

## УПРАВЛЕНИЕ ДИНАМИЧЕСКИМИ СИСТЕМАМИ

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MODEL PREDICTIVE CONTROL FOR DISCRETE SYSTEMS  
WITH STATE AND INPUT DELAYS

The paper deals with Model Predictive Control synthesis based on the system output tracking with input and state delays. Input and state constraints are taken into account in MPC problem solving. A prediction is carried out on the base of object states estimation that is obtained by the Kalman filter. The criteria function is assumed to be convex quadratic.

**Keywords:** *Model Predictive Control, discrete systems, state delay, input delay, Kalman filter.*

One of the modern formalized approaches to the system control synthesis based on mathematical methods of optimization is control methods using predictive models – Model Predictive Control (MPC).

This approach began to develop in the early 1960s. It was developed for equipment and process control in petrochemical and energy industries for which the application of traditional synthesis methods was extremely complicated according to mathematical model's complication. During the last years application was considerably extended covering technologic fields [1], economic system control [2], inventory control [3] and investment portfolio control [4].

The results of this paper extend the results of the paper [5].

## 1. Problem Statement

Suppose the object is described by the following state-space system of linear-difference equations

$$x_{t+1} = Ax_t + \sum_{i=1}^r A_i x_{t-i} + Bu_{t-h} + w_t, \quad x_k = \bar{x}_k, \quad (k = \overline{-r, 0}), \quad u_i = \bar{u}_i, \quad (i = \overline{-h, -1}), \quad (1)$$

$$\psi_t = Hx_t + v_t, \quad (2)$$

$$y_t = Gx_t. \quad (3)$$

Here  $x_t \in R^n$  ( $x_t = \bar{x}_t$ ,  $t = -r, \dots, -1, 0$ ,  $\bar{x}_t$  is considered to be given) is the object state,  $u_t \in R^m$  is the control input ( $u_t = \bar{u}_t$ ,  $t = -h, -h+1, \dots, -1$ ,  $\bar{u}_t$  is given),  $y_t \in R^p$  is the output (which is to be controlled),  $\psi_t \in R^l$  is the observation (measured output),  $r, h$  are the state and input delay values respectively. Further, the state noise  $w_t$  and meas-

urement noise  $v_t$  are assumed to be Gaussian distributed with zero mean and covariances  $W$  and  $V$  respectively, i.e.

$$M\{w_t w_k^T\} = W \delta_{t,k}, \quad M\{v_t v_k^T\} = V \delta_{t,k},$$

where  $\delta_{t,k}$  is the Kronecker delta. This model is used to make predictions about plant behavior over the prediction horizon, denoted by  $N$ , using information (measurements of inputs and outputs) up to and including the current time  $t$ .

It is supposed the plant operates under the constrained conditions:

$$a_1 \leq S_1 x_t \leq a_2, \quad (4)$$

$$\varphi_1(x_{t-h}) \leq S_2 u_{t-h} \leq \varphi_2(x_{t-h}) \quad (5)$$

Here  $S_1$  and  $S_2$  are structural matrices that are composed of zeros and units, identifying constrained components of vectors  $x_t$  and  $u_t$ ;  $a_1$ ,  $a_2$ ,  $\varphi_1(x_t)$ ,  $\varphi_2(x_t)$  are given constant vectors and vector-functions.

The problem is to determine an acting strategy on the base of the observation  $\psi_t$ , according to which the output vector of the system  $y_t$  will be close to the reference taking into account constraints on the state and input.

## 2. Prediction

With the Gaussian assumptions on state and measurement noise it is possible to make optimal (in the minimum variance sense) predictions of state and output using a Kalman filter, see e.g. [6].

