ISSN 1064-5624, Doklady Mathematics, 2016, Vol. 94, No. 3, pp. 676–680. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © V.V. Konev, 2016, published in Doklady Akademii Nauk, 2016, Vol. 471, No. 5, pp. 523–527.

MATHEMATICS

On One Property of Martingales with Conditionally Gaussian Increments and Its Application in the Theory of Nonasymptotic Inference

V. V. Konev

Presented by Academician of the RAS A.N. Shiryaev June 16, 2016

Received July 18, 2016

Abstract—A transformation of a discrete-time martingale with conditionally Gaussian increments into a sequence of i.i.d. standard Gaussian random variables is proposed as based on a sequence of stopping times constructed using the quadratic variation. It is shown that sequential estimators for the parameters in AR(1) and generalized first-order autoregressive models have a nonasymptotic normal distribution.

DOI: 10.1134/S1064562416060235

The role of sequential analysis methods in theoretical and applied studies of stochastic processes has increased in recent years. For the first time, the usefulness of sequential analysis for autoregressive processes was shown in [1, 2] as applied to the estimation of the drift coefficient of a diffusion process. Sequential analysis is also successfully used in statistical inference for discrete-time processes, thus improving the asymptotic and nonasymptotic properties of classical least squares estimators (LSE) and maximum likelihood estimators (MLE) [3, 6, 7, 9, 11]. The goal of this study is to prove one property of discrete-time martingales with conditionally Gaussian increments, which is then used to obtain nonasymptotic distributions for sequential estimators of the parameter in AR(1).

1. TRANSFORMATION OF A MARTINGALE WITH CONDITIONALLY GAUSSIAN INCREMENTS

Theorem 1. *Let* $(M_k, \mathcal{F}_k)_{k\geq 0}$ *be a square integrable martingale with a quadratic variation* $(\langle M\rangle_n)_{n\geq 1}$ *such that*

(a) $\mathrm{P}(\langle M\rangle_\infty = +\infty) = 1$;

(b) $\mathrm{Law}(\Delta M_k \mid \mathcal{F}_{k-1}) = \mathcal{N}(0, \sigma^2_{k-1}), k = 1, 2, \ldots$, *i.e., the* $\mathcal{F}_{k-1}$*-conditional distribution* $\Delta M_k = M_k - M_{k-1}$ *is a Gaussian distribution with parameters* 0 *and* $\sigma^2_{k-1} = \mathrm{E}((\Delta M_k)^2 \mid \mathcal{F}_{k-1})$.

For every $h > 0$, *define the sequence of stopping times*

$$\tau_0 = \tau_0(h) = 0,$$

$$\tau_j = \tau_j(h) = \inf\left\{n > \tau_{j-1}\colon \sum_{k=\tau_{j-1}+1}^{n} \sigma^2_{k-1} \geq h\right\}, \quad j \geq 1,$$

where $\inf\{\varnothing\} = \infty$, *and the sequence of random variables*

$$m_j(h) = \frac{1}{\sqrt{h}} \sum_{k=\tau_{j-1}+1}^{\tau_j} \alpha_k(h, j)\Delta M_k, \quad j = 1, 2, \ldots, \tag{1}$$

where

$$\alpha_k(h, j) = \begin{cases} 1 & \text{if} \quad \tau_{j-1}(h) < k < \tau_j(h), \\ \sqrt{\beta_j(h)} & \text{if} \quad k = \tau_j(h) \end{cases}$$

and $\beta_j(h)$ *are correcting multipliers*, $0 < \beta_j(h) \leq 1$, *determined by the equations*

$$\sum_{k=\tau_{j-1}+1}^{\tau_j - 1} \sigma^2_{k-1} + \beta_j(h)\sigma^2_{\tau_j(h)-1} = h.$$

Then, for any $h > 0$, $\{m_j(h)\}_{j\geq 1}$ *is a sequence of independent standard Gaussian random variables.*

