ISSN 1028-3358, Doklady Physics, 2015, Vol. 60, No. 3, pp. 135–139. © Pleiades Publishing, Ltd., 2015.
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MECHANICS

Gas Flow in a Long Tube with a Permeable Porous Wall

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Presented by Academician V.A. Levin October 17, 2014

Received September 26, 2014

Abstract—An analytical solution of the problem on the isothermal gas flow in a long tube with a permeable porous wall is derived. It is shown that this analytical solution can be used to calculate the optimal fiber length, including the nonporous permeable wall. The analysis of the application of the derived solution in studying the gas flow for various combinations of pressures p0, p1, and p2 is performed. Concrete examples of using the derived formulas, which have important practical applications, are considered. The results of the study can be used when designing the manufacturing apparatuses of various indentations.

DOI: 10.1134/S1028335815030088

Gas flows in long tubes with a porous wall are widespread in industrial production processes. For example, the possibility of applying perforated constructions is considered in the aircraft technology for improving the aerodynamic characteristics of flying vehicles by suction of perturbations from the front wing edge [1] as well as using membrane apparatuses to obtain nitrogen in order to suppress bursts or firing of the fuel tanks [2]. Hollow-fiber membranes with a permeable wall make it possible to increase considerably the operational efficiency of modern gas-separating and filtration apparatuses due to an increase in the working surface for the apparatus volume unit. To calculate such types sof facilities and optimize their designs, we should have reliable and sufficiently simple models for the description of flows of gases (and their mixtures) in hollow-fiber membranes. A characteristic feature of hollow-fiber membranes is the fact that $\frac{L}{d} = 10^3–10^4$ for them, where L and d are the total membrane length and diameter of its passage section, respectively. Such high values of this parameter allow us to attribute the hollow-fiber membranes to the category of long tubes with a permeable porous wall.

The isothermal flow of the viscous gas along a long tube with impermeable walls but allowing for variation in the gas density along the tube was investigated by Meyer and Landau [3]. The Poiseuille resistance law in the Meyer–Landau model is valid only locally in each tube section. It turns out that the gas flow rate is proportional to the difference in pressure squared at the tube input and output rather than to the pressure difference in the first degree as in the classical variant of an incompressible fluid.

STATEMENT OF THE PROBLEM

In this study, the isothermal flow of viscous ideal gas in a tube with a permeable porous wall at large ratios $\frac{L}{d}$ is considered using the Meyer–Landau model.

Let us substantiate the applicability of the Meyer–Landau model for this class of problems. It is known that the permeability coefficient of the wall material for gases in gas-separating hollow-fiber membrane apparatuses fabricated, for example, based on Graviton domestic fibers is in the range $K = 10^{-16}$ to -10^{-17} m^2/(s Pa) [4]. In this case, the ratio of the weight flow rate of gas into the wall to the flow rate in the tube section is insignificant. Consequently, the gas flow rate on longitudinal scales in tens calibers is almost invariable. Let us evaluate the length of the segment of the initial hydrodynamic stabilization of the flow by the formula [5]

$$l \simeq 0.115\mathrm{Re}d.$$

It follows from this formula that the laminar flow with a parabolic velocity profile, for example, at $\mathrm{Re} \approx 100$, is established at length l of several millimeters from the input section, while the apparatus length $L \geq 1$ m. Consequently, the influence of the initial hydrodynamic stabilization segment is negligibly small. Let us evaluate the local flow rate by the Meyer–Landau for-

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mula [3] for the case of an impermeable tube with length L and inner radius a:

$$G = \frac{\pi a^{4}}{16\eta}\frac{1}{RT}\frac{p_2^2 - p_1^2}{L}$$
$$= \frac{\pi a^4}{8\eta}\frac{p_2 - p_1}{L}\frac{p_2 + p_1}{2}\frac{1}{RT} = \langle\rho\rangle Q. \quad (1)$$

Here, G is the mass and $Q(z) = \frac{\pi a^4}{8\eta}\frac{\Delta p}{L}$ is the volume gas flow rate along the longitudinal fiber axis (the Poiseuille formula) in the section at distance z from its input, $\langle\rho\rangle = \frac{\rho_2 + \rho_1}{2}$ is the average gas density, η is the gas dynamic viscosity coefficient, R is the gas constant, and T is the temperature. Indices 1 and 2 are referred to the output and input fiber sections, respectively. At $p_2 - p_1 = 10^6$ Pa, $a = 5 \times 10^{-5}$ m, dynamic viscosity $\eta = 10^{-5}$ kg/(m s), and $L = 1$ m, we find $u \approx 30$ m/s. Thus, despite significant pressure drops at the tube (tens of atmospheres), the local gas flow in the tube will be incompressible. Consequently, the Poiseuille formula is applicable to hollow-fiber tubes with a permeable wall locally in each section.

