Quantum curves and q–deformed Painlevé equations

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ABSTRACT: We propose that the grand canonical topological string partition functions satisfy finite-difference equations in the closed string moduli. In the case of genus one mirror curve these are conjectured to be the q-difference Painlevé equations as in Sakai's classification. More precisely, we propose that the tau-functions of q-Painlevé equations are related to the grand canonical topological string partition functions on the corresponding geometry. In the toric cases we use topological string/spectral theory duality to give a Fredholm determinant representation for the above tau-functions in terms of the underlying quantum mirror curve. As a consequence, the zeroes of the tau-functions compute the exact spectrum of the associated quantum integrable systems. We provide details of this construction for the local $\mathbb{P}^1 \times \mathbb{P}^1$ case, which is related to q-difference Painlevé with affine A_1 symmetry, to $SU(2)$ Super Yang-Mills in five dimensions and to relativistic Toda system.

Contents

1   Introduction and Summary

During the last decades an intriguing relationship has been observed between the low-energy dynamics of $\mathcal{N} = 2$ four-dimensional gauge theories and integrable systems [1–4]. The use of localization techniques in the supersymmetric path integral considerably clarified and widened this relation in many directions [5, 6]. In this context, a precise connection between supersymmetric partition functions and tau functions of isomonodromic deformation problems associated to the

Seiberg-Witten geometry has been established leading to the Painlevé/ $SU(2)$ gauge correspondence [7–16]. More precisely, it was found that τ functions of differential Painlevé equations are computed by the Nekrasov–Okounkov (NO) [5] partition functions of four dimensional $SU(2)$, $\mathcal{N} = 2$ gauge theories in the self–dual phase of the Ω background. The specific matter content of the gauge theory determines the type of Painlevé equation (see Table 2 in [13] for the precise relation). Moreover the long/short distance expansions of Painlevé equations are in correspondence with the duality frames in the gauge theory [13, 17].

On the other hand, by resorting to the geometric engineering of gauge theories via topological strings [18], it has been possible to show in some cases [17, 19] that these tau functions are Fredholm (or spectral) determinants of quantum operators arising in a suitable limit of the non-perturbative topological string formulation of [20, 21]. This embedding into topological strings has allowed to take a first step through the generalisation of the Painlevé/ $SU(2)$ gauge correspondence to the higher rank case providing explicit Fredholm determinant representation for the $SU(N)$ theories [19], see also [22]. Furthermore from the string theory viewpoint it is also natural to consider the five dimensional version of this correspondence as pointed out in [13, 17, 23] and further studied in [24, 25][1]. On the Painlevé side this corresponds to a lift from differential to difference equations. The latter arises as a q–deformation of Painlevé equations and we refer to them as q-Painlevé (q–P) equations, see [28, 29] for a review and a list of references. On the gauge theory side instead this corresponds to a lift from four dimensional $SU(2)$, $\mathcal{N} = 2$ gauge theories to topological string on local CY manifolds. In particular one uses topological string theory to compute tau-functions of q-Painlevé equations as defined and studied in [28, 30].

Although we expect the correspondence between topological strings and Painlevé to hold in general for all Painlevé equations in Sakai's classification [30] (see Table 1), in this work we will focus on the ones with a *toric* topological string realization (see Fig. 1). We will give a prescription

Painlevé type	Physical theory
Elliptic	E-strings
Multiplicative	Topological string on local del Pezzo's.
Additive	4-dimensional $SU(2)$ gauge theories

Table 1. On the left: classification of Painlevé equations according to [28, 30]. The additive type correspond to the standard differential Painlevé equations plus the three finite additive cases corresponding to Minahan-Nemeshanski four-dimensional gauge theories [31]. The multiplicative cases correspond to q-difference Painlevé (see [28, 30] for more details). On the right: physical theory that we expect to compute the tau functions of Painlevé equations. In the multiplicative case one has to consider blow up of del Pezzo up to 8 blow up, while the Elliptic Painlevé makes contact with $\frac{1}{2}K3$. The analogy between Sakai's scheme for Painlevé equations and the geometry underling the above physical theories was originally suggested in [32].

to construct a Fredholm determinant representation of such tau-functions starting from the geometrical formulation of q-Painlevé presented in [33, 34]. As we will outline in section 2, the first step consists in associating a Newton polygon to these difference equations as illustrated for instance in Fig. 1. Such a polygon can then be related to the toric diagram of a corresponding Calabi–Yau (CY) manifold. We will conjecture that the Fredholm determinant of the operator arising in the quantization of its mirror curve computes the tau-function of the corresponding q-

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[1] In [25], based on [26, 27], a different type of finite dimensional determinant was considered to compute τ functions.

difference Painlevé equation. As a consequence the zeros of such tau-function compute the exact spectrum of the integrable systems associated to the underlying mirror curve [35, 36]. In this way we also provide a concrete link between q-Painlevé equations and the topological string/spectral theory (TS/ST) duality [20]. We remark that from the topological string viewpoint q-Painlevé equations control the dependence of the partition function on the *closed* string moduli. This is in line with the expectation of exact quantum background independence which should be fulfilled by a non-perturbative formulation of topological string theory [37][2].

The structure of the paper is the following. In Sect. 2 we outline the general features of the correspondence between topological strings, spectral theory and q-Painlevé. In the subsequent sections we work out explicitly the example of the q-difference Painlevé III$_3$ [3] which makes contact with topological strings on local $\mathbb{P}^1 \times \mathbb{P}^1$. In this example the TS/ST duality [20] states that

$$\Xi_{\mathbb{P}^1 \times \mathbb{P}^1}^{\mathrm{TS}}(\kappa, \xi, \hbar) = \det\left(1 + \kappa \rho_{\mathbb{P}^1 \times \mathbb{P}^1}\right), \tag{1.1}$$

where

$$\Xi_{\mathbb{P}^1 \times \mathbb{P}^1}^{\mathrm{TS}}(\kappa, \xi, \hbar) = \sum_{n \in \mathbb{Z}} e^{\mathsf{J}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu + 2\pi i n, \xi, \hbar)} \quad \kappa = e^{\mu} \tag{1.2}$$

is the grand canonical topological string partition function, $\mathsf{J}_{\mathbb{P}^1 \times \mathbb{P}^1}$ is the topological string grand potential (see appendix A) and $\rho_{\mathbb{P}^1 \times \mathbb{P}^1}$ is the operator arising in the quantisation of the mirror curve to local $\mathbb{P}^1 \times \mathbb{P}^1$ (see equation (3.1)). In section 3 we show that (1.2) satisfies the q-difference Painlevé III$_3$ equation in the τ form. As a consequence this provides a conjectural Fredholm determinant solution for the corresponding τ function whose explicit expression is given on the r.h.s. of (1.1). As shown in [17] it exists a suitable limit in which

$$\det\left(1 + \kappa \rho_{\mathbb{P}^1 \times \mathbb{P}^1}\right) \tag{1.3}$$

reduces to a well known the determinant computing the tau function of the standard Painlevé III$_3$ [39, 40]. From that perspective our result can be viewed as a generalisation of [39, 40] for the q-deformed Painlevé equations.

In section 3.4 we discuss the q-deformed algebraic solution associated to such a Fredholm determinant representation, while in section 4 and 5 we test by explicit computations that the r.h.s. of (1.1) indeed fulfils the q-difference Painlevé III$_3$ equation in the τ form. Moreover, in section 5 we connect our results with ABJ theory by relating q-Painlevé equations to Wronskian-like relations of [41]. Section 6 is devoted to conclusions and open problems. In the appendices we collect some technical results and definitions.

2 Generalities

In this work we propose that spectral determinants of operators arising in the quantization of mirror curve to CY manifolds compute τ functions of q-deformed Painlevé equations when some

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[2]We would like to thank Marcos Mariño for a discussion on this point.

[3]In Sakai's classification [30] this corresponds to surface type $A_7^{(1)'}$ and symmetry type $A_1^{(1)}$, where the superscript (1) stands for affine extension of the Dynkin algebra. Notice that this is not the unique q-P equation leading to differential Painlevé III$_3$ in the continuous limit. Indeed, as pointed out for instance in [38], also the q-P equation with surface type $A_7^{(1)}$ and symmetry type $A_1^{(1)'}$ makes contact with Painlevé III$_3$. This is perfectly consistent with the picture developed in this paper since the corresponding Newton polygon is identified with local $\mathbb{F}_1$ (see Fig. 1). By the geometric engineering construction [18] we know that topological string theory on both $\mathbb{F}_1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ reduces to pure $SU(2)$ theory in four dimensions. However in this work we denote by q-P III$_3$ only the one associated to surface type $A_7^{(1)'}$ and symmetry type $A_1^{(1)}$.
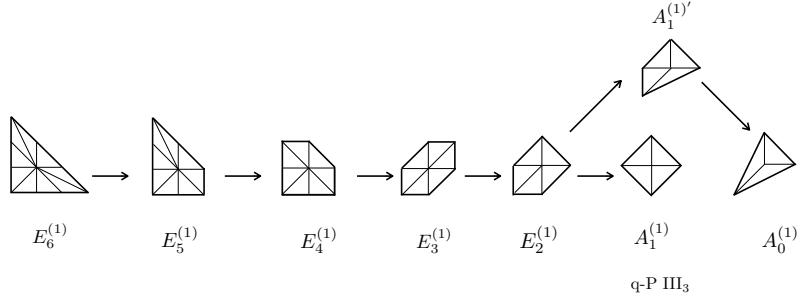
Figure 1. The letters $E^{...}_{...}$ and $A^{...}_{...}$ refers to symmetry type classification of q-Painlevé according to [28, 30]. The Newton polygons on the upper line are the ones associated to the q-Painlevé according to [33, 34]. This correspondence is not always unique as discussed in Section 2.2.

particular initial conditions are imposed. We start by reviewing some results which are relevant for this proposal.

2.1 Topological string and spectral theory

In this section we review the results of [20] in a form which is suitable for the propose of this work. Let us consider a toric CY X with genus one mirror curve [4]. By following [42, 43] the complex moduli of the mirror curve to X are divided in two classes of parameters: one "true" modulus, denoted by κ, and r_X mass parameters, denoted by

$$\boldsymbol{m}_X = \left\{ m_X^{(1)}, \cdots, m_X^{(r_X)} \right\}. \tag{2.1}$$

We introduce the rescaled mass parameters $\mathbf{m}$ as [17, 44]

$$\log m^{(i)} = \frac{2\pi}{\hbar} \log \left(m_X^{(i)} \right), \tag{2.2}$$

as well as

$$\xi^{(i)} = \log m^{(i)}. \tag{2.3}$$

The Newton polygon of X is defined as the convex hull of a set of two-dimensional vectors [18, 45–47]

$$\vec{\nu}^{(i)} = \left(\nu_1^{(i)}, \nu_2^{(i)} \right), \quad i = 1, \cdots, k, \tag{2.4}$$

from which one reads the mirror curve to X as

$$\sum_{i=1}^{k} e^{\nu_1^{(i)}x + \nu_2^{(i)}p + f_i(\mathbf{m}_X)} + \kappa = 0, \quad x, p \in \mathbb{C} \tag{2.5}$$

where f_i is a function of the mass parameters. For instance when X is the canonical bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ its Newton polygon is shown in Fig. 2. The corresponding vectors are

$$\vec{\nu}^{(1)} = \{1, 0\}, \quad \vec{\nu}^{(2)} = \{-1, 0\}, \quad \vec{\nu}^{(3)} = \{0, 1\}, \quad \vec{\nu}^{(4)} = \{0, -1\}. \tag{2.6}$$

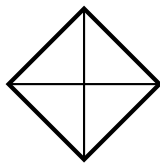[4]For the higher genus generalisation see [21].

