International School for Advanced Studies,
Mathematics Area

PhD thesis

Frobenius manifolds in critical and non-critical strings

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June 21, 2019

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Acknowledgements

First of all, I am deeply grateful to my advisors: Boris Dubrovin and Alexander Belavin. I would like to thank many people for various scientific discussions, advices and help. In particular: D. Aranha, V. Batyrev, V. Belavin, F. Benini, M. Bershtein, G. Bonelli, S. Cecotti, B. Fantechi, S.Galkin, A. Gerhardus, D. Guzzetti, K. Hori, S. Hosono, T. Hübsch, H. Iritani, S. Katz, A. Klemm, A. Litvinov, M. Liu, A. Mellit, T. Milanov, A. Okounkov, F. Quevedo, C.Rim, G. Ruzza, K. Saito, A. Tanzini, J. Teschner, V. Vasiliev and others.

Introduction.

Frobenius manifolds were introduced as a mathematical structure behind 2d topological field theories introduced by Boris Dubrovin [1].

In string theories the worldsheet of a string is a two-dimensional Riemann surface. From the worldsheet perspective the string theory is described by two-dimensional theories on worldsheets propagating in the effective space-time. To build a space-time supersymmetric theory one usually starts from a supersymmetric conformal field theory (SCFT) with extended supersymmetry (SUSY) on the worldsheet, typically $N = 2$ or $N = (2, 2)$. The vacua states of such theories can be studied in purely topological or anti-topological perspective [2, 3, 4]

Theories with extended supersymmetry in 2d allow for a topological twist [5] That is a topological theory which is canonically constructed from the original SCFT. Topological theories capture the dynamics of the chiral rings, particular ground states in field theories. Moreover, correlation numbers of the topologically twisted theory encode information about some correlation functions of the non-twisted SCFT. Describing 2d TFT, Frobenius manifolds already encode some information about superstring theories, namely topological sectors of the non-twisted CFT.

It turns out, that Frobenius manifolds also appear in non-critical string, theory. Usually in string theory one considers the target space-time to be 10-dimensional for superstring and 26-dimensional for bosonic string to get rid of the conformal anomaly. That is the theory which was conformally invariant classically may depend on conformal transformations due to a non-invariance of the measure in the path integral. This phenomenon is called a conformal anomaly. The dependence of the theory on dilations is given by the trace of the energy-momentum tensor. Classically in CFT's the trace vanishes. However, in the quantum case it is proportional to the Ricci-curvature of the surface with a numerical prefactor which is called a *central charge* (up to a universal constant). Thus, the CFT remains conformally invariant on the quantum level if and only if the total central charge of the theory vanishes. In bosonic string the total central charge is equal to the sum of the central charge of the matter which is equal to the dimension of the space-time and the central charge of the $b - c$ ghost system, which is equal to -26. Therefore the theory becomes conformal precisely in critical dimension 26. Whereas in the superstring theory the total central charge is equal to $3/2$ times the dimension of the space-time (one for each boson and $1/2$ for each superpartner) and the central charge of the $b - c - \beta - \gamma$ system that is -15.

Nevertheless, there is a way to define a theory in any dimension [6, 7]. In the critical dimension the dependence on the conformal class of the metric factorizes completely due to the cancellation of the conformal anomaly. In general, the theory also depends on the metric. The dependence on the conformal factor is given by a theory of one (pseudo)-scalar field, the Liouville field. The corresponding field theory is called a Liouville theory, because its classical equations of motion are Liouville equations for a constant curvature metric [6]. It was conjectured by Distler and Kawai in [8] that the Liouville field theory is a conformal field theory as well and its central charge compensates the conformal anomaly. The formalizm for the Liouville CFT was developed in [9, 10, 11] As an outcome, the string theory in the noncritical dimension or Liouville Gravity can be reformulated in the language on CFT.

One of the ways to escape the complications of CFT when discussing theories of gravity uses topological twists of N=2 supersymmetric field theories. In this approach all the massive modes decouple and correlation numbers of the theory are encoded in the topological field theory, whose genus zero limit is precisely a Frobenius manifold.

Another approach to two-dimensional quantum gravity or non-critical string theory was developed based on the discrete approach [12, 13, 14] This approach starts from representing generation function of triangulations of Riemann surfaces as matrix integrals or integrals over spaces of matices of some deformations of Gaussian densities. Taking proper limits in the couplings of the theory (the double-scaling limit) one can achieve that the matrix integral, which becomes infinite-dimensional in the limit, is dominated by triangulations with the huge number of triangles. In this limit the partition function of the matrix model becomes a tau-function of certain integrable hierarchy.

The famous Witten conjecture [15] proved by Kontsevich [16] (see also [17, 18]) states that matrix model approach to two-dimensional gravity is equivalent to the topological gravity. More precisely, topological field theory partition function (which is computed using intersection numbers on moduli spaces of curves) coincides with a tau function of KdV hierarchy which also satisfies the so-called *string equation*.

The first part of this thesis is devoted to unification of topological gravity and discrete approach with the Liouville gravity. That is we pursue a conjecture [19, 20, 21] that the partition function of the Liouville gravity coincides with two other partition functions after a particular change of coordinates, which is called *resonance transformations*. The main complications appear in defining and computing the Minimal Liouville Gravity correlation numbers and in computations of the resonance transformations. The latter one are conjecturally universal, that is the same for all genera of the string worldsheet. That is the most important part as in all *semisimple* TFT's is the genus zero part which is mathematically described as a Frobenius manifold. This structure plays a crucial role in the correspondence between MLG and TFT and in computations of the resonance transformations.

In the section 1.3 we formulate the results of the joint paper with V.Belavin [22] where we compute the 4-point correlation numbers in the Lee-Yang series of Minimal Liouville Gravity using certain limits from generalized Minimal Models and compare them with the discrete approach.

A few following sections are devoted to describing parts of the discrete approach. In the section 1.7 we state our results of joint works with V.Belavin and C.Rim [23, 24] on the correlation numbers on a disk which is a simplest case of a Riemann surface with boundary. We show that using the resonance transformations obtained on a sphere we reproduce the Liouville Field Theory approach values [25].

The second part, in turn, is related to critical string theories and supersymmetric 2d CFTs. We start by discussing $N = (2, 2)$ supersymmetric conformal field theories, more specifically, Landau-Ginzburg theories. It is well-known that the dynamics of the chiral rings is governed by a particular type of Frobenius manifolds arising from singularity theory. They appeared before the general notion of Frobenius manifolds as *flat structures* in the works of Kyoji Saito [26, 27, 28, 29] all the Frobenius manifolds encountered in the first chapter are isomorphic via *mirror symmetry* to Frobenius manifolds arising on universal unfoldings of A_n-type singularities. When one considers CFT's with high enough central charge (starting from structure on the unfolding space becomes complicated and starts to depend on a particular choice of a volume form in the noncompact space, so-called *primitive form.* In the Landau-Ginzburg language when there are marginal and irrelevant deformations in the chiral ring, the correlation functions are not just oscillatory integrals but receive certain corrections. We study these corrections computing primitive forms in some important cases and introducing notions of *weak primitive forms* which lead to quasi-Frobenius manifolds or F-manifolds [30] which we briefly describe in the section 2.3.

Then we turn to a more classical subject, which is critical superstring theory compactified on a Calabi-Yau variety. The moduli space of such compactifications consists of the Kähler and complex structure moduli. Frobenius manifold structure arises on both of these moduli spaces. For the Kähler moduli space this Frobenius structure is the quantum cohomology of the corresponding variety, whereas for complex structures it is a certain limit of the Frobenius manifold on the universal unfolding of a singularity which is related to the Calabi-Yau variety in question.

The mirror symmetry relates these two Frobenius manifolds for a "pair" of different Calabi-Yau varieties which is a more complicated counterpart of the mirror symmetry encountered in the first chapter. In superstring theory the moduli space corresponds to only marginal deformations of the Frobenius manifolds. Frobenius maifolds together with its tt^* geometry on this marginal deformation subspace simplifies. This simplified structure is called *special Kähler geometry* and appeared long ago in the context of 4d supersymmetric gauge theories with or without gravity.

In the corresponding section of the paper we compute the special Kähler geometry for moduli spaces of complex structures for a huge class of Calabi-Yau varieties given by *invertible* singularities. We use the connection of Calabi-Yau non-linear sigma models and N=(2,2) supersymmetric Landau-Ginzburg theories to compute special geometry for many hypersurfaces in weighted projective spaces. The exposition of this part in section 2.5 is based on the series of papers with Alexander Belavin [31, 32, 33, 34].

In the end of the chapter (section 2.6 we turn to the nice connection between special geometry on the moduli spaces of Calabi-Yau varieties and partition functions of 2d supersymmetric Gauge Linear Sigma Models (GLSM) following our paper with Alexander Belavin and Alexei Litvinov [35].

Notations

Let Σ be a Riemann surface with a holomorphic coordinate z, then

$$\mathrm{d}^2 z := \frac{1}{2i}\mathrm{d}z\overline{\mathrm{d}z}.$$

Let $\mathcal{M}$ be a complex manifold with holomorphic coordinates $\{t^i\}$.

$$\partial_i = \partial_{t^i} = \frac{\partial}{\partial t^i},$$

when it does not lead to a confusion. Moreover, we denote

$$\mathrm{d}^n t := \mathrm{d}t^1 \wedge \ldots \wedge \mathrm{d}t^n. \tag{1}$$

Let $\Phi = \Phi(z, \bar{z})$ be a primary field in a CFT. We denote a class of its descendant fields as

$$[\Phi] = \mathrm{Span}\langle L_{-n_1} \cdots L_{-n_k} \overline{L_{-m_1}} \cdots \overline{L_{-n_l}}\, \Phi\rangle_{\bar{n},\bar{m}}.$$

We use angular brackets $\langle \mathring{A} \rangle$ to denote correlation functions/numbers of observables $\mathring{A}$ in any theory. When there is a danger of confusion we specify in which theory the correlators are defined.

Consider a $\mathbb{C}^n$ with coordinates $\{x_i\}_{i=1}^n$. We denote the set of all coordinates as $\bar{x}$ or x when there is no danger of confusion.

Let $W_0(x) : \mathbb{C}^n \to \mathbb{C}$ and $W(x, \phi) : \mathbb{C}^n \times \mathbb{C}^\mu \to \mathbb{C}$ be a holomorphic function and its unfolding correspondingly. We also write them as $W_0(x) = W_0$ and $W(x, \phi) = W$. We use the following notations for their chiral (Milnor) rings

$$\mathcal{R}_0 := \frac{\mathbb{C}^n[x_1, \ldots, x_n]}{\partial_1 W_0, \ldots, \partial_n W_0},$$
$$\mathcal{R} = \mathcal{R}_\phi := \frac{\mathbb{C}^n[x_1, \ldots, x_n]}{\partial_1 W(x,\phi), \ldots, \partial_n W(x,\phi)}.$$

Let $W(x, \phi) = W_0(x) + \sum_{s=1}^h \phi^s e_s$ be a deformation family of a polynomial $W_0(x)$ where all e_s are monomials in $\bar{x}$. We use an index s for two related things:

1. $1 \leq s \leq h$ is an integer which counts the deformations e_s.
2. $\bar{s} = (s_1, \ldots, s_n)$ as an exponent vector of the corresponding deformation $e_s = x_1^{s_1} \ldots x_n^{s_n}$.

Chapter 1

Non-critical Strings

1.1 Conformal Field Theories

Two-dimensional conformal field theories in the *operator product expansion* (OPE for short) language were formulated by Belavin, Polyakov and Zamolodchikov in [36]. The paper appeared from the attempts to solve Liouville field theory which was put in the OPE formalizm much later [9, 10, 11]The mathematical language to work with 2d CFTs includes representation theory of Virasoro and other algebras, vertex operator algebras [37] and conformal blocks among other things. For the purposes of this paper we will stick to the original more physical language.

Let us give an informal introduction to Conformal Field Theories in flavor of the book [38].

Consider a Riemann surface Σ. Pick an open disc $U \subset \Sigma$ with local complex coordinates $z, \bar{z}$. In two dimensions there is an infinite-dimensional conformal algebra: namely Witt algebra of holomorphic polynomial vector fields on a punctured disc with generators $\{l_n\},\ n \in \mathbb{Z}$ and relations

$$[l_m, l_m] = (n-m)l_{n+m}, \tag{1.1}$$

or $l_n = -z^{n+1}\partial_z$. In classical conformal field theories this algebra is an automorphism algebra. We note that l_1, l_0, l_{-1} generate the sl_2-subalgebra $l_0 = -z\partial_z$ is a dilation and $l_{-1} = \partial_z$ is a shift. All the operators $l_n,\ n < -1$ are singular at the origin.

In the quantum theories it gets central extended. The central extension is essentially unique and is called a Virasoro algebra.

Definition 1.1.1. *Virasoro algebra is a Lie algebra with generators* $Vir = \{L_n, c\},\ n \in \mathbb{Z}$ *and relations*

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{1.2}$$

The element c is a central element and commutes with everything.

In representations it will be proportional to the identity operator with the coefficient which we call the same letter c and which is called a central charge of the theory. In the classical limit $c \sim 1/\hbar^2$ and $c \to \infty,\ L_n \to \hbar\, l_n$.

The main idea of the vertex operator algebras is the *state operator correspondence.* Let us explain the idea. Consider a two-dimensional quantum theory on a Riemann surface Σ. There are two main approaches one can take to work with a quantum theory. In the Lagrangian approach one considers a number of fields which belong either to a bundle over the surface $\Phi \in \Gamma(V, \Sigma)$ or to the space of maps of the surface into some target space $\Phi \in \mathrm{Maps}(\Sigma, M)$ or to a combination of such. We call $\mathcal{F}$ a space of all possible fields of our theory. Then the correlation functions of the theory are given by the path integral:

$$\langle \Phi_1(z_1) \cdots \Phi_n(z) \rangle := \int_{\mathcal{F}} \mathcal{D}\Phi\ \Phi_1(z_1) \cdots \Phi_n(z)\, e^{-\frac{1}{\hbar} S[\Phi]}, \tag{1.3}$$

where an appropriate measure $\mathcal{D}\Phi$ (mathematically ill-defined) on the space $\mathcal{F}$ and the action functional $S : \mathcal{F} \to \mathbb{C}$ are main ingredients of the theory. In the Lagrangian theory the action functional is given by

$$S[\Phi] = \int_{\Sigma} \mathrm{d}^2 z\, \mathcal{L}(\Phi, \partial\Phi), \tag{1.4}$$

where $\mathcal{L}(\Phi, \partial\Phi)$ is a local functional on the field space. Since the action functional is local, one can compute the path integral cutting Σ into a small disk D around z_1 and over $\Sigma \backslash D$. The path integral on D with the insertion of $\Phi_1(z_1)$ defines state in the Hilbert space which is attached to the boundary ∂D. The state sets a boundary condition $\Phi_i|_{\partial D} = \phi_i$ to the value of the path integral with the corresponding boundary condition for the fields Φ_i. In a CFT any Hilbert space state defines a local operator of the theory. Since a CFT is scale invariant, the path integral on a disk does not depend on its radius up to an overall factor. Sending the radius to zero we recover a local operator at the origin. In other words, there is a conformal map *exp* which sends a cyclinder to a punctured disk. The reverse time evolution on a cylinder sends a boundary state to the origin on the disk.

Therefore, for each state in the Hilbert space of the theory $v \in \mathcal{H}$ there is some local field (vertex operator) $\Phi(z)$ which can be thought of as an operator valued distribution on Σ or a formal power series in z with coefficients from $\mathrm{End}(\mathcal{H})$ and vice versa, to each nice enough vertex operator $\Phi(z)$ one can associate a state $\Phi(0)|0\rangle$, where $|0\rangle \in \mathcal{H}$ is a vacuum vector. Nice enough means that all the coefficients at $z^{-n},\ n > 0$ kill the vacuum vector. In particular, in any CFT there is a vertex operator corresponding to an energy momentum tensor

$$T(z) := \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}. \tag{1.5}$$

The vacuum vector is a free Verma module (that is a highest weight representation of the Virasoro algebra) which is Lorentz invariant, that is a subject only to the relations

$$L_n|0\rangle = 0, \ n \leq 3. \tag{1.6}$$

Then the energy-momentum tensor $T(z)$ corresponds to L_{-2} or $L_{-2}\cdot\mathrm{id}$ using the state-operator correspondence.

Since Virasoro algebra acts on $\mathcal{H}$, it decomposes into irreducible representations $\mathcal{H} = \sum_n \mathcal{H}_n$. We consider the case where all $\mathcal{H}_n$ are highest weight representations, that is [39] either Verma modules or their factors with respect to Verma submodules when possible.

Definition 1.1.2. *The Verma module $V_{c,\Delta}$ parametrised by two complex numbers c, Δ is a representation of the Virasoro algebra generated from the highest (lowest in mathematical literature) weight vector $|\Phi_\Delta\rangle$ subject only to the relations*

$$\begin{aligned} &L_n|\Phi_\Delta\rangle = 0, \ n < -1, \\ &L_0|\Phi_\Delta\rangle = \Delta\,|\Phi_\Delta\rangle \ n < -1, \\ &c(|\Phi_\Delta\rangle) = c\,|\Phi_\Delta\rangle. \end{aligned} \tag{1.7}$$

By the Poincaré Birkhoff Witt theorem (PBW) any vector in such a module is uniquely representable as a finite sum with complex coefficients of the vectors

$$v_{n_1,\ldots,n_k} = L_{n_1}\cdots L_{n_k}\,|\Phi_\Delta\rangle, \tag{1.8}$$

where $k \geq 0$ and $n_1 \leq n_2 \leq \cdots \leq n_k < 0$.

Operator Product Expansion On the other hand, we could start from the operator approach. In the operator approach one chooses a basis in the space of fields $\mathcal{F} = \langle \mathcal{A}_i(x)\rangle_{i\in I}$, where $x = (x_1, x_2)$ is a local (real) coordinate on the surface $\Sigma \in \mathcal{M}_g$. Consider a set of fields $\mathcal{A}_1(x_1), \ldots, Ac_n(x_n)$. The $n-$point correlation function

$$\langle \mathcal{A}_1(x_1)\cdots\mathcal{A}_n(x_n)\rangle_\Sigma \tag{1.9}$$

is a real analytic function of $x_1, \ldots, x_n$. Correlation functions typically have singularities when the insertion points coincide $x_i = x_j$. To define correlation functions globally we need to tell how they transform under conformal transformations on Σ. If the fields $\mathcal{A}_i(x_i)$ are *primary*, that is under a conformal map $x = x(y)$ they transform as

$$\mathcal{A}_i(x(y_i)) = \left(\frac{\partial x}{\partial y}\right)^{-\Delta_i}\Bigg|_{y=y_i}\mathcal{A}_i(y_i), \tag{1.10}$$

then the correlation function transforms as

$$\langle \mathcal{A}_1(x(y_1)) \cdots \mathcal{A}_n(x(y_n)) \rangle_\Sigma = \prod_{i=1}^{n} \left(\frac{\partial x}{\partial y} \right)^{-\Delta_i} \Bigg|_{y=y_i} \langle \mathcal{A}_1(y_1) \cdots \mathcal{A}_n(y_n) \rangle_\Sigma. \tag{1.11}$$

The points of insertion $x_1, \ldots, x_n$ together with the moduli of the Riemann surface Σ of genus g are coordinates on the moduli space of n-punctured Riemann surfaces $\mathcal{M}_{g,n}$. [1] That is the correlation functions are real analytic sections over a certain bundle over $\mathcal{M}_{g,n}$ whose gluing functions are given by (1.11).

The OPE is a way to recursively reduce n-point correlation functions to $n-1$ point functions. Using a conformal map one can bring two fields close to each other such that their contribution is equivalent to the one of a local field. The resulting local field in the correlation function (1.11) can be decomposed in the basis of fields $A_i(x_i)$:

$$\langle \cdots A_i(x_i) \cdots A_j(x_j) \cdots \rangle_\Sigma = \sum_{k \in I} C_{ij}^k(x_i, x_j) \langle \cdots A_k(x_j) \cdots \rangle_\Sigma, \tag{1.12}$$

where analytic at $x_i \neq x_j$ functions $C_{ij}^k(x_i, x_j)$ do not depend on anything but x_i, x_j and i, j, k. Due to translation invariance, these functions depend only on the difference $C_{ij}^k(x_i, x_j) = C_{ij}^k(x_i - x_j)$ Therefore one can define a product in operator space making it an algebra:

$$A_i(x_i) \cdot A_j(x_j) := \sum_{k \in I} C_{ij}^k(x_i - x_j) A_k(x_j). \tag{1.13}$$

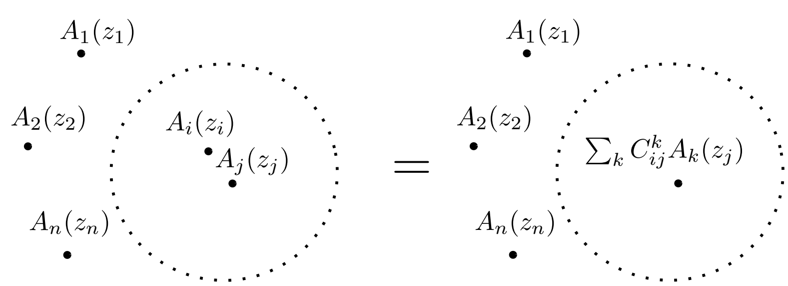

Figure 1.1: OPE

Repeating the OPE $n-1$ times one reduces the correlation function (1.11) to one-point correlation function:

$$\langle A_1(x_1) \cdots A_n(x_n) \rangle_\Sigma = \sum_{k \in I} C_n^k(x_1, \ldots, x_n) \langle A_k(x_n) \rangle_\Sigma. \tag{1.14}$$

[1] We will not study in details the complications arising from the fact that $\mathcal{M}_{0,n}$, $n < 3$ and $\mathcal{M}_{1,0}$ are Artin stacks and have negative dimensions. For our purposes we consider these spaces as one point spaces with the "gauge symmetry" given by the automorphism groups of the corresponding surfaces.

In particular, if Σ is a sphere, then there are no complex moduli, moreover, one-point correlation numbers should be shift-invariant (if we put $x_n = \infty$ on the Riemann sphere). It follows that one-point functions should be constants, which means that $\Delta_k = 0$. The identity operator id clearly has $\Delta_{id} = 0$. We will use the assumption that in CFT for any complex number Δ there could be only one primary field with conformal dimension Δ.

The correlation functions (1.11) should be independent of the order of operator product expansions. It means that the operator algebra should be associative. This requirement gives a very strong overdetermined system of constraints for functions C_{ij}^k:

$$C_{kl}^m(x_k - x_j)C_{ij}^l(x_i - x_j) = C_{ki}^l(x_k - x_i)C_{lj}^m(x_i - x_j). \tag{1.15}$$

The program of solving the equations (1.15) has the name *conformal bootstrap*.

Stress-energy tensor and primary fields In two dimensions stress-energy tensor has three independent components: $T_{zz}(z,\bar{z})$, $T_{z\bar{z}}(z,\bar{z})$, $T_{\bar{z}\bar{z}}(z,\bar{z})$. If the theory is conformal, then the trace of the tensor vanishes

$$T_z^z + T_{\bar{z}}^{\bar{z}} = T_{z\bar{z}} + T_{\bar{z}z} = 2T_{z\bar{z}} = 0. \tag{1.16}$$

The Noether conservation law reads

$$0 = \partial_\mu T_\nu^\mu \implies \partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0, \tag{1.17}$$

therefore T_ν^μ factorizes into a holomorphic and antiholomorphic parts $T(z) := T_{zz}(z)$ and $\overline{T(z)} = T_{\bar{z}\bar{z}}(\bar{z})$. In quantum theories stress-energy tensor generates transformations of the fields under coordinate changes. In the view of factorization of T_ν^μ into holomorphic and antiholomorphic components the holomorphic transformations of the fields are generated by the OPE with the holomorphic energy-momentum tensor $T(z)$ [2]

$$\delta_\epsilon \mathcal{A}_i(z,\bar{z}) := \oint_z \mathrm{d}z\, \epsilon(w) T(w) \mathcal{A}_i(z,\bar{z}), \tag{1.18}$$

where $\epsilon(w)$ is a vector field on a punctured disc centered at $w = z$ and δ_ϵ is the first order variation of the field under the coordinate change $z \to z + \epsilon(z)$. One can define operators L_n acting on $\mathcal{H}$ by the Laurent decomposition

$$T(w)A_i(z,\bar{z}) = \sum_{n\in\mathbb{Z}} \frac{L_n A_i(z,\bar{z})}{(w-z)^{n+2}}. \tag{1.19}$$

The shift in 2 is conventional in the physical literature. In mathematical notations $L_n A(0)|0\rangle$ corresponds to an $(n-1)$st bracket of the vertex operators $T(w)$ and $A(z,\bar{z})$.

[2] We omit $1/2\pi i$ factors in the residue integrals for simplicity.

From (1.18) it is clear that L_n is the operator of transformation of the field $A_i(z,\bar{z})$ by the vector field $\epsilon(z)\partial_z = z^{n+1}\partial_z$.

The space of fields $\mathcal{F}$ is generated from so-called *primary fields* $\{\Phi_{\Delta,\bar{\Delta}}(z,\bar{z})\}$ by the Virasoro algebra action. The primary fields are on the state space $\mathcal{H}$ of the theory.

The complex numbers $\Delta, \bar{\Delta}$ are called conformal dimensions, for unitary theories they is positive and the field $\Phi_{\Delta,\bar{\Delta}}$ can be thought of as a coefficient of a "differential form of rank $(\Delta,\bar{\Delta})$", that is the expression $\Phi_{\Delta,\bar{\Delta}}\,(\mathrm{d}z)^{\Delta}(\overline{\mathrm{d}z})^{\bar{\Delta}}$ is invariant under conformal transformations:

$$\Phi_\Delta(w(z),\overline{w(z)}) = \left(\frac{\partial w}{\partial z}\right)^{-\Delta}\left(\overline{\frac{\partial w}{\partial z}}\right)^{-\bar{\Delta}}\Phi_\Delta(z,\bar{z}). \tag{1.20}$$

The infinitesimal form of (1.20) reads

$$\delta_\epsilon\Phi_\Delta(z,\bar{z}) = (1-\epsilon'(z))^{-\Delta}\Phi_\Delta(z+\epsilon(z),\bar{z}) - \Phi_\Delta(z,\bar{z}) = \Delta\,\epsilon'(z)\Phi_\Delta(z,\bar{z}) + \partial_z\Phi_\Delta(z,\bar{z}). \tag{1.21}$$

This variation implies the following OPE with $T(z)$:

$$T(w)\Phi_\Delta(z,\bar{z}) = \frac{\Delta\Phi_\Delta(z,\bar{z})}{(w-z)^2} + \frac{\partial_z\Phi_\Delta(z,\bar{z})}{w-z} + reg., \tag{1.22}$$

where *reg.* means terms which do not have a singularity at $w \to z$.

The OPE of the stress-energy tensor with itself is

$$T(w)T(z) = \frac{c}{12(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} + reg., \tag{1.23}$$

from which we see that $T(z)$ is not a primary field itself and has the following anomalous transformation law:

$$\partial_\epsilon T(z) = \frac{c}{12}\epsilon'''(z) + 2\epsilon'(z)T(z) + \epsilon(z)\partial_z T(z) \tag{1.24}$$

which integrates to

$$T(w(z)) = \left(\frac{\partial w}{\partial z}\right)^{-2}\left[T(z) - \frac{c}{12}\{w,z\}\right], \tag{1.25}$$

where $\{w,z\} = w'''/w' - 3/2(w''/w')^2$ is a Schwarzian derivative. The Schwarzian derivative of any SL_2 transformation is identically zero, so T transfroms as a tensor under global conformal transformations on a Riemann sphere.

