

On a Quantization of the Classical θ -Functions^{*}

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Abstract. The Jacobi theta-functions admit a definition through the autonomous differential equations (dynamical system); not only through the famous Fourier theta-series. We study this system in the framework of Hamiltonian dynamics and find corresponding Poisson brackets. Availability of these ingredients allows us to state the problem of a canonical quantization to these equations and disclose some important problems. In a particular case the problem is completely solvable in the sense that spectrum of the Hamiltonian can be found. The spectrum is continuous, has a band structure with infinite number of lacunae, and is determined by the Mathieu equation: the Schrödinger equation with a periodic cosine potential.

Key words: Jacobi theta-functions; dynamical systems; Poisson brackets; quantization; spectrum of Hamiltonian

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Why do we attach importance to a possibility to describe the classical limit through theta-functions? . . .

Thus, when quantizing integrable models, sooner or later we should come to a quantization of theta-functions [15, § 1].

1 Introduction

Presently, the only known definition and introductory motivation to the famous theta-functions are through the corresponding Fourier series

$$\theta(z|\tau) = \sum_{k=-\infty}^{\infty} e^{\pi i(k^2\tau + 2kz)}.$$

Such series appear in numerous areas of mathematics and physics as building blocks for representation of solutions and also have a great interest in their own rights because of rich structural and applied properties. To be precise, there are some mathematical arguments to the series structure which come from considering the group and functional properties of Abelian varieties. The displayed Fourier structure then becomes, in a certain sense, unique and is generalized to multidimensional cases. In this work we shall deal only with Jacobian, i.e., 1-dimensional theta-functions.

There is one imperfection of these ‘serial’ ways of introducing the θ -functions however. These definitions are *pure functional*, not invariant, and do not even look like a definition for any special function of mathematical physics. The latter is determined usually by a certain differential equation whereas the θ -series are not; e.g., they appear (always via theta-ratios) to represent solutions of many integrable models. On the other hand, we know that property of the model

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to be algebraically integrable [17] is an invariant property (commuting operators) and is of a radically different kind from those models which are solvable in terms of various special functions [4].

A further aspect is the fact that θ -functions are the nearest generalization of the elementary ones – harmonic sine/cosine-functions – which are related to the simplest classical solvable model and extensively used, through the harmonic quantum prototype

$$\Psi'' - x^2\Psi = E\Psi,$$

in the quantum field theories. Their low energy limits and excitations are treated as quantum mechanics with the field boson particles: statements of this elementary model.

Yet another facet is a relation with θ -series quotients. It is known that basic θ -quotients are proportional to Jacobi's elliptic sn-, cn-, and dn-functions [1, 11, 18]

$$\frac{\theta_1}{\theta_4} \sim \text{sn}, \quad \frac{\theta_2}{\theta_4} \sim \text{cn}, \quad \frac{\theta_3}{\theta_4} \sim \text{dn},$$

and describe dynamics of some famous models in the rigid body theories [9, 11]. In turn, most popular of them – Eulerian top – may be quantized, thereby determining the space quantization phenomenon, so called quantum rotator [10].

These and many other aspects give rise a problem towards the direct quantization of theta-functions themselves [15] if it is possible. Fortunately this is so indeed and we shall exhibit a particular case that is completely solvable in the quantum sense: spectrum of a quantum Hamiltonian can be found. The starting point here is the fact that θ -series possesses the nice differential properties like elliptic sn-, cn-, dn-functions mentioned above. In the context above one can even say that any kind of differential definition to the θ -functions might be considered as more appropriate way of their introduction at all.

2 Definitions and base dynamical systems

Complete differential properties of Jacobian series were described very recently [5] and we reproduce them in a nutshell. These four θ -series are defined as [1, 18]

$$\begin{aligned} \theta_1(t|\tau) &= -ie^{\frac{1}{4}\pi i\tau} \sum_{k=-\infty}^{+\infty} (-1)^k e^{(k^2+k)\pi i\tau} e^{(2k+1)\pi it}, \\ \theta_2(t|\tau) &= e^{\frac{1}{4}\pi i\tau} \sum_{k=-\infty}^{+\infty} e^{(k^2+k)\pi i\tau} e^{(2k+1)\pi it}, \\ \theta_3(t|\tau) &= \sum_{k=-\infty}^{+\infty} e^{k^2\pi i\tau} e^{2k\pi it}, \quad \theta_4(t|\tau) = \sum_{k=-\infty}^{+\infty} (-1)^k e^{k^2\pi i\tau} e^{2k\pi it} \end{aligned}$$

