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Abstract

In this paper, we propose and study a first risk model in which the insurer may invest into a prevention plan which decreases claim intensity. We determine the optimal prevention investment for different risk indicators. In particular, we show that the prevention amount minimizing the ruin probability maximizes the adjustment coefficient in the classical ruin model with prevention, as well as the expected dividends until ruin in the model with dividends. We also show that the optimal prevention strategy is different if one aims at maximizing the average surplus at a fixed time horizon. A sensitivity analysis is carried out. We also prove that our results can be extended to the case where prevention starts to work only after a minimum prevention level threshold.

Keywords: Ruin theory; Prevention; Optimal prevention strategy; Insurance.

1 Introduction

An insurance company has several levers to manage risk and to optimize its risk/return profile, including reinsurance, investment and dividends. Optimal reinsurance strategies have been studied by Hesselager [Hes90], Højgaard et al. [HT98] among others. Optimal investment strategies have been studied independently from optimal reinsurance, or simultaneously like in Schmidli [Sch02].

To better reflect the strategy of the insurer, a dividend barrier has been introduced by De Finetti [DF57]. Optimal dividend strategy has become a classical topic in ruin theory, see Avanzi [Ava09] for a survey. Another way to manage risk is to update the pricing thanks to credibility techniques (see for example Afonso et al. [AdRW10]). In this paper, we propose a new risk model that takes into account the possibility for the insurer to carry out prevention plans, and we determine optimal prevention strategies in ruin theory.

Prevention has become a key, trendy topic in insurance. In 1997, Discovery was the first insurance company to implement an important prevention plan. Since then, a growing number of insurers have launched their own programs (e.g. John Hancock or OscarHealth in the USA, Generali Vitality or AXA prevention in Europe). However, according to the OECD, prevention spendings only represent 3% of the overall health expenditures in the world. Moreover, this financial effort is mainly made by public authorities (e.g. 94% of preventive spending is due to publics organism in the USA, 90% in Germany, 77% in the United Kingdom and 64% in France). These low figures suggest that nowadays, prevention is essentially a marketing tool for insurers.

In 1972, Ehrlich and Becker [EB72] proposed to separate two types of prevention: self-protection and self-insurance. If one tried to adapt this work to our insurance company, self-insurance would correspond to reinsurance. Self-protection would consist in reducing claims frequency thanks to (costly) prevention plans. Optimal self-protection strategies have not yet been studied in risk theory and determining them is an interesting problem that differs from optimal reinsurance, as we shall see in the sequel. In the sequel, the word'"prevention" refers to self-protection only.

In the present paper, we use the notation of Bowers et al. [BGH$^+$84]. We consider an insurance company with initial surplus $U(0) = u$. This company receives premiums at a rate c per unit of time and invests a fixed amount p in prevention per unit of time. Prevention strategies could be in theory dynamic and depend on the history of the claim process. In practice prevention is a long term strategy, therefore we assume in the present paper that it is defined from the beginning and that it is not a control variable. The aggregate claim amount up to time t is given by a compound Poisson process $S(t) = \sum_{i=1}^{N(t)} X_i$, where N is a Poisson process of arrival intensity $\lambda(p)$ and the $(X_i)_{i\in\mathbb{N}^*}$ are independent and identically distributed (i.i.d.) random variables, independent from N, and with cumulative distribution function F_X, such that $\mathbb{E}(X) = \mu < \infty$.

We assume that $\lambda(.)$ is a decreasing, strictly convex, positive, and $\mathcal{C}^2$ function defined on $[0, c]$. Let us further comment these assumptions.

Choosing $\lambda(.)$ positive means that one cannot prevent all risk. This hypothesis is explained by the fact that if $\lambda(.)$ could be equal to zero, it would allow some arbitrage opportunity.

Having $\lambda(.)$ decreasing means that prevention always reduces the claim arrival intensity. This is a classical economic hypothesis, done by Ehrlich and Becker [EB72] or Dionne and Eeckhoudt [DE85] for example. However, a prevention investment not ambitious enough could produce non significant effects. For example, smoking 29 cigarettes a day instead of 30 cigarettes leads to almost the same risk. Thus, this hypothesis is relaxed later on in this paper, by allowing $\lambda(.)$ to be first constant on some interval and then decreasing.

Assuming $\lambda(.)$ strictly convex means that the more one spends in prevention, the smaller the additional reduction of claims frequency is. This is also a common economic hypothesis. This assumption has been made by Ehrlich and Becker [EB72] and by Courbage [Cou01] among others.

It is also implicitly assumed that $\lambda(.)$ does not change over time. This is quite a strong hypothesis since in reality, there is no reason for prevention to be efficient from the first moment you start implementing it. In this paper, we do not treat this subject. Optimal control of prevention planning strategies in presence of varying prevention efficiency is left for further research.

The surplus process is thus given by

$$U(t,p) = u + (c-p)t - \sum_{i=1}^{N(t)} X_i. \tag{1}$$

We denote by $\varphi(u,p) = \mathbb{P}(\forall t \in \mathbb{R}^+, U(t) \geq 0)$ the non-ruin probability in infinite time ($\psi(u,p) = 1 - \varphi(u,p)$ denoting the ruin probability). In order to lighten formulas, we denote by $\varphi'(u,p) = \frac{\partial \varphi(u,p)}{\partial p}$ when there is no ambiguity. We have $\varphi(u,p) > 0$ if and only if the safety loading condition is verified, i.e.

$$1 - \frac{\lambda(p)\mu}{c-p} > 0. \tag{2}$$

We suppose that Equation (2) holds true. Thus, we only consider $p \in D = [0, p_{lim}(1-\varepsilon)]$, with p_{lim} verifying $1 - \frac{\lambda(p_{lim})\mu}{c-p_{lim}} = 0$, and $\varepsilon \in [0,1[$. The function $\lambda(.)$ being decreasing and continuous on $[0,c]$, the intermediate value theorem guarantees the existence of p_{lim}.

