Mathematical Model of a Type M/M/1/∞ Queuing System with Request Rejection: A Retail Facility Case Study

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Abstract. The model of retail outlet in form of queueing system of M/M/1/∞ type with request rejection is proposed. The output flow of the system and the rejected request flow are researched. Average number of events occurred in these flows is determined. In conditions of increasing observation time the asymptotic distributions of probabilities of number of events that occurred in studied flows are found by means of asymptotic analysis.

Keywords: Queueing system · Method of asymptotic analysis · Method of torques · Fourier transformation

1 Introduction

Due to development of economic systems, mathematical models of which could be single-line queueing systems with request rejection, the latter are pretty common in practice. The "request rejection" means customer impatience and his unwillingness to stay in queue which may lead to him refusing to stay in queue.

The subject of research is the output flow of served demands and the flow of demands which refused to stay in queue, because information about output flow properties is very useful. That way, knowing properties of output flow it is possible to draw conclusions about the quality of performance of the system and to analyze its effectiveness.

Output flows research is not getting enough attention due to lack of general approach to their research. Thus the task of modification of existing methods of output flow research and development of new ones is pretty relevant [1].

2 Mathematical Model

Mathematical models of queuing systems (QS) are widely used to investigate various systems with request inflow. The models can be used to describe the

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operation of a retail facility. In this work we consider one of these models, taking into account "impatient" customers.

Our mathematical model of this situation is a queuing system, which is fed a simple request flow with the parameter λ. In this paper, we consider the case when the service time has an exponential distribution with the parameter μ.

Our model has the following service discipline: if an incoming request encounters i requests already in the system, then the request is rejected (and leaves the system) with the probability r_i, $0 \leq r_i \leq 1$; on the other hand, the request is accepted for service with the probability $1 - r_i$.

Our notation is:

- $m(t)$ – the number of requests that were refused to be serviced during the time t (the output flow);
- $n(t)$ – the number of requests that have been serviced during the time t;
- $i(t)$ – the number of requests in the system at the time t.

In our QS, as the parameters λ, μ, r_i are specified, the process $i(t)$ is a continuous time Markov chain (birth and death process)[2]; the process is controlled by means of the flows $m(t)$ and $n(t)$. Thus both of these flows are MAP-processes [3].

3 Investigation of the Output Request Flow

Since the two dimensional random process is a Markov chain, the probability distribution of the process

$$P(i, n, t) = P\{i(T) = i, n(t) = n\}.$$

We can write down the following system of Kolmogorovs differential equations:

$$\begin{cases} \dfrac{\partial P(i,n,t)}{\partial t} = -[\lambda(1-r_i)+\mu]P(i,n,t) + \lambda(1-r_{i-1})P(i-1,n,t) \\ +\mu P(i+1,n-1,t), \\ \dfrac{\partial P(0,n,t)}{\partial t} = -\lambda(1-r_0)P(0,n,t) + \mu P(1,n-1,t), \end{cases} \tag{1}$$

To solve this system we introduce the following function [4]:

$$H(i,u,t) = \sum_{n=0}^{\infty} e^{jun} P(i,n,t),$$

where $j = \sqrt{-1}$. Then we obtain the following system of equations for these functions

$$\begin{cases} \dfrac{\partial H(i,u,t)}{\partial t} = -[\lambda(1-r_i)+\mu]H(i,u,t) + \lambda(1-r_{i-1})H(i-1,u,t) \\ +\mu e^{ju} H(i+1,u,t), \\ \dfrac{\partial H(0,u,t)}{\partial t} = -\lambda(1-r_0)H(0,u,t) + \mu e^{ju} H(1,u,t), \end{cases} \tag{2}$$

Lets introduce the following row-vector

$$\mathbf{H}(u,t) = \{H(0,u,t)H(1,u,t),\dots\}$$

and rewrite the system (2) as

$$\frac{\partial \mathbf{H}(u,t)}{\partial t} = \mathbf{H}(u,t)\{\mathbf{Q} + \mu e^{ju}\mathbf{B}\}, \tag{3}$$

where $\mathbf{Q}$ is a three-diagonal matrix for the birth-and-death process $i(t)$; the matrix looks the following way

$$\mathbf{Q} = \begin{bmatrix} -\lambda(1-r_0) & \lambda(1-r_0) & 0 & 0 & \dots \\ \mu & -[\lambda(1-r_1)+\mu] & \lambda(1-r_1) & 0 & \dots \\ 0 & \mu & -[\lambda(1-r_2)+\mu] & \lambda(1-r_2) & \dots \\ 0 & 0 & \mu & -[\lambda(1-r_3)+\mu] & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{4}$$

In the matrix $\mathbf{B}$ the lower sub-diagonal elements equal 1, while the rest equal zero. Finally, lets introduce the column-vector $\mathbf{E}$ that is the all-one column $\mathbf{E} = (1,1,1,\dots)^T$. Then it is easy to see that $\mathbf{QE} = 0$.