Let us introduce the characteristic length of the hollow-fiber membrane L_0, which is important for further description and practical applications. We will understand the length at which the mass flow rate into the wall of the hollow-fiber membrane equals 96% of the total gas mass flow rate inflowing into it as L_0. It will further be shown that

$$L_0 = \sqrt{\frac{\delta a^3}{\eta K}}. \quad (2)$$

Here, δ is the thickness of the permeable wall of the hollow-fiber membrane.

ANALYTICAL SOLUTION

Let us consider the isothermal flow of a viscous ideal gas in a long tube with a permeable porous wall. The density of the local mass flux through the wall of the permeable tube (membrane) is calculated by the formula [4]

$$j_0 = 2\pi a\tilde{\rho}\frac{\Delta p}{\delta}K. \quad (3)$$

Here, $\Delta p = p(z) - p_0$, $p(z)$ is the local gas pressure in the tube section with coordinate z, p_0 is the pressure outside the tube, and $\tilde{\rho}$ is a certain characteristic gas density, as which we accept the density of the gas under consideration under the standard conditions. The variation in the gas density should also be taken into account for the gas flow through a permeable porous wall [6]. In this case, $\tilde{\rho} = \langle\rho\rangle = \frac{\rho(z) + \rho_0}{2}$, and we derive at $T = \text{const}$:

$$j_0 = 2\pi a\tilde{\rho}\frac{\Delta p}{\delta}K = \pi a\frac{p^2(z) - p_0^2}{RT\delta}K.$$

We have from the conservation law of mass in the fiber section with longitudinal coordinate z:

$$-dG(z) = j_0 dz.$$

Using the Poiseuille formula, we will derive for the local mass gas flow rate

$$G(z) = -\rho(z)\frac{\pi a^4}{8\eta}\frac{dp(z)}{dz} = -\frac{p(z)}{RT}\frac{\pi a^4}{8\eta}\frac{dp(z)}{dz}$$
$$= -\frac{1}{RT}\frac{\pi a^4}{16\eta}\frac{dp^2(z)}{dz},$$

while for the pressure distribution along the longitudinal tube axis, we have the differential equation

$$\frac{a^3}{16\eta}\frac{d^2p^2(z)}{dz^2} = \frac{p^2(z) - p_0^2}{\delta}K,$$

which will have the following form after the passage to dimensionless coordinates by formulas $p' = \frac{p(z)}{p_0}$ and $z' = \frac{z}{L_0}$ (primes are omitted):

$$\frac{d^3}{dz^2}p^2 = 16(p^2 - 1). \quad (4)$$

Let us denote $w(z) = p^2(z)$; then we will derive the second-order linear differential equation for function $w(z)$

$$\frac{d^2}{dz^2}w = 16(w - 1),$$

the general solution of which is as follows:

$$w(z) = C_1\exp(-4z) + C_2\exp(4z) + 1. \quad (5)$$

We find integration constants C_1 and C_2 from boundary conditions $w(0) = w_2$, $w(L) = w_1$:

$$C_1 = \frac{w_2 e^{4L} - w_1 + 1 - e^{4L}}{e^{4L} - e^{-4L}},$$

$$C_2 = -\frac{-w_1 + w_2 e^{-4L} - e^{-4L} + 1}{e^{4L} - e^{-4L}}.$$

Substituting the values found for C_1 and C_2 into (5), we will find the following exact solution for the distribution of dimensional pressure $p(z)$ along the tube axis with the permeable porous wall:

$$\frac{p(z)}{p_0} = \sqrt{\frac{\left(\frac{p_2}{p_0}\right)^2 e^{4L/L_0} - \left(\frac{p_1}{p_0}\right)^2 + 1 - e^{4L/L_0}}{e^{4L/L_0} - e^{-4L/L_0}} e^{-4z/L_0} - \frac{-\left(\frac{p_1}{p_0}\right)^2 + e^{-4L/L_0}\left(\frac{p_2}{p_0}\right)^2 - e^{-4L/L_0} + 1}{e^{4L/L_0} - e^{-4L/L_0}} e^{4z/L_0} + 1}. \quad (6)$$

