Trimaximal neutrino mixing from modular A_4 invariance with residual symmetries

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A B S T R A C T

We construct phenomenologically viable models of lepton masses and mixing based on modular A_4 invariance broken to residual symmetries $\mathbb{Z}_3^T$ or $\mathbb{Z}_3^{ST}$ and $\mathbb{Z}_2^S$ respectively in the charged lepton and neutrino sectors. In these models the neutrino mixing matrix is of trimaximal mixing form. In addition to successfully describing the charged lepton masses, neutrino mass-squared differences and the atmospheric and reactor neutrino mixing angles θ_{23} and θ_{13}, these models predict the values of the lightest neutrino mass (i.e., the absolute neutrino mass scale), of the Dirac and Majorana CP violation (CPV) phases, as well as the existence of specific correlations between i) the values of the solar neutrino mixing angle θ_{12} and the angle θ_{13} (which determines θ_{12}), ii) the values of the Dirac CPV phase δ and of the angles θ_{23} and θ_{13}, iii) the sum of the neutrino masses and θ_{23}, iv) the neutrinoless double beta decay effective Majorana mass and θ_{23}, and v) between the two Majorana phases.

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1. Introduction

Understanding the origin of the flavour structure of quarks and leptons remains one of the outstanding problems in particle physics. The pattern of two large and one small neutrino (lepton) mixing angles, revealed by the data obtained in neutrino oscillation experiments (see, e.g., [1]), provides an important clue in the investigations of the lepton flavour problem, suggesting the existence of flavour symmetry in the lepton sector. The results of the recent global analyses of the neutrino oscillation data show also that a neutrino mass spectrum with normal ordering (NO) is favoured over the spectrum with inverted ordering (IO), as well as a preference for a value of the Dirac CP violation (CPV) phase δ close to $3\pi/2$ (see, e.g., [2]).

The observed 3-neutrino mixing pattern can naturally be explained by extending the Standard Theory (ST) with a flavour symmetry associated with a non-Abelian discrete symmetry group. Models based on S_3, A_4, S_4, A_5 and other groups of larger orders have been proposed and extensively studied (see, e.g., [3–9]). In particular, the A_4 flavour model attracted considerable interest because the A_4 group is the minimal one including a triplet unitary irreducible representation, which allows for a natural explanation of the existence of three families of leptons [10–15]. In all models based on non-Abelian discrete flavour symmetry, the flavour symmetry must be broken in order to reproduce the measured values of the neutrino mixing angles. This is achieved by introducing typically a large number of ST gauge singlet scalars - the so-called "flavons" - in the Lagrangian of the theory, which have to develop a set of particularly aligned vacuum expectation values (VEVs). Arranging for such an alignment requires the construction of rather elaborate scalar potentials.

An attractive approach to the lepton flavour problem, based on the invariance under the modular group, has been proposed in Ref. [16], where also models of the finite modular group $\Gamma_3 \simeq A_4$ have been presented. Although the models constructed in Ref. [16] are not realistic and make use of a minimal set of flavon fields, this work inspired further studies of the modular invariance approach to the lepton flavour problem. The modular group includes S_3, A_4, S_4, and A_5 as its principal congruence subgroups, $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$

[17]. However, there is a significant difference between the models based on the modular S_3, A_4, S_4 etc. symmetry and those based on the usual non-Abelian discrete S_3, A_4, S_4 etc. flavour symmetry. The constants of a theory based on the finite modular symmetry, such as Yukawa couplings and, e.g., the right-handed neutrino mass matrix in type I seesaw scenario, also transform non-trivially under the modular symmetry and are written in terms of modular forms which are holomorphic functions of a complex scalar field - the modulus τ. At the same time the modular forms transform under the usual non-Abelian discrete flavour symmetries. In the most economical versions of the models with modular symmetry, the VEV of the modulus τ is the only source of symmetry breaking without the need of flavon fields.

In Ref. [18] a realistic model with modular $\Gamma_2 \simeq S_3$ symmetry was built with the help of a minimal set of flavon fields. A realistic model of the charged lepton and neutrino masses and of neutrino mixing without flavons, in which the modular $\Gamma_4 \simeq S_4$ symmetry was used, was constructed in [19]. Subsequently, lepton flavour models without flavons, based on the modular symmetry $\Gamma_3 \simeq A_4$ was proposed in Refs. [20,21]. A comprehensive investigation of the simplest viable models of lepton masses and mixing, based on the modular S_4 symmetry, was performed in Ref. [22]. Necessary ingredients for constructing flavour models based, in particular, on the modular symmetries $\Delta(96)$ and $\Delta(384)$ have been obtained in [23], while for models based on A_5 symmetry they have been derived in [24].

If one of the subgroups of the considered finite modular group is preserved, this residual symmetry fixes τ to a specific value (see, e.g., [22]). Phenomenologically viable models based on the modular S_4 and A_5 symmetries, broken respectively to residual $\mathbb{Z}_3$ and $\mathbb{Z}_5$ symmetries in the charged lepton sector and to a $\mathbb{Z}_2$ symmetry in the neutrino sector, were presented in Refs. [22,24]. So far, apart from these two studies, the implications of residual symmetries have been investigated only in the framework of the usual non-Abelian discrete symmetry approach to the lepton (and quark) flavour problem. It has been shown that they lead, in particular, to specific experimentally testable correlations between the values of some of the neutrino mixing angles and/or between the values of the neutrino mixing angles and of the Dirac CP violation phase in the neutrino mixing [25–31].

In the present article we construct phenomenologically viable models of lepton masses and mixing based on residual symmetries resulting from the breaking of the A_4 modular symmetry. It is found that the weight 4 modular forms are required to obtain charged lepton and neutrino mass matrices leading to lepton masses and mixing which are consistent with the experimental data on neutrino oscillations. We also find that in these models the PMNS matrix [32–34] is predicted to be of the trimaximal mixing form [35,36].

The paper is organized as follows. In section 2, we give a brief review on the modular symmetry. In section 3, we discuss the residual symmetries of A_4 and their modular forms. In section 4, we present the lepton mass matrices in the residual symmetry. In section 5, we present models and their numerical results. Section 6 is devoted to a summary. Appendix A shows the relevant multiplication rules of the A_4 group.

2. Modular A_4 group and modular forms of level 3

The modular group $\overline{\Gamma}$ is the group of linear fractional transformations γ acting on the complex variable τ belonging to the upper-half complex plane as follows:

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \text{where} \quad a, b, c, d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1, \quad \text{Im}\,\tau > 0. \tag{2.1}$$

The group $\overline{\Gamma}$ is generated by two transformations S and T satisfying

$$S^2 = (ST)^3 = I, \tag{2.2}$$

where I is the identity element. Representing S and T as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.3}$$

one finds

$$\tau \xrightarrow{S} -\frac{1}{\tau}, \qquad \tau \xrightarrow{T} \tau + 1. \tag{2.4}$$

The modular group $\overline{\Gamma}$ is isomorphic to the projective special linear group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$, where $SL(2, \mathbb{Z})$ is the special linear group of 2×2 matrices with integer elements and unit determinant, and $\mathbb{Z}_2 = \{I, -I\}$ is its centre (I being the identity element). The special linear group $SL(2, \mathbb{Z}) = \Gamma(1) \equiv \Gamma$ contains a series of infinite normal subgroups $\Gamma(N)$, $N = 1, 2, 3, \dots$:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \tag{2.5}$$

called the principal congruence subgroups. For $N = 1$ and 2, we define the groups $\overline{\Gamma}(N) \equiv \Gamma(N)/\{I, -I\}$ with $\overline{\Gamma}(1) \equiv \overline{\Gamma}$. For $N > 2$, $\overline{\Gamma}(N) \equiv \Gamma(N)$ since $\Gamma(N)$ does not contain the subgroup $\{I, -I\}$. For each N, the associated linear fractional transformations of the form in eq. (2.1) are in a one-to-one correspondence with the elements of $\overline{\Gamma}(N)$.