Let  $\hat{x}_{i|j}$  and  $\hat{y}_{i|j}$  be estimates of the state and output at time  $i$  given information up to and including time  $j$  where  $j \leq i$ . Then

$$\hat{x}_{t+1|t} = A \hat{x}_{t|t-1} + \sum_{i=1}^r A_i \hat{x}_{t-i|t-i-1} + B u_{t-h} + K_t (\psi_t - H \hat{x}_{t|t-1}), \quad \hat{x}_{k|k-1} = \bar{x}_k, \quad k = \overline{-r, 0},$$

$$\hat{y}_{t+1|t} = G \hat{x}_{t+1|t},$$

$$K_t = A P_t H^T (H P_t H^T + V)^{-1},$$

$$P_{t+1} = W + A P_t A^T - A P_t H^T (H P_t H^T + V)^{-1} H P_t A^T, \quad P_0 = P_{x_0}, \quad (6)$$

where  $P_{x_0}$  is the given initial value of the variance matrix. Equation (6) for  $P_t$  is known as the discrete-time Riccati-equation.

MPC usually requires estimates of the state and/or output over the entire prediction horizon  $N$  from time  $t+1$  until time  $t+N$ , and can only make these predictions based on information up to and including the current time  $t$ . Equations (6) can be used to obtain  $\hat{x}_{t+1|t}$ ,  $\hat{y}_{t+1|t}$ . Optimal state/output estimates from time  $t+2$  to  $t+N$  can be obtained as follows

$$\hat{x}_{t+i+1|t} = A \hat{x}_{t+i|t} + \sum_{j=1}^r A_j \hat{x}_{t+i-j|t-j-1} + B u_{t-h+i|t}, \quad i = \overline{1, N}, \quad (7)$$

$$\hat{y}_{t+i|t} = G \hat{x}_{t+i|t}, \quad i = \overline{1, N}. \quad (8)$$

In the above the notation  $u_{t-h+i|t}$  is used to distinguish the actual input at time  $t+i$ , namely  $u_{t-h+i}$ , from that used for prediction purposes, namely  $u_{t-h+i|t}$ .

Equation (7) can be expanded in terms of the initial state  $\hat{x}_{t+|t}$  and future control actions  $u_{t-h+i|t}$  as follows

$$\hat{x}_{t+i|t} = A^{i-1}\hat{x}_{t+|t} + \sum_{k=1}^{i-1} A^{i-k-1} \sum_{j=1}^r A_j \hat{x}_{t+k-j|t-j-1} + \sum_{k=1}^{i-1} A^{i-k-1} B u_{t-h+k|t}, \quad i = \overline{1, N}. \quad (9)$$

Now in terms of predicting the output, equation (8) can be expanded in terms of the above expression for  $\hat{x}_{t+|t}$ , which results in a series of equations that provide optimal output predictions. The key point to note is that each output prediction is a function of the initial state  $\hat{x}_{t+|t}$  and future inputs  $u_{t-h+i|t}$  only:

$$\hat{y}_{t+i|t} = G A^{i-1} \hat{x}_{t+|t} + G \sum_{k=1}^{i-1} A^{i-k-1} \sum_{j=1}^r A_j \hat{x}_{t+k-j|t-j-1} + G \sum_{k=1}^{i-1} A^{i-k-1} B u_{t-h+k|t}, \quad i = \overline{1, N}. \quad (10)$$

This series of prediction equations can be stated in an equivalent manner using matrix vector notation. Denote

$$\begin{aligned} \hat{X}_t &= \begin{bmatrix} \hat{x}_{t+|t} \\ \vdots \\ \hat{x}_{t+N|t} \end{bmatrix}, \quad \hat{Y}_t = \begin{bmatrix} \hat{y}_{t+|t} \\ \vdots \\ \hat{y}_{t+N|t} \end{bmatrix}, \quad U_{t-h} = \begin{bmatrix} u_{t-h+|t} \\ \vdots \\ u_{t-h+N|t} \end{bmatrix}, \\ \Psi &= \begin{bmatrix} E_n \\ A \\ A^2 \\ \vdots \\ A^{N-1} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} G \\ GA \\ GA^2 \\ \vdots \\ GA^{N-1} \end{bmatrix}, \quad \hat{X}_i^0 = \begin{bmatrix} \hat{x}_{t+1-i|t-i} \\ \vdots \\ \hat{x}_{t+N-i|t-i} \end{bmatrix}, \quad i = \overline{1, r}, \\ \Psi_i^0 &= \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ A_i & 0 & 0 & \dots & 0 \\ AA_i & A_i & 0 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ A^{N-2} A_i & A^{N-3} A_i & \dots & A_i & 0 \end{bmatrix}, \quad \Lambda_i^0 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ GA_i & 0 & 0 & \dots & 0 \\ GAA_i & GA_i & 0 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ GA^{N-2} A_i & GA^{N-3} A_i & \dots & GA_i & 0 \end{bmatrix}, \\ P &= \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ B & 0 & 0 & \dots & 0 \\ AB & B & 0 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ A^{N-2} B & A^{N-3} B & \dots & B & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ GB & 0 & 0 & \dots & 0 \\ GAB & GB & 0 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ GA^{N-2} B & GA^{N-3} B & \dots & GB & 0 \end{bmatrix}. \quad (11) \end{aligned}$$

