NOETHER LEFSCHETZ THEORY
IN TORIC VARIETIES

Joint International ICTP/SISSA Programme in Mathematics

Ph.D. in Geometry

Candidate: William D. Montoya **Supervisor:** Prof. Ugo Bruzzo

Thesis submitted in partial fulfillment of the requirements for the degree of Philosophiæ Doctor

Abdus Salam International Centre for Theoretical Physics (ICTP)

AND

International School for Advanced Studies (SISSA)

Trieste, Italy

2019

Abstract

In [4] Batyrev and Cox proved the "Lefschetz hyper-surface theorem" for toric varieties, which claims that for a quasi-smooth hyper-surface $X = \{f = 0\}$ in a complete simplicial toric variety $\mathbb{P}^{2k+1}_{\Sigma}$ the morphism $i^* : H^p(\mathbb{P}_{\Sigma}) \to H^p(X)$ induced by the inclusion, is injective for $p = 2k$ and an isomorphism for $p < 2k$. This allows us to define NL_{β}, the main geometrical object of this work, the locus of quasi-smooth hypersurfaces of degree β such that i^* is not an isomorphism. Following the tradition we call it in [10] the Noether-Lefschetz locus, while some authors call it Hodge loci when $\mathbb{P}^{2k+1}_{\Sigma} = \mathbb{P}^{2k+1}$. This is a interesting geometrical object since it is the locus where the Hodge Conjecture is unknown [8]. The cornerstone of this thesis, a Noether-Lefschetz theorem, is a consequence of "the infinitesimal Noether-Lefschetz theorem" namely, Bruzzo and Grassi in [7] showed that if the multiplication $R(f)_{\beta} \otimes R(f)_{k\beta-\beta_0} \to R(f)_{(k+1)\beta-\beta_0}$ is surjective, where β_0 is the class of the anticanonical divisor of $\mathbb{P}^{2k+1}_{\Sigma}$, the Noether-Lefschetz locus is non-empty and each irreducible component has positive codimension. We prove in Chapter 2 that if $k\beta - \beta_0 = n\eta$ $(n \in \mathbb{N})$ where η is the class of an ample, primitive and 0-regular divisor and β is 0-regular with respect to η, then every irreducible component N of the Noether-Lefschetz locus respect to β satisfies $n + 1 \leq \operatorname{codim} N \leq h^{k-1,k+1}(X)$. The lower bound generalize to higher dimensions the work of Green in [20], Voisin in [47] and Lanza and Martino in [28] and the upper bound extend some results of Bruzzo and Grassi in [9]. In Chapter 3, continuing the study of the Noether-Lefschetz components, we prove that asymptotically the components whose codimension is bounded from above are made of hypersurfaces containing a small degree k-dimensional subvariety V. As a corollary we get an asymptotic characterization of the components of small codimension, generalizing the work of Otwinowska in [37] for $\mathbb{P}^{2k+1}_{\Sigma} = \mathbb{P}^{2k+1}$, Green in [19] and Voisin in [47] for $\mathbb{P}^{2k+1}_{\Sigma} = \mathbb{P}^3$. Finally in chapter 4 we prove asymptotically the Hodge Conjecture when V as before is smooth complete intersection. We also present a generalization of [8], proving that on a very general quasi-smooth intersection subvariety in a projective simplicial toric variety the Hodge conjecture holds. We end this work with a natural and different extension of the Noether-Lefschetz loci. Some tools that have been developed in the thesis are a generalization of Macaulay theorem for Fano, irreducible normal varieties with rational singularities, satisfying a suitable additional condition, and an extension of the notion of Gorenstein ideal to toric varieties.

Contents

Introduction

What is nowadays the Noether-Lefschetz theorem was stated in 1882 by Max Noether, and was proved in 1920 by Salomon Lefschetz using algebraic topological methods. In Lefschetz's words:

"It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry".

For a smooth complex projective variety Y, the Picard group $\operatorname{Pic} Y$ is a classical invariant. While a curve is essentially determined by its Picard group — or, to be precise, by its Jacobian as an abelian variety — this is far from true in higher dimensions. Given a variety $X \subset Y$, one can ask whether the restriction map $\operatorname{Pic} Y \to \operatorname{Pic} X$ is an isomorphism; this is in general false if $\dim Y = 2$, true if $\dim Y \geq 4$ and X is a hypersurface — this is called Grothendieck-Lefschetz theorem (see [38])— , and is a complicated issue if $\dim Y = 3$. The precise result for $Y = \mathbb{P}^3$ is that for an embedded surface of degree $d \geq 4$, the restriction map is an isomorphism for a very general surface, i.e., for all surfaces outside a countable union of proper closed subschemes of the space of degree d surfaces. This is the Noether-Lefschetz theorem, a high point in algebraic geometry and Hodge theory. It allows one to define the Noether-Lefschetz locus as the locus where the restriction map is not an isomorphism. The main geometrical object of this thesis is a generalization of the definition of this locus, and its main purpose is the study of its irreducible components.

The algebraic geometry community somehow lost interest in the Noether-Lefschetz theorem, until the late 1950s when algebraic geometry received a boost from Grothendieck's unifying theory of schemes, and mathematicians were able to look at old problems in a new perspective. Since 1980 several refinements of Noether-Lefschetz theorem have been produced, when the subject was injected with new ideas coming from infinitesimal variations of Hodge structures, as in the foundational paper [12] of Griffiths and his students Carlson, Green and Harris.

In the late 80s and early 90s, C. Voisin made interesting contributions to the theory, and since 2000 her student Otwinowska gave an asymptotic generalization of many results. Moreover, in 2009 Ravindra and Srinivas [39] provided an analogue of the Noether-Lefschetz theorem for class groups of hypersurfaces of normal varieties using a pure algebraic approach. In parallel, Bruzzo and Grassi generalized in [7] the Noether-Lefschetz theorem to toric threefolds using the Hodge theory; more specifically, they proved the theorem as a consequence of an “infinitesimal Noether-Lefschetz theorem for toric varieties". A direct consequence of that theorem is the cornerstone of this thesis, a Noether-Lefschetz theorem, allowing one to extend the definition of Noether-Lefschetz locus to higher dimensional toric varieties.

Chapter 1 is mostly not original; it introduces and motivates the topic of this thesis. We start the chapter in the toric varieties context proving "the hypersurface Lefschetz theorem", and continue showing “the infinitesimal Noether-Lefschetz theorem", and more importantly to us, we show one of its consequences, which we called the "cornerstone result". We finish the chapter generalizing the definition of the Nother-Lefschetz locus to projective simplicial toric varieties, this being the first original result of this work.

In chapter 2, we start constructing explicitly the irreducible components of the Noether-Lefschetz locus in toric varieties and then we find a lower and an upper bound for their codimension. The lower bound is a generalization of what is known in the literature as the explicit Noether Lefschetz theorem, while the upper bound is a consequence of the Griffiths transversality, which for the setting of orbifolds — and so in particular for complete simplicial toric varieties — was proved implicitly by Liu and Zhuang in [31].

In chapter 3, extending the ideas of Otwinowska in [36] and [37], we describe asymptotically the components of the Noether-Lefschetz locus by showing the existence of a subvariety V of suitable dimension and bounded degree. Moreover, we characterize those with smallest codimension for the case of toric threefolds.

Finally, in chapter 4, thanks to the description provided in chapter 3, we prove an asymptotic Hodge conjecture for a non-very general quasi-smooth hypersurface when V, as before, is a smooth complete intersection subvariety. We also show a Noether-Leschetz theorem and discuss a natural and new perspective of the Noether-Lefschetz loci inspired by the work of Mavlyutov in [33]. And we finish the thesis showing that on a very general quasi-smooth intersection subvariety in a projective simplicial toric variety, Hodge conjecture holds, generalizing the main result of Bruzzo and Grassi in [8].

Acknowledgment

My sincerest gratitude goes to ICTP and SISSA for their excellent environment to do research, to the Department of Mathematics at the University of Pennsylvania, Instituto de Matemática e Estatística of the University of São Paulo (IME-USP) and the Department of Mathematics of the University of Genoa for their hospitality while this work was developed.

Foremost, I am thankful to my supervisor Prof. Ugo Bruzzo, for introducing me to the interesting world of Algebraic Geometry, particularly toric geometry and Hodge theory, showing me their different and common flavors. I am also grateful to him for our uncountable discussions, and his patient and good disposition all the time. This work would not have been possible without his support. I also want to express my gratitude to my office mates Konstantin Aleshkin and Dhyan Aranha for interesting discussions, although from quite different perspectives of mathematics, gave me some insights into my own research. Of course my safety net, my family, has been an important motivation and support during all these years; my acknowledgment also goes to them, mainly to my parents William H. Montoya and Olga Cataño. Last, but not least, I am indebted to Sarah Jaye Oliva for being my partner, friend, confidant and always being there for me.

Chapter 0

Preliminaries

0.1 Complex orbifolds

The notion of orbifold was introduced under the name of V-manifold by Satake in [43] and it is the one we will discuss here. Nowadays, there are more general notions of orbifolds and the V- manifolds are known as effective orbifolds, see [1] for more details.

Definition 0.1. Let X be a d-dimensional variety

- X is a complex orbifold if for every $p \in X$ there exists a triple (U, G, ϕ) where $U \subset \mathbb{C}^d$ is a connected neighborhood of p, $G \subset GL(d, \mathbb{C})$ is a finite subgroup with no complex reflections other than the identity and $\phi : U \to X$ is a complex analytic map such that $\phi(gx) = \phi(x)$, $\forall x \in U$ and $\forall g \in G$. (U, G, ϕ) is called a local chart of X. A complex reflection is an element of $GL(d, \mathbb{C})$ of finite order with $d-1$ of its eigenvalues equal to 1.

- An embedding $\lambda : (U, G, \phi) \hookrightarrow (V, H, \psi)$ between two orbifold charts is an embedding $\lambda : U \hookrightarrow V$ such that $\psi \circ \lambda = \phi$.[i]

- A subvariety $Y \subset X$ is a suborbifold if for every $p \in Y$ there is a local chart (U, G, ϕ) of X such that the inverse image of Y in U is smooth at $[\phi^{-1}(p)] \in U/G$.

In general a subvariety which is an orbifold does not need to be a suborbifold. One can think about a suborbifold, roughly speaking, as a subvariety where its singular points are those coming from the ambient space.

Definition 0.2. Given $p \in X$ and a local chart $(U, G, \phi) \coloneqq U/G$.

[i] An important result for the study of orbifolds [34]: given two embeddings of orbifold charts $\lambda, \mu : (U, G, \phi) \hookrightarrow (V, H, \psi)$ there exists a unique $h \in H$ such that $\mu = j \cdot \lambda$.

- A C^∞ k-form on U/G is defined to be a C^∞ form $\omega \in \Omega^k(U)$ such that $\omega(gx) = \omega(x)$ $\forall x \in U$ and $\forall g \in G$.

- A holomorphic k-form on U/G is a G- invariant holomorphic $k-$ form on U.

Using the above definition of embedding there is a natural notion of patching $k-$forms on different charts. Holomorphic k-forms on an orbifold are called Zariski k-forms on X and they determine a sheaf $\widehat{\Omega}^k_X$ that although may fail to be locally free, the sheaf is locally free on the smooth locus of X. Moreover,

Proposition 0.3 ([17] Proposition A.3.1)**.** If X is an orbifold and $i : U_0 \hookrightarrow X$ is the inclusion of the smooth locus then $\widehat{\Omega}^k_X = i_*(\Omega^k_{U_0})$ where $\Omega^k_{U_0}$ is the classical sheaf of holomorphic k-forms on the complex manifold U_0.

0.2 Deformation of complex orbifolds

We present a generalization of Ehresmann's theorem to orbifolds following [31]. Let $U \subset \mathbb{C}^d$ be an open set and let $\mathcal{X}$ be an orbifold such that every point has a chart of the form $U \times (V_\alpha/G)$ so there exists a canonical projection $\pi_\alpha : U \times (V_\alpha/G) \to U$. These π_α's fit together to form a natural morphism $\pi : \mathcal{X} \to U$.

Definition 0.4. A smooth family of compact orbifolds over U is an orbifold $\mathcal{X}$ as before with its natural projection $\pi : \mathcal{X} \to U$, such that as a map of topological spaces, π is proper.

Remark 0.5. If $\pi : \mathcal{X} \to U$ is a smooth family of compact orbifolds each fiber of the underlying continuous map has a natural compact orbifold structure.

Lemma 0.6 ([31] Lemma 3.4)**.** A smooth family of compact orbifolds over a contractible open set U, $\pi : \mathcal{X} \to U$ is trivial.

Analogously to the classical case a Kähler form on an orbifold is a real, smooth closed $(1,1)$-form which is positive at every point, that is, its pullback in every chart is positive.

Definition 0.7. A holomorphic family of compact polarized complex orbifolds over U is a holomorphic family of compact complex orbifolds $\pi : \mathcal{X} \to U$ such that there is a Kähler form ω_u on the fiber $\pi^{-1}(u) = X_u$ which varies smoothly with respect u.

0.3 Projective simplicial toric varieties

Now we focus in projective simplicial toric varieties and we will discuss its relation with the notion of orbifold.

Definition 0.8. Let M be a free Abelian group of rank d, let $N = \mathrm{Hom}(M, \mathbb{Z})$, and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

1. A convex subset $\sigma \subset N_{\mathbb{R}}$ is a rational k-dimensional simplicial cone if there exist k linearly independent primitive elements $e_1 \ldots, e_k \in N$ such that $\sigma = \{\mu_1 e_1 + \cdots + \mu_k e_k\}$, with μ_i non negative real numbers. The generators e_i are said to be integral if for every i and any non negative rational number μ, the product μ_i is in N only if μ is an integer.

2. Given two simplicial cones σ, σ', we say that σ' is a face of σ (we then write $\sigma < \sigma$) if the set of integral generators of σ' is a subset of the set of integral generators of σ.

3. A finite set $\{\sigma_1, \ldots \sigma_r\}$ of rational simplicial cones is called a rational simplicial complete d-dimensional fan if

   - all faces of cones in Σ are in Σ;
   - if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma' < \sigma$ and $\sigma \cap \sigma' < \sigma'$;
   - $N_{\mathbb{R}} = \sigma_1 \cup \cdots \cup \sigma_r$.

A rational simplicial complete d-dimensional fan Σ defines a toric variety $\mathbb{P}^d_{\Sigma}$ of dimension d having only Abelian quotient singularities, which we will denote just $\mathbb{P}_{\Sigma}$ if the dimension is clear or not relevant. Moreover, $\mathbb{P}_{\Sigma}$ is a global orbifold (see theorem 1.9 in [4]).

In order to study hyper-surfaces in a simplicial toric variety we will introduce the notion of Cox ring in a general context, then we will see that when this ring coincides with a polynomial ring in the case of toric varieties.

Definition 0.9 (Cox ring)**.** Let Y be a complete variety with finitely generated class group $\mathrm{Cl}(Y)$, then the Cox ring associated to Y is

$$S(Y) := \bigoplus_{D \in \mathrm{Cl}(Y)} H^0(Y, \mathcal{O}_Y(D)).$$

A detailed analysis of this ring when $\mathrm{Cl}(Y)$ is free is given in Section 4 of [2].

Example 0.10 ([22] Corollary 2.10)**.** Let Y be a smooth projective variety with $\mathrm{Pic}(Y)_{\mathbb{R}} = N^1(Y)$ where $N^1(Y)$ are the classes of numerically equivalence Cartier divisors. Then, Y is a toric variety if and only if its Cox ring is a polynomial ring.

Example 0.11 ([27] Example 2.6)**.** The Cox ring need not be finitely generated; a counterexample is provided by a K3 surface with Picard number 20.

Definition-Proposition 0.12 (Irrelevant Ideal)**.** Let D be an ample Cartier divisor on Y with $S(Y)$ finitely generated and let $R_D = \bigoplus_{m=0}^{\infty} S(Y)_{mD}$. The *irrelevant ideal* is defined as

$$B(Y, D) := \sqrt{J_{Y,D}} \text{ where } J_{Y,D} = < R_D >$$

Actually $B(Y, D)$ it is independent of the choice of the ample Cartier divisor D, so we denote it $B(Y)$ (see [2]).

Example 0.13. Given a fan Σ and taking a variable x_i for each 1-dimensional cone ρ_i in Σ, the Cox ring $S(\Sigma)$ is the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. Moreover considering for every $\sigma \in \Sigma$ and $x_\sigma = \prod_{\rho_i \notin \sigma} x_i$, the irrelevant ideal $B(\Sigma)$ is generated by the x_σ's.

Let L be an ample line bundle on $\mathbb{P}_\Sigma$, and denote by $\beta \in Cl(\Sigma)$ its degree. A section of L is a polynomial in S_β.

Definition 0.14. Let f be a section of L, and let $\mathbf{V}(f) = \{f = 0\}$ in $SpecS(\Sigma)$. the hypersurface cut in $\mathbb{P}_\Sigma$ by the equation $f = 0$ is quasi-smooth if $V(f) \subset \mathbb{C}^n$ is smooth outside $Z(\Sigma)$.

Proposition 0.15 ([4] Proposition 3.5)**.** A hyper-surface $X \subset \mathbb{P}_\Sigma$ is quasi-smooth if and only if X is a sub-orbifold of $\mathbb{P}_\Sigma$.

Proposition 0.16 ([4] Proposition 4.15)**.** If f is the general section of an ample invertible sheaf, then X is a quasi-smooth hypersurface in $\mathbb{P}_\Sigma$.

Remark 0.17. Let $\mathbb{P}_\Sigma$ be a projective simplicial toric variety and let $\beta \in Cl(\Sigma)$ be a Cartier class. Let $f \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(\beta))$ such that $X_0 = \{f = 0\} \subset \mathbb{P}_\Sigma$ is quasi-smooth. Let $\mathcal{U} \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(\beta))$ be the open set parametrizing the quasi-smooth hyper-surfaces and let $\mathcal{X} \subset \mathcal{U} \times \mathbb{P}_\Sigma$ be its tautological family. Since $\mathcal{U}$ is a complex manifold and $\mathcal{X}$ a complex orbifold we have the assumptions of Ehresmann's theorem. Hence $\pi : \mathcal{X} \to \mathcal{U}$ is locally trivial.

0.4 A Lefschetz theorem in toric varieties

A very important result in the theory of Kähler complex orbifolds is the existence of a pure Hodge structure [42]. So it is pretty natural to ask when the Hodge Structure of a hyper-surface coincides with the Hodge structure of its ambient space. The next theorem is a first step in that direction, a refinement of the Lefschetz hyperplane theorem. .

Theorem 0.18 ([4] Theorem 10.8). *Let X be a quasi-smooth hypersurface of a d-dimensional complete simplicial toric variety $\mathbb{P}$, and suppose that X is defined by $f \in S^{\beta}$. If $f \in B(\Sigma)$ then the natural map $i^* : H^i(\mathbb{P}) \to H^i(X)$ is an isomorphism for $i < d-1$ and an injection for $i = d-1$.*

The above result shows that the interesting part of the cohomology of a quasi-smooth hypersurface X occurs in dimension $d-1$. Moreover by Theorem 9.3.2 in [18] $h^{p,q}(\mathbb{P}^d_{\Sigma}) = 0$ when $p \neq q$ hence the no trivial injectivity of i^* occurs (k,k)-Hodge decomposition $H^{k,k}(X)$, which makes sense when d is odd. The question raised at the beginning of this subsection will be given a complete answer, using the notion of variation of the Hodge structure.