The quotient groups $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$ are called finite modular groups. For $N \leq 5$, these groups are isomorphic to non-Abelian discrete groups widely used in flavour model building (see, e.g., [17]): $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$. We will be interested in the finite modular group $\Gamma_3 \simeq A_4$.

Modular forms of weight k and level N are holomorphic functions $f(\tau)$ transforming under the action of $\overline{\Gamma}(N)$ in the following way:

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \overline{\Gamma}(N). \tag{2.6}$$

Here k is even and non-negative, and N is natural. Modular forms of weight k and level N span a linear space of finite dimension. The dimension of the linear space of modular forms of weight k and level 3, $\mathcal{M}_k(\Gamma_3 \simeq A_4)$, is $k + 1$. There exists a basis in this space such that a multiplet of modular forms $f_i(\tau)$ transforms according to a unitary representation ρ of the finite group Γ_N:

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau), \quad \gamma \in \overline{\Gamma}. \tag{2.7}$$

In the case of $N = 3$ of interest, the three linear independent weight 2 modular forms form a triplet of A_4 [16]. These forms have been explicitly obtained [16] in terms of the Dedekind eta-function $\eta(\tau)$:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{2.8}$$

where $q = e^{2\pi i \tau}$. In what follows we will use the following basis of the A_4 generators S and T in the triplet representation:

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{2.9}$$

where $\omega = e^{i\frac{2}{3}\pi}$. The modular forms $(Y_1^{(2)}, Y_2^{(2)}, Y_3^{(2)})$ transforming as a triplet of A_4 can be written in terms of $\eta(\tau)$ and its derivative [16]:

$$Y_1^{(2)}(\tau) = \frac{i}{2\pi} \left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right),$$

$$Y_2^{(2)}(\tau) = \frac{-i}{\pi} \left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right), \tag{2.10}$$

$$Y_3^{(2)}(\tau) = \frac{-i}{\pi} \left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2 \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right).$$

The overall coefficient in eq. (2.10) is one possible choice; it cannot be uniquely determined. The triplet modular forms $Y_{1,2,3}^{(2)}$ have the following q-expansions:

$$\mathbf{Y_3^{(2)}} = \begin{pmatrix} Y_1^{(2)}(\tau) \\ Y_2^{(2)}(\tau) \\ Y_3^{(2)}(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}. \tag{2.11}$$

They satisfy also the constraint [16]:

$$(Y_2^{(2)})^2 + 2Y_1^{(2)} Y_3^{(2)} = 0. \tag{2.12}$$

3. Residual symmetries of A_4 and modular forms

Residual symmetries arise whenever the VEV of the modulus τ breaks the modular group $\overline{\Gamma}$ only partially, i.e., the little group (stabiliser) of $\langle\tau\rangle$ is non-trivial. Residual symmetries have been investigated in the case of modular S_4 invariance in [22], and of A_5 invariance in [24], where viable models of lepton masses and mixing have also been constructed. In the present work we consider models of lepton flavour based on the residual symmetries of the modular A_4 invariance.

There are only 2 inequivalent finite points with non-trivial little groups of $\overline{\Gamma}$, namely, $\langle\tau\rangle = -1/2 + i\sqrt{3}/2 \equiv \tau_L$ and $\langle\tau\rangle = i \equiv \tau_C$ [22]. The first point is the left cusp in the fundamental domain of the modular group, which is invariant under the ST transformation $\tau = -1/(\tau + 1)$. Indeed, $\mathbb{Z}_3^{ST} = \{I, ST, (ST)^2\}$ is one of subgroups of A_4 group.[2] The right cusp at $\langle\tau\rangle = 1/2 + i\sqrt{3}/2 \equiv \tau_R$ is related to τ_L by the T transformation. The $\langle\tau\rangle = i$ point is invariant under the S transformation $\tau = -1/\tau$. The subgroup $\mathbb{Z}_2^S = \{I, S\}$ of A_4 is preserved at $\langle\tau\rangle = \tau_C$. There is also infinite point $\langle\tau\rangle = i\infty \equiv \tau_T$, in which the subgroup $\mathbb{Z}_3^T = \{I, T, T^2\}$ is preserved.

It is possible to calculate the values of the A_4 triplet modular forms of weight 2 at the symmetry points τ_L, τ_C and τ_T. The results are reported in Table 1 in which the values of the modular forms at $\langle\tau\rangle = \tau_R$ are also given, to be compared with those at the other two points.

As we have noted, the dimension of the linear space $\mathcal{M}_k(\Gamma_3 \simeq A_4)$ of modular forms of weight k and level 3 is $k + 1$. The modular forms of weights higher than 2 can be obtained from the modular forms of weight 2. They transform according to certain irreducible representations of the A_4 group. Indeed, for weight 4 we have 5 independent modular forms, which are constructed by the weight 2 modular forms through the tensor product of $\mathbf{3} \otimes \mathbf{3}$ (see Appendix A). We obtain one triplet $\mathbf{3}$ and two singlets $\mathbf{1}$, $\mathbf{1'}$, while the third singlet $\mathbf{1''}$ vanishes:

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[2] In the recent publication [38] the authors obtain $\langle\tau\rangle = -1/2 + i\sqrt{3}/2 \equiv \tau_L$ in a $SU(5)$ GUT theory with modular A_4 symmetry.

Table 1
Modular forms of weight 2 and 4 and the magnitude of $Y_1^{(2)}$ at relevant τ.

τ	weight 2 $\mathbf{3}$	weight 4 $\mathbf{3}$	$\{\mathbf{1},\ \mathbf{1'}\}$	$Y_1^{(2)}$
τ_L	$Y_1^{(2)}(1, \omega, -\frac{1}{2}\omega^2)$	$3(Y_1^{(2)})^2(1, -\frac{1}{2}\omega, \omega^2)$,	$\{0,\ \frac{9}{4}(Y_1^{(2)})^2\omega\}$	0.9486...
τ_R	$Y_1^{(2)}(1, \omega^2, -\frac{1}{2}\omega)$	$3(Y_1^{(2)})^2(1, -\frac{1}{2}\omega^2, \omega)$,	$\{0,\ \frac{9}{4}(Y_1^{(2)})^2\omega^2\}$	0.9486...
τ_C	$Y_1^{(2)}(1, 1-\sqrt{3}, -2+\sqrt{3})$	$(Y_1^{(2)})^2(1, 1, 1)$,	$(Y_1^{(2)})^2\{6\sqrt{3}-9,\ \ 9-6\sqrt{3}\}$	1.0225...
τ_T	$Y_1^{(2)}(1, 0, 0)$	$(Y_1^{(2)})^2(1, 0, 0)$,	$\{(Y_1^{(2)})^2,\ \ 0\}$	1

Table 2
The charge assignment of $SU(2)$, A_4, and modular weights (k_I for fields and k for coupling Y). The right-handed charged leptons are assigned three A_4 singlets, respectively.