For this model, it is easy to extend several classical results of the classical compound Poisson model. For $u = 0$, the non-ruin probability is given by

$$\varphi(0,p) = 1 - \frac{\lambda(p)\mu}{c-p}, \tag{3}$$

and for all u, the non-ruin probability is given by the Pollaczeck-Khinchin formula

$$\varphi(u,p) = \varphi(0,p) \sum_{n=0}^{\infty} (1-\varphi(0,p))^n F_{e,X}^{*n}(u), \tag{4}$$

where $F_{e,X}$ designates the equilibrium cumulative distribution function given by

$$F_{e,X}(u) = \frac{\int_0^u 1 - F_X(x)dx}{\mu}. \tag{5}$$

As mentioned earlier, the optimal prevention problem is different from the optimal reinsurance one. Consider for example proportional reinsurance in the classical ruin model without prevention. If the insurer cedes a proportion $\alpha \in [0, 1]$ of the claims with a reinsurance loading rate $\theta > 0$, then its classical surplus process

$$U(t) = u + ct - \sum_{i=1}^{N^\gamma(t)} X_i \tag{6}$$

where N^γ is a Poisson process with constant intensity γ becomes

$$\tilde{U}(t, \alpha) = u + (c - \alpha(1 + \theta)\gamma\mu)t - (1 - \alpha) \sum_{i=1}^{N^\gamma(t)} X_i. \tag{7}$$

The ruin probability for the surplus process $\tilde{U}(t, \alpha)$ is the same as the one of the process

$$V(t, \alpha) = \frac{u}{1 - \alpha} + \frac{(c - \alpha(1 + \theta)\gamma\mu)}{1 - \alpha}t - \sum_{i=1}^{N^\gamma(t)} X_i. \tag{8}$$

If one now considers an insurer using prevention but no reinsurance, then, thanks to a time change, the ruin probability is similar to the one of the process

$$W(t, p) = u + (c - p)\frac{\gamma}{\lambda(p)}t - \sum_{i=1}^{N^\gamma(t)} X_i. \tag{9}$$

One can see that the initial surplus level remains homogeneous to the claim amounts X_i's in the prevention case (9), while it is divided by the retention rate $1 - \alpha$ in the proportional reinsurance case (8). Another way to see easily that the two problems are different is provided in Section 3.3.

The main contributions of this paper are to propose a first risk model with prevention and to determine the optimal prevention investment for different risk indicators. In particular, we show that the prevention amount minimizing the ruin probability maximizes the adjustment coefficient in the classical ruin model with prevention, as well as the expected dividends until ruin in the model with dividends. We also show that the optimal prevention strategy is different if one aims at maximizing the average surplus at a fixed time horizon. We carry out some sensitivity analysis of the optimal prevention investment level with respect to the premium income rate on the one hand, and with respect to the average claim size in the exponential case on the other hand. We also prove that our results can be extended to the case where prevention starts to work only after a minimum prevention level threshold.

The paper is organized as follows. In Section 2, two optimal prevention amounts are studied. More specifically, in Section 2.1, we discuss a prevention amount which maximizes the non ruin probability, the adjustment coefficient and the expected dividends until ruin, while in Section 2.2, we study a prevention amount maximizing the average surplus level on a given time horizon. In Section 3, we relax the assumption on λ and tackle the case where it is first constant and then decreasing. Some research perspectives are given in the conclusion.

2 Optimizing prevention strategies

An insurance company interested in prevention can pursue several purposes. In this section are presented two optimal prevention strategies. The first one consists in maximizing the non-ruin probability. Performing a time change in the model as in (9) enables to highlight that maximizing the non-ruin probability amounts to optimizing the premium rate. We show that the prevention amount maximizing the non-ruin probability also maximizes the adjustment coefficient and the expected dividend amount earned before ruin. That is why all these optimal prevention amounts are discussed in the same section (i.e. in Section 2.1). The second one consists in maximizing the average surplus on a given time horizon and is discussed in Section 2.2.

2.1 Non-ruin probability

A first goal for the insurer can be reducing its risk. In this section, we aim to determine the prevention amount that maximizes the non-ruin probability of the insurance company. We call it the “optimal prevention amount" in the following.

Let

$$p^* : \left| \begin{array}{ccc} \mathbb{R}^+ & \longmapsto & \mathbb{R}^+ \\ u & \longmapsto & min(\underset{p\in D}{argmax}(\varphi(u,p))) \end{array} \right.$$

be the function associating to an initial surplus u the prevention amount maximizing the non-ruin probability $\varphi(u,p)$. Hence, $p^*(u)$ represents the optimal prevention amount when the initial surplus is equal to u.

The next proposition shows that the optimal prevention amount does not depend on the value of the initial surplus, such that $p^*(.)$ is actually constant. We denote by p_0^* the optimal prevention amount $p^*(u)$ in the remaining of the paper. Furthermore, the next result provides an equation which enables to determine p_0^*.

Proposition 1.

The optimal prevention amount $p^*(u)$ does not depend on u.

Moreover, p_0^* is positive if and only if

$$-\lambda'(0) - \frac{\lambda(0)}{c} > 0. \tag{10}$$

In this case, we have

$$p_0^* = c + \frac{\lambda(p_0^*)}{\lambda'(p_0^*)}. \tag{11}$$

Proof.

The proof is divided into two parts. In a first time, we find p_0^* maximizing the function $\varphi(0,.)$. In a second time, we show that p_0^* also maximizes the function $\varphi(u,.)$ for all $u > 0$.