We solve the differential-matrix equation under the following initial conditions:

1. $n(0) = 0$ with probability 1.
2. Assume that at $t = 0$, the birth-and-death process $i(t)$ has a stationary probability distribution $P(i(t) = i) = R(i)$, that we will obtain later in this work. If we set $t = 0$, then $P(i,n,0) = R(i)\delta_{n0}$ and thus $H(i,u,0) = R(i)$. Lets introduce the row-vector

   $$\mathbf{R} = (R(0), R(1), R(2), \dots),$$

   then

   $$\mathbf{H}(u,0) = \mathbf{R}$$

Next, if we set $u = 0$, then

$$H(i,0,t) = \sum_{n=0}^{\infty} P(i,n,t) = P(i,t) = R(i)$$

because $R(i)$ is stationary distributed. So the following relation is true: $\mathbf{H}(0,t) = \mathbf{R}$.

Thus, the initial conditions for the system (3) are

$$\mathbf{H}(u,0) = \mathbf{H}(0,t) = \mathbf{R}.$$

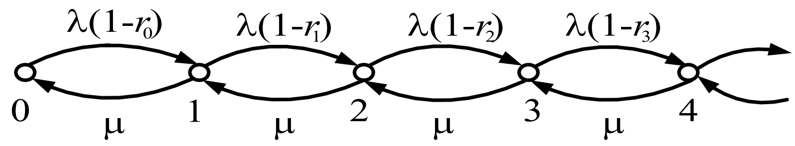

Fig. 1. The transition graph of the process $i(t)$

4 Final Probability Distribution of the Process $i(t)$

In a stationary state, the transition graph of the process $i(t)$ looks like (1).

This graph begets the following finite difference system of equations for the final probability distribution of $R(i)$

$$\begin{cases} \lambda(1-r_0)R(0) = \mu R(1), \\ \lambda(1-r_{i-1})R(i-1) - [\lambda(1-r_i)+\mu]R(i) + \mu R(i+1) = 0. \end{cases} \tag{5}$$

Notice, that this system can be written in the matrix form $\mathbf{RQ} = 0$. Lets rewrite (5) as

$$\lambda(1-r_{i-1})R(i-1) - \mu R(i) = \lambda(1-r_i)R(i) - \mu R(i+1),$$

then it follows that

$$\lambda(1-r_{i-1})R(i-1) - \mu R(i) = Const.$$

From the first equation of the system (5) it follows that $Const = 0$, so

$$\lambda(1-r_{i-1})R(i-1) = \mu R(i).$$

it comes out that

$$R(i) = \rho(1-r_{i-1})R(i-1) = \cdots = R(0)\rho^i \prod_{k=0}^{i-1}(1-r_k), \tag{6}$$

where $\rho = \lambda/\mu$.

The constant $R(0)$ can be obtained from the normalization condition $\sum_{i=0}^{\infty} R(i) = 1$, that can be written as $\mathbf{RE} = 1$. Its explicit form is $\mathbf{RE} = 1$

$$R(0) = \frac{1}{1 + \sum\limits_{i=0}^{\infty} \rho^i \prod\limits_{k=0}^{i-1}(1-r_k)}. \tag{7}$$

Specifically, from (7) it follows that a stationary probability distribution in our QS exists if

$$\sum_{i=0}^{\infty} \rho^i \prod_{k=0}^{i-1}(1-r_k) < +\infty$$

Thus for the row-vector $\mathbf{H}(u,t)$ there exists the following Cauchy problem

$$\begin{cases} \frac{\partial \mathbf{H}(u,t)}{\partial t} = \mathbf{H}(u,t)\left\{\mathbf{Q} + \mu\left(e^{ju}-1\right)\mathbf{B}\right\}, \\ \mathbf{H}(u,0) = \mathbf{R}. \end{cases} \quad (8)$$

The solution of the system $\mathbf{H}(u,t)$ specifies the characteristic function of $n(t)$. Indeed, if we expand

$$H(i,u,t) = \sum_{n=0}^{\infty} e^{jun} P(i,n,t),$$

and sum up over i, we get

$$M\left\{e^{jun(t)}\right\} = \sum_{n=0}^{\infty} e^{jun} \sum_{i=0}^{\infty} P(i,n,t) = \sum_{i=0}^{\infty} H(i,u,t) = \mathbf{H}(u,t)\mathbf{E}, \quad (9)$$

where $\mathbf{E}$ is the all-ones row-vector.