Here  $E_n$  is the  $n$ -by- $n$  identity matrix. Then the predictive model (9)-(10) can be expressed as

$$\begin{aligned} \hat{X}_t &= \Psi \hat{x}_{t+|t} + \sum_{i=1}^r \Psi_i^0 \hat{X}_i^0 + P U_{t-h}, \\ \hat{Y}_t &= \Lambda \hat{x}_{t+|t} + \sum_{i=1}^r \Lambda_i^0 \hat{X}_i^0 + \Phi U_{t-h}. \end{aligned} \quad (12)$$

### 3. Synthesis of Model Predictive Control

In order to solve the posed problem the following criterion is used as the criteria function

$$J(t) = \frac{1}{2} \sum_{k=1}^N \left\{ \left\| \hat{y}_{t+k|t} - \bar{y}_t \right\|_C^2 + \left\| u_{t-h+k|t} - u_{t-h+k-1|t} \right\|_D^2 \right\}, \quad (13)$$

where  $C \geq 0$ ,  $D > 0$  – weighing matrices.

In the case when the reference trajectory  $\bar{y}_{t+k}$  is unknown for  $k \geq 0$  it seems reasonable to assume that  $\bar{y}_{t+k} = \bar{y}_t$ , i.e. the same reference point is held throughout the prediction horizon.

The summation terms in (13) can be expanded to offer a quadratic objective function in terms of  $\hat{x}_{t+1|t}$  and  $U_t$ . Let

$$\bar{Y}_t = \begin{bmatrix} \bar{y}_{t+1} \\ \vdots \\ \bar{y}_{t+N} \end{bmatrix}.$$

Then using (12) there is the following expression

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^N \left\| \hat{y}_{t+k|t} - \bar{y}_t \right\|_C^2 &= \frac{1}{2} \left\| \hat{Y}_t - \bar{Y}_t \right\|_{\bar{C}}^2 = \\ &= \frac{1}{2} U_{t-h}^T \Phi^T \bar{C} \Phi U_{t-h} + U_{t-h}^T [\Phi^T \bar{C} \Lambda \hat{x}_{t+1|t} + \Phi^T \bar{C} \sum_{i=1}^r \Lambda_i^0 \hat{x}_i^0 - \Phi^T \bar{C} \bar{Y}_t] + c_1, \end{aligned} \quad (14)$$

where  $c_1$  is a constant term that does not depend on  $U_{t-h}$  or  $\hat{x}_{t+1|t}$  and  $\bar{C}$  is given by

$$\bar{C} = \begin{bmatrix} C & 0 & \vdots & 0 \\ 0 & C & \vdots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \vdots & C \end{bmatrix}.$$

In a similar manner

$$\frac{1}{2} \sum_{k=1}^N \left\| u_{t-h+k|t} - u_{t-h+k-1|t} \right\|_D^2 = \frac{1}{2} U_{t-h}^T \bar{D} U_{t-h} - u_{t-h+1|t}^T D u_{t-h} + c_2, \quad (15)$$

where  $c_2$  is a constant term that does not depend on  $u_{t-h+k}$  ( $k = \overline{1, N}$ ) and  $\bar{D}$  is given by

$$\bar{D} = \begin{bmatrix} 2D & -D & 0 & \vdots & 0 \\ -D & 2D & -D & \vdots & 0 \\ \dots & \ddots & \ddots & \ddots & \dots \\ 0 & \dots & -D & 2D & -D \\ 0 & \dots & 0 & -D & 2D \end{bmatrix}.$$