From (3), we directly get

$$\varphi'(0,p) = -\frac{\lambda'(p)\mu}{c-p} - \frac{\lambda(p)\mu}{(c-p)^2} \tag{12}$$

and

$$\varphi''(0,p) = \frac{2}{c-p}\varphi'(0,p) - \frac{\lambda''(p)\mu}{c-p}. \tag{13}$$

Hence, by Equation (13), thanks to the strict convexity of $\lambda(.)$, we notice that $\varphi'(0,p) \leq 0$ implies $\varphi''(0,p) < 0$ for all $p \in \mathbb{R}^+$.

Let us now prove that if $\varphi'(0,0) \leq 0$, then we have $\varphi'(0,p) < 0$ and $\varphi''(0,p) < 0$ for all $p > 0$. We can restrict the prove to the case $\varphi'(0,0) < 0$ since $\varphi'(0,0) = 0$ implies $\varphi''(0,p) < 0$ for small $p > 0$ which in turn implies that $\varphi'(0,.)$ is a decreasing function in a neighborhood of 0.

In that goal, let us define the interval $I \subset \mathbb{R}^+$ such that

- $0 \in I$;
- $\varphi''(0,p) \leq 0$ for all $p \in I$;
- If $J = [a,b] \subset \mathbb{R}^+$ such that $0 \in J$ and $\varphi''(0,p) \leq 0$ for all $p \in J$, then $J \subset I$.

Then, if $I = \mathbb{R}^+$, we have proved the desired result. Otherwise, it means that there would exist $a > 0$ such that $I = [0, a]$. But in that case, since $\varphi''(0, .)$ is continuous, the intermediate value theorem tells us that we would have $\varphi''(0, a) = 0$. However, by definition, $\varphi''(0, .)$ is negative on the interval I and so $\varphi'(0, .)$ is decreasing on I. Now, since $\varphi'(0, 0) < 0$, we necessarily have $\varphi'(0, a) < 0$ and hence $\varphi''(a) < 0$ as shown previously, which contradicts the result $\varphi''(0, a) = 0$ that we would have if $I = [0, a]$. Finally, we necessarily have $I = \mathbb{R}^+$.

In conclusion, if $\varphi'(0, 0) \leq 0$ holds, $\varphi(0, .)$ is a decreasing function meaning that we should not spend money on prevention. On the contrary, if $\varphi'(0, 0) > 0$, which is equivalent to the condition $-\lambda'(0) - \frac{\lambda(0)}{c} > 0$, then $\varphi(0, .)$ increases in a neighborhood of 0, which means that prevention reduces the risk.

In such a situation, the optimal prevention amount is solution of $\varphi'(0, p_0^*) = 0$. From (12), we directly get

$$p_0^* = c + \frac{\lambda(p_0^*)}{\lambda'(p_0^*)},$$

which is a maximum since $\varphi''(0, p_0^*) < 0$.

We now prove that the optimal prevention amount does not depend on u. Let $u \in \mathbb{R}^+$ and $p \in D$, and let us denote by N^* the Poisson process of arrival intensity $\lambda(p_0^*)$. We now consider the two surplus processes

$$U(t, p) = u + (c - p)t - \sum_{i=1}^{N(t)} X_i, \tag{14}$$

with N a Poisson process of parameter $\lambda(p)$ and

$$U(t, p_0^*) = u + (c - p_0^*)t - \sum_{i=1}^{N^*(t)} X_i. \tag{15}$$

It is well-known that changing the time scale (here multiplying the time by $\lambda(p_0^*)/\lambda(p)$) does not influence the infinite-time ruin probability, meaning that

$$\psi(u, p) = \mathbb{P}(\exists t \text{ such as } U(t, p) < 0) = \mathbb{P}(\exists t \text{ such as } U_2(t, p) < 0), \tag{16}$$

where the surplus process $U_2(t, p)$ is defined as

$$U_2(t, p) = u + (c - p)\frac{\lambda(p_0^*)}{\lambda(p)}t - \sum_{i=1}^{N^*(t)} X_i. \tag{17}$$

So, the only difference between the surplus processes $U_2(t, p)$ and $U(t, p_0^*)$ is the drift. Thus, it comes $\varphi(u, p) < \varphi(u, p_0^*)$ if and only if $(c - p)\frac{\lambda(p_0^*)}{\lambda(p)} < (c - p_0^*)$. Now, it suffices to notice that

$$(c - p)\frac{\lambda(p_0^*)}{\lambda(p)} - (c - p_0^*) = \mu\lambda(p_0^*)\left[\frac{c - p}{\lambda(p)\mu} - \frac{c - p_0^*}{\lambda(p_0^*)\mu}\right] < 0 \tag{18}$$

since p_0^* minimizes $\psi(0,p) = \left(\frac{c-p}{\lambda(p)\mu}\right)^{-1}$, which completes the proof.

■

Notice that if there is no p_0^* in D such that $\varphi'(0,p_0^*) = 0$, then the optimal prevention amount is $p_{lim}(1-\varepsilon)$ as $\varphi(0,.)$ increases with p.

In Figure 2.1, we illustrate Proposition 1. We assume $c = 10$, $\lambda(p) = \frac{0{,}5}{2} + \frac{0{,}5e^{-p}}{2}$ and we consider claim amounts that are exponentially distributed with mean $\mu = 10$. We see that the optimal prevention amount is always the same whatever the value taken by the initial surplus.

