The Second Order Asymptotic Analysis Under Heavy Load Condition for Retrial Queueing System MMPP/M/1

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Abstract. In the paper, the retrial queueing system of $MMPP|M|1$ type is studied by means of the second order asymptotic analysis method under heavy load condition. During the investigation, the theorem about the form of the asymptotic characteristic function of the number of calls in the orbit is formulated and proved. The asymptotic distribution is compared with the exact one obtained by means of numerical algorithm. The conclusion about method application area is made.

Keywords: Retrial queue · Asymptotic analysis · Heavy load

1 Introduction

In queueing theory, there are two classes of queueing systems: systems with queue and loss systems. In real systems, there are situations when queue cannot be explicitly identified, but also call is not lost if it comes when the service device is unavailable. Often primary call does not refuse to be serviced and performs repeated calls to get the service after random time intervals. Examples of these situations are telecommunication systems, cellular networks, call-centres. Thus a new class of queueing systems has been appeared: systems with repeated calls or retrial queueing systems.

The first papers about retrial queues were published in the middle of 20th century. The most of them were devoted to practical problems and influence of repeated attempts on telephone traffic, communication systems etc. [1–4]. The most comprehensive description and detailed comparison of classical queueing systems and retrial queues are contained in books and papers authored by J.R. Artalejo, A. Gomez-Corral, G.I. Falin and J.G.C. Templeton [5–7].

Today there are many papers devoted to these systems. Scientists from different countries study different types of retrial queues, develop methods of their investigation, solve practical and theoretical problems in this area. But the majority of studies of retrial queueing systems are performed numerically or via computer simulation [8–10]. Belarusian researchers A.N. Dudin and V.I. Klimenok [11] mainly use matrix methods in their works. Also matrix methods for retrial queues analysis were used by M.F. Neuts, J.R. Artalejo, A. Gomez-Corral [12],

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J.E. Diamond, A.S. Alfa [13], etc. Asymptotic and approximate methods were applied by G.I. Falin [14], V.V. Anisimov [15], T. Yang [16], J.E. Diamond [17], B. Pourbabai [18], etc. But analytical results were obtained only in cases of simple input and service processes (e.g. stationary Poisson input process or the exponential distribution of service law) [6].

In this paper, we study the retrial queueing system $MMPP|M|1$ by means of the second order asymptotic analysis method under heavy load condition. Characteristics of performance of retrial queueing systems under heavy and light loads were studied by G.I. Falin [14], V.V. Anisimov [15] and A. Aissani [19]. Also S.N. Stepanov's work [20] is devoted to investigation under "extreme" load (the intensity of primary calls tends to infinity or zero).

In the paper we use the asymptotic analysis method developed by Tomsk scientific group for investigation of all types of queueing system and networks [21,22]. Principle of the method is derivation of asymptotic equations from the systems of equations determined models states and then getting formulas for asymptotic functions.

In a number of our previous papers (eg. [23]) devoted to the study of various single-server retrial queueing system, we applied the asymptotic analysis method for retrial queueing systems under a heavy load condition. We obtained formulas for asymptotic characteristic functions of the probability distribution of the number of calls in the orbit in systems with different input processes and services laws: $M|M|1$, $M|GI|1$, $MMPP|M|1$, $MMPP|GI|1$. However, we have demonstrated that the proposed method has a fairly narrow range of applicability: for the load rate $\rho > 0.95$, Kolmogorov distance between exact and asymptotic distributions has values $\Delta \leq 0.05$. In this regard, we propose to increase the accuracy of the approximation by getting the second order asymptotic formula.

The rest of the paper is organized as follows. In the Sect. 2, the description of the mathematical model of retrial queue $MMPP|M|1$ is presented and the process of the system states is analysed. In the Sect. 3, we introduce asymptotic functions and determine the limit condition of heavy load, then the theorem about the formula for the asymptotic characteristic function is formulated and proved. The last Sect. 4 is devoted to the numerical comparison of the asymptotic distribution with exact one.

2 Mathematical Model and the Process Under Study

In the paper, retrial queueing system of $MMPP|M|1$ type is analyzed. The input process is Markov Modulated Poisson Process which is a particular case of Markovian Arrival Process (MAP) and it is defined by matrix $\mathbf{D_0}$ and $\mathbf{D_1}$ [24,25]. The underlying process $n(t)$ is Markov chain with continuous time and finite set of states $n = 1, 2, \ldots, W$.

We denote the generator of the underlying process $n(t)$ by matrix $\mathbf{Q} = \mathbf{D_0} + \mathbf{D_1}$. And the matrix $\mathbf{Q}$ has elements q_{mv} where $m, v = 1, 2, \ldots, W$.

$\mathbf{D_1}$ is a diagonal matrix with elements $\rho\lambda_n$ where $n = 1, 2, \ldots, W$ and ρ is some parameter defined below. We introduce a matrix $\mathbf{\Lambda} = \text{diag}\{\lambda_n\}$. Then the following equality holds: $\mathbf{D_1} = \rho\mathbf{\Lambda}$.

The vector-row $\boldsymbol{\theta}$ is the stationary probability distribution of the underlying process $n(t)$. $\boldsymbol{\theta}$ is defined as the unique solution of the system:

$$\begin{cases} \boldsymbol{\theta}\mathbf{Q} = \mathbf{0}, \\ \boldsymbol{\theta}\mathbf{e} = 1 \end{cases} \tag{1}$$

where $\mathbf{e}$ is unit column-vector, $\mathbf{0}$ is zero row-vector.

