Delta B_j(s,\,u)\right)^h W_j^{m-h}(u,s).
\end{aligned}
$$

Hence, we get

$$
\sum_{t<u\leq n}\left(W_j^m(u,s)-W_j^m(u-,s)\right) = -\sum_{t<u\leq n}\sum_{h=1}^{m}\binom{m}{h}\left(-\Delta B_j(s,u)\right)^h W_j^{m-h}(u,s).
$$

After substituting $h$ by $m-h$, we arrive at the statement of the proposition.

□

Differentiating the integral equation in Proposition 7.1 with respect to $t$ leads to an ordinary differential equation.

**Corollary 7.2**

*Let the derivatives* $b_j(t,t-s) := \frac{\partial}{\partial t} B_j(s,t)$ *exist. Given that the conditional moment generating functions exist in a neighborhood of* $\zeta = 0$*, the transformed conditional moment functions* $W_j^m(t,\, s)$ *satisfy the ordinary differential equation system*

$$\frac{\partial}{\partial t} W_j^m(t,s) = -m W_j^{m-1}(t,s) b_j(t,t-s) + \left( m\varphi(t) + \sum_{k:k\neq j} \mu_{jk}(t,\, t-s) \right) W_j^m(t,s)$$
$$- \sum_{k:k\neq j} \sum_{l=0}^{m} \binom{m}{l} \left( b_{jk}(t,t-s) \right)^{m-l} W_k^l(t,t)\, \mu_{jk}(t,t-s),$$

$$W_j^m(n,s) = 0$$

*for all* $j \in S,\, 0 \le r \le t \le n$*,* $m \in \mathbb{N}$.

Corollary 7.2 shows the advantage of working with the transformed reserves $W_j^m(t,\, s)$ instead of the reserves $V_j^m(t,\, r)$. If we rewrite Proposition 7.1 in terms of the reserves $V_j^m(t,\, t-s)$, then differentiation with respect to $t$ leads to the partial differential equation system of Section 6, whereas Corollary 7.2 presents an ordinary differential equation system. Analytical solutions are mostly out of reach and numerical solution methods have to be used. Solving ordinary differential equations is usually easier than solving partial differential equations.

**Remark 7.3**

*In real life applications, the function* $t \mapsto B_j(s,t)$ *can always be decomposed to a sum of a differentiable part and a pure jump part. So the integral equation of Proposition 7.1 can be solved by solving the ordinary differential equations of Corollary 7.2 in between the jump times and by connecting the continuous solutions at each jump time* $t$ *by*

$$W_j^m(t-,\, s) = W_j^m(t,\, s) + \sum_{h=1}^{m} \binom{m}{h} \left( -\Delta B_j(s,\, t) \right)^h W_j^{m-h}(t,\, s).$$

*For further details on that solution idea we recommend to read Norberg (2005).*

## 8. Solving the integral and differential equations

First, we discuss numerical solution methods, which are needed whenever analytical solutions are out of reach. Second, we demonstrate the numerical techniques by calculating a disability insurance. Third, we show how to derive analytical solutions in the special case of active-dead models.

### 8.1 Numerical solution methods

Because of Remark 7.3, it suffices to focus on the case where the derivatives $b_j(t,t-s) := \frac{\partial}{\partial t} B_j(s,t)$ exist. Corollary 7.2 defines an infinite family of ordinary differential equations (one equation for each $s \in [0,n], j \in \mathrm{S}, m \in \mathbb{N}$), which are coupled via the term $W_k^l(t,t)$ on the right hand side. Dealing with the coupling is the main difficulty. Broadly speaking, the problem of the coupling can be solved by simply solving the ordinary differential equations in the right order:

for $m$ from $1$ to $\infty$ do
  for $s$ from $n$ down to $0$ do
    solve the ODE backwards on $[s,n]$ simultaneously for all $j \in S$
  end do
end do