and satisfy the system of ordinary differential equations

$$\begin{aligned} \frac{\partial\theta_2}{\partial t} &= \frac{\theta_1'}{\theta_1}\theta_2 - \pi\vartheta_2^2 \frac{\theta_3\theta_4}{\theta_1}, & \frac{\partial\theta_1}{\partial t} &= \theta_1', \\ \frac{\partial\theta_3}{\partial t} &= \frac{\theta_1'}{\theta_1}\theta_3 - \pi\vartheta_3^2 \frac{\theta_2\theta_4}{\theta_1}, & \frac{\partial\theta_1'}{\partial t} &= \frac{\theta_1'^2}{\theta_1} - \pi^2\vartheta_3^2\vartheta_4^2 \frac{\theta_2^2}{\theta_1} - \left\{ 4\eta + \frac{\pi^2}{3}(\vartheta_3^4 + \vartheta_4^4) \right\} \theta_1, \\ \frac{\partial\theta_4}{\partial t} &= \frac{\theta_1'}{\theta_1}\theta_4 - \pi\vartheta_4^2 \frac{\theta_2\theta_3}{\theta_1}, \end{aligned} \quad (1)$$

where θ'_1 is an additional (fifth) independent function and (ϑ, η) are so called ϑ -constants. In standard theory these ‘constants’ are the values of $\theta(z|\tau)$ -functions at $z = 0$, i.e., $\vartheta_k = \theta_k(0|\tau)$, but for the differential formulation (1) such a meaning need not be implied; all the (η, θ_k) are allowed to be free external parameters [5]. In the abstract context the differential introduction of functions is much more preferable because we may consider these equations merely as a vector field and freely change coordinates on a manifold where the field acts.

For a quantization procedure it is desirable to simplify vector fields into the polynomial ones. It appears that the sought-for transformation $(\theta_1, \theta_2, \theta_3, \theta_4, \theta'_1) \rightleftharpoons (x, y, z, u, \xi)$ does exist and has the form

$$\left\{ x = -\pi\vartheta_3\vartheta_4\frac{\theta_2}{\theta_1}, y = -\pi\vartheta_2\vartheta_4\frac{\theta_3}{\theta_1}, z = -\pi\vartheta_2\vartheta_3\frac{\theta_4}{\theta_1}, u = e^{\frac{1}{2}\Lambda t^2}\theta_1, \xi = \frac{\theta'_1}{\theta_1} + \Lambda t \right\}$$

under certain value of the parameter $\Lambda(\vartheta, \eta)$. Let dot above symbol stand for a t -derivative. Then this substitution brings the θ -system above into a very simple polynomial form

$$\dot{x} = yz, \quad \dot{y} = xz, \quad \dot{z} = xy, \quad \dot{\xi} = -x^2, \quad \dot{u} = \xi u \quad (2)$$

which has an embedded structure like ‘Russian dollies’ $((x, y, z), \xi), u$. Moreover it is not difficult to show that the first three equations in (2) are nothing but the normalized (parameter-free) model of a free Euler’s rigid body [9, 11]

$$\dot{X} = \left(\frac{1}{C} - \frac{1}{B}\right)YZ, \quad \dot{Y} = \left(\frac{1}{A} - \frac{1}{C}\right)XZ, \quad \dot{Z} = \left(\frac{1}{B} - \frac{1}{A}\right)XY$$

with the standard Hamilton and Casimir functions

$$H = \frac{1}{2}\left(\frac{X^2}{A} + \frac{Y^2}{B} + \frac{Z^2}{C}\right), \quad K = X^2 + Y^2 + Z^2$$

and the well-known Poisson bracket

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{bmatrix} \begin{bmatrix} H_X \\ H_Y \\ H_Z \end{bmatrix}. \quad (3)$$

The transformation $(x, y, z) \rightleftharpoons (X, Y, Z)$ has a linear form

$$\begin{aligned} X &= -\frac{A\sqrt{BC(C-B)}}{\sqrt{(C-B)(A-C)(B-A)}}x, & Y &= -\frac{B\sqrt{CA(A-C)}}{\sqrt{(C-B)(A-C)(B-A)}}y, \\ Z &= -\frac{C\sqrt{AB(B-A)}}{\sqrt{(C-B)(A-C)(B-A)}}z \end{aligned}$$

and holds only for the full asymmetric top parameters $A \neq B \neq C \neq A$. Quantization of this dynamical system is very well known as the quantum rotator [10]. We thus arrive at a quantization problem related to a 5D-*extension* of the classical 3D quantized rotator.

