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Switch-Hysteresis Control of the Selling Times Flow in a Model with Perishable Goods

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Abstract. In this paper we obtain the probability density function of stock of perishable goods under constant production and hysteresis control of the selling price.

Keywords: Perishable goods · Hysteresis control · Probability density function · Diffusion approximation

1 Introduction

Mathematical models and methods of queueing theory [1,2] are widely used in various fields and, in particular, can be used to analyze the problems of inventory management with a limited shelf life, which have been intensively studied in recent years. Several review articles on the topic appeared during that time, for example S.K. Goyal, B.C. Giri [3], M. Bakker, J. Riezebos, R.H. Teunter [4]. Also worth noting are papers by V.K. Mishra, V.K. Mishra and L.S. Singh [5,6], R. Begum, S.K. Sahu, R.R. Sahoo [7,8], R.P. Tripathi, D. Singh, T. Mishra [9], where authors consider models of inventory management of continuously deteriorating goods under the assumption of a known demand function. In V. Sharma and R.R. Chaudhary [10] a model is considered where demand is known function of time, while the deterioration process is random and follows Weibull distribution. In K. Tripathy and U. Mishra [11] a model is considered in which demand is a known function of price. To analyse the mathematical models one can employ the methods of asymptotic analysis that are widely used in the queuing theory, for example in the mentioned above works by A.A. Nazarov [1], A.A. Nazarov and S.P. Moiseeva [2].

2 Mathematical Model of the Problem

We consider a single-line queueing system (Fig. 1) in the entrance of which applications (perishable goods) with arrival rate c come in. We assume that arrival process can be approximated in such a way that c units arrives per unit time.

The goods continuously deteriorate as they are stored. Let $S(t)$ be the amount of goods at time t. Then during a small time interval Δt a total of $kS(t)\Delta t$

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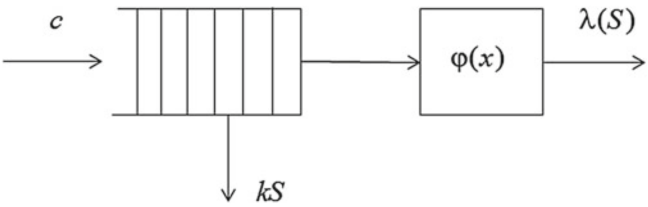

Fig. 1. Mathematical model

is lost. The service, which in this work will be called sales, is provided by parties with random size x, where the values of purchases x are independent random variable with probability density function $\varphi(x)$, mean $M\{x\} = a$ and a second moment $M\{x^2\} = a_2$. Selling times follow a Poisson process with intensity λ that depend on selling price b. We consider the case when the intensity of sales λ monotonically decreases as b grows. For a given price b and, hence, sales process intensity λ the average amount of goods $\bar{S}(t)$ is defined as

$$\bar{S}(t) = S(0)e^{-kt} + \frac{c - \lambda a}{k}(1 - e^{-kt}).$$

Thus if $c - \lambda a > 0$ and $t \gg 1$ we have a constant stock of unsold goods which is undesirable. If $c - \lambda a \leq 0$ we have unsatisfied demand. Hence we need to control either selling price b, or the pace of goods arrival c depending on current stock.

In this paper we assume that sales are controlled in the following way. First, two boundary values for the stock of goods are set, S_1 S_2, such that $S_2 > S_1$. For $S < S_1$ a selling price b_0 is established, for $S > S_2$ a selling price $b_1 < b_0$ is established. For $S_1 \leq S \leq S_2$ the selling price will be either $b = b_0$ or $b = b_1$ depending on the trajectory which the process $S(t)$ followed when it entered this domain. If it crossed the lower bound S_1 upwards then $b = b_0$, while if it crossed the upper bound S_2 downwards, then $b = b_1$. Thus the selling price $b = b_1$ is set as soon as $S(t)$ reaches S_2 and lasts until the stock falls to S_1. The domain $S_1 \leq S \leq S_2$ is in fact what we call the domain of hysteresis stock control. In accordance with this, the intensity of selling times flow at any given moment is given by

$$\lambda(S) = \begin{cases} \lambda_0, \ S < S_1, \\ \lambda_0 \ \text{ or } \qquad \lambda_1, \ S_1 \leq S \leq S_2, \\ \lambda_1, S > S_2 \end{cases} \tag{1}$$

