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SCALAR FIELD VACUUM POLARIZATION ON HOMOGENEOUS SPACES WITH AN INVARIANT METRIC

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We develop a method for calculating vacuum expectation values of the energy–momentum tensor of a scalar field on homogeneous spaces with an invariant metric. Solving this problem involves the method of generalized harmonic analysis based on the method of coadjoint orbits.

Keywords: vacuum polarization, energy–momentum tensor, harmonic analysis on homogeneous spaces

1. Introduction

Quantum field theory in a curved space–time is a sufficiently well developed theory (see [1]–[3] and [4]), which attracts interest in view of relevant applications to cosmology and astrophysics. The most important quantity characterizing matter is the energy–momentum tensor (EMT). It plays the role of a source for the gravitational field in the Einstein equations and describes the coupling of matter to the gravitational field. Expectation values of the EMT in the vacuum state characterize the effect of vacuum polarization and, if the vacuum state is not defined uniquely, also the effects of particle creation by the gravitational field.

Divergences occur in calculating quantum expectation values over any state for operators (the EMT in particular) that are bilinear in the fields because bilinear operators contain products of operator-valued generalized functions. Hence, obtaining finite values of vacuum expectation values of the EMT requires using some procedure for removing the divergences. In the case where the space is homogeneous and isotropic, using the dimensional regularization method [5] is efficient. Another way to regularize is by the method of splitting the arguments of field operators in the bilinear form of the EMT, proposed in [6]. We note that although these regularization methods do not require calculating vacuum expectation values of the EMT, these last are also interesting because it is possible to eliminate the divergences directly in several cases (for example, using the n-wave regularization method [7]).

We note that practically all the currently known models of Riemannian manifolds of general relativity are associated with various transformation groups and, not infrequently, belong to the class of homogeneous Riemannian spaces. In modern cosmology, homogeneous spaces underlie the construction of Big Bang models, initial singularities, and inflationary models. The problem of taking quantum vacuum effects on homogeneous space into account then arises naturally.

This problem is closely related to the problem of exactly integrating relativistic wave equations on manifolds with curvature and a nontrivial topology. The most universal solution method is the method of separation of variables [8], [9]. But it accounts for only a commutative algebra of the symmetry equations, while there exists a class of spaces that do not admit separation of variables. Therefore, in most cases, quantum vacuum effects can be calculated by imposing various constraints on the metric of the space (such

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as selecting conformally flat metrics [2], homogeneous isotropic spaces [1], Stäckel metrics [10], and so on) that would allow integrating the wave field equations.

This paper is devoted to calculating vacuum expectation values of the EMT of a scalar field on a homogeneous space with an arbitrary invariant metric of a static space–time.

To solve this problem, we use the method of orbits, which allows performing a noncommutative reduction (the method of noncommutative integration [11], [12]) of the Klein–Gordon wave equation to an equation with fewer independent variables on a manifold with simpler geometry and topology. This method, unlike the method of separation of variables, takes the noncommutative symmetry algebra of the Klein–Gordon equation into account. Moreover, the solution is constructed globally and is independent of the choice of local coordinates on the homogeneous space. The method of coadjoint orbits (K-orbits) was first described by Kirillov [13], [14] and then developed by Kirillov, Kostant, Souriau, and others. The main results of the method were presented in [15]–[17].

In our previous work [18], we used the orbit method to investigate particular cases where a homogeneous space is commutative (the defect of the homogeneous space is equal to zero) and where it is a Lie group (the defect is maximum). Here, we take the algebra of invariant operators on the homogeneous space into account in integrating the wave equation and assume that the defect of the space is arbitrary.

In Secs. 2 and 3, we briefly expound the method of K-orbits and the harmonic analysis on homogeneous spaces based on that method. A more detailed presentation with proofs of the main statements can be found in [19]–[22].

Section 4 is devoted to applying the orbit method to noncommutative reduction of the Klein–Gordon equation. We obtain relations that express a basis of solutions of the Klein–Gordon equation in terms of a basis of solutions of the reduced equation and satisfy the scalar field normalization condition. We note that noncommutative reduction on a homogeneous space with a nonvanishing defect essentially involves the algebra of invariant operators (the $\mathcal{F}$-algebra in what follows) on the homogeneous space. We find an expression for the generalized local zeta function of the Klein–Gordon equation operator and show that it is independent of the choice of local coordinates on the homogeneous space but is defined in terms of quantities defined on a Lagrangian submanifold to a symplectic leaf of the $\mathcal{F}$-algebra. This facilitates finding the analytic continuation of the zeta function in specific problems because the expression for the zeta function turns out to be simplified and to depend on fewer independent variables.

In Sec. 5, we consider the EMT of a scalar field in a static space–time. The EMT is considered in the *quasitetrad* components introduced in [18]. This allows proceeding without using local coordinates on the homogeneous space; we can always pass to the standard EMT components at the end of the calculation. In quasitetrad components, we find expressions for vacuum expectation values of the EMT that are defined by algebraic properties of the homogeneous space (such as a λ-representation of the Lie algebra of the Lie group of transformations and of the $\mathcal{F}$-algebra of invariant operators) and are independent of local coordinates on the homogeneous space. To obtain finite values of the vacuum expectation values of the EMT, we use a ramification of the generalized zeta function method, proposed in [23]. In [18], the generalized zeta-function method was used to calculate vacuum expectation values of the EMT on Lie groups with an invariant metric. This method for renormalizing vacuum expectation values of the EMT is based on calculating functional derivatives of a one-loop effective action over the metric and reduces the problem to finding an analytic continuation for the generalized zeta function.

As a nontrivial example, we consider a homogeneous space with the defect equal to unity and with a four-dimensional Lie algebra of transformations given by a direct product of an Abelian two-dimensional algebra and two one-dimensional ideals. For an arbitrary invariant metric of a static space–time, we find an expression for the generalized local zeta function and the renormalized value of the vacuum energy density of the scalar field due to the nontrivial topology and curvature of this homogeneous space.

2. The λ-representation of a Lie algebra and a Lie group on a K-orbit

Let G be a connected simply connected real Lie group, $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{g}^*$ be the space of linear functionals on $\mathfrak{g}$. Let the Lie group G act on the dual space $\mathfrak{g}^*$ by the coadjoint representation $\mathrm{Ad}^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$. This action foliates $\mathfrak{g}^*$ into K-orbits of dimension $\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g} - 2k$, where the number k ranges from 0 to $(\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g})/2$. The *index* $\operatorname{ind} \mathfrak{g}$ of the algebra is defined as the number of independent Casimir functions on the dual space $\mathfrak{g}^*$ with respect to the Poisson–Lie bracket $\{\,\cdot\,,\,\cdot\,\}^{\mathrm{Lie}}$. It was shown in [20] that the coalgebra $\mathfrak{g}^*$ is the union of connected invariant algebraic surfaces $M_{(s)}$, where each connected surface $M_{(s)}$ is a union of K-orbits of dimension $\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g} - 2s$.

In what follows, we use the terminology introduced in [19], [20]. Functions $K_\mu^{(s)}(f)$ that are nonconstant on $M_{(s)}$ and commute with any function on $M_{(s)}$ are called *(s)-type Casimir functions*. The number $r_{(s)}$ of functionally independent (s)-type Casimir functions coincides with the space dimension, $r_{(s)} = \dim M_{(s)}$. A K-orbit is called an *s-type orbit* if $\mathcal{O}_\lambda \in M_{(s)}$, and the number s is the *degeneration degree* of the orbit. The K-orbits with degeneration degree zero are said to be *nondegenerate* and otherwise *singular*. We let $F_\alpha^{(s)}(f)$, $\alpha = 1, \ldots, \dim \mathfrak{g} - r_{(s)}$ denote an independent tuple of functions defining the surface $M_{(s)}$.

For a Lie group G, let $\mathcal{O}_\lambda$ be an (s)-type K-orbit containing a covector λ. The Kirillov form ω_λ defines a symplectic structure on the K-orbit $\mathcal{O}_\lambda$. On the K-orbit, we introduce canonical Darboux coordinates $(p,q) \in P \times Q$, in which the Kirillov form ω_λ takes the canonical form $\omega_\lambda = dp_a \wedge dq^a$, $a = 1, \ldots, \dim \mathcal{O}_\lambda/2$. It is obvious that the domains P and Q are Lagrangian submanifolds of dimension $\dim \mathcal{O}_\lambda/2$. In accordance with [22], we define a *canonical* embedding $f\colon \mathcal{O}_\lambda \to \mathfrak{g}^*$ under which a covector $f \in \mathfrak{g}^*$ is assigned its canonical coordinates on the corresponding K-orbit. The canonical embedding is defined uniquely by functions $f_X = f_X(p,q,\lambda)$ satisfying the system of equations

$$\{f_X, f_Y\}^{\mathrm{Lie}} = f_{[X,Y]}, \qquad f_X(0,0,\lambda) = \lambda(X), \quad X, Y \in \mathfrak{g}.$$

Because $f \in M_{(s)}$, it follows that in the case of singular K-orbits, the canonical embedding must also satisfy the condition $F_\alpha^{(s)}(f) = 0$, $\alpha = 1, \ldots, \dim \mathfrak{g} - r_{(s)}$.