When designing apparatuses of the types listed, the important parameters are the mass flow rates. Using the Poiseuille formula in the input and output tube sections, we will derive the following relationships for the mass flow rates:

$$G(z = 0) = G_2 = \rho_2 Q(0) = \left[-\frac{1}{RT}\frac{\pi a^4}{16\eta}\frac{dp^2(z)}{dz}\right]_{z=0},$$

$$G(z = L) = G_1 = \rho_1 Q(L) = \left[-\frac{1}{RT}\frac{\pi a^4}{16\eta}\frac{dp^2(z)}{dz}\right]_{z=L},$$

and for the integral mass gas flow rate through the wall of the porous tube we have relationship $G_0 = G_2 - G_1$. Passing to relative mass flow rates, we will derive

$$\frac{G_0}{G_2} = 1 - \frac{G_1}{G_2} = 1 - \frac{\left[\frac{dp^2(z)}{dz}\right]_{z=0}}{\left[\frac{dp^2(z)}{dz}\right]_{z=L}}. \quad (7)$$

APPLICATION OF THE SOLUTION FOUND TO ANALYZE THE GAS FLOW AT VARIOUS COMBINATIONS OF PRESSURES p_0, p_1, AND p_2

We note that solution (6) is valid for arbitrary combinations of pressures $p_2 = p(0)$, $p_1 = p(L)$, and p_0; including the main cases used in practice:

(i) $p_2 > p_1 = p_0$ corresponds to the porous tube having one open end and immersed into the surrounding medium with pressure p_0 [9];

(ii) $p_2 = p_1 \neq p_0$ corresponds to the U-like tube ($p_0 > p_2 = p_1$ corresponds to gas inflow and $p_0 < p_2 = p_1$ corresponds to gas outflow through a porous wall) [10].

For case (i), Eqs. (6) and (7) will have the form

$$\frac{p(z)}{p_0} = \left[\frac{\left(\frac{p_2}{p_0}\right)^2 e^{4L/L_0} - e^{4L/L_0}}{e^{4L/L_0} - e^{-4L/L_0}} e^{-4z/L_0} - \frac{e^{-4L/L_0}\left(\frac{p_2}{p_0}\right)^2 - e^{-4L/L_0}}{e^{4L/L_0} - e^{-4L/L_0}} e^{4z/L_0} + 1\right]^{1/2}, \quad (8)$$

$$\frac{G_0}{G_2} = \frac{G_2 - G_1}{G_2} = 1 - \frac{2}{e^{-4L/L_0} + e^{4L/L_0}}. \quad (9)$$

It should be noted that dependence (9) for this case is universal and valid for any gases and parameters of tubes with a porous wall. Dependence (9) is a monotonically ascending function and gives $\frac{G_2 - G_1}{G_2} = 0.96$ at $L = L_0$. Thus, the relative gas flux into a wall at fiber length $L = L_0$ is close to maximally possible being independent of the pressure drop between the input and output tube sections. This allows us to consider that length $L_0 = \sqrt{\frac{\delta a^3}{\eta K}}$ is optimal for practical applications.