It is natural to assume that $C - \lambda_0 a > 0$ and $C - \lambda_1 a < 0$. Finally, there may be a situation when current demand cannot be fully satisfied by the current stock of goods. In such case we assume that $S(t) < 0$. The orders are satisfied in the order of arrival.

The main goal of this paper is to determine the probability density function of the stock of goods in this model and several additional assumptions.

Denote

$$P_i(S,t) = \frac{\Pr\{S \leq S(t) < S + dS, \ \lambda(t) = \lambda_i\}}{dS}, \quad i = 0,1. \tag{2}$$

Theorem 1. *If $P_i(S,t)$ is differentiable in t and $SP_i(S,t)$ is differentiable in S then functions $P_i(S,t)$ satisfy the following system of equations Kolmogorov*

$$\frac{\partial P_1(S,t)}{\partial t} = -\lambda_1 P_1(S,t) - \frac{\partial}{\partial S}((c-kS)P_1(S,t)) + \lambda_1 \int_0^\infty P_1(S+x)\varphi(x)dx, \; S \geq S_1, \tag{3}$$

$$\begin{aligned} &\tfrac{\partial P_0(S,t)}{\partial t} = -\lambda_0 P_0(S,t) - \tfrac{\partial}{\partial S}((c-kS)P_0(S,t)) \\ &+\lambda_0 \int_S^{S_2} P_0(x,t)\varphi(x-S)dx \;, \quad S_1 < S < S_2, \end{aligned} \tag{4}$$

$$\begin{aligned} &\tfrac{\partial P_0(S,t)}{\partial t} = -\lambda_0 P_0(S,t) - \tfrac{\partial}{\partial S}((c-kSI(S))P_0(S,t)) \\ &+\lambda_0 \int_S^{S_2} P_0(x,t)\varphi(x-S)dx + \lambda_1 \int_{S_1}^{\infty} P_1(x)\varphi(x-S)dx, \quad S \leq S_1, \end{aligned} \tag{5}$$

where $I(x)$ is a step unit function.

Proof. Consider two close moments of time t and $t+\Delta t$, where $\Delta t \ll 1$. Under given assumptions the conditional probabilities

$$\begin{gathered} P\left\{S(t+\Delta t) < z, \lambda(t+\Delta t) = \lambda_1 \,|S(t) = S, \lambda(t) = \lambda_1 \right\} = \\ (1-\lambda_1 \Delta t)I(z-S-(c-kS)\Delta t) + \lambda_1 \Delta t \int_0^{S-S_1} I(z-S+x)\varphi(x)dx + o(\Delta t), \end{gathered} \tag{6}$$

$$P\left\{S(t+\Delta t) < z, \lambda(t+\Delta t) = \lambda_1 \,|S(t) = S, \lambda(t) = \lambda_0 \right\} = 0. \tag{7}$$

Thus for $z \geq S_1$ probability

$$\begin{gathered} P\left\{S(t+\Delta t) < z, \lambda(t+\Delta t) = \lambda_1\right\} = \\ (1-\lambda_1 \Delta t) \int_{S_1}^{\infty} I(z-S-(c-kS)\Delta t)P_1(S,t)dS \\ +\lambda_1 \Delta t \int_{S_1}^{\infty} \int_0^{S-S_1} I(z-S+x)\varphi(x)dxP_1(S,t)dS + o(\Delta t). \end{gathered} \tag{8}$$