We pass from the Lie algebra $\mathfrak{g}$ to the corresponding complex extension $\mathfrak{g}_{\mathbb{C}}$ and consider the canonical embedding linear in the variables p:

$$f_X(q,p,\lambda) = \alpha_X^a(q) p_a + \chi_X(q,\lambda), \quad X \in \mathfrak{g}_{\mathbb{C}}, \quad a = 1, \ldots, \dim Q. \tag{1}$$

It was shown in [22] that for the existence of linear canonical embedding (1) of an orbit $\mathcal{O}_\lambda$, it is necessary and sufficient that the functional λ admit a *polarization* $\mathfrak{p}$. We recall that a polarization $\mathfrak{p}$ of a functional λ is a subalgebra in $\mathfrak{g}_{\mathbb{C}}$ of dimension $\dim \mathfrak{p} = \dim \mathfrak{g} - \dim \mathcal{O}_\lambda/2$ subordinated to the functional λ: $\langle \lambda, [\mathfrak{p}, p] \rangle = 0$. We note that a polarization $\mathfrak{p}$ is an isotropy subalgebra of the algebra $\mathfrak{g}_{\mathbb{C}}$ of the local group $G_{\mathbb{C}}$ acting on the local homogeneous space $Q \approx G_{\mathbb{C}}/e^p$.

We quantize K-orbits, which amounts to assigning each spectral type of orbits a special representation of the Lie algebra [20]. The canonical embedding functions $f_X(p,q,\lambda)$ then correspond to the operators $\hat{f}_X(q,\lambda) = f_X(-i\partial_q, \hat{q}, \lambda)$. This quantization procedure is unique under the condition

$$i[\hat{f}_X, \hat{f}_Y] = \hat{f}_{[X,Y]}, \quad X, Y \in \mathfrak{g}.$$

The operators $l_X(q,\lambda) \equiv i f_X(\hat{q}, \hat{p}, \lambda)$ realize an irreducible representation of the Lie algebra $\mathfrak{g}$ in the space of smooth functions $L(Q, \mathfrak{p}, \lambda)$ (the so-called λ-representation of a Lie algebra [19], [20]).

On the manifold Q, we introduce a measure $d\mu_0(q)$ and the scalar product

$$(\psi_1, \psi_2) = \int_Q \overline{\psi_1(q)} \psi_2(q)\, d\mu(q), \qquad d\mu(q) = \Delta(q)\, d\mu_0(q),$$

where $\Delta(q) = \Delta(s(q), e_H)$, $\Delta(g)$ is the modulus of the Lie group G, and $s\colon Q \to G_{\mathbb{C}}$ is a smooth section of the bundle $G_{\mathbb{C}}$ with Q as a base and $e^{\mathfrak{p}}$ as a fiber. We require that the λ-representation operators be skew-Hermitian with respect to the measure $d\mu_0(q)$. To satisfy this condition, it suffices to introduce the corresponding "quantum shift" by a real vector β in the λ-representation operators: $l_X(q, \tilde{\lambda}) = l_X(q, \lambda + i\beta)$ (see [24]).

We introduce a lift of the λ-representation of the Lie algebra $\mathfrak{g}$ to a local representation of its Lie group G:

$$T^\lambda(g)\varphi(q) = \int D^\lambda_{q\overline{q}'}(g)\varphi(q')\,d\mu(q'), \qquad \frac{\partial}{\partial t}|_{t=0} T^\lambda(e^{tX})\varphi(q) = l_X(q,\lambda)\varphi(q),$$

where $\varphi \in L(Q, d\mu(q))$. It can be shown (see [19]) that the generalized functions $D^\lambda_{q\overline{q}'}(g)$ satisfy the overdetermined system of equations

$$[\eta_X(g) + l_X(q,\lambda)]D^\lambda_{q\overline{q}'}(g) = 0, \qquad [\xi_X(g) - \overline{l^\dagger_X(q',\lambda)}]D^\lambda_{q\overline{q}'}(g) = 0, \tag{2}$$

where $\xi_X(g) = (L_g)_* X$ and $\eta_X(g) = -(R_g)_* X$, $X \in \mathfrak{g}$, $g \in G$, are left- and right-invariant vector fields on G. It was shown in [20] that the requirement that the functions $D^\lambda_{q\overline{q}'}(g)$ be well defined on a Lie group G implies the Kirillov integrality condition for the orbit $\mathcal{O}_\lambda$ [15]:

$$\frac{1}{2\pi}\int_{\gamma \in H_2(\mathcal{O}_\lambda)} \omega_\lambda = n_\gamma \in \mathbb{Z}.$$

We assign an invariant subspace $M_{(s)}$ of $\mathfrak{g}^*$ an invariant functional subspace $L_{(s)} = \{\phi \in L_2(G, d\mu(g)) \mid F^{(s)}_\alpha(\xi)\phi(g) = 0\}$ of $L_2(G, d\mu(g))$. The family of generalized functions $D^\lambda_{q\overline{q}'}(g)$ has the completeness and orthogonality properties, and the direct and inverse Fourier transformations are therefore defined for each function $\phi(g)$ in $L_{(s)}$ [19]:

$$\psi(q, q', \lambda) = \Delta^{-1}(q)\int \phi(g)\overline{D^\lambda_{q\overline{q}'}}(g^{-1})\,d\mu(g), \tag{3}$$

$$\phi(g) = \int \psi(q, q', \lambda) D^\lambda_{q\overline{q}'}(g^{-1})\,d\mu(q)\,d\mu(q')\,d\mu(\lambda), \tag{4}$$

where $d\mu(\lambda)$ is the spectral measure of Casimir operators $K_\mu(\eta)$ on the group and $d\mu(g)$ is the right Haar measure on the Lie group G. For nondegenerate orbits, direct and inverse transformations (3) and (4) are defined on the entire space $L_{(0)} = L_2(G, d\mu(g))$.

3. The λ-representation of the algebra of invariant operators and harmonic analysis on homogeneous spaces

We consider a right homogeneous space M admitting a motion group G. Any point $x \in M$ of the homogeneous space defines the isotropy subgroup $H_x \in G$ that leaves this point fixed. Let H be the closed stationary subgroup of some point $x_0 \in M$ and $\mathfrak{h}$ be its Lie algebra. The homogeneous space M is diffeomorphic to the quotient manifold G/H of right conjugacy classes of G by the isotropy subgroup H, and the group of transformations G can be regarded as the fibered manifold of the bundle (G, π, M, H) with the structure group H, base M, and canonical projection $\pi\colon G \to M$. The linear space of the Lie algebra $\mathfrak{g}$ admits a decomposition into the direct sum of subspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} \simeq T_{x_0}M$ is a complement of $\mathfrak{h}$.

It is known [25] that an associative algebra $D(M)$ of invariant differential and pseudodifferential operators commuting with the group generators corresponds to each Lie group G acting on a homogeneous space M.

In the algebra $D(M)$, we can single out a finite set of functionally independent generators $\{L_\mu\}$ that are symmetric functions of the operators $-i\hbar\eta$ and satisfy the nonlinear commutation relations

$$\frac{i}{\hbar}[L_\mu, L_\nu] = \Omega_{\mu\nu}(L), \tag{5}$$

where $\Omega_{\mu\nu}(L)$ is a symmetric function of the operators $L_{\mu\nu}$. The algebra of nonlinear commutation relations of form (5) is called the *functional algebra* (the *$\mathcal{F}$-algebra*) [12].