The service time of each call is distributed by exponential law with parameter μ. If a call arrives when a service device (server) is free, the call occupies the device for the service. If the server is busy, the call goes to the orbit (source of repeated calls) where it is staying during a random time distributed exponentially with parameter σ. After this random time, the call from the orbit makes an attempt to reach the device. If the device is free, the call occupies it, otherwise the call immediately returns to the orbit. Structure of the system is presented in Fig. 1.

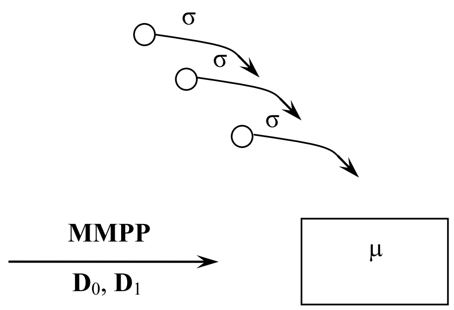

Fig. 1. Retrial queueing system $MMPP|M|1$

The rate of MMPP is defined as $\lambda = \boldsymbol{\theta} \cdot \rho\boldsymbol{\Lambda} \cdot \mathbf{e}$.

Let the system parameters be such that the following equation holds:

$$\boldsymbol{\theta} \cdot \boldsymbol{\Lambda} \cdot \mathbf{e} = \mu. \tag{2}$$

So, the parameter ρ is calculated as $\rho = \dfrac{\lambda}{\boldsymbol{\theta} \cdot \boldsymbol{\Lambda} \cdot \mathbf{e}} = \dfrac{\lambda}{\mu}$ and it is called the load of the system. Thus the stationary state of the system exists when $\rho < 1$. And the heavy load condition is determined by limit condition $\rho \uparrow 1$.

Let $i(t)$ be the random process described the number of calls in the orbit and by $k(t)$ be the random process defined the server state as follows:

$$k(t) = \begin{cases} 0, \text{if device is free}, \\ 1, \text{if device is busy at the moment } t. \end{cases}$$

The problem is to find the probability distribution of the number of calls in the orbit in this system.

However, the process $i(t)$ is not Markovian. So firstly we will consider the multidimensional process $\{k(t), n(t), i(t)\}$ which is a continuous time Markov chain.

We denote the probability that the device is in the state k, there are i calls in the orbit and the underlying process in the state n at the time moment t by $P(k, n, i, t) = P\{k(t) = k, n(t) = n, i(t) = i\}$. So the following direct system of Kolmogorov differential equations for the system states probability distribution $P(k, n, i, t)$ can be written:

$$\begin{cases} \dfrac{\partial P(0,n,i,t)}{\partial t} = -(\rho\lambda_n + i\sigma - q_{nn})P(0,n,i,t) + \mu P(1,n,i,t) \\ \quad + \sum\limits_{v \neq n} P(0,v,i,t) q_{vn}, \\ \dfrac{\partial P(1,n,i,t)}{\partial t} = -(\rho\lambda_n + \mu - q_{nn})P(1,n,i,t) \\ \quad + \rho\lambda_n P(1,n,i-1,t)(1-\delta_{i,0}) + \rho\lambda_n P(0,n,i,t) \\ \quad + (i+1)\sigma P(0,n,i+1,t) + \sum\limits_{v \neq n} P(1,v,i,t) q_{vn}, \ \text{for } i \geq 0, n = \overline{1,N} \end{cases} \tag{3}$$

where $\delta_{i,0}$ is Kronecker symbol which is defined as $\delta_{i,j} = \begin{cases} 0, \text{if } i \neq j, \\ 1, \text{if } i = j. \end{cases}$

We denote row-vectors $\mathbf{P}(k, i) = \{P(k, 1, i), P(k, 2, i), \ldots, P(k, N, i)\}$ where $P(k, n, i) = \lim\limits_{t\to\infty} P(k, n, i, t)$. Then in stationary state, the system (3) has the following matrix form:

$$\begin{cases} \mathbf{P}(0,i)(\mathbf{Q} - \rho\mathbf{\Lambda} - i\sigma\mathbf{I}) + \mu\mathbf{P}(1,i) = \mathbf{0}, \\ \mathbf{P}(1,i)(\mathbf{Q} - \rho\mathbf{\Lambda} - \mu\mathbf{I}) + \mathbf{P}(0,i)\rho\mathbf{\Lambda} + (1-\delta_{i,0})\mathbf{P}(1,i-1)\rho\mathbf{\Lambda} \\ \quad + \sigma(i+1)\mathbf{P}(0,i+1) = \mathbf{0}, \ \text{for } i \geq 0 \end{cases} \tag{4}$$

where $\mathbf{I}$ is the identity matrix.

So we have the system of matrix difference equations.

3 Asymptotic Analysis Method Under Heavy Load Condition

We introduce the partial characteristic functions:

$$\mathbf{H}(k, u) = \sum_i e^{jui}\mathbf{P}(k, i), \ \text{for } k = 0, 1$$

where $j = \sqrt{-1}$ is the imaginary unit.