Chapter 1

Infinitesimal Noether-Lefschetz theorem for toric varieties

In this chapter we prove the cornerstone of this thesis, a Noether-Lefschetz theorem, which is a consequence of the "Infinitesimal Noether-Lefschetz theorem for toric varieties". Namely, we show under certain conditions that a very general hypersurface (quasi-smooth) of a odd dimensional toric variety $\mathbb{P}^{2k+1}_{\Sigma}$ all its rational (k,k)-forms come from the rational (k,k)-forms of $\mathbb{P}^{2k+1}_{\Sigma}$, i.e., $H^{k,k}(X_u,\mathbb{Q}) = i^*(H^{k,k}(\mathbb{P}^{2k+1}_{\Sigma},\mathbb{Q}))$ where i^* is the morphism induced in cohomology by the inclusion. The proofs and the structure of the Chapter are mainly based in the papers [4] and [7] so that there is not much original, except for the definition of the Noether-Lefschetz locus in a complete simplicial toric variety $\mathbb{P}^{2k+1}_{\Sigma}$ in the last part of the Chapter. We recover the definitions given in [7] and [36] when $k = 1$ and $\mathbb{P}^{2k+1}_{\Sigma} = \mathbb{P}^{2k+1}$, respectively.

1.1 Primitive cohomology of a hypersurface.

Let L be an ample line bundle on $\mathbb{P}^d_{\Sigma}$ and let X be a hypersurface in $\mathbb{P}^d_{\Sigma}$ cut off by a section f of L then $f \in B(\Sigma)$ ([4] Lemma 9.15). Denoting by $i : X \hookrightarrow \mathbb{P}_{\Sigma}$ the inclusion and by $i^* : H^{\bullet}(\mathbb{P}_{\Sigma},\mathbb{C}) \to H^{\bullet}(X,\mathbb{C})$ the associated morphism in cohomology; by the "hypersurface Lefschetz theorem", $i^* : H^{d-1}(\mathbb{P}_{\Sigma},\mathbb{C}) \to H^{d-1}(X,\mathbb{C})$ is injective.

Definition 1.1. The primitive cohomology group $H^{d-1}_{\text{prim}}(X)$ is the quotient

$$H^{d-1}(X,\mathbb{C})/i^*\left(H^{d-1}(\mathbb{P}_{\Sigma})\right)$$

Lemma 1.2. The exact sequence

$$0 \to i^*(H^{d-1}(\mathbb{P}_{\Sigma}),\mathbb{C}) \to H^{d-1}(X,\mathbb{C}) \to H^{d-1}_{\text{prim}}(X) \to 0$$

splits orthogonally with respect to the intersection pairing in $H^{\bullet}(X,\mathbb{C})$.

Proof. The Hard Lefschetz theorem holds also for projective orbifolds [51]. Then the morphism $c_1(L) \cup \_ : H^{d-1}(\mathbb{P}_\Sigma) \to H^{d+1}(\mathbb{P}_\Sigma)$ is an isomorphism. Let $i_* : H^{d-1}(X) \to H^{d+1}(\mathbb{P}_\Sigma)$ be the Gysin map. The following commutative diagram of vector spaces

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & H^{d-1}(\mathbb{P}_\Sigma) & \longrightarrow & H^{d-1}(\mathbb{P}_\Sigma) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle l} & & \\
0 & \longrightarrow & \ker i_* & \longrightarrow & H^{d-1}(X) & \xrightarrow[i_*]{} & H^{d+1}(\mathbb{P}_\Sigma) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\uparrow{\scriptstyle s} & & \downarrow & & \\
0 & \longrightarrow & \ker i_* & \longrightarrow & H^{d-1}_{\text{prim}}(X) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

provides a straightforward splitting s of the exact sequence in the middle column. Let $\langle , \rangle$ be the intersection pairing in cohomology and since that i^* and i_* are adjoint with respect to the intersection pairing. The upper-right square commutes since by Poincaré duality

$$\langle i_* i^* \alpha, \beta \rangle = \langle i^* \alpha, i^* \beta \rangle = \langle c_1(L) \cap \alpha, \beta \rangle = \langle l(\alpha), \beta \rangle \,.$$

If $\alpha \in H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C})$ and $\beta \in H^{d-1}_{\text{prim}}(X)$, we have

$$\langle i^* \alpha, s(\beta) \rangle = \langle \alpha, i_*(s(\beta)) \rangle = 0.$$

□

Remark 1.3. Note that as $H^\bullet(X, \mathbb{C}) \simeq H^\bullet(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$, one can work indifferently with rational or complex coefficients.

As we have mentioned before $H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C})$ and $H^{d-1}(X, \mathbb{C})$ have pure Hodge structures, and the morphism i^* is compatible with them, so that $H^{d-1}_{\text{prim}}(X)$ inherits a pure Hodge structure. We shall write

$$H^{d-1}_{\text{prim}}(X) = \bigoplus_{p=0}^{d-1} H^{p,d-1-p}_{\text{prim}}(X)$$

1.2 Cohomology of the complement of an ample divisor

Proposition 1.4. There is a natural isomorphism

$$H^{p,d-p-1}_{\text{prim}}(X) \cong \frac{H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}(d-p+1)X)}{H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X)) + dH^0(\mathbb{P}_\Sigma, \Omega^{d-1}_{\mathbb{P}_\Sigma}(d-p)X)}$$

Proof. This follows from Corollaries 10.2 and 10.12 in [4]. □

The resulting projection, multiplied by the factor $(-1)^{p-1}/(d-p+1)!$ will be denoted by

$$r_p : H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}(d-p+1)X) \to H^{p,d-p-1}_{\text{prim}}(X) \tag{1.1}$$

which is called the p-th residue map in analogy with the classical case. This map will play an important role in the Chapter 3.

Definition 1.5. Let $X = \{f = 0\} \subset \mathbb{P}_\Sigma$ be a hypersurface and let $J(f)$ be the ideal of the Cox ring generated by the derivatives of f. The ring $R(f) = S(\Sigma)/J(f)$ is the Jacobian ring of X.

The Jacobian ring encodes almost all the information about the primitive cohomology of X.

Proposition 1.6 ([4] Theorem 10.13)**.** If $p \neq d/2-1$, $H^{p,d-p-1}_{\text{prim}}(X) \cong R(f)_{(d-p)\beta-\beta_0}$ where $\beta_0 = -\deg K_{\mathbb{P}_\Sigma}$, $\beta = \deg L$.

1.3 The Gauss-Manin connection

Let $\mathcal{U}_\beta$ be the open subscheme of $|L|$ parametrizing the quasi-smooth hypersurfaces with degree $\beta = \deg L$, and let $\pi : \mathcal{X}_\beta \to \mathcal{U}_\beta$ be the tautological family on $\mathcal{U}_\beta$; we denote by X_u the fiber of π at $u \in \mathcal{U}_\beta$. Let H^{d-1} be the higher direct images of the constant sheaf $\mathbb{C}$ whose fiber at u is the cohomology $H^{d-1}(X_u)$, i.e., $H^{d-1} = R^{d-1}\pi_*\mathbb{C}$ which is a local system by Ehresmann's theorem for orbifolds. We have associated a vector bundle $\mathcal{H}^{d-1} = H^{d-1} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{U}_\beta}$ and a flat connection ∇, the Gauss-Manin connection, of $\mathcal{H}^{d-1}$. Since the hypersurfaces X_u are quasi-smooth, the Hodge structure of the fibers $H^{d-1}(X_u)$ of $\mathcal{H}^{d-1}$ varies holomorphically with respect to u [45]. The corresponding filtration defines holomorphic subbundles $F^p\mathcal{H}^{d-1}$, and the graded object of the filtration defines holomorphic bundles. The bundles $\mathcal{H}^{p,d-p-1}$ given by the Hodge decomposition are not holomorphic subbundles of $\mathcal{H}^{d-1}$, but they are diffeomorphic to $Gr^p_F(\mathcal{H}^{d-1})$, thus they have a holomorphic structure. The quotient bundles $\mathcal{H}^{p,d-p-1}_{\text{prim}}$ of $\mathcal{H}^{p,d-p-1}$ correspond to the primitive cohomologies of the hypersurfaces X_u. Let $\pi_p : \mathcal{H}^{d-1} \to \mathcal{H}^{p,d-p+1}_{\text{prim}}$ be the natural projection.

We denote by $\tilde{\gamma}_p$ the cup product

$$\tilde{\gamma}_p : H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \otimes H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}(d-p)X) \to H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X)).$$

If u_0 is the point in $\mathcal{U}_\beta$ corresponding to X, the space $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))/\mathbb{C}(f)$ where $\mathbb{C}(f)$ is the 1−dimensional subspace of $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))$ generated by f, can be identified with $T_{u_0}\mathcal{U}_\beta$. The morphism $\tilde{\gamma}_p$ induces in cohomology the Gauss-Manin connection:

Lemma 1.7 ([13] Proposition 5.4.3)**.** Let σ_0 be a primitive class in $H^{p,d-p-1}_{\text{prim}}(X)$, let $v \in T_{u_0}\mathcal{U}_\beta$ and let σ be a section of $\mathcal{H}^{p,d-p-1}$ along a curve in $\mathcal{U}_\beta$ whose tangent vector at u_0 is v, such that $\sigma(u_0) = \sigma_0$. Then

$$\pi_{p-1}(\nabla_v(\sigma)) = r_{p-1}(\tilde{\gamma}_p(\tilde{v} \otimes \tilde{\sigma})) \tag{1.2}$$

where r_p, r_{p-1} are the residue morphisms defined, $\tilde{\sigma}$ is an element such that $r_p(\tilde{\sigma}) = \sigma_0$, and $\tilde{v}$ is a pre-image of v in $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))$ In particular the following diagram commutes:

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(X)) \otimes H^0(\Omega^d_{\mathbb{P}_\Sigma}(d-p)X) & \xrightarrow{\tilde{\gamma}_p} & H^0(\Omega^d_{\mathbb{P}_\Sigma}(d-p+1)X) \\ \downarrow{\scriptstyle \text{pr}\otimes r_p} & & \downarrow{\scriptstyle r_{p-1}} \\ T_{u_0}\mathcal{U}_\beta \otimes H^{p,d-1-p}_{\text{prim}}(X) & \xrightarrow{\gamma_p} & H^{p-1,d-p}_{\text{prim}}(X) \end{array} \tag{1.3}$$

where γ_p is the morphism that maps $v \otimes \alpha$ to $\nabla_v \alpha$ and pr is the projection $H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(X)) \to T_{u_0}\mathcal{U}_\beta$.

Lemma 1.8. If α and η are sections of $\mathcal{H}^{p,d-p-1}$ and $\mathcal{H}^{d-p,p-1}$ respectively, then for every $v \in T_{u_0}\mathcal{U}_\beta$,

$$\nabla_v \alpha \cup \eta = -\alpha \cup \nabla_v \eta. \tag{1.4}$$

Proof. The Gauss-Manin connection is compatible with the cup product by definition, i.e.,

$$\nabla_v(\alpha \cup \eta) = \nabla_v \alpha \cup \eta + \alpha \cup \nabla_v \eta$$

but $\alpha \cup \eta = 0$ because it is an element in $\mathcal{H}^{d,d-2}$. □

1.4 The moduli space of hypersurfaces in $\mathbb{P}_\Sigma$

This section is based in the ideas of Cox presented in [16]. We consider the moduli space $\mathcal{M}_\beta$ for general quasi-smooth hypersurfaces in $\mathbb{P}_\Sigma$ with class divisor β but in order to get a "good" moduli we have to make some restrictions because the automorphism group of a toric variety is in general non reductive. It is worth mentioning that there is a new approach to the construction of this moduli using new results in non-reductive GIT, see [11].

Definition 1.9. Given $\beta \in \text{Cl}(\Sigma)$, let $\text{Aut}_\beta(\mathbb{P}_\Sigma)$ be the subgroup of $\text{Aut}(\mathbb{P}_\Sigma)$ consisting of those automorphism which preserve β.

Remark 1.10 ([16] Section 4)**.** If $\mathrm{Aut}^0(\mathbb{P}_\Sigma)$ is the connected component of the identity of $\mathrm{Aut}(\mathbb{P}_\Sigma)$, then $\mathrm{Aut}^0(\mathbb{P}_\Sigma)$ is a subgroup of finite index in $\mathrm{Aut}_\beta(\mathbb{P}_\Sigma)$.

When we describe $\mathbb{P}_\Sigma$ as the quotient $U(\Sigma)/D(\Sigma)$, note that $\mathrm{Aut}(\mathbb{P}_\Sigma)$ does not act on $U(\Sigma)$. However in [16] it is shown that there is an exact sequence

$$1 \to D(\Sigma) \to \widetilde{\mathrm{Aut}}(\mathbb{P}_\Sigma) \to \mathrm{Aut}(\mathbb{P}_\Sigma) \to 1$$

where $\widetilde{\mathrm{Aut}}(\mathbb{P}_\Sigma)$ is the group of automorphisms of $\mathbb{C}^r$ which preserve $U(\Sigma)$ and normalize $D(\Sigma)$. An element $\phi \in \widetilde{\mathrm{Aut}}(\mathbb{P}_\Sigma)$ induces an automorphism $\phi : S \to S$ which for all $\gamma \in \mathrm{Cl}(\Sigma)$ satisfies $\phi(S_\gamma) = S_{\phi(\gamma)}$.

Remark 1.11. By differentiating the above exact sequence, we have a surjective map

$$\kappa_\beta : H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \to T_X\mathcal{M}_\beta$$

which is the analogue of the Kodaira-Spencer map.

Definition 1.12. Given $\beta \in \mathrm{Cl}(\Sigma)$, let $\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$ be the subgroup of $\widetilde{\mathrm{Aut}}(\mathbb{P}_\Sigma)$ consisting of these automorphisms that preserve β.

The group $\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$ has the following obvious properties.

Lemma 1.13. There is a canonical exact sequence

$$1 \to D(\Sigma) \to \widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma) \to \mathrm{Aut}_\beta(\mathbb{P}_\Sigma) \to 1$$

Furthermore, there is a natural action of $\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$ on S^β.

Remark 1.14. Let $\widetilde{\mathrm{Aut}}^0(\mathbb{P}_\Sigma)$ be the connected component of the identity of $\widetilde{\mathrm{Aut}}(\mathbb{P}_\Sigma)$. In [16] it is shown that $\widetilde{\mathrm{Aut}}^0(\mathbb{P}_\Sigma)$ is naturally isomorphic to the group $\mathrm{Aut}_g(S)$ of $\mathrm{Cl}(\Sigma)$-graded automorphisms of S. Then $\widetilde{\mathrm{Aut}}^0(\mathbb{P}_\Sigma) \subset \widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$, and the action of $\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$ on S^β is compatible with the action of $\mathrm{Aut}_g(S)$.

If $\beta \in \mathrm{Cl}(\Sigma)$ is an ample class, then we know that a generic element $f \in S^\beta$ is quasi-smooth. Then

$$\{f \in S_\beta \mid f \text{ is quasi-smooth}\}/\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$$

should be the coarse moduli space of quasi-smooth hypersurfaces in $\mathbb{P}_\Sigma$ in the divisor class of β. The problem is that $\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$ need not be a reductive group, so that the quotient may not exist. However it is well-known that there is a nonempty invariant open set

$$U \subset \{f \in S_\beta \mid f \text{ is quasi-smooth}\}$$

such that the geometric quotient

$$U/\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$$

exists.

Definition 1.15. We call the quotient $U/\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$ a generic coarse moduli space for hypersurfaces of $\mathbb{P}_\Sigma$ with divisor class β.

There is a relation of the Jacobian ring $R(f)$ with the generic coarse moduli space, namely.

Proposition 1.16 ([4] Proposition 13.7)**.** If β is ample and $f \in S^\beta$ is generic, then $R(f)_\beta$ is naturally isomorphic to the tangent space of the generic coarse moduli space of quasi-smooth hypersurfaces of $\mathbb{P}_\Sigma$ with divisor class β.

The local system $\mathcal{H}^{d-1}$ and its various sub-systems do not descend to the moduli space $\mathcal{M}_\beta$, because the group $\mathrm{Aut}_\beta(\mathbb{P}_\Sigma)$ is not connected. Nevertheless, perhaps after suitably shrinking U, the quotient $\mathcal{M}^0_\beta := U/\mathrm{Aut}^0_\beta(\mathbb{P}_\Sigma)$ is a finite étale covering of $\mathcal{M}_\beta$.

Proposition 1.17. There is a morphism

$$\gamma_p : T_X\mathcal{M}_\beta \otimes H^{p,d-1-p}_{\mathrm{prim}}(X) \to H^{p-1,d-p}_{\mathrm{prim}}(X) \tag{1.5}$$

such that the diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(X)) \otimes H^0(\Omega^d_{\mathbb{P}_\Sigma}(d-p)X) & \longrightarrow & H^0(\Omega^d_{\mathbb{P}_\Sigma}(d-p+1)X) \\ \downarrow & & \downarrow \\ T_X\mathcal{M}_\beta \otimes H^{p,d-1-p}_{\mathrm{prim}}(X) & \longrightarrow & H^{p-1,d-p}_{\mathrm{prim}}(X) \end{array}$$

commutes.

Proof. It suffices to prove the Proposition with $\mathcal{M}_\beta$ replaced by $\mathcal{M}^0_\beta$; in fact the tangent spaces at points $\mathcal{M}^0_\beta$ are canonically isomorphic to the tangent spaces at the image points in $\mathcal{M}_\beta$. If $\rho : \mathcal{U}_\beta \to \mathcal{M}^0_\beta$ is the induced map (where $\mathcal{U}_\beta$ has been suitable restricted), the local system H^{d-1} descend to a local system $\rho_* H^{d-1}$ on $\mathcal{M}^0_\beta$ and $\rho^*\rho_* H^{d-1} \simeq H^{d-1}$ (the natural morphism $H^{d-1} \to \rho^*\rho_* H^{d-1}$ is an isomorphism on the stalks due to the topological base change; note that ρ is proper.) Thus we obtain on $\mathcal{M}^0_\beta$ holomorphic bundles that are equipped with a Gauss-Manin connection, which is trivial in the direction of the fibers of ρ. So, if we define again γ_p by $\gamma_p(v \otimes \alpha) = \nabla_v \alpha$, the commutativity of the diagram in the statement follows from the commutativity of the diagram 1.3. □

The tangent space $T_X\mathcal{M}_\beta$ at a point X is naturally isomorphic to the degreeβsummand of the Jacobian ring of f, i.e., $T_X\mathcal{M}_\beta \simeq R(f)_\beta$ [4]. Moreover by 1.6 $H^{p,d-p-1}_{\mathrm{prim}}(X) \simeq R(f)_{(d-p)\beta-\beta_0}$.

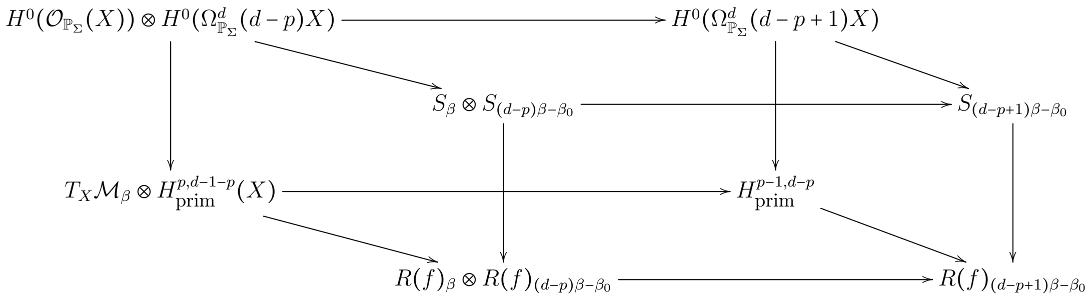
Proposition 1.18. Under these isomorphisms, γ_p coincides with the multiplication in the ring $R(f)$, i.e.,

$$R(f)_\beta \otimes R(f)_{(d-p)\beta-\beta_0} \to R(f)_{(d-p+1)\beta-\beta_0}$$

Proof. Theorem 9.7 in [4] implies

$$H^0(\Omega^d_{\mathbb{P}_\Sigma}((d-p)X))/H^0(\Omega^d_{\mathbb{P}_\Sigma}((d-p-1)X)) \simeq S_{(d-p)\beta-\beta_0},$$

and moreover, $H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(X)) \simeq S_\beta$; the cup product correspond to the product in the ring S. This implies that the "top square " of the 3- dimensional diagram



commutes. We need to show that the "bottom square" commutes as well, which will follow from the commutativity of the "side squares", and the surjectivity of the morphism $\kappa_p \otimes r_p$. The commutativity of the diagram on the right is contained in the proof of Theorem 10.6 in [4]. The commutativity of the diagram on the left follows from the commutativity of the previous diagram, with $d-p+1$ replaced by $d-p$, and the commutativity of

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(X)) & \xrightarrow{\sim} & S_\beta \\ \downarrow & & \downarrow \\ T_X\mathcal{M}_\beta & \xrightarrow{\sim} & R(f)_\beta \end{array}$$

which is shown in the proof of Proposition 13.7 in [4].