As far as we know, no regular algorithm for searching for a Poisson bracket exists. Even Euler’s top is not an exception since its Lie algebraic $\mathbb{SO}(3)$ -bracket is ad hoc; from the Lie algebra $\mathfrak{so}(3)$ to the Poisson structure (3). On the other hand, it is naturally to expect that knowing the solution in full details one could be able to find out all the associated constructions. Particularly, classical and all the other Poisson structures because they are closely related to integrals of dynamical systems. We thus need a complete set of these functions.

3 Integrals of motion

Since equations (2) is an odd dimensional system we restrict ourselves in this work to its 4-dimensional subsystem

$$\dot{x} = yz, \quad \dot{y} = xz, \quad \dot{z} = xy, \quad \dot{\xi} = -x^2. \quad (4)$$

The point here is that there are some serious foundations that even the full 5D system (2) may not be considered as closed and requires further extension [5]. Along with the elliptic functions and meromorphic elliptic integrals, the logarithmic integral should be involved to the complete theory [5]. Yet another aspect needs to be kept in mind is that we get in (4) the even dimensional vector field and may compare it locally with a straightened one having (always and at least) the canonical symplectic structure [14]. Actually, this program will be realized in this and next sections.

The functional form of solution to (4) follows from the widely known elliptic solutions of the Euler top followed by supplementing the meromorphic Legendre integral \mathbf{E} (see [1, 18] and appendix for notation and details):

$$\begin{aligned} x &= -k\alpha \operatorname{sn}(\alpha t + \varepsilon; k), & y &= i\alpha \operatorname{dn}(\alpha t + \varepsilon; k), & z &= ik\alpha \operatorname{cn}(\alpha t + \varepsilon; k), \\ \xi &= -k^2\alpha^2 \int^t \operatorname{sn}^2(\alpha s + \varepsilon; k) ds = K + \alpha \mathbf{Z}(\alpha t + \varepsilon; k) - \alpha^2 t, \end{aligned} \quad (5)$$

where $(k, \alpha, K; \varepsilon)$ are free constants. However, we are about to get some invariant constructions (Poisson bracket etc) and therefore one should transform this solution into the integrals of motion form.

Let (I, J, K) denote the three independent functions of (x, y, z, ξ) being constant on solutions of (4) and let \mathcal{N} be a function of the same variable set evolving linearly in time. Then the locally equivalent form to the system under consideration is

$$\dot{\mathcal{N}} = 1, \quad \dot{I} = 0, \quad \dot{J} = 0, \quad \dot{K} = 0. \quad (6)$$

We shall informally name the $(\mathcal{N}; I, J, K)$ -set as ‘straightened’ variables (π^1, \dots, π^4) . Clearly, the pair (I, J) is an equivalent of (α, k) and \mathcal{N} is defined up to a function of (I, J, K) , that is the constant ε . To put the scheme differently we are seeking for a formal point transformation $(x, y, z, \xi) \rightleftharpoons (\pi^1, \dots, \pi^4)$ between the original and straightened variable set in both directions. Then some routine computations with elliptic integrals yield the sought-for change:

$$\begin{aligned} x &= -I \operatorname{sn} \left(J\mathcal{N}; \frac{I}{J} \right), & y &= iJ \operatorname{dn} \left(J\mathcal{N}; \frac{I}{J} \right), & z &= iI \operatorname{cn} \left(J\mathcal{N}; \frac{I}{J} \right), \\ \xi &= K + J\mathbf{Z} \left(J\mathcal{N}; \frac{I}{J} \right) - J^2\mathcal{N}, \\ \mathcal{N} &= \frac{1}{\sqrt{x^2 - y^2}} \mathbf{F} \left(\frac{x}{\sqrt{x^2 - z^2}}; \frac{\sqrt{x^2 - z^2}}{\sqrt{x^2 - y^2}} \right), & I^2 &= x^2 - z^2, & J^2 &= x^2 - y^2, \\ K &= \xi + \sqrt{x^2 - y^2} \left\{ \mathbf{F} \left(\frac{x}{\sqrt{x^2 - z^2}}; \frac{\sqrt{x^2 - z^2}}{\sqrt{x^2 - y^2}} \right) - \mathbf{E} \left(\frac{x}{\sqrt{x^2 - z^2}}; \frac{\sqrt{x^2 - z^2}}{\sqrt{x^2 - y^2}} \right) \right\}. \end{aligned}$$

Hence the system (4) has only two independent single-valued (rational) integrals I and J and the Hamilton function H may be chosen only as their combination: $H = H(I, J)$. Differential computations with all the functions (sn, cn, dn, \mathbf{Z} , \mathbf{F} , \mathbf{E}) have been completely developed; see the well-known Jacobi calculus [1, 18] accompanied with formulas (12)–(14).