For $z \geq S_1$ and a small Δt the integral

$$\begin{gathered} \int_{S_1}^{\infty} I(z-S-(c-kS)\Delta t)P_1(S,t)dS = \int_{S_1}^{z-(c-kz)\Delta t + o(\Delta t)} P_1(S,t)dS \\ = \int_{S_1}^{z} P_1(S,t)dS - P_1(z,t)(c-kz)\Delta t + o(\Delta t), \end{gathered}$$

and the integral

$$\int_{S_1}^{\infty} \int_0^{S-S_1} I(z-S+x)\varphi(x)dxP_1(S,t)dt = \int_0^{\infty} \varphi(x) \int_{S_1+x}^{z+x} P_1(S,t)dSdx.$$

Substituting the expressions above into (8), differentiating with respect to z and taking the limit $\Delta t \to 0$ we arrive at Eq. (3).

Furthermore, the conditional probabilities

$$\begin{gathered} P\left\{S(t+\Delta t)<z, \lambda(t+\Delta t)=\lambda_0 \,|S(t)=S, \lambda(t)=\lambda_0 \right\}= \\ (1-\lambda_0\Delta t)I(z-S-(c-kS)\Delta t)+\lambda_0\Delta t\int\limits_0^\infty I(z-S+x)\varphi(x)dx+o(\Delta t), \end{gathered} \tag{9}$$

$$\begin{gathered} P\left\{S(t+\Delta t)<z, \lambda(t+\Delta t)=\lambda_0 \,|S(t)=S, \lambda(t)=\lambda_1 \right\}= \\ \lambda_1\Delta t\int\limits_{S-S_1}^\infty I(x-S+x)\varphi(x)dx+o(\Delta t). \end{gathered} \tag{10}$$

From where in the domain $z \le S_2$ probability

$$\begin{gathered} P\left\{S(t+\Delta t)<z, \lambda(t+\Delta t)=\lambda_0 \right\}= \\ (1-\lambda_0\Delta t)\int\limits_{-\infty}^{S_2} I(z-S-(c-kS)\Delta t)P_0(S,t)dS \\ +\lambda_0\Delta t\int\limits_{-\infty}^{S_2}\int\limits_0^\infty I(z-S+x)\varphi(x)dxP_0(S,t)dS \\ +\lambda_1\Delta t\int\limits_{S_1}^\infty\int\limits_{S-S_1}^\infty I(z-S+x)\varphi(x)dxP_1(S,t)dS+o(\Delta t). \end{gathered} \tag{11}$$

For $z \le S_2$ and a small Δt

$$\begin{gathered} \int\limits_{-\infty}^{S_2} I(z-S-(c-kS)\Delta t)P_0(S,t)dS=\int\limits_{-\infty}^{z-(c-kz)\Delta t+o(\Delta t)} P_0(S,t)dS= \\ \int\limits_{-\infty}^{z} P_0(S,t)dS-(c-kz)P_0(z,t)\Delta t+o(\Delta t), \end{gathered}$$

and the integral

$$\int\limits_{-\infty}^{S_2}\int\limits_0^\infty I(z-S+x)\varphi(x)dxP_0(S,t)dS=\int\limits_{-\infty}^{z} P_0(S,t)ds+\int\limits_{z}^{S_2}\int\limits_{S-z}^\infty \varphi(x)dxP_0(S,t)dS.$$

Finally, for $S_1 < S < S_2$ the integral

$$\int\limits_{S_1}^\infty\int\limits_{S-S_1}^\infty I(z-S+x)\varphi(x)dxP_1(S,t)dS=\int\limits_{S_1}^\infty\int\limits_{S-S_1}^\infty \varphi(x)dxP_1(S,t)dS,$$

while for $z \le S_1$ the integral

$$\int\limits_{S_1}^\infty\int\limits_{S-S_1}^\infty I(z-S+x)\varphi(x)dxP_1(S,t)dS=\int\limits_{S_1}^\infty\int\limits_{S-z}^\infty \varphi(x)dxP_1(S,t)dS.$$

Substituting the expressions above into (11), differentiating with respect to z and taking the limit $\Delta t \to 0$ we arrive at Eqs. (4) and (5).