□

1.5 A Noether-Lefschetz theorem, the corner-stone result

Let us recall that a property is said to be very general if it holds in the complement of a countable union of subschemes of positive codimension. Let us denote by $H^{d-1}_T(X) \subset H^{d-1}(X)$ the subspace of the cohomology classes that are annihilated by the action of the Gauss-Manin connection. Coefficients may be taken in $\mathbb{C}$ or $\mathbb{Q}$. Note that $H^{d-1}_T(X)$ has a Hodge structure.

Theorem 1.19 (Infinitesimal Noether Lefschetz Theorem). *For a given p with $1 \le p \le d-1$, assume that the morphism*

$$\gamma_p : T_X\mathcal{M}_\beta \otimes H^{d-p,p-1}_{\text{prim}}(X) \to H^{d-p-1,p}_{\text{prim}}(X)$$

is surjective. Then $H_T^{p,d-1-p}(X) = i^*(H^{p,d-1-p}(\mathbb{P}_\Sigma))$.

Proof. Replace $\mathcal{M}_\beta$ by $\mathcal{M}_\beta^0$ and consider the local systems $\mathcal{H}^{d-1}$ and $\mathcal{H}^{p,d-p-1}_{\text{prim}}$ on $\mathcal{M}_\beta^0$. Take

$$\alpha \in H_T^{p,d-1-p}(X) \cap H_{\text{prim}}^{p,d-1-p}(X).$$

We regard classes in $H_{\text{prim}}^{p,d-1-p}(X)$ as elements in the fiber of $\mathcal{H}^{p,d-p-1}$ at the point $[X] \in \mathcal{M}_\beta^0$. By assumption $\beta \in H^{d-p-1,p}(X)$ can be written as $\beta = \sum_i \gamma_p(t_i \otimes \eta_i)$ with $\eta_i \in H^{d-p,p-1}(X)$. Then by equations 1.2 and 1.4

$$\langle \alpha, \beta \rangle = \sum_i \langle \alpha, \gamma_p(t_i \otimes \eta_i) \rangle = \sum_i \langle \alpha, \nabla_{t_i} \eta_i \rangle = -\sum_i \langle \nabla_{t_i} \alpha, \eta_i \rangle = 0.$$

So α is orthogonal to $H_{\text{prim}}^{d-1-p,p}(X)$. By Lemma 1.2 , this means that α is orthogonal to the whole group $H^{d-1-p,p}(X)$, hence it is zero. Therefore $H_T^{p,d-1-p}(X) = i^*(H^{p,d-1-p}(X))$.

□

The next Lemma, a Noether-Lefschetz theorem, is the cornerstone of this thesis.

Lemma 1.20. Let $d = 2k+1 \geq 3$ and assume that the hypotheses of the previous Theorem hold for $p = k$. Then for u away from a countable union of subschemes of $\mathcal{U}_\beta$ of positive codimension one has

$$H^{k,k}(X_u, \mathbb{Q}) = i^*(H^{k,k}(\mathbb{P}_\Sigma, \mathbb{Q})).$$

Proof. Let $\widetilde{\mathcal{U}}_\beta$ be the universal cover of $\mathcal{U}_\beta$. On it the (pullback of the) local system $\mathcal{H}^{d-1}$ is trivial. Given a class $\alpha \in H^{k,k}$ we can extend it to a global section of H^{d-1} by parallel transport using the Gauss-Manin connection. Define the subset $\widetilde{\mathcal{U}}_\beta^\alpha$ of $\widetilde{\mathcal{U}}_\beta$ as the common zero locus of sections $\pi_k(\alpha)$ of $\mathcal{H}^{p,2k-p}$ for $p \neq k$, i.e., the locus where α is of type (k,k). If $\widetilde{\mathcal{U}}_\beta^\alpha = \widetilde{\mathcal{U}}_\beta$ we are done because α is in $H_T^{d-1}(X)$, hence is in the image of i^* by the previous Theorem. If $\widetilde{\mathcal{U}}_\beta^\alpha \neq \widetilde{\mathcal{U}}_\beta$, we note that $\widetilde{\mathcal{U}}_\beta^\alpha$ is a subscheme of $\widetilde{\mathcal{U}}_\beta$ and we subtract from $\mathcal{U}_\beta$ the union of the projections of the subschemes $\tilde{\mathcal{U}}_\beta$ where $\tilde{\mathcal{U}}_\beta^\alpha \neq \tilde{\mathcal{U}}_\beta$. The set of these varieties is countable because we are considering rational classes.

□

Since $T_X \mathcal{M}_\beta \cong R(f)_\beta$ and $H_{\text{prim}}^{k+1,k-1}(X) \cong R(f)_{k\beta-\beta_0}$, another way to rephrase the above Lemma is to say that if for a hyper-surface $X = \{f = 0\} \subset \mathbb{P}_\Sigma^{2k+1}$ with degree β, the multiplication morphism $R(f)_\beta \otimes R(f)_{k\beta-\beta_0} \to R(f)_{(k+1)\beta-\beta_0}$ is surjective. Then

$$NL_\beta := \{Y \subset \mathbb{P}_\Sigma^{2k+1} \text{quasi-smooth} \mid H^{k,k}(Y, \mathbb{Q}) \neq i^*(H^{k,k}(\mathbb{P}_\Sigma^{2k+1}), \mathbb{Q})\}$$

is a proper subscheme of $|L|$ and every irreducible component has positive codimension.

Definition 1.21. We call NL_β the Noether-Lefschetz locus.

Remark 1.22. We recover the definitions given in [7] and [36] when $k = 1$ and $\mathbb{P}^{2k+1}_\Sigma = \mathbb{P}^{2k+1}$ respectively.

The following chapters are dedicated to the study of the components of NL_β, the next chapter we will find upper and lower bound for the codimension of every irreducible component.

1.6 Oda Varieties

This section provides sufficient conditions for the surjectivity of the multiplication map $R(f)_\beta \otimes R(f)_{k\beta-\beta_0} \to R(f)_{(k+1)\beta-\beta_0}$

Remark 1.23. Note that $R(f)_\beta \otimes R(f)_{k\beta-\beta_0} \to R(f)_{(k+1)\beta-\beta_0}$ is surjective whenever the morphism $S_\beta \otimes S_{k\beta-\beta_0} \to S_{(k+1)\beta-\beta_0}$ is surjective.

Definition 1.24. A toric variety $\mathbb{P}_\Sigma$ is an Oda variety if the multiplication map $S_{\alpha_1} \otimes S_{\alpha_2} \to S_{\alpha_1+\alpha_2}$ is surjective whenever the classes α_1 and α_2 are ample and nef, respectively.

The question of the surjectivity of this map was posed by Oda in [35] under more general conditions. This assumption can be stated in terms of the Minkowski sum of polytopes, because the integral points of the polytope associated with a line bundle correspond to sections of the line bundle. So the above definition says that the sum $P_{\alpha_1} + P_{\alpha_2}$ of the polytopes associated with the line bundles $\mathcal{O}_{\mathbb{P}_\Sigma}(\alpha_1)$ and $\mathcal{O}_{\mathbb{P}_\Sigma}(\alpha_2)$ is equal to their Minkowski sum, that is, $P_{\alpha_1+\alpha_2}$, the polytope associated with the line bundle $\mathcal{O}_{\mathbb{P}_\Sigma}(\alpha_1 + \alpha_2)$. Three of the more relevant facts about Oda varieties are the following.

Theorem 1.25. ([9] and [23])

- *A smooth toric variety with Picard number 2 is an Oda variety.*
- *The total space of a toric projective bundle on an Oda variety is also an Oda variety.*
- *If a projective variety $\mathbb{P}_\Sigma$ has Picard number 1 and its ample generator is 0-regular then $\mathbb{P}_\Sigma$ is Oda.*

Chapter 2

Noether-Lefschetz components for toric varieties

The "Lefschetz hyper-surface theorem" claims that for a quasi-smooth hypersurface $X = \{f = 0\}$ on a complete simplicial toric variety $\mathbb{P}^{2k+1}_{\Sigma}$ the morphism $i^* : H^p(\mathbb{P}_{\Sigma}) \to H^p(X)$ induced by the inclusion is injective for $p = 2k$ and an isomorphism for $p < 2k$. This allows us to define the locus, for a fix degree β, of quasi-smooth hypersurfaces with degree β such that i^* is not an isomorphism, following the tradition we called it in [10] the Noether-Lefschetz locus .

On the other hand, we have seen an important consequence of the "Infinitesimal Noether-Lefschetz theorem for toric varieties" , the cornerstone result (1.5), which asserts that if the multiplication $R(f)_{\beta} \otimes R(f)_{k\beta-\beta_0} \to R(f)_{(k+1)\beta-\beta_0}$ is surjective, where β_0 is the anticanonical divisor of $\mathbb{P}^{2k+1}_{\Sigma}$, the Noether-Lefschetz locus is non-empty and each irreducible component has positive codimension. In the first part of the chapter we define and construct locally the Noether-Lefschetz components. We continue proving the main theorem of the chapter, let η be the class of a 0-regular ample divisor and assume that η is primitive. Let β be a Cartier class 0-regular respect to η such that $k\beta - \beta_0 = n\eta$ ($n \in \mathbb{N}$). Then every irreducible component N of the Noether-Lefschetz locus associated with β satisfies

$$n + 1 \leq \operatorname{codim} N \leq h^{k-1,k+1}(X),$$

generalizing to higher dimensions the work of Bruzzo and Grassi in [9] and Lanza and Martino in [28]. We finish the chapter showing a sufficient condition for which a Noether-Lefschetz component has maximal codimension, which we call general component.

2.1 The local Noether-Lefschetz loci

In the end of the last chapter we extended the Noether-Lefschetz locus to a complete simplicial toric variety of any odd dimension, in this section we define and characterize its components. These results have been expounded also in the paper [10].

For $f \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta)))$ a section such that $X_f = \{f = 0\}$ is a quasi-smooth hypersurface. Let $\mathcal{U}_\beta \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta)))$ be the open subset parametrizing the quasi-smooth hypersurfaces and let $\pi : \chi_\beta \to \mathcal{U}_\beta$ be its tautological family. Let $H^{2k}_{\mathbb{Q}}$ be the local system $R^{2k}\pi_\star\mathbb{Q}$ and let $\mathcal{H}^{2k}$ be the locally free sheaf $H^{2k}_{\mathbb{Q}} \otimes \mathcal{O}_{\mathcal{U}_\beta}$ over $\mathcal{U}_\beta$.

Let $0 \neq \lambda_f \in H^{k,k}(X_f, \mathbb{Q})/i^*(H^{k,k}(\mathbb{P}^{2k+1}_\Sigma))$ and let U be a contractible open subset around f, so that $\mathcal{H}^{2k}(U)$ is constant. Finally, let $\lambda \in \mathcal{H}^{2k}(U)$ be the section defined by λ_f and let $\bar{\lambda}$ its image in $(\mathcal{H}^{2k}/F^k\mathcal{H}^{2k})(U)$, where $F^k\mathcal{H}^{2k} = \mathcal{H}^{2k,0} \oplus \mathcal{H}^{2k-1,1} \oplus \cdots \oplus \mathcal{H}^{k,k}$.

Definition-Proposition 2.1 (Local Noether-Lefschetz loci)**.**

$$N^{k,\beta}_{\lambda,U} := \{G \in U \mid \bar{\lambda}_G = 0\}.$$

More explicitly,

Proposition 2.2. If $(\lambda_1, \dots \lambda_b)$ are the components of λ_f respect to a fix basis of the vector space $H^{2k}(X_f, \mathbb{Q})$, one gets

$$N^{k,\beta}_{\lambda,U} = \{G \in U \mid \lambda_f \perp F^{k+1}H^{2k}_{\text{prim}}(X_G) \iff \textstyle\sum_{i=1}^b \lambda_i \int_{\text{Tub}\,\gamma_i} \frac{K\Omega_0}{G^k} = 0 \;\forall K \in S^{N-\beta}\};$$

where N is equal to $(k+1)\beta - \beta_0$.

Proof. By Proposition 1.4 the p-th residue map

$$r_p : H^0(\mathbb{P}_\Sigma, \Omega^{2k+1}_{\mathbb{P}_\Sigma}(2k+1-p)X) \to H^{p,2k-p}_{\text{prim}}(X) \;\; for \;\; 0 \leq p \leq 2k$$

exists; it is surjective and has kernel $H^0(\mathbb{P}_\Sigma, \Omega^{2k+1}_{\mathbb{P}_\Sigma}(2k-p)X) + dH^0(\mathbb{P}_\Sigma, \Omega^{2k}_{\mathbb{P}_\Sigma}(2k-p)X)$. So

$$\text{res}\, H^0(\Omega^{2k+1}(2k+1)X) = r_{2k}H^0(\Omega^{2k+1}(X)) \oplus \cdots \oplus r_0H^0(\Omega^{2k+1}(2k+1)X))$$

by definition of $H^0(\Omega^{2k+1}(2k+1)X)$. Or, equivalently,

$$\text{res}\, H^0(\Omega^{2k+1}(2k+1)X) = H^{2k,0}_{\text{prim}}(X) \oplus \cdots \oplus H^{0,2k}_{\text{prim}}(X) = H^{2k}_{\text{prim}}(X).$$

Similarly

$$\text{res}\, H^0(\Omega^{2k+1}(kX) = F^{k+1}H^{2k}_{\text{prim}}(X).$$

On the other hand by [4, Thm 9.7] we have

$$H^0(\Omega^{2k+1}_{\mathbb{P}_\Sigma}(kX) = \{\frac{K\Omega_0}{f^k} \mid K \in S^{k\beta-\beta_0}\} = \{\frac{K\Omega_0}{f^k} \mid K \in B^{k\beta-\beta_0}_\Sigma\};$$

the last equality holds true because we are assuming that $k\beta - \beta_0$ is ample and hence $B_\Sigma^{k\beta-\beta_0} = S^{k\beta-\beta_0}$ by Lemma 9.15 in [4].

Now fixing a basis $\{\gamma_i\}_{i=1}^b$ for $H_{2k}(X,\mathbb{Q})$ we have that the components of any element in $F^{k+1}H^{2k}_{\text{prim}}(X)$ are

$$(\int_{\gamma_1} \text{res}\,\frac{K\Omega_0}{f^k}, \ldots, \int_{\gamma_b} \text{res}\,\frac{K\Omega_0}{f^k}),$$

or, equivalently,

$$(\int_{\text{Tub}(\gamma_1)} \frac{K\Omega_0}{f^k}, \ldots, \int_{\text{Tub}(\gamma_b)} \frac{K\Omega_0}{f^k})$$

where $\text{Tub}(\gamma_j)$ is the adjoint to the residue map. Now taking $0 \neq \lambda_f \in H^{k,k}(X,\mathbb{Q})$ one has $\lambda_f \perp F^{k+1}H^{2k}_{\text{prim}}(X)$ (see [47]) and since the sheaf $\mathcal{H}^{2k}$ is constant on U we have

$$NL^{k,\beta}_{\lambda,U} = \{G \in U \mid \lambda_G \in F^k H^{2k}_{\text{prim}}(X_G)\} = \{G \in U \mid \lambda_f \perp F^{k+1}H^{2k}_{\text{prim}}(X_G)\}.$$

Moreover, by the above equivalence

$$\lambda_f \perp F^{k+1}H^{2k}_{\text{prim}}(X_G) \quad \Leftrightarrow \quad \textstyle\sum_{i=1}^b \lambda_i \int_{\text{Tub}\,\gamma_i} \frac{K\Omega_0}{G^k} = 0 \;\forall K \in S^{N-\beta}$$

where N is equal to $(k+1)\beta - \beta_0$. □

Remark 2.3. Note that $NL_\beta = \bigcup_U N^{k,\beta}_{\lambda,U}$.

2.2 Explicit Noether-Lefschetz theorem

This section is a natural extension of the ideas of [28] to higher dimensional toric varieties. So starting with the study of the Noether-Lefschetz components we find a lower bound for their codimension that following the terminology in [6] and [19] we call the "Explicit Noether-Lefschetz theorem for toric varieties".

Let X be a projective variety and L be an ample and globally generated line bundle on X.

Definition 2.4. [Castelnuovo-Mumford regularity] A coherent $\mathcal{O}_X$−module $\mathcal{F}$ is m-regular with respect to L if

$$H^q(X, \mathcal{F} \otimes L^{m-q}) = 0$$

for all $q > 0$. If L is an ample and globally generated line bundle which is m−regular with respect to itself, we call it m-regular.

A line bundle on a complete toric variety is nef if and only if it is globally generated. By toric Kleiman criterion [[16] Theorem 6.3.13] every ample line bundle is globally generated.

Theorem 2.5. *[[29] Theorem 1.8.5] Let $\mathbb{P}_\Sigma$ be a projective toric variety. If a locally free $\mathcal{O}_{\mathbb{P}_\Sigma}$-module $\mathcal{F}$ is m-regular with respect to an ample line bundle L, then for all $k \geq 0$,*

i. $\mathcal{F} \otimes L^{m+k}$ is generated by global sections;

ii. The map
$$H^0(\mathcal{F} \otimes L^m) \otimes H^0(L^k) \to H^0(\mathcal{F} \otimes L^{k+m}) \tag{2.1}$$
is surjective;

iii. $\mathcal{F}$ is $(m+k)$-regular.

Proposition 2.6 ([28] Proposition 2)**.** Let X be a projective variety together with an ample line bundle L which is globally generated and 0−regular. If $\mathcal{F}$ is an m-regular locally free sheaf on X, then the $p-$ tensor power $\mathcal{F}^{\otimes p}$ is (pm)-regular. In particular, $\bigwedge^p \mathcal{F}$ and $S^p\mathcal{F}$ are (pm)-regular.

Theorem 2.7. *Let $\mathbb{P}^{2k+1}_\Sigma$ be a projective toric variety, $\beta \in \mathrm{Pic}(\mathbb{P}^{2k+1}_\Sigma)$ and η a primitive ample 0−regular Cartier class such that $k\beta - \beta_0 = n\eta$ where β_0 it is the anticanonical class of $\mathbb{P}^{2k+1}_\Sigma$. If the multiplication morphism $S_\beta \otimes S_{n\eta} \to S_{\beta+n\eta}$ is surjective and*
$$H^1(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta - \eta)) = H^2(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta - 2\eta)) = \cdots = H^{2k}(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta - 2k\eta)) = 0,$$
then
$$n + 1 \leq \operatorname{codim} NL^{k,\beta}_{\lambda,U}.$$

Proof. We take a base point free linear system W in $H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta))$ and a complete flag of linear subspaces
$$W = W_c \subset W_{c-1} \subset \cdots \subset W_1 \subset W_0 = H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta)).$$
Let M_i the kernel of the surjective map $W_i \otimes \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma} \to \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta)$ which is a vector bundle.

Step I: M_0 is 1− regular respect to η.

Equivalently we have to show that $H^q(M_0((1-q)\eta)) = 0$ for every positive q. Taking cohomology we get
$$0 \to H^0(M_0) \to H^0(W_0 \otimes \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}) \xrightarrow{\pi} H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta)) \to H^1(M_0) \to \cdots$$
since π is surjective, $H^1(M_0) = 0$. The vanishing of $H^q(M_0(1-q)\eta) = 0$ for $1 < q \leq 2k+1$ is obtained by tensoring the short exact sequence by $\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}((1-q)\eta)$, and considering that $H^q(M_0(1-q)\eta)$ is between two zeros in the long exact sequence
$$\cdots \to H^{q-1}(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta - (q-1)\eta)) \to H^q(M_0(-(q-1)\eta)) \to$$
$$\to H^q(W_0 \otimes \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(-(q-1)\eta)) \to \cdots$$

$H^{q-1}(\mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(\beta-(q-1)\eta)) = 0$ by assumption and $H^q(W_0 \otimes \mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(-(q-1)\eta)) = 0$ because η is 0-regular.