	L	(e_R, μ_R, τ_R)	H_u	H_d	Y
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1},\ \mathbf{1''},\ \mathbf{1'})$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3},\ \mathbf{1},\ \mathbf{1'}$
k_I	k_L	$(k_{e_R}, k_{\mu_R}, k_{\tau_R})$	0	0	k

$$\mathbf{Y_3^{(4)}} \equiv \begin{pmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_3^{(4)} \end{pmatrix} = \frac{2}{3} \begin{pmatrix} (Y_1^{(2)})^2 - Y_2^{(2)}Y_3^{(2)} \\ (Y_3^{(2)})^2 - Y_1^{(2)}Y_2^{(2)} \\ (Y_2^{(2)})^2 - Y_1^{(2)}Y_3^{(2)} \end{pmatrix}, \tag{3.1}$$

$$\mathbf{Y_1^{(4)}} = (Y_1^{(2)})^2 + 2Y_2^{(2)}Y_3^{(2)}, \quad \mathbf{Y_{1'}^{(4)}} = (Y_3^{(2)})^2 + 2Y_1^{(2)}Y_2^{(2)}, \quad \mathbf{Y_{1''}^{(4)}} = (Y_2^{(2)})^2 + 2Y_1^{(2)}Y_3^{(2)} \equiv 0 \tag{3.2}$$

where the vanishing $\mathbf{Y_{1''}^{(4)}}$ is due to the condition in Eq. (2.12). Using Eq. (3.2) we can calculate the values of the modular forms of weight 4, transforming as $\mathbf{3}$ and $\{\mathbf{1}, \mathbf{1'}\}$, at the symmetry points τ_L, τ_C and τ_T. We show the results also in Table 1.

4. Lepton mass matrices with residual symmetry

We will consider next modular invariant lepton flavour models with the A_4 symmetry, assuming that the massive neutrinos are Majorana particles and that the neutrino masses originate from the Weinberg dimension 5 operator. There is a certain freedom for the assignments of irreducible representations and modular weights to leptons. We suppose that three left-handed (LH) lepton doublets form a triplet of the A_4 group. The Higgs doublets are supposed to be zero weight singlets of A_4. The generic assignments of representations and modular weights k_I to the MSSM fields[3] are presented in Table 2. In order to construct models with minimal number of parameters, we introduce no flavons. For the charged leptons, we assign the three right-handed (RH) charged lepton fields for three different singlet representations of A_4, $(\mathbf{1}, \mathbf{1'}, \mathbf{1''})$. Therefore, there are three independent coupling constants in the superpotential of the charged lepton sector. These coupling constants can be adjusted to the observed charged lepton masses. Since there are three singlet irreducible representations in the A_4 group, there are six cases for the assignment of the three RH charged lepton fields. However, this ambiguity does not affect the matrix which acts on the LH charged lepton fields and enters into the expression for the PMNS matrix. Thus, effectively we have the following unique form for the superpotential:

$$w_e = \alpha e_R H_d (LY)_\mathbf{1} + \beta \mu_R H_d (LY)_{\mathbf{1'}} + \gamma \tau_R H_d (LY)_{\mathbf{1''}}, \tag{4.1}$$

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u LLY)_\mathbf{1}, \tag{4.2}$$

where the sums of the modular weights should vanish. The parameters α, β, γ and Λ are constant coefficients.

4.1. Charged lepton mass matrix with residual symmetry

By using the decomposition of the A_4 tensor products given in Appendix A, the superpotential in Eq. (4.1) leads to a mass matrix of charged leptons, which is written in terms of modular forms of A_4 triplet with weight k:

$$M_E = v_d \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} Y_1^{(k)} & Y_3^{(k)} & Y_2^{(k)} \\ Y_2^{(k)} & Y_1^{(k)} & Y_3^{(k)} \\ Y_3^{(k)} & Y_2^{(k)} & Y_1^{(k)} \end{pmatrix}_{RL}, \tag{4.3}$$

where $v_d \equiv \langle H_d^0 \rangle$. Without loss of generality the coefficients α, β, and γ can be made real positive by rephasing the RH charged lepton fields.

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[3] For the modular weights of chiral superfields we follow the sign convention which is opposite to that of the modular forms, i.e. a field $\phi^{(I)}$ transforms as $\phi^{(I)} \to (c\tau + d)^{-k_I}\rho^{(I)}(\gamma)\phi^{(I)}$ under the modular transformation γ.

We will discuss next the charged lepton mass matrix at the specific points of $\tau = \tau_L, \tau_R, \tau_C, \tau_T$ in the case of weight $k = 2$. At $\tau = \tau_L$, the matrix $M_E^\dagger M_E$, which is relevant for the left-handed mixing, is given as:

$$M_E^\dagger M_E = \frac{9}{4} v_d^2 (Y_1^{(2)})^2 \begin{pmatrix} \alpha^2 + \beta^2 + \gamma^2/4 & -\omega^2/2\alpha^2 + \omega^2\beta^2 - \omega^2/2\gamma^2 & \omega\alpha^2 - \omega/2\beta^2 - \omega/2\gamma^2 \\ -\omega/2\alpha^2 + \omega\beta^2 - \omega/2\gamma^2 & \alpha^2/4 + \beta^2 + \gamma^2 & -\omega^2/2\alpha^2 - \omega^2/2\beta^2 + \omega^2\gamma^2 \\ \omega^2\alpha^2 - \omega^2/2\beta^2 - \omega^2/2\gamma^2 & -\omega/2\alpha^2 - \omega/2\beta^2 + \omega\gamma^2 & \alpha^2 + \beta^2/4 + \gamma^2 \end{pmatrix}. \tag{4.4}$$

It is easily noticed that this matrix commutes with ST, which is guaranteed by the residual symmetry $\mathbb{Z}_3^{ST}$ at $\tau = \tau_L$, where

$$ST = \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega^2 \end{pmatrix}. \tag{4.5}$$

Both matrices $M_E^\dagger M_E$ and ST are diagonalized by the unitary matrix U_E:

$$U_E \equiv TS = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2\omega & -\omega & 2\omega \\ 2\omega^2 & 2\omega^2 & -\omega^2 \end{pmatrix}, \tag{4.6}$$

$$U_E^\dagger STU_E = T = \mathrm{diag}\,(1, \omega, \omega^2), \qquad U_E^\dagger M_E^\dagger M_E U_E = \frac{9}{4} v_d^2 (Y_1^{(2)})^2 \mathrm{diag}(\gamma^2, \alpha^2, \beta^2),$$

where U_E is independent of parameters α, β, γ.