The formula (1.24) implies Virasoro algebra commutation relations for operators L_n with the central charge equal to c:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}. \tag{1.26}$$

On the level of Virasoro operators L_n the formula (1.22) reads that

$$L_n\Phi_\Delta(z,\bar z)=0,\ n>0,\quad L_0\Phi_\Delta(z,\bar z)=\Delta\Phi_\Delta(z,\bar z). \tag{1.27}$$

Therefore, if one defines $|\Phi_\Delta\rangle := \Phi_\Delta(0,0)|0\rangle$, then primary operators correspond to Virasoro highest weight vectors of Verma modules (1.7).

Conformal blocks In two-dimensional CFT correlation functions are constrained by the so-called *Ward identities*. Due to these constraints and OPE the correlation functions can be represented as a sum of model-independent building blocks with model-dependent structure constants (three point correlation functions). The building blocks are completely determined by the Virasoro algebra and go by the name of *conformal blocks*.

Consider a CFT with a complete set of primary fields $\{\Phi_i(z,\bar z)\}_{i\in I}$ with conformal dimensions $L_0\,\Phi_i(z,\bar z)=\Delta_i\,\Phi_i(z,\bar z)$. Any field of the theory can be decomposed into a sum of the Virasoro descendents of such fields

$$L_{\bar n}\,\overline{L}_{\bar m}\,\Phi_i(z,\bar z):=L_{-n_1}\cdots L_{-n_k}\overline{L}_{-m_1}\cdots\overline{L}_{-m_p}\,\Phi_i(z,\bar z). \tag{1.28}$$

Consider OPE of two primary fields functions:

$$\Phi_1(z_1,\bar z_1)\Phi_2(z_2,\bar z_2)=\sum_{k\in I}C^k_{12}(z_1-z_2)\sum_{\bar n,\bar m\geq 0}\beta^{k,\bar n,\bar m}_{1,2}(z_1-z_2)L_{\bar n}\,\overline{L}_{\bar m}\,\Phi_k(z_2,\bar z_2). \tag{1.29}$$

Where in the right hand side we wrote the most general form of an OPE assuming the set I is at most countable. The conformal Ward identities allow to compute all the coefficients $\beta^{k,\bar n,\bar m}_{1,2}(z_1-z_2)$. The Ward identities follow from applying L_n with $n>0$ to both sides of the formula (1.29).

Rescaling both sides of the equation (or using L_0) we get z-dependence of the structure functions

$$\Phi_1(z,\bar z)\Phi_2(0,0)=\sum_{k\in I}(z\bar z)^{\Delta_k-\Delta_1-\Delta_2}\mathbb{C}^k_{12}\sum_{\bar n,\bar m\geq 0}z^{\sum_i n_i}\bar z^{\sum_j m_j}\beta^{k,\bar n}_{1,2}\beta^{k,\bar m}_{1,2}L_{\bar n}\,\overline{L}_{\bar n}\,\Phi_k(0,0), \tag{1.30}$$

where the coefficients $\beta^{k,\bar n}_{1,2}$ and $\beta^{k,\bar m}_{1,2}$ are the same because the construction is symmetric under switching of holomorphic and antiholomorphic Virasoro algebras (we assume the fields to have the same left and right conformal dimensions). It follows, that the OPE in (1.30) factorizes as

$$\begin{aligned}\Phi_1(z,\bar z)\Phi_2(0,0)&=\sum_{k\in I}(z\bar z)^{\Delta_k-\Delta_1-\Delta_2}\mathbb{C}^k_{12}\Psi_l(z)\,\overline{\Psi_l(z)}\,\Phi_k(0,0),\\ \Psi_l(z)&:=\sum_{\bar n\geq 0}z^{\sum_i n_i}\beta^{k,\bar n}_{1,2}L_{\bar n}.\end{aligned} \tag{1.31}$$

To compute the action of the general L_n on the primary field we use its definition through the OPE (1.19) and (1.22)

$$L_n(\Phi_1(z,\bar{z})\Phi_2(0,0)) = \oint_C T(w)w^{n+1}\Phi_1(z,\bar{z})\Phi_2(0,0)\mathrm{d}w, \tag{1.32}$$

where the contour C encircles the points 0 and z on the complex plane. Computing the residue we get

$$L_m(\Phi_1(z,\bar{z})\Phi_2(0,0)) = \left(z^{m+1}\frac{\partial}{\partial z} + (m+1)z^m\Delta_1\right)\Phi_1(z,\bar{z})\Phi_2(0,0)+\Phi_1(z,\bar{z})L_n\Phi_2(0,0), \tag{1.33}$$

or

$$[L_m, \Phi_1(z,\bar{z})] = \left(z^{m+1}\frac{\partial}{\partial z} + (m+1)z^m\Delta_1\right)\Phi_1(z,\bar{z}). \tag{1.34}$$

We apply (1.33) to the right hand size of (1.31) to get the constraints on $\beta_{12}^{k,\bar{n}}$

$$L_m\left[\Psi_l(z)\,\Phi_k(0,0)|0\rangle\right] = \left(z^{m+1}\partial_z + (m+1)z^m\partial_z\right)\left[\Psi_l(z)\,\Phi_k(0,0)|0\rangle\right]. \tag{1.35}$$

In the equation above one can commute L_m with $\Psi_l(z)$ in the left hand and remember that $L_m|0\rangle = 0$. The antiholomorphic part completely decouples and we are left with a set of recurrence relations for $\beta_{12}^{k,\bar{n}}$ which allow to determine them all for generic values of Δ_1, Δ_2 and Δ_k.

There is an important special case where the conformal blocks simplify significantly. If the primary field Φ_Δ defines a decomposable representation of Virasoro algebra, the conformal blocks satisfy the so-called BPZ equations, or singular vector decoupling equations. Since the Verma module of $|\Phi_\Delta\rangle$ is degenerate, there exists a subrepresentation which is a Verma submodule with the highest vector

$$\chi_\Delta = \sum_{\bar{n}\bar{m}} \chi^{\bar{n}\bar{m}} L_{\bar{n}}\,\overline{L}_{\bar{m}}\,|\Phi_\Delta\rangle. \tag{1.36}$$

Such a vector $|\chi_\Delta\rangle$ is called a singular vector in the Verma module of $|\Phi_\Delta\rangle$. Generating a subrepresentation, the singular vector is ortogonal to any vector from the factor representation. In particular, any correlation function with the primmary operator corresponding to the singular vector vanishes

$$\langle\chi_\Delta(z,\bar{z})\prod_i \Phi_i(z_i,\bar{z}_i)\rangle = 0 \tag{1.37}$$

Via (1.35) the Virasoro operators act as differential operators inside correlation functions, and we obtain a differential equation for any correlation function containing Φ_Δ:

$$\sum_{\bar{n}\bar{m}} \chi^{\bar{n}\bar{m}}\langle L_{\bar{n}}\,\overline{L}_{\bar{m}}\,\Phi_\Delta(z,\bar{z})\prod_i \Phi_i(z_i,\bar{z}_i)\rangle. \tag{1.38}$$

Let us return to the general case. Consider two-point correlation functions on a sphere. SL_2 invariance fixes them to be diagonal

$$\langle\Phi_i(z,\bar{z})\Phi_j(0,0)\rangle = \frac{1}{(z\bar{z})^{2\Delta_i}}, \tag{1.39}$$

where we normalized the fields so that the coefficients to be 1.

Similarly, the form of three-point functions is completely fixed by SL_2 invariance, since each three points can be transformed into any other three by a Moebius transform

$$\langle\Phi_i(z_1,\bar{z}_1)\Phi_j(z_2,\bar{z}_2)\Phi_k(z_3,\bar{z}_3)\rangle = \frac{\mathbb{C}_{ijk}}{|z_{12}|^{-2\Delta_3+2\Delta_1+2\Delta_2}|z_{23}|^{-2\Delta_1+2\Delta_2+2\Delta_3}|z_{31}|^{-2\Delta_2+2\Delta_3+2\Delta_1}}, \tag{1.40}$$

where $z_{ij} := z_i - z_j$ and $\mathbb{C}_{ijk} = \mathbb{C}_{ij}^k$ are the structure constants (1.30) (indices are lowered with the help of 2-point functions which are diagonal). This can be seen considering the correlation function at $0, 1, \infty$.

The first interesting correlation function thus is a 4-point correlation function. We will consider a function

$$\langle\Phi_1(z,\bar{z})\Phi_2(0,0)\Phi_3(1,1)\Phi_4(\infty,\infty)\rangle := \lim_{w\to\infty}\langle\Phi_1(z,\bar{z})\Phi_2(0,0)\Phi_3(1,1)\Phi_4(w,\bar{w})\rangle(w\bar{w})^{2\Delta_4}, \tag{1.41}$$

where the last scaling factor regularizes the correlator at infinity.

Using OPE of the fields $\Phi_1\Phi_2$ and $\Phi_3\Phi_3$ we get

$$\begin{aligned}&\langle\Phi_1(z,\bar{z})\Phi_2(0,0)\Phi_3(1,1)\Phi_4(\infty,\infty)\rangle = \\ &\qquad = \sum_k \mathbb{C}_{12}^k\mathbb{C}_{k34}\left|\sum_{\bar{n},\bar{m}} z^{\Delta_k+\sum_i n_i-\Delta_1-\Delta_2}\beta_{12}^{k,\bar{n}}\beta_{34}^{k,\bar{m}}\langle\Delta_k|L_{-\bar{n}}L_{\bar{m}}|\Delta_k\rangle\right|^2.\end{aligned} \tag{1.42}$$

The expression above factorizes into a sum of modulus square of holomorphic conformal blocks

$$\langle\Phi_1(z,\bar{z})\Phi_2(0,0)\Phi_3(1,1)\Phi_4(\infty,\infty)\rangle = \sum_k \mathbb{C}_{12}^k\mathbb{C}_{k34}\,|\mathcal{F}(\Delta_{1,2,3,4},\Delta_k;z)|^2, \tag{1.43}$$

where we defined spherical 4-point conformal blocks

$$\mathcal{F}(\Delta_{1,2,3,4},\Delta_k;z) := \sum_{\bar{n}\geq 0} z^{\Delta_k+\sum_i n_i-\Delta_1-\Delta_2}\left(\beta_{12}^{k,\bar{n}}\beta_{34}^{k,\bar{m}}\right)^{1/2}\langle\Delta_k|L_{-\bar{n}}L_{\bar{m}}|\Delta_k\rangle. \tag{1.44}$$

Conformal blocks depend only on conformal dimensions Δ_i of the fields in the OPE but not on the specific model. They are sections of "vector bundles" over the moduli space of 4-punctured spheres $\mathcal{M}_{0,4}$. In general this bundle is infinite-dimensional, because in general conformal blocks are well-defined for all generic Δ_k. However, in many interesting cases, like in Minimal Models which we shall discuss below, these bundles turn to be finite-dimensional and are isomorphic to bundles of solutions of certain ODE. These ODE are known as Belavin Polyakov Zamolodchikov equations [36] (1.38) in the case of Minimal Models and Knizhnik Zamolodchikov equations [40] in the WZW case.

Summary of CFT Conformal field theories can be defined by the following set of data:

1. A collection of Virasoro highest weight representations with highest weight vectors $\{|\Delta_i\rangle\}_{i\in I}$ such that

$$L_0|\Delta_i\rangle = \Delta_i\,|\Delta_i\rangle,\ L_n|\Delta_i\rangle = 0,\ n>0,\ \hat{c}|\Delta_i\rangle = c\,|\Delta_i\rangle. \tag{1.45}$$

   Such a collection is called a *spectrum* of primary fields of the theory (in particular, $\Delta_i + n$ for $n \in \mathbb{N}$ form a spectrum of L_0 acting in the Hilbert space of the theory.

2. A set of three-point correlation numbers or structure constants $\mathbb{C}_{ij}^k$ for $i, j, k \in I$.

Of course, this data should satisfy multiple consistency conditions. One of them is bootstrap equations (1.35), in particular the crossing relations (see e.g. [38] for a review of the bootsrap approach).

To name some other - modular bootstrap equations (consistency of the theory on a torus), in some cases unitarity (all conformal dimensions and central charge are positive).

Given such a data one can compute correlation numbers using the OPE relations. The correlation functions are computed as a combination of conformal blocks, which are holomorphic universal functions (sections of certain bundles over moduli spaces of punctured Riemann surfaces) with coefficients expressed through the structure constants of the theory.

1.1.1 Minimal Models

Minimal Models of conformal field theories [36] are particulary nice CFTs where all the matter representations are integrable Virasoro representations for central charge less then 1. There are finite number of Virasoro highest weight representations in each minimal model. Classification of integrable Virasoro representations was done in [39].

Such models can be considered as a matter for a noncritical string theory in dimension c less then 1, since in the case of non-supersymmetric sigma models the matter central charge is equal to the target space dimension.

Minimal Models are parametized by a pair of coprime positive integers $p < p'$ (or integral lines on a plane passing through the origin and different from coordinate axes). We will denote the corresponding model as $\mathcal{M}(p'/p)$.

The central charge of such a minimal model is given by

$$c_M = 1 - 6q^2, \tag{1.46}$$

where the parameter q is equal to

$$q = b^{-1} - b,\ b = \sqrt{p'/p}. \tag{1.47}$$

Letter M in c_M stands for matter (as opposed to gravity) and the convenience of the parametrization above will become clear later. The parameters b and b^{-1} are so-called momentum parameters of the screening charges in the Coulomb gas realization of the Minimal models.

For such central charges the set of integrable representations forms a Kac table on the plane, that is the primary fields $\Phi_{m,n}(x)$ are labeled by a pair of integers $1 \leq m \leq p'-1,\ \ 1 \leq n \leq p-1$. Their conformal dimensions $\Delta^M_{m,n}$ are conveniently parametrized by

$$\begin{aligned} &\lambda_{m,n} := \frac{mb^{-1} + nb}{2}, \\ &\alpha_{m,n} := \lambda_{m,-n} - \frac{q}{2}, \\ &\Delta^M_{m,n} := \alpha_{m,n}(\alpha_{m,n} - q) = \lambda^2_{m,-n} - \frac{q^2}{4}. \end{aligned} \tag{1.48}$$

In partucular, $q/2 = \lambda_{1,-1}$. All the conformal dimensions are rational, because b enters $\Delta^M_{m,n}$ only in powers $b^{\pm 2}$. There is also a $\mathbb{Z}/2$ symmetry in the Kac table. Namely

$$\lambda_{m,-n} = -\lambda_{p-m,n-p'} \implies \Delta^M_{m,n} = \Delta^M_{p-m,p'-n}. \tag{1.49}$$

As we mentioned, we consider the theories where all primaries have different conformal dimensions, therefore $\Phi_{m,n} = \Phi_{p-m,p'-n}$ and the theory has $(p'-1)(p-1)/2$ primary fields in total. These formulae have simple geometric interpretation in terms of the Kac table. Consider a line l passing through the origin and the point (p', p) in $\mathbb{R}^2$. Then $\Delta^M_{m,n}$ is a difference of oriented distance from a point (m, n) to l and from a point $(1, 1)$ to l.

Minimal models have the following fusion rules

$$\Phi_{m_1,n_1}\,\Phi_{m_2,n_2} = \sum_{r,s\in\Sigma} [\Phi_{r,s}], \tag{1.50}$$

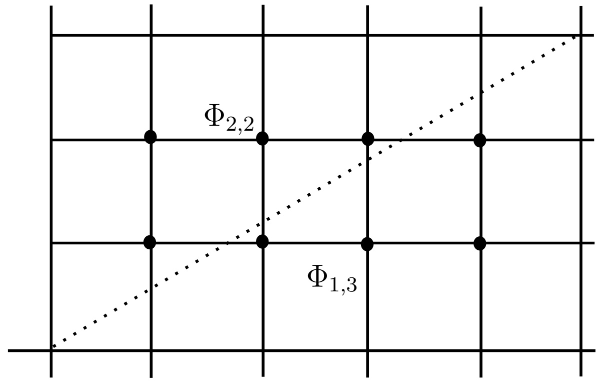

Figure 1.2: Kac table for $\mathcal{M}(3/5)$

where $[\Phi_{r,s}]$ denotes all possible contributions from the primary field $\Phi_{r,s}$ and its descendents and the summation over r, s goes with steps of 2 in the range

$$\begin{aligned} &|m_1 - m_2| + 1 \leq r \leq \min(m_1 + m_2 - 1, 2p - m_1 - m_2 - 1), \\ &|n_1 - n_2| + 1 \leq s \leq \min(n_1 + n_2 - 1, 2p' - n_1 - n_2 - 1). \end{aligned} \tag{1.51}$$

These fusion rules are equivalent to fusion rules of representations of algebra $\hat{sl}(2)_p \oplus \hat{sl}(2)_p$ with a $\mathbb{Z}/2\mathbb{Z}$ identification $(m, n) \to (p - m, p' - n)$. When the fusion rules are not satisfied, the corresponding structure constants of the OPE (1.50) vanish.

The structure constants of the theory were computed by Dotsenko and Fateev in [41] using the so-called Coulomb gas representation. We, however, will use an analytic continuation of the Dotsenko Fateev formula which was computed in [42].

$$\Phi_{m_1,n_1}(z, \bar{z})\, \Phi_{m_2,n_2}(0,0) = \sum_{r,s \in \Sigma} \mathbb{C}^{(r,s)}_{(m_1,n_1),(m_2,n_2)} (z\bar{z})^{\Delta^M_{r,s} - \Delta^M_{m_1,n_1} - \Delta^M_{m_2,n_2}} \Phi_{r,s}(0,0) + desc., \tag{1.52}$$

where desc. means contributions from descendents of the primary field and $\mathbb{C}^{(r,s)}_{(m_1,n_1),(m_2,n_2)} := \mathbb{C}^M(\alpha_{m_1,n_1}, \alpha_{m_2,n_2}, \alpha_{m_3,n_3})$,

$$\mathbb{C}^M(\alpha_1, \alpha_2, \alpha_3) = A \Upsilon(\alpha + b - q) \prod_i \frac{\Upsilon(\alpha - 2\alpha_i + b)}{[\Upsilon(2\alpha_i + b)\Upsilon(2\alpha_i + b - q)]^{1/2}}\,, \tag{1.53}$$

where $\alpha = \sum \alpha_i$ and the normalization factor

$$A = \frac{b^{b^{-2} - b^2 - 1} [\gamma(b^2)\gamma(b^{-2} - 1)]^{1/2}}{\Upsilon(b)}\,. \tag{1.54}$$

Here $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$ and special function $\Upsilon(x) = \Upsilon_b(x)$ is an entire function of complex domain with zeros in $x = -nb^{-1} - mb$ and $(n + 1)b^{-1} + (m + 1)b$, where

n, m are non-negative integers (see for example [11]). Upsilon function is subject to the following shift relations

$$\frac{\Upsilon(x+b)}{\Upsilon(x)} = b^{1-2bx}\gamma(bx), \quad \frac{\Upsilon(x+b)}{\Upsilon(x)} = 1/b^{1-2x/b}\gamma(x/b). \tag{1.55}$$

It also has an integral representation

$$\log \Upsilon(x) = \int_0^\infty \frac{\mathrm{d}t}{t}\left[(Q/2-x)^2 e^{-2t} - \frac{\sinh^2(Q/2-x)t}{\sinh(bt)\sinh(t/b)}\right], \tag{1.56}$$

which is valid for $0 < \mathrm{Re}(x) < \mathrm{Re}(Q)$ or by a product formula valid for all complex x:

$$\Upsilon_b(x) = \lambda_b^{(Q/2-x)^2} \prod_{n,m=0}^{\infty} f\left(\frac{Q/2-x}{Q/2+mb-nb^{-1}}\right), \quad f(x) = (1-x^2)e^{x^2}, \tag{1.57}$$

where λ_b is a constant.

The formula (1.53) defines a meromorphic function of $\alpha_{1,2,3}$. Perhaps, the most important data one can extract from this expression are zeros and poles, which can occur, for example, when the fields are degenerate, that is $\alpha_i = \alpha_{m,n}$. In particular, some of the fusion rules arise because of zeros of structure constants corresponding to the degenerate primary fields. The important point, however, is that not all fusion rules are satisfied automatically by zeros of the funtion (1.53). The difference between analytical vanishing of the structure constants and actual Minimal Model fusion rules will play a role in our discussion.

We will be interested in four point correlation numbers. According to the fusion rules they decompose into a finite sum of conformal blocks

$$\begin{aligned}\langle \Phi_{m_1,n_1}(x)\Phi_{m_2,n_2}(0)\Phi_{m_3,n_3}(1)\Phi_{m_4,n_4}(\infty)\rangle = \\ = \sum_{r,s} \mathbb{C}^{M,(r,s)}_{(m_1,n_1),(m_2,n_2)} \mathbb{C}^{M}_{(m_1,n_1),(m_2,n_2),(r,s)} |\mathcal{F}(\Delta^M_{m_i,n_i}; \Delta^M_{r,s}|x)|^2,\end{aligned} \tag{1.58}$$

where the sum goes over such r, s that $\Phi_{r,s}$ appears in OPE of $\Phi_{m_1,n_1}, \Phi_{m_2,n_2}$ and in OPE of $\Phi_{m_3,n_3}, \Phi_{m_4,n_4}$.

Generalized Minimal Models It is tempting to add primary fields with arbitrary conformal dimensions to Minimal Models using the structure constants (1.53). As we will see, this is very not straightforward to define such a conformal theory. Namely, the question is how to define a spectrum (which is a space of all allowed primary fields) and the fusion rules. Some insight on this very interesting problem can be found in [43].

We take a simplified approach following the older work [42]. That is we consider a theory with arbitrary central charge $c^M = 1 - 6q^2$ less then one, where b is not necessary $\sqrt{p/p'}$ and primary fields Φ_α with dimensions $\Delta^M_\alpha = \alpha(\alpha - q)$. Then for each pair (m, n) of positive integers there is a degenerate primary field $\Phi_{m,n}$ with conformal dimension $\Delta^M_{m,n} = \alpha_{m,n}(\alpha_{m,n} - q)$ where $\alpha_{m,n}$ is given by the formula (1.48). We can define spherical three-point correlators of generic fields by the formula (1.53).

$$\langle \Phi_{\alpha_1}(0)\Phi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle := \mathbb{C}^M(\alpha_1, \alpha_2, \alpha_3), \tag{1.59}$$

where the correlation function at infinity is regularized in a similar way as in (1.41).

It is also possible to define four-point correlation numbers where one of the primary fields is degenerate using the degenerate vector decoupling condition (1.38):

$$\begin{aligned}&\langle \Phi_{m,n}(x)\Phi_{\alpha_1}(0)\Phi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle := \\ &= \sum_{\substack{-m+1:r:m-1 \\ -n+1:s:n-1}} \mathbb{C}^M(\alpha_{m,n}, \alpha_1, \alpha_1 + \lambda_{r,s})\mathbb{C}^M(\alpha_1 + \lambda_{r,s}, \alpha_2, \alpha_3)|\mathcal{F}(\Delta^M_{(m,n),1,2,3}; \Delta^M_{\alpha+\lambda_{r,s}}|x)|^2.\end{aligned} \tag{1.60}$$

This definition is useful because it is simpler to evaluate correlation functions as analytic functions of parameters $\alpha_{1,2,3}$ and obtain the actual minimal model correlation functions from the limits $\alpha_i \to \alpha_{m_i,n_i}$. The naive expectation is that the limits of correlation functions in the Generalized Minimal Model coincide with correlation functions of the ordinary Minimal Model (1.58). It is, however, not always true due to the fact that some of the Minimal Model fusion rules do not follow from analytical structure constant (1.53). That is, consider an example where one of the fields becomes degenerate $\alpha_1 \to \alpha_{m_1,n_1}$. In the sum over r, s in the formula (1.60) some of the terms which should vanish in the Minimal Model due to the fusion rules of $\Phi_{m,n}(x)\Phi_{m_1,n_1}(0)$ do not vanish because not all of the fusion rules are contained in zeros of the analytical structure constants $\mathbb{C}^M(\alpha_{m,n}, \alpha_{m_1,n_1}, \alpha_{m_1,n_1} + \lambda_{r,s})$.

Even subtler phenomenon occurs in the theory of gravity, which we will discuss later.

1.1.2 Liouville Field Theory

Liouville Field Theory (LFT) appears in quantization of two-dimensional gravity due to conformal anomaly [6]. It was put in the framework of CFT by [9, 10]. It is closely related with Minimal Models but is more complicated. In particular, the spectrum and OPE of LFT are continuous. This leads to many analytical effects in a similar fashion to effects one encounters in functional analysis compared to linear algebra. (Semi)classically LFT is defined by the following action

$$S = \int_\Sigma \mathrm{d}^2x\sqrt{\hat{g}}\left[\frac{1}{4\pi}(\partial\phi(x))^2 + \frac{Q}{4\pi}\hat{R}\phi + \mu e^{2b\phi(x)}\right], \tag{1.61}$$

where $\hat{g}$ and $\hat{R}$ are background metric and scalar curvature on a Riemann surface Σ, $Q = b^{-1} + b$, b being a dimensionless parameter of the theory and $\phi(x)$ is a Liouville field, such that the "quantum metric" on Σ is given by $g = e^{2b\phi(x)}\hat{g}$. The parameter μ is called a cosmological constant and is a scale parameter of the theory.

On a sphere in an appropriate background the Lagrangian reduces to

$$L = \frac{1}{4\pi}(\partial\phi)^2 + \mu e^{2b\phi} \tag{1.62}$$

apart from a boundary term at infinity. Liouville field, being a conformal factor of the metric, has the following transformation law

$$\phi(w, \bar{w}) = \phi(z, \bar{z}) - \frac{Q}{2}\log\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|. \tag{1.63}$$

The theory is conformal [10], the holomorphic stress-energy tensor is computed to be

$$T(z) = -(\partial\phi)^2 + Q\partial^2\phi. \tag{1.64}$$

On a sphere it corresponds to a free boson theory with an additional charge $-Q$ at infinity.

It implies the formula for the central charge

$$c^L = 1 + 6Q^2. \tag{1.65}$$

The primary operators can be represented by exponents of the Liouville field:

$$V_a(x) := e^{2a\phi(x)}. \tag{1.66}$$

Their conformal dimensions are

$$\Delta_a^L := \Delta(V_a(x)) = a(Q - a). \tag{1.67}$$

In particular, $\Delta_a^L = \Delta_{Q-a}^L$, so the corresponding primary fields should coincide up to a numerical factor. In LFT this factor is called a reflection amplitude and it is convenient to not normalize it to 1.

The Liouville field itself can be obtained as a derivative of the exponential operator with respect to the parameter

$$\phi(x) = \frac{1}{2}\frac{\partial}{\partial a}V_a(x)|_{a=0} = \frac{1}{2}V_a'(x)|_{a=0}. \tag{1.68}$$

The degenerate primary fields $V_{m,n}(x) = V_{a_{m,n}}(x)$ have conformal dimensions

$$\Delta_{m,n}^L = a_{m,n}(Q - a_{m,n}), \quad a_{m,n} = \frac{Q}{2} - \lambda_{m,n}. \tag{1.69}$$

For such a field there is a singular vector of the corresponding Verma module on the mn'th level. As opposed to Minimal Models, there are no other singular vectors. One can notice, that the formulae for the central charge and for the conformal dimensions in Liouville theory can be obtained from the ones of GMM by $b \to ib$. However these two theories are not analytic continuations of each other, because they have different spectra and structure constants of these theories can not be obtained as analytic continuations of each other, in particular $\Upsilon_b(x)$ function have a natural bound of analyticity with respect to parameter b [44, 43].

As opposed to Minimal Models, spectrum of LFT does not include degenerate fields. It is continuous and consists of the fields with $a = Q/2 + iP$ for real P.

$$\mathcal{H} := \int \mathrm{d}P[V_{Q/2+iP}], \tag{1.70}$$

where $[V_{Q/2+iP}]$ denotes the Verma module of the primary field $V_{Q/2+iP}$. Dimension of the field $V_{Q/2\pm iP}$ is equal to $Q^2/4 + P^2$, in particular, dimensions of the fields in the spectrum are bounded below by $Q^2/4$.