5 Mean Number of Serviced Requests

Utilizing the properties of the characteristic function, we get the following expression for $M\{n(t)\}$

$$M\{n(t)\} = \frac{1}{j} \frac{\partial M\left\{e^{jun(t)}\right\}}{\partial u}\bigg|_{u=0} = \frac{1}{j} \frac{\partial \mathbf{H}(u,t)}{\partial u}\bigg|_{u=0} \mathbf{E}.$$

Lets denote

$$n_1(t) = \frac{1}{j} \frac{\partial \mathbf{H}(u,t)}{\partial u}\bigg|_{u=0} E,$$

so that

$$M\{n(t)\} = n_1(t)\mathbf{E}.$$

Then from (8) we get

$$\frac{dn_1(t)}{dt} = n_1(t)\mathbf{Q} + \mu\mathbf{RB},$$

hence $H(0,t) = \mathbf{R}$. Multiplying both sides on , while keeping in mind that $\mathbf{QE} = 0$, we get

$$\frac{d\mathrm{M}\{n(t)\}}{dt} = \mu\mathbf{RBE},$$

alongside the initial condition $\mathrm{M}\{n(0)\} = 0$. Thus

$$\mathrm{M}\{n(t)\} = \mu\mathbf{RBE} \cdot t,$$

From now on we denote the product $\mu\mathbf{RBE}$ as κ_1.

Since

$$\mathbf{BE} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \dots \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ \dots \end{bmatrix}$$

then

$$\mathbf{RBE} = [R(0), R(1), R(2), \dots] \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ \dots \end{bmatrix} = \sum_{i=1}^{\infty} R(i)$$

$$= 1 - R(0) = \frac{\sum\limits_{i=0}^{\infty} \rho^i \prod\limits_{k=0}^{i-1} (1 - r_k)}{1 + \sum\limits_{i=0}^{\infty} \rho^i \prod\limits_{k=0}^{i-1} (1 - r_k)}$$

so finally we obtain

$$\mathrm{M}\{n(t)\} = \kappa_1 t = \mu(1 - R(0))t. \tag{10}$$

6 A Solution by Fourier Method

Let $\mathbf{Y}(u,\alpha)$ be the Fourier transform of the vector-function $\mathbf{H}(u,t)$ over t

$$\mathbf{Y}(u,\alpha) = \int_0^{\infty} e^{j\alpha t}\mathbf{H}(u,t)dt. \tag{11}$$

Then, integrating this by parts, we get

$$\int_0^{\infty} e^{j\alpha t} \frac{\partial \mathbf{H}(u,t)}{\partial t} dt = \int_0^{\infty} e^{j\alpha t} d_t \mathbf{H}(u,t)dt = -\mathbf{R} - j\alpha\mathbf{Y}(u,\alpha),$$

and from (8) we get

$$-\mathbf{R} - j\alpha\mathbf{Y}(u,\alpha) = \mathbf{Y}(u,\alpha)\{\mathbf{Q} + \mu(e^{ju} - 1)\mathbf{B}\}. \tag{12}$$

Its solution $\mathbf{Y}(u,\alpha)$ has the following form

$$\mathbf{Y}(u,\alpha) = \mathbf{R}\sum_{n=0}^{\infty} e^{jun} \left[(\mu\mathbf{B} - \mathbf{Q} - j\alpha\mathbf{I})^{-1} \mu\mathbf{B}\right]^n (\mu\mathbf{B} - \mathbf{Q} - j\alpha\mathbf{I})^{-1}.$$

Here **I** the identity matrix. This expression, the definition of $\mathbf{H}(i,u,t)$, and the expression (12) give us the following formula for the Fourier transform of $P(n,t)$:

$$\mathbf{P}(n,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{j\alpha t}\mathbf{R}\left[(\mu\mathbf{B}-\mathbf{Q}-j\alpha\mathbf{I})^{-1}\mu\mathbf{B}\right]^{n}(\mu\mathbf{B}-\mathbf{Q}-j\alpha\mathbf{I})^{-1}\mathbf{E}d\alpha.$$

7 A Long Time Asymptotic Solution of the Problem (8)

In this part the limiting case $\rightarrow\infty$ is investigated. We call the condition $t=\tau T$ the asymptotic condition of increasing time. The problem is analyzed by means of A.A. Nazarov asymptotic method [4].

The first order asymptotics of the characteristic function

$$\mathbf{H}(u,t)\mathbf{E} = \mathrm{M}e^{jun(t)},$$

where $n(t)$ is the number of events that took place in the output flow during the time t, is a function $h_1(u,t)$ of the form

$$h_1(u,t) = \exp\{ju\kappa_1 t\},$$

where κ_1 has already been determined by the method of moment;

$$\kappa_1 = \mu\cdot\mathbf{RBE} = \mu(1-R(0)).$$

7.1 Second Order Asymptotic

To obtain the second order asymptotic $h_2(u,t)$ in Eq. (8) lets do the substitution

$$\mathbf{H}(u,t) = \mathbf{H}_2(u,t)\exp\{ju\kappa_1 t\}. \tag{13}$$

Then for $\mathbf{H}_2(u,t)$ we get the equation

$$\frac{\partial\mathbf{H}_2(u,t)}{\partial t} = \mathbf{H}_2(u,t)\left\{\mathbf{Q}+\mu\left(e^{ju}-1\right)\mathbf{B}-ju\kappa_1\mathbf{I}\right\}, \tag{14}$$

where **I** is the identity matrix.