Step II: For every $i = 0, \dots c$, $H^q(\bigwedge M_i(n\eta)) = 0$, if $q \geq 1$ and $n + q \geq p + i$.

By Theorem 2.5 one has that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_\Sigma^{2k+1}$ is m-regular with respect to η if and only if $H^q(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(n\eta)) = 0$, for all $q > 0$, $n \geq m - q$. Using induction, ascending on i and descending on p, the case $p > \operatorname{rk} M_i$ being automatic, we get the result.

Step III: If $c = \operatorname{codim} W \leq n$, then the map $W \otimes H^0(\mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(n\eta)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(\beta + n\eta))$ is surjective.

We consider the short exact sequence

$$0 \to M_c \to W \otimes \mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}} \to \mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(\beta) \to 0$$

and twist it by $\mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(n\eta)$. Taking cohomology we get

$$\cdots \to H^0(W \otimes \mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(n\eta)) \to H^0(\mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(\beta + n\eta)) \to H^1(M_c(n\eta)) \to \cdots$$

now applying Step II for $p = q = 1$, we get that $H^1(M_c(n\eta)) = 0$. Now we are ready to finish the proof of the theorem. Let T_β be the tangent space at $X = \{f = 0\}$ a point of the Noether-Lefschetz loci which can be identified with the summand R_β of the Jacobian ring of f so we may take the inverse image $\tilde{T}_\beta$ of T_β in the summand S_β of the Cox ring of X. Now $\tilde{T}_\beta$ contains J^β, which is a base point free linear system because X is quasi-smooth. Hence $\tilde{T}_\beta$ is base point free. Now, by contradiction if $\operatorname{codim} NL^{k,\beta}_{\lambda,U} \leq n$ then by Step III, $\tilde{T}_\beta \otimes S^{n\eta} \to S^{\beta+n\eta}$ is surjective hence by Infinitesimal Noether-Lefschetz Theorem $\lambda_f \notin NL^{k,\beta}_{\lambda,U}$. □

Remark 2.8. In order to find examples satisfying the assumptions of the above theorem we can use Proposition 7.3 in [4] which claims that for an ample line bundle $\mathcal{L}$ on a complete toric variety $H^i(\mathcal{L}) = 0$ for $i > 0$.

2.3 Upper bound for the codimension

The Explicit Noether-Lefschetz Theorem gave us the lower bound for the codimension of the Noether-Lefschetz components. Hodge theory in projective simplicial toric varieties will give us the upper bound. Namely, $\operatorname{codim} NL^{k,\beta}_{\lambda,U} \leq h^{k-1,k+1}(X)$.

Classically [50] or [49], the upper bound is a consequence of Griffiths Transversality which we will extend in this section to the context of projective simplicial toric varieties following [31], which contains implicitly the proof.

2.3.1 Variations of the Hodge Structure

The tautological family $\pi : \mathcal{X}_\beta \subset \mathcal{U}_\beta \times \mathbb{P}_\Sigma \to \mathcal{U}_\beta$ is of finite type and separated since $\mathcal{X}_\beta$ and $\mathcal{U}_\beta$ are varieties. Now applying Corollary 5.1 in [46] there exists a Zariski open set $\mathcal{U} \subset \mathcal{U}_\beta$ such that $\mathcal{X} := \pi^{-1}(\mathcal{U}) \to \mathcal{U}$ is a locally trivial fibration in the classical topology i.e., there exists an open cover of $\mathcal{U}$ by contractible open sets such that for every element U of the cover and every element $X_0 \in U$ we have that $\mathcal{X}_{|U} := \pi^{-1}(U) \simeq U \times X_0$. Moreover $\mathcal{X}_\beta$ is an orbifold and $\mathcal{U}_\beta$ clearly, thus by Ehresmann's theorem for orbifolds (0.6) we conclude that $\mathcal{U} = \mathcal{U}_\beta$ and $\mathcal{X}_\beta = \mathcal{X}$. So we obtain that for U a contractible open set of $\mathcal{U}_\beta$, $\forall u \in U$, $X_u \simeq X_0$ in the smooth category of orbifolds and furthermore $H^k(X_u) \simeq H^k(X_0)$.

2.3.2 The Cartan-Lie formula

The Cartan-Lie formula provides a explicit description of the Gauss-Manin connection, namely

Lemma 2.9. Let rel be the homomorphism from the space of differential forms on $\mathcal{X}$ to the space of relative differential forms on $\mathcal{X}/U$. For any smooth section $\omega : U \to \mathcal{H}^k(\mathcal{X}_{|U})$ and a smooth tangent vector field w over U, the Gauss-Manin connection can de described as:

$$\nabla_w \omega = [\mathrm{rel}(\iota_v d\Omega)],$$

where Ω is a form on $\mathcal{X}_{|U}$ such that $\mathrm{rel}(\Omega)$ represents ω_u on X_u for every $u \in U$. v any tangent vector field on $\mathcal{X}_{|U}$ such that $\pi_* v = u$ and we use $[-]$ to denote the cohomology class represented by the closed differential form and ι_v is the interior product.

Proof. The existence of Ω and v are thanks to the partition of unity subordinated to the covering by contractible open sets making π a trivial fibration ([31] Lemma 6.6). We fix a trivialization $\pi_U : \mathcal{X}_{|U} \simeq U \times X_0$ and we cover $U \times X_0$ by the charts $\{(U \times V_\alpha), G_\alpha, Id \times \phi_\alpha\}_{\alpha \in I}$, and let $\{x_\alpha, u_\alpha\} = \{x_\alpha^1, \ldots, x_\alpha^n, t_\alpha^1, \ldots, t_\alpha^m\}$ be a system of coordinates on $U \times X_0$. Now on $U \times V_\alpha$ we may write Ω as,

$$\Omega_\alpha = \Phi_\alpha + \sum_{j=1}^m dt_\alpha^j \wedge + \sum_{j=1}^m d\bar{t}_\alpha^j \wedge \psi_{\alpha j} + \Omega'_\alpha, \tag{2.2}$$

where $\Phi_\alpha, \phi_{\alpha,j}$ and $\psi_{\alpha,j}$ do not contain dt_α^i or $d\bar{t}_\alpha^i$, $i = 1, \ldots, m$ and Ω'_α is a section of $\pi^*(\bigwedge^2 \Omega_U^1) \wedge \Omega_{U \times X_0}^{k-2}$. Note that Φ_α, $\psi_{\alpha,j}$ and $\psi_{\alpha,j}$ are $G_\alpha-$ invariant forms on U_α and can be glued to be global forms on $\{u\} \times X_0$ which we denote by $\Phi(u), \phi_j(u)$ and $\psi_j(u)$ respectively. By assumption, we have $[\Phi(u)] = \omega(u)$ under the pullback of the diffeomorphism $X_u \simeq u \times X_0$. Let

$$w = \sum_{j=1}^m a^j(u) \frac{\partial}{\partial t^j} + \sum_{j=1}^m b^j(u) \frac{\partial}{\partial \bar{t}^j}$$

be a tangent vector field on U. Then,

$$\nabla_w \omega = \left[\sum_{j=1}^m a^j(u) \frac{\partial \Phi(u)}{\partial u^j} + \sum_{j=1}^m b^j(u) \frac{\partial \Phi(u)}{\partial \bar{u}^j} \right]. \tag{2.3}$$

Now taking exterior derivative of Ω_α (2.2) we get

$$d\Omega_\alpha = \sum_{j=1}^{m}\left[du_\alpha^j \wedge \frac{\partial \Phi_\alpha}{\partial u_\alpha^j} + d\bar{u}_\alpha^j \wedge \frac{\partial \Phi_\alpha}{\partial \bar{u}_\alpha^j}\right] + \sum_{j=1}^{m} du^j \wedge d\phi_{\alpha j} + \sum_{j=1}^{m} d\bar{u}^j \wedge d\psi_{\alpha j} + d\Omega'_\alpha$$

Since $\mathrm{rel}\, d\Omega = 0$, for any v lifting u, we have that

$$\mathrm{rel}\,\iota_v d\Omega_\alpha = \sum_{j=1}^{m} a^j(u_\alpha)\frac{\partial \Phi_\alpha}{\partial u_\alpha^j} + b^j(u_\alpha)\frac{\partial \Phi_\alpha}{\partial \bar{u}_\alpha^j} + a^j(t_\alpha)d\phi_{\alpha,j} + b^j(t_\alpha)d\psi_{\alpha,j}$$

then we obtain a global equality

$$rel\iota_v d\Omega = \sum_{j=1}^{m} a^j(u_\alpha)\frac{\partial \Phi_\alpha}{\partial u_\alpha^j} + b^j(u_\alpha)\frac{\partial \Phi_\alpha}{\partial \bar{u}_\alpha^j} + a^j(t_\alpha)d\phi_{\alpha,j} + b^j(t_\alpha)d\psi_{\alpha,j} \tag{2.4}$$

Combining 2.3 and 2.4 we are done. □

2.3.3 Local period map

Again we take U a contractible open set trivializing, i.e., $\mathcal{X}_{\mathcal{U}} \simeq U \times X_0$

Definition 2.10. The period map

$$\mathcal{P}^{p,k} : \mathcal{U} \to Grass(b^{p,k}, H^k(X_0, \mathbb{C}))$$

is the map which to $u \in U$ associates the subspace $F^pH^k(X_u, \mathbb{C}) \subset H^k(X_u, \mathbb{C}) \simeq H^k(X_0, \mathbb{C})$

Note that $\mathcal{P}^{p,k}$ is a map between complex manifolds. Moreover,

Proposition 2.11. $\mathcal{P}^{p,k}$ is holomorphic.

Proof. By theorem 7.9 in [24] and the fact that Hodge theorem holds also in the orbifold case (section 2.1 in [30]) and moreover the canonical isomorphism respect the Hodge filtrations because the Kähler identities on a Kähler manifold are local statements, we may apply the argument verbatim to a Kähler orbifold and conclude that all the Kähler identities remain true on a Kähler orbifold, we get that $\mathcal{P}^{p,k}$ is a C^∞ map. In order to prove that is holomorphic the strategy is to show that the $\mathbb{C}-$ linear extension of its differential to $T_uU \otimes \mathbb{C}$ vanishes in the vectors $w \otimes (0,1)$. We have that the differential has the form

$$d\mathcal{P}^{p,k}_{u_0} : T_{u_0}U \to Hom(F^pH^k(X_{u_0}), H^k(X_0)/F^p(X_u)).$$

Now, for any $w = \sum_{j=1}^m a^j \frac{\partial}{\partial u^j} + b^j \frac{\partial \omega(u)}{\partial \bar{u}^j} \in T_{u_0}U$ and any $\omega_0 \in F^p(H^k(X_{u_0}))$ since $\{F^p\mathcal{H}^k \mid u \in U\}$ is a smooth vector bundle we can find a smooth section ω of $\mathcal{H}^k$ such that $\omega(u) \in F^pH^k(X_0)$ and $\omega(u_o) = \omega_0$. Then

$$d\mathcal{P}^{p,k}_{u_0}(w)(\omega_{|0}) = \sum_{j=1}^m a^j \frac{\partial \omega(u)}{\partial u^j} + b^j \frac{\partial \omega(u)}{\partial \bar{u}^j} \mid_{u=0} mod F^p H^k(X_{u_0}) \tag{2.5}$$

Hence

$$d\mathcal{P}^{p,k}_{u_0}(w)(\omega_{|0}) = \nabla_w \omega(u) \mid_{u=0} mod F^p H^k(X_{u_0}$$

and by the Cartan-Lie formula $\nabla_w \omega(.) = \mathrm{rel}\, \iota_v d\Omega$, where v can be taken of $(0,1)-$ type and $\Omega \in \bigoplus_{i\geq p} \Omega^{i.k-i}_{\mathcal{X}}$. Hence

$$\iota_v d\Omega \in \bigoplus_{i\geq p+1} \Omega^{i,k-i}_{\mathcal{X}_{|\mathcal{U}}} \subset \bigoplus_{i\geq p} \Omega^{i,k-i}_{\mathcal{X}_{|\mathcal{U}}}.$$

□

The proof of Griffiths Transversality is implicit in the last part of the above proposition, namely

Proposition 2.12 (Griffiths Transversality)**.** $\nabla F^p\mathcal{H}^k \subset F^{p-1}\mathcal{H}^k \otimes \Omega_{\mathcal{U}}$.

Proof. Let us consider v as before but of type $(1,0)$ thus $\iota_v d\Omega \in \bigoplus_{i\geq p-1} \Omega^{i,k-i}_{\mathcal{X}_{|\mathcal{U}}}$. □

Now we are able to show the upper bound for the codimension of the Noether-Lefschetz components

Theorem 2.13. *Each $NL^{k,\beta}_{\lambda,U} \subset \mathcal{U}$ can be defined locally by at most $h^{k-1,k+1}$ holomorphic equations, where $h^{k-1,k+1} := \mathrm{rk}\, F^{k-1}\mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$.*

Proof. The proof is based on the Griffiths Transversality and follows verbatim as in classical case, see Lemma 3.1 in [50] and section 5.3 in [48]. □

When $k = 1$,that is, $\mathbb{P}^3_\Sigma$ is a threefold we can tell more

Proposition 2.14. $h^{0,2}(X) = h^0(\omega_{\mathbb{P}^3_\Sigma}(X))$.

Proof. Taking cohomology in the exact sequence

$$o \to \mathcal{O}_{\mathbb{P}_\Sigma}(-X) \to \mathcal{O}_{\mathbb{P}_\Sigma} \to \mathcal{O}_X \to 0$$

we get

$$0 \to H^2(\mathcal{O}_X) \to H^3(\mathcal{O}_{\mathbb{P}_\Sigma}(-X)) \to 0$$

because by Theorem 9.3.2 in [18] $H^2(\mathcal{O}_{\mathbb{P}_\Sigma}) = H^3(\mathcal{O}_{\mathbb{P}_\Sigma}) = 0$. Then by Serre duality $h^{2,0} = h^0(\omega_{\mathbb{P}^3_\Sigma}(X))$. □

The transversality property allows one to construct the $\mathcal{O}_U$ linear maps

$$\bar{\nabla} : \mathcal{H}^{i,j-i} \to \mathcal{H}^{i-1,j-i+1}$$

and for every $u \in U$

$$^{u}\bar{\nabla} : T_u U \to \operatorname{Hom}\left(H^{j-i}(\Omega^i_{X_u}), H^{j-i+1}(\Omega^{i-1}_{X_u})\right)$$

Proposition 2.15. The Zariski tangent space to $NL^{k,\beta}_{\lambda,U}$ at u is described as

$$T_u NL^{k,\beta}_{\lambda,U} = \ker({}^{u}\bar{\nabla}\lambda^{k,k} : T_u U \to \mathcal{H}^{k-1,k+1}_u)$$

where $\lambda^{k,k}$ is the projection of λ to $\mathcal{H}^{k,k}_u$

Proof. This follows verbatim as in Lemma 5.16 in [49]. □

Corollary 2.16. A Noether-Lefschetz component $NL^{k,\beta}_{\lambda,U}$ has codimension $h^{k-1,k+1}$ at a point u where the map ${}^{u}\bar{\nabla}\lambda^{k,k}$ is surjective.

Chapter 3

An asymptotic description of the Noether-Lefschetz components

In [19] Green and in [47] Voisin proved that if N_d is the Noether-Lefschetz locus for degree d surfaces in $\mathbb{P}^3$, with $d \geq 4$, the codimension of every component of N_d is bounded from below by $d-3$, with equality exactly for the components formed by surfaces containing a line. Otwinowska gave an asymptotic generalization of Green and Voisin's results to hypersurfaces in $\mathbb{P}^n$ [36].

In Chapter 2 we proved, in particular, that for simplicial projective toric threefolds the codimension of the Noether-Lefschetz components are also bounded from below. Bruzzo and Grassi in [9] also proved that components corresponding to surfaces containing a "line", defined as a curve which is minimal in a suitable sense, realize the lower bound. However the question whether these are exactly the components of smallest codimension was left open.

This chapter was expounded in [10], its purpose is to extend and generalize Otwinowska's ideas to odd dimensional simplicial projective toric varieties.
In section 3.1 we present a generalization of the restriction theorem due to Green [21] and we obtain an extension of the classical Macaulay theorem, while in section 3.2 we introduce a generalization of the notion of Gorenstein ideal, which we call a Cox-Gorenstein ideal; these will be the key tools in the proof of our main result. Section 3.3 is more technical; there we prove some application of Macaulay theorem to Cox-Gorenstein ideals. In section 3.4 using Hodge theory we explicitly construct the tangent space at a point in the Noether Lefschetz loci, which turns out to be a graded part of a Cox-Gorenstein ideal. In section 6 using all the machinery so far developed we prove our main result.

We shall consider a a projective simplicial toric variety $\mathbb{P}^{2k+1}_{\Sigma}$, and an ample line bundle L on $\mathbb{P}^{2k+1}_{\Sigma}$, with $\deg L = \beta \in \operatorname{Pic}(\mathbb{P}^{2k+1}_{\Sigma})$ satisfying for some $n \geq 0$ and $k \geq 1$ the condition

$$k\beta - \beta_0 = n\eta$$

where β_0 is the class of the anticanonical bundle and η is the primitive class of an ample Cartier divisor (for $k = 1$ this reduces to the condition considered in [9]). $f \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta)))$ will be a section such that $X_f = \{f = 0\}$ is quasi-smooth hypersurface in the local Noether-Lefschetz component $N^{k,\beta}_{\lambda,U}$ (2.1). The following is the main result of this Chapter.

Theorem. *3.31 For every positive ϵ there is positive δ such that for every $m \geq \frac{1}{\delta}$ and $d \in [1, m\delta]$, if $\operatorname{codim} N^{k,\beta}_{\lambda,U} \leq d\frac{m^k}{k!}$ where $m = \max\{i \mid i\eta \leq \beta\}$, then every element of $N^{k,\beta}_{\lambda,U}$ contains a k-dimensional subvariety whose degree is less than or equal to $(1+\epsilon)d$.*

3.1 A restriction theorem

Every positive integer c can be written in the form

$$\binom{k_n}{n} + \cdots + \binom{k_\delta}{\delta},$$

with $k_n > k_{n-1} > \ldots k_\delta \geq \delta > 0$. This is called the n-th Macaulay decomposition of c. Let c be the codimension of a linear subsystem $W \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$, and let $W_H \subset H^0(\mathcal{O}_H(d))$ be the restriction of W to a general hyperplane H of codimension c_H. Then the classical restriction theorem says that

$$c_H \leq c_{<n>},$$

where

$$c_{<n>} := \binom{k_n - 1}{n} + \cdots + \binom{k_\delta - 1}{\delta}.$$

We generalize this result for a Fano, irreducible, projective normal variety Y with rational singularities, satisfying a suitable additional condition. We note two elementary properties of the function $\phi : c \mapsto c_{<n>}$:

(A) If $c' \leq c$, then $c'_{<n>} \leq c_{<n>}$, i.e , the map ϕ is non-decreasing;

(B) If $k_\delta > \delta$ then $(c-1)_{<n>} < c_{<n>}$ i.e the map ϕ is increasing.