Then the system (4) is rewritten as the following system:

$$\begin{cases} \mathbf{H}(0,u)(\mathbf{Q} - \rho\mathbf{\Lambda}) + j\sigma \dfrac{\partial \mathbf{H}(0,u)}{\partial u} + \mu\mathbf{H}(1,u) = \mathbf{0}, \\ \mathbf{H}(1,u)(\mathbf{Q} - \rho\mathbf{\Lambda} - \mu\mathbf{I}) + \mathbf{H}(0,u)\rho\mathbf{\Lambda} + e^{ju}\mathbf{H}(1,u)\rho\mathbf{\Lambda} \\ \quad - j\sigma e^{-ju} \dfrac{\partial \mathbf{H}(0,u)}{\partial u} = \mathbf{0}. \end{cases} \tag{5}$$

We will solve the system (5) by the method of asymptotic analysis under heavy load condition. The heavy load condition is defined by the assumption that $\rho \uparrow 1$ or $\varepsilon \downarrow 0$ where ε is an infinitesimal variable $\varepsilon = 1 - \rho > 0$.

First of all, we introduce notations:

$$u = \varepsilon w,\ \mathbf{H}(0,u) = \varepsilon \mathbf{G}(w,\varepsilon),\ \mathbf{H}(1,u) = \mathbf{F}(w,\varepsilon).$$

Then the system (5) can be rewritten as:

$$\begin{cases} \varepsilon \mathbf{G}(w,\varepsilon)(\mathbf{Q} - (1-\varepsilon)\mathbf{\Lambda}) + j\sigma \dfrac{\partial \mathbf{G}(w,\varepsilon)}{\partial w} + \mu \mathbf{F}(w,\varepsilon) = \mathbf{0}, \\ \mathbf{F}(w,\varepsilon)(\mathbf{Q} + (1-\varepsilon)(e^{j\varepsilon w} - 1)\mathbf{\Lambda} - \mu \mathbf{I}) \\ \quad +(1-\varepsilon)\varepsilon \mathbf{G}(w,\varepsilon)\mathbf{\Lambda} - j\sigma e^{-j\varepsilon w} \dfrac{\partial \mathbf{G}(w,\varepsilon)}{\partial w} = \mathbf{0}. \end{cases} \tag{6}$$

For obtaining the second order asymptotic formula, it is necessary to consider following expansions of functions:

$$\mathbf{G}(w,\varepsilon) = \mathbf{G}(w) + \varepsilon \mathbf{g}(w) + \varepsilon^2 \mathbf{g}_2(w) + O(\varepsilon^3), \tag{7}$$

$$\mathbf{F}(w,\varepsilon) = \mathbf{F}(w) + \varepsilon \mathbf{f}(w) + \varepsilon^2 \mathbf{f}_2(w) + O(\varepsilon^3) \tag{8}$$

where $O(\varepsilon^3)$ is an infinitesimal variable of order ε^3.

The characteristic function of the number of calls in the orbit $h(u) = Me^{ju\cdot i(t)}$ can be presented by introduced notations in the following form:

$$h(u) = [\mathbf{H}(0,u) + \mathbf{H}(1,u)]\,\mathbf{e} = \left[\varepsilon \mathbf{G}\left(\frac{u}{\varepsilon}, \varepsilon\right) + \mathbf{F}\left(\frac{u}{\varepsilon}, \varepsilon\right)\right] \mathbf{e}.$$

Using expansions (7) and (8), the characteristic function of the number of calls in the orbit is presented as

$$h(u) = \mathbf{F}\left(\frac{u}{\varepsilon}\right)\mathbf{e} + \varepsilon\left[\mathbf{G}\left(\frac{u}{\varepsilon}\right) + \varepsilon \mathbf{f}\left(\frac{u}{\varepsilon}\right)\right]\mathbf{e} + O(\varepsilon^2)$$

where functions $\mathbf{F}(w), \mathbf{G}(w)$ and $\mathbf{f}(w)$ are defined in expansions (7) and (8), and the parameter $\varepsilon = 1 - \rho$.

Then we will call the function $h_1(u) = \mathbf{F}\left(\frac{u}{\varepsilon}\right)\mathbf{e}$ as the first order asymptotic characteristic function and the function

$$h_2(u) = \mathbf{F}\left(\frac{u}{\varepsilon}\right)\mathbf{e} + \left[\varepsilon \mathbf{G}\left(\frac{u}{\varepsilon}\right) + \varepsilon \mathbf{f}\left(\frac{u}{\varepsilon}\right)\right]\mathbf{e} \tag{9}$$

as the second order asymptotic characteristic function.

In the paper [23], we found that the first order asymptotic characteristic function $h_1(u)$ has the form of the characteristic function of gamma distribution:

$$h_1(u) = \mathbf{F}\left(\frac{u}{1-\rho}\right)\mathbf{e} = \left(1 - \frac{ju}{(1-\rho)\beta}\right)^{-\alpha}$$

where

$$\alpha = 1 + \frac{\mu}{\sigma}\beta, \quad \beta = \frac{\mu}{\mathbf{v\Lambda e} - \mu \mathbf{ve} + \mu}, \tag{10}$$

and the vector $\mathbf{v}$ is a solution of the inhomogeneous system $\mathbf{vQ} = \boldsymbol{\theta}(\mu\mathbf{I} - \boldsymbol{\Lambda})$.

The second order asymptotic characteristic function $h_2(u)$ is defined by the following theorem.