Figure 2. Newton polygon of local $\mathbb{P}^1 \times \mathbb{P}^1$

Therefore the mirror curve reads

$$\mathrm{e}^x + m_{\mathbb{P}^1 \times \mathbb{P}^1}\mathrm{e}^{-x} + \mathrm{e}^p + \mathrm{e}^{-p} + \kappa = 0. \tag{2.7}$$

We introduce the quantum operators x, p such that

$$[\mathsf{x}, \mathsf{p}] = \mathrm{i}\hbar, \tag{2.8}$$

and we promote the mirror curve (2.5) to a quantum operator by using Weyl quantization. The resulting operator is

$$\mathsf{O}_X = \sum_{i=1}^k \mathrm{e}^{f_i(\mathbf{m}_X)}\mathrm{e}^{\nu_1^{(i)}\mathsf{x}+\nu_2^{(i)}\mathsf{p}}. \tag{2.9}$$

An explicit list of these operators can be found in Table 1 of [20]. It was conjectured in [20], and later proved in [48, 49] for many geometries, that the inverse

$$\rho_X = \mathsf{O}_X^{-1} \tag{2.10}$$

is a self-adjoint, positive[5] and trace class operator acting on $L^2(\mathbb{R})$. Therefore its spectral, or Fredholm, determinant

$$\Xi_X^{\mathrm{ST}}(\kappa, \boldsymbol{\xi}, \hbar) = \det\left(1 + \kappa\rho_X\right) \tag{2.11}$$

is analytic in κ. The operator (2.10) is related to the Hamiltonian of a corresponding integrable system [35, 36]. The conjecture of [20] states that the spectral determinant (2.11) of operators arising in the quantization of the mirror curves to X is computed by (refined) topological string theory on X. This has led to a new and exact relation between the spectral theory of quantum mechanical operators and enumerative geometry/topological string theory. We refer to it as TS/ST duality. Let us define the grand canonical topological string partition function on X as

$$\Xi_X^{\mathrm{TS}}(\kappa, \boldsymbol{\xi}, \hbar) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathsf{J}_X(\mu+2\pi\mathrm{i}n,\boldsymbol{\xi},\hbar)}, \qquad \kappa = \mathrm{e}^\mu, \qquad \hbar \in \mathbb{R}_+ \tag{2.12}$$

where we denote by J_X the topological string grand potential studied in [50–54]. By following [21, 55] we write J_X as

$$\mathsf{J}_X(\mu, \boldsymbol{\xi}, \hbar) = \mathsf{J}_X^{\mathrm{WS}}(\mu, \boldsymbol{\xi}, \hbar) + \mathsf{J}_X^{\mathrm{WKB}}(\mu, \boldsymbol{\xi}, \hbar), \tag{2.13}$$

where $\mathsf{J}_X^{\mathrm{WS}}$ is expressed in terms of unrefined topological string while $\mathsf{J}_X^{\mathrm{WKB}}$ is determined by the Nekrasov–Shatashvili (NS) limit of the refined topological string. The precise definitions can be

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[5]Provided some positivity constraints are imposed on the mass parameters and $\hbar$.

found in appendix A. We would like to stress that, even though both J_X^{WKB} and J_X^{WS} have a dense set of poles on the real $\hbar$ axis, their sum is well defined and free of poles. This is the so–called HMO cancellation mechanism [51] which was first discovered in the context of ABJM theory and has played an important role in the TS/ST duality presented in [20]. One can also write (2.12) as [20]

$$\Xi_X^{\text{TS}}(\kappa, \boldsymbol{\xi}, \hbar) = \mathrm{e}^{J_X(\mu, \boldsymbol{\xi}, \hbar)} \Theta_X(\mu, \boldsymbol{\xi}, \hbar), \tag{2.14}$$

where Θ_X defines a quantum theta function which, for some specific values of $\hbar$, becomes a conventional theta function. This happen for instance when $\hbar = 2\pi/m$ for $m \in \mathbb{N}$ as explained in [20, 56, 57]. The conjecture [20] states that

$$\Xi_X^{\text{TS}}(\kappa, \boldsymbol{\xi}, \hbar) = \Xi_X^{\text{ST}}(\kappa, \boldsymbol{\xi}, \hbar). \tag{2.15}$$

Even though we still do not have a rigorous mathematical proof of (2.15), many aspect and consequences of this proposal have been successfully tested and proved in severals examples both numerically and analytically [17, 19–21, 44, 48, 57–75].

Originally the above construction was formulated only for real values of $\hbar$. Nevertheless it was found in [76] that when the underling geometry can be used to engineer gauge theories one can easily extend some aspects of [20] to generic complex value of $\hbar$. However in this work we will focus on the real case.

2.1.1 Self–dual point

It was pointed out in [20, 56] that there is a particular value of $\hbar$ where the grand potential simplify drastically, this occurs at $\hbar = 2\pi$ and we refer to it as self–dual point. At this point J_X is determined only by genus zero and genus one free energies (A.10), (A.13). Therefore at this point we can express the spectral determinant (2.11) in closed form in term of hypergeometric, Meijer and theta functions. Modular properties of the spectral determinant around this point have been discussed in [65]. The explicit expression for $J_X(\mu, \boldsymbol{\xi}, 2\pi)$ and $\Xi_X^{\text{TS}}(\kappa, \boldsymbol{\xi}, 2\pi)$ for generic X at the self–dual point can be found in section 3.4 of [20] or section of 3.2 of [21]. As explained in these references, the genus one free energy appears as an overall multiplication factor, while the dependence on the genus zero free energy and its derivatives is non trivial. In this section we re-write the details only for two explicit examples, which we will use later. These cases have been worked out in [56].

We first consider local $\mathbb{P}^1 \times \mathbb{P}^1$ with $\xi = 2\pi\mathrm{i}$ [6]. Then we have [56] [7]

$$\Xi^{\text{TS}}(\kappa, 2\pi, 2\pi) = \exp\left[\mathcal{A}(\mathrm{i}\kappa)\right] \vartheta_3(\xi(\mathrm{i}\kappa), \tau(\mathrm{i}\kappa)), \tag{2.16}$$

where

$$\mathcal{A}(\kappa) = \frac{\log \kappa}{4} + \mathcal{F}_1 + F_1^{\text{NS}} - \frac{1}{\pi^2}\left(\mathcal{F}_0(\lambda) - \lambda\partial_\lambda\mathcal{F}_0(\lambda) + \frac{\lambda^2}{2}\partial_\lambda^2\mathcal{F}_0(\lambda)\right). \tag{2.17}$$

We denote by F_1^{NS} the NS genus one free energy (A.13) which in the present example reads

$$F_1^{\text{NS}} = -\frac{1}{24}\log\left(16 + \kappa^2\right) - \frac{\log(\kappa)}{12}. \tag{2.18}$$

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[6]This correspond to ABJM theory with level $k = 2$ [56]. In the ABJM context the self–dual point correspond to an enhancement of the supersymmetry from $\mathcal{N} = 6$ to $\mathcal{N} = 8$.

[7]For sake of notation we will simply denote $\Xi^{\text{TS}}(\kappa, \xi, \hbar)$ instead of $\Xi_{\mathbb{P}^1 \times \mathbb{P}^1}^{\text{TS}}(\kappa, \xi, \hbar)$

Moreover $\mathcal{F}_0$ and $\mathcal{F}_1$ are the genus zero and genus one free energies in the orbifold frame. These can be obtained from F_0 and F_1 in (A.10) by using modular transformation and analytic continuation as explained in [77–79]. More precisely we have

$$\partial_\lambda \mathcal{F}_0(\lambda) = \frac{\kappa}{4} G_{3,3}^{2,3} \left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & -\frac{1}{2} \end{matrix} \middle| -\frac{\kappa^2}{16} \right) + \frac{\pi^2 \mathrm{i} \kappa}{2} \, {}_3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16} \right), \tag{2.19}$$

with

$$\mathcal{F}_0(\lambda) = -4\pi^2 \lambda^2 \left(\log(2\pi\lambda) - \frac{3}{2} - \log(4) \right) + \cdots \tag{2.20}$$

and we denote by λ the quantum Kähler parameter at $\hbar = 2\pi$ in the orbifold frame namely

$$\lambda = \frac{\kappa}{8\pi} \, {}_3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16} \right). \tag{2.21}$$

Our convention for the hypergeometric functions $G_{3,3}^{2,3}(\cdot)$ and ${}_3F_2(\cdot)$ in (2.19) and (2.21) are as in [56]. As for the genus one free energy we have

$$\mathcal{F}_1 = -\log \eta \left(2\tau \right) - \frac{1}{2} \log 2, \tag{2.22}$$

where η is the Dedekind eta function and we used

$$\tau(\kappa) = -\frac{1}{8\pi^3 \mathrm{i}} \partial_\lambda^2 \mathcal{F}_0(\lambda) = -\frac{1}{2} + \frac{\mathrm{i} K \left(\frac{\kappa^2}{16} + 1 \right)}{2K \left(-\frac{\kappa^2}{16} \right)}, \tag{2.23}$$

where $K(\kappa^2)$ is the elliptic integral of first kind. In (2.16) we denote by ϑ_3 the Jacobi theta function

$$\vartheta_3(v, \tau) = \sum_{n \in \mathbb{Z}} \exp \left[\pi \mathrm{i} n^2 \tau + 2\pi \mathrm{i} n v \right] \tag{2.24}$$

and we define

$$\xi(\kappa) = \frac{\mathrm{i}}{4\pi^3} \left(\lambda \partial_\lambda^2 \mathcal{F}_0(\lambda) - \partial_\lambda \mathcal{F}_0(\lambda) \right). \tag{2.25}$$

Noticed that many of the quantities defined above have branch cuts, however according to [56] these should cancel and the final answer (2.16) is analytic in κ.

Likewise for local $\mathbb{P}^1 \times \mathbb{P}^1$, $\hbar = 2\pi$ and $\xi = 0$ [8] we have [56]

$$\Xi^{\mathrm{TS}}(\kappa, 0, 2\pi) = \exp \left[\frac{\log 2}{2} - \frac{\log \mathrm{i} \kappa}{4} + \mathcal{A}(\mathrm{i}\kappa) \right] \vartheta_1 \left(\xi(\mathrm{i}\kappa) + \frac{1}{4}, \tau(\mathrm{i}\kappa) \right), \tag{2.26}$$

where

$$\vartheta_1(v, \tau) = \sum_{n \in \mathbb{Z}} (-1)^{n-1/2} \exp \left[\pi \mathrm{i} \left(n + 1/2 \right)^2 \tau + 2\pi \mathrm{i} \left(n + 1/2 \right) v \right]. \tag{2.27}$$

2.2 Spectral theory and q-Painlevé

In the present work we give a concrete link between q-Painlevé equations and the TS/ST duality of [20]. Our proposal is the following. As explained in [33, 34] one can associate a Newton polygon to a class of q-Painlevé equations in Sakai's classification (see Fig. 1). Such polygons represent the rational surfaces which are used to classify Painlevé equations in [30]. Once the Newton polygon
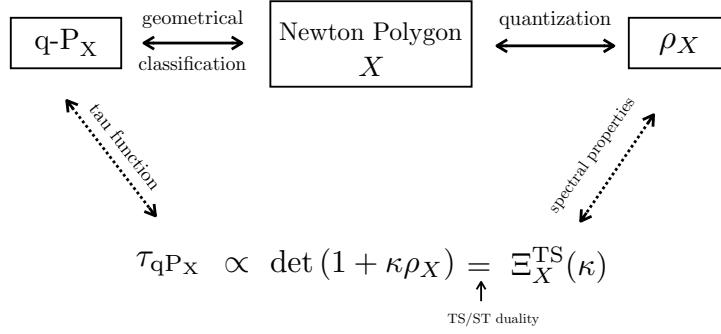
Figure 3. A schematic representation of the correspondence between q-Painlevé equations, spectral theory and topological strings. We denoted by ρ_X the operator (2.10) and by Ξ_X^{TS} the grand canonical partition function of topological string (2.12).

of a given q-Painlevé equation has been identified, we can apply the quantisation procedure presented in section 2.1. We expect the resulting Fredholm determinant (2.11) to compute the τ function of the given q–Painlevé equation. This is schematically represented on Fig. 3. In this work we will test this proposal in detail for the q–PIII$_3$, however it would be important to test this expectation for other q–Painlevé equations as well.

We would like to observe that the correspondence between q–Painlevé equations and Newton polygons is not always unique. Indeed given a q–Painlevé equation, the idea of [33, 34] is that one can naturally associate to it a Newton polygon by looking at the integral curves, or conserved Hamiltonians, in the so-called autonomous limit. However, as discussed in [33], such integral curve admits different realisations. In turn, this is related to the fact that, as explained in [28], some of the q-Painlevé admit a classification in terms of both blow ups of $\mathbb{P}^1 \times \mathbb{P}^1$ or blow ups of $\mathbb{P}^2$. For instance, in the case of q-PVI (surface type $A_2^{(1)}$ and symmetry type $E_6^{(1)}$ in Sakai classification [30]) there are two different ones leading to the two different Newton polygons depicted in Fig. 4. Analogously, for the q-Painlevé III$_3$ one can consider either $\mathbb{P}^1 \times \mathbb{P}^1$ or
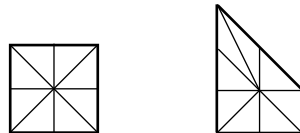

Figure 4. Two Newton polygons associated to q-PVI according to [33, 80]

$\mathbb{F}_2$ surfaces. Interestingly it was pointed out in [44, 59] that the spectral problems arising in the quantisation of the mirror curve to local $\mathbb{P}^1 \times \mathbb{P}^1$ and local $\mathbb{F}_2$ are equivalent. Therefore, upon a suitable normalisation and identification of the parameters, the Fredholm determinants associated to these two manifolds are identified. This is in perfect agreement with the proposal of this paper since they both compute the tau function of the same q-Painlevé equation. It would be interesting to see if the same identification holds also for other polygons describing the same

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[8]This correspond to ABJ theory with level $k = 2$ and gauge group $U(N) \times U(N + 1)$[56].

q-Painlevé equation such as for instance the ones in Fig. 4. [9]

As an additional comment we notice that the self-dual point introduced in section 2.1.1 has a natural meaning from the q–Painlevé perspective. Indeed the q–Painlevé equations can be studied in the so–called autonomous limit $q = 1$ [80–83] in which they are expected to be solvable by elliptic theta functions[10]. From the TS/ST perspective $q = \mathrm{e}^{4\pi^2\mathrm{i}/\hbar}$ and $q = 1$ corresponds to

$$\hbar = \frac{2\pi}{m} \quad m \in \mathbb{N}. \tag{2.28}$$

It is possible to show [20, 56, 57] that for these values the quantum theta function in (2.14) becomes a conventional theta function. Moreover for $m = 1, 2$ the full spectral determinant can be expressed simply in terms of hypergeometric, Meijer and Jacobi theta functions as illustrated in section 2.1.1. Such simplifications are expected from the q–Painlevé viewpoint since $q = 1$ correspond to the autonomous case.