The solution of the system (3)–(5) must, apparently, satisfy the following normalising condition

$$\int_{S_1}^{\infty} P_1(S,t)dS + \int_{-\infty}^{S_2} P_0(S,t)dS = 1 \qquad (12)$$

while function $P_0(S,t)$ must be continuous at point S_1

$$P_0(S_1+0,t) = P_0(S_1-0,t). \qquad (13)$$

The unconditional probability density function $P(S,t)$ of the stock of goods takes the form

$$P(S,t) = \begin{cases} P_1(S,t), & S > S_2, \\ P_1(S,t) + P_0(S,t), & S_1 \le S \le S_2, \\ P_0(S,t), & S < S_1. \end{cases} \qquad (14)$$

3 Exponential Distribution of the Sale Amount

Let us consider the simplest case when sales are distributed exponentially

$$\varphi(S) = \frac{1}{a}\exp(-\frac{S}{a}).$$

Denote

$$P_i(S) = \lim_{t\to\infty} P_i(S,t). \qquad (15)$$

In the steady state as $t \to \infty$ Eqs. (3)–(5) take the form

$$\lambda_1 P_1(S) + \frac{d}{dS}((c-kS)P_1(S)) - \frac{\lambda_1}{a}e^{\frac{S}{a}}\int_S^{\infty} P_1(x)e^{-\frac{x}{a}}dx = 0, \quad S > S_1, \qquad (16)$$

$$\lambda_0 P_0(S) + \frac{d}{dS}((c-kS)P_0(S)) - \frac{\lambda_0}{a}e^{\frac{S}{a}}\int_S^{S_2} P_0(x)e^{-\frac{x}{a}}dx = 0, \quad S_1 \le S \le S_2, \qquad (17)$$

$$\begin{gathered} \lambda_0 P_0(S) + \tfrac{d}{dS}((c-kSI(S))P_0(S)) - \tfrac{\lambda_0}{a}e^{\frac{S}{a}}\int_S^{S_2} P_0(x)e^{-\frac{x}{a}}dx \\ -\tfrac{\lambda_1}{a}e^{\frac{S}{a}}\int_{S_1}^{\infty} P_1(x)e^{-\frac{x}{a}}dx = 0, \quad S < S_1. \end{gathered} \qquad (18)$$

Equation (18) can be differentiated and represented as the following differential equation

$$\frac{d^2}{dS^2}((c-kSI(S))P_0(S)) - \frac{d}{dS}(\frac{c-kSI(S)-\lambda_0 a}{a}P_0(S)) = 0. \qquad (19)$$

From here, taking into account boundary condition $P_0(-\infty) = 0$ in the domain $S \leq 0$

$$P_0(S) = De^{\frac{c-\lambda_0 a}{ca}S}. \tag{20}$$

In the domain $0 < S < S_1$ the solution of (19) takes the form

$$P_0(S) = \left[W_1 + W_2 \int_0^S e^{-\frac{x}{a}}(c - kx)^{-\frac{\lambda_0}{k}} dx\right] e^{\frac{S}{a}}(c - kS)^{\frac{\lambda_0}{k}-1}. \tag{21}$$

The condition of continuity of the solution in $S = 0$ yields $D = W_1 c^{\frac{\lambda_0}{k}-1}$. From (18) it follows that in $S = 0$ a condition must holds:

$$cP_0'(0+0) - kP_0(0+0) = cP_0'(0-0).$$

From where $W_2 = 0$. Thus, for $0 < S < S_1$

$$P_0(S) = De^{\frac{S}{a}}(1 - \frac{k}{c}S)^{\frac{\lambda_0}{k}-1}. \tag{22}$$

Equation (17) can be differentiated and represented as the following differential equation

$$\frac{d^2}{dS^2}((c - kS)P_0(S)) - \frac{d}{dS}(\frac{c - kS - \lambda_0 a}{a}P_0(S)) = 0. \tag{23}$$