On the other hand, at $\tau = \tau_R$, we have:

$$M_E^\dagger M_E = \frac{9}{4} v_d^2 (Y_1^{(2)})^2 \begin{pmatrix} \alpha^2 + \beta^2 + \gamma^2/4 & -\omega/2\alpha^2 + \omega\beta^2 - \omega/2\gamma^2 & \omega^2\alpha^2 - \omega^2/2\beta^2 - \omega^2/2\gamma^2 \\ -\omega^2/2\alpha^2 + \omega^2\beta^2 - \omega^2/2\gamma^2 & \alpha^2/4 + \beta^2 + \gamma^2 & -\omega/2\alpha^2 - \omega/2\beta^2 + \omega\gamma^2 \\ \omega\alpha^2 - \omega/2\beta^2 - \omega/2\gamma^2 & -\omega^2/2\alpha^2 - \omega^2/2\beta^2 + \omega^2\gamma^2 & \alpha^2 + \beta^2/4 + \gamma^2 \end{pmatrix}. \tag{4.7}$$

The matrix $M_E^\dagger M_E$ in Eq. (4.7) commutes with

$$TS = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2\omega & -\omega & 2\omega \\ 2\omega^2 & 2\omega^2 & -\omega^2 \end{pmatrix}. \tag{4.8}$$

The fact that $M_E^\dagger M_E$ and TS commute is a consequence of the residual symmetry $\mathbb{Z}_3^{TS}$ at $\tau = \tau_R$. The matrices $M_E^\dagger M_E$ and ST are diagonalised by the unitary matrix:

$$U_E \equiv ST = \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega^2 \end{pmatrix}. \tag{4.9}$$

At $\tau = \tau_C$, the determinant of M_E vanishes. Indeed, this mass matrix leads to a massless charged lepton, and thus cannot be used for model building.

Finally, at $\tau = \tau_T$ we obtain the real diagonal matrix:

$$M_E = v_d Y_1^{(2)} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{4.10}$$

In the case of modular forms of weight 4 we can obtain a charged lepton mass matrix in which the modular forms transforming as **1** and **1'** do not contribute. As seen in Table 1, the weight 4 triplet modular forms coincide with weight 2 ones at $\tau = \tau_L, \tau_R$. Indeed, $M_E^\dagger M_E$ is obtained by replacing parameters (α, β, γ) of the mass matrices in Eqs. (4.4) and (4.7) with (γ, α, β), respectively. Therefore, the mixing matrices in Eqs. (4.6) and (4.9) are the same.

At $\tau = \tau_C$, the charged lepton mass matrix is of rank one, i.e., two massless charged leptons appear since the triplet modular forms are proportional to $(1, 1, 1)$. At $\tau = \tau_T$, the charged lepton mass matrix is equal to the diagonal one given in Eq. (4.10) since the triplet weight 4 modular forms coincide with the weight 2 modular forms.

4.2. Neutrino mass matrix (Weinberg operator)

The neutrino mass matrix is written in terms of A_4 triplet modular forms of weight k by using the superpotential in Eq. (4.2):

$$M_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} 2Y_1^{(k)} & -Y_3^{(k)} & -Y_2^{(k)} \\ -Y_3^{(k)} & 2Y_2^{(k)} & -Y_1^{(k)} \\ -Y_2^{(k)} & -Y_1^{(k)} & 2Y_3^{(k)} \end{pmatrix}_{LL}, \tag{4.11}$$

where $v_u \equiv \langle H_u^0 \rangle$.

In the case of weight 2 modular forms it is easily checked that two lightest neutrino masses are degenerate at $\tau = \tau_L, \tau_R$, while the determinant of M_ν vanishes at $\tau = \tau_C$. In the latter case one neutrino is massless and two neutrino masses are degenerate. The two lightest neutrino masses are degenerate also at $\tau = \tau_T$. It may be helpful to add a comment: these degeneracies of neutrino masses still hold even if we use the seesaw mechanism by introducing the three right-handed neutrino fields as A_4 triplet. Thus, the realistic neutrino mass matrix is not obtained as far as we take weight 2 modular forms at $\tau = \tau_L, \tau_R, \tau_C, \tau_T$.

In the case of weight 4 modular forms, there is one candidate that can be consistent with the observed neutrino masses. At $\tau = \tau_L, \tau_R$, the neutrino mass term $\mathbf{3_L 3_L Y_3^{(4)}}$ is similar to the case of weight 2, where two neutrino masses are degenerate. In the case of weight 4, the singlet $\mathbf{1'}$ also contributes to the neutrino mass matrix through the coupling $\mathbf{3_L 3_L Y_{1'}^{(4)}}$. However, this additional term cannot resolve the degeneracy.

It is easily noticed that two neutrino masses are degenerate also at $\tau = \tau_T$ since $\mathbf{Y_3^{(4)}} \sim (1, \ 0, \ 0)$. An additional $\mathbf{Y_1^{(4)}}$ does not change this situation.

At $\tau = \tau_C$, the triplet modular form, as seen in Table 1, is $\mathbf{Y_3^{(4)}} \sim (1, \ 1, \ 1)$, which allows to get large mixing angles. Moreover, we have $\mathbf{1}$ and $\mathbf{1'}$ modular functions. Therefore, we expect nearly tri-bimaximal mixing pattern of PMNS matrix with three different massive neutrinos. The LH weak-eigenstate neutrino fields couple to $\mathbf{Y_3^{(4)}}$. This coupling leads to the following neutrino Majorana mass matrix:

$$M_\nu = \frac{v_u^2}{\Lambda} (Y_1^{(2)})^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{4.12}$$

Moreover, the LH neutrino fields couple also to $\mathbf{Y_1^{(4)}}$ and $\mathbf{Y_{1'}^{(4)}}$, which gives the following additional contributions to the neutrino Majorana mass matrix M_ν:

$$3(2\sqrt{3} - 3) \frac{v_u^2}{\Lambda} (Y_1^{(2)})^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad -3(2\sqrt{3} - 3) \frac{v_u^2}{\Lambda} (Y_1^{(2)})^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{4.13}$$

where each of these two terms enters M_ν with its own arbitrary constant.

To summarise, the charged lepton mass matrix could be consistent with observed masses at $\tau = \tau_L, \tau_R, \tau = \tau_T$ for both cases of weight 2 and 4 modular forms. On the other hand, the neutrino Majorana mass matrix is consistent with observed masses only at $\tau = \tau_C$ for weight 4 modular forms. There is no common symmetry value of τ, which leads to charged lepton and neutrino masses that are consistent with the data.

5. Models with residual symmetry

As seen in the previous section, we could not find models with one modulus τ and with residual symmetry, which are phenomenologically viable. Therefore, we consider the case of having two moduli in the theory: one τ^ℓ, responsible via its VEV for the breaking of the modular A_4 symmetry in the charged lepton sector, and the another one τ^ν, breaking the modular symmetry in the neutrino sector. Our approach here is purely phenomenological. Constructing a model with two different moduli in the charged lepton and neutrino sectors is out of the scope of our study, it is a subject of ongoing research and work in progress. However, there are hints from the recent study [39] that this might be possible.[4] A model with two different moduli in the quark and lepton sectors, associated respectively with S_3 and A_4 modular symmetries, has been presented recently in [40].

We present next our setup. For the charged lepton mass matrix, we take weight 2 modular forms at $\tau^\ell = \tau_T$ (Case I) or at $\tau^\ell = \tau_L$ (Case II).[5] At the same time we use weight 4 modular forms at $\tau^\nu = \tau_C$ for constructing the neutrino Majorana mass term. In order for the modular weight in the superpotential to vanish, we assign the following weights to the LH lepton and RH charged lepton fields:

$$k_L = 2, \qquad k_{e_R} = k_{\mu_R} = k_{\tau_R} = 0, \tag{5.1}$$

where the notations are self-explanatory. We note that $k_L = 2$ is common in both τ^ℓ and τ^ν modular spaces.