4 Poisson bracket(s)

The Hamilton function and Poisson bracket Ω appear always in a pair and the former is subjected to our free assignment. On the other hand, every vector field is locally equivalent to the simplest one, like (6), and its Poisson structure may be assigned among the constant symplectic ones. For comparison with the standard Lie algebraic case (3) let us consider the linear in I^2, J^2 case $aI^2 + bJ^2$:

$$H = a(x^2 - z^2) + b(x^2 - y^2)$$

with some constants a, b .

The method

The scheme of finding the bracket is based on the fact that under fixed dimension any bracket for any hamiltonian system is locally equivalent to any other bracket. Hence we may take as an ansatz some ‘known and good, straightened’ bracket $\mathbf{K}^{jk}(\pi)$ for known ‘simple’ vector field $\dot{\pi}^k = v^k(\pi)$ and seek for its transformation to the initial field coordinates (x^k) . The x -coordinates $\Omega^{jk}(x)$ of the bracket tensor is then recalculated by the standard transformation law

$$\Omega^{jk}(x) = \frac{\partial x^j}{\partial \pi^n} \frac{\partial x^k}{\partial \pi^m} \mathbf{K}^{nm}(\pi). \quad (7)$$

It should be noted here that ansatz for $\mathbf{K}(\pi)$ may contain some freedom which is not restricted by the constant parameters (canonical constant bracket \mathbf{K}) because the only condition for a bracket, apart from antisymmetry, is the Jacobi identity. In terms of inverse matrix-tensor $\mathbf{K} = \mathbf{K}^{-1}$ the condition turns out to be a Maxwell (linear) equations first half $d\mathbf{K} = 0$ which, clearly, has infinitely many solutions.

To continue further quantization we have to produce at least one nontrivial solution. Some experimental computations show that there exists an unusual bracket

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\xi} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & y & -x \\ 0 & 0 & x & -y \\ -y & -x & 0 & 0 \\ x & y & 0 & 0 \end{bmatrix}}_{\Omega} \begin{bmatrix} H_x \\ H_y \\ H_z \\ H_\xi \end{bmatrix}, \quad H = \frac{1}{2}(z^2 - x^2) \quad (8)$$

and its determinant is not a zero in the generic case: $\det \Omega = (x^2 - y^2)^2$. Interestingly enough, the upper-left (3×3) -submatrix of Ω (boxed) is also a bracket and provides yet another, rather nonstandard, bracket for Euler’s top; it is related to the $\mathfrak{e}(1, 1)$ -algebra. For reference we display also the commuting vector field corresponding to the integral $I^2 = y^2 - x^2$:

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = 0, \quad \dot{\xi} = 2(y^2 - x^2), \quad \{H, I\} = 0.$$

Remark 1. Informally, we may conclude that knowing the bracket tensor is an equivalent to knowing the differential computation rules for the functions realizing complete system of integrals for the dynamics. Appearance/introduction of a specific form to the functions may be thought of as a reduction of the full system dimension to a smaller number of functions; the absence of integrals means thus just a generic – nonintegrable or not yet integrated – case. If we have differential function calculus in one direction, say, $(x) \xrightarrow{\partial} (\pi)$, then calculus in a reverse direction $(\pi) \xrightarrow{\partial} (x)$ is automatically induced.

Example 1. Consider the planar vector fields

$$\dot{x} = A(x, y), \quad \dot{y} = B(x, y),$$

where A and B are some polynomials in (x, y) over \mathbb{C} . Recently there was developed and implemented an effective algorithm [3] for computing the rational integral $H = R(x, y)$ to these equations. In this connection P. Olver put a question (October 2014): what about Hamiltonian structure for these systems? As we have seen from scheme (7), this question has a constructive answer.

First, let us straighten the field under question. One of the $(\pi^1, \pi^2) = (\mathcal{N}, H)$ -variables we've already had; this is the rational H -Hamiltonian $R(x, y)$ coming from the algorithm mentioned above. The second variable – momentum conjugated to H – is derived from equations determining trajectory of the system. Indeed,

$$t = \int \frac{dx}{A(x, y)}, \quad R(x, y) = \text{const}$$

and, hence, variable \mathcal{N} is an integral

$$\mathcal{N} = \int^x \frac{dz}{A(z, w)} =: \aleph(x; R(x, y))$$

and this integral is an Abelian one with respect to algebraic irrationality

$$R(z, w) = \text{const}. \tag{9}$$

Here $\aleph(x; \text{const})$ is a function representation to the integral as function of the x -variable and a constant being a modulus of the algebraic curve (9). Differential computations for this Abelian integral should be developed independently; including, which is an important and nontrivial point (see Remark 1), differentiation with respect to 'const' of the curve (9). Since \aleph is an Abelian integral its differential calculus will possibly involve Abelian integrals of other kinds as well as rational functions of the pair (x, y) [2]. Anyway, development of the calculus is a technical and solvable problem because derivative of the Abelian integral in a parameter is again an integral belonging to the same algebraic curve.