Theorem 1. *The second-order asymptotic characteristic function has the following form*

$$h_2(u) = \left(1 - \frac{ju}{(1-\rho)\beta}\right)^{-\alpha} \left\{1 + (1-\rho)\left[\frac{ju}{(1-\rho)}\mathbf{ve} - j\int_0^{\frac{ju}{(1-\rho)}} \frac{a(y)}{(jy-\beta)}dy\right]\right\}$$

where function $a(w)$ is presented as follows:

$$a(w) = \frac{\alpha}{\beta}\left(1 - \frac{jw}{\beta}\right)^{-1}\left[-jw\frac{2\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - \mu\mathbf{ve}}{\mu} + (jw)^2\frac{\delta}{\mu}\right]$$

$$-\frac{2\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - \mu\mathbf{ve}}{\mu} + 2jw\left(\frac{\delta}{\mu} - \frac{\mu}{\sigma}\right) - 2\left(1 + \frac{\mu}{\sigma}\right)\left(1 - \frac{jw}{\beta}\right) + jw\mathbf{ve}\frac{\mu}{\sigma},$$

α and β are described by formula (10), the constant δ is defined as

$$\delta = \mu\mathbf{ve} + \mathbf{v}_1(\boldsymbol{\Lambda}\mathbf{e} - \mu\mathbf{e}) - \frac{\mu}{2}$$

and $\mathbf{v}_1$ is a solution of the inhomogeneous system

$$\mathbf{v}_1\mathbf{Q} = \frac{\mu}{\beta}\boldsymbol{\theta} - \frac{1}{2}(\boldsymbol{\theta}\boldsymbol{\Lambda} - \mu\boldsymbol{\theta}) - (\mathbf{v}\boldsymbol{\Lambda} - \mu\mathbf{v}).$$

Proof. The proof will be carried out in several steps.

Step 1: Derivation of asymptotic equations.

Substituting expansions (7) and (8) into the system (6), performing some transformations, and equating the coefficients under the same powers of ε, we obtain the following system of equations for unknown functions $\mathbf{F}(w)$, $\mathbf{G}(w)$, $\mathbf{f}(w)$, $\mathbf{g}(w)$, $\mathbf{f_2}(w)$ and $\mathbf{g_2}(w)$:

$$\begin{cases} j\sigma\mathbf{G}'(w) + \mu\mathbf{F}(w) = \mathbf{0}, \\ \mathbf{F}(w)(\mathbf{Q} - \mu\mathbf{I}) - j\sigma\mathbf{G}'(w) = \mathbf{0}, \\ \mathbf{G}(w)(\mathbf{Q} - \boldsymbol{\Lambda}) + j\sigma\mathbf{g}'(w) + \mu\mathbf{f}(w) = \mathbf{0}, \\ jw\mathbf{F}(w)\boldsymbol{\Lambda} + \mathbf{f}(w)(\mathbf{Q} - \boldsymbol{\Lambda}) + \mathbf{G}(w)\boldsymbol{\Lambda} + j\sigma jw \cdot \mathbf{G}'(w) - \mathbf{g}'(w) = \mathbf{0}, \\ \mathbf{G}(w)\boldsymbol{\Lambda} + \mathbf{g}(w)(\mathbf{Q} - \boldsymbol{\Lambda}) + \mathbf{f}_2(w)\boldsymbol{\Lambda} + \mu\mathbf{f}_2(w) = \mathbf{0}, \\ \left(-jw + \frac{(jw)^2}{2}\right)\mathbf{F}(w)\boldsymbol{\Lambda} + jw\mathbf{f}(w)\boldsymbol{\Lambda} + \mathbf{f}_2(w)(\mathbf{Q} - \mu\mathbf{I}) - \mathbf{G}(w)\boldsymbol{\Lambda} \\ \quad +\mathbf{g}(w)\boldsymbol{\Lambda} - j\sigma\frac{(jw)^2}{2} \cdot \mathbf{G}'(w) + j\sigma jw\mathbf{g}'(w) - j\sigma\mathbf{g}_2'(w) = \mathbf{0}. \end{cases} \quad (11)$$

To get one more scalar equation, we sum equations of the system (6) and multiply the result equation by the unit column-vector $\mathbf{e}$. Taken into account that $\mathbf{Qe} = \mathbf{0}$, we obtain equation:

$$\mathbf{F}(w, \varepsilon)(1 - \varepsilon)\boldsymbol{\Lambda}\mathbf{e} + j\sigma e^{-j\varepsilon w}\frac{\partial\mathbf{G}(w, \varepsilon)}{\partial w}\mathbf{e} = 0.$$

We substitute expansions (7) and (8) into obtained equation and again equate coefficients under the same powers of ε. As the result, we write the following system:

$$\begin{cases} \mathbf{F}(w)\mathbf{\Lambda e} + j\sigma\mathbf{G}'(w)\mathbf{e} = 0, \\ -\mathbf{F}(w)\mathbf{\Lambda e} + \mathbf{f}(w)\mathbf{\Lambda e} - j\sigma jw\mathbf{G}'(w)\mathbf{e} + j\sigma\mathbf{g}'(w)\mathbf{e} = 0, \\ -\mathbf{f}(w)\mathbf{\Lambda e} + \mathbf{f}_2(w)\mathbf{\Lambda e} + j\sigma\dfrac{(jw)^2}{2}\mathbf{G}'(w)\mathbf{e} - j\sigma jw\mathbf{g}'(w)\mathbf{e} + j\sigma\mathbf{g}_2'(w)\mathbf{e} = 0. \end{cases}$$