This idea has a major flaw, namely that the family of ordinary differential equations is uncountably infinite. By discretization of the time- and duration-space, we can make the idea work. Let $M \in \mathbb{N}$ be the maximal moment that we are interested in and let $\delta > 0$ be the mesh size of the discretized time- and duration-space. Then the algorithm of above combined with Euler's method reads as follows:

for $m$ from $1$ to $M$ do
  for $s$ from $n$ by $-\delta$ to $0$ do
    for $t$ from $n$ by $-\delta$ to $s$ do
      for $j$ from $1$ to $\#S$ do

$$
\begin{aligned}
W_j^m(t-\delta,s) = W_j^m(t,s) \\
-\delta\Bigg( m W_j^{m-1}(t,s) b_j(t,t-s) + \Big( m\varphi(t) + \sum_{k:k\neq j} \mu_{jk}(t,\, t-s) \Big) W_j^m(t,s) \\
- \sum_{k:k\neq j} \sum_{l=0}^{m} \binom{m}{l} \big(b_{jk}(t,t-s)\big)^{m-l} W_k^l(t,t)\, \mu_{jk}(t,t-s) \Bigg)
\end{aligned}
$$

      end do
    end do
  end do
end do

A similar idea is used in Buchardt et al. (2014) for solving Kolmogorov's equation for the transition probabilities of a Semi-Markovian random pattern of states.

### 8.2 Example: Disability insurance

Suppose that we have a disability insurance that distinguishes between the states $1 = \text{active}$, $2 = \text{disabled}$, $3 = \text{dead}$. The contract starts at age $x$=45 and runs for $n = 15$ years. If the policyholder is in state disabled, he/she receives a continuous disability annuity with rate 1 for at maximum 5 years from the 4-th month of disability on (0.25 time units), given that the disability occurred before age 55. If the policyholder is in state active, he/she has to pay a continuous premium with rate $0.03179708\ (1.015)^t$ till the age of 55 (contract time $t = 10$). The premium was calibrated according to the net premium principle, i.e. the prospective reserve at time $t = 0$ in state $j = 1$ is zero. We assume that

$$\mu_{12}(t,s) = 0.004 + 10^{0.060(x+t)-5.46},$$
$$\mu_{13}(t,s) = 0.0005 + 10^{0.038(x+t)-4.12},$$
$$\mu_{23}(t,s) = 0.0005 + 0.001e^{-s} + 10^{0.038(x+t)-4.12}$$

and that all other transition intensities are zero. We applied the numerical algorithm from Section 8.1 for a mesh size of $\delta = 1/40$. Figure 1 and Figure 2 show the conditional first-order moments in state $1 = \text{active}$ and state $2 = \text{disabled}$ on the time- and duration space. (The axes show the mesh step numbers, starting from 0 for $s, t = 0$ and running till $n/\delta = 600$ for $s, t = n$.) Figure 3 and Figure 4 show the conditional second-order moments. As the payments in state 1 and the transition intensities out of state 1 are independent of the argument $s$, we have no duration effect in state 1, so the functions in Figure 1 and Figure 3 are constant in the argument $s$. In contrast, Figure 2 and Figure 4 show very strong duration effects in state 2 = disabled. Moreover, the first-order and second-order moments in state 2 show a similar monotonic pattern. In contrast, in state 1 the second order moment reaches its maximum for $t$ towards zero, while the first-order moment converges to zero. That means that the coefficient of variation is exploding at $t = 0$.