Lemma 3.1. Let Y be an irreducible, normal projective variety with $H^1(\mathcal{O}_Y) = 0$. Let $W \subset H^0(Y, \mathcal{O}_Y(D))$ be a sublinear system, D a generic ample Cartier divisor and let $W_D \subseteq H^0(D, \mathcal{O}_Y(D))$ be its restriction. Then

$$c_D = \operatorname{codim}(W_D, H^0(\mathcal{O}_D(D))) \leq c_{<1>} = \operatorname{codim}(W, H^0(\mathcal{O}_Y(D))) - 1$$

Proof. Taking cohomology in the fundamental short exact sequence of the divisor D we obtain

$$0 \to H^0(\mathcal{O}_Y) \to H^0(\mathcal{O}_Y(D)) \to H^0(\mathcal{O}_D(D)) \to 0 \to \cdots$$

so that

$$h^0(\mathcal{O}_Y(D)) = h^0(\mathcal{O}_Y) + h^0(\mathcal{O}_D(D)) = 1 + h^0(\mathcal{O}_D(D)). \tag{3.1}$$

Let $W_D = \{w_{|D} \mid w \in W\}$. Denoting by r the projection $W \to W_D$ one has

$$\dim W = \dim \ker r + \dim W_D. \tag{3.2}$$

so that subtracting (3.2) from (3.1) we have

$$\operatorname{codim} W = \operatorname{codim} W_D + 1 - \dim \ker r.$$

If s_D a section in $H^0(\mathcal{O}_Y(D))$ such that $D = \operatorname{div}_0(s_D)$, then

$$\ker r = \{w \in W \mid w = \lambda s_D \in W, \lambda \in \mathbb{C}\}$$

and since D is general so that $s_D \notin W$, then $\ker r = \{0\}$. □

Lemma 3.2. Let $W \subset H^0(\mathcal{O}_{\mathbb{P}^1}(n))$ $(n > 1)$ be a subsystem, D be a generic point and let $W_D \subset H^0(\mathcal{O}_D(n))$ be its restriction . Then

$$c_D \leq c_{<n>}.$$

Proof. Clearly $H^0(\mathcal{O}_D(n)) = \mathbb{C}$ and since D is generic $c_D = 0$. On the other hand because $n > 1$ we have that $k_n > 1$, so that $c_{<n>} > 0$,i.e., $c_D \leq c_{<n>}$. □

Definition 3.3. A strongly Fano variety is a pair (Y, D), where Y is an irreducible normal projective variety with rational singularities, and D is an ample ample Cartier divisor such that $-K_Y - (k-1)D$ is ample, where $k = \dim Y$.

Theorem 3.4 (Restriction Theorem)**.** *Let (Y, D) be a strongly Fano variety, let $W \subset H^0(X, \mathcal{O}_X(nD))$, with $n \geq 1$, be a subsystem, and let $W_D \subseteq H^0(D, \mathcal{O}_D(nD))$ be its restriction to D. Then*

$$c_D \leq c_{<n>}.$$

Proof. Let $l_n, \dots l_\delta$ be the coefficients of the n-th Macaulay decomposition of c_D. The inequality of the statement is equivalent to

$$\binom{l_n + 1}{n} + \binom{l_{n-1} + 1}{n-1} + \cdots + \binom{l_\delta + 1}{\delta} < c.$$

By contradiction, and recalling that $\binom{l+1}{n} = \binom{l}{n} + \binom{l}{n-1}$, we have

$$c \le \binom{l_n}{n} + \binom{l_n}{n-1} + \cdots + \binom{l_\delta}{\delta} + \binom{l_\delta}{\delta-1}$$

or equivalently

$$c - c_D \le \binom{l_n}{n-1} + \cdots + \binom{l_\delta}{\delta-1}. \tag{3.3}$$

From the exact sequence

$$0 \to W(-D) \to W \to W_D \to 0$$

one has

$$\dim W = \dim W_D + \dim W(-D). \tag{3.4}$$

By a generalized Kodaira vanishing theorem [44] applied to the divisor $(n-1)D - K_Y$ $(n \ge 1)$, we have $H^1(Y, K_Y + (n-1)D - K_Y)) = 0$, so that

$$0 \to H^0(\mathcal{O}_Y(n-1)D) \to H^0(\mathcal{O}_Y(nD)) \to H^0(\mathcal{O}_D(nD)) \to 0$$

and thus

$$h^0(\mathcal{O}_Y(nD)) = h^0(\mathcal{O}_Y(n-1)D) + h^0(\mathcal{O}_D(nD)). \tag{3.5}$$

Then (3.4) minus (3.5) yields

$$c = c_D + \operatorname{codim} W(-D).$$

Taking $D' \in |D|$ generic we are within the same assumptions of the theorem on D, i.e.,

- $D \cap D'$ is a generic Cartier divisor in D;
- moreover D is irreducible, normal with rational singularities [5];
- $-K_D - (k-2)D_{|D}$, where $k = \dim Y$, is ample because Y has rational singularities so it is Cohen-Macaulay (see e.g. [26]), and one can apply the adjunction formula [25] to get

$$\begin{aligned} -K_D - (k-2)D_{|D} &= -K_{Y|D} - D_{|D} - (k-2)D_{|D} \\ &= (-K_Y - (k-1)D)_{|D}, \quad (k-1 = \dim D); \end{aligned} \tag{3.6}$$

  by assumption the last divisor is ample.

Now we have the short exact sequence

$$0 \to W_D(-(D \cap D')) \to W_D \to W_{D|D'} \to 0$$

which gives

$$c_D = \operatorname{codim} W_{D|D'} + \operatorname{codim} W_D(-(D \cap D'))$$

Note that $W(-D')_D \subset W_D(-(D \cap D'))$, hence

$$c_D \leq \operatorname{codim} W_{D|D'} + \operatorname{codim} W(-D')_D$$

Also note that strongly Fano implies Fano, so by the generalized Kodaira vanishing theorem $H^1(\mathcal{O}_Y) = 0$; moreover since at each step of taking a successive generic divisor, the divisor is Fano, we have $h^1(\mathcal{O}_D) = 0 = h^1(\mathcal{O}_{D\cap D'})$, and so on. Now by induction on n and the dimension k the theorem is true for W_D and $W(-D)$; Lemmas 3.1 and 3.2 provide the induction basis. Applying the theorem to W_D and $W(-D)$ we get

- $(c_D)_{|D'} \leq (c_D)_{<n>} = \binom{l_n - 1}{n} + \cdots + \binom{l_\delta - 1}{\delta}$
- $(c - c_D)_{|D'} \leq (c - c_D)_{<n-1>}$

Adding the two inequalities and keeping in mind that $D' \sim D$ we have

$$c_{D'} = c_D \leq (c_D)_{<n>} + (c - c_D)_{<n-1>},$$

and by (3.3) and property (A)

$$(c - c_D)_{<n-1>} < \binom{l_n - 1}{n - 1} + \cdots + \binom{l_\delta - 1}{\delta - 1},$$

so that

$$c_D < \binom{l_n - 1}{n} + \cdots + \binom{l_\delta - 1}{\delta} + \binom{l_n - 1}{n - 1} + \cdots + \binom{l_\delta - 1}{\delta - 1} = c_D$$

which is a contradiction. □

Example 3.5. Taking $Y = \mathbb{P}^k$ and $D = H$ a generic hyperplane, we recover the classical restriction theorem [21]. Clearly

$$-K_{\mathbb{P}^{k+1}} - (k-1)H = (k+1)H - (k-1)H = 2H$$

which is ample.

More generally,

Example 3.6. Let $Y = \mathbb{P}[q_0, q_1, \ldots, q_k]$ be a weighted projective space with $gcd(q_0, \ldots, q_k) = 1$ and $\delta = lcm(q_0, \ldots, q_k)$. Then for each $0 \leq j \leq k$, by [40] $\frac{\delta}{q_j} D_j$ is a generator of $\operatorname{Pic}(Y)$ and $-K_Y = \frac{\sum_i q_i}{\delta}\left(\frac{\delta}{q_j} D_j\right)$. So taking $D = \frac{\delta}{q_j} D_j$ we get that

$$K_Y - (k-1)D \textit{ is ample if and only if } \frac{\sum_i q_i}{\delta} \geq k.$$

Lemma 3.7. Let $\mathbb{P}_\Sigma$ be a Fano projective simplicial toric 3-fold. Then every general nef D Cartier divisor with $\rho(D) \leq 4$ is toric.

Proof. By the adjunction formula D is Fano and being nef is smooth by Bertini's theorem. The smooth Fano surfaces are either $\mathbb{P}^1 \times \mathbb{P}^1$ which is toric or the projective plane blown up in at most 8 points. Since $\rho(D) < 4$, D is the blow up of $\mathbb{P}^2$ in at most 3 points. Applying an appropriate automorphism we can take these at most 3 points to the 3 toric points of $\mathbb{P}^2$, making D isomorphic to a toric variety. □

Macaulay theorem. A generalization of the classical Macaulay theorem can be obtained from the restriction Theorem 3.4. Let $W \subset H^0(\mathcal{O}_Y(nD))$ be a subsystem and let $k_n, k_{n-1}, \ldots k_\delta$ be the Macaulay coefficients of its codimension c; let W_1 be the image of the multiplication map $W \otimes H^0(\mathcal{O}_Y(D)) \to H^0(\mathcal{O}_Y(n+1)D)))$, and c_1 be the codimension of its image. Let us denote

$$c^{<n>} := \binom{k_n+1}{n+1} + \cdots + \binom{k_\delta + 1}{\delta+1}.$$

which has the following elementary properties

- if $c' \leq c$ then $c'^{<n>} \leq c^{<n>}$,i.e., the map $c \mapsto c^{<n>}$ is non-decreasing
- $(c+1)^{<n>} = \begin{cases} c^{<n>} + k_1 + 1 & if\ \delta = 1 \\ c^{<n>} + 1 & if\ \delta > 1 \end{cases}$

Theorem 3.8 (Generalized Macaulay Theorem)**.** $c_1 \leq c^{<n>}$.

Proof. Let $l_{n+1}, l_n, \ldots l_\delta$ be the $(n+1)$-th Macaulay coefficients of c_1; then

$$(c_1)_D \leq c_{<n>} = \binom{l_{n+1}-1}{n+1} + \cdots + \binom{l_\delta - 1}{\delta}$$

and by the sequence obtained by restriction it follows that

$$c_1 \leq c + (c_1)_D$$

so that

$$\binom{l_{n+1}-1}{n} + \cdots + \binom{l_\delta - 1}{\delta - 1} \leq c$$

and then

$$\binom{l_{n+1}}{n+1} + \cdots + \binom{l_\delta}{\delta} = c_1 \leq c^{<n>}.$$

□

3.2 Cox-Gorenstein ideals

Proposition 3.9 ([18]). Let $\mathbb{P}_\Sigma$ be a projective simplicial toric variety. Then the irrelevant ideal is equal to

$$B_\Sigma = \langle x_{\hat{\sigma}} \mid \sigma \in \Sigma_{max} \rangle \subset S$$

where $x_{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$ and S the Cox ring of $\mathbb{P}_\Sigma$.

Definition 3.10 (Cox-Gorenstein ideals). An ideal $I \subset B_\Sigma =\subset S$ is a *Cox-Gorenstein ideal* of socle degree $N \in Cl(\Sigma)$ if I is Artinian and there exists a nonzero linear map $\Lambda \in (S^N)^\vee$ such that for every ample class $\beta \in Cl(\Sigma)$ one has

$$I^\beta = \{P \in B_\Sigma^\beta \mid \Lambda(PQ) = 0 \text{ for all } Q \in S^{N-\beta}\}$$

Note that the linear map Λ induces a dual isomorphism

$$B_\Sigma^\beta / I^\beta \cong \left(B_\Sigma^{N-\beta} / I^{N-\beta}\right)^\vee \tag{3.7}$$

for every β such that $N - \beta$ is ample. In particular $\operatorname{codim} I^\beta = \operatorname{codim} I^{N-\beta}$.

Remark 3.11. For every projective simplicial toric variety, $S^\beta = B_\Sigma^\beta$ for every β ample class by Theorem 9.15 in [4] .

Proposition 3.12. If I and I' are two Cox-Gorenstein ideals with socle degree N and N' with $I \subset I'$, there exists $F \in B_\Sigma^{N-N'} \smallsetminus I^{N-N'}$ such that $I' = (I : F)$.

Proof. Note that N' is less than or equal to N, and Λ induces the isomorphism

$$B_\Sigma^{N-N'} / I^{N-N'} \cong \left(B_\Sigma^{N'} / I^{N'}\right)^\vee,$$

so that, as Λ' (the linear map defining the ideal I') yields a nonzero element in $(B_\Sigma^{N'}/I^{N'})^\vee$, if $[F]$is the unique element in $B_\Sigma^{N-N'}/I^{N-N'}$, taking a representative $F \in B_\Sigma^{N-N'}$, we get $\Lambda'(Q) = \Lambda(QF)$ for every $Q \in B_\Sigma^{N'}$. In particular

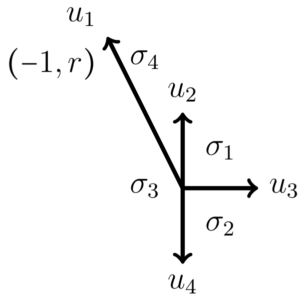
$$I' = \{Q \in B_\Sigma \mid QF \in I\}.$$

□

Remark 3.13. Artinian monomial ideals can be characterized as those whose minimal generators have the form $x_i^{a_i}$ with $a_i > 0$ for all $i \in \{1, \dots r\}$ ([41], Def. 2.2.13).

Example 3.14. If $\mathbb{P}_\Sigma = \mathbb{P}^k$ one recovers the classical Gorenstein ideals. Other natural examples are the Artinian base point free ideals.

Example 3.15. The Hirzebruch surface $\mathcal{H}_r$ $(r \geq 1)$ has fan



Denoting by D_i the toric divisor corresponding to u_i the are the equivalences $D_1 \sim D_3$ $D_4 \sim rD_1 + D_2$, so that $\mathrm{Pic}(\mathcal{H}_r) = \langle D_1, D_2 \rangle$. There generators of the irrelevant ideal are

$$x^{\hat{\sigma_1}} = x_1x_4, \quad x^{\hat{\sigma_2}} = x_1x_2, \quad x^{\hat{\sigma_3}} = x_2x_3, \quad x^{\hat{\sigma_4}} = x_3x_4.$$

Introducing variables

- $w := x^{\hat{\sigma_1}} = x_1x_4$ with $\deg w = (r+1, 1)$
- $x := x^{\hat{\sigma_2}} = x_1x_2$ with $\deg x = (1, 1)$
- $y := x^{\hat{\sigma_3}} = x_2x_3$ with $\deg y = (1, 1)$
- $z := x^{\hat{\sigma_4}} = x_3x_4$ with $\deg z = (r+1, 1)$

one can write

$$B(\Sigma) = \langle w, x, y, z \rangle .$$

Let us consider a monomial ideal I with minimal generator elements of the form $w^{d_1}, x^{d_2}.y^{d_3}, z^{d_4}$ with $d_i > 0$, i.e,

$$I = \left\langle w^{d_1}, x^{d_2}, y^{d_3}, z^{d_4} \right\rangle \; with \; d_i > 0.$$

Let us check that I is Cox-Gorenstein with socle degree

$$N = \deg\left(\frac{w^{d_1}x^{d_2}y^{d_3}z^{d_4}}{wxyz}\right) = (d_1-1)\deg w + (d_2-1)\deg x + (d_3-1)\deg y + (d_4-1)\deg z.$$

Let $F = \frac{w^{d_1}x^{d_2}y^{d_3}z^{d_4}}{wxyz} = w^{d_1-1}x^{d_2-1}y^{d_3-1}z^{d_4-1}$, which can be seen as one of the generators of S^N, and denote by $G_1, \ldots, G_s$ the other generators, i.e, $P \in S^N$ is $\sum_i a_iG_i + aF$. We define $\Lambda : P \mapsto a$. Note that, if $R \in B(\Sigma)^\beta$,

$$\begin{aligned}&\Lambda(RQ) \neq 0 \; \forall Q \in S^{N-\beta} \Leftrightarrow \\ &R = \sum_{k_1,k_2,k_3,k_4} a_{k_1k_2k_3k_4} w^{k_1}x^{k_2}y^{k_3}z^{k_4} \text{ such that there exists } k_1, k_2, k_3, k_4 \text{ with } 0 < k_i < d_i,\end{aligned} \tag{3.8}$$

or equivalently,

$$\Lambda(RQ) = 0 \Leftrightarrow R = \sum_{k_1,k_2,k_3,k_4} a_{k_1k_2k_3k_4} w^{k_1}x^{k_2}y^{k_3}z^{k_4} \text{ such that } k_i \geq d_i \ \forall k_1,k_2,k_3,k_4,$$

i.e, $R \in I$.

Remark 3.16. Note that in the above example $wy = xz$ thus $F = w^{d_1-1}x^{d_2-1}y^{d_3-1}z^{d_4-1}$ has different "representations", factorizations, in the ring generated by w,x,y,z. So for the construction of the linear map Λ is very important to fix the "representation" ,i.e., the factorization.

Example 3.17. If $f \in B^\beta \subset S = \mathbb{C}[x_1,\dots,x_r]$ is a very ample quasi-smooth hypersurface then $J(f) = \langle \frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_r} \rangle$ is a Cox-Gorenstein ideal with socle degree

$$N = \deg \frac{\prod_{i=1}^r \partial f/\partial x_i}{\prod_{\sigma\in\Sigma_{max}} x_{\hat{\sigma}}}$$

3.3 Applications of Macaulay theorem

In this section we prove some applications of Macaulay theorem to Cox-Gorenstein ideals. This generalizes some of the results in [36, 37] to the more general setting of odd-dimensional toric varieties, as opposed to odd-dimensional projective spaces, which is the case considered in [36, 37]. We assume that $(\mathbb{P}_\Sigma, D)$ is a strongly Fano variety and we denote $\deg D = \eta \in \mathrm{Pic}(\mathbb{P}_\Sigma)$.

Lemma 3.18. Let $W \subset H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(n\eta))$ be a linear subspace whose base locus has dimension k and degree d. Then

$$\mathrm{codim}(W) \geq \binom{n+k+1}{k+1} - \binom{n-d+k+1}{k+1}$$

Proof. Let Z be the base-locus of W and I_Z its ideal. Since $W \subset I_Z$ and $\mathrm{codim}\, W \geq \mathrm{codim}\, I_Z^n$ we can just prove that the result holds true for $\mathrm{codim}\, I_Z^n$. We shall prove that by induction over n and k. For $n = 0$ it is clear. For $k = 0$ and $n > 0$ we need to show that $\mathrm{codim}\, I_Z^n \geq d$. Taking cohomology in the exact sequence

$$0 \to \mathcal{I}_Z(rD) \to \mathcal{O}_{\mathbb{P}_\Sigma}(rD) \to \mathcal{O}_Z(rD) \to 0$$

we have

$$0 \to H^0(\mathcal{I}_Z(rD)) \to H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(rD)) \to H^0(\mathcal{O}_Z(rD)) \to H^1(\mathcal{I}_Z(r)) \to \cdots$$

where by Serre vanishing theorem $H^1(\mathcal{I}(rD)) = 0$ for $r >> 0$. Thus

$$c := \mathrm{codim}\, I_Z^{rD} = h^0(\mathcal{O}_{\mathbb{P}_\Sigma}(rD)) - h^0(rD) = h^0(\mathcal{O}_Z(rD)) = d$$

as Z has degree d. Taking $n > d$ and reasoning by contradiction we have $c < d < n$, so that

$$d = \binom{n}{n} + \cdots + \binom{n-(d-1)}{n-(d-1)} = \underbrace{1 + \cdots + 1}_{d\text{-}times}.$$

By applying the generalized Macaulay theorem and using the fact that the map ${}^{<n>} : c \mapsto c^{<n>}$ is increasing, we have

$$c_1 \leq c^{<n>} < d \text{ where } c_1 = \operatorname{codim} I_Z^{(n+1)D};$$

repeating the same argument replacing c with c_1 we have

$$c_2 \leq c_1^{<n+1>} \leq (c^{<n>})^{<n+1>} < d \text{ where } c_2 = \operatorname{codim} I_Z^{(n+2)D},$$

so that

$$c_r \leq (c^{<n>})^{<n+1>\cdots<n+r-1>} < d$$

which implies $c_r \leq d-1$. This is a contradiction as $c_r = d$.

Now let us assume that the result is true for $n-1$ and $k-1$. To easy the notation we write I_Z^n instead of I_Z^{nD}.

Claim: Since D is general, the multiplication for x_D

$$\mu_D : B^{(n-1)}/I_Z^{(n-1)} \to B^n/I_Z^n,$$

where $D = \operatorname{div}_0(x_D)$, is injective.