The first two equations are linearly dependent on the first four equations of the system (11), so we will use for further derivations only the last equation:

$$-\mathbf{f}(w)\mathbf{\Lambda e} + \mathbf{f}_2(w)\mathbf{\Lambda e} + j\sigma\frac{(jw)^2}{2}\mathbf{G}'(w)\mathbf{e} - j\sigma jw\mathbf{g}'(w)\mathbf{e} + j\sigma\mathbf{g}_2'(w)\mathbf{e} = 0. \quad (12)$$

Six matrix equations in the system (11) and one scalar equation (12) are enough to find functions $\mathbf{F}(w), \mathbf{G}(w)$ and $\mathbf{f}(w)$ which are necessary for obtaining the second order asymptotic characteristic function $h_2(u)$.

Step 2: Multiplicative form of functions $\mathbf{F}(w), \mathbf{G}(w)$.

Obviously, summing the first and second equations of the system (11), we can write:

$$\mathbf{F}(w) = \boldsymbol{\theta}\Phi(w) \quad (13)$$

where the unknown scalar function $\Phi(w)$ is defined as $\Phi(w) = \mathbf{F}(w)\mathbf{e}$.

Then the first equation of the system (11) has the form:

$$\mathbf{G}'(w) = j\frac{\mu}{\sigma}\mathbf{F}(w) = j\frac{\mu}{\sigma}\boldsymbol{\theta}\Phi(w). \quad (14)$$

Step 3: Determination of functions $\mathbf{G}(w)$ *and* $\mathbf{f}(w)$.

Summing up the third and the fourth equations of the system (11), we obtain

$$\{\mathbf{G}(w) + \mathbf{f}(w)\}\,\mathbf{Q} + jw\mathbf{F}(w)\mathbf{\Lambda} + j\sigma jw\mathbf{G}'(w) = \mathbf{0}.$$

Given the formula (14), it is easy to show that

$$\{\mathbf{G}(w) + \mathbf{f}(w)\}\,\mathbf{Q} = -jw\Phi(w)\boldsymbol{\theta}\,\{\mathbf{\Lambda} - \mu\mathbf{I}\}\,. \quad (15)$$

Let the solution of the Eq. (14) with respect to the vector $\mathbf{G}(w) + \mathbf{f}(w)$ has the form:

$$\mathbf{G}(w) + \mathbf{f}(w) = jw\Phi(w)\mathbf{v} + \varphi(w)\boldsymbol{\theta} \quad (16)$$

where $\varphi(w)$ is an arbitrary scalar function and vector $\mathbf{v}$ is a solution of the following system:

$$\mathbf{vQ} = \boldsymbol{\theta}(\mu\mathbf{I} - \mathbf{\Lambda}). \quad (17)$$

For existence of the solution of the system (15), it is necessary that the rank of the augmented matrix be equal to the rank of the matrix $\mathbf{Q}$. Because the determinant $\det(\mathbf{Q}) = 0$ the rank of the augmented matrix must be less than the dimension of the system. Then it is sufficient that the following condition should hold:

$$(\mu\boldsymbol{\theta} - \boldsymbol{\theta}\boldsymbol{\Lambda})\mathbf{e} = 0,$$

what is true due to the condition (2).

So from the Eq. (15), it follows that

$$\mathbf{f}(w) = jw\Phi(w)\mathbf{v} + \varphi(w)\boldsymbol{\theta} - \mathbf{G}(w). \tag{18}$$

Step 4: Obtaining of expression for the function $\mathbf{g}'(w)$.

From the third equation of the system (11), it follows that:

$$j\sigma\mathbf{g}'(w) = \mathbf{G}(w)(\boldsymbol{\Lambda} - \mathbf{Q}) - \mu\mathbf{f}(w).$$

By substituting the expression (17) into this formula, we get:

$$j\sigma\mathbf{g}'(w) = \mathbf{G}(w)(\boldsymbol{\Lambda} - \mathbf{Q} + \mu\mathbf{I}) - \mu jw\Phi(w)\mathbf{v} - \mu\varphi(w)\boldsymbol{\theta}. \tag{19}$$

Step 5: Derivation of the explicit expression for the scalar function $\Phi(w)$ *and calculation of functions* $\mathbf{F}(w)$ *and* $\mathbf{G}(w)$.

Summing up the fifth and the sixth equations of the system (11) and multiplying the result by the vector $\mathbf{e}$, we can write:

$$\mathbf{f}(w)\boldsymbol{\Lambda}\mathbf{e} + j\sigma\mathbf{g}'(w)\mathbf{e} + j\sigma(1 - jw)\mathbf{G}'(w)\mathbf{e} = 0.$$

We substitute formulas (14) and (19) into the last expression and take into account the expression (2). So, the following equation is derived:

$$jw\Phi(w)(\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - \mu) + j\sigma(1 - jw)\mathbf{G}'(w)\mathbf{e} + \mu\mathbf{G}(w)\mathbf{e} = 0.$$