Combining the above the criteria function can be expressed as

$$J(t) = \frac{1}{2} U_{t-h}^T F U_{t-h} + U_{t-h}^T f + c_3. \quad (16)$$

Here  $c_3$  is the combination of previous constant terms  $c_1$  and  $c_2$  and may be safely ignored. The terms  $F$  and  $f$  are given by

$$F = \Phi^T \bar{C} \Phi + \bar{D}, \quad f = \Gamma \begin{bmatrix} \hat{x}_{t+1|t} \\ \sum_{i=1}^r \Lambda_i^0 \hat{X}_i^0 \\ \bar{Y}_t \end{bmatrix} - \begin{bmatrix} Du_{t-h} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \Gamma = [\Phi^T \bar{C} \Lambda \quad \Phi^T \bar{C} \quad -\Phi^T \bar{C}].$$

In the absence of constraints an analytical solution of the posed problem can be obtained from the condition  $\frac{dJ}{dU_{t-h}} = 0$  using vector derivative formulas, see e.g. [7]:

$$\begin{aligned} \frac{\partial J}{\partial U_{t-h}} &= \frac{\partial}{\partial U_{t-h}} \left[ \frac{1}{2} U_{t-h}^T F U_{t-h} + U_{t-h}^T f + c \right] = \\ &= \frac{1}{2} \frac{\partial(\text{tr} F U_{t-h} U_{t-h}^T)}{\partial U_{t-h}} + \frac{\partial(U_{t-h}^T f)}{\partial U_{t-h}} = \frac{1}{2} [F^T U_{t-h} + F U_{t-h}] + f = 0. \end{aligned} \quad (17)$$

As matrix  $F$  is symmetric the equation (17) can be expressed as follows

$$F U_{t-h} + f = 0.$$

So, the criteria function can be expressed as

$$U_{t-h}^* = -(\Phi^T \bar{C} \Phi + \bar{D})^{-1} (\Phi^T \bar{C} \Lambda \hat{x}_{t+1|t} - \Phi^T \bar{C} \bar{Y}_t) - \begin{pmatrix} Du_{t-h} \\ 0 \\ \dots \\ 0 \end{pmatrix},$$

and the optimal predictive control has the form:

$$u_{t-h+1|t}^* = (E_n \quad 0 \quad \dots \quad 0) U_{t-h}^*.$$

Optimization of the constrained model (1)-(5) can be realized numerically using Matlab's function named as «quadprog».

#### 4. Economic system control modelling

Consider the economic system control intended for goods production, storage and delivery to consumers

$$\begin{aligned} q_{t+1} &= \bar{A} q_t + \sum_{i=1}^r \bar{A}_i q_{t-i} + \varphi_{t-h} + \xi_t, \quad q_k = \bar{q}_k, \quad k = \overline{-r, 0}, \\ z_{t+1} &= z_t + \bar{B} \omega_{t-h} - \varphi_{t-h} + \zeta_t, \quad z_0 = \bar{z}_0, \end{aligned} \quad (18)$$

where  $q_t \in R^s$ ,  $q_{i,t}$  is the  $i$ -typed consumer's goods amount at the moment  $t$  ( $t = \overline{1, T}$ ,  $i = \overline{1, s}$ ),  $z_{i,t}$  is the  $i$ -typed goods amount in the producer's store,  $\omega_{i,t}$  is the production of the  $i$ -typed goods,  $\varphi_{i,t}$  is the delivery volume of the  $i$ -typed goods. Vector Gaussian sequences  $\xi_t$ ,  $\zeta_t$  have the following characteristics:  $M\{\xi_t\} = 0$ ,  $M\{\zeta_t\} = 0$ ,  $M\{\xi_t \xi_k^T\} = \Sigma \delta_{t,k}$ ,  $M\{\zeta_t \zeta_k^T\} = \Xi \delta_{t,k}$ ,  $M\{\xi_t \zeta_k^T\} = 0$ . The last vectors take into account errors arisen from the model definition inaccuracy. Matrices  $\bar{A}$  and  $\bar{B}$  define the production and consumption dynamics. It is supposed the time delays  $r$  and  $h$  are integer.