Proof. First, we show that $m_1(h)$ is a Gaussian random variable with parameters (0, 1), i.e., it has the characteristic function

$$\varphi(u) = \mathrm{E}e^{im_1(h)u} = e^{-\frac{u^2}{2}}. \tag{2}$$

Introducing the sequence of truncated times $\bar{\tau}_1(h, N) = \tau_1(h) \wedge N$, $N = 1, 2, \ldots$, we define

National Research Tomsk State University, Tomsk, 634050 Russia
e-mail: vvkonev@mail.tsu.ru

$$\xi_N(h) = \frac{1}{\sqrt{h}} \sum_{k=1}^{\bar{\tau}_1(h,N)} \alpha_k(h,1)\Delta M_k.$$

Since $\lim_{N\to\infty} \xi_N(h) = m_1(h)$ a.s., we have

$$\varphi(u) = \lim_{N\to\infty} \mathrm{E}e^{iu\xi_N(h)}.$$

Here, $e^{iu\xi_N(h)}$ can be represented as

$$\mathrm{E}e^{iu\xi_N(h)} = e^{-\frac{u^2}{2}}\mathrm{E}e^{\xi_N^{(1)}(h,u)} + R_N, \quad (3)$$

where

$$R_N = \mathrm{E}e^{\xi_N^{(1)}(h,u)}\left(e^{-\xi_N^{(2)}(h,u)} - e^{-\frac{u^2}{2}}\right);$$

$$\xi_N^{(1)}(h,u) = \sum_{k=1}^{N}\left[\frac{iu\alpha_k(h,1)\Delta M_k}{\sqrt{h}}\chi_{\{k\le\tau(h)\}} + \frac{u^2\alpha_k^2(h,1)\sigma_{k-1}^2}{2h}\chi_{\{k\le\tau(h)\}}\right],$$

$$\xi_N^{(2)}(h,u) = \frac{u^2}{2h}\sum_{k=1}^{N}\alpha_k^2(h,1)\sigma_{k-1}^2\chi_{\{k\le\tau(h)\}}.$$

Since $|\exp(\xi_N^{(1)(h,u)})| \le e^{\frac{u^2}{2}}$ and $\lim_{N\to\infty}\xi_N^{(2)}(h,u) = \frac{u^2}{2}$, we have

$$\lim_{N\to\infty} R_N = 0. \quad (4)$$

Calculating the repeated conditional expectations and taking into account that the increments ΔM_k have a conditionally Gaussian distribution yields

$$\mathrm{E}e^{\xi_N^{(1)}(h,u)} = \mathrm{E}[E(e^{\xi_N^{(1)}(h,u)} \mid \mathcal{F}_{N-1})] = \mathrm{E}e^{\xi_{N-1}^{(1)(h,u)}} = \dots = 1.$$

Combining this equality and limit relation (4) with (3), we derive formula (2), i.e., $m_1(h) \sim N(0,1)$.

Now let us show that the characteristic function of the random vector $(m_1(h), \dots, m_l(h))$ for all $l = 2, 3, \dots$ has the form

$$\varphi_l(u) = \varphi(u_1, \dots, u_l) := \mathrm{E}e^{i\sum_{j=1}^{l} m_j(h)u_j} = e^{-\frac{1}{2}\sum_{j=1}^{l}u_j^2}.$$

Since

$$\varphi_l(u) = \mathrm{E}\left\{e^{i\sum_{j=1}^{l-1} m_j(h)u_j}\mathrm{E}[e^{im_l(h)u_l} \mid \mathcal{F}_{\tau_{l-1}}]\right\},$$

it is sufficient to check that

$$\mathrm{E}[e^{im_l(h)u_l} \mid \mathcal{F}_{\tau_{l-1}}] = e^{-\frac{u_l^2}{2}}, \quad l = 2, 3, \dots. \quad (5)$$