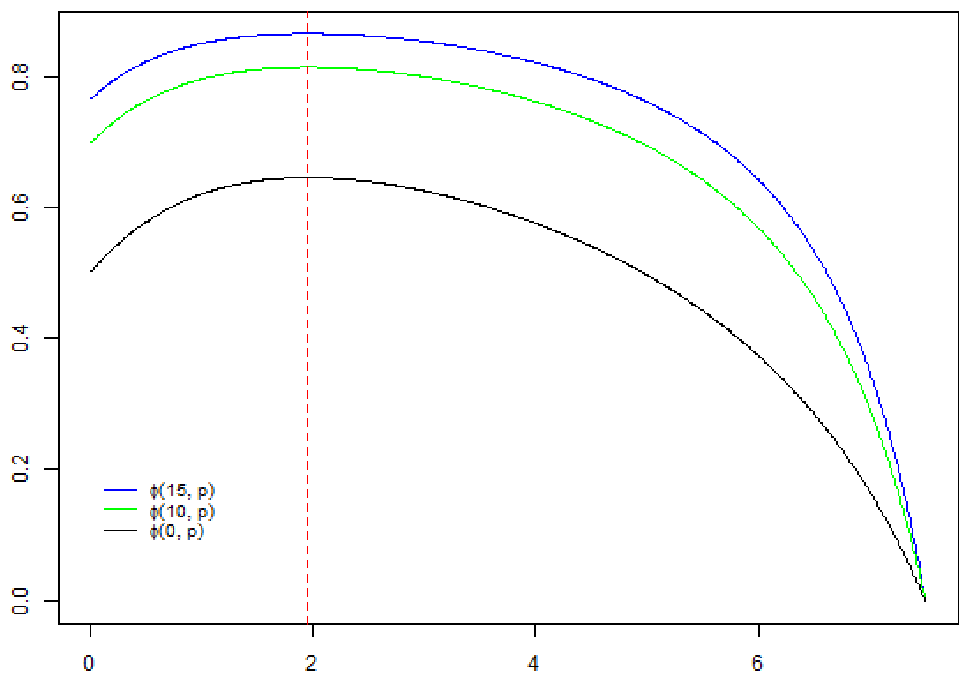

Figure 1: Non-ruin probabilities $\varphi(0,p)$, $\varphi(10,p)$ and $\varphi(15,p)$ for $c = 10$, $\lambda(p) = \frac{0{,}5}{2} + \frac{0{,}5e^{-p}}{2}$ and claim amounts that are exponentially distributed with mean $\mu = 10$

From Proposition 1, we observe that maximizing $\varphi(u,.)$ amounts to find p minimizing the loss ratio $\frac{\lambda(p)\mu}{c-p}$, which is the expected claim amount $\lambda(p)\mu$ over one year divided by the annual net premium $c - p$. In other words, it amounts to find p that maximizes the implicit safety loading.

Moreover, notice that equations (10) and (11) imply

$$\lambda'(p) \leq \frac{-\lambda(p)}{c-p} \quad \text{for all } p \in [0, p_0^*)], \tag{19}$$

which yields

$$\frac{\lambda(p)}{\lambda(0)} \leq \frac{c-p}{c} \quad \text{for all } p \in [0, p_0^*]. \tag{20}$$

In words, investing in prevention makes sense as long as the claim frequency decreases more than the premium rate.

It is also interesting to remark that p_0^* can be found with a geometric construction. Indeed, Equation (11) shows that the tangent line of the function $\lambda(.)$ at point p_0^* passes through the point $(c, 0)$, as illustrated in Figure 1.

2.1.1 Adjustment coefficient

In this section, we assume that the moment generating function $M_X(.)$ for the claim amount is well defined. Namely, we assume that

$$M_X(s) < \infty \quad \text{for all } s \in \mathbb{R} \tag{21}$$

or that there exists $s^* > 0$ such that

$$M_X(s) < \infty \quad \text{for all } s < s^* \qquad \text{and} \qquad M_X(s) = \infty \quad \text{for all } s \geq s^*. \tag{22}$$

For all $p \in D$, we denote by $\kappa(p)$ the adjustment coefficient defined as the positive solution of the equation

$$\lambda(p) + (c-p)s = \lambda(p) M_X(s). \tag{23}$$

The adjustment coefficient is thus well defined regardless of the prevention amount.

It is well known that if the adjustment coefficient exists, inequality $1 - \varphi(u, p) \leq e^{-\kappa(p)u}$ holds true, which means that the ruin probability decreases exponentially with the initial surplus, the adjustment coefficient controlling the slope of the decrease. Therefore, the adjustment coefficient can be seen as an indicator of the initial surplus efficiency in reducing the ruin probability.

A prevention amount can be defined as optimal if it maximizes the adjustment coefficient of the insurance company. The next result shows that this optimal prevention amount coincides with the one that maximes the non-ruin probability. It is a direct consequence of Proposition 1 which highlights the fact that maximizing the non-ruin probability with respect to p amounts to find p maximizing the implicit safety loading.