It follows from (13) that the initial condition for the solution $\mathbf{H}_2(u,t)$ is the same as the initial condition for the function $\mathbf{H}(u,t)$ in the problem (8)

$$\mathbf{H}_2(u,t) = \mathbf{R}.$$

Lets introduce ε such that $\varepsilon^2 = 1/T$; then we plug

$$t\varepsilon^2 = \tau, u = \varepsilon w, \mathbf{H}_2(u,t) = \mathbf{F}_2(w,\tau,\varepsilon). \tag{15}$$

into equation (14). The substitution begets the following differential equation

$$\varepsilon^2\frac{\partial\mathbf{F}_2(w,\tau,\varepsilon)}{\partial t} = \mathbf{F}_2(w,\tau,\varepsilon)\left\{\mathbf{Q}+\mu\left(e^{j\varepsilon w}-1\right)\mathbf{B}-j\varepsilon w\kappa_1\mathbf{I}\right\}, \tag{16}$$

We solve this equation in two steps.

Step 1. The solution $\mathbf{F}_2(w,\tau,\varepsilon)$ of Eq. (16) can be written in the following form

$$\mathbf{F}_2(w,\tau,\varepsilon) = \Phi_2(w,\tau)\left\{\mathbf{R} + j\varepsilon w\mathbf{f}\right\} + O(\varepsilon^2). \tag{17}$$

At first lets find the vector $\mathbf{f}$, while the scalar function $\Phi_2(w,\tau)$ will be obtained on the next step. Equation (16) can be rewritten as

$$O(\varepsilon^2) = j\varepsilon w\left\{\mathbf{fQ} + \mathbf{R}(\mu\mathbf{B} - \kappa_1\mathbf{I})\right\},$$

where we took into account that $\mathbf{RQ} = 0$.

It follows that the vector $\mathbf{f}$ is a solution of the inhomogeneous system of equations

$$\mathbf{fQ} + \mathbf{R}(\mu\mathbf{B} - \kappa_1\mathbf{I}) = 0. \tag{18}$$

As the matrix $\mathbf{Q}$ is degenerate, we have to impose additional restrictions on $\mathbf{f}$ for the vector to be determined uniquely. Let this restriction be

$$\mathbf{fE} = 0. \tag{19}$$

Step 2. Multiplying the matrix differential Eq. (16) on $\mathbf{E}$, we get

$$\varepsilon^2\frac{\partial\mathbf{F}_2(w,\tau,\varepsilon)}{\partial t}\mathbf{E} = \mathbf{F}_2(w,\tau,\varepsilon)\left\{\mathbf{QE} + \mu\left(e^{j\varepsilon w} - 1\right)\mathbf{BE} - j\varepsilon w\kappa_1\mathbf{E}\right\}$$
$$= \mathbf{F}_2(w,\tau,\varepsilon)\left\{j\varepsilon w(\mu\mathbf{B} - \kappa_1\mathbf{I})\mathbf{E} + \mu\frac{(j\varepsilon w)^2}{2}\mathbf{BE}\right\} + O(\varepsilon^3).$$

Substituting the expansion (17) into this equation, it comes out that

$$\varepsilon^2\frac{\partial\Phi_2(w,\tau)}{\partial t}\mathbf{RE} = \Phi_2(w,\tau)\frac{(j\varepsilon w)^2}{2}\left\{j\varepsilon w(\mu\mathbf{RBE} + 2\mu\mathbf{fBE}\right\} + O(\varepsilon^3).$$

Let the $\varepsilon \to 0$ to zero in the last equation. This expression gives the equation to determine the scalar function $\Phi_2(w,\tau)$

$$\frac{\partial\Phi_2(w,\tau)}{\partial t} = \Phi_2(w,\tau)\frac{(jw)^2}{2}\kappa_2).$$

where

$$\kappa_2 = \mu\mathbf{RBE} + 2\mu\mathbf{fBE} = \kappa_1 + 2\mu\mathbf{fBE}. \tag{20}$$

Here the vector $\mathbf{f}$ is the solution of (18)-(19).

Obviously, $\Phi_2(w,\tau)$ has the following form

$$\Phi_2(w,\tau) = \exp\left\{\frac{(jw)^2}{2}\kappa_2)\tau\right\}$$

Substituting this expression in (17) and multiplying by $\mathbf{E}$, we obtain

$$\mathbf{F}_2(w,\tau,\varepsilon)\mathbf{E} = \Phi_2(w,\tau)\left\{\mathbf{RE} + j\varepsilon w\mathbf{fE}\right\} + O(\varepsilon^2)$$
$$= \Phi_2(w,\tau) + O(\varepsilon^2) = \exp\left\{\frac{(jw)^2}{2}\kappa_2)\tau\right\} + O(\varepsilon^2).$$

We get

$$\mathbf{H}_2(u,t)\mathbf{E} = \exp\left\{\frac{(ju)^2}{2}\kappa_2)t\right\} + O\left(\frac{1}{T}\right). \tag{21}$$

Plugging this expression into (13), we get the second order asymptotic $h_2(u,t)$ for the characteristic function of $n(t)$

$$h_2(u,t) = \exp\left\{ju\kappa_1 t + \frac{(ju)^2}{2}\kappa_2)t\right\}. \tag{22}$$

Here κ_2 is determined from (20).