Its solution takes the form

$$P_0(S) = \left[W_1 + W_2 \int_{S_1}^S e^{-\frac{x}{a}}(1 - \frac{k}{c}x)^{-\frac{\lambda_0}{k}} dx\right] e^{\frac{S}{a}}(1 - \frac{k}{c}S)^{\frac{\lambda_0}{k}-1}. \tag{24}$$

The condition of continuity in the point S_1 of (13) gives

$$W_1 = D. \tag{25}$$

Furthermore, solution (24) must satisfy the initial Eq. (17). Then

$$W_2 = -D[ae^{-\frac{S_2}{a}}(1 - \frac{k}{c}S_2)^{-\frac{\lambda_0}{k}} + \int_{S_1}^{S_2} e^{-\frac{x}{a}}(1 - \frac{k}{c}x)^{-\frac{\lambda_0}{k}} dx]^{-1}. \tag{26}$$

Finally, given that in the model considered the amount of goods is always $S \leq \frac{c}{k}$, the solution of (16) takes the form

$$P_1(S) = Ae^{\frac{S}{a}}(1 - \frac{k}{c}S)^{\frac{\lambda_1}{k}-1}. \tag{27}$$

The relationship between constants A and D is obtained from the condition that the set of found solutions must satisfy (18). Then

$$A = -ae^{-\frac{S_1}{a}}(1 - \frac{k}{c}S_1)^{-\frac{\lambda_1}{k}} W_2, \tag{28}$$

where W_2 is determined by the ratio (26). The last constant D is obtained from the normalising condition (12).

To sum up, the probability density function of the stock of goods is determined by (20), (22), (24), (27), while constants in these expressions are obtained from conditions (25), (26), (28) and (12).

For $S_2 = S_1$ we get the case of switch (threshold) control of the selling price and the probability density function $P(S)$ takes the form

$$P(S) = \begin{cases} De^{\frac{c-\lambda_0 a}{ca}S}, & S < 0, \\ D(1-\frac{k}{c}S)^{\frac{\lambda_0}{k}-1}e^{\frac{S}{a}}, & 0 \le S \le S_1, \\ D(1-\frac{k}{c}S_1)^{\frac{\lambda_0-\lambda_1}{k}}(1-\frac{k}{c}S)^{\frac{\lambda_1}{k}-1}e^{\frac{S}{a}}, & S_1 < S \le \frac{c}{k}, \end{cases} \tag{29}$$

where D is determined by the normalising condition.

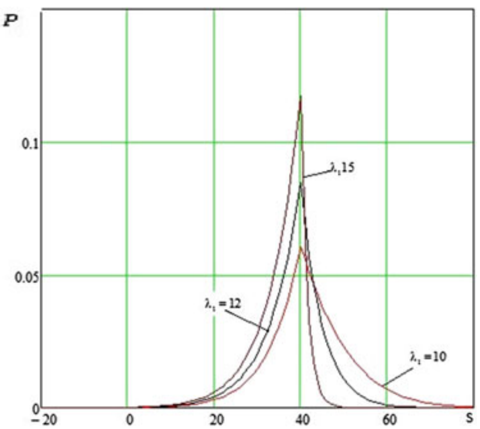

Fig. 2. Relationship between the probability density $P(S)$ and the stock size S.

The relationship between the probability density function $P(S)$ and the stock size S is given on Fig. 1. Threshold $S_1 = 40$, $\lambda_0 = 8$, $c = 10, k = 0.02, a = 1$ (Fig. 2).