The following constraints expressed as linear inequalities should be satisfied at each moment:

$$z_{\min} \leq z_t \leq z_{\max}, 0 \leq \omega_{t-h} \leq \omega_{\max}, 0 \leq \varphi_{t-h} \leq z_t. \quad (19)$$

The variables  $\omega_t$  and  $\varphi_t$  are considered to be the controlling inputs. The problem is to determine an optimal control strategy for goods production, storage and delivery on the base of the observation according to which the consumer's goods amount  $q_t$  will be close to given one taking into account constraints (19).

The model of the economic system (18) with constraints (19) can be transformed and expressed in terms of the model (1) with constraints (4) – (5) assuming  $n=2s$ . Let

$$x_t = \begin{bmatrix} q_t \\ z_t \end{bmatrix}, u_{t-h} = \begin{bmatrix} \varphi_{t-h} \\ \omega_{t-h} \end{bmatrix}, A = \begin{bmatrix} \bar{A} & 0 \\ 0 & E_s \end{bmatrix}, A_i = \begin{bmatrix} \bar{A}_i & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} E_s & 0 \\ -E_s & \bar{B} \end{bmatrix},$$

$$w_t = \begin{bmatrix} \xi \\ \zeta \end{bmatrix}, W = \begin{bmatrix} \Sigma & 0 \\ 0 & \Xi \end{bmatrix}, a_1 = z_{\min}, a_2 = z_{\max}, S_1 = [0 \quad E_s], \varphi_1(x_t) = 0,$$

$$S_2 = \begin{bmatrix} E_s & 0 \\ 0 & E_s \end{bmatrix}, \varphi_2(x_t) = \begin{bmatrix} z_t \\ \omega_{\max} \end{bmatrix}.$$

The optimization problem is solved at each time interval. In order to solve the problem of criterion (16) minimization numerically in the Matlab code it is necessary to express constraints in terms of matrixes and vectors. Then the constraint on the production  $\omega_{t-h+i|t} \leq \omega_{\max}$  for the expanded system is the following one

$$R_1 U_{t-h} \leq \bar{E} \omega_{\max}. \quad (20)$$

The constraint on the delivery volume  $\varphi_{t-h+i|t} \leq \hat{z}_{t-h+i|t-h}$  is expressed as

$$R_2 U_{t-h} \leq R_1 (\Psi \hat{x}_{t-h+1|t-h} + \sum_{i=1}^r \Psi_i^0 \hat{X}_i^0) + R_1 P U_{t-2h}. \quad (21)$$

As  $\omega_{t-h+i|t} \geq 0$  and  $\varphi_{t-h+i|t} \geq 0$ , then

$$U_{t-h} \geq 0. \quad (22)$$

The constraints  $\hat{z}_{t+i|t} \leq z_{\max}$ ,  $\hat{z}_{t+i|t} \geq z_{\min}$  can be expressed in the form:

$$R_1 P U_{t-h} \leq \bar{E} z_{\max} - R_1 (\Psi \hat{x}_{t+1|t} + \sum_{i=1}^r \Psi_i^0 \hat{X}_i^0), \quad (23)$$

$$-R_1 P U_{t-h} \leq -\bar{E} z_{\min} + R_1 (\Psi \hat{x}_{t+1|t} + \sum_{i=1}^r \Psi_i^0 \hat{X}_i^0). \quad (24)$$

Matrixes  $R_1, R_2, \bar{E}$  are assumed to be as follows

$$R_1 = \begin{bmatrix} 0 & E_s & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & E_s & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & E_s & \end{bmatrix}, R_2 = \begin{bmatrix} E_s & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & E_s & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & E_s & 0 & \end{bmatrix}, \bar{E} = \begin{bmatrix} E_s \\ E_s \\ \cdots \\ E_s \end{bmatrix}.$$