It follows that for large enough t we have $\mathrm{D}t = \kappa_2 t$. Obviously that if $\mathbf{fBE} = -f(0) \neq 0$, then the flow $n(t)$ is not Poisson since the necessary condition $\mathrm{M}\{n(t)\} \neq \mathrm{D}\{n(t)\}$ is violated.

Thus, we find that the asymptotic probability distribution of the number of applications, have completed service in the system during the time t in a growing period of observation is normal with parameters $\kappa_1 t$ and $\kappa_2 t$ [5].

7.2 Determination of the Vector F

The explicit expression of (19) is

$$\begin{gathered}
-\lambda(1-r_0)f(0) + \mu f(1) + \mu R(1) - \kappa_1 R(0) = 0,\\
\lambda(1-r_0)f(0) - [\lambda(1-r_1)+\mu]\, f(1) + \mu f(2) + \mu R(2) - \kappa_1 R(1) = 0,\\
\dots\\
\lambda(1-r_{i-1})f(i-1) - [\lambda(1-r_i)+\mu]\, f(i) + \mu f(i+1) + \mu R(i+1) - \kappa_1 R(i) = 0,\\
\dots
\end{gathered} \tag{23}$$

Summing up the first i equations of this system, we get

$$\begin{gathered}
-\lambda(1-r_0)f(0) + \mu f(1) + \mu R(1) - \kappa_1 R(0) = 0,\\
\lambda(1-r_1)f(0) + \mu f(2) + \mu\,[R(1)+R(2)] - \kappa_1\,[R(0)+R(1)] = 0,\\
\dots\\
\lambda(1-r_{i-1})f(i-1) + \mu f(i) + \sum_{\nu=1}^{i} R(\nu) - \kappa_1 \sum_{\nu=0}^{i-1} R(\nu) = 0,\\
\dots
\end{gathered} \tag{24}$$

So, there are the following recurrent relations for $f(i)$, where $\rho = \lambda/\mu$:

$$f(i) = \rho(1-r_{i-1})f(i-1) + \frac{\kappa_1}{\mu}\sum_{\nu=0}^{i-1} R(\nu) - \sum_{\nu=1}^{i} R(\nu). \tag{25}$$

Lets introduce the function $b(i)$ such that

$$\begin{aligned}
b(i) &= \frac{\kappa_1}{\mu}\sum_{\nu=0}^{i-1} R(\nu) - \sum_{\nu=1}^{i} R(\nu) = (1-R(0))\sum_{\nu=0}^{i-1} R(\nu) - \sum_{\nu=1}^{i} R(\nu)\\
&= R(0) - R(i) - R(0)\sum_{\nu=0}^{i-1} R(\nu) = R(0)\sum_{\nu=i}^{\infty} R(\nu) - R(i).
\end{aligned} \tag{26}$$

Hence, (25) becomes

$$f(i) = \rho(1 - r_{i-1})f(i-1) + b(i), \quad \rho = \lambda/\mu, \quad b(i) = R(0)\sum_{\nu=i}^{\infty} R(\nu) - R(i).$$

It follows that

$$\begin{gathered} f(1) = \rho(1 - r_0)f(0) + b(1), \\ f(2) = \rho(1 - r_1)\rho(1 - r_0)f(0) + \rho(1 - r_1)b(1) + b(2), \\ f(3) = \rho(1 - r_2)\rho(1 - r_1)\rho(1 - r_0)f(0) + \\ + \rho(1 - r_2)\rho(1 - r_1)b(1) + \rho(1 - r_2)b(2) + b(3). \end{gathered}$$

The general form of these expressions is

$$f(i) = f(0)\rho^i \prod_{k=0}^{i-1}(1 - r_k) + \sum_{\nu=1}^{i-1} b(\nu) \prod_{k=\nu}^{i-1}(1 - r_k) + b(i). \tag{27}$$

To get $f(0)$, we use the condition $\mathbf{fE} = 0$, that is $\sum_{i=0}^{\infty} f(i) = 0$. Then

$$\begin{aligned} 0 &= \sum_{i=0}^{\infty} f(i) \\ &= f(0) + \sum_{i=0}^{\infty}\left\{ f(0)\rho^i \prod_{k=0}^{i-1}(1 - r_k) + \sum_{\nu=1}^{i-1} b(\nu) \prod_{k=\nu}^{i-1}(1 - r_k) + b(i)\right\} \\ &= f(0)\left\{1 + \sum_{i=0}^{\infty} \rho^i \prod_{k=0}^{i-1}(1 - r_k)\right\} + \sum_{i=0}^{\infty}\left\{\sum_{\nu=1}^{i-1} b(\nu) \prod_{k=\nu}^{i-1}(1 - r_k) + b(i)\right\}. \end{aligned} \tag{28}$$