We have (see [4])

$$\mathrm{E}[e^{im_l(h)u_l} \mid \mathcal{F}_{\tau_{l-1}}] = \lim_{n\to\infty}\mathrm{E}[e^{im_l(h)u_l} \mid \mathcal{F}_{\tau_{l-1}\wedge n}];$$

moreover,

$$\begin{aligned}\mathrm{E}[e^{im_l(h)u_l} \mid \mathcal{F}_{\tau_{l-1}\wedge n}] &= \sum_{t=0}^{n}\mathrm{E}[e^{im_l(h)u_l} \mid \mathcal{F}_t]\chi_{\{\tau_{l-1}=t\}} \\ &= \sum_{t=0}^{n}\mathrm{E}[e^{\xi(h,l,t)}g \mid \mathcal{F}_t]\chi_{\{\tau_{l-1}=t\}};\end{aligned} \quad (6)$$

$$\xi(h,l,t) = \frac{i}{\sqrt{h}}\sum_{k=t+1}^{\tau_l}\alpha_k(h,l)\Delta M_k u_l.$$

Introducing the truncated times

$$\tau_l \wedge N, \quad N = t+1, t+2, \dots$$

and the sequence of random variables

$$\xi_N(h,l,t) = \frac{i}{\sqrt{h}}\sum_{k=t+1}^{\tau_l\wedge N}\alpha_k(h,l)\Delta M_k u_l, \quad N = t+1, \dots,$$

we find

$$\mathrm{E}[e^{\xi(h,l,t)} \mid \mathcal{F}_t] = \lim_{N\to\infty}\mathrm{E}[e^{\xi_N(h,l,t)} \mid \mathcal{F}_t] = e^{-\frac{u_l^2}{2}}. \quad (7)$$

Combining this relation with (6) yields (5). Theorem 1 is proved.

Remark 1. The Gaussian property of the random variable $m_1(h)$ seems to be a discrete analogue of a well-known property of Ito stochastic integrals stopped at a special time (see [2, Theorem 17.6]).

2. SEQUENTIAL ESTIMATION OF THE PARAMETER IN AR(1)

Consider some applications of Theorem 1 to sequential estimation problems.

Let an observed process $\{x_n\}_{n\ge1}$ be a first-order autoregressive process AR(1)

$$x_k = \theta x_{k-1} + \varepsilon_k, \quad k \ge 1, \quad x_0 = 0, \quad (8)$$

where θ is the unknown parameter, $\theta \in R$, and $\{\varepsilon_k\}_{k\ge1}$ is a sequence of independent standard Gaussian random variables. The MLE (LSE) of θ from the observations $x_1, \dots, x_n$ has the form

$$\hat{\theta}_n = \frac{\sum_{k=1}^{n}x_{k-1}x_k}{\sum_{k=1}^{n}x_{k-1}^2}. \quad (9)$$

An unbiased sequential estimator for θ in (8) with guaranteed mean-square accuracy was constructed in [8]. The sequential design is defined by the pair $(\tau(h), \hat{\theta}_{\tau(h)})$, where

$$\tau(h) = \inf\left\{n \geq 1: \sum_{k=1}^{n} x_{k-1}^2 \geq h\right\}, \quad \inf\{\phi\} = \infty, \quad (10)$$

and the estimator $\hat{\theta}_{\tau(h)}$ was a modified LSE of the form

$$\theta^*_{\tau(h)} = \frac{1}{h}\sum_{k=1}^{\tau(h)} \alpha_k x_{k-1} x_k, \quad (11)$$

where

$$\alpha_k = \begin{cases} 1 & \text{if } 1 \leq k < \tau(h), \\ \beta(h) & \text{if } k = \tau(h) \end{cases} \quad (12)$$

with $\beta(h)$ determined by the equation

$$\sum_{k=1}^{\tau(h)-1} x_{k-1}^2 + \beta(h) x_{\tau(h)-1}^2 = h.$$

Intending to apply Theorem 1, we define a somewhat different sequential design for the estimation of θ in (8). Let

$$\hat{\theta}_{\tau(h)} = \frac{1}{\tilde{h}}\sum_{k=1}^{\tau(h)} \sqrt{\alpha_k} x_{k-1} x_k, \quad (13)$$

where

$$\tilde{h} = \sum_{k=1}^{\tau(h)} \sqrt{\alpha_k} x_{k-1}^2,$$

while the stopping time $\tau(h)$ and the weighting coefficients α_k remain unchanged.