The authors of [7, 8] derived the dimensionless equation for the pressure distribution in the hollow-fiber permeable membrane with a nonporous wall (the diffusion permeability mechanism), which in the notations used in this article has the form

$$\frac{d^2}{dz^2}p^2(z) = 32(p(z) - 1). \quad (10)$$

Equation (10) was numerically investigated in the statement of a two-point boundary problem $p(0) = \frac{p_2}{p_0}$, $p(L) = \frac{p_1}{p_0} = 1$. In particular, the plot for the relative

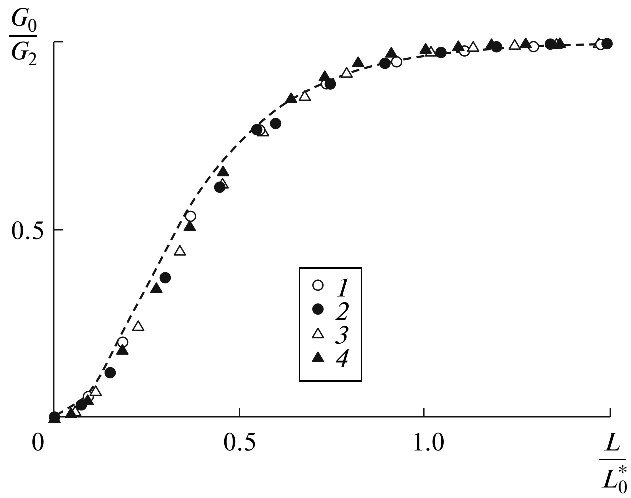

Fig. 1. Relative mass gas flow $\frac{G_0}{G_2}$ into the wall of a long porous tube depending on its reduced length $\frac{L}{L_0^*}$. $\frac{p_2}{p_0}$ = (*1*) 2, (*2*) 5, (*3*) 10, and (*4*) 15 atm.

mass flow into the membrane wall $\frac{G_2 - G_1}{G_2} = \frac{G_0}{G_2}$ depending on dimensionless fiber length $\frac{L}{L_0^*}$ for various input pressures p_2 is presented in Fig. 1. Here, L_0^* was calculated from the condition $\frac{G_0(L_0^*)}{G_2} = 0.96$ for input pressures $\frac{p_2}{p_0}$ = [2.0, 5.0, 10.0, 15.0] and was $\frac{L_0^*}{L_0} = 1 + 0.0858\frac{p_2 - p_0}{p_0}$. The dashed line in Fig. 1 shows analytical dependence (9). It is seen that the calculated points are described by analytical dependence (9) with good accuracy. Based on our description, we can recommend to producers of hollow-fiber permeable membranes to use the notion of the characteristic fiber length $L = L_0^*$ that we introduced above as one of its important fabrication characteristics.

Let us consider the second case when gas is simultaneously supplied (or withdrawn) from two ends of a long porous tube—the case of the so-called U-like tube. Now $p_2 = p_1 \neq p_0$ and formulas (6) and (7) will take the form

$$\frac{p_z}{p_0} = \left[\frac{\left(\frac{p_2}{p_0}\right)^2 e^{4L/L_0} - \left(\frac{p_2}{p_0}\right)^2 + 1 - e^{4L/L_0}}{e^{4L/L_0} - e^{-4L/L_0}} e^{-4z/L_0} - \frac{-\left(\frac{p_2}{p_0}\right)^2 + e^{-4L/L_0}\left(\frac{p_2}{p_0}\right)^2 - e^{-4L/L_0} + 1}{e^{4L/L_0} - e^{-4L/L_0}} e^{4z/L_0} + 1\right]^{1/2},$$

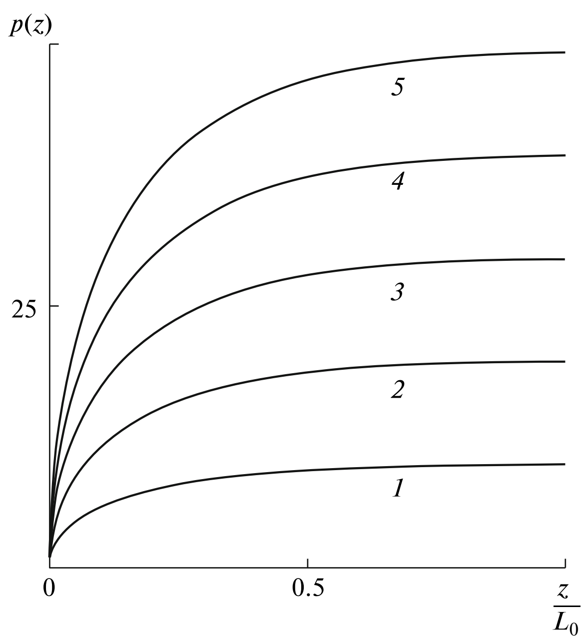

Fig. 2. Gas pressure distribution in a porous U-like tube depending on p_0 = (*1*) 10, (*2*) 20, (*3*) 30, (*4*) 40, and (*5*) 50 atm. $L = 2L_0$, $p_2 = p_1 = 1$ atm.