Then, the charged lepton mass matrix is obtained by using as input the expressions for the weight 2 modular forms given in Table 1. At τ_T, it is a diagonal matrix:

$$M_E = v_d \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} : \quad \text{Case I}. \tag{5.2}$$

At $\tau = \tau_L$, the charged lepton mass matrix has the form:

$$M_E = v_d \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & \omega^2 & -\frac{1}{2}\omega \\ -\frac{1}{2}\omega & 1 & \omega^2 \\ \omega^2 & -\frac{1}{2}\omega & 1 \end{pmatrix}_{RL} : \quad \text{Case II}. \tag{5.3}$$

The matrix $M_E^\dagger M_E$, which is relevant for the calculation of the left-handed mixing, is given in Eq. (4.4).

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[4] The authors of [39] write in the Conclusions: "As we find different flavor symmetries at different points in moduli space (in particular in six compact dimensions), fields that live at different locations in moduli space feel a different amount of flavor symmetry. (...) This could lead to a different flavor- and CP -structure for the various sectors of the standard model like up- or down-quarks, charged leptons or neutrinos."

[5] The same numerical results are obtained at τ_R for weight 2 modular forms. Weight 4 modular forms lead also to the same results at τ_L and τ_R.

The neutrino mass matrix represents a sum of the contributions of modular forms of **3**, **1** and **1**′, with the terms involving the two singlet modular forms entering the sum with arbitrary complex coefficients A and B:

$$M_\nu = \frac{v_u^2}{\Lambda} (Y_1^{(2)})^2 \left\{ \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \left[A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - B \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] \right\}, \tag{5.4}$$

where the constants of the two terms in Eq. (4.13) are absorbed in the parameters A and B.

The two models with charged lepton mass matrix M_E specified in Eqs. (5.2) and (5.3) and neutrino mass matrix M_ν given in Eq. (5.4), as we will show, lead to the same phenomenology.

As an alternative to the models with two moduli τ^ℓ and τ^ν, we present next a model with one modulus τ^ν and one flavon, breaking the modular symmetry to $\mathbb{Z}_2^S$ and $\mathbb{Z}_3^T$ in the neutrino and charged lepton sectors respectively and leading to the charged lepton and neutrino mass matrices given in Eqs. (5.2) and (5.4). We introduce an A_4 triplet flavon ϕ with modular weight $k_\phi = -3$. In contrast to Eq. (5.1), the modular weights of the LH lepton doublet and RH charged lepton fields are chosen as follows:

$$k_L = 2, \qquad k_{e_R} = k_{\mu_R} = k_{\tau_R} = 1. \tag{5.5}$$

As a consequence, the modular functions $Y^{(i)}$ do not couple to the charged lepton sector, but couple to the neutrino sector because $Y^{(i)}$ have positive even modular weights. On the other hand, the flavon ϕ couples only to the charged lepton sector because of its odd weight.[6] The corresponding terms of the superpotential are the same as given in Eq. (4.1) with the modular form Y replaced by the flavon ϕ. Moreover, we can easily obtain the requisite VEV $\phi = v_E(1, 0, 0)^T$ preserving $\mathbb{Z}_3^T$, v_E being a constant parameter, from the potential analysis as seen in Refs. [12,13]. Finally, we get the charged lepton and neutrino mass matrices given in Eqs. (5.2) and (5.4). This flavon model with one modulus τ^ν leads to the same phenomenology as the models considered earlier with two different moduli τ^ℓ and τ^ν.

5.1. The neutrino mixing

In case I, only the neutrino mass matrix contributes to the PMNS matrix since the charged lepton mass matrix is diagonal. The neutrino mass matrix in this case leads to the so called TM$_2$ mixing form of PMNS matrix U_{PMNS} [35,36] where the second column of U_{PMNS} is trimaximal:

$$U_{\text{PMNS}}^{\text{I}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & e^{i\phi}\sin\theta \\ 0 & 1 & 0 \\ -e^{-i\phi}\sin\theta & 0 & \cos\theta \end{pmatrix} P. \tag{5.6}$$

Here θ and ϕ are arbitrary mixing angle and phase, respectively, and P is a diagonal phase matrix containing contributions to the Majorana phases of U_{PMNS}. Employing the standard parametrisation of U_{PMNS} (see, e.g., [1]), it is possible to show that the trimaximal mixing pattern leads to the following relation between the reactor angle θ_{13} and θ, between the atmospheric neutrino mixing angle θ_{23} and θ_{13} and θ, and sum rules for the solar neutrino mixing angle θ_{12} and for the Dirac phase δ [35,36] (see also [9,29]):

$$\sin^2\theta_{13} = \frac{2}{3}\sin^2\theta, \tag{5.7}$$

$$\sin^2\theta_{12} = \frac{1}{3\cos^2\theta_{13}}, \tag{5.8}$$

$$\sin^2\theta_{23} = \frac{1}{2} + \frac{s_{13}}{2}\frac{\sqrt{2 - 3s_{13}^2}}{1 - s_{13}^2}\cos\phi, \tag{5.9}$$

$$\cos\delta = \frac{\cos 2\theta_{23}\cos 2\theta_{13}}{\sin 2\theta_{23}\sin\theta_{13}(2 - 3\sin^2\theta_{13})^{\frac{1}{2}}}. \tag{5.10}$$

Using the 3σ allowed range of $\sin^2\theta_{13}$ from [2] and Eq. (5.7) we get the following constraints on $\sin\theta$:

$$0.17 \lesssim |\sin\theta| \lesssim 0.19. \tag{5.11}$$

To leading order in s_{13} we obtain from Eq. (5.9):

$$\frac{1}{2} - \frac{s_{13}}{\sqrt{2}} \lesssim \sin^2\theta_{23} \lesssim \frac{1}{2} + \frac{s_{13}}{\sqrt{2}}, \quad \text{or} \quad 0.391\,(0.390) \lesssim \sin^2\theta_{23} \lesssim 0.609\,(0.611), \tag{5.12}$$

where the numerical values correspond to the maximal allowed value of $\sin^2\theta_{13}$ at 3σ C.L. for NO (IO) neutrino mass spectrum [2]. The interval of possible values of $\sin^2\theta_{23}$ in eq. (5.12) is somewhat wider than the 3σ ranges of experimentally allowed values of $\sin^2\theta_{23}$ for NO and IO spectra given in [2]. Using the 3σ allowed ranges of $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ for NO (IO) spectra from [2] and Eq. (5.9) we also get:

$$-0.640\,(-0.508) \lesssim \cos\phi \leq 1. \tag{5.13}$$

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[6] A similar construction in the charged lepton sector was presented in Ref. [16].