Theorem 2. *Let the sequential design* $(\tau(h), \hat{\theta}_{\tau(h)})$ *be defined by formulas* (10) *and* (13). *If* $(\varepsilon_k)_{k\geq 1}$ *is an i.i.d.* $\mathcal{N}(0,1)$, *then, for all* $h > 0$ *and* $\theta \in R$,

$$P_\theta\left\{\frac{\tilde{h}}{\sqrt{h}}[\hat{\theta}_{\tau(h)} - \theta] \leq z\right\} = \Phi(z), \quad -\infty < z < \infty,$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{u^2}{2}} du$.

Proof. Substituting x_k from Eq. (8) into (13) yields

$$\begin{aligned}\theta_{\tilde{\tau}(h)} &= \frac{\theta}{\tilde{h}}\sum_{k=1}^{\tau(h)} \sqrt{\alpha_k} x_{k-1}^2 + \frac{1}{\tilde{h}}\sum_{k=1}^{\tau(h)} \sqrt{\alpha_k} x_{k-1}\varepsilon_k \\ &= \theta + \frac{\sqrt{h}}{\tilde{h}}\frac{1}{\sqrt{h}}\sum_{k=1}^{\tau(h)} \sqrt{\alpha_k} x_{k-1}\varepsilon_k = \theta + \frac{\sqrt{h}}{\tilde{h}} m(h),\end{aligned} \quad (14)$$

where $m(h) = \frac{1}{\sqrt{h}}\sum_{k=1}^{\tau(h)} \sqrt{\alpha_k} x_{k-1}\varepsilon_k$. Introducing the martingale $M_n = \sum_{j=1}^{n} x_{j-1}\varepsilon_j$, we represent $m(h)$ in the form

$$m(h) = \frac{1}{\sqrt{h}}\sum_{k=1}^{\tau(h)} \sqrt{\alpha_k}\Delta M_k.$$

By Theorem 1, $m(h) \sim \mathcal{N}(0, 1)$.

From this and (14), we obtain the assertion of Theorem 2.

Corollary 1. *For all* $-\infty < \theta < \infty$ *and* $h > 0$,

$$E_\theta\left[\frac{\tilde{h}}{\sqrt{h}}(\hat{\theta}_{\tau(h)} - \theta)\right]^{2n} = (2n-1)!!,$$

$$E_\theta[\hat{\theta}_{\tau(h)} - \theta]^{2n} < \frac{(2n-1)!!}{h^n}, \quad n = 1, 2, \ldots .$$

This inequality holds, since $\frac{\tilde{h}}{h} \geq 1$.

Remark 2. The nonasymptotic estimates for the moments of estimate (9) are obtained in [3, 5].

3. SEQUENTIAL ESTIMATION OF THE PARAMETER IN AR(1): THE CASE OF VARIABLE VARIANCE

Let $\{x_k\}_{k\geq 0}$ be an AR(1) process with a variable noise variance:

$$x_k = \theta x_{k-1} + d_k \varepsilon_k, \quad (15)$$

where $(\varepsilon_k)_{k\geq 1}$ is an i.i.d. $\mathcal{N}(0, 1)$, the random variable x_0 is independent of $(\varepsilon_k)_{k\geq 1}$, $\theta \in R$, $(d_k)_{k\geq 1}$ is a given sequence of constant numbers, and $d_k \neq 0$.