Finally, we obtain

$$-f(0) = \frac{\displaystyle\sum_{i=0}^{\infty}\left\{\sum_{\nu=1}^{i-1} b(\nu) \prod_{k=\nu}^{i-1}(1 - r_k) + b(i)\right\}}{\displaystyle 1 + \sum_{i=0}^{\infty} \rho^i \prod_{k=0}^{i-1}(1 - r_k)} = \mathbf{fBE}. \tag{29}$$

This expression determines κ_2 and, accordingly, $\mathrm{D}\{n(t)\}$.

7.3 A Specific Case

Let

$$r_i = \begin{cases} 0, & \text{if } i < N, \\ 1, & \text{if } i = N, \end{cases}$$

In this case

$$R(i) = R(0)\rho^i, \quad R(0) = \frac{1}{\sum_{k=0}^{N} \rho^k} = \frac{1 - \rho}{1 - \rho^{N+1}}.$$

Next

$$f(i) = f(0)\rho^i + \sum_{\nu=1}^{i-1} b(\nu) + b(i) = f(0)\rho^i + \sum_{\nu=1}^{i-1} b(\nu), \quad i \le N,$$

$$0 = \sum_{i=0}^{N} f(i) = f(0) \sum_{i=0}^{N} \rho^i + \sum_{\nu=1}^{i-1} (N+1-\nu) b(\nu).$$

It follows that

$$-f(0) = \frac{\sum\limits_{\nu=1}^{i-1} (N+1-\nu) b(\nu)}{\sum\limits_{i=0}^{N} \rho^i}.$$

Lets compute $b(i)$. So, finally,

$$-f(0) = \frac{\sum\limits_{\nu=1}^{i-1} (N+1-\nu) b(\nu)}{\sum\limits_{i=0}^{N} \rho^i}, \quad b(i) = R(0)(\rho^i - 1)\frac{\rho^{N+1}}{\rho^{N+1}}. \tag{30}$$

If $f(0) \neq 0$, then the output flow is not Poisson.

8 Specific Cases of the Flow $m(t)$

Lets investigate two simplest specific cases of the output flow $m(t)$. Recall that the flow deals with rejected requests.

1. Let $\forall i \quad r_i = r$. Then the flow $m(t)$ is the simplest one with the parameter λr (simplest sifted flow).
2. Let

$$r_i = \begin{cases} 0, & \text{if } i < N, \\ 1, & \text{if } i = N, \end{cases}$$

this means that we deal with a M/M/1/N system. It follows that the flow $m(t)$ is a recurrent phase flow, since the lengths of its intervals match the time it takes the process $i(t)$ to return back to the state N before the first request is lost.

Indeed, lets take a look at the interval between the time t and the time t when the request leaves the system.

We adopt the following notation for $i \leq N$

$$g_i(\alpha, t) = \mathrm{M}\{e^{j\alpha(t_n - t)} \mid i(t) = i\}.$$

Then if $i < N$, we have

$$\begin{gathered} g_0(\alpha, t - \Delta t) = (1 - \lambda \Delta t) e^{j\alpha \Delta t} g_0(\alpha, t) + \lambda \Delta t g_1(\alpha, t) + o(\Delta t), \\ g_i(\alpha, t - \Delta t) = [1 - (\lambda + \mu)\Delta t)]\, e^{j\alpha \Delta t} g_i(\alpha, t) + \lambda \Delta t g_{i+1}(\alpha, t) \\ + \mu \Delta t g_{i-1}(\alpha, t) + o(\Delta t), \\ g_N(\alpha, t - \Delta t) = [1 - (\lambda + \mu)\Delta t)]\, e^{j\alpha \Delta t} g_N(\alpha, t) + \lambda \Delta t \cdot 1 \\ + \mu \Delta t g_{N-1}(\alpha, t) + o(\Delta t). \end{gathered}$$

Set $g_i(\alpha, t) \equiv g_i(\alpha)$, then

$$\begin{gathered} (\lambda + j\alpha) g_0(\alpha) = \lambda g_1(\alpha), \\ (\lambda + \mu + j\alpha) g_i(\alpha) = \lambda g_{i+1}(\alpha, t) + \mu g_{i-1}(\alpha), \quad 0 < i < N, \\ (\lambda + \mu + j\alpha) g_N(\alpha) = \lambda + \mu g_{N-1}(\alpha). \end{gathered}$$

From this system we get the conditional characteristic function

$$g_N(\alpha, t) = \mathrm{M}\{e^{j\alpha(t_n - t)} \mid i(t) = N\}.$$

The function is of the length of the time interval between t (when our QS is in the state N) and t_n (when the request leaves the system). Since the exponential distribution has no long-term memory, the distribution of the length of the remaining interval matches that of the length of the full interval.