The simulation is based on the following initial data:

$$\bar{A} = \begin{bmatrix} 0,75 & 0 \\ -0,25 & 0,9 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 0 & 0 \\ -0,1 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0,3 & 0,1 \\ 0,2 & 0,8 \end{bmatrix}, z_{\min} = \begin{bmatrix} 0,1 \\ 0,1 \end{bmatrix}, z_{\max} = \begin{bmatrix} 1,5 \\ 1,5 \end{bmatrix},$$

$$\omega_{\max} = \begin{bmatrix} 0,8 \\ 0,7 \end{bmatrix}, z_0 = \begin{bmatrix} 0,2 \\ 0,2 \end{bmatrix}, q_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{q} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, r=1, h=1, N=8, H=E_4, W=0,$$

$$V = \text{diag}\{0,0005; 0,0005; 0,0005; 0,0005\}.$$

The simulation results are shown in Figures 1 – 3 as the plots of processes.

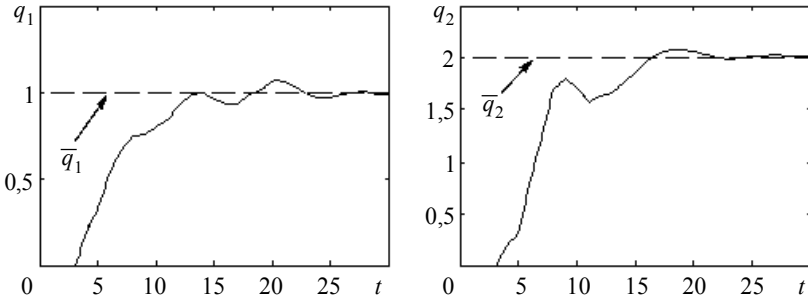


Fig. 1. The consumer's amount of the good.

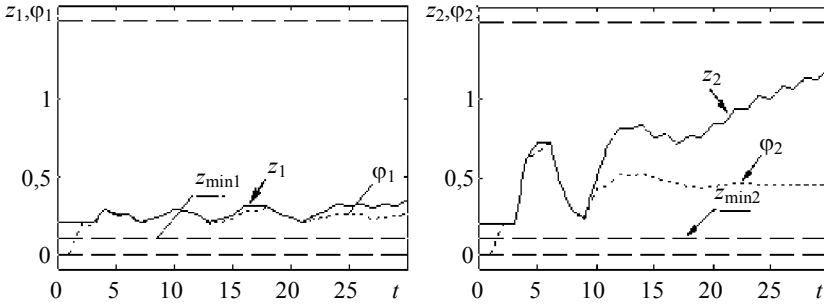


Fig. 2. The goods amount in the producer's store and the delivery volume of the goods.

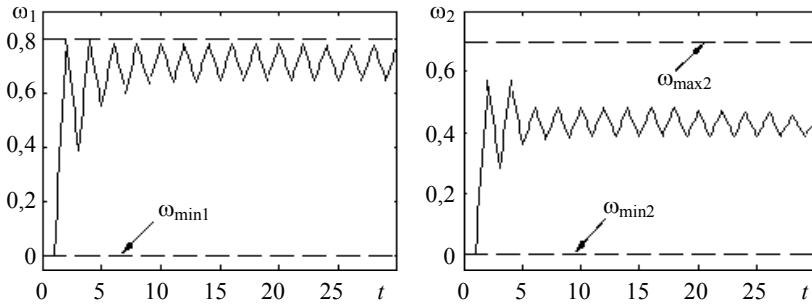


Fig.3. The goods production.

Economic system modelling proved algorithm efficiency. It is shown the goal is achieved; state and input constraints are satisfied under time delay condition.

## 5. Conclusion

The Model Predictive Control of the system with state and input delays has been developed, guaranteeing constraints satisfaction and feasibility. The solution of the MPC synthesis problem is obtained. The extrapolator is offered to use in order to obtain predicted values of the system output.

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