The number $s_M = \dim\mathfrak{g} - \operatorname{ind}\mathfrak{g} - d_{\max}$, where $d_{\max}$ is the maximal dimension of (s)-type K-orbits that have a nonvanishing intersection with the subspace $\mathfrak{h}^\perp = \{f \in \mathfrak{g}^* \mid f(X) = 0, X \in \mathfrak{h}\}$, is called the *degeneration degree* of the homogeneous space M. It was shown in [21] that a set of i_M independent identities (functional relations for the transformation group generators) on a homogeneous space M consists of the functions $\Gamma(f) = \{F_\alpha^{(s_M)}(f), \tilde{K}_\mu^{(s_M)}(f)\}$, where $\tilde{K}_\mu^{(s_M)}(f)$ are *trivial Casimir functions* of the (s_M) type (trivial Casimir functions are equal to zero for all $f \in \mathfrak{h}^\perp$). The number i_M is called the *homogeneous space index*. The index and the degeneration degree of a homogeneous space are defined by structure constants of the transformation group algebra $\mathfrak{g}$ and of the isotropy subalgebra $\mathfrak{h}$ [21]:

$$s_M = \frac{1}{2}\sup_{\lambda\in\mathfrak{h}^\perp}\operatorname{rank}\langle\lambda, [\mathfrak{g},\mathfrak{g}]\rangle - \frac{1}{2}\operatorname{ind}\mathfrak{g}, \qquad i_M = \dim\mathfrak{h} - \operatorname{rank}\langle\lambda, [\mathfrak{g},\mathfrak{h}]\rangle,$$

where λ is a generic element of $\mathfrak{h}^\perp$. The positive integer

$$d(M) = \dim M + i_M - s_M - \frac{1}{2}(\dim\mathfrak{g} + \operatorname{ind}\mathfrak{g})$$

is called the *defect* of the homogeneous space. It was introduced in [19], [21]. The defect of a homogeneous space characterizes the properties of the algebra of invariant operators. For commutative spaces ($d(M) = 0$), such as, in particular, symmetric and weakly symmetric spaces, the algebra of invariant operators is commutative and consists of Casimir operators $K_\mu^{(s_M)}(X)$.

To the algebra of invariant operators $D(M)$, there corresponds a Poisson algebra $\mathcal{F}$ with the commutation relations

$$\{a_\mu, a_\nu\}^{\mathrm{Lie}} = \Omega^{\mathrm{cl}}_{\mu\nu}(a), \qquad \Omega^{\mathrm{cl}}_{\mu\nu}(a) \equiv \lim_{\hbar\to0}\Omega_{\mu\nu}(a), \tag{6}$$

where $a_\mu(f) \equiv \lim_{\hbar\to0} L_\mu(f)$ are functions on the dual space $\mathfrak{g}^*$ (the symbol of the operator L_μ). Let $\mathcal{F}^*$ be the linear space conjugate to $\mathcal{F}$. The infinite-dimensional linear space of smooth functions on $\mathcal{F}^*$ is a Poisson algebra with respect to the Poisson bracket

$$\{\varphi(a), \psi(a)\}^{\mathcal{F}} \equiv \Omega^{\mathrm{cl}}_{\mu\nu}(a)\partial^\mu\varphi(a)\partial^\nu\psi(a), \qquad \partial^\mu \equiv \frac{\partial}{\partial a_\mu}, \quad a \in \mathcal{F}^*. \tag{7}$$

Poisson bracket (7) is degenerate in general, and the space $\mathcal{F}^*$ foliates into symplectic leaves

$$O_\sigma = \{a \in \mathcal{F}^* \mid Z_t(a) = \sigma_t = \mathrm{const},\ t = 1, \dots, \operatorname{ind}\mathcal{F}\}, \tag{8}$$

where the functions $Z_t(a)$ are called *Casimir functions of the $\mathcal{F}$-algebra* and can be found from the system of equations $\Omega^{\mathrm{cl}}_{\mu\nu}(a)\partial^\mu Z_t(a) = 0$. The number $\operatorname{ind}\mathcal{F}$ is equal to the number of independent Casimir functions $Z_t(a)$ and is called the *index* of the $\mathcal{F}$-algebra. The dimension and the index of the $\mathcal{F}$-algebra are defined by the formulas [21],

$$\dim\mathcal{F} = i_M + 2\dim M - \dim\mathfrak{g}, \qquad \operatorname{ind}\mathcal{F} = \operatorname{ind}\mathfrak{g} + 2s_M - i_M.$$

The dimension of a symplectic leaf O_σ is determined by the defect of the homogeneous space: $\dim O_\sigma = \dim \mathcal{F} - \operatorname{ind} \mathcal{F} = 2d(M)$.

We note that the number of Casimir functions $Z_t(a)$ on $\mathcal{F}^*$ coincides with the number of nontrivial Casimir functions $K_\mu(f)$, and there exist functions $D_t(Z)$ such that

$$D_t(Z(a(f))) = K_t^{(s_M)}(f), \quad f \in \mathfrak{h}^\perp, \quad t = 1, \dots, \operatorname{ind} \mathcal{F}.$$

Hence, the method of coadjoint orbits allows studying the structure of the $\mathcal{F}$-algebra of invariant operators in detail. In particular, based on the foregoing constructions, it is relatively easy to provide an explicit form of these operators.

On symplectic leaves (8) of the dual space $\mathcal{F}^*$, we pass to canonical Darboux coordinates (u, v). As in the case of K-orbits, we can define the canonical embedding $a_\mu \colon O_\sigma \to \mathcal{F}^*$, which is given by the functions $a_\mu = a_\mu(u, v, \sigma)$ satisfying the system of equations $\{a_\mu, a_\nu\} = \Omega^{\mathrm{cl}}_{\mu\nu}(a)$, $\mu, \nu = 1, \dots, \dim \mathcal{F}$.

We proceed with the quantization of symplectic leaves. For this, we replace the functions $a_\mu(u, v, \sigma)$ with differential operators $\hat{a}_\mu(v, i\partial_v, \tilde{\sigma})$ acting in the space of functions defined on a Lagrangian submanifold of the symplectic leaf (functions of the variables $v \in V$) and impose the commutation relations $-i[\hat{a}_\mu, \hat{a}_\nu] = \Omega_{\mu\nu}(\hat{a})$. In the operators $\hat{a}_\mu(v, i\partial_v, \tilde{\sigma})$, we introduce the "quantum shift" $\sigma \to \tilde{\sigma}$, which occurs in quantizing (s_M)-type K-orbits corresponding to the homogeneous space. We have the identities

$$Z_t(\hat{a}) = \tilde{\sigma}_t, \qquad D_t(\tilde{\sigma}) = \kappa_t^{(s_M)}(\sigma). \tag{9}$$

The realization of the $\mathcal{F}$-algebra by the operators $\hat{a}$ such that conditions (9) are satisfied is called a *λ-representation of the $\mathcal{F}$-algebra* [19].

We consider the procedure for constructing a harmonic analysis for functions on a homogeneous space M that belongs to the Hilbert space $L_2(M, d\mu(x))$, where $d\mu(x)$ is the Riemann measure constructed from an invariant metric.

Over a trivialization domain $U \in P$ in the fibered space G, we introduce coordinates $g^A = (x^a, h^\alpha)$ of the direct product $U \times H$ ($a = 1, \dots, \dim M, \alpha = \dim M + 1, \dots, \dim G$). The coordinates of an arbitrary point $g \in G$ can then be represented as $g = hs(x)$, where $s \colon M \to G$ is a local smooth section of the bundle G. Any smooth function $\varphi(g)$ on the Lie group G that is constant on the fibers H of the principal bundle (G, π, M, H) uniquely corresponds to a function $(\pi^*\varphi)(x) = \varphi(s(x))$ on the homogeneous space M (the projection of φ onto M). In other words, there is an isomorphism $C^\infty(M) \simeq \mathcal{F}_G$, where the functional space $\mathcal{F}_G$, in view of the connectedness of the Lie group H, is defined by the equality

$$\mathcal{F}_G \equiv \{\varphi \in C^\infty(G) \mid \eta_X \varphi(g) = 0,\ X \in \mathfrak{h},\ g \in G\}.$$

Because $F^{(s_M)}(\xi) = 0$, the space $L_2(M, d\mu(x))$ is isomorphic to the space

$$L_{(s_M)}(M) = \{\varphi \in L_{(s_M)} \mid \eta_X \varphi = 0, X \in \mathfrak{h}\}.$$

Hence, harmonic analysis on a homogeneous space M reduces to harmonic analysis in the space $L_{(s_M)}$.