9 Investigation of the Output Request Flow

The two dimensional random process $\{i(t), m(t)\}$ is a Markov chain. For the process probability distribution function

$$P(i, m, t) = P\{i(t) = i, m(t) = m\}$$

we can write the following expression

$$\begin{cases} \dfrac{\partial P(i, m, t)}{\partial t} = -[\lambda + \mu] P(i, m, t) + \lambda r_i P(i, m - 1, t) \\ + \lambda(1 - r_{i-1}) P(i - 1, m, t) + \mu P(i + 1, m, t), \\ \dfrac{\partial P(0, m, t)}{\partial t} = -\lambda P(0, m, t) + \lambda r_0 P(0, m - 1, t) + \mu P(1, m, t). \end{cases} \tag{31}$$

Denote the sum

$$H(i, u, t) = \sum_{n=0}^{\infty} e^{jum} P(i, m, t), \tag{32}$$

then we get the following system of equations

$$\begin{cases} \dfrac{\partial H(0,u,t)}{\partial t} = -\lambda(1-r_0)H(0,u,t) + \lambda r_0(e^{ju}-1)H(0,u,t) + \mu H(1,u,t), \\ \dfrac{\partial H(i,u,t)}{\partial t} = \lambda(1-r_{i-1})H(i-1,u,t) - [\lambda(1-r_i)+\mu]H(i,u,t) \\ +\mu H(i+1,u,t) + (e^{ju}-1)\lambda r_i H(i,u,t). \end{cases}$$

We can combine $H(i,u,t)$ into the row-vector

$$\mathbf{H}(u,t) = \{H(0,u,t)H(1,u,t),\dots\}$$

so that the system becomes

$$\frac{\partial \mathbf{H}(u,t)}{\partial t} = \mathbf{H}(u,t)\{\mathbf{Q} + \lambda(e^{ju}-1)\mathbf{r}\}, \tag{33}$$

where $\mathbf{r}$ is the diagonal matrix with elements r_i, the matrix $\mathbf{Q}$ is the three-diagonal infinitesimal matrix of the birth-and-death process $i(t)$; the matrix is shown in (4).

Just as we did for the serviced request flow, we take the following initial condition for the differential-matrix equation (33)

$$\mathbf{H}(u,0) = \mathbf{R} = \mathbf{H}(0,t),$$

where $\mathbf{R}$ is the row-vector of the stationary probability distribution of the Markov chain of the process $i(t)$; recall that R was obtained already and has the following properties: $\mathbf{RQ} = 0$, $\mathbf{RE} = 1$.

Thus, for the row-vector $= \mathbf{H}(u,t)$ we have the following Cauchy problem

$$\begin{cases} \dfrac{\partial \mathbf{H}(u,t)}{\partial t} = \mathbf{H}(u,t)\{\mathbf{Q} + \lambda(e^{ju}-1)\mathbf{r}\}, \\ \mathbf{H}(u,0) = \mathbf{R}. \end{cases} \tag{34}$$

The solution of this problem uniquely determines the characteristic function of $m(t)$ by means of the relation

$$\mathrm{M}\left\{e^{ium(t)}\right\} = \mathbf{H}(u,t)\mathbf{E}. \tag{35}$$

9.1 Method of Moments

Lets denote

$$\mathbf{m}_1(t) = \frac{1}{j}\frac{\partial \mathbf{H}(u,t)}{\partial u}\bigg|_{u=0}$$

then from (34) we get

$$\frac{d\mathbf{m}_1(t)}{dt} = \mathbf{m}_1(t)\mathbf{Q} + \lambda \mathbf{R}\mathbf{r}.$$

Since

$$\mathrm{M}\{m(t)\} = \mathbf{m}_1(t)\mathbf{E},$$

then

$$\mathrm{M}\{m(t)\} = \lambda \mathbf{RrE} \cdot t = \kappa_1 t \tag{36}$$

and

$$\kappa_1 = \lambda \mathbf{RrE} = \lambda \sum_{i=0}^{\infty} r_i R(i).$$

This expression can be simplified. As we derived the stationary distribution $R(i)$, we obtained the following relation

$$\kappa_1 = \lambda - \mu(1 - R(0)), \quad \mathrm{M}\{m(t)\} = [\lambda - \mu(1 - R(0))] \cdot t. \tag{37}$$

Notice that the following relation is true

$$\mathrm{M}\{n(t)\} + \mathrm{M}\{m(t)\} = \lambda t$$

that is quite natural.