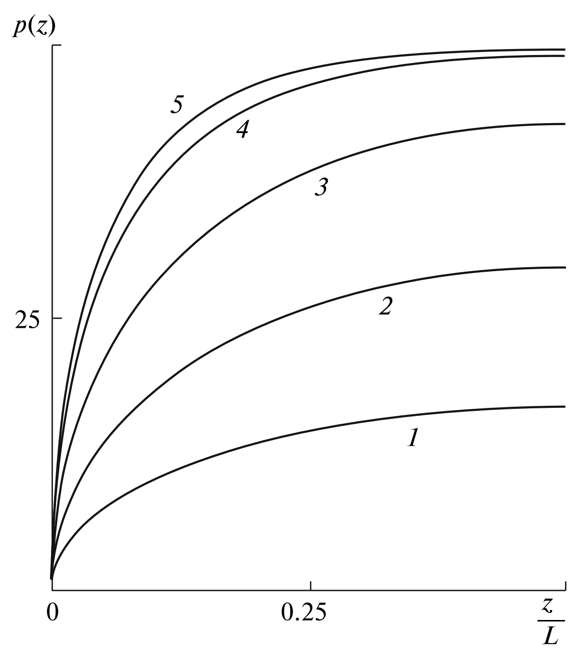

Fig. 3. Pressure distribution in a U-like porous tube for the case of gas inflow through its wall depending on its reduced length $\frac{L}{L_0}$: (*1*) $L = L_0/4$, (*2*) $L_0/2$, (*3*) $L = L_0$, (*4*) $L = 2L_0$, and (*5*) $L = 2.5L_0$. $p_2 = p_1 = 1$ atm, $p_0 = 50$ atm.

$$\frac{G_0}{G_2} = 1 - \frac{G_1}{G_2} = 1 - \frac{\left[\frac{dp^2(z)}{dz}\right]_{z=0}}{\left[\frac{dp^2(z)}{dz}\right]_{z=L}} = 2.$$

Figure 2 shows the pressure distribution plots inside such a *U*-like porous tube for the case of gas inflow through its wall.

Figure 3 shows the pressure distribution plots inside the U-like porous tube for the case of gas inflow through its wall depending on its reduced length $\frac{L}{L_0}$. The mass flow of gas flowing into the tube in the case under consideration is directly proportional to the derivative

$$\left[\frac{dp^2(z)}{dz}\right]_{z=0} = -\left[\frac{dp^2(z)}{dz}\right]_{z=L}.$$

It is seen from Fig. 3 that an increase in the tube working length above introduced characteristic length L_0 does not lead to a substantial rise of this derivative. The importance of the introduced notion of characteristic length L_0 here manifests itself from the mathematical viewpoint as well. Point $z = 0$ becomes singular at tube length $L \gg L_0$ since there is a finite limiting value of derivative $\lim\limits_{L \to \infty}\left[\frac{dp^2(z)}{dz}\right]_{z=0}$. Therefore, the numerical solution of the Cauchy problem and the two-point boundary problem of differential equation (10) becomes complicated (the problem is stated incorrectly since integral curves degenerate into one limiting curve, which characterize the pressure distribution in an infinitely long porous tube).

Thus, we present the derivation of the analytical solution of the problem on gas flow in a long tube with a permeable porous wall. The formula for the calculation of the optimal fiber length at which the relative gas flow into the wall is 96% of the maximally possible length and does not depend on the pressure drop between the input and output tube sections is derived. The application of the found solution for studying the gas flow is analyzed at various combinations of pressure p_0, p_1, and p_2. Concrete examples of using the derived formulas, which have important practical applications, are considered. These results can be used when designing production apparatuses of various indentations.

ACKNOWLEDGMENTS

This study was supported by the Russian Foundation for Basic Research, project no. 14-08-01048-a and in part by the Program for Increasing Competitiveness of Tomsk State University.

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Translated by N. Korovin