Now, we assign the function $R(x, y)$ as a Hamiltonian $H = R(x, y)$ and take the simplest canonical local (2×2) -bracket $\mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for the straightened vector field

$$\dot{\mathcal{N}} = 1, \quad \dot{H} = 0.$$

Then the sought-for (x, y) -coordinates $\Omega(x, y)$ of the bracket is calculated explicitly by (7). Suffice it to display only the Ω^{12} -component:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \Omega \nabla R, \quad (\Omega^{12})^{-1} = \nabla_y R \cdot \nabla_x \aleph - \nabla_x R \cdot \nabla_y \aleph. \tag{10}$$

Rationality or the global single-valuedness of the found bracket tensor is established (if any) on the base of differentiation rules for the Abelian integrals that arise. Then it remains only to reassign the Hamiltonian $H \rightarrow f(H)$ for a simplification of the final answer.

Let us consider a further specific subexample: the vector field

$$\dot{x} = y, \quad \dot{y} = 6x^2$$

defining the equianharmonic elliptic Weierstrass function $x = \wp(t; 0, a)$ and its derivative $y = \wp'(t; 0, a)$ [18]. According to the scheme above, we have

$$H = \frac{1}{2}(y^2 - 4x^3), \quad \mathcal{N} = \int^x \frac{ds}{\sqrt{4s^3 - \text{const}}} \Rightarrow \mathcal{N} = \mathfrak{A}(x; 0, 4x^3 - y^2),$$

where $\mathfrak{A}(x; g_2, g_3)$ is a notation for a holomorphic integral in the Weierstrass elliptic theory [18]. Although differential calculus with Weierstrassian integrals is also absent in standard literature, it can be developed in the same manner as Legendrian formulas (12), (13). Computation (10) with rational function $R(x, y)$ and transcendental one $\mathfrak{A}(x; 0, 4x^3 - y^2)$ yields: $\Omega^{12} = 1$. This (global) unity corresponds exactly to the standard canonical bracket $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It should be emphasized here that the seed ‘straightened’ bracket \mathbb{K} is not well-defined but is just a ‘local unity’ since $\mathcal{N} = \mathfrak{A}(x; 0, 4x^3 - y^2)$ is a multivalued function. Meanwhile the (x, y) -coordinates of the bracket are the global well-defined ‘object unity’. Clearly, the (2×2) -case under consideration has an additional convenience. Reassignment of the Hamilton function exhausts all the possible brackets and there is no need to investigate nonconstant ‘straightened’ brackets $\mathbb{K}(\mathcal{N}, H)$.

Example 2. In a slightly different but equivalent manner the Hamilton τ -dynamics was found in work [6] for the elliptic $\theta(0|\tau)$ -constants; the very ϑ , η -coefficients of system (1).

Example 3 (exercise). Extend the method into odd dimensional cases and reproduce the standard Poisson $\mathbb{S}\mathbb{O}(3)$ -description (3).

5 Quantization: operators and spectrum

In this section we propose a complete canonical quantization procedure for the model (8) described above. As a byproduct, we get also some nonstandard quantization for the classical Euler equations (3).

As usual, we should, according to the correspondence principle

$$\{x, y\} \dashrightarrow [\hat{x}, \hat{y}],$$

find representations for the phase observable operators $(x, y, z, \xi) \dashrightarrow (\hat{x}, \hat{y}, \hat{z}, \hat{\xi})$, where bracket $\{x, y\}$ is given by the Ω -matrix (8). Taking into account that Euler’s (3×3) -subsystem (3) is realized by the 1st order differential operators with linear coefficients (the classical momentum operators [10]), we can try the same structure ansatz for our (4×4) -extension

$$(\text{phase variable operator}) = (\text{linear function}) \times \partial_{\text{variable}} + (\text{linear function}).$$

Note, the standard generators $\partial_x, \partial_y, x\partial_y + y\partial_x$ of the aforementioned $\mathfrak{e}(1, 1)$ -algebra are also nicely fit this ansatz. Such a solution – combination of dilatations, rotations, and ∂ -operators – does exist and we find

$$\hat{x} := x, \quad \hat{y} := y, \quad \hat{z} := -x\partial_y - y\partial_x, \quad \hat{\xi} := x\partial_x + y\partial_y \quad (11)$$