4 Diffusion Approximation of the Production/Sales Process Under Switch Control of the Selling Price

In a general case the solution of the system (3)–(5) cannot be obtained even in the stationary mode. Hence in the following we focus on constructing an approximate solution. Consider the case of the switch control of the selling price when thresholds $S_2 = S_1$. The system (3)–(5) can be rewritten to yield

$$\begin{aligned} \frac{\partial P(S,t)}{\partial t} &= \frac{\partial}{\partial S}\left[(kSI(S)-c)P(S,t)\right] \\ &-\lambda(S)P(S,t) + \int_0^\infty \lambda(S+y)P(S+y,t)\varphi(y)dy, \end{aligned} \tag{30}$$

where

$$\lambda(S) = \begin{cases} \lambda_0, \ S \le S_1, \\ \lambda_1, \ S > S_1 \ . \end{cases} \tag{31}$$

Let us assume that the production speed $c = CN$, purchase process' intensities $\lambda_0 = \Lambda_0 N,\ \lambda_1 = \Lambda_1 N$, threshold $S_1 = S_0 N$, where $N \gg 1$. Let us analyse the behaviour of the solution of (30) as $N \to \infty$. Denote $\varepsilon^2 = 1/N$. Let us introduce a function

$$F(S, t, \varepsilon) = P(\frac{S}{\varepsilon}, t). \tag{32}$$

Consider first the domain $S > S_0$. Equation (30) in this domain takes the form

$$\begin{aligned} &\varepsilon^2 \tfrac{\partial F(y,t,\varepsilon)}{\partial t} + \Lambda_1 F(y, t, \varepsilon) = \\ &\varepsilon \tfrac{\partial}{\partial y} \left[(k\varepsilon y - C) F(y, t, \varepsilon) \right] + \Lambda_1 \int\limits_0^\infty F(y + \varepsilon z, t, \varepsilon) \varphi(z) dz. \end{aligned} \tag{33}$$

Taking Taylor expansion of $F(y+\varepsilon z, t, \varepsilon)$ with respect to the first argument and focusing our analysis on the first three member of the sum we get

$$\varepsilon^2 \frac{\partial F(y,t,\varepsilon)}{\partial t} = \varepsilon \frac{\partial}{\partial y} \left[(k\varepsilon y - C + \Lambda_1 a) F(y, t, \varepsilon) \right] + \Lambda_1 \frac{a_2}{2} \varepsilon 2 \frac{\partial^2 F(y,t,\varepsilon)}{\partial y^2} + o(\varepsilon^2). \tag{34}$$

Introduce new variables

$$t = t, \quad u = y - \frac{1}{\varepsilon} x(t), \tag{35}$$

where the function $x(t)$ will be determined later on, and a function $Q(u, t, \varepsilon)$ such that

$$F(y, t, \varepsilon) = Q(y - \frac{1}{\varepsilon} x(t), t, \varepsilon). \tag{36}$$

We impose an additional condition on $x(t)$ to satisfy equation

$$\dot{x}(t) = -kx(t) + C - \Lambda_1 a. \tag{37}$$

Then for $Q(u, t, \varepsilon)$ we have

$$\frac{\partial Q(u,t,\varepsilon)}{\partial t} = \frac{\partial}{\partial u} \left[kuQ(u, t, \varepsilon) \right] + \frac{\Lambda_1 a_2}{2} Q(u, t, \varepsilon) + \frac{o(\varepsilon^2)}{\varepsilon^2}. \tag{38}$$

Let

$$Q(u, t) = \lim_{\varepsilon \to 0} Q(u, t, \varepsilon). \tag{39}$$

Then

$$\frac{\partial Q(u,t)}{\partial t} = \frac{\partial}{\partial u} \left[kuQ(u, t) \right] + \frac{\Lambda_1 a_2}{2} \frac{\partial^2 Q(u,t)}{\partial u^2}. \tag{40}$$

The stochastic differential equation that satisfies (40) for the process $u(t)$ is of the form

$$du(t) = -ku(t)dt + \sqrt{\Lambda_1 a_2} dW(t), \tag{41}$$

where $W(t)$ – is a standard Wiener process.