In principle the base locus Z may contain D but since D is general we may assume by Bertini's theorem that $Z \cap D = \emptyset$, i.e., $\mu_D \neq 0$. Now, if $\mu(f) = 0$ then $f.x_D = 0$ and since $x_D \neq 0$ then $f = 0$.

We have a well defined surjective restriction map (D is general), $B^n/I_Z^n \xrightarrow{r} B^n/I_{Z\cap D}^n$. There is a short exact sequence

$$0 \to \ker r \xrightarrow{\mu_D} B^n/I_Z^n \xrightarrow{r} B^n/I_{Z\cap D}^n \to 0.$$

It is clear that $\ker r$ contains B^{n-1}/I_Z^{n-1}. By induction we have

$$\operatorname{codim} I_Z^{n-1} \geq \binom{n+k}{k+1} - \binom{n-d+k}{k+1} \tag{3.9}$$

and

$$\operatorname{codim} I_{Z\cap D}^n \geq \binom{n+k}{k} - \binom{n-d+k}{k}; \tag{3.10}$$

thus adding (3.9) and (3.10), and keeping in mind that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, we get the result. □

Corollary 3.19. Let $W \subset H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(n\eta))$ a subsystem whose base locus has dimension and degree greater than or equal to k and d, respectively. Then for every $x \leq min(k,n)$ one has

$$\operatorname{codim} W \geq x\frac{(n-x)^k}{k!}.$$

Proof. Since

$$\binom{n+k+1}{b+1} - \binom{n-d+k+1}{k+1} = \sum_{j=1}^{d} \binom{k+1+n-j}{n-j+1}$$

applying the above lemma we get

$$\sum_{j=1}^{d} \binom{k+1+n-j}{n-j+1} \geq \sum_{j=1}^{d} \frac{(k+1+n-j)\dots(k-(k-1)+1+n-j)}{k!}$$
$$\geq \sum_{j=1}^{d} \frac{(n-j)^k}{k!} \geq d\frac{(n-d)^k}{k!} \geq x\frac{(n-x)^k}{k!} \quad (3.11)$$

□

Since $\mathbb{P}_\Sigma$ is $\mathbb{Q}$-factorial, i.e., for every Weil divisor D there is an integer number m such that mD is Cartier. We establish a preorder in $N^1(\mathbb{P}_\Sigma) = \text{Pic}(\mathbb{P}_\Sigma) \otimes \mathbb{Q}/\sim$ by letting $N < N'$ when $N' - N$ is numerically effective.

Proposition 3.20. For every $\epsilon_1 > 0$ there exists $\delta_1 > 0$ such that for every $m \geq \frac{1}{\delta_1}$ and every real $d \in [1, \delta_1 m]$, if a Cox-Gorenstein ideal I with socle degree N satisfies

- $\beta - \beta_0 \leq N - \beta = n\eta$ with $n \geq 1$
- $\operatorname{codim} I^\beta \leq d\frac{m^k}{k!}$ where $m = \max\{i \in \mathbb{N}^+ \mid i\eta \leq \beta\}$

then

1. For every integer $i \in \{0, \dots, \lfloor \delta_1 m \rfloor\}$ one has

$$\operatorname{codim} I^{\beta - i\eta} \leq (1+\epsilon_1)d\frac{m^k}{k!}$$

2. For every $i \in \{0, \dots m\}$ one has

$$\operatorname{codim} I^{\beta - i\eta} \leq 4^k d\frac{m^k}{k!}$$

Proof. First note that since I is Gorenstein of socle degree N,

$$\operatorname{codim} I^{\beta - i\eta} = \operatorname{codim} I^{N-(\beta - i\eta)} = \operatorname{codim} I^{(n+i)\eta}.$$

So by the generalized Macaulay theorem (3.8)

$$\operatorname{codim} I^{\beta - i\eta} \leq (\operatorname{codim} I^{n\eta})^{<n>\cdots<n+i-1>}$$

and since for a fixed c the map $c^{<->}$ is decreasing, and for a fixed n the map $c \mapsto c^{<n>}$ is increasing, for every natural number $x \leq n$

$$\operatorname{codim} I^{\beta - i\eta} \leq (\operatorname{codim} I^n)^{<x>\cdots<x+i-1>} \tag{3.12}$$

Also note that if

$$\operatorname{codim} I^{\beta} \leq \binom{\tau + x}{x} + \cdots + \binom{\tau + x - \upsilon}{x - \upsilon} \text{ where } \tau, \upsilon \in \mathbb{N} \tag{3.13}$$

as the map $c \mapsto c^{<n>}$ is increasing, (3.12) and (3.13) imply

$$\operatorname{codim} I^{\beta - i\eta} \leq \binom{\tau + x + i}{x + i} + \cdots + \binom{\tau + x - \upsilon + i}{x - \upsilon + i} \tag{3.14}$$

Suppose that δ_1 is small enough that $d \leq \frac{m-2r}{2^{k+1}}$ for $r = \min\{i \mid \beta \leq i\eta\}$. By assumption $\beta - \beta_0 \leq n\eta$ i.e, $(m - r)\eta \leq n\eta$, so that

$$\lfloor \frac{m}{2} \rfloor + 2^k d \leq \lfloor \frac{m}{2} \rfloor + \frac{m - 2r}{2} \leq m - r \leq n.$$

Let γ be the smallest positive real number such that $(2 + \gamma)^k d$ is an integer and

$$\lfloor \frac{m}{2 + \gamma} \rfloor + (2 + \gamma)^k d \leq n;$$

then the inequality (3.12) is true for $x = \lfloor \frac{m}{2+\gamma} \rfloor + (2 + \gamma)^k d$. On the other hand,

$$m^k \leq (\gamma + 2 + m)^k = (1 + \frac{m}{2 + \gamma})^k \leq (1 + \lceil \frac{m}{2 + \gamma} \rceil)^k =$$
$$(2 + \lfloor \frac{m}{2 + \gamma} \rfloor)^k \leq (k + \lfloor \frac{m}{2 + \gamma} \rfloor) \ldots (2 + \lfloor \frac{m}{2 + \gamma} \rfloor) = \frac{(k + \frac{m}{2+\gamma})!}{(\frac{m}{2+\gamma} + 1)!} \tag{3.15}$$

so that

$$\frac{m^k}{k!} \leq \binom{k + \lfloor \frac{m}{2+\gamma} \rfloor}{\lfloor \frac{m}{2+\gamma} \rfloor + 1}$$

and

$$d\frac{m^k}{k!} \leq \underbrace{\binom{k + \lfloor \frac{m}{2+\gamma} \rfloor + (2 + \gamma)^k d - 1}{\lfloor \frac{m}{2+\gamma} \rfloor + (2 + \gamma)^k d} + \cdots + \binom{k + \lfloor \frac{m}{2+\gamma} \rfloor}{1 + \lfloor \frac{m}{2+\gamma} \rfloor}}_{(2+\gamma)^k d\text{-}terms}$$

Then by the second assumption we have that the inequality (3.13) is true for

- $x = \lfloor \frac{m}{2+\gamma} \rfloor + (2 + \gamma)^k d$,
- $\tau = k - 1$,

- $\upsilon = (2+\delta)^b t - 1$;

thus inequality (3.14) holds, i.e.,

$$\begin{aligned}
\operatorname{codim} I^{\beta - i\eta} &\le \binom{\lfloor \frac{m}{2+\gamma} \rfloor + (2+\gamma)^k d + k - 1 + i}{\lfloor \frac{m}{2+\gamma} \rfloor + (2+\gamma)^k d + i} + \cdots + \binom{\lfloor \frac{m}{2+\gamma} \rfloor + k + i}{\lfloor \frac{m}{2+\gamma} \rfloor - 1 + i} \\
&\le (2+\gamma)^k d \frac{(\frac{m}{2+\gamma} + (2+\gamma)^k d + k + i)^k}{k!} \\
&\le (m + (2+\gamma)^{k+1} d + (2+\gamma)k + (2+\gamma)i)^k \frac{d}{k!} \\
&\le (1 + \frac{(2+\gamma)^{k+1} d + (2+\gamma)k + (2+\gamma)i}{m})^k d \frac{m^k}{k!}
\end{aligned}$$

Now if $0 \le i \le \lfloor m\delta_1 \rfloor$ we have

$$\operatorname{codim} I^{\beta - i\eta} \le (1 + \big((2+\gamma)^{k+1} + (2+\gamma)k + (2+\gamma)\big)\delta_1)^k d, \frac{m^k}{k!}$$

so that, given $\epsilon_1 > 0$, we take $\delta_1 > 0$ small enough so that

$$\big((2+\gamma)^{k+1} + (2+\gamma)k + (2+\gamma)\big)\delta_1 < \epsilon_1,$$

i.e., one gets claim 1 and taking $0 \le i \le m$ one gets claim 2. □

Definition 3.21. Let $I \subset B_\Sigma$ be an ideal. For $i \in \{0, \ldots, 2k\}$ and a fixed $n \in \mathbb{N}^+$ we define

$$l_i^n(I) := \min\{l \in \mathbb{N} \cup \infty \mid dim V(I^{(n+l)\eta}) \le 2k - i\},$$

or, equivalently,

$$l_i^n(I) := \max\{l \in \mathbb{N} \cup \infty \mid \dim V(I^{(n+l-1)\eta}) > 2k - i\}.$$

We let $\dim \varnothing = -1$, and $l_i = \infty$ when this number does not exist.

Remark 3.22.
- We shall write $l_i(I)$ instead of $l_i^n(I)$.
- Note that $l_0(I) \le \cdots \le l_{2k}(I)$.
- If I is base point free, then $l_{2k}(I) \in \mathbb{N}$.

Lemma 3.23. For every $\epsilon_2 > 0$ there exists $\delta_2 > 0$ such that for every $m \ge \frac{1}{\delta_2}$ and $d \in [1, \delta_2 m]$, if a Cox-Gorenstein ideal $I \subset B_\Sigma$ with socle degree N satisfies

- $N - \beta = n\eta$
- $\operatorname{codim} I^\beta \le d\frac{m^k}{k!}$, where $m = \max\{i \in \mathbb{N}^+ \mid i\eta \le \beta\}$,

then

$$l_i(I) - 1 \le \epsilon_2(m-2) \ \ \forall i \in \{k, \ldots, 2k\}.$$

Proof. Note that it is enough to prove the Lemma for $i = k$, so we apply the previous Proposition for $\epsilon_1 = 1$, and the Corollary for $x = 1$. Then for $l = \min(l_k(I) - 1, m)$ we have

$$\frac{(l-1)^{k+1}}{(k+1)!} \leq \operatorname{codim} I^{l\eta} \leq 4^k d \frac{m^k}{k!}$$

so that

$$l \leq 1 + \left(4^k d m^k (k+1)\right)^{\frac{1}{k+1}} \leq \left(\frac{1}{m} + (4^k(k+1)\frac{d}{m})^{\frac{1}{k+1}}\right)m \leq (\delta_2 + (4^k(k+1)\delta_2)^{\frac{1}{k+1}})m$$

and since $2 \leq 2m\delta_2$,

$$l \leq (3\delta_2 + (4^k(k+1)\delta_2)^{\frac{1}{k+1}})m - 2.$$

So, given $\epsilon_2 > 0$, we take δ_2 small enough to have $3\delta_2 + (4^k(k+1)\delta_2)^{\frac{1}{k+1}} < \min\{1, \epsilon_2\}$; then $l < m$ i.e $l = l_k(I) - 1$ or, in other words, $l_k(I) - 1 < \epsilon_2 m - 2$, and taking $\epsilon_2 \leq 1$ we get that $l_k(I) - 1 < \epsilon_2(m-2)$ as desired. □

The following Proposition will be the technical core of what follows.

Proposition 3.24. *For every $\epsilon > 0$ there exists $\eta > 0$ such that for every integer $m > \frac{1}{\delta}$ and for every $d \in [1, \delta m]$, if a Cox-Gorenstein ideal $I \subset B_\Sigma \subset S = \mathbb{C}[x_1, \ldots, x_r]$ with socle degree N satisfies*

i) *$N = (k+1)\beta - \beta_0$ and $N - \beta = n\eta$;*

ii) *I contains r polynomials in complete intersection $\{F_i\}_{i=1}^r$ with $\deg F_i = \beta - \deg x_i$ and whose associated ideal is base point free;*

iii) *$\operatorname{codim} I^\beta \leq d\frac{m^k}{k!}$ where $m = \max\{i \in \mathbb{N}^+ \mid i\eta \leq \beta\}$,*

then I contains the ideal I_V of a closed scheme $V \subset \mathbb{P}_\Sigma$ of pure dimension k and degree less than or equal to $(1+\epsilon)d$. Moreover, I and I_V coincide in degree less than or equal to $\big(m - 2 - (r-j)\deg V\big)\eta$.

Proof. By definition $\dim V(I^{l_k(I)}) \leq k$, so that there exist $j \in \mathbb{N}^+$ and $f_1, f_2, \ldots f_{r-j} \in I^{l_k(I)}$ such that $\dim V(< f_1, \ldots, f_{r-j} >) = k$; more precisely, note that $j = k+1$. Moreover, as I satisfies the assumptions of the previous Lemma, $f_1, f_2, \ldots f_{r-j} \in I^{\leq \frac{\epsilon_2}{2}(m-2)+1}$, and by the second assumption it is possible to find $r - j$ polynomials $f_{r-j+1}, \ldots, f_r$,where $\deg(f_i) = \beta - deg(x_i)$ ($i > j$), so that the ideal $< f_1, \ldots f_r >$ is base point free and is a Cox-Gorenstein ideal of socle degree less or equal to

$$\sum_{i=1}^{r-j} \deg(f_i) - \deg(x_i) + \sum_{i=r-j+1}^{r} \deg(f_i) - \deg(x_i) \leq (r-j)\big((m-2)\frac{\epsilon_2}{2} + 1\big)\eta + j\beta - \beta_0$$

Now, by Proposition 3.6 there exists a polynomial P with

$$\deg P \leq (r-j)\big((m-2)\frac{\epsilon_2}{2}+1\big)\eta + j\beta - \beta_0 - N = (r-j)\big((m-2)\frac{\epsilon_2}{2}+1\big)\eta$$

and $I = \big((f_1,\dots f_r):P\big)$. Moreover I and $J = \big((f_1,\dots f_{r-j}):P\big)$ coincide in degree less than or equal to

$$\beta - 2\eta - \deg P \geq (m-2)\eta - (r-j)\big((m-2)\frac{\epsilon_2}{2}+1\big)\eta \leq (m-2)\eta - (r-j)\big((m-2)\epsilon_2\big)\eta;$$

the last inequality is true when for $\delta_2 < \frac{\epsilon_2}{2}$ and $\frac{1}{\delta_2}+2 \leq m$. Now let us consider $l = \lfloor\big(1-(r-j)\epsilon_2\big)(m-2)\rfloor$ and let us apply the previous results to $I^{l\eta}$. Then for every $x \leq \min(degV, (r-j)\epsilon_2 m) \leq \min(k,l)$

$$x\frac{(l-x)^k}{k!} \leq \operatorname{codim} I^l \leq (1+\epsilon_1)d\frac{m^k}{k!}$$

and

$$x\big(1-\frac{\lfloor\epsilon_2(r-t)m\rfloor + x}{m}\big)^k \leq (1+\epsilon_1)d$$

so that

$$x \leq \frac{(1+\epsilon_1)}{\big(1-2\epsilon_2(r-j)\big)^k}d;$$

then, given $0<\epsilon<1$ and taking ϵ_1 and ϵ_2 so that

$$\frac{(1+\epsilon_1)}{\big(1-2\epsilon_2(r-j)\big)^k}d \leq (1+\epsilon)d,$$

one has $x \leq (1+\epsilon)d < 2d < 2\delta m$. Thus taking $\eta < \frac{\epsilon_2}{2}$ we have $x < \epsilon_2 m \leq (r-j)\epsilon_2 m$, i.e., $x = \deg V$ and $\deg V \leq (1+\epsilon)d$. Moreover, I and I_V coincide in degree less than or equal to

$$\big(m-2-(r-j)\deg V\big)\eta$$

□

3.4 The tangent space at a point of the Noether Lefschetz locus

Since $\mathbb{P}^{2k+1}_\Sigma$ has a pure Hodge structure [42, 51], there is a well defined residue map for it, and we can use it to construct the tangent space at a point of the Noether-Lefschetz locus. This is again basically done as in [37], however we provide more details, and use the properties of the residue map as developed in [4] for simplicial toric varieties.

Let $X = \{f = 0\}$ be a quasi-smooth hypersurface in $\mathbb{P}_\Sigma$, with $\deg f = \beta$. Denote by $i : X \to \mathbb{P}_\Sigma$ the inclusion, and by $i^* : H^\bullet(\mathbb{P}^{2k+1}_\Sigma, \mathbb{Q}) \to H^\bullet(X,\mathbb{Q})$ the associated morphism in cohomology; $i^* : H^{2k}(\mathbb{P}^{2k+1}_\Sigma,\mathbb{Q}) \to H^{2k}(X,\mathbb{Q})$ is injective by Proposition 10.8 in [4].

Definition 3.25. The primitive cohomology group $H^{2k}_{\text{prim}}(X)$ is the quotient

$$H^{2k}(X,\mathbb{Q})/i^*(H^{2k}(\mathbb{P}^{2k+1}_\Sigma,\mathbb{Q})$$

Both $H^{2k}(\mathbb{P}^{2k+1}_\Sigma,\mathbb{Q})$ and $H^{2k}(X,\mathbb{Q})$ have pure Hodge structures, and the morphism i^* is compatible with them, so that H^{2k}_{prim} inherits a pure Hodge structure.

Also, we shall denote by M the dual lattice of the lattice N which contains the fan Σ, i.e., $\Sigma \subset N \otimes \mathbb{R}$.

Definition 3.26. Fix an integral basis $m_1,\dots m_{2k+1}$ for the lattice M. Then given a subset $\iota = \{i_1,\dots,i_{2k+1}\} \subset \{1,\dots,\#\rho(1)\}$, where $\#\rho(1)$ is the number of rays, we define

$$\det(e_\iota) \coloneqq \det\big(< m_j, e_{i_h} >_{1\leq j,h\leq 2k+1} \big);$$

moreover, $dx_\iota = dx_{i_1} \wedge \dots \wedge dx_{i_{2k+1}}$ and $\hat{x}_\iota = \Pi_{i\notin\iota} x_\iota$.

Definition 3.27. The $(2k+1)$−form $\Omega_0 \in \Omega^{2k+1}_S$ is defined as

$$\Omega_0 \coloneqq \sum_{|\iota|=2k+1} \det(e_\iota)\hat{x}_\iota dx_\iota$$

where the sum is over all subsets $\iota \subset \{1,\dots,2k+1\}$ with $2k+1$ elements.

For more details about these definitions see [4].