The phase ϕ is related to the Dirac phase δ [9]:

$$\sin 2\theta_{23} \sin \delta = \sin \phi. \tag{5.14}$$

The Majorana phase $\alpha_{31}/2$ of the standard parametrisation of U_{PMNS} [1] receives contributions from the phase ϕ via [9]

$$\frac{\alpha_{31}}{2} = \frac{\xi_{31}}{2} + \alpha_2 + \alpha_3, \tag{5.15}$$

where the phase ξ_{31} will be specified later,

$$\alpha_2 = \arg\left(-\frac{c}{\sqrt{2}} - \frac{s}{\sqrt{6}} e^{i\phi}\right), \quad \alpha_3 = \arg\left(\frac{c}{\sqrt{2}} - \frac{s}{\sqrt{6}} e^{i\phi}\right), \tag{5.16}$$

$$\sin \alpha_2 = -\frac{s}{\sqrt{6}} \frac{\sin\phi}{s_{23} c_{13}} = -\tan\theta_{13} \cos\theta_{23} \sin\delta, \tag{5.17}$$

$$\sin \alpha_3 = -\frac{s}{\sqrt{6}} \frac{\sin\phi}{c_{23} c_{13}} = -\tan\theta_{13} \sin\theta_{23} \sin\delta. \tag{5.18}$$

We also have [9]:

$$\sin(\phi - \alpha_2 - \alpha_3) = -\sin\delta. \tag{5.19}$$

For further discussion of phenomenology of the neutrino trimaximal mixing (5.6), see, e.g., [9,14,30,37].

In case II, the contribution of the rotation of the charged lepton sector is added to the trimaximal mixing, which is derived from the neutrino mass matrix in Eq. (5.4). The mixing matrix in the charged lepton sector is the matrix U_E in Eq. (4.6). The PMNS matrix is given by:

$$U_{\text{PMNS}}^{\text{II}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2\omega & -\omega & 2\omega \\ 2\omega^2 & 2\omega^2 & -\omega^2 \end{pmatrix}^{\dagger} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & e^{i\phi}\sin\theta \\ 0 & 1 & 0 \\ -e^{-i\phi}\sin\theta & 0 & \cos\theta \end{pmatrix} P. \tag{5.20}$$

It is straightforward to check that after a substitution $\theta \to \theta - \pi/2$, $\phi \to -\phi$, the PMNS matrix (5.20) can be rewritten as

$$U_{\text{PMNS}}^{\text{II}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & e^{i\pi/3} & 0 \\ 0 & 0 & e^{-i\pi/3} \end{pmatrix} U_{\text{PMNS}}^{\text{I}} \begin{pmatrix} e^{i(\phi-\pi/2)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i(\phi+\pi/2)} \end{pmatrix}. \tag{5.21}$$

The leftmost phase matrix does not contribute to the mixing, since its effect can be absorbed into the charged lepton field phases. The rightmost phase matrix contributes only to the Majorana phases, therefore the numerical predictions in this case are the same as in Case I, apart possibly from the corresponding shift of the Majorana phases. However, as can be shown analytically, and we have confirmed numerically, also the predictions for the Majorana phases in Case II coincide with the predictions in case I.

5.2. The neutrino masses and Majorana phases

It follows from (5.4) that the neutrino mass matrix M_ν is a linear combination of three basis matrices:

$$M_1 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.22}$$

To diagonalize M_ν, it is convenient to rewrite it in a different basis:

$$M_1' = \frac{1}{\sqrt{3}}(M_2 + 2M_3) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix},$$

$$M_2' = M_2 + \frac{1}{3}M_1 = \frac{1}{3}\begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & 2 & 2 \end{pmatrix}, \tag{5.23}$$

$$M_3' = M_2 - \frac{1}{3}M_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 4 & -2 \end{pmatrix},$$

$$M_\nu = c\left(M_1' + aM_2' + bM_3'\right),$$

where a and b are arbitrary complex coefficients and c is the overall scale factor which can be rendered real positive. M_ν is diagonalized by a unitary matrix U_ν° of the following form:

$$U_\nu^\circ = V_{\text{TBM}}\, U_{13}(\theta,\phi)\,, \tag{5.24}$$

so that $M_\nu = (U_\nu^\circ)^* M_\nu^{\text{diag}} (U_\nu^\circ)^\dagger$, with $M_\nu^{\text{diag}} = \text{diag}\left(m_1 e^{-i2\phi_1}, m_2 e^{-i2\phi_2}, m_3 e^{-i2\phi_3}\right)$, where $m_i e^{-i2\phi_i}$ are complex eigenvalues and $m_i \geq 0$ are the neutrino masses.[7] Extracting the phases ϕ_i from M_ν^{diag}, we find:

$$M_\nu^{\text{diag}} = e^{-i2\phi_1}\, P^*\, \text{diag}\,(m_1, m_2, m_3)\, P^*\,, \quad P = \text{diag}\left(1, e^{i(\phi_2 - \phi_1)}, e^{i(\phi_3 - \phi_1)}\right), \tag{5.25}$$

where the phases $(\phi_2 - \phi_1)$ and $(\phi_3 - \phi_1)$ contribute to the Majorana phases $\alpha_{21}/2$ and $\alpha_{31}/2$ of the standard parametrisation of the PMNS matrix [1]. Thus, the PMNS matrix has the form:

$$U_{\text{PMNS}} = U_\nu^\circ\, P = e^{-i2\phi_1}\, V_{\text{TBM}}\, U_{13}(\theta,\phi)\, P\,, \tag{5.26}$$

where the common phase factor $e^{-i2\phi_1}$ is unphysical. The phase $\xi_{31}/2$ in Eq. (5.15) can be identified now with $(\phi_3 - \phi_1)$: $\xi_{31}/2 = \phi_3 - \phi_1$. Thus, the Majorana phases $\alpha_{21}/2$ and $\alpha_{31}/2$ are given by:

$$\frac{\alpha_{21}}{2} = \phi_2 - \phi_1\,, \quad \frac{\alpha_{31}}{2} = \phi_3 - \phi_1 + \alpha_2 + \alpha_3\,. \tag{5.27}$$

The complex rotation parameters θ and ϕ are fixed by a choice of a and b, which we will now show explicitly. We find by direct calculation that

$$U_\nu^{\circ\, T} M_1' U_\nu^\circ = \begin{pmatrix} -e^{-i\phi}\sin 2\theta & 0 & \cos 2\theta \\ 0 & \sqrt{3} & 0 \\ \cos 2\theta & 0 & e^{i\phi}\sin 2\theta \end{pmatrix},$$

$$U_\nu^{\circ\, T} M_2' U_\nu^\circ = \begin{pmatrix} 2\cos^2\theta & 0 & e^{i\phi}\sin 2\theta \\ 0 & 1 & 0 \\ e^{i\phi}\sin 2\theta & 0 & 2e^{2i\phi}\sin^2\theta \end{pmatrix}, \tag{5.28}$$

$$U_\nu^{\circ\, T} M_3' U_\nu^\circ = \begin{pmatrix} -2e^{-2i\phi}\sin^2\theta & 0 & e^{-i\phi}\sin 2\theta \\ 0 & 1 & 0 \\ e^{-i\phi}\sin 2\theta & 0 & -2\cos^2\theta \end{pmatrix}.$$

Thus, the neutrino mass matrix M_ν is diagonalized when the corresponding linear combination of the off-diagonal entries vanishes, which leads to

$$\cos 2\theta + a e^{i\phi}\sin 2\theta + b e^{-i\phi}\sin 2\theta = 0 \quad \Leftrightarrow \quad a e^{i\phi} + b e^{-i\phi} = -\cot 2\theta. \tag{5.29}$$

The above condition is equivalent to:

$$e^{i\phi} = \pm \frac{a^* - b}{|a^* - b|}, \quad \cot 2\theta = \mp \frac{|a|^2 - |b|^2}{|a^* - b|}. \tag{5.30}$$