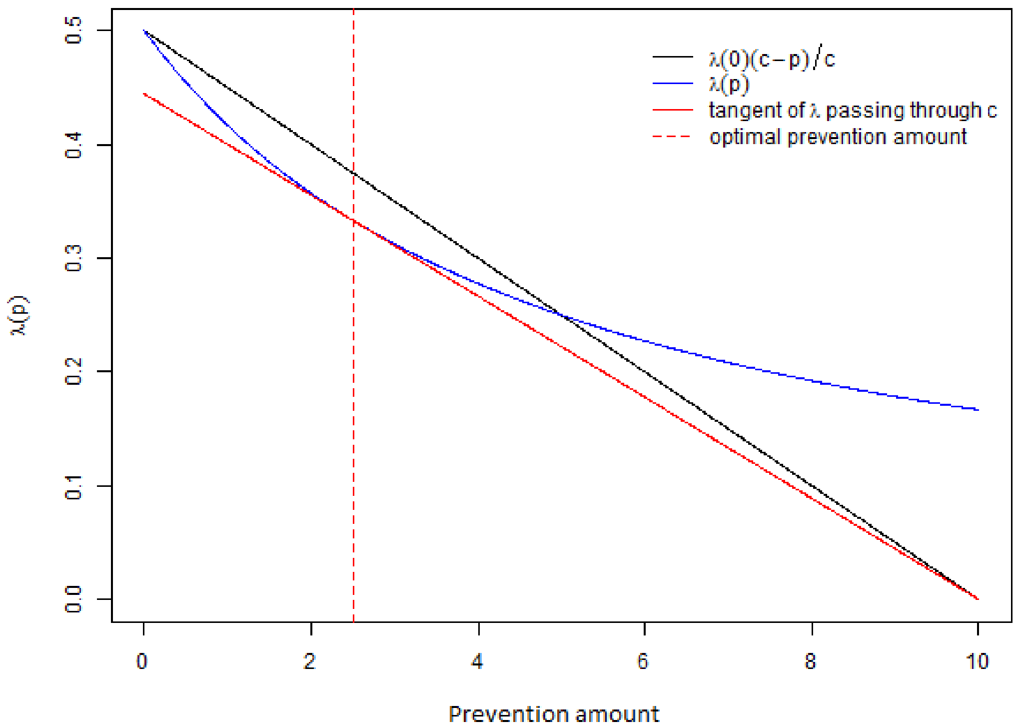

Figure 2: Geometric construction to find the optimal prevention amount (with $c = 10$)

Corollary 2.

If Equation (10) holds, then p_0^* *maximizes the function* $\kappa(.)$.

Proof. Let $p \in D$. Equation (23) can be rewritten as

$$1 + \frac{(c-p)s}{\lambda(p)} = M_X(s). \tag{24}$$

Geometrically, Equation (24) tells us that the adjustment coefficient is the point where the line passing through (0,1) with slope $\frac{(c-p)}{\lambda(p)}$ crosses the moment generating function $M_X(.)$, as depicted in Figure 3. So, one can see that the adjustment coefficient increases with the drift of the surplus process $U(t,p)$. Let us now change the time scale to obtain the surplus process described in (17). Of course, the adjustment coefficient is not impacted by such a transformation.

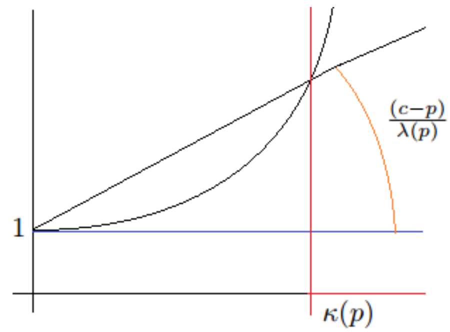

Figure 3: Geometric construction of $\kappa(p)$

Now, if we compare the drifts of surplus processes (17) and (15), one sees by Equation (18) that p_0^* leads to the largest drift for (17) (which corresponds in that case to the drift of (15)) and hence maximizes $\kappa(.)$. ■

Hence, the prevention amount that maximizes the adjustment coefficient is equivalent to the prevention amount that minimizes the ruin probability.

2.1.2 Dividends

In 1957, De Finetti introduced a dividend parameter in the classical Cramer-Lundberg model [DF57]. In this paper, we consider the case of a constant dividend barrier as explored by Segerdahl [Seg70] and Dickson and Gray [DG84], among others.

Denote the constant dividend barrier level by $K > u$. When the surplus process hits K, it remains constant until a new claim happens and dividends are being transferred to shareholders. We denote by L_t the total amount of dividends distributed up to time t. In such a setting, ruin occurs almost surely.

In Segerdahl [Seg70], we can find an expression for the probability of being ruined before reaching dividend barrier K, which is

$$\xi(u, K, p) = \frac{\varphi(K, p) - \varphi(u, p)}{\varphi(K, p)}. \tag{25}$$

From that expression, we can easily obtained an expression for the expected amount of dividends earned before being ruined, namely

$$\mathbb{E}(L_t)(p) = \frac{(1 - \xi(u, K, p))(c - p)}{\lambda(p)\theta}, \tag{26}$$

where $\theta = \int_0^K \xi(K - x, K, p) dF_X(x)$. We are now in position to prove the following result, where we use the same change of time than in the proof of Proposition (1).

Corollary 3. *The function $\mathbb{E}(L_t)(.)$ reaches its maximum in p_0^*.*

Proof. Let $u < K \in \mathbb{R}^+$ and $p \in D$. Using the same change of time than in (17) and comparing again the deterministic parts of the surplus processes (17) and (15), one easily sees that p_0^* minimizes the function $\xi(u, K, .)$.

Now, combining (25) and (26), we get

$$\mathbb{E}(L_t)(p) = \frac{\varphi(u,p)}{\int_0^K 1 - \varphi(K-x,p)dF_X(x)} \frac{c-p}{\lambda(p)}. \tag{27}$$

We have seen in the proof of Proposition 1 that p_0^* maximizes $\frac{c-p}{\lambda(p)}$. Furthermore, we also know by Proposition 1 that p_0^* maximizes $\varphi(u,p)$ and therefore minimizes $\int_0^K 1 - \varphi(K-x,p)dF_X(x)$ as well, which ends the proof. ∎

Thus, p_0^* introduced in Proposition 1 is also the prevention amount that maximizes the expected amount of dividends.

2.1.3 Impact of c on p_0^*

We denote the optimal prevention amount by $p_0^*(c)$ to make explicit the dependence with the premium rate c and we investigate the impact of c on $p_0^*(c)$. We get the following result.

Proposition 4. *If condition (10) is fulfilled, then $p_0^*(.)$ increases with c.*

Proof. Recalling Equation (11),

$$p_0^*(c) = c + \frac{\lambda(p_0^*(c))}{\lambda'(p_0^*(c))} \tag{28}$$

yields

$$p_0^{*\prime}(c) = \frac{\lambda'(p_0^*(c))^2}{\lambda''(p_0^*(c))\lambda(p_0^*(c))} > 0. \tag{29}$$

Equation (29) guarantees that $p_0^*(.)$ increases with c. ∎

So, if the insurer gets a higher premium amount, it is always optimal to invest a part of this premium increase for prevention. In the extreme case where the premium rate c goes to infinity, the next result shows that $p_0^*(c)$ tends to infinity as well.