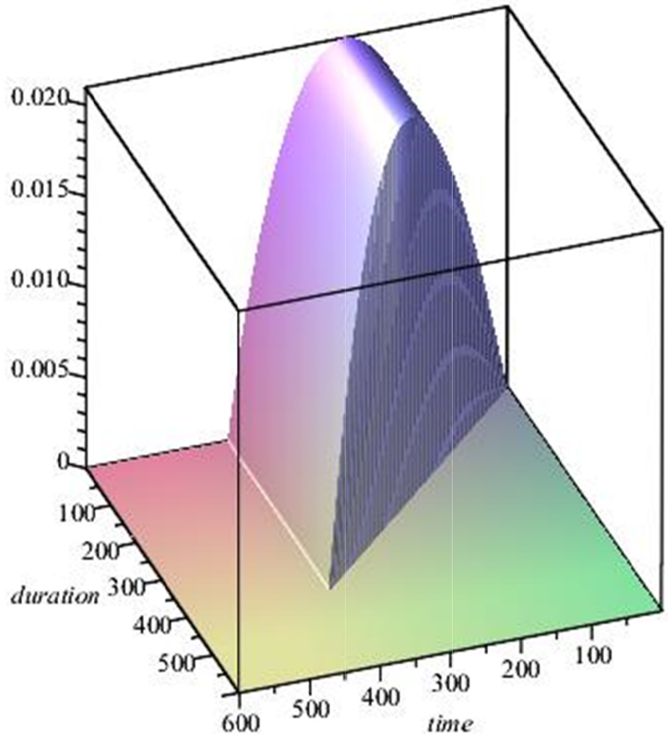


**Figure 1:** $(t,s) \mapsto W_1^1(t\delta, s\delta)$

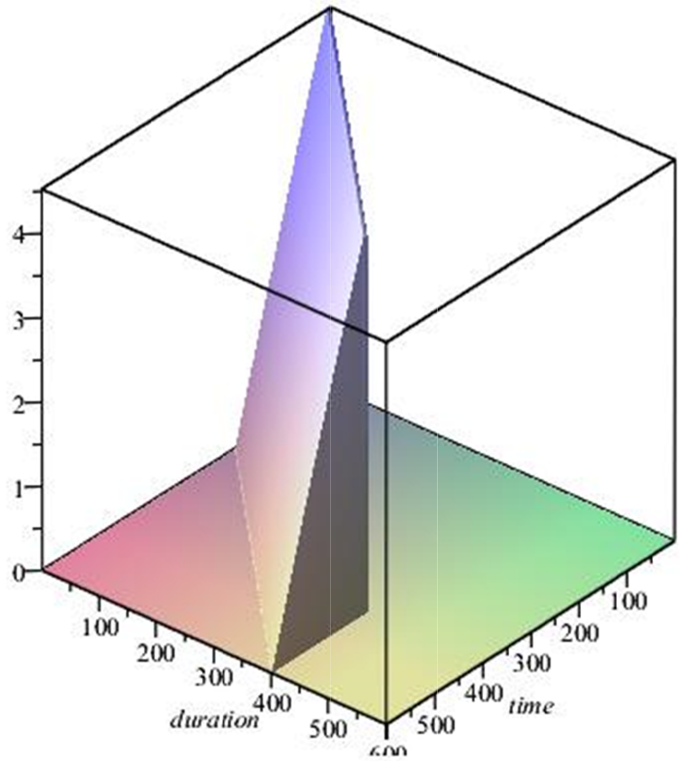


**Figure 2:** $(t,s) \mapsto W_2^1(t\delta, s\delta)$

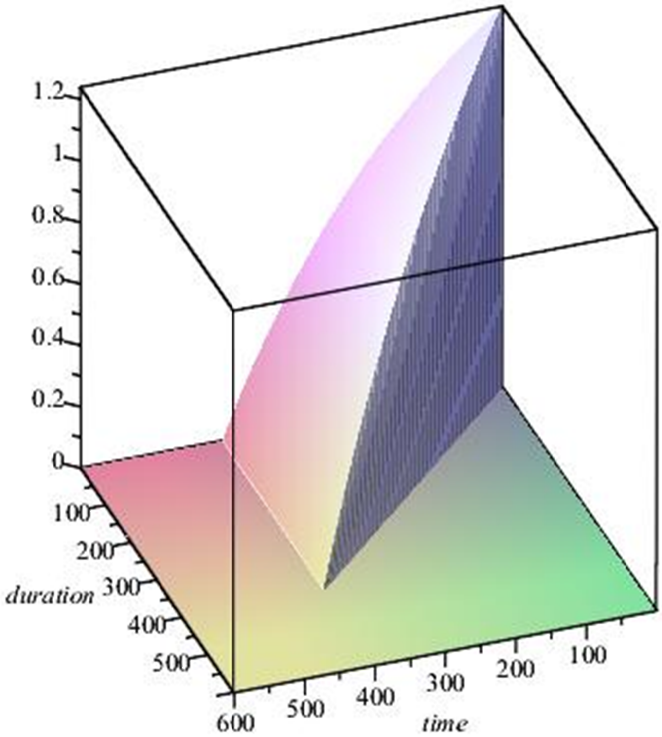


**Figure 3:** $(t,s) \mapsto W_1^2(t\delta, s\delta)$

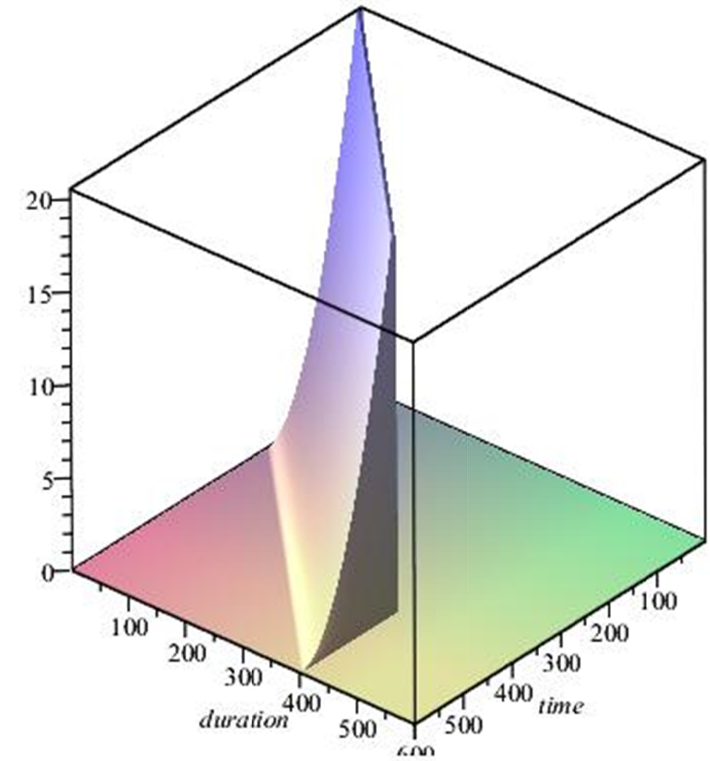


**Figure 4:** $(t,s) \mapsto W_2^2(t\delta, s\delta)$

Suppose that we have a portfolio of $Q$ identical and stochastically independent disability insurance contracts as described above. With the notation $A_1(0),...,A_Q(0)$ for the corresponding present values of the single contracts at time zero, the total discounted portfolio cash flow equals $A_1(0)+\ldots+A_Q(0)$. The Normal Power Approximation suggests to approximate this random variable in distribution by