It proves convenient to introduce the complex parameter

$$z = a e^{i\phi} - b e^{-i\phi} = \pm \frac{|a|^2 + |b|^2 - 2ab}{|a^* - b|}. \tag{5.31}$$

Using (θ, ϕ, z) is a reparametrisation of (a, b) determined by (5.30) and (5.31). The inverse parameter transformation is given by

$$\begin{aligned} a &= \frac{e^{-i\phi}}{2}\,(z - \cot 2\theta)\,, \\ b &= \frac{e^{i\phi}}{2}\,(-z - \cot 2\theta)\,. \end{aligned} \tag{5.32}$$

The neutrino mass matrix eigenvalues are the corresponding linear combinations of the diagonal entries in (5.28):

$$\begin{aligned} m_1 e^{-i(2\phi_1 - \phi)} &= c\left(z - \frac{1}{\sin 2\theta}\right), \\ m_2 e^{-i2\phi_2} &= c\left(\sqrt{3} - iz\sin\phi - \cot 2\theta\cos\phi\right), \\ m_3 e^{-i(2\phi_3 + \phi)} &= c\left(z + \frac{1}{\sin 2\theta}\right). \end{aligned} \tag{5.33}$$

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[7] In general, the standard labelling of the neutrino masses [1] corresponds to some permutation of the neutrino mass matrix eigenvalues, which affects the order of the PMNS matrix columns. However, the only non-trivial permutation of the TM$_2$ matrix columns consistent with the experimental data is (321), which is equivalent to a shift $\theta \to \theta - \pi/2$ up to an unphysical overall column sign. Hence, we can assume that the order of neutrino mass matrix eigenvalues coincides with the standard labelling without loss of generality.
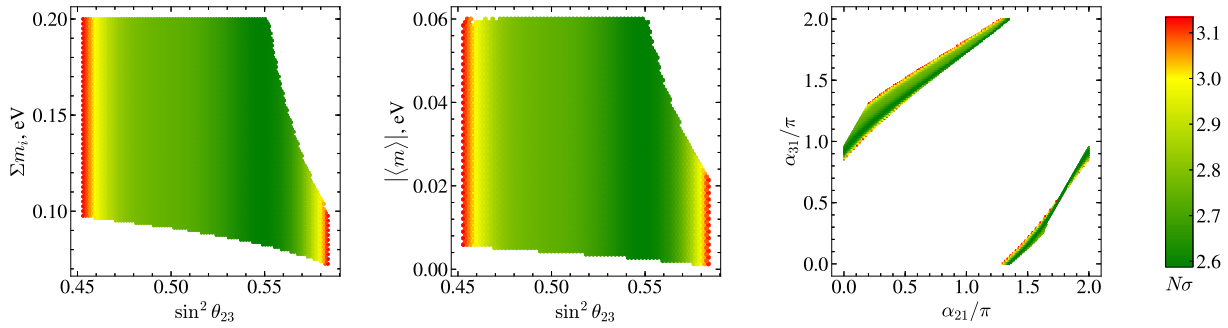
Fig. 1. Correlations between $\sin^2\theta_{23}$ and the sum of neutrino masses $\sum m_i$, between $\sin^2\theta_{23}$ and the effective Majorana mass $|\langle m\rangle|$, and between the Majorana phases α_{31} and α_{21} in the case of NO neutrino mass spectrum. See text for further details.

Fitting the mass-squared differences to experimentally observed values, we find the following constraint on z in terms of θ, ϕ and $r \equiv \Delta m_{21}^2/\Delta m_{31}^2$:

$$|z - z_0|^2 = R^2, \quad \mathrm{sign}\,(\mathrm{Re}\,z) = \pm\,\mathrm{sign}\,(\sin 2\theta)\,, \tag{5.34}$$

where the plus (minus) sign corresponds to NO (IO) spectrum of neutrino masses, and

$$z_0(\theta,\phi,r) = \frac{1-2r}{\cos^2\phi\,\sin 2\theta} + \tan\phi\left(\frac{\sqrt{3}}{\cos\phi} - \cot 2\theta\right)i,$$

$$R^2(\theta,\phi,r) = \left[\left(\sqrt{3} - \cot 2\theta\,\cos\phi\right)^2 + \frac{(1-2r)^2 - \cos^2\phi}{\sin^2 2\theta}\right]\Big/\cos^4\phi. \tag{5.35}$$

Since θ and r are tightly constrained by the experimental data, the set of phenomenologically viable models is effectively described by two angles ϕ and ψ, with the latter being the angle parameter on the circle (5.34), i.e. $z = z_0 + R\,e^{i\psi}$. Scanning through ϕ and ψ numerically, we find that to each set of the experimentally allowed values of the mixing angles and the mass-squared differences corresponds a range of models (parameterised by ψ) with different values of the neutrino masses and the Majorana phases.

We report the numerical results in the case of NO spectrum in Fig. 1. The allowed range of the sum of neutrino masses depends on the value of $\sin^2\theta_{23}$. The lower bound slightly decreases from 0.097 to 0.074 eV as $\sin^2\theta_{23}$ runs through its 3σ confidence interval of $[0.46, 0.58]$.[8] On the other hand, the upper bound is highly dependent on the value of $\sin^2\theta_{23}$, and tends to infinity as $\sin^2\theta_{23}$ approaches 0.5, which corresponds to $\delta = \phi = 3\pi/2$. This means that at this point the sum of neutrino masses is allowed to take any value greater than its lower bound of 0.093 eV. The dependence of the effective Majorana mass $|\langle m\rangle|$ on $\sin^2\theta_{23}$ is qualitatively similar to that of the sum of neutrino masses. The maximal value of $|\langle m\rangle| \cong 0.059$ eV is practically independent of $\sin^2\theta_{23}$ for $0.46 \le \sin^2\theta_{23} \le 0.55$. The lower bound of $|\langle m\rangle|$ varies from 0.0015 eV to 0.0059 eV for $\sin^2\theta_{23}$ in its 3σ range. However, for values of $\sin^2\theta_{23}$ from its 3σ range, $0.46 \le \sin^2\theta_{23} \le 0.58$, $|\langle m\rangle|$ can have values in the interval $[0.0059, 0.059]$ eV (see Fig. 1). Most (if not all) of these values may be probed in the future neutrinoless double beta decay experiments.

There is also a strong correlation between the Majorana phases. The set of best-fit models corresponds to $\phi = 1.664\pi$ and leads to the following values of observables:

$$r = 0.0299, \quad \delta m^2 = 7.34 \cdot 10^{-5}\ \mathrm{eV}^2, \quad \Delta m^2 = 2.455 \cdot 10^{-3}\ \mathrm{eV}^2,$$

$$\sin^2\theta_{12} = 0.3406, \quad \sin^2\theta_{13} = 0.02125, \quad \sin^2\theta_{23} = 0.5511,$$

$$m_1 = 0.0143 - 0.0612\ \mathrm{eV}, \quad m_2 = 0.0166 - 0.0618\ \mathrm{eV}, \quad m_3 = 0.0519 - 0.079\ \mathrm{eV},$$

$$\sum_i m_i = 0.0828 - 0.2019\ \mathrm{eV}, \quad |\langle m\rangle| = 0.0029 - 0.0589\ \mathrm{eV}, \quad \delta/\pi = 1.339, \tag{5.36}$$

consistent with the experimental data at 2.59σ C.L.