Proposition 5.

We have $\lim_{c\to\infty} p_0^*(c) = +\infty$. Furthermore,

(i) If $\lim_{c\to\infty} \lambda'(c)c \neq 0$, then $\lim_{c\to\infty} \frac{p_0^*(c)}{c} \neq 0$;

(ii) If $\lim_{c\to\infty} \lambda'(c)c = 0$ *and* $\lim_{c\to\infty} \lambda(c) > 0$*, then* $\lim_{c\to\infty} \frac{p_0^*(c)}{c} = 0$*;*

(iii) If $\lim_{c\to\infty} \lambda(c) = 0$*, then* $\lim_{c\to\infty} \frac{p_0^*(c)}{c} = 0$ *if and only if* $\lim_{c\to\infty} \frac{\lambda(c)-\lambda'(c)c}{\lambda(c)} = 1$*.*

Proof. Let us first prove that $\lim_{c\to\infty} p_0^*(c) = +\infty$. Equation (11) can be rewritten as

$$\frac{p_0^*(c)}{c} = \frac{\lambda(p_0^*(c))}{\lambda'(p_0^*(c))c} + 1, \tag{30}$$

where $c \in \mathbb{R}^+$ is such that the safety loading condition is fulfilled. Because $0 \leq \frac{p_0^*(c)}{c} < 1$, we know that $\frac{\lambda(p_0^*(c))}{\lambda'(p_0^*(c))c}$ is bounded.

Let us assume that $\lim_{c\to\infty} p_0^*(c) = \alpha \in \mathbb{R}^+$. We have $\lambda'(\alpha) < 0$ since $\lambda(.)$ is a decreasing function. It comes

$$\lim_{c\to\infty} \frac{p_0^*(c)}{c} = 1 + \lim_{c\to\infty} \frac{\lambda(p_0^*(c))}{\lambda'(p_0^*(c))c} = 1 \tag{31}$$

as $\lim_{c\to\infty} \frac{\lambda(p_0^*(c))}{\lambda'(p_0^*(c))c} = \lim_{c\to\infty} \frac{\lambda(\alpha)}{\lambda'(\alpha)c} = 0$. Now, since $\lim_{c\to\infty} p_0^*(c) = \alpha$, we also get that $\lim_{c\to\infty} \frac{p_0^*(c)}{c} = 0$, which contradicts our previous result. Hence, as $p_0^*(.)$ is an increasing function, we necessarily have $\lim_{c\to\infty} p_0^*(c) = +\infty$.

Let us now turn to items (i)-(iii). Since $p_0^*(.)$ is strictly increasing, the inverse function $p_0^{*-1}(.)$ exists. Hence, Equation (30) evaluated in $p_0^{*-1}(c)$ gives

$$p_0^{*-1}(c) = c - \frac{\lambda(c)}{\lambda'(c)}. \tag{32}$$

Obviously, we have $\lim_{c\to\infty} p_0^{*-1}(c) = +\infty$, so that $\lim_{c\to\infty} \frac{\lambda(p_0^*(c))}{\lambda'(p_0^*(c))c}$ exists and is equal to l if and only if

$$\lim_{c\to\infty} \frac{\lambda(c)}{\lambda'(c)p_0^{*-1}(c)} = \lim_{c\to\infty} \frac{\lambda(c)}{\lambda'(c)c - \lambda(c)} = l. \tag{33}$$

Moreover, by assumption on $\lambda(.)$, we know that $\lim_{c\to\infty} \lambda(c) = l_1$.

In a first time, let us assume that $l_1 > 0$. Since $0 \leq \frac{p_0^*(c)}{c} < 1$, Equation (30) directly yields $\lim_{c\to\infty} \lambda'(p_0^*(c))c \neq 0$. Furthermore, Equation (29) leads to

$$(\lambda'(p_0^*(c))c)' = \lambda'(p_0^*(c)) \left[1 + \frac{\lambda'(p_0^*(c))c}{\lambda(p_0^*(c))}\right] > 0 \tag{34}$$

since $\lambda(.)$ is decreasing and condition $0 \leq \frac{p_0^*(c)}{c} < 1$ is equivalent to

$$\frac{\lambda'(p_0^*(c))c}{\lambda(p_0^*(c))} < -1. \tag{35}$$

Thus, $\lambda'(p_0^*(c))c$ is negative and increases with c. Therefore, there exists $l_2 < 0$ such that $\lim_{c\to\infty} \lambda'(p_0^*(c))c = l_2$, which in turn implies that $\lim_{c\to\infty} \frac{p_0^*(c)}{c} = 1 + \frac{l_1}{l_2}$. Hence, from Equation (33), one sees that we have $l_2 = -l_1$ if and only if $\lim_{c\to\infty} \lambda'(c)c = 0$, which proves item (ii). Also this proves item (i) in the case where $l_1 > 0$.