Assume that

$$\sum_{k\geq 1}\frac{d_k^2}{d_{k+1}^2} = \infty. \quad (16)$$

The sequential design for the MLE-based estimation of θ is defined by the formulas

$$\tau(h) = \inf\left\{n \geq 1: \sum_{k=1}^{n}\frac{x_{k-1}^2}{d_k^2} \geq h\right\},$$

$$\hat{\theta}_{\tau(h)} = \frac{\sum_{k=1}^{\tau(h)} \sqrt{\alpha_k}\frac{x_{k-1}x_k}{d_k}}{\tilde{h}}, \quad (17)$$

where $\tilde{h} = \sum_{k=1}^{\tau(h)} \sqrt{\alpha_k}\frac{x_{k-1}^2}{d_k^2}$; $\alpha_k = 1$ if $k < \tau(h)$; and $\alpha_{\tau(h)} = \beta(h)$, where $\beta(h)$ is determined by the equation

$$\sum_{k=1}^{\tau(h)-1} \frac{x_{k-1}^2}{d_k^2} + \beta(h)\frac{x_{\tau(h)-1}^2}{d_{\tau(h)}^2} = h.$$

Theorem 3. *Under condition* (16), *for all* $\theta \in R$ *and* $h > 0$

$$\mathrm{P}_\theta\left\{\frac{\tilde{h}}{\sqrt{h}}[\theta_{\tau(h)} - \theta] \le z\right\} = \Phi(z), \quad -\infty < z < \infty.$$

Proof. Substituting x_k from Eq. (15) into (17) yields

$$\hat{\theta}_{\tau(h)} = \theta + \frac{\sqrt{h}}{\tilde{h}}m(h), \quad m(h) = \frac{1}{\sqrt{h}}\sum_{k=1}^{\tau(h)} \frac{\sqrt{\alpha_k}x_{k-1}\varepsilon_k}{d_k}. \quad (18)$$

Consider the martingale

$$M_n = \sum_{k=1}^{n} \frac{x_{k-1}\varepsilon_k}{d_k}.$$

It is well known (see [4]) that condition (16) is necessary and sufficient for

$$\langle M\rangle_\infty = \infty, \quad \mathrm{P}_\theta \text{ is a.s.,} \quad \theta \in R.$$

Expressing $m(h)$ in terms of M_n and applying Theorem 1, we obtain the claim of Theorem 3.

4. SEQUENTIAL ESTIMATION OF THE PARAMETER IN A GENERALIZED REGRESSION MODEL

Suppose that an observed process $\{x_k\}_{k\ge 0}$ satisfies the generalized regression model

$$x_k = (\theta + \eta_k)x_{k-1} + \varepsilon_k, \quad k \ge 1, \quad (19)$$

where $(\varepsilon_k)_{k\ge1}$ and $(\eta_k)_{k\ge1}$ are independent sequences of i.i.d. normal random variables, $\varepsilon_k \sim \mathcal{N}(0, 1)$, $\eta_k \sim \mathcal{N}(0, \sigma^2)$, $\sigma^2 > 0$; the random variable x_0 is independent of the processes (ε_k) and (η_k), and σ^2 is known. The parameter θ is unknown, and the task is to estimate it from the observations $x_0, x_1, x_2, \ldots$.

Equation (19) is reduced to the form

$$x_k = \theta x_{k-1} + \sqrt{1 + \sigma^2 x_{k-1}^2}\,\xi_k,$$

where ξ_k is an i.i.d. $\mathcal{N}(0, 1)$. Let $h > 0$. The sequential design $(\tau(h), \hat{\theta}_{\tau(h)})$ is defined as

$$\tau(h) = \inf\left\{n \ge 1: \sum_{k=1}^{n} \frac{x_{k-1}^2}{1 + \sigma^2 x_{k-1}^2} \ge h\right\}, \inf\{\varnothing\} = \infty,$$

$$\hat{\theta}_{\tau(h)} = \frac{1}{\tilde{h}}\sum_{k=1}^{\tau(h)} \sqrt{\alpha_k}\,\frac{x_{k-1}x_k}{1 + \sigma^2 x_{k-1}^2}, \quad (20)$$

where

$$\alpha_k = \begin{cases} 1 & \text{if } k < \tau(h), \\ \beta(h) & \text{if } k = \tau(h). \end{cases}$$