The OPE of the Liouville theory is continuous as opposed to Minimal Models. It means that LFT is not a *rational* CFT.

$$V_{a_1}(z,\bar{z})V_{a_2}(0,0) = \int' \frac{\mathrm{d}P}{4\pi}(z\bar{z})^{\Delta_{Q/2+iP}-\Delta_{a_1}-\Delta_{a_2}}\mathbb{C}^{Q/2+iP}_{a_1,a_2}[V_{Q/2+iP}(0)], \tag{1.71}$$

where the basic structure constants $\mathbb{C}^{Q/2+iP}_{a_1 a_2} = \mathbb{C}^L(a_1, a_2, Q/2 - iP)$ [9, 10] (derived from the crossing symmetry in [45]) have the explicit form (here a denotes $a_1 + a_2 + a_3$)

$$\mathbb{C}^L(a_1,a_2,a_3) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{(Q-a)/b} \frac{\Upsilon_b(b)}{\Upsilon_b(a-Q)} \prod_{i=1}^{3} \frac{\Upsilon_b(2a_i)}{\Upsilon_b(a-2a_i)}\ , \tag{1.72}$$

where Υ_b is the same "upsilon" function as the one, which appears in the expression for GMM structure constants (see [46, 10]).

The OPE (1.71) is continuous and involves integration over the "momentum" P. The prime on the integral indicates possible discrete terms, In our computations such extra terms do appear and give an important contribution.

The four-point function of Liouville fields is defined using OPE (1.71)

$$\begin{aligned}\langle V_{a_1}(x)V_{a_2}(0)V_{a_3}(1)V_{a_4}(\infty)\rangle &= \\ &= \int' \frac{\mathrm{d}P}{4\pi}\mathbb{C}^L(a_1,a_2,Q/2+iP)\mathbb{C}^L(a_1,a_2,Q/2-iP)|\mathcal{F}(\Delta_i;\Delta_{Q/2+iP}|x)|^2. \end{aligned} \tag{1.73}$$

The prime at the integral has the same meaning as in the OPE and we are going to discuss it in the end of this section.

Liouville OPE discrete terms It turns out, that the fields of interest in Liouville theory usually do not belong to the spectrum of the theory, so one cannot use OPE literally as in minimal models. However, correlation numbers are real analytic functions (with singularities) of conformal dimensions and it is possible to continue them to compute correlators of the fields which do not belong to the spectrum. ? Thus, one has to use OPE (1.71) carefully if the fields V_{a_i} are not in the spectrum. For example, if one computes 3-point function with the naive OPE, one often gets zero, which is inconsistent with DOZZ formula (1.72) and the 4-point function is inconsistent with conformal bootstrap.

Four-point correlation function (1.73) involves integration in the Liouville momentum parameter P. The integrand is a product of LFT structure constants (1.72)

$$\mathbb{C}^L(a_1, a_2, p)\mathbb{C}^L(Q-p, a_3, a_4), \quad p = iP \tag{1.74}$$

and the conformal blocks. In the case, where $\mathrm{Re}(|Q/2-a_i|)+\mathrm{Re}(|Q/2-a_j|) < Q/2$ for $i \neq j$ the contour of integration goes along the real axis. This corresponds to the fact that in this case the correlator is a sum over intermediate states in the Hilbert space of Liouville theory. When this condition is not satisfied, meromorphic continuation of the correlation functions is required. It can be achieved by deforming the integration contour (see, e.g. [11, 47]). Basically, in this case poles of structure constants intersect the real line and one needs to add corresponding residues to the total integral, as depicted in figure 1.3. These residues are called discrete terms. If, for example,

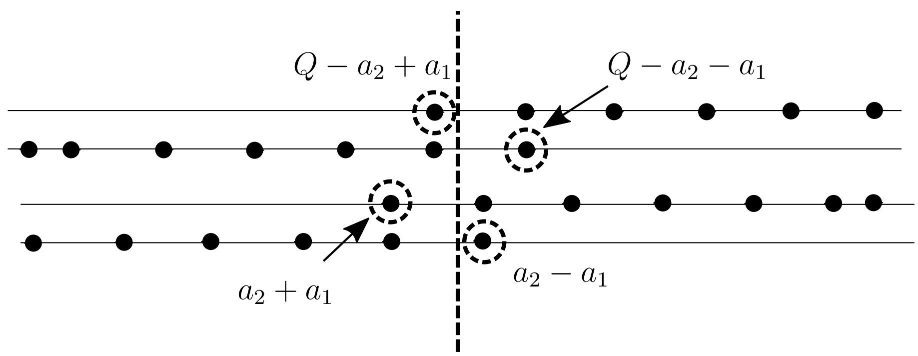

Figure 1.3: Poles of structure the constant and discrete terms.

$Q/2 - a_i > 0$ then the corresponding poles come from zeros in

$$\Upsilon(a_i + a_j - p) = \Upsilon(p + (Q/2 - a_i + Q/2 - a_j)),$$
$$\Upsilon(a_i + a_j + p - Q) = \Upsilon(p - (Q/2 - a_i + Q/2 - a_j))\ .$$

In this case one can easily see that the corresponding residues are to be taken at

$$p = (Q/2 - a_i + Q/2 - a_j) - r/b - sb,\ p > Q/2$$

and in the reflected positions $Q - p$ with the same residues.

We note that expression (1.74) in principle may have a second order pole if both of the structure constants have poles for the momentum p. In what follows we assume that this is not the case, then the residues are computed easily using quasiperiodicity of Υ-function and the fact that $\Upsilon(\varepsilon) = \Upsilon(b)\varepsilon + O(\varepsilon^2)$.

1.2 Liouville Gravity

In this section we discuss the Minimal Liouville Gravity correlation numbers on a sphere.

Minimal Liouville Gravity (MLG) is a theory of two-dimensional quantum gravity where matter is represented by a Minimal Model. It can also be considered as noncritical string theory in dimension $c^M < 1$. To gauge diffeomorphisms of two-dimensional system one introduces a $b - c$ ghost system (see, e.g., [48, 49, 50]), consisting of two anticommuting fields (b, c) of spins $(2, -1)$, is the conformal field theory with central charge $c^{\text{gh}} = -26$.

When taking path integral due to conformal anomaly the metric field does not decouple completely and the remaining degree of freedom is described by Liouville Field Theory of central charge $-c^M - c^{gh} = 26 - c^M$. In particular, if $c^M = 1 - 6q^2 = 1 - 6(b^{-1} - b)^2$, then $c^L = 1 + 6Q^2 = 1 + 6(b^{-1} + b)^2$.

In the framework of the so-called DDK approach [51, 8], Liouville Gravity is a tensor product of the conformal matter (M), represented by ordinary or generalized Minimal Models, Liouville theory (L), and the ghost system (gh).

$$A_{\text{LG}} = A_{\text{M}} + A_{\text{L}} + A_{\text{gh}} \ ,$$

with the "interaction" between them via the construction of the physical fields and the conformal anomaly cancellation condition

$$c^{\text{M}} + c^{\text{L}} + c^{\text{gh}} = 0 \ . \tag{1.75}$$

Physical Fields and Correlation Numbers. As in all gauge theories in the BRST formulartion the physical fields form a space of cohomology classes with respect to the nilpotent BRST charges $Q_{BRST}, \overline{Q}_{BRST}$,

$$Q_{BRST} = \sum_m : \left[L_m^{\text{M+L}} + \frac{1}{2} L_m^{\text{g}} \right] c_{-m} : - c_0 \ , \tag{1.76}$$

where L_m^{M+L} denotes a sum of Virasoro algebra elements of matter and Liouville CFTs. The BRST cohomologies decompose into sectors with fixed ghost numbers. Each of the physical fields is generally covariant, because the diffeomorphism group is gauged.

In the sector with the ghost number $(1,1)$ physical fields are

$$\mathbb{W}_{m,n}(z,\bar{z}) = C\bar{C}\cdot\Phi_{m,n}(z,\bar{z})V_{m,-n}(z,\bar{z}), \tag{1.77}$$

where

$$V_{m,-n} := V_{a_{m,-n}}, \quad a_{m,-n} = Q/2 + \lambda_{m,-n} \tag{1.78}$$

and Minimal Model and Liouville fields are chosen in a way that $\Delta^M_{m,n} + \Delta^L_{m,-n} = 1$, i.e. the field $\mathbb{W}_a$ has total conformal dimension $(0,0)$ and transforms as a scalar. Here we note, that the fields $V_{m,-n}$:

1. do not belong to the spectrum of the LFT, so their correlation function require analytic continuation as discussed before.

2. are not degenerate, therefore their OPE is not subject to the singular vector decoupling equations (1.38) and are continuous.

The $(1,1)$-forms $\mathbb{U}_{m,n}$ are closely related to $\mathbb{W}_{m,n}$ and can be integrated over the worldsheet Riemann surface:

$$\mathbb{U}_{m,n}(z,\bar{z}) = \Phi_{m,n}(z,\bar{z})V_{m,-n}(z,\bar{z}) = B_{-1}\overline{B}_{-1}\cdot\mathbb{W}_{m,n}. \tag{1.79}$$

The fields $\mathbb{U}_{m,n}$ are not BRST invariant themselves, but their integrals are.

The n-point correlation number on a sphere for these observables [47] is

$$I\left((m_1,n_1),\ldots,(m_k,n_k)\right) := \\ = \int\prod_{i=4}^{k} d^2z_i\left\langle\prod_{i=4}^{k}\mathbb{U}_{m_i,n_i}(z_i)\mathbb{W}_{m_3,n_3}(z_3)\,\mathbb{W}_{m_2,n_2}(z_2)\,\mathbb{W}_{m_1,n_1}(z_1)\right\rangle, \tag{1.80}$$

where angular brackets denote correlation function in all three CFTs involved and we put 3 fields with ghost numbers $(1,1)$ to fix SL_2 symmetry on a sphere and integrated over all other insertions.

In principle, one could replace Minimal Models with Generalized Minimal Models and consider the BRST closed fields

$$\mathbb{W}_a(z,\bar{z}) = C\,\bar{C}\cdot\Phi_{a-b}(z,\bar{z})V_a(z,\bar{z}), \tag{1.81}$$

where the parameter a can take generic values with the usual caution that one needs to check that the corresponding correlation function is well-defined.

The ghost number zero sector consists of the so-called ground ring states [52, 53, 47]

$$\mathbb{O}_{m,n}(z,\bar{z}) = \bar{H}_{m,n}H_{m,n}\Phi_{m,n}(z,\bar{z})V_{m,n}(z,\bar{z})\ . \tag{1.82}$$

The operators $H_{m,n}$ are composed of Virasoro generators in all three theories and are defined uniquely modulo $\mathbb{Q}$ exact terms.

It turns out that the ground ring states $\mathbb{O}_{m,n}$ and ghost number $(1,1)$ states $\mathbb{W}_{m,n}$ are related. As customary in cohomology, the nontrivial de Rham cohomology elements quite often can be represented as differentials of certain singular differential forms. Similarly, in MLG the fields $\mathbb{W}_{m,n}$ can be represented as Q_{BRST}-images of certain singular operators connected with $\mathbb{O}_{m,n}$. This relation goes under the name of Higher Equations of Motion of LFT.

Consider logarithmic counterparts of the ground ring states $\mathbb{O}_{m,n}$,

$$\mathbb{O}'_{m,n} := \bar{H}_{m,n} H_{m,n} \Phi_{m,n}(V'_a)|_{a=a_{m,n}} \ .$$

Higher Equations of Motion read [54], [47]

$$\mathbb{W}_{m,n} = B^{-1}_{m,n} Q_{BRST} \overline{Q}_{BRST} \mathbb{O}'_{m,n}, \tag{1.83}$$

or using the commutation relations

$$\mathbb{U}_{m,n} = B^{-1}_{m,n} \bar{\partial}\partial \mathbb{O}'_{m,n} \mod \mathbb{Q}\ , \tag{1.84}$$

where $B_{m,n}$ are numerical coefficients arising in the higher equations of motion of LFT [54]. Consider the four-point correlation function

$$I(a_{m,-n}, a_2, a_3, a_4) = \int d^2 z \Big\langle \mathbb{U}_{m,n}(z) \mathbb{W}_{a_2}(0)\, \mathbb{W}_{a_3}(1)\, \mathbb{W}_{a_4}(\infty) \Big\rangle \tag{1.85}$$

with generic momenta parameters a_2, a_3 and a_4. Relation (1.84) applied to $\mathbb{U}_{m,n}(z)$ allows to reduce the moduli integral in (1.85) to the boundary integrals and a curvature term. This was done in [55] with the following result

$$I(a_{m,-n}, a_2, a_3, a_4) = \kappa N(a_{m,-n}) \left(\prod_{i=2}^{4} N(a_i) \right) \Sigma^{(m,n)}(a_2, a_3, a_4)\ , \tag{1.86}$$

where

$$\Sigma^{(m,n)}(a) = -mn\lambda_{m,n} + \sum_{i=1}^{3} \sum_{r,s}^{(m,n)} |\lambda_i - \lambda_{r,s}|_{\mathrm{Re}}\ , \tag{1.87}$$

$\lambda_i = Q/2 - a_i$ are the "momentum parameters" and the fusion set is governed by OPE with $\Phi_{m,n}\ :\ (r,s) \in \{1-m : 2 : m-1, 1-n : 2 : n-1\}$. The prefactor κ in (1.86) is

$$\kappa = -(b^{-2}+1)b^{-3}(b^{-2}-1)Z_L,\ Z_L = \left[\pi\mu\gamma(b^2)\right]^{Q/b} \frac{1-b^2}{\pi^3 Q \gamma(b^2)\gamma(b^{-2})} \tag{1.88}$$

and the "leg" factors are

$$N(a) = \frac{\pi}{(\pi\mu)^{(a/b)}} \left[\frac{\gamma(2ab - b^2)\gamma(2ab^{-1} - b^{-2})}{\gamma^{2a/b-1}(b^2)\gamma(2 - b^{-2})} \right]^{1/2} .$$

The expression (1.86) was derived under the assumption that the number of conformal blocks in the expansion of the matter sector correlation function is maximally possible, i.e. the number of conformal blocks $= mn$. We discuss this point in more details in the sections 1.3, 1.3.1.

In what follows, we focus on the four-point correlators in the Lee-Yang series of Minimal Models, that is the series $M(2/p)$ for arbitrary odd number $p > 3$. In the Lee-Yang series $b = \sqrt{2/p}$ and $a_i = a_{1,-n_i}$. We also denote

$$\mathcal{I}_4(n_i) := \int\limits_{\mathcal{M}_{0,3}} d^2z \langle \mathbb{U}_{1,n_1}(z) \mathbb{W}_{1,n_2}(0)\, \mathbb{W}_{1,n_3}(1)\, \mathbb{W}_{1,n_4}(\infty) \rangle. \tag{1.89}$$

Taking into account the explicit form of the correlation functions in the ghost sector

$$\langle C(0)C(1)C(\infty) \rangle = 1 ,$$

we obtain

$$\begin{aligned} \mathcal{I}_4(n_i) = \int\limits_{\mathcal{M}_{0,3}} & d^2z \langle \Phi_{1,n_1}(z)\Phi_{1,n_2}(0)\, \Phi_{1,n_3}(1)\, \Phi_{1,n_4}(\infty) \rangle \times \\ & \times \langle V_{1,-n_1}(z) V_{1,-n_2}(0)\, V_{1,-n_3}(1)\, V_{1,-n_4}(\infty) \rangle . \end{aligned} \tag{1.90}$$

For further purposes this expression can be conveniently written in more explicit form, for details, see Appendix 1.A.

1.3 Four point numbers

We call the formula (1.85) the HEM formula for correlation numbers. In the original paper [55] it was derived assuming that a_2, a_3, a_4 are generic, in particular they correspond to nondegenerate matter fields. It was assumed that the formula is correct in the more general case when all a_i are degenerate but the number of conformal blocks in the matter sector is maximal, that is equal to mn.

Let us make a comment on what do we mean by number of conformal blocks. Let $a_i = a_{m_i,n_i}$. Then the Minimal Models correlation function

$$\langle \Phi_{m,n}(z)\Phi_{m_2,n_2}(0)\Phi_{m_3,n_3}(0)\Phi_{m_4,n_4}(\infty) \rangle \tag{1.91}$$

is computed using the formula (1.58). In the formula the summand with $|\mathcal{F}(\Delta_i; \Delta_{r,s}|z)|^2$ appears with nonzero coefficient if and only if $\Phi_{r,s}$ appears in the OPE of $\Phi_{m,n}\Phi_{m_2,n_2}$

and in OPE of $\Phi_{m_3,n_3}\Phi_{m_4,n_4}$. From the general form of OPE with degenerate field (or from the decoupling condition (1.38)) it follows that the number of conformal blocks is not greater then $\min(mn, m_i n_i)$.

As an example consider a Lee-Yang model $M(2/p)$, where $p > 7$. Then the correlator

$$\langle \Phi_{1,2}(z)\Phi_{1,2}(0)\Phi_{1,2}(0)\Phi_{1,4}(\infty)\rangle \tag{1.92}$$

has only one conformal block in the decomposition. Indeed, $\Phi_{1,2}\Phi_{1,2} = [\Phi_{1,1}] + [\Phi_{1,3}]$ whereas $\Phi_{1,2}\Phi_{1,4} = [\Phi_{1,3}] + [\Phi_{1,5}]$. Therefore the only conformal block appears in the intermediate channel $[\Phi_{1,3}]$ as shown on the picture 1.3. The fields on top of the central line come from OPE of the fields on the left, the fields below the central line appear in the OPE of the fields on the right. The only nontrivial contributions to the correlation function come from the intermediate fields which appear both on top and bottom of the line.

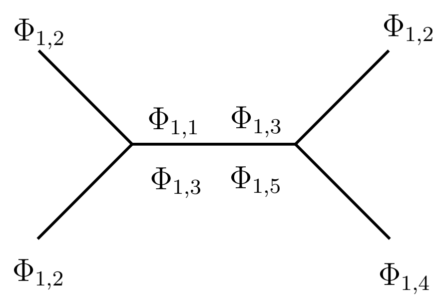

Figure 1.4: Conformal block diagram for $\langle \Phi_{1,2}\Phi_{1,2}\Phi_{1,2}\Phi_{1,4}\rangle$

As we find out, the HEM formula holds true in more general cases. Its failure is connected with appearance of particular discrete terms in the LFT correlation function. For example, the HEM formula for the correlator (1.92) holds true in $M(2/9)$ but fails in $M(2/11)$ and higher.

Most of our discussion holds true for the general Minimal Model case, so we will keep the notations general. Our modification of the HEM formula for the Lee-Yang series reads:

$$\Sigma_{MHEM} = \Sigma_{HEM} - \sum_{i=2}^{4} \sum_{(r,s)\in F_i \cap R_i} 2\lambda_{r,s}\ , \tag{1.93}$$

where Σ_{HEM} is given by (1.87), and F_i is the fusion set of $\Phi_{m,n}\Phi_{m_i,n_i}$ ($\Phi_{m,n}\Phi_{m_i,n_i} \to \Phi_{r,s}$) and R_i is the set of discrete terms in the OPE of $V_{m_j,-n_j}V_{m_k,-n_k}$ ($V_{m_j,n_j}V_{m_k,-n_k} \to V_{r,-s}$) and $\{i,j,k\} = \{2,3,4\}$.

To compute the correlator of degenerate fields we start with the correlator of one degenerate and three generic nondegenerate fields $\mathbb{W}_{a_i}$ and carefully study the limit $a_i \to a_{m_i,-n_i}$. Then the HEM formula reads:

$$\langle \mathrm{U}_{a_{m_1,n_1}} \mathbb{W}_{a_2} \mathbb{W}_{a_3} \mathbb{W}_{a_4} \rangle = 2mn\lambda_{m,n} + \sum_{i=2}^{4} \sum_{r,s}^{(m,n)} \left(|\lambda_i - \lambda_{r,s}|_{\mathrm{Re}} - \lambda_{m,n} \right) . \tag{1.94}$$

Instead of applying higher equations of motion formula (1.84) we will consider more direct upproach to study this limit. In the Appendix 1.A we show that conformal block decomposition for the integral (1.90) gives the formula for the correlator (1.94):

$$\begin{aligned} \mathcal{I}_4(a_{m_1,n_1}, a_i) = 2 \sum_k \mathbb{C}_{12}^k \mathbb{C}_{k34} \\ \pi^2 \int' \frac{dP}{4\pi} \mathbb{C}^L(a_{m_1,n_1}, a_2, Q/2 - iP) \mathbb{C}^L(Q/2 + iP, a_3, a_4) \\ \sum_l \sum_j \Big(b_j(P) b_{l-j}(P) \Phi(\Delta^L(P) + \Delta_k^M - 1, j, l-j) \Big) + \circlearrowright , \end{aligned} \tag{1.95}$$

where $\circlearrowright$ denotes two similar summands differing by cyclic permutations of a_2, a_3, a_4 (see Appendix 1.A for details). Let us focus on one of these terms. In (1.95) some Minimal Model structure constants $\mathbb{C}_{12}^k$ and $\mathbb{C}_{k34}$ become zero in the desired limit. Moreover, as is not difficult to see from the zeros of Υ function in Lee-Yang series structure constants in (1.95) always have zeros when the fusion rules are not satisfied. [3]. Let us denote the corresponding terms in (1.95) as

$$\langle \mathrm{U}_{a_{m_1,n_1}} \mathbb{W}_{a_{m_2,n_2}} \mathbb{W}_{a_{m_3,n_3}} \mathbb{W}_{a_{m_4,n_4}} \rangle^k + \circlearrowright . \tag{1.96}$$

In the limit these terms do not necessary vanish, because zeros of structure constants can get cancelled by poles on from conformal blocks. From another side, when the matter in Minimal Models correlation numbers, these terms do not appear in the expression because of the fusion rules.

Let us study when these terms do not vanish automatically in the limit, so that to get an answer for Minimal Models we take the limit of (1.94) and then subtract the

[3]There are some complications when these structure constants do not vanish even if they should do according to the fusion rules [44], but this is not the case for Lee-Yang series and is not discussed here.

terms (1.96)

$$\langle \mathrm{U}_{a_{m_1,n_1}} \mathbb{W}_{a_{m_2,n_2}} \mathbb{W}_{a_{m_3,n_3}} \mathbb{W}_{a_{m_4,n_4}} \rangle = \lim_{a_i \to a_{m_i,n_i}} \left[\langle \mathrm{U}_{a_{m_1,n_1}} \mathbb{W}_{a_2} \mathbb{W}_{a_3} \mathbb{W}_{a_4} \rangle \right] - \left(\sum_k \langle \mathrm{U}_{a_{m_1,n_1}} \mathbb{W}_{a_{m_2,n_2}} \mathbb{W}_{a_{m_3,n_3}} \mathbb{W}_{a_{m_4,n_4}} \rangle^k + \circlearrowleft \right) . \tag{1.97}$$

Let us compute the contribution of (1.96). Some of these terms do not vanish because $\Phi(\Delta^L(P) + \Delta_k^M - 1, j, l - j)$, arising from the x-integration, has a pole and annihilates zero appearing in the structure constant (1.53). Explicitly one has (see (1.188))

$$\Phi(A, r, l) = \frac{(16)^{2A}}{\pi(2A + r + l)} \int_{-1/2}^{1/2} \cos(\pi(r - l)x) e^{-\pi\sqrt{1-x^2}(2A+r+l)} dx ,$$

so that it has a pole when $2A + r + l = 0$ and $r - l$ is odd or zero. In our case it implies $r = l = 0$ and $A = 0$, which leads us to the conclusion that in the intermediate channel the Liouville dimension $\Delta^L(P)$ should be dressing for the matter dimension Δ_k^M in the sense that $\Delta^L(P) + \Delta_k^M = 1$. This can be possible only if Liouville correlation function has specific discrete terms, i.e. $iP \to \lambda_{m,-n}$. The first thing to notice is that nonzero terms (1.96) appear precisely if $k = (r, s) \in F_i \cap R_i$ as in the formula (1.93).

Let us compute the value of each of these terms. We have

$$\lim_{a_i \to a_{m_i,n_i}} \langle \mathrm{U}_{a_{m_1,n_1}} \mathbb{W}_{a_2} \mathbb{W}_{a_3} \mathbb{W}_{a_4} \rangle_F^k = \mathbb{C}^{G,p}_{a_{m_1,n_1}, a_{m_2,n_2}} \mathbb{C}^G_{p, a_{m_3,n_3}, a_{m_4,n_4}} = \mathbb{C}^G_{a_{m_1,n_1}, a_{m_2,n_2}, p} (D^G_{p,p})^{-1} \mathbb{C}^G_{p, a_{m_3,n_3}, a_{m_4,n_4}} = 2\lambda_k \kappa \prod_{j=1}^4 N(a_{m_i,n_i}) , \tag{1.98}$$

where D^G and $\mathbb{C}^G$ denote MLG two- and three-point functions, κ is given in (1.88) and λ_k is $Q/2 - a_k$. Taking (1.97), (1.98) into account one derives modified HEM formula (1.93).

Let us now accurately prove (1.98). We start from the formula (1.95). Taking residue in the discrete terms and using reflection relation in LFT, $\mathbb{C}^L(a_{m_1,n_1}, a_2, p_k) R_L(p_k)^{-1} = \mathbb{C}^L(a_{m_1,n_1}, a_2, Q - p_k)$, we have

$$\langle \mathrm{U}_{a_{m_1,n_1}} \mathbb{W}_{a_2} \mathbb{W}_{a_3} \mathbb{W}_{a_4} \rangle_F^k = 2\, \mathbb{C}^M(\alpha_{m_1,n_1}, \alpha_2, \alpha_k) \mathbb{C}^M(\alpha_k, \alpha_3, \alpha_4) \\ \pi^2 \mathbb{C}^L(a_{m_1,n_1}, a_2, p_k) R_L(p_k)^{-1} \mathrm{Res}_{p \to p_k} [\mathbb{C}^L(p_k, a_3, a_4)] \\ \sum_l \sum_j \Big(b_j(p_k) b_{l-j}(p_k) \Phi(A_k(p_k), j, l - j) \Big) + \circlearrowleft \ . \tag{1.99}$$

In the last formula we used the notations $p = Q/2 + iP$, $A_k(P) = \Delta^L_{a_k} + \Delta^M_{\alpha_k} - 1$ and p_k is the value of p corresponding to the discrete term of interest. In (1.99) we also took into account two equivalent symmetric residues, which produces the factor of 2. Now we denote $\varepsilon = p_k - a_k$, where $\Delta^L_{a_k} + \Delta^M_{\alpha_k} = 1$ and $\Delta^M_{\alpha_k}$ is the dimension of the intermediate field in the MM conformal block. We ignore terms of order $o(\varepsilon)$ and multiply Minimal Model structure constants by Liouville ones to get MLG three-point functions. In this way we obtain

$$\begin{aligned}\langle \mathrm{U}_{a_{m_1,n_1}} \mathbb{W}_{a_2} \mathbb{W}_{a_3} \mathbb{W}_{a_4} \rangle^k_{\mathbf{F}} \sim 2\, \mathbb{C}^M(\alpha_{m_1,n_1}, \alpha_2, \alpha_k) \cdot \mathbb{C}^L(a_{m_1,n_1}, a_2, p_k) \\ \mathbb{C}^M(\alpha_k, \alpha_3, \alpha_4) \cdot (-\varepsilon) \mathbb{C}^L(\alpha_k, a_3, a_4) \\ \pi^2 R_L(p_k)^{-1} \sum_l \sum_j \Big(b_j(p_k) b_{l-j}(p_k) \Phi(A_k(p_k), j, l-j) \Big) + \circlearrowleft \ . \end{aligned} \tag{1.100}$$

To compute this expression we note that Φ has a pole in ε only if $j = l = 0$, so that we can ignore other terms. Using the explicit formula for Φ we find

$$\Phi(A_k(p_k), 0, 0) \sim \frac{1}{2\pi A_k(p_k)} \sim \frac{1}{2\pi \Delta^L(p_k)' \varepsilon}. \tag{1.101}$$

Now we expand the value of $R_L(p_k)$:

$$R_L(a) = (\pi \mu \gamma(b^2))^{(Q-2a)/b} \frac{\gamma(2ab - b^2)}{b^2 \gamma(2 - 2ab^{-1} + b^{-2})} \tag{1.102}$$

and two- and three-point functions in MLG are correspondingly:

$$\begin{aligned} D^G_{a,a} &= \frac{\kappa}{2\lambda_a} N(a)^2, \\ \mathbb{C}^G_{a_1,a_2,a_3} &= b\kappa \prod_{i=1}^3 N(a_i). \end{aligned} \tag{1.103}$$

Using these expressions we finally arrive to the formula (1.98) and thus prove (1.93).
□

For the Lee-Yang series we can further simplify (1.93). Without loss of generality let $n_1 \le n_2 \le n_3 \le n_4 \le s,\ p = 2s+1$. Then only the term

$$\sum_{(1,s) \in R_4} 2\lambda_{1,s} \tag{1.104}$$

survives in the sum. If $\sum_i n_i$ is even, then the last expression is equal to

$$\sum_{s=n_2+n_3+1\,:\,2}^{\min(n_1+n_4-1,s)} 2\lambda_{1,s} = \frac{1}{2\sqrt{2p}} \Big(\hat{F}(\min(n_1+n_4, n_2+n_3)) - \hat{F}(n_1+n_4) \Big),$$

where $\hat{F}(n) = (s+1-n)(s-n)\theta(n \le s)$. If $\sum_i n_i$ is odd, then (1.104) equals to

$$\sum_{s=n_2+n_3+1\,:\,2}^{s} 2\lambda_{1,s} = \frac{1}{2\sqrt{2p}}\hat{F}(n_2+n_3) = \frac{1}{2\sqrt{2p}}\Big(\hat{F}(\min(n_1+n_4, n_2+n_3)) - \hat{F}(n_1+n_4)\Big),$$

where the last equality is due to $n_1+n_4 > s$ and $n_2+n_3 < s$.