In a second time, let us assume that $l_1 = 0$ in order to end the proof for item (i) and to prove item (iii). We learn from Equation (33) that if $\lim_{c\to\infty} \lambda'(c)c \neq 0$, then we have $l = 0$ and $\lim_{c\to\infty} \frac{p_0^*(c)}{c} = 1$, which ends the proof of item (i). Finally, turning to item (iii), we know that if $\lim_{c\to\infty} \frac{\lambda(c)-\lambda'(c)c}{\lambda(c)} = 1$, then there exists $c^* \in \mathbb{R}$ such that for all $c > c^*$, one has $\lambda(c) - \lambda'(c)c > 0$. Hence, $\lim_{c\to\infty} \frac{\lambda(c)-\lambda'(c)c}{\lambda(c)} = 1$ is equivalent to $\lim_{c\to\infty} \frac{\lambda(c)}{\lambda'(c)c-\lambda(c)} = -1$, so that from Equations (31) and (33), one sees that $\lim_{c\to\infty} \frac{\lambda(c)-\lambda'(c)c}{\lambda(c)} = 1$ is equivalent to $\lim_{c\to\infty} \frac{p_0^*(c)}{c} = 0$.
■

Notice that the case $\lim_{c\to\infty} \lambda(c) = 0$ is of practical relevance if one thinks about vaccination campaigns for instance. Indeed, in this setting, investing enough money in prevention to vaccinate the entire population will enable one to eradicate the associated disease, as it was the case for the smallpox virus.

Remark 6. When $\lim_{c\to+\infty} \{(\lambda(c) - \lim_{c\to+\infty} \lambda(c))c\}$ exists[1] (which is generally true in our setting), condition $\lim_{c\to\infty} \lambda'(c)c \neq 0$ cannot be fulfilled. Indeed, let us show that the existence of $\lim_{c\to+\infty} \{(\lambda(c) - \lim_{c\to+\infty} \lambda(c))c\}$ necessarily implies $\lim_{c\to+\infty} \lambda'(c)c = 0$. Notice that since $\lambda'(c) \leq 0$, we get $\lim_{c\to+\infty} \lambda'(c)c \leq 0$ when it exists.

In a first time, we consider the case where $\lim_{c\to+\infty} \lambda(c) = 0$.We have

$$\lambda'(c)c = (\lambda(c)c)' - \lambda(c). \tag{36}$$

One distinguish two situations. Firstly, we suppose that $\lim_{c\to\infty} \lambda(c)c = +\infty$. Hence, we cannot find a constant $\tilde{c}$ such that $\lambda(c)c$ is decreasing on $[\tilde{c}, +\infty[$. However, if $\lim_{c\to+\infty} \lambda'(c)c < 0$, then Equation (36) leads to $\lim_{c\to+\infty} (\lambda(c)c)' < 0$, which contradicts the previous sentence. Thus, we necessarily have $\lim_{c\to+\infty} \lambda'(c)c = 0$. Secondly, we assume that $\lim_{c\to\infty} \lambda(c)c = \gamma \geq 0$. Since for all $c > 0$, we have $\lambda(c)c = \int_0^c (\lambda(x)x)'dx$ and $\lim_{c\to\infty} (\lambda(c)c)' = 0$, one sees that Equation (36) yields $\lim_{c\to+\infty} \lambda'(c)c = 0$.

[1] including here the case where $\lim_{c\to+\infty} \{(\lambda(c) - \lim_{c\to+\infty} \lambda(c))c\} = +\infty$

In a second time, we consider the case where $\lim_{c\to+\infty} \lambda(c) = \beta > 0$. Let

$$\lambda_2(c) = \lambda(c) - \beta. \tag{37}$$

One sees that $\lambda_2(.)$ is decreasing, convex, $\mathcal{C}_2$ and such that $\lim_{c\to+\infty} \lambda_2(c) = 0$. The assumptions on $\lambda(.)$ implies that $\lim_{c\to+\infty} \lambda_2(c)c$ exists. Applying the same arguments to $\lambda_2(.)$ than in the first case leads to $\lim_{c\to+\infty} \lambda_2'(c)c = 0$, which in turn implies that $\lim_{c\to+\infty} \lambda'(c)c = 0$ by deriving Equation (37).

Remark 7. Equation (11) tells us that p_0^* does not depend on μ while the non-ruin probability does. If the claim amounts are exponentially distributed, so that their distribution is entirely characterized by its mean, then we have $\frac{\partial \frac{\varphi(u,p_0^*)}{\varphi(u,0)}-1}{\partial\ \mu} > 0$, which means that prevention is still important when you deal with claim amounts that are higher in average. Indeed, in the exponential case, the non-ruin probability writes

$$\varphi(u,p) = 1 - \frac{\lambda(p)\mu}{c-p} e^{-(\frac{1}{\mu} - \frac{\lambda(p)}{c-p})u} \tag{38}$$

for all $u > 0$ and $p \in [0, p_{lim}]$, so that

$$\frac{\partial \varphi(u,p)}{\partial \mu} = -\left(1 + \frac{u}{\mu}\right) \frac{\lambda(p)}{c-p} e^{-\left(\frac{1}{\mu} - \frac{\lambda(p)}{c-p}\right)u}. \tag{39}$$

Now, as $\frac{\partial \frac{\varphi(u,p_0^*)}{\varphi(u,0)}-1}{\partial\mu} = \frac{\frac{\partial\varphi(u,p_0^*)}{\partial\mu}\varphi(u,0) - \frac{\partial\varphi(u,0)}{\partial\mu}\varphi(u,p_0^*)}{\varphi^2(u,0)}$, Equation (39) leads to

$$\frac{\partial \frac{\varphi(u,p_0^*)}{\varphi(u,0)} - 1}{\partial\mu} = \frac{\left(1 + \frac{u}{\mu}\right)}{\varphi^2(u,0)} \left(\frac{\lambda(0)}{c} e^{-\left(\frac{1}{\mu} - \frac{\lambda(0)}{c}\right)u} - \frac{\lambda(p_0^*)}{c - p_0^*} e^{-\left(\frac{1}{\mu} - \frac{\lambda(p_0^*)}{c-p_0^*}\right)u} \right). \tag{40}$$

Therefore, since $\frac{\lambda(p_0^*)}{c-p_0^*} < \frac{\lambda(0)}{c}$ by definition of p_0^*, it comes $\frac{\partial \frac{\varphi(u,p_0^*)}{\varphi(u,0)}-1}{\partial\mu} > 0$, as previously announced.