3 Fredholm determinant solution for q-Painlevé III$_3$

Let us consider the canonical bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. Its mirror curve is given by (2.7) and the corresponding operator is

$$\rho_{\mathbb{P}^1 \times \mathbb{P}^1} = \left(\mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}} + m_{\mathbb{P}^1 \times \mathbb{P}^1}\mathrm{e}^{-\mathsf{x}}\right)^{-1}, \quad [\mathsf{x}, \mathsf{p}] = \mathrm{i}\hbar. \tag{3.1}$$

The rescaled mass parameter m (2.2) is given by

$$\log m_{\mathbb{P}^1 \times \mathbb{P}^1} = \frac{\hbar}{2\pi}\log m = \frac{\hbar}{2\pi}\xi. \tag{3.2}$$

Then according to the proposal of [20] we have

$$\det\left(1 + \kappa\rho_{\mathbb{P}^1 \times \mathbb{P}^1}\right) = \sum_{n \in \mathbb{Z}}\mathrm{e}^{\mathsf{J}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu + 2\pi\mathrm{i}n, \xi, \hbar)}, \quad \kappa = \mathrm{e}^{\mu} \tag{3.3}$$

where $\mathsf{J}_{\mathbb{P}^1 \times \mathbb{P}^1}$ is given in appendix A. In this section by using the results of [23] we show that, up to a normalisation factor, the term

$$\sum_{n \in \mathbb{Z}}\mathrm{e}^{\mathsf{J}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu + 2\pi\mathrm{i}n, \xi, \hbar)} \tag{3.4}$$

satisfies the q-deformed Painlevé III$_3$ in the τ form. As a consequence equation (3.3) provides a Fredhom determinant representation for the τ function of q-PIII$_3$.

3.1 The Bershtein–Shchechkin approach

It was observed in [23] that the τ function of q-deformed Painlevé III$_3$ can be written as

$$\tau_{\mathrm{C}}(u, Z, q) = \sum_{n \in \mathbb{Z}}s^n C(uq^{2n}, q, Z)\frac{\mathcal{Z}(uq^{2n}, Z, q^{-1}, q)}{(uq^{2n+1}, q, q)_\infty\,(u^{-1}q^{-2n+1}, q, q)_\infty}, \tag{3.5}$$

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[9] After the preliminary version of this work was sent to Yasuhiko Yamada, he proved the equality between the two spectral problems arising in the quantisation of the two polygons in Fig. 4.

[10] We thank Yasuhiko Yamada for discussions on this point.

where $\mathcal{Z}$ are the $SU(2)$ q-deformed conformal blocks at $c = 1$. As explained in appendix B these correspond to the instanton partition function of topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover $(\cdot, \cdot, \cdot)_\infty$ denotes the Pochhammer symbol and C is a quite generic function. The variable s characterises the initial condition of the tau function and for the pourpouse of this paper we will always consider $s = 1$. In [23] it was conjectured and tested that $\tau_C(u, Z, q)$ satisfies the q–deformed Painlevé III$_3$ equation in the τ form, namely

$$Z^{1/4} \tau_C(u, q, qZ) \tau_C(u, q, q^{-1}Z) = \tau_C(u, q, Z)^2 + Z^{1/2} \tau_C(uq, q, Z) \tau_C(uq^{-1}, q, Z) \qquad (3.6)$$

provided C fulfils [23]

$$\frac{C(uq, q, Z)C(uq^{-1}, q, Z)}{C(u, q, Z)^2} = -Z^{1/2},$$

$$\frac{C(uq, q, qZ)C(uq^{-1}, q, q^{-1}Z)}{C(u, q, Z)^2} = -uZ^{1/4}, \qquad (3.7)$$

$$\frac{C(u, q, qZ)C(u, q, q^{-1}Z)}{C(u, q, Z)^2} = Z^{-1/4}.$$

The solution to these difference equations is clearly not unique and in [23] several C-functions have been proposed. In the next section we show that it is possible to chose C in such a way that $\tau_C(u, Z, q)$ is the spectral determinant of (3.1) in the form conjectured in [20]. This choice express C in terms of q-deformed conformal blocks at $c = \infty$. Notice also that in [23], in order to ensure good convergence properties of (3.5) for a generic function C, they had to assume $| q | \neq 1$. However, as we will see in the following, by using an appropriate definition of C one can make sense of (3.5) when $| q |= 1$, which is the case studied in this paper [11]. This is important in order to study the autonomous limit of qP equations.

3.2 Tau function and quantum curve

In this section we explain how to relate the results of [23] to the TS/ST conjecture of [20]. As discussed previously the relevant geometry which makes contact with the q-Painlevé III$_3$ studied in [23] is the canonical bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. In particular to connect (3.5) and the r.h.s. of (3.3) we use the following dictionary

$$Z^{-1} = e^\xi, \quad q = e^{i4\pi^2/\hbar}, \qquad u = e^\xi Q_b, \quad Q_b = e^{-\frac{2\pi}{\hbar} t(\mu, \xi, \hbar)}, \qquad (3.8)$$

where $t(\mu, \xi, \hbar)$ denotes the quantum mirror map namely

$$t(\mu, \xi, \hbar) = 2\mu - 2(m_{\mathbb{P}^1 \times \mathbb{P}^1} + 1)\kappa^{-2} + \kappa^{-4} \left(-3m_{\mathbb{P}^1 \times \mathbb{P}^1}^2 - \frac{2m_{\mathbb{P}^1 \times \mathbb{P}^1} \left(e^{2i\hbar} + 4e^{i\hbar} + 1 \right)}{e^{i\hbar}} - 3 \right) + O\left(\kappa^{-6} \right),$$

$$\kappa = e^\mu, \quad m_{\mathbb{P}^1 \times \mathbb{P}^1} = e^{\frac{\hbar}{2\pi}\xi}. \qquad (3.9)$$

When $\hbar$ is real, one can check numerically that the series (3.9) has a finite radius of convergence (see for instance [54]). For generic complex values of $\hbar$ one has to perform a partial resumation in $m_{\mathbb{P}^1 \times \mathbb{P}^1}$ but it is still possible to organise (3.9) into a convergent series as discussed in [76]. Notice that

$$t(\mu, \xi \pm \frac{4\pi^2 i}{\hbar}, \hbar) = t(\mu, \xi, \hbar). \qquad (3.10)$$

---

[11] On the operator side our result can be generalised straightforwardly for any $\hbar$, however on the topological string side one has to be more careful to ensure the good convergence properties of Ξ_X^{TS} (see [76] for more details).

By using the dictionary (3.8) we have that shifting

$$u \to q^n u, \quad n \in \mathbb{Z} \tag{3.11}$$

while leaving Z invariant in (3.6), (3.7) is equivalent to

$$\mu \to \mu - \mathrm{i} n \pi, \quad n \in \mathbb{Z}, \tag{3.12}$$

in the language of section 2.1, where we used

$$t(\mu, \xi, \hbar) + 2n\pi \mathrm{i} = t(\mu + n \mathrm{i} \pi, \xi, \hbar). \tag{3.13}$$

Likewise shifting

$$Z \to q^n Z, \quad n \in \mathbb{Z} \tag{3.14}$$

while leaving u invariant is equivalent to

$$\mu \to \mu - \mathrm{i} n \pi, \quad \text{and} \quad \xi \to \xi - 4 n \mathrm{i} \pi^2 / \hbar \quad n \in \mathbb{Z}. \tag{3.15}$$

Therefore by using the dictionary (3.8) together with the above considerations we can write (3.6) as

$$e^{-\xi/4} \tau_{\mathrm{C}}(\mu - \mathrm{i}\pi, \hbar, \xi - \frac{4\mathrm{i}\pi^2}{\hbar}) \tau_{\mathrm{C}}(\mu + \mathrm{i}\pi, \hbar, \xi + \frac{4\mathrm{i}\pi^2}{\hbar}) = \tau_{\mathrm{C}}(\mu, \hbar, \xi)^2$$
$$+ e^{-\xi/2} \tau_{\mathrm{C}}(\mu + \mathrm{i}\pi, \hbar, \xi) \tau_{\mathrm{C}}(\mu - \mathrm{i}\pi, \hbar, \xi). \tag{3.16}$$

In the rest of the paper we will use the notation $\tau_C(\mu, \hbar, \xi)$ and $\tau_C(u, q, Z)$ interchangeably. We define $C_0(u, q, Z)$ such that

$$\frac{C_0(u, q, Z) \mathcal{Z}(u, Z, q^{-1}, q)}{(uq; q, q)_\infty (u^{-1}q; q, q)_\infty} = e^{\mathsf{J}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu, \xi, \hbar) + \mathsf{J}_{\mathrm{CS}}\left(\mathrm{i}\pi + \frac{1}{2}\xi, \frac{2\pi^2}{\hbar}\right)}, \tag{3.17}$$

where the variables on the two sides are related by the dictionary (3.8). Moreover $\mathsf{J}_{\mathbb{P}^1 \times \mathbb{P}^1}$ is the topological string grand potential (2.13) for the canonical bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ and we denoted by J_{CS} the non–perturbative Chern–Simons free energy, which can be identified with the topological string grand potential of the resolved conifold [84] (see appendix A for the full definition). We also denote

$$Z_{\mathrm{CS}}(\hbar, \xi) = \exp\left[J_{\mathrm{CS}}\left(\mathrm{i}\pi + \frac{1}{2}\xi, \frac{2\pi^2}{\hbar}\right)\right]. \tag{3.18}$$

By using the results of appendix C it is easy to see that $C_0(u, q, Z)$ defined as in (3.17) fulfils (3.7). Therefore

$$\tau_{\mathrm{C}_0}(\mu, \hbar, \xi) = Z_{\mathrm{CS}}(\hbar, \xi) \sum_{n \in \mathbb{Z}} e^{\mathsf{J}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu + 2\pi \mathrm{i} n, \xi, \hbar)} \tag{3.19}$$

satisfies the q- Painlevé III$_3$ equation given in (3.16). By using the conjectural expression for the spectral determinant given in (3.3) together with (3.19), it follows that

$$\tau_{\mathrm{C}_0}(\mu, \hbar, \xi) = Z_{\mathrm{CS}}(\hbar, \xi) \det\left(1 + \kappa \rho_{\mathbb{P}^1 \times \mathbb{P}^1}\right). \tag{3.20}$$

Hence this choice of $C_0(u, q, Z)$ provides a Fredholm determinant representation for the τ function of the q-Painlevé III$_3$ equation. This representation can be thought as a generalisation of [39, 40]

for the q-difference equation since in the limit $\hbar \to \infty$ (3.20) reproduces the solution to differential Painlevé III$_3$ presented in [39, 40] (see section 3.5).

At this point the following question arises: can we prove directly that the r.h.s of (3.20) satisfy (3.16) without using the TS/ST duality namely without using the expression of

$$\det\left(1 + \kappa \rho_{\mathbb{P}^1 \times \mathbb{P}^1}\right) \tag{3.21}$$

in terms of enumerative invariants given in (3.3)? Even though we do not have a proof of it, in sections 4 and 5 we will test this by direct analytical an numerical computations.