The invariant space $L_{(s_M)}$ corresponds to the invariant subspace $M_{(s_M)}$ consisting of K-orbits $\mathcal{O}_\lambda^{(s_M)}$ whose representatives are in $\mathfrak{h}^\perp$. We quantize the K-orbit and construct a λ-representation of the algebra $\mathfrak{g}$ corresponding to the K-orbit $\mathcal{O}_\lambda^{(s_M)}$. Because $\lambda \in \mathfrak{h}^\perp$, the condition $\tilde{K}_\mu^{(s_M)}(\lambda) = 0$, $\mu = 1, \dots, r_{(s_M)}$ is imposed on the parameters λ. In [19], the λ-representation satisfying this condition is said to *correspond to the homogeneous space M*.

We lift the λ-representation of the (s_M)-type algebra $\mathfrak{g}$ corresponding to the homogeneous space M to a representation of the Lie group G. By completeness and orthogonality, the set of functions $D^{\lambda}_{q\overline{q}'}(g^{-1})$ constitutes a basis of the functional space $L_{(s_M)}$.

We define a parametric family of functions $D^{\lambda}_{qv}(x)$ that form a basis in the functional space

$$L_2(M, d\mu(x)) \simeq L_{(s_M)} \cap \mathcal{F}_G$$

as the decomposition in terms of the functions $D^{\lambda}_{q\overline{q}'}(g^{-1})$:

$$D^{\lambda}_{qv}(x) = \int c_{\lambda}(q', v) D^{\lambda}_{q\overline{q'}}(g^{-1})\, d\mu(q'), \qquad g = (x, h). \tag{10}$$

We impose the conditions

$$l_{\alpha}(q', \lambda) c_{\lambda}(q', v) = 0, \tag{11}$$

$$[L_{\mu}(-il(q', \lambda)) - \hat{a}_{\mu}(v, \lambda)] c_{\lambda}(q', v) = 0 \tag{12}$$

on the decomposition coefficients. In view of condition (11), the right-hand side of (10) belongs to the class of functions $\mathcal{F}_G$. If (12) is satisfied, then the functions $D^{\lambda}_{qv}(x)$ constitute a basis of the λ-representation of the $\mathcal{F}$-algebra of invariant functions.

By relations (2), the functions $D^{\lambda}_{qv}(x)$ satisfy the system of equations

$$[X_A(x) + l_A(q, \lambda)] D^{\lambda}_{qv}(x) = 0, \qquad [L_{\mu}(x) - \hat{a}_{\mu}(v, \lambda)] D^{\lambda}_{qv}(x) = 0,$$

where $A = 1, \dots, \dim \mathfrak{g}$ and $\mu = 1, \dots, \dim \mathcal{F}$. We define a family of generalized functions $\tilde{D}^{\lambda}_{qv}(x)$ as solutions of the system

$$[X^{\dagger}_A(x) + l^{\dagger}_A(q, \lambda)] \tilde{D}^{\lambda}_{qv}(x) = 0, \qquad [L^{\dagger}_{\mu}(x) - \hat{a}^{\dagger}_{\mu}(v, \lambda)] \tilde{D}^{\lambda}_{qv}(x) = 0.$$

Because the generalized functions $D^{\lambda}_{q\overline{q'}}(g^{-1})$ have the properties of completeness and orthogonality in the functional space L_{s_M}, it follows that the family of generalized functions $D^{\lambda}_{qv}(x)$ and $\tilde{D}^{\lambda}_{qv}(x)$, where λ is a nondegenerate covector of the (s_M) type, constitute a complete and orthogonal set in the functional space $L_2(M, d\mu(x))$:

$$\int_M \overline{\tilde{D}^{\tilde{\lambda}}_{\tilde{q}\tilde{v}}}(x) D^{\lambda}_{qv}(x)\, d\mu(x) = \delta(q, \overline{\tilde{q}}) \delta(\tilde{v}, v) \delta(\tilde{\lambda}, \lambda), \tag{13}$$

$$\int_{Q \times V \times J} \overline{\tilde{D}^{\lambda}_{qv}(\tilde{x})} D^{\lambda}_{qv}(x)\, d\mu(q)\, d\mu(v)\, d\mu(\lambda) = \delta(\tilde{x}, x). \tag{14}$$

Relations (13) and (14) allow introducing analogues of direct and inverse Fourier transformations on the homogeneous space M. Let $\Phi(M) \subset L_2(M, d\mu(x)) \subset \Phi'(M)$ be a Gelfand triplet. The space $\Phi(M)$ is a linear space of functions $\varphi(x) \in L_2(M, d\mu(x))$ for which a generalized Fourier transformation is defined:

$$\psi_{\lambda}(q, v) = \int_M \varphi(x) \overline{\tilde{D}^{\lambda}_{qv}(x)}\, d\mu(x). \tag{15}$$

The functions $\psi_{\lambda}(q, v)$ constitute a linear space $\widehat{\Phi}(M)$ dual to $\Phi(M)$. The inverse transformation is given by

$$\varphi(x) = \int_{Q \times V \times J} \psi_{\lambda}(q, v) D^{\lambda}_{qv}(x)\, d\mu(q)\, d\mu(v)\, d\mu(\lambda). \tag{16}$$

Transformations (15) and (16) define a continuous one-to-one map of the spaces $\Phi(M)$ and $\widehat{\Phi}(M)$. By analogy with harmonic analysis on Lie groups, we assume that the differential operators X_A and L_μ acting in $\Phi(M)$ are also differential in the dual space $\widehat{\Phi}(M)$:

$$X_A(x) \Longleftrightarrow -\overline{l_A^\dagger(q,\lambda)}, \qquad L_\mu(x) \Longleftrightarrow \overline{a_\mu^\dagger(q,\lambda)}.$$

We note that the Casimir operators $K_t^{(s_M)}(iX) = D_t(Z(L))$ in the space $\widehat{\Phi}(M)$ are operators of multiplication by $\kappa_t^{(s_M)}(\lambda)$.

4. Noncommutative reduction of the Klein–Gordon equation

We introduce an invariant metric on the homogeneous space M. Let $\mathbf{G}$ be a nondegenerate quadratic form on a subspace $\mathfrak{m} \subset \mathfrak{g}$ satisfying the Ad_H-*invariance* condition:

$$\mathbf{G}(\overline{[X,Y]},\overline{Z}) + \mathbf{G}(\overline{Y},\overline{[X,Z]}) = 0, \quad X \in \mathfrak{h}, \quad Y,Z \in \mathfrak{g}, \tag{17}$$

where the overline denotes the projection of an element of $\mathfrak{g}$ onto the subspace $\mathfrak{m}$. The quadratic form $\mathbf{G}$ defines an Ad_H-invariant scalar product on the tangent space $T_{x_0}M \simeq \mathfrak{m}$. The Ad_H-invariance requirement in (17) allows using the group G action to make this scalar product well defined on the entire homogeneous space M:

$$g_M(\tau,\sigma)(x) = \mathbf{G}((R_{g^{-1}})_*\tau,(R_{g^{-1}})_*\sigma), \quad \tau,\sigma \in T_xM, \quad x=\pi(g), \quad g \in G. \tag{18}$$

Scalar product (18) defines an *invariant metric* g_M on the homogeneous space M [26].

We give expressions for the metric tensor of the invariant metric g_M in local coordinates on M:

$$(g_M)_{ij}(x) = G_{ab}\sigma_i^a(x)\sigma_j^b(x), \qquad G_{ab} \equiv \mathbf{G}(e_a,e_b).$$

Here, $\{e_a\}$ is a fixed basis in the space $\mathfrak{m}$, $\sigma^b(x)$ are basis right-invariant 1-forms, $\sigma^b(y) \equiv (R_g)^*e^b$, and $\{e^b\}$ is a basis in $\mathfrak{m}^*$, $\langle e_a, e^b\rangle = \delta_a^b$. For contravariant components of the metric tensor, we then have

$$(g_M)^{ij} = G^{ab}\eta_A^i(x)\eta_B^j(x), \qquad G^{ab} = (G_{ab})^{-1}.$$

Summation over the repeated upper and lower indices is understood.

We assume that an invariant metric g_M with Lorentzian signature is defined on the homogeneous space M. We consider the model of a *static* space–time, where a global timelike vector Killing $X_0 = \partial_{x_0}$ orthogonal to constant-time hypersurfaces Σ: $x_0 = \mathrm{const}$ exists on M. Then $x = (\overline{x},t)$, where $\overline{x} = (x^1,\dots,x^{\dim M-1})$ are local coordinates on the hypersurface Σ and $t = x_0$ is a variable playing the role of time. We assume that the metric tensor depends only on local coordinates on the hypersurface Σ: $g_M = g_M(\overline{x})$.