The quantity $0 < \beta(h) \le 1$ is determined by solving the equation

$$\sum_{k=1}^{\tau(h)-1} \frac{x_{k-1}^2}{1 + \sigma^2 x_{k-1}^2} + \beta(h)\frac{x_{\tau(h)-1}^2}{1 + \sigma^2 x_{\tau(h)-1}^2} = h;$$

$$\tilde{h} = \sum_{k=1}^{\tau(h)} \sqrt{\alpha_k}\,\frac{x_{k-1}^2}{1 + \sigma^2 x_{k-1}^2}.$$

Theorem 4. *For any* $\theta \in R$, $\sigma^2 \ge 0$, *and* $h > 0$,

$$\mathrm{P}_\theta\left\{\frac{\tilde{h}}{\sqrt{h}}\left(\tilde{\theta}_{\tau(h)} - \theta\right) \le z\right\} = \Phi(z), \quad -\infty < z < \infty.$$

The proof of Theorem 4 is similar to that of Theorem 3.

Corollary 2. *Corollary* 1 *to Theorem* 2 *holds for the moments of estimator* (20).

5. SEQUENTIAL ESTIMATION OF THE PARAMETER IN AR(1): THE CASE OF UNKNOWN VARIANCE

Suppose that an observed process AR(1) satisfies the equation

$$x_k = \theta x_{k-1} + \sigma\varepsilon_k, \quad k \ge 1, \quad (21)$$

where $\{\varepsilon_k\}$ is an i.i.d. $\mathcal{N}(0, 1)$, $\sigma \ne 0$, and the parameters θ and σ are unknown. To estimate θ in the case of an unknown σ, the following sequential design $(\tau(n, h), \tilde{\theta}(h, n))$ is introduced for every $h > 0$:

$$\tau = \tau(n, h) = \inf\left\{m \ge n + 1: \sum_{k=n+1}^{m} x_{k-1}^2 \ge hA_nR_n\right\},$$

$$\tilde{\theta}(h, n) = \frac{1}{\tilde{h}A_nR_n}\sum_{k=n+1}^{\tau} \sqrt{\alpha_k}\,x_{k-1}x_k. \quad (22)$$

Here, R_n is a statistic replacing the unknown variance σ^2. Let $R_n = \sum_{j=1}^{n} x_{j-1}^2$, and let $\alpha_k = 1$ if $n < k < \tau$ and $\alpha_\tau = \beta(h)$. The multiplier $0 < \beta(h) \le 1$ is determined by the equation

$$\sum_{k=n+1}^{\tau-1} x_{k-1}^2 + \beta(h)x_{\tau-1}^2 = hA_nR_n;$$

$$\tilde{h}A_nR_n = \sum_{k=n+1}^{\tau} \sqrt{\alpha_k}\,x_{k-1}^2.$$

Let $A_n = \frac{1}{2}\left(\frac{\Gamma\left(\frac{n-1}{2}-p\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^{1/p}$, $n > 2p+1$. Define the filtration $\mathcal{F}_0 = \sigma(x_0)$, $\mathcal{F}_n = \sigma(x_0, \eta_1, \ldots, \eta_n, \varepsilon_1, \ldots, \varepsilon_n)$, $n \geq 1$.

Theorem 5. *For any* $-\infty < \theta < \infty$ *and* $\sigma \neq 0$, *estimator* (22) *has the following properties for all* $h > 0$ *and* $n > 2p+1$:

$$\mathrm{Law}\left(\frac{\tilde{h}}{\sqrt{h}}(\hat{\theta}(h,n)-\theta) \mid \mathcal{F}_{n-1}\right) = \mathcal{N}\left(0, \frac{\sigma^2}{A_n R_n}\right),$$

$$\mathrm{E}_{\theta,\sigma}(\tilde{\theta}(h,n)-\theta)^{2p} \leq \frac{(2p-1)!!}{h^p}. \tag{23}$$

Proof. In view of (21), estimator (22) is transformed into

$$\hat{\theta}(h,n) = \theta + \frac{\sigma\sqrt{h}}{\tilde{h}\sqrt{A_n R_n}} m(h,n), \tag{24}$$

where $m(h,n) = (hA_nR_n)^{-1/2}\sum_{k=n+1}^{\tau}\sqrt{\alpha_k}x_{k-1}\varepsilon_k$.