Theorem 3.28. *$T_{[f]}\big(NL^{k,\beta}_{\lambda,U}\big) \cong E^\beta$, where*

$$E = \{K \in B(\Sigma)^\bullet \mid \sum_{i=1}^{b} \lambda_i \int_{\text{Tub}\,\gamma_i} \frac{KR\Omega_0}{f^{k+1}} = 0 \; for\ all\ R \in S^{N-\bullet}\},$$

and Tub(−) *is the adjoint of the residue map.*

Proof. By [7, Prop. 2.10] the p-th residue map

$$r_p : H^0(\mathbb{P}_\Sigma, \Omega^{2k+1}_{\mathbb{P}_\Sigma}(2k+1-p)X) \to H^{p,2k-p}_{\text{prim}}(X) \;\; for \;\; 0 \leq p \leq 2k$$

exists; it is surjective and has kernel

$$H^0(\mathbb{P}_\Sigma, \Omega^{2k+1}_{\mathbb{P}_\Sigma}(2k-p)X) + dH^0(\mathbb{P}_\Sigma, \Omega^{2k}_{\mathbb{P}_\Sigma}(2k-p)X).$$

So

$$\text{res}\, H^0(\Omega^{2k+1}(2k+1)X) = r_{2k}H^0(\Omega^{2k+1}(X)) \oplus \dots \oplus r_0 H^0(\Omega^{2k+1}(2k+1)X))$$

by definition of $H^0(\Omega^{2k+1}(2k+1)X)$. Or, equivalently,

$$\text{res}\, H^0(\Omega^{2k+1}(2k+1)X) = H^{2k,0}_{\text{prim}}(X) \oplus \dots \oplus H^{0,2k}_{\text{prim}}(X) = H^{2k}_{\text{prim}}(X).$$

Similarly

$$\operatorname{res} H^0(\Omega^{2k+1}(kX) = F^{k+1}H^{2k}_{\text{prim}}(X).$$

On the other hand by [4, Thm 9.7] we have

$$H^0(\Omega^{2k+1}_{\mathbb{P}_\Sigma}(kX) = \{\frac{K\Omega_0}{f^k} \mid K \in S^{k\beta-\beta_0}\} = \{\frac{K\Omega_0}{f^k} \mid K \in B_\Sigma^{k\beta-\beta_0}\};$$

the last equality holds true because we are assuming that $k\beta - \beta_0$ is ample and hence $B_\Sigma^{k\beta-\beta_0} = S^{k\beta-\beta_0}$ by Lemma 9.15 in [4]. Now fixing a basis $\{\gamma_i\}_{i=1}^b$ for $H_{2k}(X, \mathbb{Q})$ we have that the components of any element in $F^{k+1}H^{2k}_{\text{prim}}(X)$ are

$$(\int_{\gamma_1} \operatorname{res} \frac{K\Omega_0}{f^k}, \dots, \int_{\gamma_b} \operatorname{res} \frac{K\Omega_0}{f^k}),$$

or, equivalently,

$$(\int_{\text{Tub}(\gamma_1)} \frac{K\Omega_0}{f^k}, \dots, \int_{\text{Tub}(\gamma_b)} \frac{K\Omega_0}{f^k})$$

where $\text{Tub}(\gamma_j)$ is the adjoint to the residue map. Now taking $0 \neq \lambda_f \in H^{k,k}(X, \mathbb{Q})$ one has $\lambda_f \perp F^{k+1}H^{2k}_{\text{prim}}(X)$ (see [47]) and since the sheaf $\mathcal{H}^{2k}$ is constant on U we have

$$NL^{k,\beta}_{\lambda,U} = \{G \in U \mid \lambda_G \in F^k H^{2k}_{\text{prim}}(X_G)\} = \{G \in U \mid \lambda_f \perp F^{k+1}H^{2k}_{\text{prim}}(X_G)\}.$$

More explicitly, if $(\lambda_1, \dots \lambda_b)$ are the components of λ_f, one gets

$$\lambda_f \perp F^{k+1}H^{2k}_{\text{prim}}(X_G) \quad \Leftrightarrow \quad \textstyle\sum_{i=1}^b \lambda_i \int_{\text{Tub}\,\gamma_i} \frac{K\Omega_0}{G^k} = 0 \ \forall K \in S^{N-\beta}$$

where N is equal to $(k+1)\beta - \beta_0$.Thus we can characterize the local Noether-Lefschetz locus in the following way: Let us consider the differentiable map ψ which assigns to every homogeneous polynomial $G \in B_\Sigma^\beta$ a linear map $\psi_G \in (B_\Sigma^{N-\beta})^\vee$, i.e., $\psi : B_\Sigma^\beta \longrightarrow (B_\Sigma^{N-\beta})^\vee$ sends G to

$$\begin{aligned} \psi_G\colon\ B^{N-\beta} &\to \mathbb{C} \\ K &\mapsto \sum_i \lambda_i \int_{\text{Tub}(\gamma_i)} \frac{K\Omega_0}{G^k}; \end{aligned}$$

then $NL^{k,\beta}_{\lambda,U} = \psi^{-1}_{|U}(0)$, hence the tangent space at f is the kernel of $d\psi_f$. Now $T_{[f]}U \simeq S_\beta$ and since β is ample, $S^\beta = B^\beta$. Thus we can identified canonically $T_{[f]}(NL^{k,\beta}_{\lambda,U})$ with the subspace $E^\beta \subset B_\Sigma^\beta$, which is the β-summand of the Cox-Gorenstein ideal

$$E = \{K \in B_\Sigma^\bullet \mid \forall R \in S^{N-\bullet}, \ \sum_{i=1}^b \lambda_i \int_{\text{Tub}\,\gamma_i} \frac{KR\Omega_0}{f^{k+1}} = 0\}$$

whose socle degree is $N = (k+1)\beta - \beta_0$. $\square$

Remark 3.29. Note that E contains the Jacobian ideal $J(f)$ which is Cox-Gorenstein.

We also consider the Cox-Gorenstein ideals

$$E_s := \{K \in B_\Sigma^\bullet \mid \forall R \in S^{N+r\beta-\bullet}, \sum_{i=1}^{b} \lambda_i \int_{\mathrm{Tub}\,\gamma_i} \frac{KR\Omega_0}{f^{k+r+1}} = 0\},$$

with $s \in \mathbb{N}^+$, which have socle degree $N + r\beta$. For a fixed s, the ideal E_s describes the deformation of order $s+1$ of $NL_{\lambda,U}^{k,\beta}$ in a neighborhood of f.

Proposition 3.30. The Cox-Gorenstein ideals E_s have the following properties:

i. $E_s = (E_{s+1} : f)$;
ii. If f is a generic point of $NL_{\lambda,U}^{red}$ then $(E_r)^2\Theta \subset E_{s+1}$, where $\Theta \subset S_\beta$ is the image of the tangent space $T_f(N_{\lambda,U})^{red}$
iii. For all $K \in E_s$ and for every $j \in \{1,\dots,r\}$, $\frac{\partial K}{\partial x_j} f - (k+s+1)K; \frac{\partial f}{\partial x_j} \in E_{s+1}$.

Proof. 1. Clear.

2. For every $G \in NL_{\lambda,U}^{k,\beta}$ and for every $i \in \mathbb{N}^+$ such that $N + r\beta - i\eta$ is ample, consider the bilinear map

$$\begin{array}{rcl} \mathcal{Q}_i(G): \ B_\Sigma^{i\eta} \times B_\Sigma^{N+r\beta-i\eta} & \to & \mathbb{C} \\ (K,R) & \mapsto & \sum_{i=1}^{b} \lambda_i \int_{\mathrm{Tub}\,\gamma_i} \frac{KR\Omega_0}{G^{k+r+1}} \end{array}$$

For a fixed R we have $\ker \mathcal{Q}_i(G) = E_s^{i\eta}(G)$, and for a fixed K we have $\ker \mathcal{Q}_i(G) = E_s(G)^{N+rL-iD}$, where $E_s(G)$ is the Cox-Gorenstein ideal associated to the class λ_G. Since f is a quasi-smooth point of $(NL_{\lambda,U}^{k,\beta})^{red}$, the map $G \mapsto \mathcal{Q}_i(G)$ has constant rank for every G close to f. So for each $\vec{v} \in T_f(N_{\lambda,U})^{red}$ associated to $M \in \Theta$ the differential of the bilinear map

$$\begin{array}{rcl} d\mathcal{Q}_i(f)(\vec{v}): \ B_\Sigma^{i\eta} \times B_\Sigma^{N+r\beta-i\eta} & \to & \mathbb{C} \\ (K,R) & \mapsto & -(k+s+2)\sum_{i=1}^{t} \lambda_i \int_{\mathrm{Tub}\,\gamma_i} \frac{KRM\Omega_0}{f^{k+s+2}} \end{array}$$

is zero on $E_s^{i\eta} \times E_s^{\eta+r\beta-i\eta}$, or, in other words, $E_s^{i\eta} E_s^{N+r\beta-i\eta}\Theta \subset E_{s+1}^{N+(s+1)\beta}$.

3. Given $K \in E_s$, for every $R \in B_\Sigma^{N+s\beta+\eta-deg(K)}$ we have

$$R\left(\frac{\partial K}{\partial x_i} f - (k+s+1)K\frac{\partial f}{\partial x_i}\right) = \underbrace{\frac{\partial (KR)}{\partial x_i} f - (k+r+1)KR\frac{\partial f}{\partial x_i}}_{A} - \underbrace{KF\frac{\partial R}{\partial x_i}}_{B}.$$

Note that $\frac{A\Omega_0}{f^{n+r+2}}$ is an exact form in the kernel of the residue map, so that $A \in E_{s+1}$. By assumption $K\frac{\partial R}{\partial x_j} \in E_s$ so $B \in E_{s+1}$ by the first property. Thus $R(\frac{\partial K}{\partial x_i} f - (k+r+1)K\frac{\partial f}{\partial x_i}) \in E_{s+1}$ and since R is arbitrary we get the result. □

3.5 Proof of the chapter main theorem

Now we have all the machinery necessary to prove the main result of this chapter.

Theorem 3.31. *For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $m \geq \frac{1}{\delta}$ and for all $d \in [1, m\delta]$, if $\operatorname{codim} N^{k,\beta}_{\lambda,U} \leq d\frac{m^k}{k!}$ where $m = \max\{i \mid i\eta \leq \beta\}$ and if $G \in N^{k,\beta}_{\lambda,U}$, then there exists a $k-$dimensional subvariety $V \subset X_G$ with degree less than or equal to $(1+\epsilon)d$.*

Proof. If f is a generic point in $(NL^{k,\beta}_{\lambda,U})^{red}$, by Proposition 3.24 there exists a subscheme $V \subset \mathbb{P}_\Sigma$ of pure dimension k and degree $d' \leq (1+\epsilon)d \leq 2\delta m$ such that $I_V \subset E$; the two ideals agree in degree less or equal to $\big(m-2-(r-j)d'\big)\eta$, so it is enough to prove that $f \in \sqrt{I_V}$. Moreover

Step 1: $\left(I_V^{\leq d'\eta}\right)^2 \subset E_1$. Let $R \in (I_V^{\leq d'\eta})^2$, then the partial derivatives of R belong to E, and by items (i) and (iii) of Proposition 3.30, the partial derivatives of f belong to $(E_1 : R)$. Since f is quasi-smooth, its Jacobian is base point free, and $(E_1 : R)$ contains a base point free ideal whose socle degree is less than or equal to

$$(r-(k+1))(\epsilon_2(m-2))\eta + (k+1)\beta - \beta_0.$$

By contradiction $R \notin E_1$ then $(E_1 : R)$ has socle degree greater than or equal to

$$N + \beta - \deg R \geq N + \beta - 2d'\eta \geq N + \big((1-4\delta)m\big)\eta.$$

Now by (ii) in Proposition 3.30 we have $\Theta \subset (E_1 : R)$, and by assumption $\operatorname{codim}(\Theta) \leq d\frac{m^k}{k!}$, so that $\operatorname{codim}(E_1 : R)^\beta \leq d\frac{m^k}{k!}$, i.e, $(E_1 : R)$ satisfies the assumptions of Lemma 3.23. Then taking $\epsilon_2 = \frac{1}{2(r-(k+1))}$ and $\delta_2 = \delta < \frac{1}{4(r-(k+1))}$ we get

$$\frac{m-2}{2}\eta + N \geq N + \big((1-4\delta)m\big)\eta,$$

which implies $\delta > \frac{1}{8}$. Since

$$r-(k+1) \geq k+1 \Leftrightarrow \frac{1}{4(k+1)} \geq \frac{1}{4(r-(k+1))}$$

so that $\delta < \frac{1}{8}$, which is a contradiction. So one has $R \in E_1$ as desired.

Step 2: $f \in \sqrt{I_V}$. Since V is of pure dimension k, it is enough to show that $f \in \sqrt{I_W}$ for every irreducible subscheme W of V associated to the primary ideal decomposition of I_V. Let W' be the smallest subscheme of V such that $I_V = I_W \cap I_{W'}$, and let $P \subset \mathbb{P}_\Sigma$ be a projective linear space of dimension $k-1$, for which we can suppose without loss of generality that it has equations $x_1 =, \ldots, x_{r-k} = 0$ and we set $B_P = \mathbb{C}[x_1, \ldots, x_{r-k}]$. Since W and W' are of pure dimension k, the homogeneous ideals $I_W \cap B_P \subset B_P$ and $I_{W'} \cap B_P \subset B_P$ are of pure codimension 1 for P generic; therefore they are principal. Let

$K_{P,W}$ and $K_{P,W'}$ be the images of the generators in B_Σ. Let $\kappa = \deg K_{P,W}$ and $\kappa' = degK'_{P,W}$; by construction we have that $\kappa \le \deg W$ and $\kappa' \le degW'$. Considering $K_P = K_{P,W}K^2_{P,W'}$, we have $K_P \in E$, $K_P \notin E_1$, so that the ideal $(E_1 : K_P)$ has socle degree $N + \beta - (\kappa + 2\kappa')$ and moreover contains the ideal

$$J_P = \left\langle f, I_W^{\deg W}, \frac{\partial f}{\partial x_{r-k+1}}, \dots, \frac{\partial f}{\partial x_r} \right\rangle.$$

More precisely, the following facts hold true:

- $K_P \in E$ as $\kappa + 2\kappa' \le m - 2 - (r - j)d'$;
- $K_P \notin E_1$. Otherwise, $(k + r + 1)K_P\frac{\partial f}{\partial X_i} \in E_1$ and then, using property (iii) of Proposition 3.30, $\frac{\partial K_P}{\partial X_i} f \in E_1$ and by property (i) in Proposition 3.30, $\frac{\partial K_P}{\partial x_i} \in E$ for all i; however, by construction not all partial derivatives of K_P are in E, so this is a contradiction.
- $J_P \subset (E_1 : K_P)$; indeed, as $\frac{\partial K_P}{\partial x_{r-k+1}} = 0, \dots, \frac{\partial K_P}{\partial x_r} = 0$ then $(E_1 : K_P)$ contains $\frac{\partial f}{\partial x_{r-k+1}}, \dots, \frac{\partial f}{\partial x_r}$ by property 3 of proposition 2. On the other hand by lemma 2 we have $\left((I_V)^{\le d'}\right)^2 \subset E_1^{\le 2d'}$ and since $I_W^{\deg W} K_P \subset \left((I_V)^{\le d'}\right)^2$, we have $I_W^{\deg W} \subset (E_1 : K_P)^{degW}$.

Now by contradiction, if $f \notin I_W$, then $\dim V(f, I_W^{\deg W}) \le k - 1$, and moreover J_P contains a Cox-Gorenstein ideal with socle degree less than or equal to $N + (k + 1)d'\eta$. On the other hand, $(E_1 : K_P)$ has socle degree greater than or equal to $N + \beta - 2d'\eta$, so that

$$N + (r - (k + 1))2\delta m\eta \ge N + (r - (k + 1))d'\eta \ge N + \beta - 2d'\eta \ge N + (1 - 4\delta)m\eta$$

which implies that $\delta \ge \frac{1}{2(r-(k+1)+2)} \ge \frac{1}{2(k+3)}$, contradicting our choice of δ. Thus $f \in I_W$. □

Chapter 4

On the Hodge Conjecture in toric varieties

In this last chapter we apply the previous results to establish the Hodge conjecture in some special cases. In section 1 we show that for a quasi-smooth hypersurface in the Noether-Lefschetz locus containing a suitable complete intersection subvariety, the Hodge Conjecture is true asymptotically, i.e., when the degree of the hypersurface is "big" enough. In section 2 we study quasi-smooth intersection subvarieties in a projective simplicial toric variety, which is a right notion to generalized complete intersection subavarieties in the toric world, and we show that under appropriate conditions, on a very general quasi-smooth intersection subvariety Hodge Conjecture holds, generalizing the work on quasi-smooth hypersurfaces of Bruzzo and Grassi in [8].

4.1 An asymptotic argument for Hodge Conjecture

The notation and assumptions are the same as Chapter 3, that is, as in Proposition 3.24 or Theorem 3.31, i.e., we have a quasi-smooth hypersurface $X_f \subset \mathbb{P}^{2k+1}_{\Sigma}$ in the Noether-Lefschetz locus and there exists a k-dimensional subvariety V satisfying:

- $V \subset X_f \subset \mathbb{P}^{2k+1}_{\Sigma}$
- $\deg V \leq 2\delta m$ with $\delta < \frac{1}{4(r-(k+1))}$ and r the number of rays of Σ.
- I_V and E coincide in degree less than or equal to $(m-2-(r-j)d')\eta$ for some $(0 < j < r)$.

Since V is k-dimensional by Poincaré duality there exists $\lambda_V \in H^{k,k}(\mathbb{P}^{2k+1}_{\Sigma}, \mathbb{Q})$ the cohomology class associated to $[V]$ and let us denote $\lambda_{V_{\text{prim}}} := i^*(\lambda_V) \in H^{k,k}(X_f)$.

Theorem 4.1. *If V is a smooth complete intersection subvariety, then there exists $c \in \mathbb{C}^*$ such that $\lambda_f = c\lambda_{[V]_{\text{prim}}}$.*

Proof. We divide the proof in three steps.

Step I: $\lambda_{[V]_{\text{prim}}} \neq 0$.

Since $V \subset X_f$ is a regular embedding we have that

$$\begin{aligned}
[V]^2_{X_f} &= \int_V c_n(N_{V/X_f}) \\
&= \int_V c_n(N_{V/\mathbb{P}_\Sigma})/c_n(N_{X_f/\mathbb{P}_{\Sigma|V}}) \\
&= degV\left(\text{coefficient } t^n \text{of } \frac{\prod_i(1+deg(A_i)t)}{1+deg(X_f)t}\right)
\end{aligned}$$

By contradiction if $\deg(X_f)[V] = \deg V c_1^k(\mathcal{O}_{X_f}(\eta))$ then $\deg(X_f)^2[V]^2_{X_f} = (\deg V)^2 c_1^{2k}(\mathcal{O}_{X_f}(\eta)) = (\deg V)^2 \deg(X_f)$ which implies that $\deg(X_f)$ divides $\deg V$ proving the Step I.

Let E_{alg} be the Cox-Gorenstein ideal associated to $\lambda_{[V]_{\text{prim}}}$ and E as in Chapter 3 the Cox-Gorenstein ideal associated to λ_f. So to prove the theorem, it is enough to show that

Step II: $E = E_{\text{alg}}$. Note that $I_V + J(f)$ is contained in E and E_{alg}. Moreover, since $V \subset X_f$, f is of the form $A_1K'_1 + \cdots + A_{k+1}K'_{k+1}$. Now, because f is quasi-smooth, there exist $K_1, \dots K_{k+1} \in B_\sigma$ dividing $K'_1 \dots, K_{k+1}$ respectively, such that $(A_1, \dots, A_{k+1}, K_1, \dots K_{k+1})$ is a Cox-Gorenstein ideal with socle degree N. So that to conclude the theorem is enough the following step.

Step III : the ideal $I_V + J(f)$ coincides in degree N with the ideal $(A_1, \dots, A_{k+1}, K_1, \dots K_{k+1})$. It is enough to show that every Cox-Gorenstein ideal $\mathcal{I}$ of degree N containing $I_V + J(f)$ also contains $(A_1, \dots, A_{k+1}, K_1, \dots K_{k+1})$. By assumption

$$\left(A_j, j \in \{1, \dots, k+1\}, \sum_{j=1}^{k+1} \frac{\partial A_i}{\partial x_i} K_j, i \in 1, \dots, r\right) \subset \mathcal{I}$$

Let us see that $K_j \in \mathcal{I}$ for every $j \in \{1, \dots, k+1\}$. Let M_{rxk+1} be the matrix $[\frac{\partial A_j}{\partial x_i}]$ and K the column $(K_j)_{j\in\{1,\dots,k+1\}}$. For each $I \subset \{1, \dots r\}$ with cardinal $k+1$ and let M_I be the matrix extracting the $i \in I$-arrows of M. We have that $\sum_{j=1}^{k+1} \frac{A_j}{x_i} K_j = (MK)_i = (M_I K)_i$; multiplying by the adjunct of M_I we get that $\det(M_I)K_j \in \mathcal{I}$ for all $j \in \{1, \dots k+1\}$. On one hand the ideal $(\mathcal{I}, K_j)$ contains the ideal

$$\mathcal{J} = I_V + \langle \det M_I \rangle$$

Hence $(\mathcal{I} : K_j)$ contains a Cox-Gorenstein ideal with socle degree less or equal to

$$r \deg V\eta - \beta_0 \leq 2m\delta\eta - \beta_0$$

On the other hand if $K_j \notin \mathcal{I}$ then $(\mathcal{I} : K_j)$ contains a Cox-Gorenstein ideal with socle degree

$$N - \deg K_j \geq N - \beta = k\beta - \beta_0$$

then comparing the above two inequalities and keeping in mind that $r \geq 2(k+1)$, we get that $\delta > \frac{1}{2r} > \frac{1}{4(r-(k+1))}$.