The Klein–Gordon equation is the Euler–Lagrange equation for the action $S = \int \mathcal{L}^0\,d\mu(x)$ of a scalar field $\varphi(x)$ on the homogeneous space M with the Lagrangian

$$\mathcal{L}^0\{\overline{\varphi},\varphi\} = \frac{\sqrt{-\mathrm{G}}}{2}\big(G^{ab}\eta_a\overline{\varphi(x)}\eta_b\varphi(x) - [m^2+\zeta R]|\varphi(x)|^2\big), \tag{19}$$

where R is the scalar curvature of M, $\zeta = (\dim M - 2)/4(\dim M - 1)$ is the conformal factor ensuring the coupling of the gravitational field to curvature, and $\mathrm{G} = \det(G_{ab})$. The Klein–Gordon equation on M then has the form

$$(\Delta_M + m^2 + \zeta R)\varphi(x) = 0, \tag{20}$$

where Δ_M is the Laplace operator of the invariant metric on M:

$$\Delta_M = G^{ab}(\eta_a\eta_b - 2c_a\eta_b), \qquad c_a \equiv \frac{1}{2}\,\mathrm{Sp}(\mathrm{ad}_a\,|_{\mathfrak{m}}). \tag{21}$$

Our aim is to construct a basis of solutions $\varphi_\sigma(x)$ of Eq. (20) labeled by a collective index σ and satisfying the scalar field normalization condition:

$$-\,i\int \overline{\varphi_\sigma(x)}\,\overleftrightarrow{X_0}\varphi_{\sigma'}(x)\sqrt{-g_M(\overline{x})}\,d\overline{x} = \delta(\sigma,\sigma'), \qquad g_M(\overline{x}) = \det((g_M)_{ij}(\overline{x})). \tag{22}$$

Positive- and negative-frequency solutions $\varphi_\sigma^\pm(x)$ are defined as eigenfunctions of a timelike Killing vector X_0:

$$X_0(x)\varphi_\sigma^{(\pm)}(x) = \mp i\omega\varphi_\sigma^{(\pm)}(x), \quad \omega\in\sigma. \tag{23}$$

We note that Eq. (20) can be regarded as a wave equation on the Lie group G of transformations in the class of functions $\mathcal{F}_G$. We therefore reduce Eq. (20) similarly to how quantum equations on Lie groups are reduced. It follows from (23) that

$$D^\lambda_{qv}(\overline{x},t) = e^{-i\omega t}D^{\lambda'}_{qv}(\overline{x},0), \qquad l_0 = i\omega, \quad \hat{a}_0 = \omega,$$

where the parameter ω enters the set of parameters $\lambda = (\omega,\lambda')$ labeling the orbits.

We introduce operators $\eta'_X \equiv \eta_X - c_X$ that are a *trivial continuation* of the vector fields η_X: $[\eta'_X,\eta'_Y] = \eta'_{[X,Y]}$, $X,Y\in\mathfrak{g}$. The vector fields η'_X are Hermitian with respect to the Riemannian measure, and operator (21) can be represented as a symmetric form of η'_X:

$$\Delta_M = H'(\eta') = G^{ab}\eta'_a\eta'_b - G^{ab}c_ac_b.$$

Because the generators X of the Lie group of transformations are Killing vectors of the invariant metric $g_M(x)$, the Laplace operator Δ_M on the homogeneous space M commutes with X and can therefore be represented as a differential operator expressed polynomially in terms of the invariant operators $L_\mu(-i\eta')\equiv L_\mu(-i\eta+ic)$: $\Delta_M = H(L(x))$. In contrast to $L_\mu(-i\eta)$, the operators $L_\mu(-i\eta')$ are Hermitian with respect to the Riemannian measure.

Equation (12) in our case has the form

$$[L_\mu(-il(q',\lambda)+ic) - \hat{a}_\mu(v,\lambda)]c_\lambda(q',v) = 0.$$

We choose the measure $d\mu(v)$ with respect to which the operators $\hat{a}_\mu(v,\lambda)$ are Hermitian. Then $\tilde{D}^\lambda_{qv}(x) = \Delta^{-1}(q)D^\lambda_{qv}(x)$.

Using transformations (16), we reduce Eq. (20) to an equation on a symplectic leaf of the $\mathcal{F}$-algebra of invariant functions. The variables q enter the reduced equation as parameters, and the set

$$\varphi_\sigma(\overline{x},t) = \frac{1}{\sqrt{2\omega\Delta(q)\sqrt{-G}}}e^{-i\omega t}\int\psi_\omega(v,\lambda')D^{\lambda'}_{qv}(\overline{x},0)\,d\mu(v), \qquad \sigma=(q,\lambda), \tag{24}$$

therefore constitutes a basis of solutions of Eq. (20). The functions $\psi_\omega(v,\lambda')$ can be found from the reduced equation with $d(M)$ independent variables on a symplectic leaf:

$$\big(H(\overline{\hat{a}(v,\lambda)}) + m^2 + \zeta R\big)\psi_\omega(v,\lambda') = 0. \tag{25}$$

Equation (25) is an ordinary differential equation whenever $d(M) < 2$.

Because the operator of Eq. (25) is Hermitian, the eigenfunctions $\psi_\omega(v, \lambda)$ satisfy the orthogonality condition

$$\int_Q \overline{\psi_\omega(q', \lambda')}\psi_{\omega'}(q', \lambda')\, d\mu(q') = \delta(\omega, \omega'). \tag{26}$$

This relation implies normalization conditions (22) for the complete set of solutions (24). We note that Eq. (26) is a normalization condition for the functions $\psi_\omega(v, \lambda')$ and defines the measure following from ω.

We consider the generalized zeta function of the operator $\widehat{F} = \Delta_M + \zeta R + m^2$ of the Klein–Gordon equation:

$$\zeta(s) = \int \theta_\sigma^{-s}\, d\mu(\sigma),$$

where θ_σ are the eigenvalues of $\widehat{F}$ labeled by a collective index σ. The generalized zeta function $\zeta(s)$ admits an analytic continuation to the complex plane that is regular at the point $s = 0$, and can be defined as an integral over M of the *local* zeta function:

$$\zeta(\overline{x}, t; s) = \int \theta_\sigma^{-s}\overline{\phi_\sigma(\overline{x}, t)}\phi_\sigma(\overline{x}, t)\, d\mu(\sigma), \qquad \zeta(s) = \sqrt{-G}\int \zeta(\overline{x}, t; s)\, d\mu(x),$$

where $\phi_\sigma(\overline{x}, t)$ is a complete orthogonal set of eigenfunctions of $\widehat{F}$ such that the normalization condition

$$\sqrt{-G}\int \overline{\phi_\sigma(\overline{x}, t)}\phi_{\sigma'}(\overline{x}, t)\, d\mu(x) = \delta(\sigma, \sigma')$$

is satisfied.

As in the case of the Klein–Gordon equation reduction, we seek the eigenfunctions $\phi_\sigma(\overline{x}, t)$ in the form

$$\phi_\sigma(\overline{x}, t) = (\Delta(q)\sqrt{-G})^{-1/2}\int \tilde{\psi}_\nu(v, \lambda) D^\lambda_{qv}(\overline{x}, t)\, d\mu(v), \qquad \sigma = (q, \lambda, \nu). \tag{27}$$

The eigenvalue problem $\widehat{F}\phi_\sigma(\overline{x}, t) = \theta_\sigma\phi_\sigma(\overline{x}, t)$ then reduces to the equation

$$H(\overline{\hat{a}(v, \lambda)})\tilde{\psi}_\nu(v, \lambda) = (\theta_{\nu\lambda} - \zeta R - m^2)\tilde{\psi}_\nu(v, \lambda) \tag{28}$$

on the symplectic leaf. The spectrum of the operator $H(\overline{\hat{a}(v, \lambda)})$ is determined by an additional quantum number ν and is independent of q. We impose the condition

$$\int \overline{\tilde{\psi}_\nu(v, \lambda)}\tilde{\psi}_{\nu'}(v, \lambda)\, d\mu(v) = \delta(\nu, \nu')$$

on $\tilde{\psi}_\nu(v, \lambda)$. For the local zeta function, we obtain the expression

$$\begin{aligned} \zeta_{\rm loc}(s) &= \frac{1}{\sqrt{-G}}\int \theta_{\nu\lambda}^{-s}\left(\int \overline{\tilde{\psi}_\nu(v, \lambda)}\tilde{\psi}_\nu(\tilde{v}, \lambda)\chi^\lambda(v, \tilde{v})\, d\mu(v)\, d\mu(\tilde{v})\right) d\mu(\lambda)\, d\mu(\nu), \\ \chi^\lambda(v, \tilde{v}) &= \int \overline{D^\lambda_{qv}(x)} D^\lambda_{q\tilde{v}}(x)\, d\mu_0(q). \end{aligned} \tag{29}$$