(first pointed out to the author by P. Kazinsky). Commuting relations

$$[\hat{x}, \hat{z}] = \hat{y}, \quad [\hat{y}, \hat{z}] = \hat{x}, \quad [\hat{\xi}, \hat{x}] = \hat{x}, \quad [\hat{\xi}, \hat{y}] = \hat{y}$$

are the same as the bracket ones for base variables (x, y, z, ξ) , i.e., (8):

$$\{x, z\} = y, \quad \{y, z\} = x, \quad \{\xi, x\} = x, \quad \{\xi, y\} = y$$

(all the absent pairs are zeroes). For simplicity we omit here the physical multipliers like $i\hbar$ in front of commutators. From (11) it follows immediately that quantum Hamiltonian is represented by the operator

$$\widehat{\mathcal{H}} := \frac{1}{2}(\hat{z}^2 - \hat{x}^2) = \frac{1}{2}\{(x\partial_y + y\partial_x)^2 - x^2\}$$

and we readily check that quantum equations of motion (Heisenberg's dynamics) also holds with respect to the canonical Weyl ordering

$$\begin{aligned}\widehat{x} &= \frac{1}{2}(\widehat{y}\widehat{z} + \widehat{z}\widehat{y}) = [\widehat{x}, \widehat{\mathcal{H}}], \\ \widehat{y} &= \frac{1}{2}(\widehat{x}\widehat{z} + \widehat{z}\widehat{x}) = [\widehat{y}, \widehat{\mathcal{H}}], \\ \widehat{z} &= \frac{1}{2}(\widehat{x}\widehat{y} + \widehat{y}\widehat{x}) = [\widehat{z}, \widehat{\mathcal{H}}], \\ \widehat{\xi} &= -\frac{1}{2}(\widehat{x}\widehat{x} + \widehat{x}\widehat{x}) = [\widehat{\xi}, \widehat{\mathcal{H}}].\end{aligned}$$

Availability of a (complex) rotation operator $x\partial_y + y\partial_x$ in $\widehat{\mathcal{H}}$ prompts us to make a variable change to the radius-angle (r, γ) -variables

$$x = 2r \cos\left(\frac{1}{2}\gamma\right), \quad y = 2ir \sin\left(\frac{1}{2}\gamma\right),$$

whereupon we get $\widehat{z} = -2\partial_\gamma$ and the following equivalent of the eigen-value problem

$$\widehat{\mathcal{H}}\Psi = E\Psi \quad \Rightarrow \quad \Psi_{\gamma\gamma} + \frac{1}{2}r^2 \cos(\gamma) \Psi = -\frac{1}{2}(E + r^2)\Psi.$$

Variable r (commuting observable I) has been separated, i.e., became a parameter, and we may finally normalize the spectral problem into the Schrödinger equation with a trigonometric cos-type potential

$$\Psi'' = (A \cos \gamma - E)\Psi$$

and two extra parameters E and A . This is the very well-known equation – Mathieu equation [1, 18] – and excellent book on its theory has hitherto remained the MacLachlan treatise [12].

Until now, we were dealing with integrable but complex equations. Their general integrability is related essentially to the complex phase space \mathbb{C}^4 but for physical motivation we need a real reduction of the space as long as the reduction was a non-contradictory one; see also some arguments on reality and complexification at the end of § 1 in [15]. A rational question at this point is: Whether this model is realistic? In other words, is it possible to ‘adjust’ our formal (yet complexified) quantities in system (4) to be real physical observables and their operators to be (real) self-adjointing ones? The question has a positive answer. Indeed, we want to have real H and, hence, $x^2, z^2, \gamma, r^2 \in \mathbb{R}$ under $y^2 - x^2 \neq 0$ ($\det \Omega \neq 0$). These requirements can be satisfied if we put $x, \xi \in \mathbb{R}$, $y \in i\mathbb{R}$, $z \in i\mathbb{R}$ and $\gamma, r \in \mathbb{R}$. So, changing $z \rightarrow iz$, $y \rightarrow iy$ we keep the system (4) to be pure real and Hamiltonian

$$\widehat{\mathcal{H}} \dashrightarrow \frac{1}{2}(\widehat{z}^2 + \widehat{x}^2)$$

then becomes not only the real operator but also a lower bounded one. It immediately follows that background state and quantum mechanics do exist and therefore this Hamiltonian may be ‘not a bad’ candidate for the realistic one with a physical spectrum.