$$A_\Sigma(0):=\sum_{q=1}^{Q} A_q(0) \overset{d}{\approx} E[A_\Sigma(0)]+\sqrt{Var[A_\Sigma(0)]}\,Y+\frac{E\left[(A_\Sigma(0)-E[A_\Sigma(0)])^3\right]}{6\,Var[A_\Sigma(0)]}\left(Y^2-1\right),$$

where $Y$ is a standard normal random variable. For $t>0$ a similar formula applies for $A_\Sigma(t)$ but with conditional expectations and conditional variances. Since

$$\begin{aligned} E[A_\Sigma(t)\,|\,X_t=j,U_t=r]&=Q\,V_j^1(t,r),\\ Var[A_\Sigma(t)\,|\,X_t=j,U_t=r]&=Q\,C_j^2(t,r),\\ E\left[(A_\Sigma(t)-E[A_\Sigma(t)\,|\,X_t=j,U_t=r])^3\,|\,X_t=j,U_t=r\right]&=Q\,C_j^3(t,r) \end{aligned}$$

for $C_j^2(t,r)$ and $C_j^3(t,r)$ defined according to Remark 5.3, we obtain that

$$\sum_{q=1}^{Q} A_q(t)\;|\;(X_t=j,U_t=r) \overset{d}{\approx} Q\,V_1^1(t,r)+\sqrt{Q\,C_1^2(t,r)}\,Y+\frac{C_1^3(t,r)}{6\,C_1^2(t,r)}\left(Y^2-1\right).$$

For our example we get at $t=0$ the approximation

$$\sum_{q=1}^{Q} A_q(0)\;|\;(X_0=1,U_0=0) \overset{d}{\approx} 1.113105\sqrt{Q}\;Y+0.619855\left(Y^2-1\right).$$