Now for Lee-Yang series we can rewrite (1.93) as

$$\Sigma_{MHEM} = \Sigma_{HEM} - \frac{1}{2\sqrt{2p}}\Big(\hat{F}(\min(n_1+n_4, n_2+n_3)) - \hat{F}(n_1+n_4)\Big). \qquad (1.105)$$

1.3.1 Comparison with Douglas equation approach

In this section we compare our results with the results of the Douglas equation approach based on matrix models and Frobenius manifolds (see section 1.6.1 below).

Using identification $\Phi_{1,n} = \Phi_{1,p-n}$ in Lee-Yang series we will study fields $\mathrm{U}_{1,n}$ with $n \le s$, where $p = 2s+1$. Our modified HEM approach gives formula (1.105). For comparison purposes we consider a normalization independent version of this formula:

$$\frac{\langle\langle \mathrm{U}_{m_1,n_1}\mathrm{U}_{m_2,n_2}\mathrm{U}_{m_3,n_3}\mathrm{U}_{m_4,n_4}\rangle\rangle}{\left(\prod_{i=1}^4 \langle\langle \mathrm{U}^2_{m_i,n_i}\rangle\rangle\right)^{1/2}} =$$
$$\frac{\prod_{i=1}^4 |m_i p - n_i p'|^{1/2}}{2p(p+p')(p-p')}\Big(\sum_{i=2}^{4}\sum_{r=-(m_1-1)}^{m_1-1}\sum_{t=-(n_1-1)}^{n_1-1}|(m_i-r)p-(n_i-t)p'| - m_1 n_1(m_1 p + n_1 p')\Big) =$$
$$\frac{\prod_{i=1}^4 |m_i p - n_i p'|^{1/2}}{2p(p+p')(p-p')}(-2\sqrt{pp'}\Sigma_{MHEM}(m_i,n_i)), \qquad (1.106)$$

where $p'=2$ and $n_i = 1$.[4] We denote $\Sigma'(m_i,n_i) = -2\sqrt{pp'}\Sigma(m_i,n_i)$ and expect it to be an integer number, so that in the comparison it will be the most convenient quantity.

The numerical quantity to be compared with Σ'_{MHEM} is

$$\Sigma'_{NUM}(m_i,n_i) = -2\sqrt{pp'}\frac{I_4(m_i,n_i)}{\prod_{i=1}^4 N(m_i,n_i)\kappa}. \qquad (1.107)$$

In the framework of the Douglas equation approach there are two formulae for the four-point correlation numbers. First of them [56, 21] after renormalization can be

[4] We do not specify p' and m_i in (1.106) in order to make the structure of this formula more clear.

written as

$$\Sigma'_{DSE}(n_i) = -\hat{F}(0) + \sum_{i=1}^{4} \hat{F}(n_i)$$
$$- \hat{F}(\min(n_1+n_2, n_3+n_4)) - \hat{F}(\min(n_1+n_3, n_2+n_4)) - \hat{F}(\min(n_1+n_4, n_3+n_2)) \ , \tag{1.108}$$

where $\hat{F}(n) = (s+1-n)(s-n)\theta(n \leq s)$.

The second one is proposed in [57] and coincides with the above one when the number of conformal blocks is maximal and does not otherwise.

Proposition 1.3.1. *The formula for four-point correlation numbers in Douglas equation approach is equivalent to the modified HEM formula:*

$$\Sigma_{MHEM}(n_i) = \Sigma_{DSE}(n_i) \ .$$

Moreover, if there are no discrete terms in the operator product expansion $V_{1,n_2}V_{1,n_3}$, then we also have $\Sigma_{HEM} = \Sigma_{MHEM}$. The proof can be found in Appendix 1.B.

All our numerical computations of correlation numbers in various models confirm that the formula (1.105) is correct. In order to give some reference points we list some of the numerical results compared with Douglas equation approach and with the old HEM formula in tables 1.1,1.2. In the tables correlator 12 12 12 14 means $\langle \mathrm{U}_{1,2}\mathbb{W}_{1,2}\mathbb{W}_{1,2}\mathbb{W}_{1,4}\rangle$ and so on. Sign * after the correlator means that there are discrete terms in Liouville OPE of any of the four fields, sign † means that there is a discrepancy between different approaches ($\Sigma'_{NUM,DSE,HEM}$ correspond to numerical computation, Douglas equation approach and higher equations of motion approach respectively).

In the table 1.1 we give some results on correlation numbers in different models. Note that in the table we also presented the results for the Minimal Model $\mathcal{M}(4/15)$, which does not belong to the Lee-Yang series. We list a larger set of correlation numbers in the table 1.2 for the model $\mathcal{M}(2/15)$.

1.3.2 Discussion

We have considered the direct approach to Liouville Minimal Gravity. Our main result is the formula (1.93) for four-point correlation numbers in the Lee-Yang series. This formula generalizes the old one (1.87) proposed in [47]. We show that our modified HEM formula is equivalent to the DSE formula (1.108). We also performed numerical checks, which confirm our results in the region of parameters where the old formula was not applicable.

Below we state some questions which naturally arise from the present considerations.

If the matter sector is represented by the Minimal Model with $p' > 2$, in Douglas equation approach it is impossible to fulfil all the Minimal Model fusion rules as was shown in [21, 58]. So it would be interesting to see how does the correspondence between DSE and conformal field theory approaches extends to other Minimal Models.

In [44] there was obtained a formula for three-point functions in GMM. It coincides with the one obtained by Dotsenko and Fateev in [41] when it is not forbidden by fusion rules. But for some reason this formula gives a nonzero result for some structure constants which should vanish according to the fusion rules. Taking into account this fact would clearly lead to further complications for four-point correlation numbers in general Minimal Models, as mentioned in Section 1.3. In [44] the prescription to obtain MM from GMM is to multiply the GMM structure constants by fusion algebra constants. As far as we know, there is no good understanding of this phenomenon, but it can also be connected with the previous question and with the fusion rules problem in MLG. For instance, without this additional restriction MLG three-point functions are always nonzero, that requires a better understanding. Some insight to this problem can be found in [43], where Liouville theory with $c \leq 1$ (GMM in our language) is discussed.

1.4 Discrete approach to Minimal Gravity

In the previous sections we discussed non-critical string theory or 2d quantum gravity where integration over two-dimensional surfaces was realized using BRST quantization and CFT formalism applied to the resulting path integral. Another approach to integration over all surfaces is to discretize the surfaces and to sum over all possible discretizations. This idea was implemented by various people [12, 13, 14] using integration over spaces of matrices.

The most classical realization of this idea is called Gaussian Unitary Ensemble uses interal of deformed Gaussian measure over space of Hermitian matrices of some fixed size. Expansion in deformation parameters have a combinatorial interpretation as a sum over Feynman diagrams. The dual to each diagram represents a graph which defines a triangulation of a Riemann surface. The conformal structure appears if one thinks of each triangle as a perfect metric triangle. In a particular limit where size of matrices goes to infinity the matrix integral is dominated by configurations with many triangles and thus, it is a good candidate for description of integration over all possible surfaces [5].

[5] In some cases this was done rigorously from the probability theory point of view, e.g. [59]. In such cases the typical surface turns out to be a fractal.

It turns out that such matrix integrals and their limits can be computed explicitly. This is connected with the fact that matrix integrals of such type compute tau functions of integrable hierarchies. The simplest GUE for every finite size of matrix corresponds to finite-dimensional Toda lattice [60] whereas in the large matrix size limit it computes a particular Korteweg-de Vriez (KdV) equation.

It follows that in the discrete approach the correlation numbers can be computed as derivatives of tau functions of integrable hierarchies. Tau functions arise in many ways in the theory of integrable hierarchies and related subjects such as isomonodromy deformations, Riemann-Hilbert problems and such. One way to think of tau functions is as follows. Given a solution to all equations of the hierarchy, its tau function is a generating function of all Hamiltonian densities of the hierarchy on this particular solution. The condition which fixes a solution of the hierarchy and thus the corresponding tau function is a celebrated string equation.

The genus zero part of the discretized string partition function corresponds to dispersionless limit of the integrable hierarchy, that is a limit where in all equations the terms which have more then one derivatives are sent to zero. Such genus zero limits usually have a nice geometrical structure which is a Frobenius manifold.

Witten conjecture proved by Konstsevich [16] and others later [17, 18] states that the discrete apporach should coincide with the topological approach to two-dimensional quantum gravity. In particular, tau functions of integrable hierarchies should coincide with generating functions of intersection numbers like Gromov-Witten invariants.

$$\begin{aligned} \log Z_{MM}(T_i(t_i)) &= \log Z_{top}(t_i), \\ T_i &= t_i * const \end{aligned} \tag{1.109}$$

It was also proposed long ago that Liouville gravity approach to two-dimensional quantum gravity isi equivalent to matrix models approach and topological gravity [20, 19, 61]. The correspondence, however, is more subtle then in the Witten conjecture. The first important result is that gravitational dimensions of natural observables in Minimal Liouville Gravity match scaling dimensions of observables in particular matrix models [19].

Under the naive identification of observables the correlation function do not match, in particular the Minimal Models fusion rules are not satisfied. The correspondence between operators of the theories is nontrivial and are called *resonance transformations* [20]. The resonance transformations were interpreted as contact terms in OPE of the fields when their insertion points coincide in the original paper. In particular, these resonance transformations should be local and should not depend on topology of the surface. With that said it is possible to compute resonance transformations in genus zero cases.

Another important aspect of MLG is that under the conjectural correspondence between it and Matrix Models approach the partition function of MLG is different from topological gravity one, in particular, it satisfies a different string equation. String equations differ by shift in the coupling constants which are deformation parameters of the observables in the theory. Therefore, partition function of Liouville gravity can be (conjecturally) thought of as a coherent state in the topological gravity and vice versa, where the coordinate change is given by resonance transformations.

Part of spherical correlation numbers and resonance transformations was computed in [61] for the Lee-Yang series of MLG (matter sector in Lee-Yang series of MLG is represented by 2/p Minimal Models) using study of the string equation and Minimal Models fusion rules. Lee-Yang series corresponds to KdV hierarchy and thus can be fairly well understood. The main problem is a lack of computations of correlation numbers on the MLG side beind 4-point numbers on a sphere.

The next important step was made in [21] where it became possible to compute resonance transformations and four-point correlation numbers in the 3/p series of Minimal models using the connection with A_2 Frobenius manifold which is dispersionless limit of the Boussinesq integrable hierarchy. The Frobenius manifold prepotential is the restriction of the specific tau-function of the dispersionless hierarchy to the so-called small phase space. Given a string equation the tau function is fiven as an integral of an explicit differential form on the Frobenius manifold.

The same approach was successfully applied to Unitary and more general Minimal Models in [62, 63]. In this approach the resonance transformations are computed using the fusion rules of Minimal Models which imply orthogonal polynomials-like constraints on the resonance transformations and determine them at least up to four-point numbers.

After discussing in more details matrix models and the Frobenius manifold approach we proceed to the study of two-dimensional gravity on surfaces with boundaries or open non-critical strings. Each of the three approaches to two-dimensional gravity can be applied to theories with boundaries.

For the Liouville Gravity it involves conformal field theories with conformal boundary conditions. On the matter side Minimal Models on surfaces with boundaries were studied by Cardy [64] and Liouville theory in the CFT approach by Fateev and Zamolodchikov brothers [25].In both theories in addition to the bulk operators there are boundary operators which can be inserted on the boundary and separate regions with different boundary conditions. The boundary conditions of Minimal Models which we will study are called Cardy states and are in one-to-one correspondence with the bulk fields. In the Liouville theory the boundary conditions are numbered by a continuous parameter which is called *boundary cosmological constant* μ_B and is Laplace dual to the length of the boundary in a similar way that the original cosmological constant μ is Laplace dual to the area of the Riemann surface. Since there are two scaling constants in the Liouville theory, correlation functions can depend on their ratio μ/μ_B^2 which is dimensionless under the scaling. In theory with boundary even one point correlation functions (where the operator is inserted in the bulk) are nontrivial and were computed in [25].

In the Minimal Liouville Gravity we consider boundary conditions which are called FZZT branes: they are tensor products of Cardy states from Minimal Models and Liouville states with fixed boundary cosmological constant or boundary length. Correlation functions are defined as integrals of products of correlation functions of Minimal Models and Liouville theory over the moduli space of curves with boundary. At this point one has to be careful, because moduli spaces of curves with boundaries are more complicated then for the closed curves. Main complications arise from two facts. First, the boundary length is a real parameter, and the moduli space might be a real manifold. In particular, the moduli space might be (real) odd-dimensional.

Secondly, the moduli space can be a non-orientable manifold since there is no natural orientation coming from a complex structure. Definitions of the moduli spaces can be found in the literature (e.g. [65, 66]). For physical expositon one can look in the paper [67]. The problems with moduli spaces do not appear in the case of correlation numbers of bulk operators on a disk.

In the topological gravity approach the correlation numbers are defined as integrals of certain cohomology classes on moduli spaces of curves with boundaries. The moduli spaces are usually studied in this context.

In the Matrix Models approach one can modify matrix integrals to include triangulations of surfaces with boundaries. This can be done by introducing new vector integration variables. After integration over these additional variables the boundary contribution is equivalent to insertion of some operators in the integral corresponding to closed surfaces. In particular, the one-boundary contribution is called a loop operator and plays an important role in theory of matrix models. This allows to compute disk correlation numbers using correlation numbers on a sphere. The important result that we get in this computation is that under the same resonance transformations of couplings as on the sphere, the μ and μ_B-dependence of disk correlation numbers is the same as in one-point correlation numbers with FZZT brane boundary condition in the Minimal Liouville Gravity.

In the matrix models approach we compute disk correlation functions by two different methods following our papers [23, 24]. The equality of the results has a nice interpretation as Mirror Symmetry for A_n singularity or generalization of the heat kernel asymptotic expansion for the A_1 case.

1.5 Matrix Models

1.5.1 One-matrix model and Virasoro constraints

The perturbative expansion of matrix models (MM) in terms of ribbon Feynman diagrams gives an interpretation of MM as a discrete version of 2D quantum gravity [12, 13, 14].

Let us give an idea how it works (see e.g. [68].Let $\mathcal{H}_n$ be a space of $N \times N$ Hermitian matrices. Consider asymptotic expansion in the formal variable λ of the following integral:

$$Z_N(\lambda) := \mathrm{Vol}_n^{-1} \int_{\mathcal{H}_n} \mathrm{d}M \; e^{-N\mathrm{Tr}\left[M^2/2-\lambda M^3/3\right]} = \sum_k \frac{\lambda^k}{k!} \int_{\mathcal{H}_n} \mathrm{d}M \; \mathrm{Tr}\, M^{3k} \; e^{-N\,\mathrm{Tr}\, M^2/2}, \tag{1.110}$$

where $\mathrm{d}M = \prod_{i \leq n} M_{ii} \prod_{i<j} \mathrm{Re} M_{ij} \prod_{i<j} \mathrm{Im} M_{ij}$ is a natural Haar measure on $\mathcal{H}_n$ and $\mathrm{Vol}_n = Z_N(0)$.

Notation 1.5.1. *Let us introduce the correlation numbers or correlators as averages with respect to the measure given by the matrix integral*

$$\langle M_{i_1 j_1} \cdots M_{i_k j_k} \rangle := \mathrm{Vol}_n^{-1} \int_{\mathcal{H}_n} \mathrm{d}M \; M_{i_1 j_1} \cdots M_{i_k j_k} \; e^{-N\mathrm{Tr}\, M^2/2}. \tag{1.111}$$

The correlators in the right hand side of (1.5.1) are Gaussian and thus can be computed as a sum of weighted Feynman diagrams. To compute them consider a Gaussian integral with a source matrix $S \in \mathcal{H}_n$:

$$Z_{N,S} := \mathrm{Vol}_n^{-1} \int_{\mathcal{H}_n} \mathrm{d}M \; e^{-N\mathrm{Tr}\, M^2/2 + \mathrm{Tr}\, SM} = \langle e^{\mathrm{Tr}\, SM} \rangle = e^{\mathrm{Tr}\, S^2/2N}. \tag{1.112}$$

Then the integrals in the RHS of can be computed as derivatives of the integral with a source

$$\langle (\mathrm{Tr}\, M^3)^k \rangle = \sum_{a_1, b_1, c_1, \cdots, a_k, b_k, c_k} \langle (M^{a_1}_{b_1} M^{b_1}_{c_1} M^{c_1}_{a_1}) \cdots (M^{a_k}_{b_k} M^{b_k}_{c_k} M^{c_k}_{a_k}) \rangle =$$

$$= \sum_{a_1, b_1, c_1, \cdots, a_k, b_k, c_k} \frac{\partial}{\partial S^{b_1}_{a_1}} \cdots \frac{\partial}{\partial S^{b_k}_{a_k}} Z_{N,S}. \tag{1.113}$$

Since $\partial S^k_l e^{\mathrm{Tr}\, S^2/2N} = \frac{1}{N} S^l_k e^{\mathrm{Tr}\, S^2/2N}$, the expression above vanishes at $S = 0$ unless for each $\partial/\partial S^k_l$ there is a derivative $\partial/\partial S^l_k$ which kills the prefactor of the exponent.

To get a Feynman diagram expansion let us represent M^k_l as a half-edge consisting af two oriented lines where one is oriented outward and is labeled by k and another is directed inward and is labeled by l.

A correlator $\langle (M^{a_1}_{b_1} M^{b_1}_{c_1} M^{c_1}_{a_1}\rangle$ is pictorially represented by a union of such half-edges where we glue two half-edges whenever they have the same index.

Nonvanishing correlators correspond precisely to diagrams where all the edges are paired.

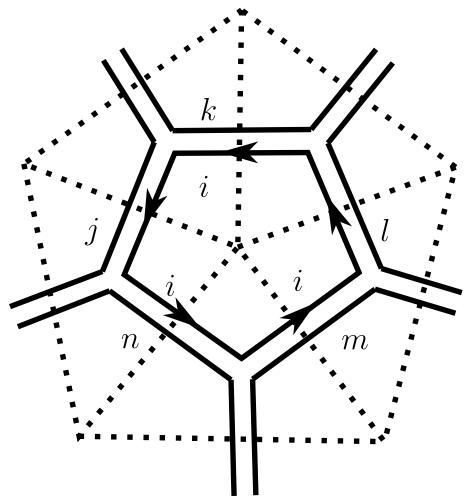

Figure 1.5: A part of a fatgraph and a dual triangulation

Such a diagram is called a (marked) fatgraph $\tilde{\Gamma}$. In the correlators $\langle (\mathrm{Tr}\, M^3)^k \rangle$ there are no free indices, meaning that there is a summation over all indexed fatgrahps. By construction, all edges of the same face of a fatgraph have the same index and summation over all indices gives a factor N.

Therefore, pictorially the correlator $\langle (\mathrm{Tr}\, M^3)^k \rangle$ is equal to

$$\sum_{\Gamma} N^{-E(\Gamma)+F(\Gamma)}, \tag{1.114}$$

where the summation is over all fatgraphs consisting of k trivalent vertices and $E(\Gamma) = 3k/2$ is the number of edges in Γ and $F(\Gamma)$ is the number of faces. Returning to the original integral (1.5.1) we obtain

$$Z_N(\lambda) = \sum_{\Gamma} \frac{\lambda^{V(\Gamma)}}{3^{V(\Gamma)} V(\Gamma)!} N^{V(\Gamma)-E(\Gamma)+F(\Gamma)} = \sum_{\Gamma} \frac{\lambda^{V(\Gamma)}}{3^{V(\Gamma)} V(\Gamma)!} N^{\chi(\Gamma)}, \tag{1.115}$$

where the summation now is over all possible fatgraphs consisting of trivalent vertices. The expression can be further reduced with the help of combinatorics:

$$Z_N(\lambda) = \sum_{\Gamma} \frac{1}{\sharp \mathrm{Aut}(\Gamma)} \lambda^{V(\Gamma)} N^{\chi(\Gamma)}, \tag{1.116}$$

where $\sharp\mathrm{Aut}(\Gamma)$ is the number of automorphisms of the fatgraph Γ.

For a fatgraph Γ consider a dual fatgraph $\check{\Gamma}$ whose faces are triangles dual to vertices of Γ. Each such dualgraph defines a triangulation of some Riemann surface. Moreover, it has a canonical orientation given by orientations of the lines of Γ and defines a conformal structure if we think of each triangle as a perfect triangle. Then the formula (1.116) is a sum over triangulations of Riemann surfaces Σ with weights $\lambda^{Area(\Sigma)}N^{\chi(\Sigma)}$, where area of each triangle is 1. We see that the matrix integral (1.5.1) represents a discretized version of path integral approach to two-dimensional quantum gravity.

This construction has a simple generalization. By a similar argument,

$$\mathrm{Vol}_n^{-1}\int_{\mathcal{H}_n}\mathrm{d}M\, e^{-N\mathrm{Tr}\left[M^2/2-\sum_{s>2}t_sM^s/s\right]}=\sum_{\Gamma}\frac{1}{\sharp\mathrm{Aut}(\Gamma)}N^{\chi(\Gamma)}\prod_{s>2}t_s^{V_s(\Gamma)}, \qquad (1.117)$$

where the summation is over all fatgraphs Γ with $V_s(\Gamma)$ vertices of valency s. Equivalently, a dual graph $\check{\Gamma}$ is a tesselation of a Riemann surface with $V_s(\Gamma)$ s-edged faces. The addition of parameters t_s refines pure gravity partition function and is interpreted as addition of matter. As we will see, this matter is equivalent to the Lee-Yang series of Minimal Models on the Liouville Gravity side.

Virasoro constraints Let us recall celebrated Virasoro constraintswhich appear as equations which are satisfied by the Matrix Models partition functions.

Consider a general Hermitian one-matrix model (which is also known under the name of Gaussian Unitary Ensemble or GUE for short):

$$\begin{aligned}Z_N(t)&=\frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N}\mathrm{d}M\cdot e^{-\frac{1}{g}\mathrm{Tr}V(M)},\\ V(N)&=\sum_{s\geq 0}t_sM^s.\end{aligned} \qquad (1.118)$$

One way to understand Virasoro constraints is as conditions which follow from the coordinate invariance of matrix integrals. Consider a coordinate change in the integral $M\to M+\epsilon M^{n+1}$. Then in the first order in ϵ we have:

$$\begin{aligned}\mathrm{d}M\to\mathrm{d}M\left(1+\epsilon\sum_{i,j\leq N}\frac{\partial\left(M^{n+1}\right)^i_j}{\partial M^i_j}\right)&=\mathrm{d}M(1+\epsilon\sum_{a+b=n}\mathrm{Tr}M^a\mathrm{Tr}M^b),\\ V(M)\to V(M)+\epsilon M^{n+1}V'(M)&=V(M)+\epsilon\sum_{s\geq 0}s\,t_sM^{n+s}.\end{aligned} \qquad (1.119)$$

Then the constraint for the partition function is

$$0=\frac{\partial}{\partial\epsilon}Z_N(t)=\frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N}\mathrm{d}M\cdot e^{-\frac{1}{g}\mathrm{Tr}V(M)}\left[\sum_{a+b=n}\mathrm{Tr}M^a\,\mathrm{Tr}M^b-\frac{1}{g}\sum_{s\geq 0}s\,t_s\mathrm{Tr}M^{n+s}\right]. \tag{1.120}$$

We now use that correlation numbers of trace operators are derivatives of the partition function $Z_N(t)$ with respect to the "times" t_s:

$$-\frac{1}{g}\frac{\partial}{\partial t_s}Z_N(t)=\frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N}\mathrm{d}M\cdot e^{-\frac{1}{g}\mathrm{Tr}V(M)}\mathrm{Tr}M^s \tag{1.121}$$

to rewrite as a differential equation on the partition function:

$$\begin{aligned}&L_nZ_N(t)=0,\\&L_n=\sum_{s\geq 0}s\,t_s\frac{\partial}{\partial t_{s+n}}+g^2\sum_{0\leq a\leq n}\frac{\partial^2}{\partial t_a\partial t_{n-a}},\quad n\geq -1.\end{aligned} \tag{1.122}$$

A particulary important example is the so-called string equation which we will discuss in much more details later

$$L_{-1}=\sum_{s\geq 0}s\,t_s\frac{\partial}{\partial t_{s-1}}Z_N(t)=0. \tag{1.123}$$

The operators L_n span half of the Virasoro algebra

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{1}{12}(n^3-n)\delta_{n+m,0}. \tag{1.124}$$

In fact, they form a well-known representation of Virasoro algebra representation as differential operators acting on a Fock space of a free boson. To see this define a free bosonic current:

$$\begin{aligned}&\partial\phi(z):=\sum_{n\in\mathbb{Z}}a_nz^{-n-1},\\&a_n=g\sqrt{2}\frac{\partial}{\partial t_n},\ n\geq 0,\\&a_{-n}=\frac{1}{g\sqrt{2}}nt_n,\ n>0.\end{aligned} \tag{1.125}$$

The energy-momentum tensor of a free boson is defined as

$$\sum_{n\in\mathbb{Z}}\frac{L_n}{z^{n+2}}=T(z)=\frac{1}{2}:\partial\phi(z)\partial\phi(z):=\frac{1}{2}\sum_{n\in\mathbb{Z}}\frac{1}{z^{n+2}}\sum_{s\in\mathbb{Z}}:a_sa_{n-s}:, \tag{1.126}$$

where z is a "worldsheet coordinate" [6] formal variable and $:XY:$ means a normal ordering in the ring of polynomial differential operators. Normal ordering differential operators X and Y which puts all the derivatives in X to the right of Y treating them as commuting variables. In particular, if $X = \sum_n f_n(\bar{t})\partial/\partial t_n,\ Y = \sum_m g_m(\bar{t})\partial/\partial t_m$, then $: XY := \sum_{n,m}\ f_n(\bar{t})g_n(\bar{t})\,\partial^2/\partial t_n\partial t_m$.

Plugging in the definition of a_n we conclude that L_n in (1.126) are indeed the same as in (1.122) for $n \geq -1$.