We differentiate this equation:

$$j\Phi(w)(\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - \mu) + jw\Phi'(w)(\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - \mu) + \sigma\mathbf{G}'(w)\mathbf{e} + j\sigma(1 - jw)\mathbf{G}''(w)\mathbf{e} + \mu\mathbf{G}'(w)\mathbf{e} = 0.$$

So the following equation can be obtained by performing some transformations:

$$\begin{aligned} &\Phi(w)\left[j\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - j\mu\mathbf{v}\mathbf{e} + j\mu + j\frac{j\mu^2}{\sigma}\right] \\ &= \Phi'(w)\left[-jw\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} + jw\mu\mathbf{v}\mathbf{e} + \mu - jw\mu\right]. \end{aligned} \tag{20}$$

Denote $\beta = \dfrac{\mu}{\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - \mu\mathbf{v}\mathbf{e} + \mu}$, $\alpha = 1 + \dfrac{\mu^2}{\sigma(\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - \mu\mathbf{v}\mathbf{e} + \mu)}$. Then the formula (20) can be rewritten as:

$$\Phi(w)j\alpha = \Phi'(w)(\beta - jw).$$

The solution of this equation has the form:

$$\Phi(w) = c(w + j\beta)^{-\alpha} \tag{21}$$

where c is an arbitrary constant and it is equal to $(j\beta)^{\alpha}$ from the initial condition $\Phi(0) = 1$.

So, the formula (21) is rewritten as

$$\Phi(w) = \left(1 + \frac{jw}{\beta}\right)^{-\alpha}. \tag{22}$$

Turning to expressions (13) and (14), we can obtain functions $\mathbf{F}(w)$, $\mathbf{G}(w)$:

$$\begin{cases} \mathbf{F}(w) = \boldsymbol{\theta}\left(1 + \dfrac{jw}{\beta}\right)^{-\alpha}, \\ \mathbf{G}(w) = \boldsymbol{\theta}\left(1 + \dfrac{jw}{\beta}\right)^{-\alpha+1}. \end{cases} \tag{23}$$

Step 6: Getting of the expression for the function $\mathbf{f}_2(w)$.

From the fifth equation of system (11), we obtain the following expression:

$$j\sigma\mathbf{g}_2'(w) = \mathbf{g}(w)(\boldsymbol{\Lambda} + \mathbf{Q}) - \mathbf{G}(w)\boldsymbol{\Lambda} - \mu\mathbf{f}_2(w). \tag{24}$$

Substituting the expressions (18), (23) and (24) in the sixth equation of the system (11), the following equation is obtained:

$$[\mathbf{g}(w) + \mathbf{f}_2(w)]\mathbf{Q} = jw\Phi(w)(\boldsymbol{\theta\Lambda} - \mu\boldsymbol{\theta})$$
$$+(jw)^2\Phi(w)\left[\frac{\mu}{\beta}\boldsymbol{\theta} - \frac{1}{2}(\boldsymbol{\theta\Lambda} + \mu\boldsymbol{\theta}) - (\mathbf{v}\boldsymbol{\Lambda} - \mu\mathbf{v})\right] - jw\varphi(w)(\boldsymbol{\theta\Lambda} - \mu\boldsymbol{\theta}).$$

Let the solution of this equation with respect to the vector $\mathbf{g}(w) + \mathbf{f}_2(w)$ has the form:

$$\mathbf{g}(w) + \mathbf{f}_2(w) = (jw)^2\Phi(w)\mathbf{v}_1 - jw\Phi(w)\mathbf{v} + jw\varphi(w)\mathbf{v} + \varphi_2(w)\boldsymbol{\theta} \tag{25}$$

where $\varphi_2(w)$ is an arbitrary scalar function, $\mathbf{v}$ is a solution of the system (16) and vector $\mathbf{v}_1$ is a solution of the following system:

$$\mathbf{v}_1\mathbf{Q} = \frac{\mu}{\beta}\boldsymbol{\theta} - \frac{1}{2}(\boldsymbol{\theta\Lambda} + \mu\boldsymbol{\theta}) - (\mathbf{v}\boldsymbol{\Lambda} - \mu\mathbf{v}).$$

For existence of a solution of the system (17), it is necessary that the rank of the augmented matrix be equal to the rank of the matrix $\mathbf{Q}$. Because the determinant $\det(\mathbf{Q}) = 0$, the rank of the augmented matrix must be less than the dimension of the system. Then it is sufficient that the following condition should hold:

$$\left[\frac{\mu}{\beta}\boldsymbol{\theta} - \frac{1}{2}(\boldsymbol{\theta\Lambda} + \mu\boldsymbol{\theta}) - (\mathbf{v}\boldsymbol{\Lambda} - \mu\mathbf{v})\right]\mathbf{e} = 0.$$

It is easy to show that this condition is satisfied.

Then from the Eq. (25), we have

$$\mathbf{f}_2(w) = (jw)^2\Phi(w) \cdot \mathbf{v}_1 - jw\Phi(w)\mathbf{v} + jw\varphi(w)\mathbf{v} + \varphi_2(w)\boldsymbol{\theta} - \mathbf{g}(w). \tag{26}$$

Step 7: Derivation of the explicit expression for the scalar function $\varphi(w)$.