(Note that $U_0=0$ is always true and that $X_0=1=\text{active}$ is always satisfied in real life applications.) The right hand side can be rearranged to

$$0.619855\left(Y+\frac{1.113105\sqrt{Q}}{2\cdot 0.619855}\right)^2-\frac{1.113105^2\,Q}{4\cdot 0.619855}-0.619855$$

and has the cumulative distribution function

$$y\mapsto \Phi\left(\sqrt{\frac{y}{0.619855}+\frac{1.113105^2\,Q}{4\cdot 0.619855^2}+1}-\frac{1.113105\sqrt{Q}}{2\cdot 0.619855}\right),$$

where $\Phi$ is the cumulative distribution function of the standard normal random variable $Y$. Given that $Q$ is sufficiently large such that the relative approximation error is small, it is now possible to calculate risk margins and solvency requirements. For example, the Value-at-Risk of the Normal Power Approximation equals here

$$0.619855\left(\Phi(\alpha)^{-1}\right)^{2}+1.113105\sqrt{Q}\,\Phi(\alpha)^{-1}-0.619855,$$

where $\alpha\in(0,1)$ is the safety level.

Our example uses a run-off perspective, so all randomness till expiration of the insurance contracts is considered. The solvency capital requirements according to Solvency II and the Swiss Solvency Test use a one-year perspective. A simple trick to achieve a one-year perspective is to set the contract time $n$ equal to 1 and to use the prospective reserves at time 1 as terminal conditions (differential equations) or to add the prospective reserves at time 1 as terminal lump sum sojourn payments at time 1 (integral equations). The new differential/integral equations will then only consider the randomness in the first year, and the Normal Power Approximation allows for calculating quantiles (Value-at-Risk) and further risk measures, e.g. the Tail-Value-at-Risk.

In our example the state space $\{1,2,3\}$ relates only to unsystematic biometric risk and does not account for systematic biometric risk (trend risk) or financial risk. So our numerical results are just measuring the unsystematic biometric risk. In order to capture also systematic biometric risk and financial risk, one could follow the concept in Norberg (1999), where the state space is extended in such a way that the financial and demographical environment is stochastic as well.

**Acknowledgments**

This research was supported by the International Mathematical Union via the Abel Visiting Scholar Program and by the German Academic Exchange Service (DAAD) via a Research Stay Scholarship.

**References**

Amsler, M.H., 1968. Les chaines de Markov des assurances vie, invalidité et maladie. In: Transactions of the 18th International Congress of Actuaries, München, Volume 5, 731–746.

Buchardt, K., Møller, T., Schmidt, K.B., 2014. Cash flows and policyholder behaviour in the semi-Markov life insurance setup. Scandinavian Actuarial Journal, (ahead-of-print), 1–29.

Casasnovas, G.L., Nicodemo, C., 2012.Transition probabilities and duration analysis among disability states: Some evidence from Spanish data. Barcelona GSE Working Paper No. 643.

Christiansen, M.C., 2012.Multistate models in health insurance. Advances in Statistical Analysis 96, 155–186.

Christiansen, M.C., Niemeyer, A., Teigiszerova, L., 2015. Modeling and forecasting duration-dependent mortality rates. Computational Statistics and Data Analysis 83, 65-81.

Helwich, M., 2008. Durational effects and non-smooth semi-Markov models in life insurance. Doctoral dissertation, University of Rostock. urn:nbn:de:gbv:28-diss2008-0056-4. http://rosdok.uni-rostock.de/

Hoem, J.M., 1969. Markov chain models in life insurance. Blätterder DGVFM 9, 91–107.

Hoem, J.M., 1972. Inhomogeneous semi-Markov processes, select actuarial tables, and duration-dependence in demography. In: Population Dynamics, 251–296, Academic Press, New York.

Norberg, R., 1991. Reserves in life pension insurance, Scandinavian Actuarial Journal, 3–24.

Norberg, R., 1992. Hattendorf's theorem and Thiele's differential equation generalized. Scandinavian Actuarial Journal, 2–14.

Norberg, R., 1995. Differential equations for moments of present values in life insurance. Insurance Mathematics and Economics 17, 171–180.

Norberg, R., 1999. A theory of bonus in life insurance. Finance and Stochastics 3, 373–390.

Norberg, R., 2005. Anomalous PDEs in Markov chains: Domains of validity and numerical solutions. Finance and Stochastics 9, 519–537.

Segerer, G., 1993. The actuarial treatment of the disability risk in Germany, Austria and Switzerland. Insurance: Mathematics and Economics 13, 131–140.