3.3 The self–dual point

As we will see in section 3.4, $Z_{\text{CS}}(\hbar, \xi)$ satisfy (3.38). In particular this means that we can replace (3.16) with a more explicit equation for [12]

$$\Xi^{\text{TS}}(\kappa, \xi, \hbar) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{\mathsf{J}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu + 2\pi \mathrm{i} n, \xi, \hbar)}. \tag{3.22}$$

More precisely (3.16) becomes

$$\Xi^{\text{TS}}\left(-\kappa, \xi - \frac{4\mathrm{i}\pi^2}{\hbar}, \hbar\right)\Xi^{\text{TS}}\left(-\kappa, \xi + \frac{4\mathrm{i}\pi^2}{\hbar}, \hbar\right)(1 + \mathrm{e}^{-\xi/2}) = \Xi^{\text{TS}}(\kappa, \xi, \hbar)^2 + \mathrm{e}^{-\xi/2}\Xi^{\text{TS}}(-\kappa, \xi, \hbar)^2 \quad . \tag{3.23}$$

At the self dual point $\hbar = 2\pi$ and for $\xi = 0$ this reads

$$2\Xi^{\text{TS}}(-\kappa, -2\pi\mathrm{i}, 2\pi)\Xi^{\text{TS}}(-\kappa, 2\pi\mathrm{i}, 2\pi) = \Xi^{\text{TS}}(\kappa, 0, 2\pi)^2 + \Xi^{\text{TS}}(-\kappa, 0, 2\pi)^2. \tag{3.24}$$

By using the results of section 2.1.1 it is easy to see that (3.24) becomes simply an identity between theta functions (we take $\text{Re}(\kappa) > 0$)

$$\begin{aligned}
\vartheta_3\left(\xi(\mathrm{i}\kappa), \tau(\mathrm{i}\kappa)\right)\vartheta_4\left(\xi(\mathrm{i}\kappa), \tau(\mathrm{i}\kappa)\right) =& \frac{\mathrm{i}}{\sqrt{\mathrm{i}\kappa}}\left(\vartheta_1(\xi(\mathrm{i}\kappa) + \frac{1}{4}, \tau(\mathrm{i}\kappa))\right)^2 \\
&+ \frac{\mathrm{i}}{\sqrt{\mathrm{i}\kappa}}\left(\vartheta_1(\xi(\mathrm{i}\kappa) - \frac{1}{4}, \tau(\mathrm{i}\kappa))\right)^2,
\end{aligned} \tag{3.25}$$

where

$$\vartheta_4(v, \tau) = \sum_{n \in \mathbb{Z}}(-1)^n \exp[\mathrm{i}\pi n^2 \tau + 2\pi \mathrm{i} n v] \tag{3.26}$$

while the others quantities have been defined in section 2.1.1. We also used

$$\Xi^{\text{TS}}(-\kappa, -2\pi\mathrm{i}, 2\pi) = \Xi^{\text{TS}}(\kappa, 2\pi\mathrm{i}, 2\pi). \tag{3.27}$$

Hence in this particular case the q-Painlevé equation leads to

$$\frac{\eta^8\left(4\tau(\kappa)\right)}{\eta^8(\tau(\kappa))} = \frac{\kappa^2}{256}. \tag{3.28}$$

where we used

$$\eta(\tau) = \mathrm{e}^{\mathrm{i}\pi\tau/12}\prod_{n=1}^{\infty}\left(1 - \mathrm{e}^{2\pi \mathrm{i} n\tau}\right) \tag{3.29}$$

---

[12]For sake of notation we omit the subscript $\mathbb{P}^1 \times \mathbb{P}^1$ in Ξ^{TS}

as well as the following theta function identity [13]

$$\frac{\vartheta_1(v+1/4,\tau)^2 + \vartheta_1(v-1/4,\tau)^2}{4\mathrm{e}^{\mathrm{i}\pi\tau/2}\vartheta_3(v,\tau)\vartheta_4(v,\tau)} = \prod_{n=1,n\notin 4\mathbb{N}}^{\infty}(1-\mathrm{e}^{2ni\pi\tau})^{-2}. \qquad (3.30)$$

Let us look at (3.28) as an equation for τ. We can write it as as

$$j(2\tau) = \frac{\left(\kappa^4 + 16\kappa^2 + 256\right)^3}{\kappa^4\left(\kappa^2+16\right)^2} \qquad (3.31)$$

where

$$j(2\tau) = \frac{\left(256\Delta_\eta^{16} + 16\Delta_\eta^8 + 1\right)^3}{\Delta_\eta^{16}\left(16\Delta_\eta^8 + 1\right)^2}, \qquad \Delta_\eta = \frac{\eta\left(4\tau(\kappa)\right)}{\eta(\tau\left(\kappa\right))}. \qquad (3.32)$$

This is the well known expression for the j-invariant function as quotient of η functions and can be easily derived by using the identities in appendix D. Hence (3.31) is the well known relation between j-invariant and the modular parameter of the elliptic curve describing the mirror curve to local $\mathbb{P}^1 \times \mathbb{P}^1$ whose solution is known to be (2.23). [14] Therefore in the self-dual case the q-Painlevé equation reduces to the well known relation (3.31) which define the prepotential of the underling geometry. To capture all the gravitational corrections instead one should consider q-Painleve with $q \neq 1$.

3.4 The q-deformed algebraic solution

One of the immediate consequences of representing the τ function as a spectral determinant is that one can easily obtain the corresponding algebraic solution studied for instance in [85, 86]. In the case of differential Painlevé III$_3$ the τ function is characterised by the initial conditions [15]

$$(\sigma, \eta) \qquad (3.33)$$

which correspond to the monodromy data of the related Fuchsian system. When $\eta = 0$ the τ function admits the following spectral determinant representation [17, 39]

$$\tau(\sigma, T) = \mathrm{e}^{-\frac{\log(2)}{12} - 3\zeta'(-1)}T^{1/16}\mathrm{e}^{-4\sqrt{T}}\det\left(1 + \frac{\cos(2\pi\sigma)}{2\pi}\rho_{4\mathrm{D}}\right) \qquad (3.34)$$

where

$$\rho_{4\mathrm{D}} = \mathrm{e}^{-4T^{1/4}\cosh(\mathsf{x})}\frac{4\pi}{\left(\mathrm{e}^{\mathsf{p}/2} + \mathrm{e}^{-\mathsf{p}/2}\right)}\mathrm{e}^{-4T^{1/4}\cosh(\mathsf{x})}, \qquad [\mathsf{x},\mathsf{p}] = 2\pi\mathrm{i}. \qquad (3.35)$$

In particular when $\sigma = 1/4$ we have

$$\tau(1/4, 0, T) = \mathrm{e}^{-\frac{\log(2)}{12} - 3\zeta'(-1)}T^{1/16}\mathrm{e}^{-4\sqrt{T}}. \qquad (3.36)$$

which reproduces the well known algebraic solution for Painlevé III$_3$ [23, 85].

Similarly for the q-Painlevé equations the Fredholm determinant representation makes contact with the q-analogue of the algebraic solution when $\kappa = 0$. From (3.20) it follows that

$$\log\tau_{C_0}(\mu, \hbar, \xi)\mid_{\kappa=0} = \log Z_{\mathrm{CS}}(\hbar, \xi). \qquad (3.37)$$

---

[13]We thank Yasuhiko Yamada for bringing our attention on this identity.
[14]We thank Jie Gu for discussions on this point.
[15]We follow the notation of [8].

Hence the q-difference Painlevé III$_3$ at $\kappa = 0$ reads

$$Z^{\mathrm{CS}}\left(\hbar, \xi + 4\pi^2\mathrm{i}/\hbar\right) Z^{\mathrm{CS}}\left(\hbar, \xi - 4\pi^2\mathrm{i}/\hbar\right) = Z^{\mathrm{CS}}\left(\hbar, \xi\right)^2 \left(\mathrm{e}^{\xi/4} + \mathrm{e}^{-\xi/4}\right). \tag{3.38}$$

By using (A.24), (A.25) together with (C.13), (C.14) it is easy to verify that (3.38) is indeed satisfied. As expected, up to an overall $\hbar$ dependent normalisation, the solution (3.37) reproduces the q-deformed algebraic solution of [23] provided we choose C in (3.5) as in (3.17). At the self–dual point $\hbar = 2\pi$ we have a particularly nice expression (we suppose $0 < \mathrm{e}^{-\xi} < 1$) (A.26)

$$\begin{aligned}
\log Z^{\mathrm{CS}}\left(2\pi, \xi\right) = &\frac{\mathrm{Li}_3(\mathrm{e}^{-\xi})}{8\pi^2} + \frac{\mathrm{Li}_2(\mathrm{e}^{-\xi})\xi}{8\pi^2} - \frac{\xi^3}{96\pi^2} - \frac{\log(1 - \mathrm{e}^{-\xi})\xi^2}{16\pi^2} \\
&- \frac{\xi}{16} - \frac{1}{8}\log(1 - \mathrm{e}^{-\xi}) - \frac{1}{8}\log\left(\frac{\mathrm{e}^{-\xi/2} + 1}{1 - \mathrm{e}^{-\xi/2}}\right) + \frac{A_c(4)}{2},
\end{aligned} \tag{3.39}$$

where A_c is defined in (A.22). It is easy to see that (3.39) fulfils (3.38).

We note that, to our knowledge, the generic Riemann–Hilbert problem behind q-deformed Painlevé equations has not been found so far. However the algebraic solution to q-III$_3$ is the grand potential of the resolved conifold as defined in [54]. We observe that in [87, 88] the partition function of the resolved conifold arises as a tau function of a given Riemann–Hilbert problem. It would be interesting to see if it exists a generalisation for the Riemann–Hilbert problem of [87, 88] whose tau function makes contact with the Fredholm determinants appearing in [20] namely (2.11),(2.15) and therefore with q-deformed Painlevé equations.

3.5 The continuous limit

We are now going to explain how to obtain the standard differential Painlevé III$_3$ starting from the q-Painlevé III$_3$ written in the form (3.23). It is more convenient to work with the variables

$$\xi, \quad Q_f = \mathrm{e}^{\xi}\mathrm{e}^{-\frac{2\pi}{\hbar}t(\mu,\xi,\hbar)}. \tag{3.40}$$

Then we write (3.23) as

$$\begin{aligned}
\Xi(Q_f, \xi - \frac{4\mathrm{i}\pi^2}{\hbar}, \hbar)\Xi(Q_f, \xi + \frac{4\mathrm{i}\pi^2}{\hbar}, \hbar)(1 + \mathrm{e}^{-\xi/2}) = {}&\Xi(Q_f, \xi, \hbar)^2 \\
&+ \mathrm{e}^{-\xi/2}\Xi(\mathrm{e}^{4\pi^2\mathrm{i}/\hbar}Q_f, \xi, \hbar)\Xi(\mathrm{e}^{-4\pi^2\mathrm{i}/\hbar}Q_f, \xi, \hbar).
\end{aligned} \tag{3.41}$$

We omit the overscripts $\cdot^{\mathrm{TS}}$ or $\cdot^{\mathrm{ST}}$ because what follows holds both for Ξ^{TS} and for Ξ^{ST}. We write

$$\begin{aligned}
\Xi(Q_f, \xi - \frac{4\mathrm{i}\pi^2}{\hbar}, \hbar)\Xi(Q_f, \xi + \frac{4\mathrm{i}\pi^2}{\hbar}, \hbar) = {}&\Xi(Q_f, \xi, \hbar)^2 + \\
\left(\frac{4\pi^2\mathrm{i}}{\hbar}\right)^2 \Big(-(\partial_\xi\Xi(Q_f, \xi, \hbar))^2 &+ \Xi(Q_f, \xi, \hbar)\partial_\xi^2\Xi(Q_f, \xi, \hbar)\Big) + \mathcal{O}(\hbar^{-4}).
\end{aligned} \tag{3.42}$$

The continuous limit leading to differential Painlevé III$_3$ in the gauge theory language corresponds to the dual 4d limit introduced in [17, 19]. Let us introduce a new set of variable (T, a) such that

$$\xi = a\beta\epsilon - \log\left((4\pi^2)^4\beta^4 T\epsilon^4\right), \quad Q_f = \mathrm{e}^{a\beta\epsilon}. \tag{3.43}$$

and

$$\hbar = \frac{1}{\beta\epsilon}. \tag{3.44}$$

Hence

$$\partial_\xi = -T\partial_T \tag{3.45}$$

The continuous limit is obtained by sending

$$\beta \to 0. \tag{3.46}$$

More precisely it was shown in [17] that in this case one has

$$\Xi^{\mathrm{TS}}(\mathrm{e}^{\alpha 4\pi^2 \mathrm{i}/\hbar}Q_f, \xi, \hbar) \quad \xrightarrow{\beta \to 0} \quad \Xi_{4\mathrm{d}}^{\mathrm{TS}}(\sigma, T) = \mathrm{e}^{\frac{\log(2)}{12} + 3\zeta'(-1)} T^{-1/16} \mathrm{e}^{4\sqrt{T}} Z^{\mathrm{NO}}(\sigma + \frac{\alpha}{2}, T) \tag{3.47}$$

where $a = 8\pi^2 \mathrm{i}\sigma$ and $Z^{\mathrm{NO}}(\sigma, T)$ is the Nekrasov–Okounkov partition function for pure $\mathcal{N} = 2$, $SU(2)$ gauge theory in the selfdual Ω background. More precisely

$$Z^{\mathrm{NO}}(\sigma, T) = \sum_{n\in\mathbb{Z}} \frac{T^{(\sigma+n)^2} B(T, \sigma+n)}{G(1 - 2(\sigma+n))G(1 + 2(\sigma+n))},$$

$$B(T, \sigma) = \left(1 + \frac{T}{2\sigma^2} + \frac{(8\sigma^2 + 1)T^2}{4\sigma^2(1 - 4\sigma^2)^2} + \mathcal{O}(T^3)\right). \tag{3.48}$$