We show that integral (29) is independent of local coordinates. Substituting decomposition (10), we obtain

$$\chi^\lambda(v, \tilde{v}) = \int \overline{c_\lambda(q', v)} c_\lambda(\tilde{q}', \tilde{v}) I^\lambda_{q'\tilde{q}'}\, d\mu(q')\, d\mu(\tilde{q}'), \qquad I^\lambda_{q'\tilde{q}'} = \int \overline{D^\lambda_{q\overline{q'}}(g^{-1})} D^\lambda_{q\overline{\tilde{q}'}}(g^{-1}) d\mu_0(q).$$

It follows from system of equations (2) that $\overline{D^{\lambda}_{q\overline{q'}}(g^{-1})} = (\Delta(q)/\Delta(q'))D^{\lambda}_{q'\overline{q}}(g)$, and the integral $I^{\lambda}_{q'\tilde{q}'}$ is therefore equal to

$$I^{\lambda}_{q'\tilde{q}'} = \frac{1}{\Delta(q')}\int D^{\lambda}_{q'\overline{q}}(g)D^{\lambda}_{q\overline{\tilde{q}'}}(g^{-1})\,d\mu(q) = \frac{1}{\Delta(q')}D^{\lambda}_{q'\overline{\tilde{q}'}}(0) = \delta_0(q',\overline{\tilde{q}'}).$$

We then have an expression for $\chi^{\lambda}(v,\tilde{v})$ that is free of local coordinates:

$$\chi^{\lambda}(v,\tilde{v}) = \int \overline{c_{\lambda}(q',v)}c_{\lambda}(q',\tilde{v})\,d\mu_0(q'). \tag{30}$$

If the homogeneous space is a Lie group, then $c(q',v) = \delta(q',v)$, and expression (30) gives a Dirac delta-function [18]. We note that the generalized zeta function can be expressed in terms of the set of solutions of the reduced equation and is defined in terms of algebraic characteristics of the homogeneous space.

5. Vacuum expectation value of the scalar field EMT on a homogeneous space

The metric EMT of the scalar field on a homogeneous space M is defined by varying the scalar field action with Lagrangian (19) over the metric [1]

$$\delta S = -\frac{1}{2}\int \sqrt{-g_M(x)}\langle T,\delta g_M(x)\rangle\,d\mu(x)\,dt,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the contraction of the EMT T with the variation of the metric tensor $g_M(x)$. The expression for the EMT scalar field in local coordinates is well known and in our notation has the form [1]

$$T_{ij}\{\varphi,\overline{\varphi}\} = (1-2\zeta)\overline{\varphi_{(,i}}\varphi_{,j)} + \left(2\zeta-\frac{1}{2}\right)g_{ij}g^{kl}\overline{\varphi_{,k}}\varphi_{,l} - \zeta[(\nabla_i\nabla_j\overline{\varphi})\varphi + \overline{\varphi}(\nabla_i\nabla_j\varphi)] -$$
$$-\left[\zeta R_{ij} + \left(2\zeta-\frac{1}{2}\right)g_{ij}(m^2+\zeta R)\right]|\varphi|^2, \tag{31}$$

where $(a,b) \equiv (ab+ba)/2$, Δ_i is the covariant derivative of the Levi-Civita connection on the homogeneous space M with an invariant metric, and R_{ij} is the Ricci tensor.

We multiply expression (31) by $\eta^i_X(x)\eta^j_Y(x)$ and use the properties of an invariant metric on a homogeneous space to obtain

$$T'(X,Y)\{\overline{\varphi},\varphi\} = T(\eta_X,\eta_Y)\{\overline{\varphi},\varphi\} = T_{ij}\{\overline{\varphi},\varphi\}\eta^i_X(x)\eta^j_Y(x) =$$
$$= \left(2\zeta-\frac{1}{2}\right)\overline{\eta_{(X}\varphi}\eta_{Y)}\varphi + \left(2\zeta-\frac{1}{2}\right)\mathbf{G}(X,Y)G^{AB}\overline{\eta_A\varphi}\eta_B\varphi -$$
$$-\left[\zeta\mathbf{Ric}(X,Y) + \left(2\zeta-\frac{1}{2}\right)\mathbf{G}(X,Y)(m^2+\zeta R)\right]\overline{\varphi}\varphi -$$
$$-\zeta\eta^i_X\eta^j_Y[(\nabla_i\nabla_j\overline{\varphi})\varphi + \overline{\varphi}(\nabla_i\nabla_j\varphi)], \quad X,Y\in\mathfrak{m}, \tag{32}$$

where we introduce the notation $\mathbf{Ric}(X,Y) = R_{ij}\eta^i_X(x)\eta^j_Y(x)$. Using the expression for Christoffel symbols of the Levi-Civita connection on the homogeneous space M in local coordinates, by a simple but cumbersome calculation, we can obtain

$$\eta^i_X\eta^j_Y\nabla_i\nabla_j\varphi = \eta_X\eta_Y - \eta_{\mathbf{\Gamma}(X,Y)}, \tag{33}$$

where the bilinear map $\mathbf{\Gamma}\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ defines the Levi-Civita connection and is uniquely defined by the system of equations

$$2\mathbf{G}(\overline{\mathbf{\Gamma}(X,Y)},Z)=\mathbf{G}(\overline{[X,Y]},Z)+\mathbf{G}(\overline{[Y,Z]},X)-\mathbf{G}(\overline{[Z,X]},Y),\quad X,Y,Z\in\mathfrak{m}.$$

We note that the quantity $\mathbf{Ric}(X,Y)$ is independent of local coordinates and can be defined as the trace of the linear operator $Z\to\mathbf{R}(Z,Y)X,\ X,Y,Z\in\mathfrak{m}$, where

$$\mathbf{R}(X,Y)Z=\mathbf{\Gamma}(Y,\mathbf{\Gamma}(X,Z))-\mathbf{\Gamma}(X,\mathbf{\Gamma}(Y,Z))+\mathbf{\Gamma}(\overline{[X,Y]},Z),\quad X,Y,Z\in\mathfrak{m}.$$

As a result, we obtain

$$T'(X,Y)\{\overline{\varphi},\varphi\}=(1-2\zeta)\overline{\eta_{(X}\varphi}\eta_{Y)}\varphi+(4\zeta-1)\mathbf{G}(X,Y)\frac{1}{\sqrt{-G}}\mathcal{L}^0\{\overline{\varphi},\varphi\}-\zeta\mathbf{Ric}(X,Y)|\varphi|^2-$$
$$-\zeta\big[\overline{(\eta_{(X}\eta_{Y)}-\eta_{Z(X,Y)})\varphi}\varphi+\overline{\varphi}(\eta_{(X}\eta_{Y)}-\eta_{Z(X,Y)})\varphi\big],\quad X,Y\in\mathfrak{m},\tag{34}$$

where $Z(X,Y)=(\mathbf{\Gamma}(X,Y)+\mathbf{\Gamma}(Y,X))/2$. We call components (34) the *quasitetrad components* of scalar field EMT (31). We note that expression (34) is independent of local coordinates on the homogeneous space; by the relation

$$\sigma^a_b(x)\eta^c_a(x)=\delta^a_b,\qquad \sigma^a_c(x)\eta^c_b(x)=\delta^a_b,\quad a,b,c=1,\dots,\dim M,$$

we can always pass to the usual EMT components:

$$T_{ij}=T'(e_a,e_b)\sigma^a_i(x)\sigma^b_j(x).\tag{35}$$

Relations (31)–(35) are an analogue of the tetrad formalism for tensor fields on a homogeneous space and can be generalized in an obvious way to the case of arbitrary-rank tensor fields.