From (37) and (41), accounting for the variable changes been made, we have for the process $\xi(t) = \varepsilon^2 S(t)$ when $\varepsilon \ll 1$ that

$$d\xi(t) = -k\xi(t)dt + (C - \Lambda_1 a)dt + \sqrt{\Lambda_1 a_2}\varepsilon dW(t). \tag{42}$$

Let

$$h(z,t) = \frac{\partial \Pr\{\xi(t) < z\}}{\partial z}. \tag{43}$$

According to (42) probability density function $h(z,t)$ satisfies

$$\frac{\partial h(z,t)}{\partial t} = -\frac{\partial}{\partial z}\left[(C - \Lambda_1 a - kz)h(z,t)\right] + \frac{\Lambda_1 a_2}{2}\varepsilon^2 \frac{\partial^2 h(z,t)}{\partial z^2}. \tag{44}$$

In a steady state we get for probability density function

$$h(z) = \lim_{t\to\infty} h(z,t)$$

$$\frac{\Lambda_1 a_2 \varepsilon^2}{2}\frac{d^2 h(z)}{dz^2} + \frac{d}{dz}\left[(\Lambda_1 a - C + kz)h(z)\right] = 0. \tag{45}$$

From where accounting for the boundary condition $h(\infty) = 0$ we obtain

$$h(z) = Be^{-\frac{(\Lambda_1 a - C + kz)^2}{\Lambda_1 a_2 \varepsilon^2 k}}. \tag{46}$$

Consider now the domain $S < S_0$. Equation (30) with respect to function $F(S,t,\varepsilon)$ (32) now takes the form

$$\begin{gathered}\varepsilon^2 \tfrac{\partial F(y,t,\varepsilon)}{\partial t} + \Lambda_0 F(y,t,\varepsilon) = \\ \varepsilon \tfrac{\partial}{\partial y}\left[(k\varepsilon y I(y) - C)F(y,t,\varepsilon)\right] + \Lambda_0 \int\limits_0^\infty F(y+\varepsilon z,t,\varepsilon)\varphi(z)dz + R(y,\varepsilon)\end{gathered}, \tag{47}$$

where

$$R(y,\varepsilon) = (\Lambda_1 - \Lambda_0)\int\limits_{S_0 - \frac{y}{\varepsilon}}^{\infty} F(y+\varepsilon z,t,\varepsilon)\varphi(z)dz = o(\varepsilon^2),$$

since the function $F(y,t,\varepsilon)$ is bounded and the second moment a_2 exists. Hence we do not account for the last member of the sum in (47). Taking Taylor expansion of $F(y+\varepsilon z,t,\varepsilon)$ with respect to the first argument we get

$$\varepsilon^2 \frac{\partial F(y,t,\varepsilon)}{\partial t} = \varepsilon\frac{\partial}{\partial y}\left[(k\varepsilon y I(y) - C + \Lambda_0 a)F(y,t,\varepsilon)\right] + \Lambda_0 \frac{a_2}{2}\varepsilon^2 \frac{\partial^2 F(y,t,\varepsilon)}{\partial y^2} + o(\varepsilon^2). \tag{48}$$

Considery $y < 0$. Making substitutions (35) and (36) and assuming

$$\dot{x}(t) = C - \Lambda_0 a, \tag{49}$$

we have for $\varepsilon \to 0$ for the function (13)

$$\frac{\partial Q(u,t)}{\partial t} = \frac{\Lambda_0 a_2}{2}\frac{\partial^2 Q(u,t)}{\partial u^2}. \tag{50}$$

Let $y > 0$. Making substitutions (36) and (37) and assuming

$$\dot{x}(t) = -kx(t) + C - \Lambda_0 a, \tag{51}$$

we have for $\varepsilon \to 0$ for the function $Q(u,t)$ (39)

$$\frac{\partial Q(u,t)}{\partial t} = \frac{\partial}{\partial u}\left[kuQ(u,t)\right] + \frac{\Lambda_0 a_2}{2}\frac{\partial^2 Q(u,t)}{\partial u^2}. \tag{52}$$