Substituting all found expressions in the Eq. (12), the following equation can be obtain:

$$\Phi(w)\left[-jw(2\mathbf{v\Lambda e}-\mu\mathbf{ve})+(jw)^2\left(\mu\mathbf{ve}+\mathbf{v}_1(\mathbf{\Lambda e}-\mu\mathbf{e})-\frac{\mu}{2}\right)\right]$$
$$+\varphi(w)\left[-\boldsymbol{\theta}\mathbf{\Lambda e}+jw(\mu+\mathbf{v\Lambda e}-\mu\mathbf{ve})\right]-jw\mathbf{G}(w)(\mathbf{\Lambda e}+\mu\mathbf{e})+\mu\mathbf{g}(w)\mathbf{e}=0.$$

We denote $\delta=\mu\mathbf{ve}+\mathbf{v}_1(\mathbf{\Lambda e}-\mu\mathbf{e})-\frac{\mu}{2}$.

Differentiating the equation, we obtain:

$$\Phi'(w)\left[-jw(2\mathbf{v\Lambda e}-\mu\mathbf{ve})+(jw)^2\delta\right]+\Phi(w)\left[-j(2\mathbf{v\Lambda e}-\mu\mathbf{ve})+2j^2w\delta\right]$$
$$+\varphi'(w)\left[-\boldsymbol{\theta}\mathbf{\Lambda e}+jw\frac{\mu}{\beta}\right]+\varphi(w)j\frac{\mu}{\beta}-jw\mathbf{G}'(w)(\mathbf{\Lambda e}+\mu\mathbf{e})$$
$$-j\mathbf{G}(w)(\mathbf{\Lambda e}+\mu\mathbf{e})+\mu\mathbf{g}'(w)\mathbf{e}=0.$$

Taking into account formulas (2), (10), (18) and (19), the following differential equation is obtained:

$$\varphi'(w)\left(1-\frac{jw}{\beta}\right)-j\varphi(w)\frac{\alpha}{\beta}=j\Phi(w)a(w) \tag{27}$$

where

$$a(w)=\frac{\alpha}{\beta}\left(1-\frac{jw}{\beta}\right)^{-1}\left[-jw\frac{2\mathbf{v\Lambda e}-\mu\mathbf{ve}}{\mu}+(jw)^2\frac{\delta}{\mu}\right]-\frac{2\mathbf{v\Lambda e}-\mu\mathbf{ve}}{\mu}$$
$$+2jw\left(\frac{\delta}{\mu}-\frac{\mu}{\sigma}\right)-2\left(1+\frac{\mu}{\sigma}\right)\left(1-\frac{jw}{\beta}\right)+jw\mathbf{ve}\frac{\mu}{\sigma}.$$

The solution of the inhomogeneous differential equation (27) has form:

$$\varphi(w)=e^{j\int_0^w\frac{\alpha/\beta}{1-\frac{jx}{\beta}}dx}\left\{\varphi(0)+j\int_0^w e^{-j\int_0^y\frac{\alpha/\beta}{1-\frac{jx}{\beta}}dx}\frac{\Phi(y)a(y)}{1-jy/\beta}dy\right\}. \tag{28}$$

Given normalization condition for the function $\mathbf{F}(w)$: $\mathbf{F}(0)\mathbf{e}=1$, from the expression(16) we have $\varphi(0)=0$.

It is easy to show that

$$\int_0^w\frac{\alpha/\beta}{1-\frac{jx}{\beta}}dx=j\alpha\ln\left(1-\frac{jw}{\beta}\right)$$

and

$$\int_0^w\left(1-\frac{jy}{\beta}\right)^{\alpha}\frac{\Phi(y)a(y)}{1-\frac{jy}{\beta}}dy=\int_0^w\frac{a(y)}{1-\frac{jy}{\beta}}dy.$$

So, the solution (28) has the following form:

$$\varphi(w) = j\left(1 - \frac{jw}{\beta}\right)^{-\alpha} \int_0^w \frac{a(y)}{1 - \frac{jy}{\beta}} dy = -j\Phi(w) \int_0^w \frac{a(y)}{jy - \beta} dy.$$

Step 8: Obtaining of the final formula for the function $h_2(u)$.

Turning to the formula (16), we have

$$\begin{aligned} \{\mathbf{G}(w) + \mathbf{f}(w)\}\mathbf{e} &= jw\Phi(w)\mathbf{ve} + \varphi(w) \\ &= \left(1 - \tfrac{jw}{\beta}\right)^{-\alpha} \left\{jw\mathbf{ve} - j\int_0^w \tfrac{a(y)}{jy-\beta} dy\right\}. \end{aligned} \tag{29}$$

From the formula (9), the second order asymptotic characteristic function for retrial queueing system $MMPP|M|1$ is represented as

$$h_2(u) = \mathbf{F}\left(\frac{u}{1-\rho}\right)\mathbf{e} + (1-\rho)\left[\mathbf{G}\left(\frac{u}{1-\rho}\right)\mathbf{e} + \mathbf{f}\left(\frac{u}{1-\rho}\right)\mathbf{e}\right].$$

Taking into account expressions (23) and (29), we obtain that the function $h_2(u)$ has the following form:

$$h_2(u) = \left(1 - \frac{ju}{(1-\rho)\beta}\right)^{-\alpha} \left\{1 + (1-\rho)\left[\frac{ju}{1-\rho}\mathbf{ve} - j\int_0^{\frac{u}{1-\rho}} \frac{a(y)}{jy-\beta} dy\right]\right\}.$$

So the theorem is proved.