Mention should be made of relations between proposed scheme and some closely related models. On the one hand, the Mathieu equation can be treated as a quantum physical 1-dimensional pendulum and its limit $A \cos \gamma = A(1 - \frac{1}{2}\gamma^2 + \dots)$ reproduces locally a quadratic oscillator with an infinite discrete spectrum. Parameter A may play here a role of the potential well depth. The model (4) does thus inherit properties of these limit cases (a dimension reduction $4 \dashrightarrow 2$ and approximation $\cos \gamma \dashrightarrow \gamma^2$) and, on the other hand, is an alternative to the standard $\mathbb{S}\mathbb{O}(3)$ -quantization of a rotator [10]. It is interesting to observe that quantum statements

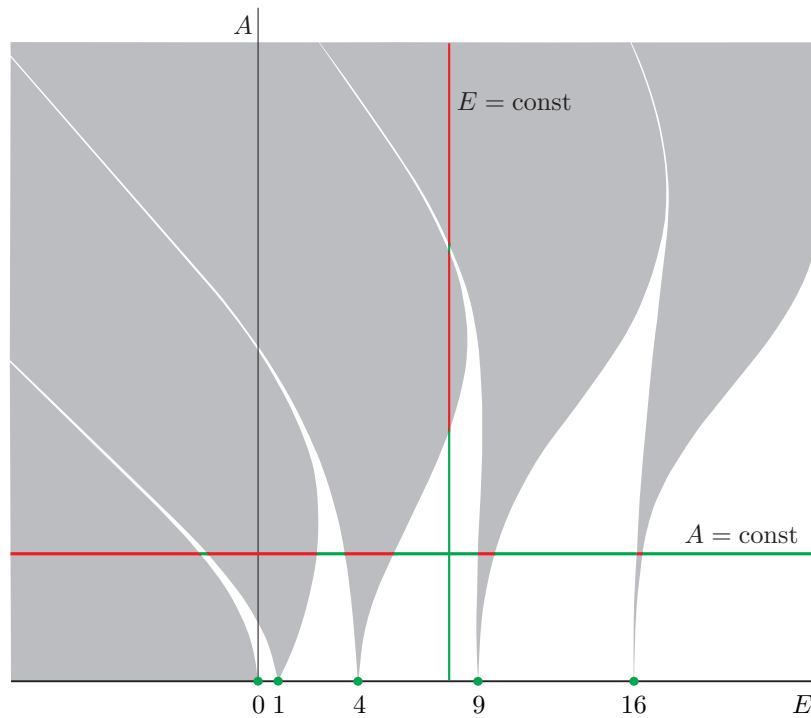


Figure 1. The gap structure spectrum for the Mathieu equation $\Psi'' = (A \cos \gamma - E)\Psi$. Shaded domains are the spectral lacunae ('forbidden' zones).

in both this quantization scheme and proposed new one are determined by the complete set of compatible/commuting observables which are classically evolving as *meromorphic functions* of t ; not as meromorphic/logarithmic elliptic integrals. Recall, meromorphic (elliptic) functions are the single-valued (well-defined) functions on torus, whereas integrals, in general, are not. Indeed, the standard momentum square L^2 and L_z for the rigid Euler's body are combinations of sn-, cn-, and dn-functions [10]. In the model under consideration the variable ξ , which is a meromorphic elliptic integral (5), does not get to the commuting pair of observables and all the other phase variables are also the meromorphic functions on a torus. We observe in passing that integral structure of the classical solutions to this new model is much simpler than the standard one. One of the Euler classical position (ψ, θ, φ) -angles of a body – the angle $\varphi(t)$ – is a cumbersome combination of meromorphic and logarithmic elliptic integrals; see [9, Chapter VI, § 37, equations (37.17)–(37.20)] and [11, p. 139]. Meanwhile the only variable of an integral type in the model (4) – variable ξ – is just a simple Z-function or, which is the same, some θ'_1/θ -combination. One may be noted in this connection that complete quantization procedure of theta functions will perhaps inevitably involve, the holomorphic integrals excepted, the full set of Abelian objects on Abelian variety: meromorphic functions, meromorphic and logarithmic integrals, and, which is the most nontrivial point, the θ itself as integral of an Abelian meromorphic integral. Such a nature of the theta-function is explained in [5].

Finally, Fig. 1 exhibits the typical spectra structure for the Mathieu equation [13] in both the quantum numbers E and A . The number of lacunae (colored in red) is infinite and bold green dots on the horizontal E -axis is the sequence $0^2, 1^2, 2^2, \dots$. It is a gap boundary set of the limit case $A \rightarrow 0$ following from the fact that the zero/constant potential has pure 2π -periodic solutions when spectral parameter is an integer perfect square $E = n^2$. Exhaustive theory, numerous applications, and wealth of numeric information to the Fig. 1 can be found in [8, 12, 19]. Note that number of gaps in A -direction is also infinite as in the E -axis; for proof see [19, Chapter 7, § 3.5].