Likewise it was shown in [17] that

$$\Xi^{\mathrm{ST}}(\mathrm{e}^{\alpha 4\pi^2 \mathrm{i}/\hbar}Q_f, \xi, \hbar) \quad \xrightarrow{\beta \to 0} \quad \Xi_{4\mathrm{d}}^{\mathrm{ST}}(\sigma, T) = \det\left(1 + \kappa(\sigma + \frac{\alpha}{2})\rho_{4\mathrm{d}}\right) \tag{3.49}$$

where

$$\kappa(\sigma) = \frac{\cos(2\pi\sigma)}{2\pi} \tag{3.50}$$

and $\rho_{4\mathrm{D}}$ are defined in (3.35). In particular both Ξ^{ST} and Ξ^{TS} are well defined in the limit $\beta \to 0$. In this limit (3.42) reads

$$\Xi_{4\mathrm{d}}(\sigma, T)^2 + (4\pi^2 \mathrm{i}\epsilon\beta)^2 \left(-(T\partial_T\Xi_{4\mathrm{d}}(\sigma, T))^2 + \Xi_{4\mathrm{d}}(\sigma, T)(T\partial_T)^2\Xi_{4\mathrm{d}}(\sigma, T)\right) + \mathcal{O}(\beta^4). \tag{3.51}$$

Hence (3.23) becomes

$$16\pi^4\sqrt{T}\epsilon^2 \left(\Xi_{4\mathrm{d}}(\sigma, T)^2 - \Xi_{4\mathrm{d}}(\sigma + \frac{1}{2}, T)^2 - \Xi_{4\mathrm{d}}(\sigma, T)T^{3/2}\partial_T^2\Xi_{4\mathrm{d}}(\sigma, T)\right.$$

$$\left. + T^{3/2}(\partial_T\Xi_{4\mathrm{d}}(\sigma, T))^2 - \Xi_{4\mathrm{d}}(\sigma, T)\sqrt{T}\partial_T\Xi_{4\mathrm{d}}(\sigma, T)\right) + \mathcal{O}(\beta) = 0 \tag{3.52}$$

By keeping the leading order in β and defining

$$\tau(\sigma, T) = \mathrm{e}^{-4\sqrt{T}}\Xi_{4\mathrm{d}}(\sigma, T) \tag{3.53}$$

we can write (3.52) as

$$-\tau(\sigma, T)^2 \left(T\frac{\mathrm{d}}{\mathrm{d}T}\right)^2 \log\tau(\sigma, T) - \sqrt{T}\tau(\sigma + \frac{1}{2}, T)^2 = 0, \tag{3.54}$$

which is the Painlevé III$_3$ in the τ form. More precisely equation (3.54) is also called the Toda-like form of Painlevé III$_3$, see for instance [86]. Notice that if we take

$$\tau(\sigma, T) = \mathrm{e}^{-4\sqrt{T}}\Xi_{4\mathrm{d}}^{\mathrm{TS}}(\sigma, T), \tag{3.55}$$

then we recover the results of [8] relating Painlevé III$_3$ to pure $SU(2)$ gauge theory. Likewise by setting

$$\tau(\sigma, T) = e^{-4\sqrt{T}} \Xi_{4d}^{ST}(\sigma, T), \tag{3.56}$$

we make contact with the solution to Painlevé III$_3$ of [39, 40]. As explained in [17] the expressions (3.55) and (3.56) are different representation of the same function, namely the τ function of differential Painlevé III$_3$.

4 The q-deformed Painlevé III$_3$ and matrix models

In this section we focus on the operator side of the TS/ST duality and we formulate the results of section 3 by using the operator theory/matrix models point of view. We first recall that by using (3.38) and (3.19) we can write the q-deformed Painlevé III$_3$ (3.16) in the form (3.23). This means that at the level of the spectral determinant

$$\Xi^{ST}(\kappa, \xi, \hbar) = \det(1 + \kappa \rho_{\mathbb{P}^1 \times \mathbb{P}^1}) \tag{4.1}$$

the q-deformed Painlevé III$_3$ equation reads

$$\Xi^{ST}(-\kappa, \xi - \frac{4i\pi^2}{\hbar}, \hbar) \Xi^{ST}(-\kappa, \xi + \frac{4i\pi^2}{\hbar}, \hbar)(1 + e^{-\xi/2}) = \Xi^{ST}(\kappa, \xi, \hbar)^2 + e^{-\xi/2} \Xi^{ST}(-\kappa, \xi, \hbar)^2. \tag{4.2}$$

In the forthcoming sections we will perform several tests of (4.2), even though a proof is still missing.

4.1 Matrix model representation

By using standard results in Fredholm theory we express the determinant (4.1) in terms of fermionic spectral traces

$$Z(N, \hbar, \xi) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int d^N x \prod_{i=1}^N \rho_{\mathbb{P}^1 \times \mathbb{P}^1}(x_i, x_{\sigma(i)}), \tag{4.3}$$

as

$$\Xi^{ST}(\kappa, \hbar, \xi) = \sum_{N \geq 0} \kappa^N Z(N, \hbar, \xi) \tag{4.4}$$

where S_N in (4.3) is the group of permutations of N elements and $\rho_{\mathbb{P}^1 \times \mathbb{P}^1}(x, y)$ is the kernel of (3.1). Furthermore by using the Cauchy identity we can write (4.3) as an $O(2)$ matrix model [44]

$$Z(N, \hbar, \xi) = \frac{e^{-\frac{\hbar}{8\pi} N\xi}}{N!} \int \frac{d^N z}{(2\pi)^N} e^{-\sum_{i=1}^N (V(z_i, \hbar, \xi))} \frac{\prod_{i<j}(z_i - z_j)^2}{\prod_{i,j}(z_i + z_j)}, \tag{4.5}$$

where the integral is over the positive real axis and the potential is given by

$$e^{-V(z, \hbar, \xi)} = e^{\frac{b^2 \log z}{2}} \frac{\Phi_b(\frac{b \log z}{2\pi} - \frac{b}{8\pi}\xi + ib/4) \, \Phi_b(\frac{b \log z}{2\pi} + \frac{b}{8\pi}\xi + ib/4)}{\Phi_b(\frac{b \log z}{2\pi} + \frac{b}{8\pi}\xi - ib/4) \, \Phi_b(\frac{b \log z}{2\pi} - \frac{b}{8\pi}\xi - ib/4)}. \tag{4.6}$$

We use

$$\hbar = \pi b^2 \tag{4.7}$$

and Φ_b denotes the Fadeev quantum dilogarithm [89, 90]. Our conventions for Φ_b are as in [44]. Let us define

$$
\begin{aligned}
V_\pm(z, \hbar, \xi) &= V(z, \hbar, \xi \pm 4\pi^2 i/\hbar), \\
Z_\pm(N, \hbar, \xi) &= Z(N, \hbar, \xi \pm 4\pi^2 i/\hbar).
\end{aligned}
\tag{4.8}
$$

Then we have

$$
Z_\pm(N, \hbar, \xi) = e^{-\frac{\hbar}{8\pi} N \xi} \frac{(\mp i)^N}{N!} \int \frac{d^N z}{(2\pi)^N} e^{-\sum_{i=1}^N (V_\pm(z_i, \hbar, \xi))} \frac{\prod_{i<j}(z_i - z_j)^2}{\prod_{i,j}(z_i + z_j)}.
\tag{4.9}
$$

It follows that the q-difference Painlevé equation (4.2) is equivalent to the following relation between the matrix models $Z_\pm(N, \hbar, \xi)$ and $Z(N, \hbar, \xi)$

$$
\begin{aligned}
&\sum_{N_1=0}^N Z(N_1, \hbar, \xi) Z(N - N_1, \hbar, \xi) \left(1 + e^{-\xi/2}(-1)^N\right) \\
&= \sum_{N_1=0}^N Z_+(N_1, \hbar, \xi) Z_-(N - N_1, \hbar, \xi) \left((-1)^N + e^{-\xi/2}(-1)^N\right).
\end{aligned}
\tag{4.10}
$$

Let us check this equation in the simplest example, namely $N = 1$. We write (4.10) as

$$
2(1 - e^{-\xi/2}) Z(1, \hbar, \xi) = -(1 + e^{-\xi/2})(Z_+(1, \hbar, \xi) + Z_-(1, \hbar, \xi)).
\tag{4.11}
$$

By using the properties of the quantum dilogarithm (see for instance appendix A in [91]) it is easy to see that

$$
\begin{aligned}
e^{-V_-(z, \hbar, \xi)} &= e^{-V_-(z^{-1}, \hbar, \xi)} \\
e^{-V_+(z, \hbar, \xi)} &= -e^{-V_-(z, \hbar, \xi)} \left(\frac{\left(e^{\xi/4} + iz\right)\left(e^{\xi/4} z + i\right)}{\left(e^{\xi/4} - iz\right)\left(e^{\xi/4} z - i\right)}\right) \\
e^{-V(e^{i\pi/b^2} z, \hbar, \xi)} &= \frac{1 - ie^{\xi/4} z}{e^{\xi/4} z - i} e^{-V_-(z, \hbar, \xi)}.
\end{aligned}
\tag{4.12}
$$

Moreover

$$
\int_{\mathbb{R}_+} \frac{dz}{z} e^{-V(e^{i\pi/b^2} z, \hbar, \xi)} = \int_{\mathbb{R}_+} \frac{dz}{z} e^{-V(z, \hbar, \xi)}, \qquad \xi, b \in \mathbb{R}.
\tag{4.13}
$$

Hence (4.11) is equivalent to

$$
\int_{\mathbb{R}_+} \frac{dz}{z} H(z, \xi) e^{-V_-(z, \hbar, \xi)} = 0,
\tag{4.14}
$$

where

$$
H(z, \xi) = \left(-\frac{2 e^{-\frac{\xi}{4}}\left(e^{\xi/2} - 1\right)\left(z^2 - 1\right)}{\left(e^{\xi/4} - iz\right)\left(e^{\xi/4} z - i\right)}\right).
\tag{4.15}
$$

Since $H(z, \xi) = -H(1/z, \xi)$ it follows that (4.14) indeed holds.

4.2 Comment on TBA

In the previous sections we found that the spectral determinant of the operator (3.1) fulfils the q-deformed Painlevé III$_3$ equation in the τ form. Moreover it is possible to show that such a determinant is determined by a TBA system [39, 92]. Let us review how this goes. We consider a trace class operator whose kernel is of the form

$$\rho(x, y) = \frac{\mathrm{e}^{-u(x)-u(y)}}{4\pi \cosh\left(\frac{x-y}{2}\right)}. \tag{4.16}$$

Then the Fredholm determinant of (4.16) is determined by a set of TBA equations whose explicit form can be found in [39]. If we set

$$u(x) = t \cosh(x) \tag{4.17}$$

we can make contact with the solution (3.34) of Painlevé III$_3$ as explained in [39]. Let us take instead

$$u(x) = -\log\left| f\left(\frac{bx}{2\pi}\right)\right| + \frac{b^2}{16}\xi \tag{4.18}$$

where

$$f(x) = \mathrm{e}^{\pi x b/2} \frac{\Phi_{\mathsf{b}}(x - \frac{b}{8\pi}\xi + \mathrm{i}b/4)}{\Phi_{\mathsf{b}}(x + \frac{b}{8\pi}\xi - \mathrm{i}b/4)}, \qquad \hbar = \pi b^2 \tag{4.19}$$

and Φ_{b} denotes the Faddeev's quantum dilogarithm. With the choice (4.18) the kernel (4.16) is related by unitary transformation to the one of (3.1) (see [44]) and therefore to $q-$P III$_3$. This connection with TBA system can be exploited to compute $Z(N, m, \hbar)$ in (4.5) exactly for finite values of $N, m, \hbar$. Some examples are provided in the next sections.

We also observe that in [60] the authors constructed a TBA system which determines the Fredholm determinant of the operator arising by quantizing the mirror to local $\mathbb{P}^2$. It would be interesting to understand if one can systematically construct a TBA for each q-deformed Painlevé equation.