We proceed with quantizing the scalar field with Lagrangian (19) on the homogeneous space M. We decompose the field operator $\hat{\phi}(\overline{x},t)$ with respect to the basis of solutions of wave equation (20):

$$\hat{\phi}(\overline{x},t)=\int d\mu(\sigma)\,[\varphi(\overline{x},t)\hat{a}_\sigma+\overline{\varphi(\overline{x},t)}\hat{a}^\dagger_\sigma],$$

where $d\mu(\sigma)$ is a measure for all quantum numbers and $\hat{a}^\dagger_\sigma$ and $\hat{a}_\sigma$ are the respective creation and annihilation operators. Covariant quantization is realized by imposing the commutation relations $[\hat{a}_\sigma,\hat{a}^\dagger_{\sigma'}]=\delta(\sigma,\sigma')$. The vacuum state corresponding to this quantization procedure is defined by the conditions $\hat{a}_\sigma|0\rangle=0$. The vacuum expectation values of the EMT are then defined by an integral over all quantum numbers:

$$\langle\widehat{T}'(X,Y)\rangle=\int T'(X,Y)\{\varphi_\sigma,\overline{\varphi}_\sigma\}\,d\mu(\sigma),\quad X,Y\in\mathfrak{m}.\tag{36}$$

To calculate integral (36), we find an expression for the vacuum expectation values of the form

$$\Phi(X,Y)=\langle\eta_X\hat{\varphi},\eta_Y\hat{\varphi}\rangle=\int\overline{(\eta_X\varphi_\sigma)}\eta_Y\varphi_\sigma\,d\mu(q)\,d\mu(\lambda),\quad X,Y\in\mathfrak{m}.$$

Substituting (24) and using the equality

$$\eta_X D^\lambda_{qv}(x,t)=\int(l_X(q',\lambda)c_\lambda(q',v))D^\lambda_{\overline{qq'}}(g^{-1})\,d\mu(q'),$$

we have a chain of equalities for the expectation values $\Phi(X,Y)$:

$$\Phi(X,Y) = \frac{1}{2\sqrt{-G}}\int \frac{1}{\omega}\overline{(\psi_\omega(v,\lambda')l_X(q',\lambda)c_\lambda(q',v))}(\psi_\omega(\tilde v,\lambda')l_X(\tilde q',\lambda)c_\lambda(\tilde q',\tilde v))\times$$
$$\times I^\lambda_{q'\tilde q'}\,d\mu(q')\,d\mu(\tilde q')\,d\mu(v)\,d\mu(\tilde v)\,d\mu(\lambda) =$$
$$= \frac{1}{2\sqrt{-G}}\int \frac{1}{\omega}\overline{(\psi_\omega(v,\lambda')l_X(q',\lambda)c_\lambda(q',v))}(\psi_\omega(\tilde v,\lambda')l_Y(q',\lambda)c_\lambda(q',\tilde v))\times$$
$$\times d\mu_0(q')\,d\mu(v)\,d\mu(\tilde v)\,d\mu(\lambda). \tag{37}$$

Similarly, for vacuum expectation values of the squared field operator $\langle\hat\varphi^2\rangle \equiv \langle\hat\varphi,\hat\varphi\rangle$, we obtain

$$\langle\hat\varphi^2\rangle = \frac{1}{2\sqrt{-G}}\int\frac{1}{\omega}\overline{\psi_\omega(v,\lambda')}\psi_\omega(\tilde v,\lambda')\chi^\lambda(v,\tilde v)\,d\mu(v)\,d\mu(\tilde v)\,d\mu(\lambda). \tag{38}$$

We note that in (37), the λ-representation operators are skew-Hermitian with respect to $d\mu_0(q')$, whence it follows that $\langle\eta_X\eta_Y\hat\varphi,\hat\varphi\rangle = -\Phi(X,Y)$ and $\langle\eta_X\hat\varphi,\hat\varphi\rangle = -\langle\hat\varphi,\eta_X\hat\varphi\rangle$. We can then express the vacuum expectation values of the EMT in terms of $\Phi(X,Y)$ and $\langle\hat\varphi^2\rangle$:

$$\langle\widehat{T}'(X,Y)\rangle = \frac{1}{2}(\Phi(X,Y)+\Phi(Y,X)) - \zeta\mathbf{Ric}(X,Y)\langle\hat\varphi^2\rangle +$$
$$+\left(2\zeta-\frac{1}{2}\right)\mathbf{G}(X,Y)[G^{ab}\Phi(e_a,e_b)-(m^2+\zeta R)\langle\hat\varphi^2\rangle].$$

Using (37) and (38), after some simplifications, we finally obtain

$$\langle\widehat{T}'(X,Y)\rangle = -\frac{1}{2\sqrt{-G}}\int\frac{1}{\omega}\overline{(\psi_\omega(\tilde v,\lambda)c_\lambda(q',\tilde v))}\psi_\omega(v,\lambda)\widehat{T}^l(X,Y)c_\lambda(q',v)\times$$
$$\times d\mu_0(q')\,d\mu(v)\,d\mu(\tilde v)\,d\mu(\lambda),$$
$$\widehat{T}^l(X,Y) = \frac{1}{2}\{l_X(q',\lambda),l_Y(q',\lambda)\}_+ + \zeta\mathbf{Ric}(X,Y)\times \tag{39}$$
$$+\left(2\zeta-\frac{1}{2}\right)\mathbf{G}(X,Y)G^{ab}\,\mathrm{Sp}(\mathrm{ad}_a|_{\mathfrak{m}})l_b(q',\lambda),$$

where $\{\hat a,\hat b\}_+ \equiv \hat a\hat b+\hat b\hat a$ denotes the anticommutator of operators.

A separate problem is given by performing the renormalization procedure and obtaining finite values of vacuum expectation values (39) characterizing the effect of vacuum polarization by the gravitational field. Because the vacuum expectation values of the EMT are defined by taking functional derivatives of the effective action over the metric, this problem is equivalent to renormalizing the effective action W of the quantized field.

In the ζ-regularization method, the effective action is expressed in terms of a generalized zeta function of the Klein–Gordon equation operator $\widehat{F} = \Delta_M + \zeta R + m^2$:

$$W(s) = -\frac{i}{2}(\zeta'(s)+\zeta(s)\log(-2\pi i\mu^2)), \qquad W_{\mathrm{ren}} = W(s)|_{s=0}, \tag{40}$$

where μ is a normalization constant independent of the metric and having the dimension of mass. The renormalized vacuum expectation value can be obtained by evaluating the functional derivatives of effective action (40) over the metric and then finding an analytic continuation at $s=0$:

$$\langle\widehat{T}_{ij}\rangle_{\mathrm{ren}} = \frac{2}{\sqrt{-g_M(x)}}\frac{\delta W(s)}{\delta g_{ij}}|_{s=0} = -\frac{i}{2}\left(\frac{d}{ds}|_{s=0}Z_{ij}(x,t;s)+Z_{ij}(x,t;0)\log(-2\pi i\mu^2)\right),$$

where the function $Z_{ij}(x,t;s)$ is an analytic continuation with respect to the variable s of the variation of the zeta function $\zeta(s)$ over the metric,

$$Z_{ij}(x,t;s) = \frac{2}{\sqrt{-g_M(x)}}\frac{\delta\zeta(x,t;s)}{\delta g^{ij}(x)} = 2s\zeta_{ij}(x,t;s+1) + s\zeta_{\rm loc}(s)g_{ij}(x),$$

$$\zeta_{ij}(x,t;s) \equiv \int \lambda_\sigma^{-s} T_{ij}\{\overline{\phi_\sigma}, \phi_\sigma\}\, d\mu(\sigma).$$

In [23], this function was called the *EMT zeta-function.* The quantities $\zeta_{\rm loc}(0)$ and $\zeta'_{\rm loc}(0)$ must be finite, and therefore

$$\lim_{s\to 0} s\zeta'_{\rm loc}(s) = 0, \qquad \lim_{s\to 0} s\zeta_{\rm loc}(s) = 0.$$

For the renormalized EMT, we can then obtain the expression [23]

$$\langle \widehat{T}_{ij}\rangle_{\rm ren} = i\biggl(\zeta_{ij}(x,t;s+1) + \frac{1}{2}g_{ij}(x)\zeta_{\rm loc}(s) + \\ + s[\zeta'_{ij}(x,t;s+1) + \zeta_{ij}(x,t;s+1)\log(-2\pi i\mu^2)]\biggr)\bigg|_{s=0}.$$

This approach to the problem of obtaining finite values of vacuum expectation values was proposed in [23] and was used in our previous work [18] in the particular case where the homogeneous space is a Lie group with an invariant metric.