Similar analysis can be performed in the case of IO neutrino mass spectrum. However, in that case the minimal value of the sum of the three neutrino masses is 0.63 eV, and we do not analyse this case further.

6. Summary

We have investigated models of lepton masses and mixing based on modular A_4 flavour symmetry broken to residual symmetries in the charged lepton and neutrino sectors. The standard case of three lepton families was considered. In a theory based on finite modular flavour symmetry not only the matter fields, but also the constants such as the Yukawa couplings transform non-trivially under the modular symmetry. These constants are written in terms of modular forms which are holomorphic functions of a complex scalar field – the modulus τ. The modular forms have specific transformation properties under the modular symmetry transformations, which are characterised by a positive even number k called "weight", and depend on the order of the finite modular group via their "level" N. In the

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[8] We define the number of standard deviations from the χ^2 minimum as $N\sigma = \sqrt{\Delta\chi^2}$, where $\Delta\chi^2$ is a sum of one-dimensional projections $\Delta\chi_j^2$, $j = 1, 2, 3, 4$ from [2] for the accurately known dimensionless observables $\sin^2\theta_{12}$, $\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and r.

case of modular A_4 symmetry we have $N = 3$ and for the lowest weight modular forms $k = 2$. The modular forms transform under the usual non-Abelian discrete flavor symmetries as well. Modular forms of weight k and level N span a linear space of finite dimension. There exists a basis in this space such that the modular forms form multiplets transforming according to unitary irreducible representations of the finite modular group. In the case of modular A_4 symmetry, the dimension of the linear space of modular forms of weight $k = 2$ is 3, and one can employ modular forms transforming as the triplet irreducible representation of A_4. Modular forms of higher weights can be obtained as direct products of the modular forms of weight $k = 2$.

In lepton flavour models with finite modular symmetry, the modular symmetry must be broken in order to distinguish between the electron, muon and tauon, generate three different neutrino masses and reproduce the measured values of the three neutrino mixing angles. In the most economical versions of the flavour models the only source of breaking of the modular symmetry is the VEV of the modulus τ, $\langle \tau \rangle \neq 0$, and there is no need to introduce flavon fields. In the present article we consider both a model without flavons, in which the A_4 symmetry is broken only by $\langle \tau \rangle$, and a model with one triplet flavon field, in which the A_4 symmetry is broken by $\langle \tau \rangle$ in the neutrino sector and by the VEV of the flavon in the charged lepton sector.

The modular group A_4 has two generators S and T satisfying the presentation rules: $S^2 = (ST)^3 = T^3 = I$, where I is the unit operator. Residual symmetries arise whenever the VEV of the modulus τ breaks the considered finite modular group Γ_N, $\Gamma_3 \simeq A_4$, only partially, i.e., the little group (stabiliser) of $\langle \tau \rangle$ is non-trivial. There are only 2 inequivalent finite points with non-trivial little groups, namely, $\langle \tau \rangle = -1/2 + i\sqrt{3}/2 \equiv \tau_L$ and $\langle \tau \rangle = i \equiv \tau_C$ [22]. The first one is the left cusp in the fundamental domain of the modular group, and corresponds to a residual symmetry associated with the subgroup $\mathbb{Z}_3^{ST} = \{I, ST, (ST)^2\}$ of the A_4 group. The $\langle \tau \rangle = i$ point is invariant under the S transformation ($\tau = -1/\tau$) of the $\mathbb{Z}_2^S = \{I, S\}$ subgroup of A_4. There is also infinite point $\langle \tau \rangle = i\infty \equiv \tau_T$, in which the subgroup $\mathbb{Z}_3^T = \{I, T, T^2\}$ of A_4 is preserved.

We have constructed phenomenologically viable models of lepton masses and mixing based on modular A_4 invariance broken to residual symmetries $\mathbb{Z}_3^T$ or $\mathbb{Z}_3^{ST}$ and $\mathbb{Z}_2^S$ respectively in the charged lepton and neutrino sectors. The neutrino Majorana mass term is assumed to be generated by the dimension 5 Weinberg operator. We found that there is no common symmetry value of τ, which leads to charged lepton and neutrino masses that are consistent with the data. For the construction of the charged lepton mass matrix, we used weight 2 modular forms at $\tau^\ell = \tau_T$ (Case I) or at $\tau^\ell = \tau_L$ (Case II). At the same time we used weight 4 modular forms at $\tau^\nu = \tau_C$ for constructing the neutrino Majorana mass term. Since at present we are not aware of a mechanism that can lead to different values of $\langle \tau \rangle$ in the neutrino and the charged lepton sectors, our analysis only suggests that having trimaximal neutrino mixing in models with modular A_4 symmetry without flavons might be possible. We also show that, alternatively, trimaximal neutrino mixing can be obtained by assuming that $\langle \tau \rangle$ breaks the A_4 symmetry to $\mathbb{Z}_2^S$ in the neutrino sector, while a VEV of a triplet flavon field breaks the A_4 symmetry to $\mathbb{Z}_3^T$ in the charged lepton sector. This alternative construction requires only one value of $\langle \tau \rangle$, but leads to the same form of the mass matrices, hence the same phenomenology.

The so constructed models involve three real parameters fixed by the values of the three charged lepton masses. The three neutrino masses, three neutrino mixing angles and three CPV phases are functions of altogether 2 real constants and two phases. In these models the neutrino mixing matrix is of trimaximal mixing form. In Case I it is given by the tri-bimaximal mixing matrix multiplied on the right by a unitary rotation in the 1-3 plane, which depends on one angle and one phase. In addition to successfully describing the charged lepton masses, neutrino mass-squared differences and the atmospheric and reactor neutrino mixing angles θ_{23} and θ_{13}, these models predict the values of the lightest neutrino mass (i.e., the absolute neutrino mass scale), of the Dirac and Majorana CP violation (CPV) phases and correspondingly of the effective neutrinoless double beta decay Majorana mass, as well as the existence of specific correlations between i) the values of the solar neutrino mixing angle θ_{12} and the angle θ_{13}, ii) the values of the Dirac CPV phase δ and of the angle θ_{23}, iii) the sum of the neutrino masses and θ_{23}, iv) the neutrinoless double beta decay effective Majorana mass and θ_{23}, and v) between the two Majorana phases (Fig. 1). These predictions will be tested with future more precise neutrino oscillation data, with results from direct neutrino mass and neutrinoless double beta decay experiments, as well as with improved cosmological measurements.

Acknowledgements

Appendix A. Multiplication rule of A_4 group

We take

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{A.1}$$

where $\omega = e^{i\frac{2}{3}\pi}$ for a triplet. In this base, the multiplication rule of the A_4 triplet is

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1 b_1 + a_2 b_3 + a_3 b_2)_{\mathbf{1}} \oplus (a_3 b_3 + a_1 b_2 + a_2 b_1)_{\mathbf{1'}}$$

$$\oplus (a_2 b_2 + a_1 b_3 + a_3 b_1)_{\mathbf{1''}}$$

$$\oplus \frac{1}{3} \begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2} \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}} ,$$

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1} , \qquad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'' , \qquad \mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}' , \qquad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1} . \tag{A.2}$$

More details are shown in the review [4,5].

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