4.3 The self–dual point

At the self–dual point $\hbar = 2\pi$ and for $\xi = 0$ equation (4.10) gives a relation between the following $O(2)$ matrix integrals:

$$Z(N, 0, 2\pi) = \frac{1}{N!} \int \frac{\mathrm{d}^N z}{(4\pi)^N} \left(\prod_{i=1}^{N} \frac{1}{(z_i + 1)^2}\right) \prod_{i<j} \left(\frac{z_i - z_j}{z_i + z_j}\right)^2 \tag{4.20}$$

$$Z_{\pm}(N, 0, 2\pi) = \frac{(\mp\mathrm{i})^N}{N!} \int \frac{\mathrm{d}^N z}{(4\pi)^N} \left(\prod_{i=1}^{N} \frac{1}{(z_i^2 + 1)}\right) \prod_{i<j} \left(\frac{z_i - z_j}{z_i + z_j}\right)^2. \tag{4.21}$$

Both these matrix integrals have been evaluated exactly for various values of N in [51, 93] . For the first few values we have

$$Z(1, 0, 2\pi) = \frac{1}{4\pi}, \quad Z(2, 0, 2\pi) = \frac{1}{128}\left(1 - \frac{8}{\pi^2}\right), \quad Z(3, 0, 2\pi) = \frac{5\pi^2 - 48}{4608\pi^3}, \tag{4.22}$$

$$Z_{+}(1, 0, 2\pi) = -\frac{\mathrm{i}}{8}, \quad Z_{+}(2, 0, 2\pi) = -\frac{1}{32\pi^2}, \quad Z_{+}(3, 0, 2\pi) = \mathrm{i}\frac{10 - \pi^2}{512\pi^2}, \tag{4.23}$$

and similar expression can be obtained for higher N as well. By using these exact values we checked that (4.10) indeed holds for $N = 1, \cdots 13$.

5 Connection with ABJ theory

The q-deformed Painlevé written in the form (4.2) is similar to Wronskian like relations that have been found experimentally in ABJ theory [41]. This link is not surprising since topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ and ABJ theory at level k with gauge group $U(N) \times U(N+M)$ are closely related [78, 94, 95]. In order to connect these two theories one has to use the following dictionary [93, 96]

$$\log m_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathrm{i}\hbar - 2\pi\mathrm{i}M, \quad \hbar = \pi k. \tag{5.1}$$

Therefore shifting

$$\xi \quad \to \quad \xi \pm 4\mathrm{i}\pi^2/\hbar \tag{5.2}$$

in topological string is equivalent to a shift of the rank of the gauge group in the ABJ theory by

$$M \to M \pm 1. \tag{5.3}$$

Let us denote the grand canonical partition function of ABJ theory by

$$\Xi_{\mathrm{ABJ}}(\kappa, k, M) = \sum_{N \geq 0} \kappa^N Z_{\mathrm{ABJ}}(N, M, k) \tag{5.4}$$

where $Z_{\mathrm{ABJ}}(N, M, k)$ is the partition function of ABJ theory at level k and with gauge group $U(N) \times U(N+M)$. According to [94, 95, 97] we have

$$Z_{\mathrm{ABJ}}(N, M, k) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{4\pi k} V_M(x_i) \prod_{i<j} \left(\tanh\left(\frac{x_i - x_j}{2k}\right) \right)^2, \tag{5.5}$$

where

$$V_M(x) = \frac{1}{\mathrm{e}^{x/2} + (-1)^M \mathrm{e}^{-x/2}} \prod_{s=-\frac{M-1}{2}}^{\frac{M-1}{2}} \tanh \frac{x + 2\pi\mathrm{i}s}{2k}. \tag{5.6}$$

It follows that

$$\Xi_{\mathrm{ABJ}}(\kappa, k, M) = \det\left(1 + \kappa\rho_{\mathrm{ABJ}}\right) = \prod_{n \geq 0} \left(1 + \kappa\mathrm{e}^{-E_n}\right) \tag{5.7}$$

where

$$\rho_{\mathrm{ABJ}} = \frac{1}{2\cosh(\mathsf{v}/2)} \frac{1}{\mathrm{e}^{\frac{\mathsf{u}}{2}} + (-1)^M \mathrm{e}^{-\frac{\mathsf{u}}{2}}} \prod_{s=\frac{-M+1}{2}}^{\frac{M-1}{2}} \tanh\left(\frac{\mathsf{u} + 2\pi\mathrm{i}s}{2k}\right), \qquad [\mathsf{u}, \mathsf{v}] = 2\pi\mathrm{i}k. \tag{5.8}$$

This is a trace class operator acting on $L^2(\mathbb{R})$ and we denote by e^{-E_n} its eigenvalues. It was shown in [44], that by using the dictionary (5.1) we have

$$Z_{\mathrm{ABJ}}(N, M, k) = \mathrm{e}^{N\mathrm{i}\pi k/4 - N\mathrm{i}\pi M/2} Z(N, \hbar, m). \tag{5.9}$$

5.1 Wronskian-like relations and q-Painlevé

The q–Painlevé equation in the form (4.2) and in the ABJ dictionary (5.1) reads

$$\Xi_{\text{ABJ}}(-\kappa\mathrm{i}, k, M+1)\Xi_{\text{ABJ}}(\mathrm{i}\kappa, k, M-1)(1 - \mathrm{e}^{2\pi\mathrm{i}M/k})$$
$$= \Xi_{\text{ABJ}}(\kappa, k, M)^2 - \mathrm{e}^{2\pi\mathrm{i}M/k}\Xi_{\text{ABJ}}(-\kappa, k, M)^2. \tag{5.10}$$

In the following we show that (5.10) can be derived also by using the Wronskian-like relations of [41]. Let us recall the result of [41]. We factorise the determinant (5.7) according to the parity of the eigenvalues of ρ_{ABJ}, namely

$$\Xi_{\text{ABJ}}(\kappa, k, M) = \Xi_+(\kappa, k, M)\Xi_-(\kappa, k, M), \tag{5.11}$$

where

$$\Xi_+(\kappa, k, M) = \prod_{n\geq 0}\left(1 + \kappa\mathrm{e}^{-E_{2n}}\right), \quad \Xi_-(\kappa, k, M) = \prod_{n\geq 0}\left(1 + \kappa\mathrm{e}^{-E_{2n+1}}\right). \tag{5.12}$$

Then in [41] it was found experimentally [16] that the following relations hold

$$\mathrm{e}^{\frac{M}{2k}\pi\mathrm{i}}\Xi_+\left(\mathrm{i}\kappa, k, M+1\right)\Xi_-\left(-\mathrm{i}\kappa, k, M-1\right)$$
$$- \mathrm{e}^{-\frac{M}{2k}\pi\mathrm{i}}\Xi_+\left(-\mathrm{i}\kappa, k, M+1\right)\Xi_-\left(\mathrm{i}\kappa, k, M-1\right) = 2\mathrm{i}\sin\left(\frac{M\pi}{2k}\right)\Xi\left(\kappa, k, M\right), \tag{5.13}$$

and

$$\mathrm{e}^{-\frac{M}{2k}\pi\mathrm{i}}\Xi_+\left(\mathrm{i}\kappa, k, M-1\right)\Xi_-\left(-\mathrm{i}\kappa, k, M+1\right)$$
$$+ \mathrm{e}^{\frac{M}{2k}\pi\mathrm{i}}\Xi_+\left(-\mathrm{i}\kappa, k, M-1\right)\Xi_-\left(\mathrm{i}\kappa, k, M+1\right) = 2\cos\left(\frac{M\pi}{2k}\right)\Xi\left(\kappa, k, M\right). \tag{5.14}$$

These relations are similar to the Wronskian-like relations of [98]. Let us denote by

$$W_1[\kappa] = \mathrm{e}^{\frac{M}{2k}\pi\mathrm{i}}\Xi_+\left(\mathrm{i}\kappa, k, M+1\right)\Xi_-\left(-\mathrm{i}\kappa, k, M-1\right) - \mathrm{e}^{-\frac{M}{2k}\pi\mathrm{i}}\Xi_+\left(-\mathrm{i}\kappa, k, M+1\right)\Xi_-\left(\mathrm{i}\kappa, k, M-1\right) \tag{5.15}$$

and

$$W_2[\kappa] = \mathrm{e}^{-\frac{M}{2k}\pi\mathrm{i}}\Xi_+\left(\mathrm{i}\kappa, k, M-1\right)\Xi_-\left(-\mathrm{i}\kappa, k, M+1\right) + \mathrm{e}^{\frac{M}{2k}\pi\mathrm{i}}\Xi_+\left(-\mathrm{i}\kappa, k, M-1\right)\Xi_-\left(\mathrm{i}\kappa, k, M+1\right). \tag{5.16}$$

By using these definitions it is easy to verify that

$$\frac{1}{2}\mathrm{i}\csc\left(\frac{\pi M}{k}\right)\left(W_1[\kappa] + \mathrm{e}^{\frac{\mathrm{i}\pi M}{k}}W_1[-\kappa]\right)\left(-W_2[\kappa] + \mathrm{e}^{\frac{\mathrm{i}\pi M}{k}}W_2[-\kappa]\right)$$
$$= \left(1 - \mathrm{e}^{\frac{2\mathrm{i}\pi M}{k}}\right)\Xi_{\text{ABJ}}(-\mathrm{i}\kappa, k, M+1)\Xi_{\text{ABJ}}(\mathrm{i}\kappa, k, M-1). \tag{5.17}$$

On the other hand by using (5.13) and (5.14) we have

$$\frac{1}{2}\mathrm{i}\csc\left(\frac{\pi M}{k}\right)\left(W_1[\kappa] + \mathrm{e}^{\frac{\mathrm{i}\pi M}{k}}W_1[-\kappa]\right)\left(-W_2[\kappa] + \mathrm{e}^{\frac{\mathrm{i}\pi M}{k}}W_2[-\kappa]\right)$$
$$= \Xi_{\text{ABJ}}(\kappa, k, M)^2 - \mathrm{e}^{\frac{2\mathrm{i}\pi M}{k}}\Xi_{\text{ABJ}}(-\kappa, k, M)^2. \tag{5.18}$$

The combination of (5.17) with (5.18) leads to (5.10). Since (5.13) and (5.14) have been tested in detail both numerically and analytically (see [41]), this provides a further strong evidence for the conjecture that the Fredholm determinant of the operator (3.1) indeed computes the tau function of q-PIII$_3$.

---

[16] By a detailed numerical analysis of the spectrum of the operator (5.8).

5.2 Additional tests

In this section we perform further tests of (4.10) by using several results obtained in the context of ABJ theory. By using (5.1) and (5.9) we write (4.10) as

$$\sum_{N_1=0}^{N} e^{-Ni\pi k/4+Ni\pi M/2} Z_{ABJ}(N_1, M, k) Z_{ABJ}(N - N_1, M, k) \left(1 - e^{2\pi i M/k}(-1)^N\right) =$$

$$\sum_{N_1=0}^{N} e^{-Ni\pi k/4+i\pi(-N+MN+2N_1)/2} Z_{ABJ}(N_1, M + 1, k) Z_{ABJ}(N - N_1, M - 1, k)(-1)^N \left(1 - e^{2\pi i M/k}\right).$$

$$(5.19)$$

This equation can be tested in detail for several values of N, M, k thanks to the exact results for $Z_{ABJ}(N, M, k)$ obtained in [51, 93, 99, 100] by using TBA like techniques. Let us illustrate this in one example. We consider (5.19) for $M = 1, k = 3$. We have

$$\sum_{N_1=0}^{N} e^{-Ni\pi/4} Z_{ABJ}(N_1, 1, 3) Z_{ABJ}(N - N_1, 1, 3) \left(1 - e^{2\pi i/3}(-1)^N\right) =$$

$$\sum_{N_1=0}^{N} e^{-Ni\pi 3/4}(-1)^{N_1+N} Z_{ABJ}(N_1, 1, 3) Z_{ABJ}(N - N_1, 0, 3) \left(1 - e^{2\pi i/3}\right).$$

$$(5.20)$$

where we used Seiberg-like duality of ABJ theory [101] to set

$$Z_{ABJ}(N, 2, 3) = Z_{ABJ}(N, 1, 3). \tag{5.21}$$

The quantities

$$Z_{ABJ}(N, 0, 3), \quad Z_{ABJ}(N, 1, 3), \tag{5.22}$$

have been computed exactly in [51, 93] for $N = 1, \cdots, 10$. For instance for the first few values of N we have

$$Z_{ABJ}(1, 0, 3) = \frac{1}{12}, \quad Z_{ABJ}(1, 1, 3) = \frac{1}{12}\left(2\sqrt{3} - 3\right),$$

$$Z_{ABJ}(2, 0, 3) = \frac{\pi - 3}{48\pi}, \quad Z_{ABJ}(2, 1, 3) = \frac{1}{432}\left(-27 + 14\sqrt{3} + \frac{9}{\pi}\right).$$

$$(5.23)$$

The exact values of (5.22) for higher N can be found in [51, 93, 100]. By using these results we have explicitly checked that (5.20) indeed holds for $N = 1, \cdots 10$. Similar tests can be done for other values of M, k providing in this way additional evidence for (4.2) and as a consequence for the conjecture that the Fredholm determinant of the operator (3.1) computes the τ function of q-Painlevé III$_3$.

6 Conclusions and open questions

In this work we conjecture that Fredholm determinants of operators associated to mirror curves on suitable Calabi-Yau backgrounds compute τ functions of q–Painlevé equations. We test this proposal in detail for the case of q–Painlevé III$_3$ which is related to topological strings on local $\mathbb{P}^1 \times \mathbb{P}^1$. However, it would be important to test, and eventually prove, our proposal in the other cases, see Fig. 1.