Using expression (34), we obtain the quasitetrad components of the function $\zeta_{ij}(x,t;s)$ in the form

$$\zeta(X,Y)(s) = \overline{\zeta(X,Y)}(s) - \zeta\mathbf{Ric}(X,Y)\zeta_{\rm loc}(s) - \left(2\zeta - \frac{1}{2}\right)\mathbf{G}(X,Y)(\zeta_{\rm loc}(s-1) + \zeta_{\rm un}(s)),$$

$$\overline{\zeta(X,Y)}(s) = \int \theta_{\nu\lambda}^{-s}\eta_{\{X}\overline{\phi_\sigma(x,t)}\eta_{Y\}}\phi_\sigma(x,t)\, d\mu(\sigma),$$

$$\zeta_{\rm un}(s) = 2G^{ab}c_a\int \theta_{\nu\lambda}^{-s}\overline{\phi_\sigma(x,t)}\eta_b\phi_\sigma(x,t)\, d\mu(\sigma).$$

Using inverse Fourier transformation (27), we can easily see that $\zeta(X,Y)(s)$ is independent of local coordinates:

$$\overline{\zeta(X,Y)}(s) = -\frac{1}{2\sqrt{-G}}\int \theta_{\nu\lambda}^{-s}\overline{(\tilde{\psi}_\nu(v,\lambda)c_\lambda(q',v))}\tilde{\psi}_\nu(\tilde{v},\lambda)\{l_X(q',\lambda), l_Y(q',\lambda)\}_+ c_\lambda(q',\tilde{v}) \times \\ \times d\mu_0(q')\, d\mu(v)\, d\mu(\tilde{v})\, d\mu(\lambda)\, d\mu(\nu),$$

$$\zeta_{un}(s) = \frac{1}{\sqrt{-G}}G^{ab}c_a\int \theta_{\nu\lambda}^{-s}\overline{(\tilde{\psi}_\nu(v,\lambda)c_\lambda(q',v))}\tilde{\psi}_\nu(\tilde{v},\lambda)l_b(q',\lambda)c_\lambda(q',\tilde{v}) \times \\ \times d\mu_0(q')\, d\mu(v)\, d\mu(\tilde{v})\, d\mu(\lambda)\, d\mu(\nu).$$

As a result, the quasitetrad components of the renormalized vacuum expectation values of the EMT become

$$\langle \widehat{T}'(X,Y)\rangle_{\rm ren} = i\biggl(\zeta(X,Y)(s+1) + \frac{1}{2}\mathbf{G}(X,Y)\zeta_{\rm loc}(s) + \\ + s[\zeta'(X,Y)(s+1) + \zeta(X,Y)(s+1)\log(-2\pi i\mu^2)]\biggr)\bigg|_{s=0}. \qquad (41)$$

Hence, calculating renormalized vacuum expectation values (41) reduces to solving reduced equation (28) and finding an analytic continuation of the function $\zeta(X,Y)(s)$.

It can be shown that if the massless scalar field in a four-dimensional homogeneous space M has no conformal anomaly of the EMT, in other words, $(\mathrm{Sp}\langle\widehat{T}'(X,Y)\rangle_{\mathrm{ren}}=0)$, then

$$\lim_{s\to 0} s\zeta(X,Y)(s+1)=0, \qquad \lim_{s\to 0} s\zeta'(X,Y)(s+1)=0,$$

and the expression for the renormalized EMT is independent of the parameter μ.

For example, we consider a four-dimensional homogeneous space with a five-dimensional Lie group of transformations whose Lie algebra $\mathfrak{g}$ with the generators $\{e_0,\dots,e_5\}$ is defined by the commutation relations

$$[e_1,e_3]=e_1, \qquad [e_1,e_4]=e_1, \qquad [e_2,e_3]=e_2, \qquad [e_2,e_4]=-e_2, \qquad [e_0,e_A]=0$$

and the isotropy subalgebra $\mathfrak{h}=\{e_4\}$. The index and degeneration degree of the homogeneous space are equal to zero, and the defect is equal to unity.

The algebra $\mathfrak{g}$ is a solvable Lie algebra of index 1, and the coalgebra $\mathfrak{g}^*$ admits a Casimir function $K_0(f)=f_0$ and is the union of nonintersecting invariant surfaces

$$M_0^{(\pm)}=\{\pm f_1 f_2>0\},$$

$$M_{1a}^{(\pm)}=\{f_1=0,\pm f_2>0\}, \qquad M_{1b}^{(\pm)}=\{f_1=0,\pm f_2<0\}, \qquad M_2=\{f_1=f_2=0\}.$$

Hence, each nondegenerate K-orbit belongs to one of the spaces $M_0^{(\pm)}$ and passes through the parameterized covector $\lambda(\omega)=(\omega,\pm1,\pm1,0,0)$, $\omega\in\mathbb{R}$. The algebra $\mathfrak{g}$ admits a real polarization $\mathfrak{p}=\{e_0,e_1,e_2\}$, and the λ-representation corresponding to a nondegenerate orbit $\mathcal{O}^0_{(\pm)}$ has the form

$$l_0=i\omega, \qquad l_1=\pm ie^{-q_1-q_2}, \qquad l_2=\pm ie^{q_1-q_2}, \qquad l_3=\partial_{q_2}, \qquad l_4=\partial_{q_1}, \tag{42}$$

where $(q_1,q_2)\in Q$, $q_1\in[0,2\pi)$, and $q_2\in\mathbb{R}$. Operators (42) are skew-Hermitian with respect to the measure $d\mu_0(q)=dq_1\,dq_2$.

The homogeneous space admits a three-dimensional Poisson algebra of invariant functions $a_0=f_0$, $a_1=f_3$, and $a_2=f_1\circ f_2$ with the commutation relation $\{a_1,a_2\}=-2a_2$ and the Casimir function $Z(f)=f_0$. The λ-representation operators of the $\mathcal{F}$-algebra of invariant functions

$$\hat{a}_0=\omega, \qquad \hat{a}_1=i(\partial_v-1), \qquad \hat{a}_2=e^{2v}$$

are Hermitian with respect to the measure $d\mu(v)=e^{-2v}dv$, $v\in\mathbb{R}$. Solving system (11), (12), we obtain $c(q',v)=\delta(q'_2,v)$.

The homogeneous space admits a G-invariant metric of a static space–time

$$G^{ab}=\begin{pmatrix}1&0&0&0\\0&0&-c_2&0\\0&-c_2&0&0\\0&0&0&c_1\end{pmatrix}, \qquad R=6c_1, \qquad R_{ab}=-\frac{2c_1}{c_2}\delta_{a2}\delta_{b3}, \quad c_1\geq 0, \quad c_2\geq 0.$$

The reduced Klein–Gordon equation is a second-order ordinary differential equation,

$$c_1(\psi''(v)-2\psi'(v))+(2c_2e^{2v}-\omega^2+m^2+c_1)\psi(v)=0. \tag{43}$$

The functions

$$\psi_n(v) = e^v J_{2n}\left(\sqrt{\frac{2c_2}{c_1}}e^v\right), \tag{44}$$

where $J_\nu(x)$ is a Bessel function of the first kind, constitute a complete and orthogonal set of solutions of reduced equation (43) with respect to the measure

$$\int (\,\cdot\,)\, d\mu(n) = 4\sum_{n=1}^{\infty} n(\,\cdot\,)$$

if the spectral parameter satisfies the condition $\omega_n = 4c_1 n^2 + m^2$. Functions (44) constitute a set of eigenfunctions of operator (28) with the eigenvalues $\theta_{\omega n} = 4c_1 n^2 + m^2 - \omega^2$.

The generalized zeta-function of the Klein–Gordon equation is given by

$$\zeta(s) = -\frac{i\sqrt{\pi}}{\sqrt{c_1}}\frac{\Gamma(s-1/2)}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{(4c_1 n^2 + m^2)^{1/2-s}}{4n^2-1}. \tag{45}$$

A simple calculation shows that

$$\zeta_{\mathrm{un}}(s) = 0, \qquad \overline{\zeta_{00}(s)} = -\frac{1}{2}\frac{\zeta(s-1)}{s-1}.$$

We then use (41) to find the energy of the scalar field as

$$\epsilon = \langle \widehat{T}_{00}\rangle_0 = -\frac{i}{2}\lim_{s\to 0}\frac{\zeta(s)}{s}.$$

Using the Abel–Plana formula for regularizing zeta function (45), we obtain the renormalized value of the scalar field energy density

$$\epsilon = \frac{\pi c_2^2}{\sqrt{c_1}}\left[\frac{m}{2} - 2\int_{m/(2\sqrt{c_1})}^{\infty}\frac{\sqrt{4c_1 t^2 - m^2}\, dt}{(e^{2\pi t}-1)(4t^2-1)}\right]. \tag{46}$$

Expression (46) describes the effect of the scalar field vacuum polarization on a homogeneous space with a Lie algebra of transformations $\mathfrak{g}$ and an isotropy subalgebra $\mathfrak{h}$.

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