Having the second order asymptotic characteristic function $h_2(u)$, we can construct the asymptotic probability distribution $P_2(i)$ of the number of calls in the orbit by means of the formula of inverse Fourier transform.

4 Numerical Analysis of the Results

To determine the applicability range of the proposed method, we compare the obtained asymptotic distribution and the distribution obtained by numerical solution of the system of linear algebraic equations (4) and calculate Kolmogorov distance between distributions.

Consider an example. Let the system parameters be the following:

$$\mu = 1, \sigma = 1,$$

$$\mathbf{\Lambda} = \begin{pmatrix} 0.588 & 0 & 0 \\ 0 & 0.980 & 0 \\ 0 & 0 & 1.373 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -0.5 & 0.2 & 0.3 \\ 0.1 & -0.3 & 0.2 \\ 0.3 & 0.2 & -0.5 \end{pmatrix}.$$

In Figs. 2 and 3, comparison of distributions are shown for different value of the load ρ (D_n is exact distribution which is obtained numerically, $P1_n$ and $P2_n$ are the first order and the second order asymptotic distributions, respectively).

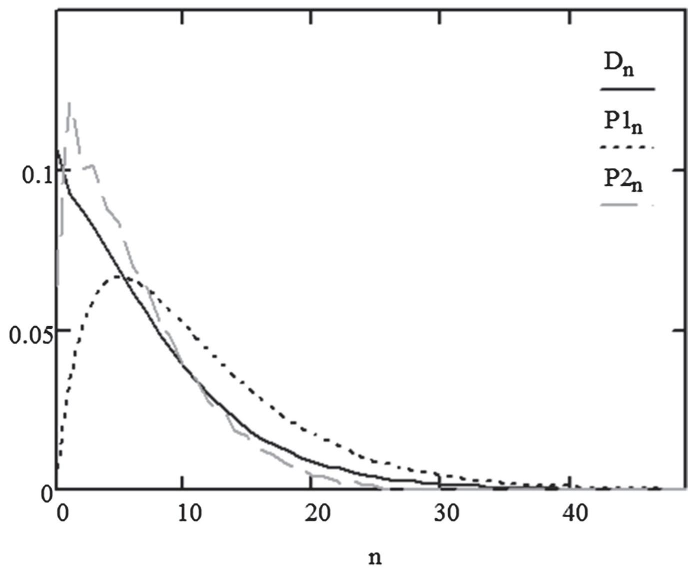

Fig. 2. Comparison of asymptotic and exact distributions for $\rho = 0.8$

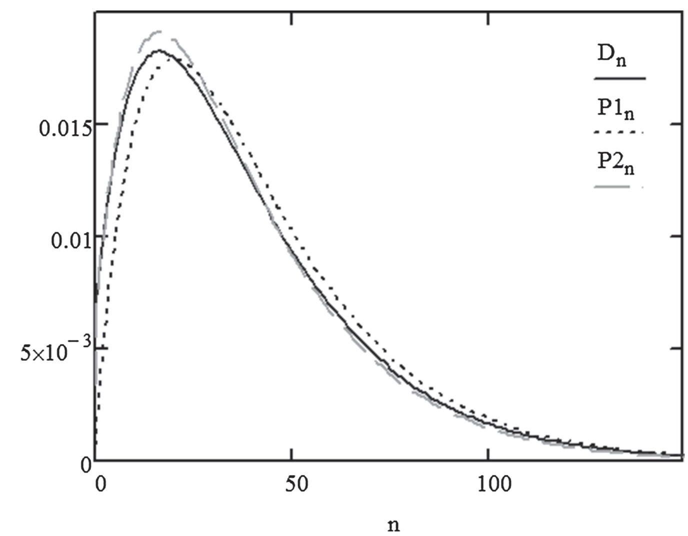

Fig. 3. Comparison of asymptotic and exact distributions for $\rho = 0.95$

Table 1. Kolmogorov distance between asymptotic and exact distributions

Values of the load rate	First-order asymptotic distribution	Second-order asymptotic distribution
$\rho = 0.7$	0.350	0.118
$\rho = 0.8$	0.235	0.050
$\rho = 0.9$	0.114	0.026
$\rho = 0.95$	0.050	0.018

In the Table 1 we show the Kolmogorov distance between asymptotic and exact distributions:

$$\Delta = \max_{0 \le i \le N} \left| \sum_{n=0}^{i} D_n - \sum_{n=0}^{i} P_n \right|$$

for different values of the parameter of load ρ.

We chose the condition $\Delta \le 0.05$ as the criteria of method application. So the second order asymptotic analysis method is applied for $\rho \ge 0.8$.

5 Conclusion

In the paper, we study the retrial queueing system $MMPP|M|1$ by means of the second order asymptotic analysis method under heavy load condition. During the investigation, the asymptotic characteristic function of the number of calls in the orbit is obtained. Numerical comparison of asymptotic distributions (of the 1st and the 2nd orders) with the exact one is performed. The comparison shows that the application area of the second order asymptotic method increases by 4 times than first order asymptotic results: for load rate $\rho > 0.8$ Kolmogorov distance has values $\Delta \le 0.05$.

In this regard, there is the question about the increasing the accuracy of the method by means of obtaining the third order asymptotic formula. However, studies have shown that this approximation does not increase the range of the method applicability. So for $\rho \le 0.8$ it need to develope other methods of system studying.

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