At the self dual point $\hbar = 2\pi$ the spectral determinant reduces to a classical theta function up to a normalisation factor and the q-Painlevé equations to the well known relation between the j-invariant and the modular parameter of the elliptic curve describing the mirror curve to local $\mathbb{P}^1 \times \mathbb{P}^1$. Hence in this particular limit q-Painlevé equations determine the tree-level prepotential F_0 of the underling geometry. It would be interesting to determine all the higher genus free energies F_g explicitly starting from the q-Painlevé at $q \neq 1$. In addition it would be interesting to understand better the relation between the self-dual point, which is determined by F_0 and its derivatives, and the autonomous limit of q-Painlevé equations or QRT maps [81–83]. This would provide an interesting new link between supersymmetric gauge theories/topological strings and dynamical systems, presenting the deformation of five dimensional Seiberg-Witten theory induced by a self-dual Ω-background as a deformation of integrable mappings in two-dimensions, which appear in soliton theory and statistical systems [81, 82].

A special role in the tests that we perform is played by ABJ theory. Actually, q–Painlevé III$_3$ equation gives a relation between two ABJ theories with different ranks of the gauge group. In this case equation (4.2) can be derived from the Wronskian-like relations of [41]. It would be interesting to see if it is always possible to express the q–Painlevé equations by using Wronskian-like relations. In the case of other q–Painlevé equations we expect the role of ABJ to be played by the other superconformal gauge theories which make contact with topological string on del Pezzo's surfaces as displayed in Fig. 1 (see for instance [102, 103]).

Another related question is the connection with tt* equations. Indeed it is well known that differential Painlevé III$_3$ arises in the context of 2d tt* equations [104]. It would be interesting to see if and how the q-deformed case is related to the 3d tt* equations of [105]. Likewise it would be interesting to understand if it exists a deformation of the 2d Ising model which makes contact with q-deformed Painlevé equation and the determinant (1.3).

A last interesting point would be the generalisation to mirror curves with higher genus. Indeed the conjecture of [20] can be generalized to higher genus mirror curves by introducing the notion of generalised spectral determinant [21]. However, the equations obeyed by these generalised determinants both in five and in four dimensions are not known to us. Some first results in this direction are presented in [19] were a relation with 2d tt* geometry and Toda equation was found in a particular limit. It would be interesting to see whether this continues to hold also in five dimensions at the level of relativistic Toda hierarchy.

Acknowledgements

A   The grand potential: definitions

In this section we review the definition of the topological string grand potential J_X associated to a toric CY X with genus one mirror curve. We mainly follow the notation of [21, 55]. As in

section 2.1 we denote

$$\kappa = e^\mu \tag{A.1}$$

the "true" modulus of X, $\mathbf{m_X}$ the set of mass parameters, $\mathbf{m}$ the rescaled mass parameters (2.2) and

$$\boldsymbol{\xi} = \log \mathbf{m}. \tag{A.2}$$

The Kähler parameters of X are denoted by t_i and can be expressed in terms of the complex moduli through the mirror map:

$$t_i = c_i\mu + \sum_{j=1}^{r_X} a_{ij} \log m_X^{(j)} + \Pi(\kappa^{-1}, \mathbf{m}_X), \tag{A.3}$$

where Π is a series in κ^{-1} and $\mathbf{m}_X$ while c_i, a_{ij} are constants which can be read from the toric data of the CY [42, 43]. For instance for local $\mathbb{P}^1 \times \mathbb{P}^1$ $c_2 = c_2 = 2$, $a_{11} = 0$, $a_{21} = -1$. By using the quantum curve (2.9) one promotes the Kähler parameters to quantum Kähler parameters [106] which we denote by

$$t_i(\hbar) = c_i\mu + \sum_{j=1}^{r_X} a_{ij} \log m_X^{(j)} + \Pi(\kappa^{-1}, \mathbf{m}_X, \hbar). \tag{A.4}$$

For instance when X is the canonical bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ we have

$$t_1(\hbar) = t(\mu, \xi, \hbar), \quad t_2(\hbar) = t(\mu, \xi, \hbar) - \frac{\hbar}{2\pi}\xi, \tag{A.5}$$

where (see also equation (3.9))

$$t(\mu, \xi, \hbar) = 2\mu - 2(m_{\mathbb{P}^1 \times \mathbb{P}^1} + 1)z + z^2\left(-3m_{\mathbb{P}^1 \times \mathbb{P}^1}^2 - \frac{2m\left(e^{2i\hbar} + 4e^{i\hbar} + 1\right)}{e^{i\hbar}} - 3\right) + O\left(z^3\right), \tag{A.6}$$

$$z = e^{-2\mu}, \quad m_{\mathbb{P}^1 \times \mathbb{P}^1} = e^{\frac{\hbar}{2\pi}\xi}.$$

We introduce the topological string free energy

$$F_X^{\text{top}}(\mathbf{t}, g_s) = \frac{1}{6g_s^2}a_{ijk}t_it_jt_k + b_it_i + F_X^{\text{GV}}(\mathbf{t}, g_s) \tag{A.7}$$

with

$$F_X^{\text{GV}}(\mathbf{t}, g_s) = \sum_{g \geq 0}\sum_{\mathbf{d}}\sum_{w=1}^{\infty} \frac{1}{w}n_g^{\mathbf{d}}\left(2\sin\frac{wg_s}{2}\right)^{2g-2}e^{-w\mathbf{d}\cdot\mathbf{t}}, \tag{A.8}$$

where $n_g^{\mathbf{d}}$ are the Gopakumar–Vafa invariants of X and g_s is the string coupling constant. The coefficients a_{ijk}, b_i are determined by the classical data of X . In the limit $g_s \to 0$ we have

$$F_X^{\text{top}}(\mathbf{t}, g_s) \sim \sum_{g \geq 0} F_g(\mathbf{t})g_s^{2g-2}, \tag{A.9}$$

where $F_g(\mathbf{t})$ are called the genus g free energies of topological string. For instance we have

$$F_0(\mathbf{t}) = \frac{1}{6}a_{ijk}t_it_jt_k + \sum_{\mathbf{d}} N_0^{\mathbf{d}}e^{-\mathbf{d}\cdot\mathbf{t}},$$

$$F_1(\mathbf{t}) = b_it_i + \sum_{\mathbf{d}} N_1^{\mathbf{d}}e^{-\mathbf{d}\cdot\mathbf{t}}, \tag{A.10}$$

where $N_g^{\mathbf{d}}$ are the genus g Gromov–Witten invariants. When X is a toric CY one has explicit expressions for (A.10) in terms of hypergeometric and standard functions (see for instance [54, 78, 107]). Let us discuss briefly the convergence properties of (A.8). Let us first note that (A.8) has poles for $\pi^{-1}g_s \in \mathbb{Q}$ which makes ill defined on the real g_s axis. If instead $g_s \in \mathbb{C}/\mathbb{R}$, then (A.8) diverges as a series in $\mathrm{e}^{-\mathbf{t}}$. Nevertheless by using instanton calculus it is possible to partially resumm it and organise it into a convergent series [23].

Similarly we define the Nekrasov–Shatahsvili free energy as

$$F^{\mathrm{NS}}(\mathbf{t}, \hbar) = \frac{1}{6\hbar} a_{ijk} t_i t_j t_k + b_i^{\mathrm{NS}} t_i \hbar + \sum_{j_L, j_R} \sum_{w, \mathbf{d}} N_{j_L, j_R}^{\mathbf{d}} \frac{\sin \frac{\hbar w}{2}(2j_L + 1) \sin \frac{\hbar w}{2}(2j_R + 1)}{2w^2 \sin^3 \frac{\hbar w}{2}} \mathrm{e}^{-w\mathbf{d}\cdot\mathbf{t}}, \quad (A.11)$$

where $N_{j_L, j_R}^{\mathbf{d}}$ are the refined BPS invariants of X. Moreover it exists a constant vector $\mathbf{B}$, called the B-field , such that

$$N_{j_L, j_R}^{\mathbf{d}} \neq 0 \quad \leftrightarrow \quad (-1)^{2j_L + 2j_R + 1} = (-1)^{\mathbf{B}\cdot\mathbf{d}}. \quad (A.12)$$

For local $\mathbb{P}^1 \times \mathbb{P}^1$ this can be set to zero [20, 54]. When $\hbar \to 0$ we recover the following genus expansion

$$F^{\mathrm{NS}}(\mathbf{t}, \hbar) = \sum_{g \geq 0} F_g^{\mathrm{NS}}(\mathbf{t}) \hbar^{2g-2}. \quad (A.13)$$

The convergent properties for the NS free energy are analogous to the ones (A.8).

The topological string grand potential is defined as

$$\mathsf{J}_X(\mu, \boldsymbol{\xi}, \hbar) = \mathsf{J}_X^{\mathrm{WKB}}(\mu, \boldsymbol{\xi}, \hbar) + \mathsf{J}_X^{\mathrm{WS}}(\mu, \boldsymbol{\xi}, \hbar), \quad (A.14)$$

where

$$\mathsf{J}_X^{\mathrm{WS}}(\mu, \boldsymbol{\xi}, \hbar) = F_X^{\mathrm{GV}}\left(\frac{2\pi}{\hbar}\mathbf{t}(\hbar) + \pi\mathrm{i}\mathbf{B}, \frac{4\pi^2}{\hbar}\right). \quad (A.15)$$

Moreover

$$\mathsf{J}_X^{\mathrm{WKB}}(\mu, \boldsymbol{\xi}, \hbar) = \frac{t_i(\hbar)}{2\pi} \frac{\partial F^{\mathrm{NS}}(\mathbf{t}(\hbar), \hbar)}{\partial t_i} + \frac{\hbar^2}{2\pi} \frac{\partial}{\partial \hbar}\left(\frac{F^{\mathrm{NS}}(\mathbf{t}(\hbar), \hbar)}{\hbar}\right) + \frac{2\pi}{\hbar} b_i t_i(\hbar) + A(\boldsymbol{\xi}, \hbar), \quad (A.16)$$

where $A(\boldsymbol{\xi}, \hbar)$ denotes the so–called constant map contribution. It is important to notice that, even tough both $\mathsf{J}_X^{\mathrm{WS}}$ and $\mathsf{J}_X^{\mathrm{WKB}}$ have a dense set of poles on the real $\hbar$ axis, their sum (A.14) is free of poles. In particular J_X is well defined for any value of $\hbar$. Moreover, we have

$$\mathsf{J}_X^{\mathrm{WKB}}(\mu, \boldsymbol{\xi}, \hbar) = \frac{1}{12\pi\hbar} a_{ijk} t_i(\hbar) t_j(\hbar) t_k(\hbar) + \left(\frac{2\pi b_i}{\hbar} + \frac{\hbar b_i^{\mathrm{NS}}}{2\pi}\right) t_i(\hbar) + \mathcal{O}\left(\mathrm{e}^{-t_i(\hbar)}\right) + A(\boldsymbol{\xi}, \hbar). \quad (A.17)$$

Hence it is convenient to split

$$\mathsf{J}_X^{\mathrm{WKB}}(\mu, \boldsymbol{\xi}, \hbar) = \mathrm{P}_X(\mu, \boldsymbol{\xi}, \hbar) + \mathsf{J}_X^{\mathrm{WKB, inst}}(\mu, \boldsymbol{\xi}, \hbar) + A(\boldsymbol{\xi}, \hbar) \quad (A.18)$$

where P_X encodes the polynomial part in t_i of $\mathsf{J}_X^{\mathrm{WKB}}$. For local $\mathbb{P}^1 \times \mathbb{P}^1$ we have

$$\mathrm{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu, \xi, \hbar) = -\frac{\xi t(\mu, \xi, \hbar)^2}{16\pi^2} + \frac{t(\mu, \xi, \hbar)^3}{12\pi\hbar} - \frac{\hbar t(\mu, \xi, \hbar)}{24\pi} + \frac{\pi t(\mu, \xi, \hbar)}{6\hbar} - \frac{\xi}{24}. \quad (A.19)$$

The constant map contribution for local $\mathbb{P}^1 \times \mathbb{P}^1$ reads [108]

$$A(\xi, \hbar) = A_p(\xi, \hbar) - J_{\mathrm{CS}}\left(\frac{2\pi^2}{\hbar}, \mathrm{i}\pi + \frac{1}{2}\xi\right), \tag{A.20}$$

where

$$A_p(\xi, \hbar) = \frac{\hbar^2}{(4\pi^2)^2}\left[\frac{\xi^3}{24} + \frac{\pi^2\xi}{6}\right] + A_c\left(\frac{\hbar}{\pi}\right), \tag{A.21}$$

with

$$A_{\mathrm{c}}(k) = \frac{2\zeta(3)}{\pi^2 k}\left(1 - \frac{k^3}{16}\right) + \frac{k^2}{\pi^2}\int_0^\infty \frac{x}{\mathrm{e}^{kx} - 1}\log(1 - \mathrm{e}^{-2x})\mathrm{d}x. \tag{A.22}$$