It follows from (49)–(52) that for $\varepsilon \ll 1$ the process $\xi(t) = \varepsilon^2 S(t)$ satisfies a stochastic differential equation

$$d\xi(t) = -k\xi(t)I(\xi(t))dt + (C - \Lambda_0 a)dt + \sqrt{\Lambda_0 a_2}\varepsilon dW(t). \tag{53}$$

Thus the probability density function (43) satisfies the following equation

$$\frac{\partial h(z,t)}{\partial t} = -\frac{\partial}{\partial z}\left[(C - \Lambda_0 a - kzI(z))h(z,t)\right] + \frac{\Lambda_1 a_2}{2}\varepsilon^2\frac{\partial^2 h(z,t)}{\partial z^2},$$

whereas in a steady state for the probability density function $h(z)$ we have

$$\frac{\Lambda_0 a_2 \varepsilon^2}{2}\frac{d^2 h(z)}{dz^2} + \frac{d}{dz}\left[(\Lambda_0 a - C + kzI(z))h(z)\right] = 0. \tag{54}$$

Taking in account boundary condition $h(-\infty) = 0$ for $z < 0$ we obtain

$$h(z) = De^{\frac{2(C-\Lambda_0 a)}{\Lambda_0 a_2 \varepsilon^2}z}. \tag{55}$$

For $0 = z = s_0$ the solution of (54) takes the form

$$h(z) = (D_1 + D_2\int\limits_0^z e^{-\frac{(kx+C-\Lambda_0 a)^2}{k\Lambda_0 a_2 \varepsilon^2}}dx)e^{\frac{(kz+C-\Lambda_0 a)^2}{k\Lambda_0 a_2 \varepsilon^2}}. \tag{56}$$

In$z = 0$ the continuity conditions $h(0-0) = h(0+0),\ h'(0-0) = h'(0+0)$ must hold since function $h(z)$ satisfies a second-order differential equation. Hence $D_2 = 0$ and $D = D_1 e^{\frac{(c-\Lambda_0 a)^2}{\Lambda_0 a_2 \varepsilon^2}}$.

Thus the probability density function $h(s)$ is determined by the following expression

$$h(s) = \begin{cases} Ae^{\frac{(C-\Lambda_0 a)^2}{\Lambda_0 a_2 \varepsilon^2 k}}e^{\frac{2(C-\Lambda_0 a)}{\Lambda_0 a_2 \varepsilon^2}s}, & s < 0, \\ Ae^{\frac{(ks+C-\Lambda_0 a)^2}{\Lambda_0 a_2 \varepsilon^2 k}}, \ 0 \le s \le s_0, \\ Be^{-\frac{(ks+C-\Lambda_1 a)^2}{\Lambda_1 a_2 \varepsilon^2 k}}, & s > s_0\ . \end{cases} \tag{57}$$

The relationship between A and B follows, firstly, from the normalising condition

$$\int_{-\infty}^{0} h(s)ds + \int_{0}^{s_0} h(s)ds + \int_{s_0}^{\infty} h(s)ds = 1 ,$$

and, secondly, from Eq. (30) when $S = S_0$ in a steady state, which under the above takes the form

$$(kS_0 - c)\frac{\partial P(S_0,\infty)}{\partial S} + (k - \lambda_0)P(S_0,\infty) + \lambda_1 \int_{0}^{\infty} P(S_0 + y,\infty)\varphi(y)dy = 0. \quad (58)$$

Substituting the probability density $P(S,\infty)$ with its approximation (57) we get the second equation that describes the relationship between A and B. To obtain the final expressions one must, evidently, know the explicit form of the probability density function $\varphi(y)$.

5 Conclusion

In this paper we obtain expressions for the probability density function of the stock of perishable goods under constant arrival speed and switch-hysteresis control of the purchase process intensity. We also obtain the explicit solutions for the case of exponentially distributed purchase amounts and a diffusion approximation of the goods production/selling process under switch control of selling intensity. A similar approach can be used when considering other models of control for production and sales of perishable goods, in particular, a model with switch-hysteresis control of the production speed.

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