The current have a simple expression when plugged in the partition function:

$$\frac{g}{\sqrt{2}}\partial\phi(z) = V'(z)/2 - g\mathrm{Tr}\frac{1}{z-M}. \tag{1.127}$$

Partition function (1.118) is also a tau-function of a Toda integrable hierarchy.

Open Liouville problem Matrix integrals of the type (1.118) have combinatorial interpretation as sums over discretizations of closed surfaces. There is a simple way to modify the integral so that its Feynman diagram representation includes also surfaces with boundaries. This is done by introducing new vector degrees of freedom, namely considering integrals of the type

$$Z_N^o(t) = \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} \mathrm{d}M \cdot e^{-\frac{1}{g}\mathrm{Tr}V(M)} \int \mathrm{d}\Psi \mathrm{d}\bar{\Psi}\, e^{-z\bar{\Psi}^T\Psi + \bar{\Psi}^T M\,\Psi}, \tag{1.128}$$

where $\Psi \in \mathbb{C}^N$. This integral is also Gaussian and can be taken by the Feynman diagram technic. This integral adds new type of vertices to the diagrams. These are also trivalent vertices with one half-edge corresponding to the matrix M_i^j and two other corresponding to vectors Ψ_i which must be paired with half-edges of the same type. Since Ψ_i carries only one index i these half-edges are represented by one line which carries the index i. In the Feynman diagram one interpretes closed loops of such half-edges as boundary components, since interiors of such loops do not contribute factror N to the total area of the Riemann surface. Instead, each edge corresponding to $\Psi_i\bar{\Psi}_i$ has a factor of z^{-1} so that the partition function (1.129) decomposed according to Feynman diagrams has a prefactor z^{-L} at each diagram with total boundary length L.

Computing the Gaussian integral in $\Psi, \bar{\Psi}$ we get

$$Z_N^o(t,z) = \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} \mathrm{d}M\ \det(z-M)\, e^{-\frac{1}{g}\mathrm{Tr}V(M)} \tag{1.129}$$

up to a constant and assuming that the variables $\Psi, \bar{\Psi}$ are odd. It turns out, that computation of open and closed surfaces partition function reduces to computation of the correlation number of the operator $\det(z-M)$ in the closed surfaces case.

[6] It is not connected with the string worldsheet

Such an integral counts discretizations of closed and open surfaces with arbitrary number of connected and boundary components. In particular, expanding $\det(z-M)$ in traces of powers of M we get products $\mathrm{Tr}M^{a_1}\cdots\mathrm{Tr}M^{a_k}$ which combinatorically correspond to insertion of k boundary components to the Riemann surface.

Similarly to the closed case one can compute the one-boundary contributon taking the logarithm of the boundary term:

$$w(z) := \log\det(z-M) = \mathrm{Tr}\log(z-M). \tag{1.130}$$

The operator $w(z)$ is called a loop operator and plays an important role in the theory. Using this interpretation we can introduce a parameter N_b to count the number of boundary components in the theory. The matrix integral we use to sum over open and closed Riemann surfaces is thus

$$Z_N^o(t,z,N_b) := \frac{e^{V(z)/2}}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} \mathrm{d}M\ \det(z-M)^{N_b}\ e^{-\frac{1}{g}\mathrm{Tr}V(M)}. \tag{1.131}$$

The prefactor $e^{V(z)/2}$ is a convenient choice of normalization which we now explain.

Both partition function $Z_N^o(t,z,N_b)$ and the loop operator $w(z)$ have transparent interpretations in terms of the free boson $\partial\phi(z)$ (1.125). Explicitly we have

$$\begin{aligned}
&\phi(z) = -\tilde{a}_0 + a_0\log z + \sum_{n\neq 0} a_n \frac{z^{-n}}{n},\\
&\frac{1}{\sqrt{2}}\phi(z)Z_N(t) = -\frac{1}{2}V(z)Z_N(t) - \langle \mathrm{Tr}\log(z-M)\rangle,\\
&:e^{-\frac{N_b}{\sqrt{2}}\phi(z)}: Z_N(t) = \langle \det(z-M)^{N_b}\rangle e^{N_b V(z)/2},
\end{aligned} \tag{1.132}$$

where $\langle A(M)\rangle := \int_M \mathrm{d}M\, A(M)\, e^{-\frac{1}{g}\mathrm{Tr}V(M)}$ and normal ordering is defined as before. It follows that

$$Z_N^o(t,z,N_b) =: e^{-\frac{N_b}{\sqrt{2}}\phi(z)}: Z_N(t). \tag{1.133}$$

Open Virasoro constraints The closed surfaces partition function satisfies Virasoro constraints in the form (1.122). Using these Virasoro constraints and the formula (1.133) for open surfaces partition function one can derive so-called *open Virasoro constraints* [65, 69]. One way to derive them is commuting L_n with the operator $:e^{-\frac{N_b}{\sqrt{2}}\phi(z)}:$ in the formula (1.132). However, there is a simpler method to get them. Using the stress-energy tensor of a free boson $\phi(z)$. $T(x) = \sum_n L_n x^{-n-2}$ ordinary Virasoro constraints can be rewritten as

$$T(x)\, Z_N(t) = P(x,t), \tag{1.134}$$

where $P(x)$ is a polynomial in x. The operator $: e^{-\frac{N_b}{\sqrt{2}}\phi(z)} :$ is a well-known primary operator for free bosons with conformal dimension $N_b^2/4$. Using the OPE of conformal field theory of a free field $\phi(z)$ we write

$$T(x) : e^{-\frac{N_b}{\sqrt{2}}\phi(z)} : Z_N(t) = \frac{N_b^2}{4(x-z)^2} : e^{-\frac{N_b}{\sqrt{2}}\phi(z)} : Z_N(t) + \\ + \frac{1}{x-z}\frac{\partial}{\partial z} : e^{-\frac{N_b}{\sqrt{2}}\phi(z)} : Z_N(t) + \tilde{P}(x,t,z), \quad (1.135)$$

where $\tilde{P}(x,t,z)$ is regular in x. Let us expand the singular part of the operator in their right hand side of the previous equation in x:

$$\frac{N_b^2}{4(x-z)^2} + \frac{1}{x-z}\frac{\partial}{\partial z} = \sum_{n=0}^{\infty}\left(z^n \frac{\partial}{\partial z} + \frac{N_b^2}{4} n z^{n-1}\right) x^{-n-1}. \quad (1.136)$$

If we collect the terms at x^{-n}, $n > 0$ and move singular terms from the left hand side to the right hand sidee we get open Virasoro constraints:

$$\begin{aligned} &\mathcal{L}_n Z_N^o(t,z,N_b) = 0, \ n \geq -1, \\ &\mathcal{L}_n = L_n - z^{n+1}\frac{\partial}{\partial z} - \frac{N_b^2}{4}(n+1)z^n, \end{aligned} \quad (1.137)$$

where L_n are closed Virasoro constraints.

Double scaling limit For any finite matrix size N partition function (1.118) counts finite triangulations of surfaces and thus is a discrete approximation of a sum over all Riemann surfaces. The continuum limit is achieved through the so-called *double scaling limit*. Namely, one takes a limit where size of matrices N goes to infinity and simultaniously coupling constants t_i undergo a coordinate change. The limit is nontrivial and involves several steps which depend on a particular matrix model. The main idea is that in the limit the Virasoro constraints should survive. We will use the approach which starts from the continuum Virasoro constraints but we outline the idea how to perform double scaling limit in the case of the Hermitian one matrix model for the sake of completeness.

The procedure of taking the limit which is known in the literature [70, 71] consists of three steps:

1. Reduction of the integral (1.118) to only even times. The corresponding function is a tau-function of the Volterra or discrete KdV hierarchy.

2. Coordinate change in the even times of the hierarchy and introduction of reduced Volterra tau-function to restore Virasoro invariance.

3. Taking the continuum limit where the reduced Volterra tau-function becomes KdV tau function satisfying Virasoro constraints.

When the limiting procedure is done, the limiting tau function $Z_{KdV}(t)$ satisfies Virasoro constraints which can be written in the form

$$T(x)Z_{KdV}(t) = P(x,t), \tag{1.138}$$

where $P(x,t)$ is regular in x and the stress-energy tensor is defined in terms of the free boson:

$$T(x) = \sum \frac{L_n}{x^{n+2}} := -\frac{1}{2} : \partial\phi(x)\partial\phi(x) : -\frac{1}{16x^2}, \tag{1.139}$$

$$\frac{1}{2}g\partial\phi(x) = x^{1/2} - \sum_{n\geq 0}(n+1/2)t_n x^{n-1/2} - \frac{1}{4}g^2\sum_{n\geq 0}\frac{\partial}{\partial t_n}x^{-n-3/2} . \tag{1.140}$$

The boson in the expression (1.140) is twisted, that is it is expanded in the half-integer modes of x.

In the expressions above t_n stand for new (re-scaled) KdV couplings, which are some functions of the “bare” couplings of the underlying matrix model (1.118). The derivatives ∂/∂_{t_k} are interpreted as insertions of operators $\mathcal{O}_k$ in the correlation function. In order to motivate this change we note that in the semiclassical limit the eq. (1.138) becomes

$$y^2 = P(x) , \tag{1.141}$$

where $y := \lim_{N\to\infty}\langle J(x)\rangle$, and $P(x)$ is a polynomial, which arises from the RHS of (1.138). This can be interpreted as an equation for a so-called spectral curve. The boson $\phi(x)$ is then defined on this curve rather then on the x-plane (for more details, see e.g. [67]).

In the double scaling limit the Virasoro constraints, which arise for the twisted free boson (1.140), become:

$$\begin{aligned} L_{-1} &= \sum_{k\geq 1}(k+1/2)t_k\frac{\partial}{\partial t_{k-1}} + \frac{1}{2g^2}t_0^2 , \\ L_0 &= \sum_{k\geq 0}(k+1/2)t_k\frac{\partial}{\partial t_k} + \frac{1}{16} , \\ L_n &= \sum_{k\geq 0}(k+1/2)t_k\frac{\partial}{\partial t_{k+n}} + \frac{1}{8}g^2\sum_{i+j=n-1}\frac{\partial^2}{\partial t_i\partial t_j} , \qquad n>0 . \end{aligned} \tag{1.142}$$

Finally we note, that any function of KdV parameters, which is annihilated by all these operators, is uniquely defined and represents in fact a (square root of a) tau-function of a KdV hierarchy [72].

1.5.2 $\mathcal{W}$-constraints and loop operator in general (q,p)-case

With the insight from the one-matrix model now we consider the general situation. Our goal will be to construct the generalization of the open generating function with the loop operator (1.155) and to check it against the results of the worldsheet approach.

1.5.3 From twisted bosons to loop operator

General (q, p) models have several matrix model descriptions, some of which like conventional multi-matrix model are appropriate for orthogonal polynomials method and for the double scaling limit consideration. It turns out, however, that the $\mathcal{W}$-symmetry which is present in the general (q, p) case is not manifest in this setup. What we use here is the integrable systems approach, which can be also obtained from the conformal matrix models approach [73]. In this setting various quantities of the theory are expressed by analogy with the one-matrix case. The results of this approach can be briefly formulated as follows (see, e.g., [74]).

We consider q twisted bosons $\phi_l(x)$, with $l = 1, ..., q$:

$$\partial\phi_l(x) = \sum_{k=-\infty}^{\infty} \alpha_{k+l/q} x^{-(k+l/q+1)} \ , \tag{1.143}$$

where the modes

$$\alpha_{k+l/q} = g\frac{\partial}{\partial t_{l,k}} \text{ and } \alpha_{-k-l/q} = \frac{1}{g}(k + l/q)t_{l,k} \ . \tag{1.144}$$

The energy-momentum tensor of a system of twisted bosons is

$$T(x) = \sum_{r=1}^{q-1} : \frac{1}{2}\partial\phi_r\partial\phi_{q-r} : + \frac{q^2-1}{24qx^2} = \sum_{n=-\infty}^{\infty} L_n x^{-n-2}. \tag{1.145}$$

The system obeys an extended $\mathcal{W}_{q-1}$ symmetry, and the other $\mathcal{W}$-currents, $W^{(n)}(x)$, may be constructed explicitly using the standard bosonization methods. The closed string partition function $Z(t)$ of the q-th model is uniquely defined by the condition that it is annihilated by all the $W^{(n)}$-currents [75]. More precisely,

$$W_k^{(n)} Z(t) = 0 \ , \quad n \geq 2 \ , \quad k \geq 1 - n \ , \tag{1.146}$$

where $W_k^{(2)}$ is L_k.

The correlation numbers are given as usual as derivatives of connected partition function with respect to deformation parameters:

$$\langle \mathcal{O}_{\alpha_1,k_1} \cdot \ldots \cdot \mathcal{O}_{\alpha_n,k_n} \rangle := \frac{\partial}{\partial t_{\alpha_1,k_1}} \cdots \frac{\partial}{\partial t_{\alpha_n,k_n}} \log Z(t) \ , \tag{1.147}$$

where the derivatives are computed at the corresponding critical point.

For later purposes, we write explicitly the string and the dilation equations, $L_{-1}Z(t) = 0$ and $L_0 Z(t) = 0$, with the generators:

$$
\begin{aligned}
L_{-1} &= \sum_{\alpha,k} (\alpha/q + k) t_{\alpha,k} \frac{\partial}{\partial t_{\alpha,k-1}} + \frac{1}{2g^2} \sum_{\beta} \beta(q-\beta)/q^2 t_{\beta,0} t_{q-\beta,0} \ , \\
L_0 &= \sum_{\alpha,k} (\alpha/q + k) t_{\alpha,k} \frac{\partial}{\partial t_{\alpha,k}} + \frac{q^2-1}{24q} \ .
\end{aligned}
\tag{1.148}
$$

Another equivalent description of the system is based on the statement that its partition function is the (q-th root of the) tau-function of the q-th Gelfand-Dickey hierarchy, which satisfies the string equation, *i.e.*, the L_{-1} Virasoro constraint.

Similarly to the KdV case, the boundary partition functions are obtained as the exponential vertex operators constructed from the bosons $\phi_l(x)$,

$$
Z_{open}(t,z) = \langle \exp(N_b \sum_{\alpha=1}^{q-1} \phi_\alpha(z)) \rangle \ , \tag{1.149}
$$

where N_b is a boundary component number counting parameter [7].

To define open correlation numbers we subtract all purely closed surfaces and contributions and take the connected part. The open correlation numbers are then given by

$$
\langle \mathcal{O}_{\alpha_1,k_1} \cdot \ldots \cdot \mathcal{O}_{\alpha_n,k_n} \rangle_{open} := \frac{\partial}{\partial t_{\alpha_1,k_1}} \ldots \frac{\partial}{\partial t_{\alpha_n,k_n}} \log(Z_{open}(t,z)/Z(t)) \ . \tag{1.150}
$$

From this expression one gets [74] the open $\mathcal{W}$-constraints as the Ward identities for the vertex operator

$$
L_n^{full} = L_n - \frac{N_b^2}{2}(n+1) z^n - z^{n+1} \frac{\partial}{\partial z} \ . \tag{1.151}
$$

Let us take a formal Laplace transform of this operator in the variable z:

$$
\hat{L}_n^{full} = L_n - \frac{N_b^2}{2}(n+1) \frac{\partial^n}{\partial s^n} - s \frac{\partial^{n+1}}{\partial s^{n+1}} \ . \tag{1.152}
$$

Having in mind the interpretation of Z_{open} and Z as disconnected surfaces partition functions, or tau-functions, let us now represent

$$
Z_{open}(t,s) = \exp(F_c + F_o), \quad F_c = \log(Z(t)) \ . \tag{1.153}
$$

[7] In principle, there can be independent boundary parameters for each of the bosons.

Using the ordinary string equations (1.148) for $Z(t)$, the open string equation and the dilation equations for F_o correspondingly read:

$$\begin{aligned}\sum_{\alpha,k}(\alpha/q+k)t_{\alpha,k}\frac{\partial}{\partial t_{\alpha,k-1}}F^o &= s\ ,\\ \sum_{\alpha,k}(\alpha/q+k)t_{\alpha,k}\frac{\partial}{\partial t_{\alpha,k}}F^o &= N_b^2/2+s\frac{\partial}{\partial s}F^o\ .\end{aligned}\tag{1.154}$$

After some renormalization these equations coincide with the equations obtained recently in [76] for the generating function $F^{1/q,o}$ of the open correlation numbers in the topological gravity of q-spin curves.[8]

Loop operator. The analogue of the formula for the loop operator is easily obtained from the bosonic representation [74] together with the Laplace transform:

$$\begin{aligned}w_r(l) &= \sum_{k=0}^{\infty}\frac{l^{k+r/q}}{\Gamma(k+r/q+1)}\mathcal{O}_{r,k}\ ,\\ w_r(l) &\sim \int_l \frac{\mathrm{d}l}{l}\, e^{-lz}\langle\phi_r(z)\rangle\ .\end{aligned}\tag{1.155}$$

The loop insertion is the one-boundary part of the open partition function (1.149). In this case there are in general $q-1$ linearly independent loop operators. They can be interpreted as corresponding to different boundary conditions. However, the precise identification with the FZZT branes is yet to be clarified. The general loop operator can be written as

$$w(l) := \sum_{\alpha=1}^{q-1} c_\alpha w_\alpha(l)\ .\tag{1.156}$$

Below we omit the coefficients c_α as it will be trivial to restore them in the final answer.

The analogs of open-Virasoro and $\mathcal{W}$-constraints are then obtained from the Ward identities for the fields $\phi_r(z)$, for instance:

$$\langle T(x)\phi_r(z)\rangle \sim \frac{\partial\phi_r(z)}{z-x}\ ,\tag{1.157}$$

from which we get

$$L_n^{loop}\langle\phi_r(z)\rangle = \left(L_n - z^{n+1}\frac{\partial}{\partial z}\right)\langle\phi_r(z)\rangle = 0,\quad n\geq -1\ ,\tag{1.158}$$

[8]The generating function $F^{1/q,o}$ has many interesting properties. In particular, in [76] the authors give an expression for $F^{1/q,o}$ in terms of the wave function of the KP hierarchy.

where L_n are the closed Gelfand-Dickey Virasoro constraints.

Let us denote by $\hat{\mathcal{O}}_{\alpha,k}$ the bulk insertion operator $\partial/\partial\lambda_{\alpha,k}$ in the MLG frame. Then using the resonance transformations, that is change of the couplings $t_{\alpha,k} \to \lambda_{\alpha,k}$ from KdV to MLG frame, we obtain

$$\langle \hat{\mathcal{O}}_{\alpha,k} \sum_r w_r(l) \rangle = u^{-k-\alpha/q} I_{k+\alpha/q}(2lu) \ , \tag{1.159}$$

which is precisely the singular part of the Liouville one-point boundary (FZZ) function found in [77]. This is the main result of the present paper. Moreover, if we include the regular part of bosons in the definition of the loop, we correctly restore also the regular part of FZZ formula. The important point is that the resonance transformations have been computed from the condition of diagonality of two-point functions in the MLG frame, corresponding to the Liouville couplings $\lambda_{\alpha,k}$, in the spherical topology.

Let us sketch the derivation of the formula (1.159). The detailed computation is given in Appendix 1.C. First, we expand the sum $\sum_r w_r(l)$ as

$$\sum_{\alpha=1}^{q-1} \sum_{k \geq 0} \frac{l^{\alpha/q+k+1}}{\Gamma(\alpha/q+k+1)} \frac{\partial}{\partial t_{\alpha,k}} \ . \tag{1.160}$$

Then we express the left hand side of (1.159) in terms of the two point functions

$$\sum_{\beta,m} \frac{l^{\alpha/q+k+1}}{\Gamma(\alpha/q+k+1)} \langle \hat{\mathcal{O}}_{\alpha,k} \mathcal{O}_{\beta,m} \rangle \tag{1.161}$$

and compute this expression using the results of [78] and [79], namely the formula for the generating function of the correlators and the explicit expression for the resonance transformation.

We shall now briefly describe the computation method of the correlation numbers, based on the Frobenius manifold structure, leading to the results (1.159).

1.6 Frobenius Manifolds

1.6.1 Dual approach and Frobenius manifolds

Here we formulate the result for the spherical partition function, corresponding to the tau-function of the $q-$th Gelfand-Dickey integrable hierarchy. The L_{-1} constraint (which uniquely fixes the tau-function) is written in the form of Douglas string equation, conveniently formulated as the action principle [80], $\partial S(u)/\partial u_i = 0$, and

$$S(u,t) = \mathrm{Res}_{y=\infty} \sum_{\alpha=1}^{q-1} \sum_{k=0}^{\infty} t_{\alpha,k} Q^{\frac{\alpha}{q}+k} \ . \tag{1.162}$$

Here $Q = Q(y,u)$ is the symbol of the Lax operator of the corresponding $q-$th Gelfand-Dickey hierarchy,

$$Q(y,u) = y^q + u_1 y^{q-2} + u_2 y^{q-3} + ... + u_{q-1} \ . \tag{1.163}$$

The parameters u_α, $(\alpha = 1, ..., q-1)$, can be regarded as coordinates on the Frobenius manifold on the unfolding space of the A_{q-1} singularity, see sections 2.2, 2.3 for definitions and details on such manifolds. The tangent space of the Frobenius manifold at a point u is a Frobenius algebra $\mathbb{C}[y] \mod \frac{\partial Q(y,u)}{\partial y}$ (for more details, see, e.g., [81]). We note that for the $(q,p)-$model the time parameter in (1.162) in front of $Q^{p/q}$ is equal to 1 and the time parameter in front of $Q^{p/q-1}$ is equal to the Liouville cosmological constant μ.

In order to consider the general MLG(q,p) case it is convenient [81] to use another parametrization, s and p_0, such that $p = sq + p_0$ and $0 < p_0 < q$. As described before, the physical fields $\mathcal{O}_{m,n}$ are labeled by pairs (m,n), where $1 \geq m \geq q-1$ and $1 \geq m \geq q-1$. Equivalently, in the "KdV frame", we use the parameters (α, k):

$$\alpha = p_0 m \mod q \,, \qquad k = sm - n + [p_0 m/q] \ . \tag{1.164}$$

The action can be rewritten as

$$S(u, t[\lambda]) = \text{Res}_{y=\infty} \left(Q^{\frac{p+q}{q}} + \sum_{(m,n)\in Kac(q,p)} t^{(m,n)} Q^{\frac{|pm-qn|}{q}} \right) \ . \tag{1.165}$$

The KdV times $t^{(m,n)}$ and the Liouville couplings λ_{mn} are related through the resonance transformation,

$$t^{(m,n)} = \lambda_{m,n} + \sum A^{(m,n)}_{(m_1,n_1),(m_2,n_2)} \lambda_{m_1,n_1} \lambda_{m_2,n_2} + \cdots \ , \tag{1.166}$$

with the coefficients $A^{(m,n)}_{(m_1,n_1),(m_2,n_2)}$ constraint by the scaling properties and fixed by the underlying CFT selection rules [81].

We will perform the computations in the $t_{\alpha,k}$ frame, as it makes the connection with the integrable structure more transparent. Following [81], we introduce the deformed flat coordinated on the Frobenius manifold

$$\theta_\alpha(z) := \sum_{k\geq 0} \theta_{\alpha,k} z^k \ , \tag{1.167}$$

where

$$\theta_{\alpha,k} = -c_{\alpha,k} \operatorname*{res}_{y=\infty} Q^{k+\frac{\alpha}{q}}(y) \tag{1.168}$$

and

$$c_{\alpha,k} = \frac{\Gamma(\frac{\alpha}{q})}{\Gamma(\frac{\alpha}{q}+k+1)} \ . \tag{1.169}$$

Then the action takes the form

$$S(u,t[\lambda]) = -\left[\frac{\theta_{p_0,s}}{c_{p_0,s}} + \sum_{\sigma,k} t_{\sigma,k}\frac{\theta_{\sigma,k}}{c_{\sigma,k}}\right] , \tag{1.170}$$

where $t(\lambda)$ stands for the resonance transformation (1.166). The generating function of the spherical correlators is

$$Z[t(\lambda)] = \int_0^{v_*^1} dv^1 C_1^{\beta\gamma}\frac{\partial S}{\partial v^\beta}\frac{\partial S}{\partial v^\gamma} , \tag{1.171}$$

where v_α and $C_\sigma^{\beta\gamma}$ are correspondingly flat coordinates and structure constants of the Frobenius manifold A_{q-1}, and $v_* \in A_{q-1}$ is the special solution of the string action.

1.7 Correlation numbers on a disk

In this section we compare the loop formulae (1.155), (1.156) with the results in [23], where a different way to compute open MLG correlators was chosen. Namely, the following expression for the loop operator in the (q,p) model has been proposed there:

$$\tilde{w}(l) := \int_{t_{1,0}}^{\infty} \mathrm{d}x \int_\gamma \mathrm{d}y \, e^{lQ(y,v(x))} \ . \tag{1.172}$$

Here $Q(y,v)$ is defined in (1.163) and $v(x)$ is the solution of the string equation, corresponding to the closed string partition function. Let us compare the formula (1.172) with the formula (1.155), and recall the difficulties encountered in [23].

First of all it is more convenient to write the derivative of the normalized loop

$$\partial_{t_{1,0}}\sum_\beta c_\beta w_\beta(l) = \sum_{\beta,m} c_\beta \frac{l^{\beta/q+m}}{\Gamma(\beta/q+m+1)}\partial_{t_{1,0}}\partial_{t_{\beta,m}}\log Z(t) \ , \tag{1.173}$$

where $Z(t)$ is the spherical partition function. It is known, see, e.g., [78], that the second derivative $\partial_{t_{1,0}}\partial_{t_{\beta,m}}\log Z(t) = \mathrm{Res}_{y=\infty} Q(y)^{\alpha/q+k}$.[9]

To analyze the difference with the earlier approach [23], we compare (1.173) with the derivative of (1.172):

$$\int_\gamma \mathrm{d}y \, e^{lQ(y,v(x))} \ \stackrel{?}{=} \ \sum_{\beta=1}^{q-1} c_\beta \sum_{m\geq 0}\frac{l^{\beta/q+m}}{\Gamma(\beta/q+m+1)}\mathrm{Res}_{y=\infty} Q(y)^{\beta/q+m} \ . \tag{1.174}$$

[9]Here we do not have a prefactor as in [78] due to the different normalization of the action $S = \sum_{\alpha,k} t_{\alpha,k}\mathrm{Res}\, Q(y)^{\alpha/q+k}$.

We note, that if satisfied the relation (1.174) would reflect the classical genus zero mirror symmetry. Indeed, according to the extended Witten's conjecture, the RHS is an analytic continuation of a certain power series, counting the intersection numbers on the moduli space of curves with $q-$spin structure, namely an A-model expression. Whereas the left hand side is a period integral for the dual B-model, which is an oscillating integral of a Landau-Ginzburg model $W(y) = Q(y)$, or an A_{q-1} singularity. In our case we can simply establish the explicit connection. From this point of view, the genus zero loop is a period of the mirror model with the deformation parameters v_i, as functions of couplings t governed by the string equation.

We note, that for $q = 2$, that is in the KdV case, the equality above is known as an asymptotic expansion of the heat kernel operator, where the residues in the RHS are dispersionless analogues of the Seeley coefficients [82]. A very similar phenomenon occurs in the arbitrary q case.