9.2 Solution of the Problem (34) by Means of Fourier Transform

Lets do the Fourier transform of $\mathbf{H}(u,t)$ over t

$$\mathbf{Y}(u, \alpha) = \int_0^{\infty} e^{j\alpha t} \mathbf{H}(u,t) dt. \tag{38}$$

Then, similar to the findings of paragraph Sect. 6 of this article, we get

$$\mathbf{P}(m,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\alpha t} \mathbf{R} \left[(\lambda \mathbf{r} - \mathbf{Q} - j\alpha \mathbf{I})^{-1} \lambda \mathbf{r} \right]^m (\lambda \mathbf{r} - \mathbf{Q} - j\alpha \mathbf{I})^{-1} \mathbf{E} d\alpha.$$

It is problematic to compute this formula numerically since we need to compute the product of the inverse of the infinitely large matrices and to compute the improper integrals. So we seek an approximate asymptotic solution of (34) as $t \to \infty$.

9.3 Asymptotic Solution of (34)

Let be large enough. We investigate the limit $T \to \infty$. We call the condition $t = \tau T$, where $0 \leq \tau < \infty$ the asymptotic condition of increasing time. Recall that $m(t)$ is the number of events that appeared in the unserviced request flow during the time t. Also recall that

$$\mathbf{H}(u,t) \cdot \mathbf{E} = \mathrm{M}\left\{ e^{ium(t)} \right\}$$

is the characteristic function of $m(t)$. Lets call $h_1(u,t)$ the first order asymptotic of $\mathbf{H}(u,t)$

$$h_1(u,t) = \exp\{ju\kappa_1 t\},$$

where the constant κ_1 was already determined and it value is

$$\kappa_1 = \lambda \mathbf{RrE} = \lambda - \mu(1 - R(0)).$$

9.4 Second Order Asymptotic

To obtain the second order asymptotic in the equation of the problem (34) lets make the substitution

$$\mathbf{H}(u,t) = \mathbf{H}_2(u,t)\exp\{ju\kappa_1 t\}. \tag{39}$$

Then for the function $\mathbf{H}_2(u,t)$ we get the equation

$$\frac{\partial \mathbf{H}_2(u,t)}{\partial t} = \mathbf{H}_2(u,t)\left\{\mathbf{Q} + \lambda\left(e^{ju} - 1\right)\mathbf{r} - ju\kappa_1 \mathbf{I}\right\},$$

In this equation we make the substitutions

$$t\varepsilon^2 = \tau, u = \varepsilon w, \mathbf{H}_2(u,t) = \mathbf{F}_2(w,\tau,\varepsilon). \tag{40}$$

where $\varepsilon^2 = 1/T$, to get

$$\varepsilon^2 \frac{\partial \mathbf{F}_2(w,\tau,\varepsilon)}{\partial t} = \mathbf{F}_2(w,\tau,\varepsilon)\left\{\mathbf{Q} + \lambda\left(e^{j\varepsilon w} - 1\right)\mathbf{r} - j\varepsilon w\kappa_1 \mathbf{I}\right\}, \tag{41}$$

This equation is similar to the solution of Eq. (16) is solved in two steps, then you can write the asymptotic $h_2(u,t)$ of the second order for the characteristic function of $m(t)$

$$h_2(u,t) = \exp\left\{\frac{(ju)^2}{2}\kappa_2)t\right\}.$$

Here, the value κ_2 is defined

$$\kappa_2 = \lambda \mathbf{RrE} + 2\lambda \mathbf{f_2 rE} = \kappa_1 + 2\lambda \mathbf{f_2 rE},$$

where $\mathbf{f_2}$ is defined by

The general form of these expressions is

$$f_2(i) = f_2(0)\rho^i \prod_{k=0}^{i-1}(1 - r_k) + \sum_{\nu=1}^{i-1} b(\nu) \prod_{k=\nu}^{i-1}(1 - r_k) + b(i).$$

$$-f(0) = \frac{\sum_{i=0}^{\infty}\left\{\sum_{\nu=1}^{i-1} b(\nu)\prod_{k=\nu}^{i-1}(1 - r_k) + b(i)\right\}}{1 + \sum_{i=0}^{\infty}\rho^i \prod_{k=0}^{i-1}(1 - r_k)},$$

where

$$R(0)\sum_{\nu=i}^{\infty}R(\nu)-R(i)=b(i).$$

If follows that as $t\to\infty$, the variable $m(t)$ is asymptotically normal with the mean $\{m(t)\}=\kappa_1 t$ and variance $\mathrm{D}\{m(t)\}=\kappa_2 t$.

Obviously, if **f2rE** $\neq 0$, then the flow $m(t)$ is not Poisson, since the necessary condition $\mathrm{M}\{m(t)\}=\mathrm{D}\{m(t)\}$ is violated.

10 Conclusion

1. So, we presented a new model of a retail facility. The model is a type M/M/1/∞ queuing system with request rejection.
2. The output flow of the system and the flow of flow of rejected requests are researched. Exact formulas for average number of events that occurred in both flows are determined. Prelimit distributions of probabilities of number of events that occurred in these flows in form of integral transformation are found.
3. The asymptotic distributions of probabilities of number of events that occurred in flows $n(t)$ and $m(t)$ are found by means of asymptotic analysis proposed by A.A. Nazarov.

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