By Theorem 1,

$$\mathrm{Law}(m(h,n) \mid \mathfrak{F}_{n-1}) = \mathcal{N}(0, 1). \tag{25}$$

Combining this relation with (24) yields the first claim of the theorem. From (24) and (25), for the conditional moments, we obtain the formula

$$E_{\theta,\sigma}\left\{\left[\hat{\theta}(h,n)-\theta\right]^{2p}\left(\frac{\tilde{h}}{h}\right)^{2p} \mid \mathcal{F}_{n-1}\right\} = \frac{(2p-1)!!\sigma^{2p}}{h^p(A_nR_n)^p}.$$

Since $\tilde{h} \geq h$,

$$\mathrm{E}_{\theta,\sigma}\left[\hat{\theta}(h,n)-\theta\right]^{2p} \leq \frac{(2p-1)!!}{h^p A_n^p}\mathrm{E}_{\theta,\sigma}\frac{\sigma^{2p}}{R_n^p}. \tag{26}$$

Applying Anderson's lemma [10], we obtain

$$\mathrm{E}_{\theta,\sigma}\frac{\sigma^{2p}}{R_n^p} \leq \mathrm{E}_{\theta,\sigma}\frac{\sigma^{2p}}{\sum_{l=1}^{n-1}x_l^2} \leq \mathrm{E}_{\theta,\sigma}\frac{\sigma^{2p}}{\left(\sigma^2\sum_{l=1}^{n-1}\varepsilon_l^2\right)^p} = \mathcal{J}_{n,p},$$

where $\mathcal{J}_{n,p} = \frac{\Gamma\left(\frac{n-1}{2}-p\right)}{2^p\Gamma\left(\frac{n-1}{2}\right)}$ and $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$ is the Euler gamma function. Combining this relation with (21) and taking into account the choice of A_n yields inequality (23). Theorem 5 is proved.

ACKNOWLEDGMENTS

This work was supported by the Science Foundation of Tomsk State University in 2015–2016 (grant no. 8.1.55.2015) and by the Russian Foundation for Basic Research (project no. 16-01-00121A).

REFERENCES

1. A. A. Novikov, Teor. Veroyatn. Ee Primen. **16** (2), 394–396 (1971).
2. R. S. Liptser and A. N. Shiryaev, *Statistics of Random Processes* (Nauka, Moscow, 1974; Springer-Verlag, Berlin, 1977/1978).
3. A. N. Shiryaev and V. G. Spokoiny, *Statistical Experiments and Decision: Asymptotic Theory* (World Scientific, Singapore, 2000).
4. A. N. Shiryaev, *Probability* (Nauka, Moscow, 1984; Springer-Verlag, Berlin, 1994).
5. P. W. Mikulski and M. J. Monsour, J. Time Ser. Anal. **12** (3), 235–253 (1991).
6. T. L. Lai and D. Siegmund, Ann. Stat. **11**, 478–485 (1983).
7. P. E. Greenwood and A. N. Shiryaev, Stoch. Stoch. Rep. **38**, 49–56 (1992).
8. V. Z. Borisov and V. V. Konev, Autom. Remote Control **38** (10), 1475–1480 (1977).
9. S. M. Pergamenshchikov, Theory Probab. Appl. **36** (1), 36–49 (1991).
10. T. W. Anderson, Proc. Am. Math. Soc. **8** (2), 170–176 (1955).
11. S. M. Pergamentschikov and A. N. Shiryaev, Ann. Acad. Sci. Fenn. Ser. A.I. Math. **17**, 111–116 (1992).

Translated by I. Ruzanova