□

4.2 Very general quasi-smooth intersection varieties and Hodge Conjecture

A projective simplicial toric variety $\mathbb{P}^d_\Sigma$ satisfies the Hodge Conjecture, i.e., every cohomology class in $H^{p,p}(\mathbb{P}^d_\Sigma, \mathbb{Q})$ is a linear combination of algebraic cycles. On one hand by the Lefschetz theorem in toric varieties, the Hodge conjecture holds true for every hypersurface and $p < \frac{d-1}{2}$ and by Poincarè duality, also for $p > \frac{d-1}{2}$ and on the other hand by Theorem 1.1 in [8] when, $d = 2k+1$ and $\mathbb{P}^{2k+1}_\Sigma$ is an Oda variety with an ample class β such that $k\beta - \beta_0$ is nef, where β_0 is the anticanonical class, the Hodge conjecture with rational coefficients holds for a very general hypersurface in the linear system $|\beta|$.

The main purpose of this chapter is to generalize the above results to "good" complete intersections between quasi-smooth hypersurfaces. Let $f_1, \ldots, f_s$ homogeneous polynomials in the Cox ring of $\mathbb{P}^d_\Sigma$. Then they define a zero locus $V(f_1, \ldots, f_s)$ which has associated a closed subvariety $X \subset \mathbb{P}^d_\Sigma$.

Definition 4.2. We say that X is a quasi-smooth intersection if $V(f_1, \ldots, f_s) \cap U(\Sigma)$ is either empty or a smooth subvariety of codimension s in $U(\Sigma)$.

Remark 4.3. This notion generalizes smooth complete intersection in a projective space. In fact a quasi-smooth intersection $X = X_{f_1} \cap \cdots \cap X_{f_s}$ defined by $f_1, \ldots f_s \in B_\Sigma$ has pure dimension $d - s$.

Again as in the case of quasi-smooth hypersurfaces we can relate the above definition with the notion of orbifold, namely

Proposition 4.4 ([33] Proposition 1.3)**.** If $X \subset \mathbb{P}^d_\Sigma$ is a closed subset of codimension s defined by the homogeneous polynomials $f_1, \ldots f_s$, then X is quasi-smooth intersection if and only if X is a suborbifold of $\mathbb{P}^d_\Sigma$.

We also have a Lefschetz type theorem in this context.

Proposition 4.5 ([33] Proposition 1.4)**.** Let $X \subset \mathbb{P}^d_\Sigma$ be a closed subset, defined by homogeneous polynomials $f_1, \ldots f_s \in B_\Sigma$ of ample hypersurfaces. The the natural map $i^* : H^i(\mathbb{P}^d_\Sigma) \to H^i(X)$ is an isomorphism for $i < d-s$ and an injection for $i = d - s$.

Hence if $p \neq \frac{d-s}{2}$ every cohomology class in $H^{p,p}(X)$ is a linear combination of algebraic cycles. So let us see what happens when $p = \frac{d-s}{2}$. The idea will be to relate the Hodge structure of a quasi-smooth intersection variety $X = X_{f_1} \cap \cdots \cap X_{f_s}$ in $\mathbb{P}^d_\Sigma$ with the Hodge structure of a quasi-smooth hypersurface Y in a toric variety $\mathbb{P}^{d+s-1}_{X,\Sigma}$ whose fan depends of X and Σ.

Proposition 4.6. Let $X = X_1 \cap \cdots \cap X_s$ be quasi-smooth intersection subvariety in $\mathbb{P}^d_\Sigma$ cut off by homogeneous polynomials $f_1 \dots f_s$ respectively . Then there exists a projective simplicial toric variety $\mathbb{P}^{d+s-1}_{X,\Sigma}$ and a quasi-smooth hypersurface $Y \subset \mathbb{P}^{d+s-1}_{X,\Sigma}$ such that for $p \neq \frac{d+s-1}{2} \neq \frac{d+s-3}{2}$

$$H^{p-1,d+s-1-p}_{\text{prim}}(Y) \simeq H^{p-s,d-p}_{\text{prim}}(X).$$

Proof. The way to construct $\mathbb{P}^{d+s-1}_{X,\Sigma}$ is through of what it is known as the "Cayley trick". Let $L_1, \dots, L_s$ be the line bundles associated to the quasi-smooth hypersurfaces $X_1, \dots X_s$ so let $\mathbb{P}(E)$ be the projective bundle associated to the vector bundle $E = L_1 \oplus \cdots \oplus L_s$ it turns out that $\mathbb{P}(E)$ is $d+s-1$- dimensional projective simplicial toric variety whose Cox ring is

$$\mathbb{C}[x_1, \dots, x_n, y_1, \dots y_s]$$

where $\mathbb{C}[x_1, \dots, x_n]$ is the Cox ring of $\mathbb{P}^d_\Sigma$. The hypersurface Y is cut off by the polynomial $F = y_1 f_1 + \cdots + y_s f_s$ and is quasi-smooth by Lemma 2.2 in [33]. Moreover combining Theorem 10.13 in [4] and Theorem 3.6 in [33] we have that

$$H^{p-1,d+s-1-p}_{\text{prim}}(Y) \cong R(F)_{(d+s-p+1)\beta-\beta_0} \cong H^{p-s,d-p}_{\text{prim}}(X)$$

for $p \neq \frac{d+s-1}{2} \neq \frac{d+s-3}{2}$ as we wanted. □

Corollary 4.7. If X is a quasi-smooth intersection variety and its associated quasi-smooth hypersurface satisfies the Hodge conjecture, then the Hodge conjecture also holds on X.

Proof. The isomorphism of Proposition 4.6 preserves algebraic classes.

□

Remark 4.8. With the same notation of Propostion 4.6, note that we have a well defined map,

$$\begin{array}{rcl} \phi : |\beta_1| \times \cdots \times |\beta_s| & \to & |\beta| \\ (f_1, \dots, f_s) & \mapsto & f_1 y_1 + \cdots + f_s y_s. \end{array}$$

Moreover, by Noether-Lefschetz theorem $|\beta| \setminus NL_\beta$ is a countable union of closed sets $\bigcup_i C_i$ and hence $\bigcup \phi^{-1}(C_i)$ too.

We have an extension of the Noether-Lefschetz theorem, namely.

Lemma 4.9. Let $\mathbb{P}^d_\Sigma$ be an Oda projective simplicial toric variety. Then for a very general quasi-smooth intersection subvariety X cut off by $f_1, \dots f_s$ such that $d + s = 2(k+1)$ one has that,

$$H^{k+1-s,k+1-s}(X, \mathbb{Q}) = i^* \left(H^{k+1-s,k+1-s}(\mathbb{P}^d_\Sigma) \right)$$

So we get a natural generalization of the main geometrical object of the thesis, the Noether-Lefschetz loci.

Definition 4.10. We called *the Noether-Lefschetz locus of a quasi-smooth intersection variety*, the locus of $s-$tuples $(f_1, \dots, f_s)$ such that $X = X_{f_1} \cap \dots X_{f_s}$ is quasi-smooth intersection with $f_i \in |\beta_i|$ such that $H^{k+1-s,k+1-s}(X, \mathbb{Q}) \neq i^* (H^{k+1-s,k+1-s}(\mathbb{P}^d_\Sigma))$ and we denote it by $NL_{\beta_1,\dots,\beta_s}$.

Now we transfer what we already know about Hodge conjecture on $\mathbb{P}^d_\Sigma$ to quasi-smooth intersection subvarieties.

Theorem 4.11. *Let $\mathbb{P}^d_\Sigma$ be a Oda projective simplicial toric variety, then on a very general quasi-smooth intersection subvariety X cut off by $f_1, \dots f_s$ such that $d + s = 2(k+1)$, the Hodge Conjecture holds .*

Proof. First note that by Corollary 4.2 in [23] the projective simplicial toric variety $\mathbb{P}^{2k+1}_{X,\Sigma}$ is Oda and since X is very general the quasi-smooth hypersurface Y is very general as well. So applying the Noether-Lefschetz theorem one has that $h^{k,k}_{\text{prim}}(Y) = 0 = h^{k+1-s,k+1-s}_{\text{prim}}(X)$ or equivalently every $(k+1-s, k+1-s)$ cohomology class comes from a linear combination of algebraic cycles. □

Developments

Along the thesis we extended some classical results, machinery and ideas known for projective spaces to a more general setting, i.e., to projective simplicial toric varieties. Pushing forward those developments I expect to get some new results in different topics, mainly related with Noether-Lefschetz theory, namely:

- Definition 4.10 is a new and quite natural perspective of the Noether-Lefschetz loci. So I would like to transfer the results of my thesis presented along the previous sections to that context.

- The main tool in [37] is Macaulay theorem, which I generalized to normal "strongly Fano" varieties 3.3 with rational singularities. I expect that studying the Hodge structure developed by Steenbrik in [45] to some varieties, non necessarily toric, I will be able to extend Theorem 3.1 to some normal, non necessarily toric, varieties.

- Macaulay theorem is also an important key in order to understand Hilbert polynomials. So pushing forward the ideas of Green in [21] I hope to generalize to "strongly Fano" varieties with rational singularities the Gotzmann's Regularity theorem, which say that for a graded ideal $I^\bullet$ of the Cox ring associated to the projective space $\mathbb{P}^r$ with Hilbert polynomial $P(k)$, i.e., $P(k) = \operatorname{codim}(I^k, H^0(\mathcal{O}_{\mathbb{P}^r}(k)$ for $k >> 0$, that is, if $\mathcal{I}$ is an ideal sheaf corresponding to $I^\bullet$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^r}/\mathcal{I}$ we have that $P(k) = \chi(\mathcal{F}(k))$. Then the Hilbert polynomial has the form

$$P(k) = \binom{k+a_1}{a_1} + \binom{k+a_2-1}{a_2} + \cdots + \binom{k+a_s-(s-1)}{a_s},\ a_1 \geq a_2 \geq a_s \geq 0.$$

  Furthermore, the associated ideal sheaf $\mathcal{I}$ is $s-$ regular.

- Chapter 2 gives bounds for the codimension of a Noether-Lefschetz component. In [15] Ciliberto and Lopez constructed explicitly some Noether-Lefschetz components for a given codimension when $\mathbb{P}^{2k+1}_\Sigma = \mathbb{P}^3$. This result is based on the determination of generators of the Picard group for a general surface containing a fix curve, what was done by Lopez in [29]. Subsequently [29] was based on the Kronecker-Castelnuovo theorem which says that, an irreducible surface $S \subset \mathbb{P}^3$ has a 2-dimensional family of reducible plane sections if and only if S is either ruled by lines or the Roman surface. I expect that the result can be generalized to others toric threefolds as a first step to extend the ideas in [15].

- [14] and [3] proved the algebraicity of the Noether-Lefschetz components for the projective space. This is a very hard theorem and seems to be true also for the Noether-Lefschetz loci on a simplicial projective toric variety. I would like to tackle this problem.
- I studied in wide generality Cox rings not only associated to a toric variety. Thus it is natural to think that some of well-known properties for toric varieties (see [18]) can be extended to more general normal varieties.
- There exists a connection between Noether-Lefschetz theory and Gromov-Witten theory [32], more precisely the Noether-Lefschetz divisors in the moduli of $K3$ surfaces are the loci corresponding to Picard rank at least 2. Maulik and Pandharipande relate the degrees of the Noether-Lefschetz divisors in 1-parameter families of $K3$ surfaces to the Gromov-Witten theory of the 3-fold total space. I would like to study this, to look for inspiration and motivation.

Bibliography

[1] A. Adem, J. Leida, and Y. Ruan. *Orbifolds and Stringy Topology*. Cambridge University Press, 2007.

[2] Arzhantsev, I. Defenthal, U. Hausen, J. Laface, A. *Cox Rings*. Cambridge Studies in Advanced Mathematics 144, 2015.

[3] B. Bakker, B. Klinger and J. Tsimerman. *Tame topology of arithmetic quotients and algebricity of Hodge loci*. arXiv:1810.04801, *Developments in Noether-Lefschetz theory*, 2018.

[4] D. A. Batyrev, V. V. Cox. *On the Hodge structure of projective hypersurfaces in toric varieties*. Duke Math. J., 75: pp. 293–338, 1994.

[5] M. Beltrametti and A. Sommese. *The adjunction theory of complex projective varieties*. Walter de Gruyter, Berlin, 2000.

[6] S. Brevik, J. Nollet. *Hodge Theory, Complex Geometry, and Representation Theory: NSF-CBMS Regional Conference in Mathematics*. Texas Christian University, Fort Worth, Texas, *Developments in Noether-Lefschetz theory*, June 18, 2012,.

[7] U. Bruzzo and A. Grassi. *Picard group of hypersurfaces in toric 3-folds*. Internat. J. Math., 23. No.2, 2012.

[8] U. Bruzzo and A. Grassi. *On the Hodge conjecture for hypersurfaces in toric varieties*. arXiv:1708.04944, 2017.

[9] U. Bruzzo and A. Grassi. *The Noether-Lefschetz locus of surfaces in toric threefolds*. Commun. Contemp. Math, 20. No.5, 2018.

[10] U. Bruzzo and W. D. Montoya. *An asymptotic description of the Noether-Lefschetz components in toric varieties*. arXiv:1905.01570 [math.AG], 2019.

[11] D. Bunnet. *Moduli of hypersurfaces in toric orbifolds*. arXiv:1906.00272 [math.AG], 2019.

[12] J. Carlson, M. Green, P. Griffiths, and J. Harris. *Infinitesimal variations of Hodge structure.* I, Composio Math, 50: pp. 109–205, 1983.

[13] J. Carlson, S. Müller-Stach, and C. Peters. *Period mappings and period domains.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, 85, 2003.

[14] E. Cattani, P. Deligne, and A. Kaplan. *On the locus of Hodge classes.* J. Amer. Math Soc, 8: pp. 483–506, 1995.

[15] C. Ciliberto and A.F Lopez. *On the existence of components of the Noether-Lefschetz locus with given codimension.* J. Manuscripta Math, 73: pp. 341–357, 1991.

[16] D. Cox. *The homogeneous coordinate ring of a toric variety.* J. Algebraic Geom., 4, 1995.

[17] D. Cox and S. Katz. *Mirror symmetry and algebraic geometry.* Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 68, 1999.

[18] D. Cox, J. Little, and H. Schenck. *Toric varieties.* Graduate Studies in Mathematics, 124. American Mathematical Society, Providence, RI, 2011.

[19] M. L. Green. *A new proof of the explicit Noether-Lefschetz theorem.* J. Diff. Geom, 27: pp. 155–159, 1988.

[20] M. L. Green. *Components of maximal dimension in the Noether-Lefschtez locus.* J. Diff. Geom, 29: pp. 295–302, 1989.

[21] M. L. Green. *Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann, Algebraic curves and projective geometry.* Commun. Contemp. Math, 20(5):20, Trento, 1988.

[22] S. Hu, Y. Keel. *Mori dream spaces and GIT.* Michigan Math. J., 48: pp. 331–348, 2000.

[23] A. Ikeda. *Subvarieties of generic hypersurfaces in a non-singular projective toric variety.* Math. Z., 263: pp. 923–937, 2009.

[24] K. Kodaira. *Complex manifolds and deformations of complex strutures.* Grundlehren de Math. Wiss 283, Springer, 1986.

[25] S. Kollár, J. Mori. *Birational geometry of algebraic varieties.* Cambridge Tracts in Mathematics. Cambridge University Press, 134(5):254, 1998.

[26] S. Kovács. *A characterization of rational singularities.* Duke Math. J, 102(2):187–191, 2000.

[27] M. Laface, A. Velasco. *A survey on Cox rings.* Geom. Dedicata, 139: pp. 269–287, 2009.

[28] V. Lanza and I. Martino. *On the codimension of Noether-Lefschetz loci for toric threefolds.* arXiv:1807.11426 [math.AG], 2018.

[29] R. Lazarsfeld. *Positivity in algebraic geometry I*, volume Vol. 48. Ergebnisse de Mathematik und ihrer Grenzgebiete. Springer-Verlag, Berlin, 2004.

[30] C. LeBrun. *Edges, orbifolds and Seiber-Witten theory.* J. Math. Soc. Japan, Vol. 67, 2015.

[31] K F. Liu and Zhuang X B. *Deformation of complex orbifolds and the period maps.* Sci. China Math `https://doi.org/10.1007/s11425-017-9245-3`, 62, 2019.

[32] D. Maulik and R. Pandharipande. *Gromov-Witten and Noether-Lefschetz theory.* In a celebration of algebraic geometry, Clay Mathematics, 18: pp. 469–507, 2010.

[33] A. Mavlyutov. *Cohomology of Complete Intersections In Toric Varieties.* Pacific J. Math, 191 No.1: pp. 133–144, 1999.

[34] I. Moerdijk and D. A. Pronk. *Orbifolds, sheaves and groupoids. K-Theory*, 12: 1: pp. 3–21, 1997.

[35] T. Oda. *Problems on Mokowski sums of convex lattice polytopes.* arXiv:0812.1418 [math.AG], 2008.

[36] A. Otwinowska. *Composantes de dimension maximale d'un analogue du lieu de Noether-Lefschetz.* Compositio Math., 131: pp. 31–50, 2002.

[37] A. Otwinowska. *Composantes de petite codimension du lieu de Noether-Lefschez: un argument asymptotique en faveur de la conjecture de Hodge.* J. Alg. Geom., 12: pp. 307–320, 2003.

[38] G.V. Ravindra and V. Srinivas. *The Grothendieck-Lefschetz theorem for normal projective varieties.* J. Algebraic Geom., 15: pp. 563–590, 2006.

[39] G.V. Ravindra and V. Srinivas. *The Noether-Lefschetz theorem for the divisor class group.* J. Algebra, 322: pp. 3373–3391, 2009.

[40] L. Rossi, R. Terracini. *Weighted projective spaces from the toric point of view with computational applications.* Rend. Sem. Mat. Univ. Politec. Torino, 70:469–495, 2012.

[41] E. Sáenz de Cabezón Irigaray. *Combinatorial Koszul homology, computations and applications.* PhD Thesis, Universidad de La Rioja, Spain, 2008.

[42] M. Saito. *Mixed Hodge modules*. Publ. Res. Inst. Math. Sci., 26: pp. 221–333, 1990.

[43] I. Satake. *On a generalization of the the notion of manifold.* Proc. Nat. Acad. Sci. U.S.A., 42: pp. 359–363, 1956.

[44] B. Shiffman and A. Sommese. *Vanishing theorems on complex manifolds*. Birkhäuser, Basel, 2003.

[45] J. Steenbring. *Intersection form for quasi-smooth singularieties*. Compositio Math., 34: pp. 211–223, 1977.

[46] J. Verdier. *Stratification de Whitney et théoréme de Bertini-Sard*. Springer-Verlag, 1976.

[47] C. Voisin. *Composantes de petite codimension du lieu de Noether-Lefschetz.* Comment. Math. Helv, 64. No. 5: pp. 515–526, 1989.

[48] C. Voisin. *Hodge Theory and Complex Algebraic Geometry I.* Cambridge University Press, 2002.

[49] C. Voisin. *Hodge Theory and Complex Algebraic Geometry II.* Cambridge University Press, 2002.

[50] C. Voisin. *Hodge loci, Handbook of Moduli.* Institute de Mathematiques de Jussieu, 2010.

[51] Z. Z. Wang and D. Zaffran. *A remark on the hard Lefschetz theorem for Kähler orbifolds.* Proc. Amer. Math. Soc., 137:2497–2501, 2009.