A Appendix. Differential calculus of Legendre integrals

The standard Legendrian elliptic integrals [1] are the holomorphic and meromorphic integrals

$$\mathbf{F}(x; k) := \int_0^x \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}, \quad \mathbf{E}(x; k) := \int_0^x \frac{(1-k^2s^2)ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

(1st and 2nd kind respectively) and the logarithmic integral (3rd kind)

$$\mathbf{\Pi}_\alpha(x; k) := \int_0^x \frac{1}{1-\alpha s^2} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}.$$

The latter has a parameter α .

The triple $(\mathbf{F}, \mathbf{E}, \mathbf{\Pi}_\alpha)$ forms a linear vector space and the theory above requires complete set of its partial differentiations ∂_x , ∂_k , and ∂_α . Somewhat surprising facet is the fact that no self-contained differential calculus of these objects with respect to *all* the arguments (x, k, α) seems to have hitherto been tabulated in handbooks; see, e.g., [1, 7, 16]. Bearing in mind a remark about logarithmic integral at the beginning of Section 3, we make up this deficiency.

Denote $y^2 := (1-x^2)(1-k^2x^2)$. Then one can show that the three objects $(\mathbf{F}, \mathbf{E}, \mathbf{\Pi}_\alpha)$ have the following *closed* rules of differential computations

$$\frac{\partial \mathbf{F}}{\partial x} = \frac{1}{y}, \quad \frac{\partial \mathbf{F}}{\partial k} = -\frac{\mathbf{F}}{k} - \frac{\mathbf{E}}{k(k^2-1)} + \frac{k}{k^2-1} \frac{xy}{1-k^2x^2}, \quad (12)$$

$$\frac{\partial \mathbf{E}}{\partial x} = (1-k^2x^2) \frac{1}{y}, \quad \frac{\partial \mathbf{E}}{\partial k} = -\frac{\mathbf{F}}{k} + \frac{\mathbf{E}}{k},$$

$$\frac{\partial \mathbf{\Pi}_\alpha}{\partial x} = \frac{1}{1-\alpha x^2} \frac{1}{y},$$

$$\frac{\partial \mathbf{\Pi}_\alpha}{\partial k} = \frac{-k}{(k^2-\alpha)(k^2-1)} \left(\mathbf{E} + (k^2-1)\mathbf{\Pi}_\alpha - \frac{k^2xy}{1-k^2x^2} \right), \quad (13)$$

$$\frac{\partial \mathbf{\Pi}_\alpha}{\partial \alpha} = \frac{1}{2\alpha(\alpha-1)(k^2-\alpha)} \left((k^2-\alpha)\mathbf{F} + \alpha\mathbf{E} - (k^2-\alpha^2)\mathbf{\Pi}_\alpha - \frac{\alpha^2xy}{1-\alpha x^2} \right).$$

Note that these equations are compatible; the cross derivatives like

$$\frac{\partial}{\partial x} \frac{\partial \mathbf{E}}{\partial k} = \frac{\partial}{\partial k} \frac{\partial \mathbf{E}}{\partial x}, \quad \frac{\partial}{\partial \alpha} \frac{\partial \mathbf{\Pi}_\alpha}{\partial k} = \frac{\partial}{\partial k} \frac{\partial \mathbf{\Pi}_\alpha}{\partial \alpha}, \quad \dots$$

turn out to be identities. If we introduce also a Jacobi analog of Weierstrassian ζ -function, i.e., \mathbf{u} -representation

$$\mathbf{Z}(\mathbf{u}; k) := \mathbf{E}(\operatorname{sn}(\mathbf{u}; k); k)$$

of the \mathbf{E} -integral, then it is differentiated as follows

$$\frac{\partial \mathbf{Z}}{\partial \mathbf{u}} = 1 - k^2 \operatorname{sn}^2, \quad \frac{\partial \mathbf{Z}}{\partial k} = \frac{k}{k^2-1} (\mathbf{Z} \operatorname{cn}^2 - (k^2-1) \mathbf{u} \operatorname{sn}^2 - \operatorname{sn} \operatorname{cn} \operatorname{dn}). \quad (14)$$

If required, partial k -derivatives of functions sn , cn , $\operatorname{dn}(\mathbf{u}; k)$ are calculated by formulae which are known [11, pp. 82–83]; we do not display them here.

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