Let us first expand the left hand side of the equation (1.174). We introduce a notation $Q(y) = y^q + Q_0(y)$, where $Q_0(y)$ is of degree $q-2$ in y. Then

$$\begin{aligned} &\int_\gamma \mathrm{d}y\, e^{lQ(y,v(x))} = \int_\gamma \mathrm{d}y\, e^{ly^q} \sum_n \frac{l^n Q_0(y)^n}{n!} = \\ &= \sum_{\alpha=0}^{q-2} \sum_{k,n\geq 0} c_\alpha^\gamma e^{\pi i k} \frac{[y^{\alpha+kq}]Q_0(y)^n}{n!} \Gamma\left(\frac{\alpha+1}{q}+k\right) l^{n-k-\frac{\alpha+1}{q}} \ , \end{aligned} \tag{1.175}$$

where we introduced a notation $[y^n](\sum p_k y^k) := p_n$ and in the second line we computed the integral term by term in powers $\alpha + kq$ of y as

$$\int_\gamma \mathrm{d}y\, e^{y^q} y^{\alpha+kq} = c_\alpha^\gamma \Gamma\left(\frac{\alpha+1}{q}+k\right) \ , \tag{1.176}$$

where c_α^γ are some (in general complex) coefficients of the expansion of the cycle γ in a certain basis $(\Gamma_\alpha)_\alpha$ in homology $H_1(\mathbb{C}, \Re y^q << 0; \mathbb{C})$. This basis is defined by duality

$$\int_{\Gamma_\alpha} e^{-y^q}\, y^\beta = \delta_{\alpha,\beta} \ , \quad \beta \in [0, q-2] \ . \tag{1.177}$$

Now we turn to the right hand side of (1.174),

$$\operatorname{Res} Q(y)^{\beta/q+m} = \sum_n [y^{-\beta-1+(n-m)q}] Q_0(y)^n \frac{\Gamma(\beta/q+m+1)}{\Gamma(\beta/q+m+1-n)\, n!} \ . \tag{1.178}$$

Then the right hand side of the equation (1.174) becomes

$$\sum_\beta c_\beta \sum_{n,m} \frac{[y^{-\beta-1+(n-m)q}]Q_0(y)^n}{n!} \Gamma(\beta/q+m+1-n)^{-1} l^{\beta/q+m} \ . \tag{1.179}$$

Using reflection relation for gamma function and changing summation variables we get

$$\sum_{\beta} c_\beta \sum_{k\geq 0,\, n\geq 0} \frac{[y^{\alpha+kq}]Q_0(y)^n}{n!}\Gamma\left(\frac{\alpha+1}{q}+k\right)\frac{\sin\pi((\alpha+1)/q+k)}{\pi} l^{n-k-\frac{\alpha+1}{q}}\ . \tag{1.180}$$

Now it is clear, that the formulae (1.175) and (1.180) differ by some constants and coefficients c_β and c^γ_α. Basically, c^γ_α in the expression (1.172) is defined by the cycle γ, whereas in the approach of the present paper it is a matter of choice of a particular linear combination $w(l) := \sum_\beta c_\beta\, w_\beta(l)$. It is tempting to interpret c_β as a boundary condition of the minimal model, however it requires further investigation. From this point of view the vanishing of one-point correlators of the form $\langle\hat{\mathcal{O}}_{2\alpha,m}\rangle^{\rm disk}$ for even q, encountered in [23], is just related to the fact that the corresponding cycle γ does not contain some of Γ_α in the expansion over this basis. Another problem encountered in [23] was the problem in the computation of one-point functions in the non-unitary (q,p) models (i.e. $p > q+1$), which was due to inappropriate solution of the string equation in $\mathrm{d}x$ integration.

$m_i n_i$	$\lvert\Sigma'_{NUM}(m_i,n_i)\rvert/2$ num.	$\Sigma'_{DSE}(m_i,n_i)/2$ exact	$\Sigma'_{HEM}(m_i,n_i)/2$ exact
2/9	-	-	-
12 12 12 12	2.00002	2	2
13 13 12 12	2.00031	-2	-2
12 14 12 12	1.00003	-1	-1
13 12 13 13	4.00016	-4	-4
2/11	-	-	-
13 15 13 13	5.99976	-6	-6
12 14 12 12*†	1.000001	1	2
2/13	-	-	-
12 14 12 12*†	3.0001	3	6
4/15	-	-	-
13 17 13 13	2.00009	N/A	-2
13 15 13 13	10.9998	N/A	-11

Table 1.1: Numerical data for Σ'. * - means discrete terms. † - discrepancies.

$m_i n_i$	$\lvert\Sigma'_{NUM}(m_i,n_i)\rvert/2$ num.	$\Sigma'_{DSE}(m_i,n_i)/2$ exact	$\Sigma'_{HEM}(m_i,n_i)/2$ exact
2/15	-	-	-
12 12 13 15*†	3.08	3	6
12 12 14 16*†	1.025	1	2
12 12 15 15*	1.98	2	2
12 12 15 17*	1.03	-1	-1
12 12 16 16*	2.06	-2	-2
12 12 17 17*	4.09	-4	-4
12 13 13 16*†	1.01	1	2
12 13 14 15*	1.99	2	2
12 13 14 17*	1.015	1	1
12 13 15 16*	2.03	-2	-2
12 13 16 17*	5.07	-5	-5
12 14 14 14*	1.995	2	2
12 14 14 16*	2.02	-2	-2
12 14 15 15*	2.01	-2	-2
12 14 15 17*	5.04	-5	-5
12 14 16 16*	6.05	-6	-6
12 14 17 17*	8.04	-8	-8
13 13 13 15*†	1.995	2	3
13 13 13 17*†	0.999	-1	0
13 13 14 14*	2.93	3	3
13 13 14 16*	2.01	-2	-2
13 13 15 15*	3.03	-3	-3
13 13 15 17*	6.05	-6	-6
13 13 16 16*	7.04	-7	-7
13 13 17 17*	9.05	-9	-9

Table 1.2: Numerical data for Σ'. * - means discrete terms. † - discrepancy.

Appendix

1.A Conformal block decomposition in correlation numbers

In this appendix we derive convenient representation for the correlation numbers.

The following considerations are known in the literature, see e.g. [83]. We start from the formula (1.90) for the correlation numbers and use the symmetry of the integrals under modular transformations in order to reduce the integration from the whole complex plane to the fundamental domain. The modular subgroup of projective transformations divides the complex plane into six regions. The fundamental region is defined as $\mathbf{G} = \{\mathrm{Re}\, x < 1/2;\ |1-x| < 1\}$. The other five regions are mapped to the fundamental one using one of the transformations $\mathcal{A}, \mathcal{B}, \mathcal{AB}, \mathcal{BA}, \mathcal{ABA}$, where $\mathcal{A}$: $z \to 1/z$ and $\mathcal{B}$: $z \to 1-z$. Combining the projective transformations of the fields and the corresponding change of the variables in the integrals, we reduce the integration to the fundamental region. We note that the Jacobian of the transformation exactly cancels the transformation of the fields because their total conformal dimension is 1. Then,

$$\mathcal{I}_4(n_i) = 2\int_{\mathbf{G}} d^2z \Big(\langle \mathbb{W}_1(0)\mathrm{U}_2(z)\mathbb{W}_3(1)\mathbb{W}_4(\infty)\rangle + \langle \mathbb{W}_3(0)\mathrm{U}_2(z)\mathbb{W}_1(1)\mathbb{W}_4(\infty)\rangle + \\ + \langle \mathbb{W}_4(0)\mathrm{U}_2(z)\mathbb{W}_3(1)\mathbb{W}_1(\infty)\rangle \Big), \tag{1.181}$$

where the factor 2 in front counts the equivalent projective images (the order of the last two fields is not relevant) and $\mathrm{U}_i, \mathbb{W}_i$ stand for $\mathrm{U}_{a_i}, \mathbb{W}_{a_i}$.

Conformal block decomposition. For a while, we omit some arguments that are easily reconstructed in the final expressions. In the matter sector,

$$
\begin{aligned}
\langle \Phi_1(0)\Phi_2(z)\Phi_3(1)\Phi_4(\infty)\rangle &= \sum_k c_k^{(1)} |\mathcal{F}_k^{(1)}(z)|^2 \ , \\
\langle \Phi_3(0)\Phi_2(z)\Phi_1(1)\Phi_4(\infty)\rangle &= \sum_k c_k^{(1)} |\mathcal{F}_k^{(3)}(z)|^2 \ , \\
\langle \Phi_4(0)\Phi_2(z)\Phi_3(1)\Phi_1(\infty)\rangle &= \sum_k c_k^{(1)} |\mathcal{F}_k^{(4)}(z)|^2 \ .
\end{aligned} \tag{1.182}
$$

Here the index k corresponds to the channels in the degenerate OPE of the fields Φ_i and the coefficients c_k are related to the basic structure constants [41, 44]:

$$
c_k^{(1)} = \mathbb{C}_{12}^k \mathbb{C}_{34}^k, c_k^{(3)} = \mathbb{C}_{32}^k \mathbb{C}_{14}^k, c_k^{(4)} = \mathbb{C}_{42}^k \mathbb{C}_{31}^k \ .
$$

In (1.182), $\mathcal{F}_k^{(i)}$ denotes the conformal blocks appearing in the k-channel for the given correlation function. In the Liouville sector we have

$$
\begin{aligned}
\langle V_1(z)V_2(0)V_3(1)V_4(\infty)\rangle &= \mathcal{R}\int' \frac{dP}{4\pi} r^{(1)}(P)|\mathcal{F}^{(1)}(P,z)|^2 \ , \\
\langle V_3(z)V_2(0)V_1(1)V_4(\infty)\rangle &= \mathcal{R}\int' \frac{dP}{4\pi} r^{(3)}(P)|\mathcal{F}^{(3)}(P,z)|^2 \ , \\
\langle V_4(z)V_2(0)V_3(1)V_1(\infty)\rangle &= \mathcal{R}\int' \frac{dP}{4\pi} r^{(4)}(P)|\mathcal{F}^{(4)}(P,z)|^2 \ ,
\end{aligned} \tag{1.183}
$$

where

$$
\mathcal{R} r^{(1)}(P) = \mathbb{C}^L(a_1, a_2, Q/2 + iP)\mathbb{C}^L(Q/2 - iP, a_3, a_4)
$$

and so on. Here $\mathcal{R}$ stands for the momentum independent part of the product. In what follows we omit upper subscripts pointing permutations of the fields and summation with respect to them in the correlators.

The Modular Integral. It is efficient [84] to go to the universal cover of the moduli space $\mathcal{M}_{0,3} = S^2 \backslash \{0, 1, \infty\}$, that is to use elliptic transformation in the integration. We use the map

$$
\tau = i\frac{K(1-z)}{K(z)} \ ,
$$

where $K(z)$ is the complete elliptic integral of the first kind

$$
K(z) = \frac{1}{2}\int_0^1 \frac{dt}{y}
$$

and $y^2 = t(1-t)(1-zt)$. It can be verified that

$$dz = \pi z(1-z)\theta_3^4(q)d\tau \ ,$$

where the elliptic nome parameter

$$q = e^{i\pi\tau}$$

and theta constant

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \ .$$

Following [84] we can write

$$\mathcal{F}(\Delta_i, \Delta|q) = (16q)^{\Delta_p - \Delta_0} z^{\Delta_0 - \Delta_1 - \Delta_2}(1-z)^{\Delta_0 - \Delta_2 - \Delta_3}\theta_3^{12\Delta_0 - 4\sum\Delta_i}(q)H(\Delta_i, \Delta|q) \ , \tag{1.184}$$

in order to represent integral (1.181) in the following form

$$\mathcal{I}_4(a_i) = 2\sum_i^{1,3,4} \pi^2 \mathcal{R}\int' \frac{dP}{4\pi}\sum_k r^{(i)}(P)c_k^{(i)}\int_{\mathbf{F}} |16q^{A_k(P)}H_k^{(i)}(q|\Delta_p^L)H_k^{(i)}(q)|^2 \mathrm{d}^2\tau \ , \tag{1.185}$$

where $\mathbf{F} = \{|\tau| > 1;\ |\mathrm{Re}\,\tau| < 1/2\}$, $A_k(P) = \Delta^L(P) + \Delta_k^M - 1$ is sum of conformal dimensions in the intermediate channel minus 1 and $H(\Delta_i, \Delta|q)$ is a series in q of the form $1 + O(q)$, which is computed using recurrence relation [84].

Numerics. With (1.185), the calculation reduces to the numerical integration of several integrals of the general form

$$\int_{\mathbf{F}} |z(1-z)\theta_3^4(q)\mathcal{F}_P(z)|^2 d^2\tau \ , \tag{1.186}$$

where $\mathcal{F}_P(z)$ is some Liouville conformal block like in (1.182) or some more complicated composite expression like in (1.183). The integrand can be developed as a power series in q according to

$$z(1-z)\theta_3^4(q)\mathcal{F}_P(z) = (16q)^\alpha \sum_{r=0}^{\infty} b_r(P)q^r \tag{1.187}$$

and the same for $\bar{q}$. In each term, we can integrate in $\tau_2 = \mathrm{Im}\,\tau$ explicitly with the result conviniently represented in terms of the function

$$\Phi(A, r, l) = \int_{\mathbf{F}} d^2\tau\, |16q|^{2A}\, q^r \bar{q}^l = \frac{(16)^{2A}}{\pi(2A+r+l)}\int_{-1/2}^{1/2} \cos(\pi(r-l)x)e^{-\pi\sqrt{1-x^2}(2A+r+l)}dx \ . \tag{1.188}$$

Using explicit formulae (1.185), (1.187), (1.188) we finally obtain the following expression for (1.185):

$$\mathcal{I}_4(a_i) = \\ 2\sum_{i}^{1,3,4}\sum_{k} c_k^{(i)}\pi^2\mathcal{R}\int' \frac{dP}{4\pi} r^{(i)}(P)\sum_{l}\sum_{j}\Big(b_j(P)b_{l-j}(P)\Phi(A_k(P),j,l-j)\Big)\ , \quad (1.189)$$

where $b_j(P) = [q^j](H^{(i)}(q|\Delta_p^L)H_k^{(i)}(q))$ is a q^jth term in the expansion of the elliptic conformal blocks and $A_k(P) = \Delta^L(P) + \Delta_k^M - 1$ as above. Each term in (1.189) is suppressed by a factor $\max_{\mathbf{F}} |q|^{2l}$ and the series in l converges very rapidly.

Main source of numerical errors in these computations is a method of computing product of Liouville structure constants, namely functions $r^{(i)}(P)$.

1.B Proof of the proposition 1.3.1

Here we prove that

$$\Sigma'_{MHEM}(n_i) = \Sigma'_{DSE}(n_i). \quad (1.190)$$

Let $n_1 \leq n_2 \leq n_3 \leq n_4 \leq s$. For our purposes we write Σ'_{DSE} and Σ'_{MHEM} as

$$\Sigma'_{DSE} = -\hat{F}(0)+\sum_{i=1}^{4}\hat{F}(n_i)-\hat{F}(n_1+n_2)-\hat{F}(n_1+n_3)-\hat{F}(\min(n_1+n_4,\ n_2+n_3)) \quad (1.191)$$

and

$$\Sigma'_{MHEM} = \sum_{i=2}^{4}\sum_{t}^{(n_1)}|p-2(n_i+t)|-n_1(p+2n_1)+-\hat{F}(\min(n_1+n_4,n_2+n_3))+\hat{F}(n_1+n_4). \quad (1.192)$$

First, we need to show that the old formula Σ'_{HEM} coincides with Σ^*_{DSE}, where we have introduced

$$\Sigma^*_{DSE} = -\hat{F}(0) + \sum_{i=1}^{4}\hat{F}(n_i) - \hat{F}(n_1+n_2) - \hat{F}(n_1+n_3) - \hat{F}(n_1+n_4)\ . \quad (1.193)$$

If $n_1 + n_i \leq s+1$, in Σ'_{HEM} all expressions under modules are positive and in Σ'_{DSE} all $\hat{F}$ are equal to $\hat{F}_0$, so that both Σ' simplify to

$$2n_1(p - \sum_i n_i)\ . \quad (1.194)$$

When $n_1 + n_i \geq s + 2$ for some i, Σ'_{HEM} gets a correction to (1.194) because of the modules equal to

$$-2 \sum_{t\,:\,p-2(n_i+t)<0} (p - 2(n_i + t)) = (s + 1 - n_1 - n_i)(s - n_1 - n_i) \ .$$

Σ^*_{DSE} gets a correction because of the Heaviside theta function equal to $\hat{F}(n_1 + n_i) = (s + 1 - n_1 - n_i)(s - n_1 - n_i)$, which finishes the proof.

Now the initial statement (1.190) follows immediately from the definitions.

1.C Computation of one-point correlation numbers

As explained in Section 1.5.2, the macroscopic loop $w(l)$, where l is the length of the loop, is created by the operator

$$w(l) = \sum_{\beta,j} \frac{l^{\beta/q+j}}{\Gamma(\beta/q + j + 1)} \frac{\partial}{\partial t_{\beta,j}} \ , \tag{1.195}$$

where $t_{\beta,j}$ are KdV times. The one-point function of the bulk operator $\hat{\mathcal{O}}_{\alpha,k}$ on a disk

$$\langle \hat{\mathcal{O}}_{\alpha,k} \rangle^{\text{disk}} = \langle \hat{\mathcal{O}}_{\alpha,k} \cdot w(l) \rangle^{\text{sphere}} \tag{1.196}$$

is obtained from the generating function (1.171) as follows

$$\langle \hat{\mathcal{O}}_{\alpha,k} w(l) \rangle^{\text{sphere}} = \sum_{\beta,j} \frac{l^{\beta/q+j}}{\Gamma(\beta/q + j + 1)} \frac{\partial}{\partial t_{\beta,j}} \cdot \frac{\partial}{\partial \lambda_{\alpha,k}} \log Z[t(\lambda)] \ . \tag{1.197}$$

Note that the second derivative is taken with respect to $\lambda_{\alpha,k}$, since we are interested in the correlation functions in the Liouville frame. Here comes non-trivial dependence on the resonance transformations (1.166).

Using (1.196),(1.197), one gets

$$\langle \mathcal{O}_{m,n} \rangle^{\text{disk}} = \sum_{\beta,j} \frac{l^{\beta/q+j}}{\Gamma(\beta/q + j + 1)} \int_0^{v_*} dv^\sigma C_\sigma^{\beta\gamma} \left(-\frac{1}{c_{\beta,j}} \right) \frac{\partial \theta_{\beta,j}}{\partial v^\beta} \frac{\partial \hat{S}_{\alpha,k}}{\partial v^\gamma} \ . \tag{1.198}$$

It is convenient to take the integration contour[10] along v_1-axis and to use the properties of the derivatives $\frac{\partial S^{(m,n)}}{\partial v^\gamma}$ and of the structure constants on the line v_1, obtained in [79]. Namely, using expressions for structure constants, one gets

$$\langle \hat{\mathcal{O}}_{\alpha,k} \rangle^{\text{disk}} = \sum_{\beta,j} \frac{l^{\beta/q+j}}{\Gamma(\beta/q + j + 1)} \left(-\frac{1}{c_{\beta,j}} \right) \sum_{\gamma=1}^{q-1} \int_0^{v_1^0} dv_1 \left(-\frac{v_1}{q} \right)^{\gamma-1} \frac{\partial \theta_{\beta,j}}{\partial v_\gamma} \frac{\partial \hat{S}_{\alpha,k}}{\partial v_\gamma} \ . \tag{1.199}$$

[10] This is possible due to specific properties of the integral representation and of the special solution v_* of the string equation, for more details, see [79].

Because the expressions of $\hat{S}_{\alpha,k}$ and $\theta_{\beta,j}$ differ for odd and even k, j we consider four computations separately.

First case. Here we compute the correlation function for a field with even k.

$$\sum_{\beta,j} \frac{l^{\beta/q+j+1}}{\Gamma(\beta/q+j+1)} \langle \hat{\mathcal{O}}_{\alpha,2k} \cdot \mathcal{O}_{\beta,j} \rangle =$$

$$= \sum_{\beta,m} \frac{l^{\beta/q+2m+1}}{\Gamma(\beta/q+2m+1)} \langle \hat{\mathcal{O}}_{\alpha,2k} \cdot \mathcal{O}_{\beta,2m} \rangle + \sum_{\beta,m} \frac{l^{\beta/q+2m+2}}{\Gamma(\beta/q+2m+2)} \langle \hat{\mathcal{O}}_{\alpha,2k} \cdot \mathcal{O}_{\beta,2m+1} \rangle \ . \tag{1.200}$$

To compute the first summand in (1.200) we expand it using (1.199)

$$\langle \hat{\mathcal{O}}_{\alpha,2k} \cdot \mathcal{O}_{\beta,2m} \rangle = \sum_{\gamma=1}^{q-1} \int_0^{v_1^*} \mathrm{d}(-v_1/q)^\gamma \, \frac{\partial S_{\beta,2m}}{\partial v_\gamma} \frac{\partial \hat{S}_{\alpha,2k}}{\partial v_\gamma} \ . \tag{1.201}$$

We use expressions from [81]:

$$\begin{aligned} \frac{\partial S_{\beta,2m}}{\partial v_\gamma} &= -\delta_{\beta,\gamma}/c_{\beta,2m} \frac{\partial \theta_{\beta,2m}}{\partial v_\gamma} = \frac{\Gamma(\alpha/q+2m+1)}{\Gamma(\alpha/q+m)m!} x^m \ , \\ \frac{\partial \hat{S}_{\alpha,2k}}{\partial v_\gamma} &= \delta_{\alpha,\gamma} \, x_0^k \, P_k^{(0,\alpha/q-1)}(2x/x_0 - 1) \ , \end{aligned} \tag{1.202}$$

where $x := (-v_1/q)^q$ and $x_0 := (-v_1^*/q)^q$, and explicit formula for Jacobi polynomial:

$$P_k^{(0,\beta)}(2z-1) = \frac{1}{k!} z^{-\beta} \partial_z^k \left[z^{\beta+k}(1-z)^k \right] \ . \tag{1.203}$$

When we plug all this expressions into (1.208), we get:

$$\frac{\Gamma(\alpha/q+2m+1)}{\Gamma(\alpha/q+m)m!} \frac{x_0^{m+k+\alpha/q}}{k!} \int_0^{x_0} \mathrm{d}(x/x_0)\,(x/x_0)^m \partial_{\frac{x}{x_0}} \left[(x/x_0)^{\alpha/q+k} \left(1-(x/x_0)^k\right) \right] \ . \tag{1.204}$$

After using Leibniz rule k times, the last integral becomes beta function integral

$$\int_0^1 \mathrm{d}x \partial_x \left[x^{\beta+k}(x-1)^k \right] = \frac{k!(m-k+1)_k}{(\beta+m)_{k+1}} \ . \tag{1.205}$$

Inserting it into the formula (1.204) we obtain

$$\langle \hat{\mathcal{O}}_{\alpha,2k} \cdot \mathcal{O}_{\beta,2m} \rangle = \frac{\Gamma(\alpha/q+2m+1) x_0^{m+k+\alpha/q}}{\Gamma(\alpha/q+m+k+1)\,(m-k)!} \ . \tag{1.206}$$

Finally, summing over m with weight $l^{\alpha/q+2m}/\Gamma(\alpha/q+2m+1)$ and changing the summation variable $m \to m+k$ we get

$$\langle \hat{\mathcal{O}}_{\alpha,2k} \rangle^{\rm disk} = (x_0^{1/2})^{\alpha/q+2k} \sum_{m=0}^{\infty} \frac{(l x_0^{1/2})^{\alpha/q+2k+2m}}{\Gamma(\alpha/q+2k+m+1)\, m!} = (x_0^{1/2})^{\alpha/q+2k} I_{\alpha/q+2k}(2l x_0^{1/2}) \ . \tag{1.207}$$

Now we compute the second summand from (1.200):

$$\langle \hat{\mathcal{O}}_{\alpha,2k} \cdot \mathcal{O}_{\beta,2m+1} \rangle = \sum_{\gamma=1}^{q-1} \int_0^{v_1^*} \mathrm{d}(-v_1/q)^{\gamma} \, \frac{\partial S_{\beta,2m+1}}{\partial v_\gamma} \frac{\partial \hat{S}_{\alpha,2k}}{\partial v_\gamma} \ , \tag{1.208}$$

where

$$\frac{\partial S_{\beta,2m+1}}{\partial v_\gamma} = -\delta_{q-\beta,\gamma} \, \frac{\Gamma(\alpha/q+(2m+1)+1)}{\Gamma(\alpha/q+m+1)\, m!} x^{m+\alpha/q} \ . \tag{1.209}$$

Analogous to the previous case we get:

$$\frac{\Gamma(\alpha/q+(2m+1)+1)}{\Gamma(\alpha/q+m+1)\, m!} \frac{x_0^{m+k+1}}{k!} \int_0^{x_0} \mathrm{d}(x/x_0)\, (x/x_0)^{m+1-\alpha/q} \partial_{\frac{x}{x_0}} \left[(x/x_0)^{\alpha/q+k} \left(1-(x/x_0)^k\right) \right] \ . \tag{1.210}$$

We notice, that this expression is analytic in μ and therefore should be disregarded in the expression as non-universal. However we proceed with computation of this non-universal part because it gives interesting results. Computing the integral above with same formula we obtain:

$$\langle \hat{\mathcal{O}}_{\alpha,2k} \cdot \mathcal{O}_{\beta,2m+1} \rangle = -\frac{\Gamma(1-\alpha/q+2m+2)\, x_0^{m+k+1}}{\Gamma(-\alpha/q+m-k+1)\, (m+k+1)!} \ . \tag{1.211}$$

Finally we sum over m and perform a variable shift $m \to m-k-1$:

$$\sum_{m=k+1}^{\infty} -(x_0^{1/2})^{-\alpha/q-2k-1} \frac{(l x_0^{1/2})^{-\alpha/q-2k-1+2m}}{\Gamma(-\alpha/q-2k-1+2m+1)\, m!} \ . \tag{1.212}$$

We note, that if in the definition of the loop (1.195) we add regular terms, that is to consider the summation range from $-\infty$ to ∞ treating differentiation wrt negative times as multilication by conjugated time, or as a pseudodifferential equation, then the result in (1.207) will not change whereas the formula (1.212) the summation will be from 0 to ∞, yielding Bessel function

$$(1.212) = -(x_0^{1/2})^{\alpha/q+2k}\, I_{-\alpha/q-2k}(2l x_0^{1/2}) \ . \tag{1.213}$$

When we add up both the contributions we get

$$\langle \hat{\mathcal{O}}_{\alpha,2k} \rangle^{\rm disk} = \frac{2\sin(\alpha\pi)}{\pi} (x_0^{1/2})^{\alpha/q+2k} K_{\alpha/q}(2l x_0^{1/2}) \ , \tag{1.214}$$

which coincides with the FZZ expression.