Moreover $J_{\mathrm{CS}}(g_s, t)$ is the non–perturbative Chern–Simons free energy [84, 109]. As explained in [84] this also coincides with the grand potential of the resolved conifold as defined in [54] namely

$$J_{\mathrm{CS}}(g_s, T + \mathrm{i}\pi) = -g_s^{-2}\frac{T^3}{12} - g_s^{-2}\frac{\pi^2 T}{12} - \frac{1}{24}T + \frac{1}{2}A_c(4\pi/g_s) + \sum_{n\geq 1}\frac{1}{n}\left(2\sin\frac{ng_s}{2}\right)^{-2}(-1)^n\mathrm{e}^{-nT}$$
$$- \sum_{n\geq 1}\frac{1}{4\pi n^2}\csc\left(\frac{2\pi^2 n}{g_s}\right)\left(\frac{2\pi n}{g_s}T + \frac{2\pi^2 n}{g_s}\cot\left(\frac{2\pi^2 n}{g_s}\right) + 1\right)\mathrm{e}^{-\frac{2\pi n}{g_s}T}. \tag{A.23}$$

We also denote

$$J_{\mathrm{CS}}^{\mathrm{pert}}(g_s, T + \mathrm{i}\pi) = -g_s^{-2}\frac{T^3}{12} - g_s^{-2}\frac{\pi^2 T}{12} - \frac{1}{24}T + \frac{1}{2}A_c(4\pi/g_s) + \sum_{n\geq 1}\frac{1}{n}\left(2\sin\frac{ng_s}{2}\right)^{-2}(-1)^n\mathrm{e}^{-nT}, \tag{A.24}$$

$$J_{\mathrm{CS}}^{\mathrm{np}}(g_s, T + \mathrm{i}\pi) = -\sum_{n\geq 1}\frac{1}{4\pi n^2}\csc\left(\frac{2\pi^2 n}{g_s}\right)\left(\frac{2\pi n}{g_s}T + \frac{2\pi^2 n}{g_s}\cot\left(\frac{2\pi^2 n}{g_s}\right) + 1\right)\mathrm{e}^{-\frac{2\pi n}{g_s}T}. \tag{A.25}$$

For instance when $g_s = \pi$ and $T > 0$ we have [84]

$$J_{\mathrm{CS}}(\pi, T + \mathrm{i}\pi) = -(\pi)^{-2}\frac{T^3}{12} - \frac{T}{12} - \frac{1}{24}T + \frac{1}{2}A_c(4) + \frac{1}{8\pi^2}\mathrm{Li}_3(\mathrm{e}^{-2T}) + \frac{T}{4\pi^2}\mathrm{Li}_2(\mathrm{e}^{-2T})$$
$$- \left(\frac{T^2}{4\pi^2} + \frac{1}{8}\right)\log\left(1 - \mathrm{e}^{-2T}\right) - \frac{1}{4}\mathrm{arctanh}(\mathrm{e}^{-T}). \tag{A.26}$$

B   Conformal blocks and topological strings

We follow the convention of [23] and write the $c = 1$ q-conformal blocks as

$$\mathcal{Z}(u, Z, q_1, q_2) = \sum_{\lambda, \mu} Z^{|\lambda| + |\mu|}\frac{1}{N_{\lambda,\lambda}(1, q_1, q_2)N_{\mu,\mu}(1, q_1, q_2)N_{\lambda,\mu}(u, q_1, q_2)N_{\mu,\lambda}(u^{-1}, q_1, q_2)} \tag{B.1}$$

where the sum runs over all pairs (μ, λ) of Young diagrams. Moreover we use

$$N_{\lambda,\mu}(u, q_1, q_2) = \prod_{s\in\lambda}(1 - uq_2^{-a_\mu(s)-1}q_1^{\ell_\lambda(s)}) \cdot \prod_{s\in\mu}(1 - uq_2^{a_\lambda(s)}q_1^{-\ell_\mu(s)-1}) \tag{B.2}$$

where $a_\lambda(s), l_\lambda(s)$ are the arm length and the leg length of the box $s \in \lambda$. The leading order in Z is obtain by considering the couples $(\Box, 0)$ and $(0, \Box)$ which gives

$$\mathcal{Z}(u, Z, q, q^{-1}) = \frac{Z}{(1-q)(1-q^{-1})}\frac{2}{(1-u)(1-u^{-1})} + \mathcal{O}(Zu)^2 \tag{B.3}$$

Hence by identifying

$$Zu = Q_B, \quad u = Q_F, \quad q = e^{ig_s} \tag{B.4}$$

we obtain

$$\mathcal{Z}(u, Z, q, q^{-1}) = \frac{2qQ_B}{(1-Q_F)^2(1-q)^2} + \mathcal{O}(Q_B)^2 \tag{B.5}$$

which corresponds to the instanton partition function (without the one loop contribution) of topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore we have

$$\exp\left[\mathrm{J}^{\mathrm{WS}}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu, \xi, \hbar)\right] = \frac{1}{(Q_f q, q, q)^2_\infty}\mathcal{Z}(Q_f, \frac{Q_b}{Q_f}, e^{4\pi^2 i/\hbar}, e^{-4\pi^2 i/\hbar}), \tag{B.6}$$

where

$$Q_b = e^{-\frac{2\pi}{\hbar}t(\mu,\xi,\hbar)}, \quad Q_f = e^\xi Q_b, \tag{B.7}$$

and $t(\mu, \xi, \hbar)$ is defined in (A.6).

C   Some relevant shifts

We recall the dictionary (3.8)

$$Q_b = e^{-\frac{2\pi}{\hbar}t(\mu,\xi,\hbar)}, \quad Q_f = e^\xi Q_b, \qquad \xi = \log m = \frac{2\pi}{\hbar}\log m_{\mathbb{P}^1 \times \mathbb{P}^1}, \quad q = e^{4\pi^2 i/\hbar} \tag{C.1}$$

$$t_1(\hbar) = t(\mu, \xi, \hbar), \quad t_2(\hbar) = t(\mu, \xi, \hbar) - \frac{\hbar}{2\pi}\xi, \tag{C.2}$$

We define

$$F_1(\mu, \xi, \hbar) = e^{J^{\mathrm{WKB,inst}}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu,\xi,\hbar)}. \tag{C.3}$$

Then we have

$$\frac{F_1(\mu + i\pi, \xi, \hbar)F_1(\mu - i\pi, \xi, \hbar)}{F_1(\mu, \xi, \hbar)^2} = 1,$$

$$\frac{F_1(\mu - 2\pi i, \xi - 4i\pi^2/\hbar, \hbar)F_1(\mu + 2\pi i, \xi + 4i\pi^2/\hbar, \hbar)}{F_1(\mu, m, \hbar)^2} = 1, \tag{C.4}$$

$$\frac{F_1(\mu - i\pi, \xi - 4i\pi^2/\hbar, \hbar)F_1(\mu + i\pi, \xi + 4i\pi^2/\hbar, \hbar)}{F_1(\mu, \xi, \hbar)^2} = 1.$$

Similarly we define

$$F_2(\mu, \xi, \hbar) = \frac{(Q_f^{-1}q, q, q)_\infty}{(Q_f q, q, q)_\infty}. \tag{C.5}$$

Then we have

$$\frac{F_2(\mu + i\pi, \xi, \hbar)F_2(\mu - i\pi, \xi, \hbar)}{F_2(\mu, \xi, \hbar)^2} = -\frac{1}{Q_f},$$

$$\frac{F_2(\mu - 2\pi i, \xi - 4i\pi^2/\hbar, \hbar)F_2(\mu + 2\pi i, \xi + 4i\pi^2/\hbar, \hbar)}{F_2(\mu, \xi, \hbar)^2} = -\frac{1}{Q_f}, \tag{C.6}$$

$$\frac{F_2(\mu - \mathrm{i}\pi, \xi - 4\mathrm{i}\pi^2/\hbar, \hbar) F_2(\mu + \mathrm{i}\pi, \xi + 4\mathrm{i}\pi^2/\hbar, \hbar)}{F_2(\mu, \hbar, \hbar)^2} = 1, \tag{C.7}$$

where we used

$$(uq; q, q)_\infty = \prod_{i,j \geq 0} \left(1 - uqq^{i+j}\right) = \exp\left[-\sum_{s \geq 1} \frac{u^s}{s \left(q^{\frac{s}{2}} - q^{-\frac{s}{2}}\right)^2} \right]. \tag{C.8}$$

Similarly we define

$$F_3(\xi, \hbar) = \exp[A_p(\xi, \hbar)] \tag{C.9}$$

and we have

$$\frac{F_3(\xi - 4\mathrm{i}\pi^2/\hbar, \hbar) F_3(\xi + 4\mathrm{i}\pi^2/\hbar, \hbar)}{F_3(m, \hbar)^2} = \mathrm{e}^{-\xi/4}. \tag{C.10}$$

Also for

$$F_4(\mu, \xi, \hbar) = \exp[\mathrm{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mu, \xi, \hbar)] \tag{C.11}$$

we have

$$\frac{F_4(\mu + \mathrm{i}\pi, \xi, \hbar) F_4(\mu - \mathrm{i}\pi, \xi, \hbar)}{F_4(\mu, \xi, \hbar)^2} = \mathrm{e}^{\xi/2} Q_b,$$

$$\frac{F_4(\mu - 2\pi\mathrm{i}, \xi - 4\mathrm{i}\pi^2/\hbar, \hbar) F_4(\mu + 2\pi\mathrm{i}, \xi + 4\mathrm{i}\pi^2/\hbar, \hbar)}{F_4(\mu, \xi, \hbar)^2} = \mathrm{e}^{2\xi} Q_b^2, \tag{C.12}$$

$$\frac{F_4(\mu - \mathrm{i}\pi, \xi - 4\mathrm{i}\pi^2/\hbar, \hbar) F_4(\mu + \mathrm{i}\pi, \xi + 4\mathrm{i}\pi^2/\hbar, \hbar)}{F_4(\mu, \xi, \hbar)^2} = \mathrm{e}^{\xi/2}.$$

Moreover we have

$$J_{\mathrm{CS}}^{\mathrm{np}} \left(\frac{2\pi^2}{\hbar}, \frac{1}{2}\xi + \frac{2\pi^2\mathrm{i}}{\hbar} + \mathrm{i}\pi \right) + J_{\mathrm{CS}}^{\mathrm{np}} \left(\frac{2\pi^2}{\hbar}, \frac{1}{2}\xi - \frac{2\pi^2\mathrm{i}}{\hbar} + \mathrm{i}\pi \right) = 2 J_{\mathrm{CS}}^{\mathrm{np}} \left(\frac{2\pi^2}{\hbar}, \frac{1}{2}\xi + \mathrm{i}\pi \right). \tag{C.13}$$

Similarly

$$J_{\mathrm{CS}}^{\mathrm{pert}} \left(\frac{2\pi^2}{\hbar}, \frac{1}{2}\xi + \frac{2\pi^2\mathrm{i}}{\hbar} + \mathrm{i}\pi \right) + J_{\mathrm{CS}}^{\mathrm{pert}} \left(\frac{2\pi^2}{\hbar}, \frac{1}{2}\xi - \frac{2\pi^2\mathrm{i}}{\hbar} + \mathrm{i}\pi \right) - 2 J_{\mathrm{CS}}^{\mathrm{pert}} \left(\frac{2\pi^2}{\hbar}, \frac{1}{2}\xi + \mathrm{i}\pi \right)$$

$$= \frac{1}{4}\xi + \log\left(\mathrm{e}^{-\xi/2} + 1 \right). \tag{C.14}$$

D   Some identities for η function

We denote

$$\eta(\tau) = \mathrm{e}^{\mathrm{i}\pi\tau/12} \prod_{n=1}^{\infty} \left(1 - \mathrm{e}^{2\pi\mathrm{i}n\tau}\right) \tag{D.1}$$

the Dedekind η function. The Weber modular functions are defined as

$$f(\tau) = \frac{\eta^2(\tau)}{\eta(\tau/2)\eta(2\tau)},$$

$$f_1(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)}, \tag{D.2}$$

$$f_2(\tau) = \sqrt{2}\frac{\eta(2\tau)}{\eta(\tau)}.$$

Standard identities of Weber modular functions are

$$f_1(\tau)^8 + f_2(\tau)^8 = f(\tau)^8, \tag{D.3}$$

$$8j(\tau) = (f_1(\tau)^{16} + f_2(\tau)^{16} + f^{16}(\tau))^3, \tag{D.4}$$

where j is the j-invariant:

$$j(\tau) = \frac{1}{\bar{q}} + 744 + 196884\bar{q} + \mathcal{O}(\bar{q}^2), \quad \bar{q} = \mathrm{e}^{2\pi\mathrm{i}\tau} \tag{D.5}$$

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