2.2 Surplus expectancy

An insurance company may also want to maximize its expected surplus on a given time horizon t, which amounts to find the prevention amount maximizing the expectation of $U(t)$, denoted $\tilde{p}$ in the following.

Obviously, we have

$$\mathbb{E}(U(t))(p) = u + [c - p - \lambda(p)\mu]t. \tag{41}$$

Hence, by deriving (41) with respect to p, we easily see that the optimal prevention amount $\tilde{p}$ is positive if and only if $-\lambda'(0) > \frac{1}{\mu}$. In such a case, $\tilde{p}$ solves

$$-\lambda'(\tilde{p}) = \frac{1}{\mu}. \tag{42}$$

This time, the optimal prevention amount $\tilde{p}$ is different from p_0^*. One sees that $\tilde{p}$ does not depend on the premium rate c but well on the expected claim amount μ. As a consequence, an insurance company has to make a choice between maximizing its expected wealth on a given time horizon and decreasing its risk on the long run.

Note that this particular optimisation result provides an additional way to demonstrate that, as stated in the introduction, the present problem differs from the optimal reinsurance problem: in the proportional reinsurance context where the surplus process is given by

$$V(t,\alpha) = u + (c - \alpha(1+\theta)\gamma\mu)t - \sum_{i=1}^{N^\gamma(t)} X_i, \tag{43}$$

as in Equation (8), maximizing the expected surplus at the end of the first period leads to maximizing

$$u + c - \gamma\mu(1+\alpha\theta),$$

which is achieved for $\alpha = 0$ as long as $\theta > 0$.

In the sequel, we focus our study on the optimal prevention amount p_0^*.

3 $\lambda(.)$ constant then decreasing

Assuming $\lambda(p)$ decreasing with p can be too restrictive in practice. Indeed, as mentioned in the introduction, we can have situations where we must at least invest a certain amount P in prevention to start to observe an impact on $\lambda(.)$. That is why, in this section, we consider the case where $\lambda(.)$ is first constant for prevention amounts smaller than a certain threshold P. Beyond P, $\lambda(.)$ is still assumed to be decreasing and strictly convex.

Formally, let $0 < P < c$ and let

$$\lambda : \left| \begin{array}{ccl} [0,c] & \longmapsto & \mathbb{R}^+ \\ p & \longmapsto & \lambda(0) \; if \; p \leq P \\ & & \lambda(p) < \lambda(0) \; otherwise. \end{array} \right.$$

Moreover, for all $p \geq P$, we suppose $\lambda \in \mathcal{C}^2$, $\lambda'(p) < 0$ and $\lambda''(p) > 0$.

We look for the prevention amount p_0^* maximizing $\varphi(0,p) = 1 - \frac{\lambda(p)\mu}{c-p}$. The reasoning is conducted in two steps. Firstly, we look for the optimal prevention amount $\overline{p^*}$ on the interval $[P, p_{lim}[$, where prevention is useful. Secondly, we compare the ruin probability obtained by investing this amount in prevention with the one obtained without prevention.

Proposition 8. *If condition* $\frac{c-\overline{p^*}}{c}\lambda(0) > \lambda(\overline{p^*})$ *is verified, then we have* $p_0^* = \overline{p^*}$. *Otherwise, we have* $p_0^* = 0$.

Proof. In a first time, we consider $p \in [P, p_{lim}]$. In this case, $\lambda(.)$ verifies all the assumptions stated in Section 2 and condition (10) becomes

$$-\lambda'(P) - \frac{\lambda(P)}{c-P} > 0. \tag{44}$$

Thus, when condition (44) is fulfilled, Propositions 1 and **??** tell us that $\overline{p^*}$ does not depend on u and is given by $\overline{p^*} = c + \frac{\lambda(\overline{p^*})}{\lambda'(\overline{p^*})}$. Otherwise, if (44) is not verified, it comes $\overline{p^*} = P$ and hence $p_0^* = 0$.

In a second time, when condition (44) is fulfilled, we have to compare $\varphi(0,0)$ and $\varphi(0,\overline{p^*})$. It comes

$$\begin{aligned} \varphi(0,0) - \varphi(0,\overline{p^*}) < 0 \quad &\Leftrightarrow \quad -\frac{\lambda(0)\mu}{c} + \frac{\lambda(\overline{p^*})\mu}{c-\overline{p^*}} < 0 \\ &\Leftrightarrow \quad \frac{c-\overline{p^*}}{c}\lambda(0) > \lambda(\overline{p^*}). \end{aligned}$$

Hence, if condition $\frac{c-\overline{p^*}}{c}\lambda(0) > \lambda(\overline{p^*})$ holds true, then we have $p_0^* = \overline{p^*}$. Otherwise, we have $p_0^* = 0$. ■

Let us notice that condition $\frac{c-\overline{p^*}}{c}\lambda(0) > \lambda(\overline{p^*})$ is closed to (10). It can be rewritten as $\frac{\lambda(\overline{p^*})}{\lambda(0)} < \frac{c-\overline{p^*}}{c}$, such that investing $\overline{p^*}$ must decrease more the claim frequency than the premium rate. Moreover, the proofs of Proposition 1 and Corollaries 2 and 3 can easily be adapted to this section.

4 Conclusion

In this paper, a first risk model with prevention has been introduced. The optimal prevention strategies have been identified for several optimisation problems. In further works, it would be interesting to consider different impacts of prevention for different categories of claims. The uncertainty on prevention efficiency, its evolution over time, as well as the impact of impulses of prevention strategies (represented by an instantaneous, one-shot drop in the surplus instead of a steady prevention effort which decreases the premium income rate) could be taken into account. It would also be interesting to consider optimal prevention as a stochastic control problem, and to jointly use optimal prevention and reinsurance. This is left for further research.

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