Second case. In this paragraph we compute

$$\langle\hat{\mathcal{O}}_{\alpha,2k+1}\rangle^{\text{disk}} = = \sum_{\beta,m} \frac{l^{\beta/q+2m+1}}{\Gamma(\beta/q+2m+1)}\langle\hat{\mathcal{O}}_{\alpha,2k+1}\cdot\mathcal{O}_{\beta,2m}\rangle + \sum_{\beta,m} \frac{l^{\beta/q+2m+2}}{\Gamma(\beta/q+2m+2)}\langle\hat{\mathcal{O}}_{\alpha,2k+1}\cdot\mathcal{O}_{\beta,2m+1}\rangle \ . \tag{1.215}$$

The situation is analogous to the even case: pairings with fields $\mathcal{O}_{\beta,2m+1}$ are non-analytic and the ones with $\mathcal{O}_{\beta,2m+1}$ are analytic. First we compute the non-analytic part.

$$\langle\hat{\mathcal{O}}_{\alpha,2k+1}\cdot\mathcal{O}_{\beta,2m+1}\rangle = \sum_{\gamma=1}^{q-1}\int_0^{v_1^*} \mathrm{d}(-v_1/q)^{\gamma}\,\frac{\partial S_{\beta,2m+1}}{\partial v_\gamma}\frac{\partial \hat{S}_{\alpha,2k+1}}{\partial v_\gamma} \ , \tag{1.216}$$

We again use the formula derived in [81] from the diagonality condition for two point functions:

$$\frac{\partial \hat{S}_{\alpha,2k+1}}{\partial v_\beta} = \delta_{\alpha,q-\beta}\, x_0^k x^{\alpha/q} P_k^{0,\alpha/q}(2x/x_0-1) \ . \tag{1.217}$$

Formula (1.216) becomes:

$$\frac{-\Gamma(\alpha/q+2m+2)}{\Gamma(\alpha/q+m+1)\,m!\,k!}\int_0^{x_0} x_0^{\alpha/q+m+1}\mathrm{d}x\, x^m\partial_x^k\left[x^{\alpha/q+1+k}(1-x)^k\right] \ . \tag{1.218}$$

Non-analytic part the correlator becomes

$$-(x_0^{1/2})^{\alpha/q+2k+1}\sum_{m=0}^{\infty}\frac{(lx_0^{1/2})^{\alpha/q+2k+1+2m}}{\Gamma(\alpha/q+2k+1+m+1)\,m!} = -(x_0^{1/2})^{\alpha/q+2k+1} I_{\alpha/q+2k+1}(2lx_0^{1/2}) \ . \tag{1.219}$$

By the same argument, the nonanalytic part is equal to

$$\sum_{m\geq 0}\frac{\Gamma(1-\alpha/q+2m+1)}{\Gamma(1-\alpha/q+m)\,m!\,k!}x_0^{m+k+1}\int_0^1 \mathrm{d}x\, x^{m-\alpha/q}\partial_x^k\left[x^{\alpha/q+k}(1-x)^k\right] \ , \tag{1.220}$$

which evaluates to

$$(x_0^{1/2})^{\alpha/q+2k+1}\sum_{m\geq k+1}\frac{(lx_0)^{-\alpha/q-2k-1_2 m}}{\Gamma(m-\alpha/q-2k-1+1)\,m!} \ . \tag{1.221}$$

Similarly to the even case we see that up to a small mismatch in first $k+1$ terms this coincides with the Bessel function. If we add regular terms to the loop definition, we get

$$\langle\hat{\mathcal{O}}_{\alpha,2k+1}\rangle^{\text{disk}} = \frac{2\sin(\alpha\pi)}{\pi}(x_0^{1/2})^{\alpha/q+2k+1}K_{\alpha/q+2k+1}(2lx_0^{1/2}) \ . \tag{1.222}$$

Chapter 2

Critical Strings

2.1 Introduction

In this chapter we discuss the questions related with the critical string theory. Superstring theory in 10 dimensions is free of conformal anomaly and the path integral over metrics in the worldsheet formulation reduces to finite dimension integration over conformal classes of the metric. In particular, the critical string theory does not contain Liouville CFT which makes the theory significantly simpler. It is critical string theory which has nice low-energy target space supergravity limits which can be understood and are the common way to relate to phenomenology.

In the worldsheet formulation the critical superstring theory one starts from a N=(2,2) supersymmetric "matter" CFT with a total central charge 15 which is cancelled by the N=2 (b, c, β, γ) ghost system central charge -15. In the classical compactification scenario the matter theory is a product of a linear sigma model into the flat Minkowski space and a Calabi-Yau threefold nonlinear sigma model. In principle, any N=2 superconformal theory with appropriate central charge could give a superstring background.

In this chapter we study a bunch of $N = (2, 2)$ superconformal field theories. We start our study with the discussion of super Landau-Ginzburg theories. Ground states dynamics of N=2 Landau-Ginzburg theories is governed by a holomorphic function W called a superpotential. When W is weighted homogeneous, the theory can be made conformal and is a candidate for a superstring background. In the first section we study not only conformal Landau-Ginzburg theories but also their massive deformations which are still $N = 2$ supersymmetric theories.

$N = 2$ supersymmetric theories can be topologically twisted [5].Topologically twisted theory is related with the dynamics of the ground states of the theory and can be computed exactly in principle. Topological theories have topological deformations. Deformed topological correlation functions are expanded into series of topological gravity correlation functions of the undeformed theory. All these topological correlation finctions satisfy many constraints. Among them the most important ones are the celebrated WDVV equations [85].Mathematical formalization of the structure of deformations of topological field theories (TFT) and, in particular, WDVV equations led to the introduction of Frobenius manifolds in [1].

A Frobenius manifold is flat Riemann manifold with multiplication structure in its tangent spaces. Tangent spaces are identified with the state spaces of TFTs. The flat Riemann metric is a topological two-point function which is invariant under deformations and therefore flat. The multiplication structure is the operator product in the state spaces identified with operators. The WDVV equations are associativity of this multiplication and integrability of the structure constants of the mutltiplication. Frobenius manifolds also have a pair of distinguished vector fields: a unit vector field which corresponds to an indentity operator of the TFT and the Euler vector field which corresponds to a scaling vector field in the conformal case.

Remarkably, for the Landau-Ginzburg theories essentially the Frobenius manifold structure was known under a name of flat structure and appeared in the works [26, 27, 28, 86]dedicated to complex geometry of Milnor fibrations of singularities.

Being a natural example of Frobenius manifolds, flat structures correspond to Frobenius manifold structures on families of topologically twisted Landau-Ginzburg theories. The singularity in the flat structure formulation is a Landau-Ginzburg superpotential. The deformations of topological LG are deformations of the superpotential by the chiral fields. Via mirror symmetry this class of Frobenius manifolds is connected to a second important class which arises from enumerative geometry: quantum cohomology, FJRW and related theories.

In the first sections of this chapter we explain how to construct a Frobenius manifold from a topological LG theory. The main result of these sections is a construction of weak primitive forms or primitive forms without metric ([30], in preparation) which correspond to F-manifolds [87] or flat structures without metric [88].

Another important quantity from N=2 supersymmetric theories which can be compted exactly is quasi-topological. It is called tt^* geometry or topological-antitopological fusion. There is a choice in topological twisting of the N=2 theory which leads to either topological or antitopological theory. The tt^* metric is a two-point function of topologically twisted and anti-topologically twisted observables on a stretched sphere. It is related with a N=2 Hilbert space two-point function of the N=2 theory.

tt^* geometry appeared in [89, 4] and was put in the mathematical context in [90]. It is an additional Hermitian metric on a holomorphic Frobenius manifold which is integrable (in a sense of zero-curvature equations) and is compatible with the Frobenius manifold structure. It is a very imprtant object since it computes the natural Hermitian pairing in the Hilbert space of the N=2 theory. However, it is usually difficult to compute, more complicated then the Frobenius manifold structure itself.

In the case of conformal only deformations both Frobenius manifold structure and the tt^* metric significantly simplify and quite often can be computed explicitly. The following sections of this chapter are devoted to computations of these structures in the special cases connected with 4d superstring compactifications. Particular Landau-Ginzburg orbifolds of central charge 9 lead to such compactifications. Restriction of the Frobenius manifold structure with tt^* geometry onto the conformal deformation spaces of such Landau-Ginzburg orbifolds defines another nice mathematical structure which is called special Kähler geometry and which appeared as the geometry of coupling constants of 4d N=2 supergravity.Special Kähler geometry is related to variations of polarized Hodge structures of particular type. We study special geometry on nonlinear Calabi-Yau sigma models moduli spaces which comes from superstring compactifications and on conformal deformations of Landau-Ginzburg orbifolds which come from the restriction of Frobenius manifold structure with tt^* geometry for Landau-Ginzburg theories deformation spaces.

We use a version of Landau-Ginzburg Calabi-Yau correspondence and compute special geometries on many Calabi-Yau complex structures moduli spaces using simpler computations of special geometries on Landau-Ginzburg orbifolds deformation spaces. Mathematically this correspondence is the relation between period integrals and complex oscillatory exponential integrals or, more generally, the relation between singularity theory and complex geometry.

We finish the chapter with a mirror symmetric computation. Certain supersymmetric Gauge Linear Sigma Models (GLSM) interpolate between non-linear sigma models and Landau-Ginzburg theories. Such theories have supersymmetric backgrounds on a round sphere. In [91, 92] the authors computed partition functions for such theories in the round sphere backgrounds. It was conjectured [93] and physically proved [94] that the sphere partition function computes the exponent of the Kähler potential of the quantum corrected special geometry metric on a Kähler deformations moduli space of a vacuum manifold of the GLSM (when it is in the geometric phase). In the last section we check the mirror version of this conjecture by direct construction of the mirror pair, mirror map and computations of the both sides. The localization formulas give a convenient tool to analyze special geometry.

In all these cases the physical correlation functions of the ground ring operators are computed using Frobenius manifold structure with tt^* metric or its particular cases and limits. Frobenius manifolds compute the string genus zero contributions to correlation functions. The all genera contributions should be connected with full integrable hierarchies of which the Frobenius manifolds are dispersionless limits. In the semisimple (massive) cases the whole hierarchy can be uniquely reconstructed from the genus zero part [82] by various constructions. In the interesting conformal case, however, the higher genera contributions are more subtle. There are partial results in special cases such as higher genus Gromow-Witten invariants and holomorphic anomaly equations [95].The author is not aware of systematic approaches to the higher genera reconstruction problem in the conformal case (see [96, 97, 98] for some partial results).

2.2 Landau-Ginzburg theories

N=(2,2) Landau-Ginzburg theories form an important class of 2d supersymmetric QFT and are connected to many interesting mathematical structures. This class of theories is quite well understood an it is relatively simple to compute the ground ring correlaiton functions. Mathematically, Landau-Ginzburg vacua are described by the singularity theory of the LG superpotential.

Consider a d=2 superalgebra on a Minkowski spacetime with coordinates t and s. It has four supercharges $Q_\pm, \overline{Q}_\pm,\ Q^*_\pm = \overline{Q}_\pm$ in addition to the d=2 Poincaré generators. The supercharges are spinors on the worldsheet and their commutation relations are

$$\begin{aligned}
&\{Q_\pm, \overline{Q}_\pm\} = H \pm P = -2i\partial_\pm, \\
&\{Q_+, Q_-\} = Z, \\
&\{\overline{Q}_-, \overline{Q}_+\} = \tilde{Z},
\end{aligned} \tag{2.1}$$

where $Z, \tilde{Z}$ are the central charges and $\partial_\pm = \partial_t \pm \partial_s$. The Landau-Ginzburg theory can be conveniently formulated in the superfield language. The worldsheet is transformed into a supermanifold with 4 additional odd coordinates $\theta^\pm, \overline{\theta^\pm},\ (\theta^\pm)^* = \overline{\theta^\pm}$.

In the superfield formalizm the supercharges are

$$\begin{aligned}
Q_\pm &= \frac{\partial}{\partial\theta^\pm} + i\overline{\theta^\pm}\partial_\pm, \\
\overline{Q}_\pm &= -\frac{\partial}{\partial\overline{\theta^\pm}} - i\theta^\pm\partial_\pm.
\end{aligned} \tag{2.2}$$

Their (anti-)commutator is

$$\{Q_\pm, \overline{Q}_\pm\} = -2i\partial_\pm. \tag{2.3}$$

The supercharges commute with the superderivatives

$$\begin{aligned}
D_\pm &= \frac{\partial}{\partial\theta^\pm} - i\overline{\theta^\pm}\partial_\pm, \\
\overline{D_\pm} &= -\frac{\partial}{\partial\overline{\theta^\pm}} + i\theta^\pm\partial_\pm.
\end{aligned} \tag{2.4}$$

The superderivatives are convenient to write irreducible representations of N=2 superalgebra (supermultiplets). The chiral and antichiral operators are defined by the super(anti-)holomorphicity conditions [1]

$$\begin{aligned} \overline{D_\pm}\Phi_i &= 0, \\ D_\pm\overline{\Phi_i} &= 0. \end{aligned} \tag{2.5}$$

The N=(2,2) supersymmetric Landau-Ginzburg theory action is built from chiral and antichiral fields:

$$S_{LG} = \int_{S^+S^-\Sigma} \mathrm{d}^2 z\, \mathrm{d}^2\theta \overline{\mathrm{d}^2\theta}\ K(X^i, \overline{X^j}) + \int_{S^+\Sigma} \mathrm{d}^2 z\, \mathrm{d}^2\theta\ W(X^i) + \text{h.c.}, \tag{2.6}$$

where $K(x, \bar{x})$ is a real function of its arguments which is interpreted as a Kähler potential of the metric on the target space of scalars of the chiral fields. $W(x)$ is a holomorphic function called a superpotential. The first term in the Lagrangian is called a D-term and is not universal (it does not affect the ground ring dynamics) and the superpotential term is called an F-term.

If the superpotential is weighted homogeneous

$$W(\lambda^{q_i} x^i) = \lambda W(x^i), \tag{2.7}$$

then there exists a D-term such that the theory is superconformal. We will consider only the theories with a finite number of vacua, that is all critical points $\mathrm{d}W = 0$ are isolated.

The chiral superfield can be expanded in the odd coordinates

$$X^i(z) = \phi(y^\pm) + \theta^\alpha \psi_\alpha(y^\pm) + \theta^2 F(y^\pm), \tag{2.8}$$

where $y^\pm := x^\pm - i\theta^\pm\overline{\theta^\pm}$ is a superholomorphic coordinate, that is $\overline{D^\pm} y^\pm = 0$.

After integrating out odd coordinates and auxilary terms $F(x)$ one computes the component Lagrangian. The potential term of the scalars is $|\mathrm{d}W|^2$, so the critical points of the superpotential are classical vacua of the scalars of the chiral fields.

The chiral fields form a ring with respect to operator multiplication due to the Leibniz rule. This ring is called a chiral ring $\mathcal{R}$ of the theory. The equations of motion are

$$\partial_i W(X_j) = -\overline{D^- D^+}\partial_i K(X, \bar{X}). \tag{2.9}$$

In particular, $\partial_i W(X)$ vanish in the chiral ring so that $\mathcal{R}$ is isomorphic to a factor of the polynomial ring generated by the elementary chiral fields X_i with respect to derivatives of the superpotential:

$$\mathcal{R} = \frac{\mathbb{C}[X_1, \ldots, X_n]}{(\partial_1 W, \ldots, \partial_n W)}. \tag{2.10}$$

[1]There are different conventions for supecharges and superderivatives in the literature.

Mathematically, the chiral ring $\mathcal{R}$ is a Milnor or Jacobi ring of the singularity $W(x)$. Elements of the chiral ring are deformations of the superpotential which cannot be undone by coordinate transformations.

In the conformal case every chiral field has a U(1) R-charge and $W(x)$ is weighted homogeneous with respect to this R-charge. The smallest degree element of the chiral ring is $e_0 := 1$ and the largest degree element is $e_\rho := \operatorname{Hess} W = \det(\partial_i \partial_j W)$. Let us normalize the weights q_i of the fields X_i so that $wt(W) = 1$. Then the central charge of the corresponding CFT is

$$c = 3 \sum_{i=1}^{n} (1 - 2q_i). \tag{2.11}$$

The weight of e_ρ is one third of the central charge.

Topological twist Topological twist can be done in any d=2 N=2 theory. Let Q_+ and Q_- be a pair of supercharges with the commutation relations

$$\begin{aligned} (Q_+)^2 = (Q_-)^2 = 0, \\ \{Q_+, Q_-\} = H. \end{aligned} \tag{2.12}$$

The topological twist shifts the spins of $Q_\pm$ by 1/2 so that one of them, say Q_+, becomes a scalar and Q_- becomes a 1-form. Since $Q_+^2 = 0$ it can be interpreted as a BRST operator, so that the physical fields are defined to be Q_+ cohomology.

$$Q_+ \Phi = 0, \quad \Phi \sim \Phi + Q_+ \chi. \tag{2.13}$$

In each class there is a harmonic representative fixed by the condition $Q_- \Phi = 0$. The chiral ring is precisely a state space of the topologically twisted theory.

$$|i\rangle = \Phi_i |0\rangle + Q_+(\cdots). \tag{2.14}$$

Energy-momentum tensor is Q_+ exact, thus the correlators of the twisted theory do not depend on the metric. In particular, the correlation functions can be defined on any Riemann surface.

One can pick an appropriate D-term [99] such that the Landau-Ginzurg theory effectively becomes one-dimensional or just supersymmetric quantum mechanics. The two-point function of the twisted theory can be computed in this regime

$$\eta_{ij} = \langle i | j \rangle = \operatorname{Res} \frac{\Phi_i(x)\, \Phi_j(x)\, \mathrm{d}^n x}{\partial_1 W \cdots \partial_n W}, \tag{2.15}$$

where the residue is a Grothendieck residue symbol which denotes a contour integral around zeros of the denominator.

The Milnor ring endowed with the bilinear pairing η_{ij} is a Frobenius algebra.

The structure constants with lowered indices are given by

$$C_{ijk} = \langle i|\Phi_j|k\rangle = \mathrm{Res}\frac{\Phi_i(x)\,\Phi_j(x)\,\Phi_j(x)\,\mathrm{d}^n x}{\partial_1 W\cdots\partial_n W}. \tag{2.16}$$

The theory can be deformed by arbitrary chiral fields

$$\mathcal{L}_{lg} \to \mathcal{L}_{lg} + \sum_i t^i \int \mathrm{d}^2\theta\,\Phi_i + \mathrm{h.c.}, \tag{2.17}$$

where the deformation parameters t^i define a point in the moduli space $\mathcal{M}$. Such a deformation is equivalent to a deformation of the superpotential $W(x,t)$ such that $W(x,0) = W(x)$. In what follows we will denote the undeformed superpotential by $W(x,0) =: W_0(x) = W_0$ and save W for the deformed one.

The ground ring form a vector bundle $\mathcal{H} \to \mathcal{M}$ over the deformation space. Each fiber of this bundle is isomorphic to the chiral ring through

$$\frac{\partial}{\partial t^i} \to \frac{\partial}{\partial t^i} W(x,t) =: \partial_i W. \tag{2.18}$$

The deformation space $\mathcal{M}$ becomes a Frobenius manifold with tt^* structure. The topological metric and multiplication are given by the topological pairing and chiral operator product in $\mathcal{H} \simeq T\mathcal{M}$. The exact formulas for these structure can be complicated and require a more careful analysis. The flat basis in $\mathcal{H}$ does not coincide with the standart linear deformations basis and its computation leads to a notion of *primitive forms* by K. Saito [26, 27, 28, 29].

Vacuum bundle and Berry connection It turns out convenient to perform the dimensional reduction of the LG theory and proceed in the supersymmetric quantum mechanics language. Consider a deformation family of Landau-Ginzburg theories with the superpotentials

$$W(x,t) = W_0(x) + \sum_{i=1}^{\mu} t^i \Phi_i, \tag{2.19}$$

where $\Phi_i = \Phi_i(x)$ form a basis of the Milnor ring of W_0, $\mu = \dim\mathcal{R}$ is a Milnor number, and the deformation parameters $\{t_i\}$ define a point in the deformation space $\mathcal{M}$.

Let $\{|a(t,\bar{t})\rangle\}_{a=1}^{\mu}$ form a basis of the ground states of the theory for each t. The full Hilbert space is a trivial bundle over $\mathcal{M}$, whereas the vacuum bundle $\mathcal{H} := \mathrm{span}(|a(t,\bar{t})\rangle)_{a=1}^{\mu}$ varies inside the Hilbert space. There is an induced connection on $\mathcal{H}$ which is called a *Berry connection*:

$$\frac{\partial}{\partial t^i}|a(t,\bar{t})\rangle = A_{ia}^b|b(t,\bar{t})\rangle, \tag{2.20}$$

which can be written as a covariant derivative $D_i e_j = \partial_i e_j - A^k_{ij} e_k$. The Berry connection is a projection of the deformed ground state back to the ground state space.

Chiral and antichiral rings define two natural "holomorphic" and "antiholomorphic" bases in the vacuum bundle $\mathcal{H}$ by the following.

Consider a disk with the geometry of a long stretched sigar and insert a (anti-)chiral field Φ_i ($\overline{\Phi_i}$) in the tip of the sigar. One can define a path integral of (anti-)topologically twisted theory in this background which is equivalent to a physical theory on the flat space. The path integral determines a state on the boundary of the sigar and this state is projected to a ground state when the sigar becomes infinitely long.

We call the corresponding states in the ground ring $|i\rangle$ ($|\bar{j}\rangle$). In particular, there is always a distinguished ground state $|0\rangle$ which corresponds to an identity operator. The action of the chiral ring on the vacuum bundle in this basis coincides with the action of the chiral ring on itself

$$\Phi_i \Phi_j = C^k_{ij} \Phi_k, \quad \Phi_i |j\rangle = C^k_{ij} |k\rangle. \tag{2.21}$$

The Berry connection is holomorphic in the basis $\{|i\rangle\}$.

The topological pairing is

$$\eta_{ij} = \langle i|j\rangle. \tag{2.22}$$

It can be computed using the path integral over a stretched sphere which is glued out of two infinite sigars with Φ_i and Φ_j inserted on the tips of the sigars. Since the correlator is purely topological, the precise metric on the sphere does not matter.

Since holomorphic and antiholomorphic bases both span the same groud state space, they are connected by an invertible matrix

$$\overline{|j\rangle} = M^i_{\bar{j}} |i\rangle. \tag{2.23}$$

This matrix of complex conjugation is called a real structure matrix. It accounts for the fact that the actual real structure in the Hilbert space does not coincide with the naive real structure in the basis $|i\rangle$.

The actual Hilbert space two-point function is

$$g_{i\bar{j}} := \langle i|\bar{j}\rangle = \langle i|M^k_{\bar{j}}|k\rangle = \eta_{ik} M^k_{\bar{j}}. \tag{2.24}$$

is computed by a path integral on a stretched sphere which is glued out of two infinite sigars 2.2. However, in this case one sigar is topologically twisted and the other is antitopologically twisted, so the geometry of the sphere does matter.

One of the ways to compute η_{ij} and $g_{i\bar{j}}$ uses the quantum mechanics reduction of the Landau-Ginzburg theory. The Hilbert space is a space of L^2 functions of bosonic and fermionic variables

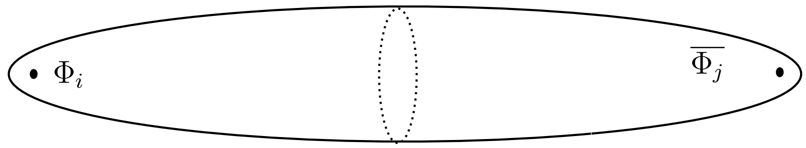

Figure 2.2.1: Topological-antitopological fusion

$$\omega(x)_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_q} \psi^{i_1} \cdots \psi^{i_p} \overline{\psi^{j_1}} \cdots \overline{\psi^{j_q}}, \tag{2.25}$$

which are identified with the differential forms on the target space $\mathbb{C}^n$ by the rule $\psi^i \to \mathrm{d}x^i$ and $\overline{\psi^j} \to \overline{\mathrm{d}x^j}$. The 1d theory has four supercharges $Q_\pm, \bar{Q}_\pm$ which are identified with the twisted Dolbeault operators

$$\begin{aligned} \bar{Q}_+ &= \bar{\partial} + \partial W \wedge \quad Q_+ = Q_+^\dagger, \\ Q_- &= \partial + \overline{\partial W} \wedge \quad \bar{Q}_- = Q_-^\dagger. \end{aligned} \tag{2.26}$$

The Hamiltonian coincides with the twisted Laplace-Beltrami operator

$$H = \Delta_W = \{\bar{Q}_+, Q_+\} = \{\bar{Q}_-, Q_-\}. \tag{2.27}$$

N=2 superalgebra commutation relations translate into the Kähler identities for the twisted Dolbeault operators.

The ground states are annihilated by all these supercharges and, therefore, are harmonic forms. The vacuum bundle $\mathcal{H}$ is identified with the kernel on Δ_W.

If $\Phi_i \in \mathbb{C}[\bar{x}]/(\partial W)$ is a representative of the chiral ring, then the corresponding chiral $|i\rangle$ ground state is represented by a harmonic form

$$\omega_i = \Phi_i \, \mathrm{d}^n x + \bar{Q}_+ \eta_i = \Phi_i \, \mathrm{d}^n x + (\bar{\partial} + \partial W \wedge) \eta_i \tag{2.28}$$

for a 4-form η_i. From this representation it is clear that the real structure on the ground ring (which is induced from the real structure on the Hilbert space) is computed as $\omega_i = M_i^{\bar{j}} \omega_{\bar{j}}$, where $\omega_{\bar{j}}$ are harmonic forms representing antichiral ground states.

To compute the correlation functions we note that if ω is harmonic with respect to Δ_W, then the following forms are d-closed

$$\begin{aligned} &e^{W+\bar{W}} \omega, \\ &e^{-W-\bar{W}} \star \omega, \end{aligned} \tag{2.29}$$

where $\star\omega_k$ is the Hodge dual form. The d-closed differential forms can be integrated over cycles such that the integral converges and the integrals are invariant under deformations of the cycles. Such cycles are relative cohomology elements $\Gamma^\pm \in H_n(\mathbb{C}^n, \mathrm{Re}(\pm W) \ll 0)$, that is the unbounded cycles which go to infinity in the regions where the exponent $e^{\pm W \pm \bar{W}}$ vanishes.

The integrals

$$\int_{\Gamma^+} e^{W+\bar{W}}\omega \tag{2.30}$$

are identified with one-point functions on a disk with a boundary condition given by the brane Γ^+.

In the SUSY QM formalizm the two-point functions in the QM setting are

$$\begin{aligned} \eta_{ij} &= \int_{\mathbb{C}^n} \star\omega_j \wedge \omega_i, \\ g_{i\bar{j}} &= \int_{\mathbb{C}^n} \star\omega_j^* \wedge \omega_i. \end{aligned} \tag{2.31}$$

The topological metric is computed to coincide with the residue pairing of the holomorphic parts:

$$\eta_{ij} = \mathrm{Res}\frac{\Phi_i \Phi_j \mathrm{d}^n x}{\partial_1 W \cdots \partial_n W}. \tag{2.32}$$

Both metrics, being spherical two-point functions, can be decomposed into a pairing of disk one-point functions

$$\begin{aligned} \eta_{ij} &= \sum_{a=1}^{\mu} \int_{\Gamma_+^a} e^{W+\bar{W}}\omega_i \int_{\check{\Gamma}_-^a} e^{-W-\bar{W}} \star \omega_j, \\ g_{i\bar{j}} &= \sum_{a=1}^{\mu} \int_{\Gamma_+^a} e^{W+\bar{W}}\omega_i \overline{\int_{\check{\Gamma}_-^a} e^{-W-\bar{W}} \star \omega_j}, \end{aligned} \tag{2.33}$$

where Γ_+^a and $\check{\Gamma}_-^a$ are dual cycles which represent branes which preserve different linear combinations of supercharges. There is an intersection pairing on the relative homology groups which is given by the geometrical intersection and $\Gamma_+^a \cap \check{\Gamma}_-^b = \delta^{ab}$.

2.2.1 Frobenius manifolds and tt^* geometry

The holomorphic metric η_{ij}, Hermitian metric $g_{i\bar{j}}$ and multiplication in the chiral ring which is identified with the tangent space $T\mathcal{M}$ defines a structure of Frobenius manifold with a tt^* metric on $\mathcal{M}$. In fact, topological field theories and WDVV equations were the original motivation to introduce Frobenius manifolds.

Definition 2.2.1. *Frobenius manifold is a flat Riemannian manifold $(\mathcal{M}, \eta)$ such that each of its tangent spaces is a Frobenius algebra and the following integrability condition is satisfied:*

$$[\nabla_i - z^{-1}C_i, \nabla_j - z^{-1}C_j] = 0, \tag{2.34}$$

where ∇_i is a flat Levi-Civita connection for η_{ij} and $C_i = \partial_i *$ is a multiplication operator in the tangent spaces: $\partial_i * \partial_j = C_{ij}^k \partial_k$. The variable z is a formal parameter.

In addition, there should exist two special vector fields on $\mathcal{M}$:

1. *The identity vector field ∂_1 such that $C_{1i}^j = \delta_i^j$ and $\nabla(\partial_1) = 0$.*

2. *An Euler vector field E with the following properties:*

   1) The grading operator $Q := \nabla E$ is covariantly constant, that is $\nabla\nabla E = 0$.

   2) The homogeneity property: the one-parametric diffeomorphism group generated by E should act as conformal transformations of the metric η and rescale the Frobenius algebras in the tangent spaces.

The connection $\nabla_i - z^{-1}C_i$ can be understood as a deformation family of flat connections with the parameter z or as a flat connection on $\pi^*T\mathcal{M}$, where $\pi : \mathcal{M} \times \mathbb{C}^* \to \mathcal{M}$ is a natural projection.

The flat sections of the connection $\nabla_i - z^{-1}C_i$ can be identified with the brane amplitudes in the holomorphic limit. The parameter z is related with the particular choice of supercharges which is preserved by the brane.

The integrability condition can be rewritten in the classical WDVV form:

$$\begin{aligned} \partial_i C_{jk}^l &= \partial_j C_{ik}^l, \\ C_{ij}^k C_{kl}^m &= C_{jl}^k C_{ik}^m, \end{aligned} \tag{2.35}$$

that is associativity and integrability equations. The Frobenius algebra condition is equivalent to that

$$C_{ijk} := C_{ij}^l \eta_{lk} \tag{2.36}$$

is symmetric in all indices. The integrability condition implies that locally there exists a prepotential F for the structure constants:

$$C_{ijk} = \nabla_i \nabla_j \nabla_k F. \tag{2.37}$$

The Euler vector field conditions can be written in coordinates as

$$\begin{aligned} \nabla_i(\nabla_j E^k) &= 0, \\ \mathcal{L}_E c_{ij}^k &= c_{ij}^k, \\ \mathcal{L}_E \partial_1 &= -\partial_1, \\ \mathcal{L}_E \eta_{ij} &= D\eta_{ij} \end{aligned} \tag{2.38}$$

for some constant $D = 2 - d$. $\mathcal{L}_E$ denotes the Lie derivative along the vector field E. In the interesting examples the Euler vector field expresses the (weighted) homogeneity of the Frobenius manifold.

Two important classes of Frobenius manifolds include Frobenius manifold structures on unfoldings of singularities and quantum cohomology. The first class is the one which is studied in this paper and examples of this class are also called B-models. The quantum cohomology, FJRW theory and related enumerative geometry theories, in turn, are called A-models. The mirror symmetry identifies B-models with A-models. In the language of Frobenius manifolds the mirror symmetry states that Frobenius manifolds appearing on deformation spaces of singularities are connected with Frobenius manifolds appearing in enumerative geometry.

tt^* **geometry** Frobenius manifolds themselves are purely holomorphic. The B-model Frobenius manifolds apriory contain only information about chiral sectors of Landau-Ginzburg theories. The tt^* metric is a Hermitian metric on a Frobenius manifold which is an actual ground ring metric in the underlying N=2 supersymmetric theory. We briefly describe the mathematical structure following [90].

Consider a Hermitian metric $g_{a\bar{b}}$ on a Frobenius manifold $(\mathcal{M}, \eta)$. Let D_a be a Chern connection for $g_{a\bar{b}}$, that is

$$\begin{aligned} &D_a v^b = \partial_a v^b + \Gamma^b_{ac} v^c, \\ &D_{\bar{a}} v^b = \overline{\partial_a} v^b, \\ &D_{\bar{c}} = \overline{D_c}, \\ &D_a g_{a\bar{b}} = 0 \end{aligned} \tag{2.39}$$

The metric $g_{a\bar{b}}$ is called compatible with η_{ab} if the Chern connection D_a annihilates η_{ab}:

$$D_a \eta_{bc} = 0. \tag{2.40}$$

The Christoffel coefficients are given by the formula:

$$D_a = \partial_a + \Gamma_a, \quad \Gamma_a = g^{-1} \partial_a g, \quad g = (g_{\bar{a}b}). \tag{2.41}$$

The real structure tensor is defined as

$$M^b_{\bar{a}} := g_{\bar{a}c} \eta^{cb}. \tag{2.42}$$

This tensor satisfies the condition

$$M\bar{M} = const\, 1. \tag{2.43}$$

If the constant is equal to 1, the compatible pair $(g_{\bar{a}b}, \eta_{ab})$ is called a normalized compatible pair. The tensor M defines a complex conjugation in tangent space of $\mathcal{M}$ by the rule

$$\overline{v^a \partial_a} = M^b_{\bar{a}} \partial_b \overline{v^a}. \tag{2.44}$$

Definition 2.2.2. *The normalized compatible pair* $(g_{\bar{a}b}, \eta_{ab})$ *defines a topological- antitopological fusion or a tt* structure on the Frobenius manifold* $(\mathcal{M}, \eta)$ *if the following pensil of connections (tt* Lax connection) is flat:*

$$\begin{aligned} D^\lambda_v w &= D_v w - \lambda v \star w, \\ D^\lambda_{\bar{v}} w &= D_{\bar{v}} w - \lambda^{-1}\overline{v \star \overline{w}}, \end{aligned} \tag{2.45}$$

where the complex conjugation is defined using the formula (2.44)*, in particular*

$$C_{\bar{b}} = M\overline{C_b}\bar{M}. \tag{2.46}$$

The integrability conditions can be rewritten in the form

$$\begin{aligned} D_a C_b &= D_b C_a, \\ [D_a, \overline{D_b}] &= -[C_a, C_{\bar{b}}], \end{aligned} \tag{2.47}$$

2.3 Primitive forms

The theory of primitive forms physically can be thought of as a holomorphic description of deformations of Landau-Ginzburg theories. Originally it appeared in the works of K.Saito [26, 27, 28, 29] in the study of period mappings related with Milnor fibrations of singularities.

Let us briefly describe this theory in the Landau-Ginzburg language. We consider a *holomorphic limit* of the constructions in the end of the previous section which turns out to capture all the structure of the chiral sector but loses information about the *tt** metric.

We follow the discussion in [100]. Consider a nonunitary deformation of the LG superpotential $(W, \bar{W}) \to (\lambda^{-1}\beta W, \lambda\beta\bar{W})$ and send $\lambda \to 0$ while keeping $z = \lambda/\beta$ finite. Then the relevant supercharges become

$$\bar{Q}_+ = \bar{\partial} + z^{-1}\partial W \wedge \quad Q_- = \partial \tag{2.48}$$

The vacum wave-forms also depend on z and are annihilated by these operators. For all z we consider a vacuum form of the type

$$\omega = \Omega_1 + (\bar{\partial} + \partial W/z\wedge)\eta_1, \tag{2.49}$$

where $\Omega_1 = \sum_{i=1}^{\mu} c^i\, \Phi_i\, \mathrm{d}^n x$ is a holomorphic top form such that c^i are λ-independent and $\{[\Phi_i]\}_{i=1}^{\mu}$ form a basis of the chiral ring $\mathcal{R}$. The one-point function in the holomorphic limit is then

$$\begin{aligned} \lim_{z\to 0} \int_\gamma e^{W/z}\omega &= \int_\gamma e^{W/z}(\Omega_1 + (\bar{\partial} + \partial W/z)\eta_1 = \\ &= \int_\gamma e^{W/z}\Omega_1 + \int_\gamma \left[\mathrm{d}\left(e^{W/z}\eta_1\right) + e^{W/z}\partial\eta_1\right]. \end{aligned} \tag{2.50}$$

In the right hand side the total derivative vanish and the second term can be recursively rewritten as an integral of a purely holomorphic form. Indeed, $\omega = \Omega_1 + \bar{Q}_+\eta_1$ is killed by $Q_- = \partial$. Since Ω_1 is a top holomorphic form $\partial\Omega_1 = 0$ and therefore $\partial\bar{Q}_+\eta_1 = 0$. The operators $Q_- = \partial$ and $\bar{Q}_+$ anticommute, therefore $\bar{Q}_+\partial\eta_1 = 0$, that is $\partial\eta_1$ is an element of $\bar{Q}_+$-cohomology and is killed by ∂, so we can recursively apply the procedure above to

$$\partial\eta_1 = \Omega_2 + \bar{Q}_+\eta_2. \tag{2.51}$$

Moreover, since $\partial\eta_1$ does not have a $(0,n)$ component, we can choose η_2 to not have $(0,n-1)$ component and the recursion terminates after n steps. We get $\partial\eta_i = \Omega_{i+1} + \bar{Q}_+\eta_{i+1}$. The one point function becomes

$$\lim_{z\to 0}\int_\gamma e^{W/z}\omega = \int_\gamma e^{W/z}\omega^h, \quad \omega \to \omega^h, \tag{2.52}$$

where $\omega^h := \sum_{i\le n}\Omega_i$ is a holomorphic form. The deformation parameter z which can be interpreted as a generalized $U(1)_V$ R-symmetry rotation.

The holomorphic limit of the Landau-Ginzburg theory and its deformations is reduced to a study of complex oscillatory integrals

$$\int_\gamma e^{W/z}\omega^h, \tag{2.53}$$

where ω^h is a holomorphic $n-$form which makes the integral convergent.

2.3.1 Filtered de-Rham cohomology module and Gauss-Manin connection

The Stokes formula implies

$$0 = \int_{\Gamma_z} \mathrm{d}(e^{W/z}\alpha) = \int_{\Gamma_z} e^{W/z}(\mathrm{d}\alpha + z^{-1}\mathrm{d}W\wedge\alpha). \tag{2.54}$$

The oscillatory integrands naturally belong to the cohomology group of the twited de-Rham operator

$$D_z := z\mathrm{d} + \mathrm{d}W = e^{-W/z}\mathrm{d}e^{W/z}. \tag{2.55}$$

It is convenient to treat z as a formal variable or as a pseudo-differential operator. Let us explain our mathematical setting.

Let $Z = X\times S \subset \mathbb{C}^n\times\mathbb{C}^\mu$ be an open subset in the space with coordinates $(x_1,\dots,x_n,t^1,\dots,t^\mu)$. The Landau-Ginzburg superpotential $W_0(x)$ is a function $\mathbb{C}^n\to\mathbb{C}$ and the deformed superpotential $W(x,t)$ is called a universal unfolding $W : \mathbb{C}^n\times\mathbb{C}^\mu\to\mathbb{C}$ of W_0. In particular, $\partial_{t^i}W$ span the corresponding Milnor ring. We consider the case where at each t the critical points of W are isolated so the Milnor ring is finite-dimensional.

The separation of coordinates into x and t is somewhat arbitrary, because it depends on a choice of basis in the chiral ring for each t. This implies that the de-Rham differential d_X is not invariant and the correct differential is the *relative de-Rham differential* $\mathrm{d} := \mathrm{d}_{Z/S}$. The corresponding complex which substitutes $(\Omega^*_X, \mathrm{d}_X)$ is $(\Omega^*_{Z/S}, \mathrm{d})$, where the relative differential forms are defined as differential forms on Ω_Z modulo $\mathrm{d}t^i$

$$\Omega^*_{Z/S} := \frac{\Omega^*_Z}{\mathrm{d}t^i \wedge \Omega^{*-1}_Z}. \tag{2.56}$$

The filtered relative de-Rham cohomology group is defined as [2]

$$\mathcal{H}_W = \frac{\Omega^n_{Z/S}((z))}{D_z \Omega^{n-1}_{Z/S}((z))}. \tag{2.57}$$

This space has a semi-infinite filtration in the formal variable z:

$$\cdots \subset \mathcal{H}^{(i)}_W \subset \mathcal{H}^{(i+1)}_W \subset \cdots \tag{2.58}$$

such that

$$\bigcup_{i\in\mathbb{Z}} \mathcal{H}^{(i)}_W = \mathcal{H}_W, \quad \bigcap_{i\in\mathbb{Z}} \mathcal{H}^{(i)}_W = \varnothing, \tag{2.59}$$

where each filter is

$$\mathcal{H}^{(0)}_W := \frac{\Omega^n_{Z/S}[[z]]}{D_z \Omega^{n-1}_{Z/S}[[z]]}, \mathcal{H}^{(i)}_W := z^{-i}\mathcal{H}^{(0)}_W. \tag{2.60}$$

Since the differential $D_z = z\mathrm{d} + \mathrm{d}W$ mixes differential forms with different powers of z, there is no natural splitting of this filtration.

Each graded component is isomorphic to the chiral ring of the singularity:

$$\mathcal{H}^{(0)}_W / \mathcal{H}^{(-1)}_W \simeq \Omega_W = \frac{\Omega^n_{Z/S}}{\mathrm{d}W \wedge \Omega^{n-1}_{Z/S}} \simeq \mathcal{R}\, \mathrm{d}^n x, \tag{2.61}$$

where the first isomorphism is given by setting $z = 0$.

The splittng of the filtration (2.58) is an injective reverse map

$$\Omega_W \overset{\iota}{\hookrightarrow} \mathcal{H}^{(0)}_W \tag{2.62}$$

such that its image $\mathcal{B} := \iota(\Omega_W)$ projects back $\mathcal{B} \overset{z=0}{\simeq} \Omega_W$ isomorphically.

[2] All other cohomology groups vanish for isolated singularities.

The subspace $\mathcal{B}$ is called a *section*[3] in the theory of primitive forms. In the presence of a section the filtration (2.58) splits:

$$\mathcal{H}_W = \bigoplus_{\in\mathbb{Z}} \mathcal{B} z^i. \tag{2.63}$$

From the point of view of Landau-Ginzburg theory the splitting is given by the ground states wave-forms. Namely, if $\{\omega_k\}_{k\le\mu}$ form a chiral basis of the ground states, then the formula (2.52) which sends a wave form ω_k to a holomorphic form ω_k^h with the same oscillatory integral integral defines a map from the chiral ring $\mathcal{R}\,\mathrm{d}^n x \simeq \Omega_W$ to a space of oscillatory integrands $\mathcal{H}_z^{(0)}$ which is a section.

From the holomorphic perspective apriory there is no canonical choice of a section. It turns out that there is a finite dimensional family of so-called *good sections* [29] which correspond to topological field theories or allow to define a Frobenius manifold on the base space S of the unfolding. In particular, a section coming from the Landau-Ginzburg theory is certainly a good section.

Gauss-Manin connection Consider the Landau-Ginzburg theory again.

For the brane amplitudes (or oscillatory integrals) the deformations with respect to chiral ring elements are related to the product in the chiral ring. This is expressed as flatness with respect to the tt^* Lax connection. In the holomorphic limit the condition reads

$$\frac{\partial}{\partial t_i} \int_{\Gamma_z} e^{W/z} \omega_j^h = z^{-1} C_{ij}^k \int_{\Gamma_z} e^{W/z} \omega_k^h. \tag{2.64}$$

The equation above implies that the one-parametric family of connections $\nabla_i - z^{-1}C_i$ is flat on the space S of deformation parameters. According to the definition (2.2.1), such a family (with some additional assumptions) define a Frobenius manifold structure on S. The brane amplitudes $\int_{\Gamma_z} e^{W/z}\omega_j^h = \sum_{k\ge 0} \theta_{j,k} z^{-k}$ are called the deformed flat coordinates on S. In particular, $\theta_{j,0}$ are the flat coordinates for the Frobenius manifold metric.

The differentiation of the oscillatory integrals can be rephrased in terms of the so-called Gauss-Manin connection. The Gauss-Manin connection is a connection on the filtered de-Rham cohomology group $\mathcal{H}_W$ which can be considered as a bundle of topological gravity states over S. The Gauss-Manin (GM) connection on the dual space of cycles $H_n(\mathbb{C}^n, \mathrm{Re}(W/z) \ll 0)$ is a connection which is flat on the families of integral cycles over S. In the homology group z should be considered as an analytic variable.

GM connection acts on the oscillatory integrals as partial derivatives with respect to the deformation variables t^i. From this description one can derive the formula in

[3] There is a related notion of an opposite filtration $\mathcal{L}$ which is equivalent to a section via $\mathcal{L}_{\mathcal{B}} = z^{-1}\mathcal{B}[z^{-1}]$ and $\mathcal{B}_{\mathcal{L}} = z\mathcal{L} \cap \mathcal{H}_W^{(0)}$. In this text we will use the language of sections.

coordinates. Let $\omega = [\phi(x,t,z)\,\mathrm{d}^n x] \in \mathcal{H}_W$, then:

$$\begin{aligned}\nabla_i \omega &= e^{-W/z}\partial_{t^i}\, e^{W/z}\omega = (\partial_{t^i}\phi + z^{-1}(\partial_{t^i}W)\,\phi)\,\mathrm{d}^n x,\\ \nabla_z \omega &= e^{-W/z}\partial_z\, e^{W/z}\omega = (\partial_z \phi - z^{-2} W\,\phi)\,\mathrm{d}^n x.\end{aligned} \tag{2.65}$$

A defining property of good sections $\mathcal{B}$ will be that the analogue of the equation (2.64) is satisfied. Good sections can be generated from Gauss-Manin derivatives of a form which is called a primitive form ζ. A choice of a good section [4] is equivalent to a choice of a primitive form.

Let us try to pick a naive section to show what we need from a good section. Let $W(x,t) = W_0(x) + \sum_{i\leq\mu} t^i \Phi_i$ and pick a naive "primitive form" $\mathrm{d}^n x$. Then the section $\mathcal{B}$ is spanned by $\nabla_i \mathrm{d}^n x = \Phi_i\,\mathrm{d}^n x$. The derivative of the exponential period (brane amplitude) is given by

$$\partial_i \int_{\Gamma_z} e^{W/z}\Phi_j\,\mathrm{d}^n x = z^{-1}\int_{\Gamma_z} e^{W/z}\Phi_i\,\Phi_j\,\mathrm{d}^n x = (*) \tag{2.66}$$

We use the relation in the function ring

$$\Phi_i\,\Phi_j = C_{ij}^k \Phi_k + \sum_{\alpha\leq n} B_{ij}^{\alpha}(x,t)\partial_{x_\alpha} W \tag{2.67}$$

to compute (2.66)

$$(*) = z^{-1} C_{ij}^k \int_{\Gamma_z} e^{W/z}\Phi_k\,\mathrm{d}^n x + z^{-1}\int_{\Gamma_z} e^{W/z} B_{ij}^{\alpha}\partial_{x_\alpha} W\,\mathrm{d}^n x. \tag{2.68}$$

The second integral is equal to

$$z^{-1}\int_{\Gamma_z} e^{W/z}\partial_\alpha B_{ij}^{\alpha}\,\mathrm{d}^n x \tag{2.69}$$

via the Stokes formula. We can decompose

$$\partial_\alpha B_{ij}^{\alpha}\,\mathrm{d}^n x = \sum_{k\leq\mu}(\Gamma^0)_{ij}^k \Phi_k + (B^1)_{ij}^{\alpha}\partial_\alpha W = \sum_{a\geq 0} z^a \sum_{j\leq\mu}(\Gamma^a)_{ij}^k \Phi_k. \tag{2.70}$$

using our section $\mathcal{B}$. If the only nonvanishing component in the series is $(\Gamma^0)_{ij}^k$, then it can be interpreted as Christoffel symbols of a connection $\nabla\!\!\!/$ on TS by the formula

$$\nabla\!\!\!/_i e_j = \partial_{t^i} e_j - (\Gamma^0)_{ij}^k e_k. \tag{2.71}$$

[4] In the case where $W(x,t)$ is not weighted homogeneous there is no canonical choice of a volume form for $W_0(x)$ and the notion of good section is replaced with a notion of a good pair. A good pair consists of a good section and a volume form on $W_0(x)$ satisfying some condition.

This connection is flat. Indeed, the tangent space to S is identified with $\mathcal{B}$ by the rule $\partial_i \to z\nabla_i \mathrm{d}^n x$ and by the construction

$$[(\nabla\!\!\!/_i + z^{-1}\partial_{t^i} W)\Phi_j \, \mathrm{d}^n x] = 0. \tag{2.72}$$

This equation in cohomology can be integrated over a Lefschetz thimble Γ_z for any fixed z:

$$\partial_i \int_{\Gamma_z} e^{W/z}\Phi_j \, \mathrm{d}^n x + (\Gamma_{ij}^k + z^{-1}C_{ij}^k) \int_{\Gamma_z} e^{W/z}\Phi_k \, \mathrm{d}^n x = 0, \tag{2.73}$$

that is the connection $\nabla\!\!\!/_i + z^{-1}\partial_{t^i} W$ is integrable. Expanding the integrability condition in z we check that $\nabla\!\!\!/$ is also integrable.

In the case where $(\Gamma^a)_{ij}^k \neq 0$ for $a > 0$ the construction of a flat connection above does not work and one has to consider a different section, a so-called good section to compensate the "higher Christoffel symbols" $(\Gamma^{(a)})_{ij}^k$. From this perspective the primitive form can be thought of as a renormalized form $\mathrm{d}^n x$ where the correction terms are respondible for annihilating all the higher coefficients $(\Gamma^a)_{ij}^k$.

Good sections Let us understand the condition on a good section if we want an analogue of (2.73) to be satisfied. Let $\{\Phi_i \, \mathrm{d}^n x\}_{i \leq \mu}$ be a basis of a section $\mathcal{B}$, where $\Phi_i(x,t,z) \, \mathrm{d}^n x \in \}$ are t and z-dependent cohomology classes. We want our section to satisfy (2.73) to be a good section such that the osillatory integrals define a Frobenius manifold structure on the space S of deformations. This condition is satisfied if there are no "higher Christoffel symbols" $(\Gamma^a)_{ij}^k$, that is if

$$\partial_i \int_{\Gamma_z} e^{W/z}\Phi_j \, \mathrm{d}^n x \in \mathbb{C}[\bar{t}] + \mathbb{C}[\bar{t}]z^{-1} \tag{2.74}$$

or, in the cohomological notation:

$$\nabla_i \mathcal{B} \subset \mathcal{B} \oplus z^{-1}\mathcal{B}, \tag{2.75}$$

that is all the positive powers of z vanish. The first coefficient is responsible for the Christoffel symbols and the second contains the multiplication structure constants. This condition on $\mathcal{B}$ is called a holonomicity condition and is the most nontrivial one.

It will be convenient to rewrite the holonomicity condition in yet another form. The space $\mathcal{H}_W^{(0)} = \bigoplus_{k \geq 0} \mathcal{B} z^k$ plays a role Let us define an negative subspace $\mathcal{H}_W \supset \mathcal{L} := \bigoplus_{k>0} \mathcal{B} z^{-k}$. Obviously, there is a direct sum decomposition

$$\mathcal{H}_W = \mathcal{H}_W^{(0)} \oplus \mathcal{L}. \tag{2.76}$$

Then the holonomicity condition can be rewritten as a stability condition on $\mathcal{L}$:

$$\nabla_i \mathcal{L} \subset \mathcal{L}. \tag{2.77}$$

The construction below solves this condition, that is we show how to build a section $\mathcal{B}$ satisfying the holonomicity condition. We start from a (weighted homogeneous) section $\mathcal{R}_0 \simeq \mathcal{B}_0 \subset \mathcal{H}_{W_0}^{(0)}$ whose different choices define different solutions to (2.73).

We build $\mathcal{B}$ from $\mathcal{B}_0$ using the Gauss-Manin connection. Let $\omega_0 \in \mathcal{H}_{W_0}$. Then $e^{-\sum_s t^s e_s/z}\omega_0 \in \mathcal{H}_W$ is a Gauss-Manin flat section:

$$\partial_i \int_{\Gamma_z} e^{W(x,t)/z} e^{-\sum_s t^s e_s/z}\omega_0 = \partial_i \int_{\Gamma_z} e^{W_0(x)/z}\omega_0 = 0, \tag{2.78}$$

that is

$$\nabla_i \left(e^{-\sum_s t^s e_s/z}\mathcal{B}_0\right) \subset e^{-\sum_s t^s e_s/z}\mathcal{B}_0. \tag{2.79}$$

The Gauss-Manin flat sections $\omega := e^{-\sum_s t^s e_s/z}\omega_0$ have essential singularities at $z = 0$ and cannot be used as sections, since the sections must be regular at $z = 0$ (semiclassical approximation).

To solve this problem we decompose the space of oscillatory integrands $\mathcal{H}_W$ into the positive and negative parts with respect to the z-grading:

$$\mathcal{H}_W = \mathcal{H}_W^{(0)} \oplus \mathcal{L}, \tag{2.80}$$

where $\mathcal{L}$ is called an opposite filtration and is defined as a Gauss-Manin flat continuation of the negative part of the decomposition at 0:

$$e^{-\sum_S t^s e_s/z}\left(\bigoplus_{k>0} \mathcal{B}_0 z^{-k}\right). \tag{2.81}$$

This is $\mathcal{L}$ consists of differential forms whose integrals do not contain non-negative powers of z:

$$\int_{\Gamma_z} e^{W/z}\,\omega \in \bigoplus_{k>0} \mathbb{C}[\bar{t}]z^{-k}, \quad \omega \in \mathcal{L}. \tag{2.82}$$

The important property of $\mathcal{L}$ is that by its definition it is stable under the Gauss-Manin connection ∇_i:

$$\nabla_i \mathcal{L} = \nabla_i(e^{-\sum_S t^s e_s/z}\bigoplus_{k>0} \mathcal{B}_0 z^{-k}) = (e^{-\sum_S t^s e_s/z} z^{-1}\partial_i \bigoplus_{k>0} \mathcal{B}_0 z^{-k}) \subset \mathcal{L}. \tag{2.83}$$

This is an analogue of the holonomicity condition in the form (2.77). To finish the construction we need to build a section $\mathcal{B} \subset z\mathcal{L}$ such that $\mathcal{L} = \bigoplus_{k>0} \mathcal{B}z^{-k}$.

Using the decomposition (2.80) we can define the positive parts of the Gauss-Manin flat sections ω:

$$e^{-\sum_S t^s e_s/z}\omega_0 = \omega_+ + \omega_-, \tag{2.84}$$

where $\omega_+ \in \mathcal{H}_W^{(0)}$ and $\omega_- \in \mathcal{L}$. This equation is equivalent to

$$\int_{\Gamma_z^0} e^{W_0/z}\omega_0 = \int_{\Gamma_z} e^{W/z}\omega_+ + O(z^{-1}). \tag{2.85}$$

We define a section $\mathcal{B}$ to be a positive part of the Gauss-Manin flat continuation of $\mathcal{B}_0$:

$$\mathcal{B} := \left(e^{-\sum_s t^s e_s}\mathcal{B}_0\right)_+, \tag{2.86}$$

where $(\cdot)_+$ stands for the positive part in the decomposition (2.80). Indeed, the condition

$$\mathcal{L} = \bigoplus_{k>0} \mathcal{B} z^{-k} \tag{2.87}$$

holds true, since $\left(e^{-\sum_s t^s e_s}\mathcal{B}_0\right)_-$ is already an element of $\mathcal{L}$ and vanishes when $\bar{t} \to 0$.

Let $\{\Phi_i^+ \, \mathrm{d}^n x\}_{i\leq\mu}$ form a basis of a good section $\mathcal{B}$. The condition (2.83) implies that

$$\partial_i \int_{\Gamma_z} e^{W/z}\, \Phi_j^+ \, \mathrm{d}^n x + (\Gamma_{ij}^k + z^{-1} C_{ij}^k) \int_{\Gamma_z} e^{W/z}\, \Phi_k^+ \, \mathrm{d}^n x = 0 \tag{2.88}$$

for some coefficients Γ_{ij}^k which define a flat connection on S.

The most difficult part in computations of Φ_i^+ is the decomposition (2.80).

Primitive forms A primitive form ζ_+ is an image of 1 in the isomorphism $\mathcal{R} \to \mathcal{B}$ of the Milnor ring and the section. All other elements of $\mathcal{B}$ can be obtained as Gauss-Manin derivatives of ζ_+. In the weighted homogeneous case the primitive form is also weighted homogeneous. It is equal to (up to a constant)

$$\zeta_+ = \left(e^{-\sum_s t^s e_s} \mathrm{d}^n x\right)_+ \tag{2.89}$$

since the restriction of a primitive form to the weighted homogeneous case $\bar{t} = 0$ is proportional to the unique volume form of the minimal weight $\mathrm{d}^n x$. Using the formula (2.88) we recover all other elements of $\mathcal{B} = \{\Phi_i^+ \, \mathrm{d}^n x\}_{i\leq\mu}$:

$$\Phi_i^+ \, \mathrm{d}^n x = z\nabla_i \zeta_+. \tag{2.90}$$

This property is called primitivity and hence the name *primitive form*.

Let us list the properties a primitive form should have to define a Frobenius manifold structure or a flat structure of the deformation space S [26].

A form $\zeta \in \mathcal{H}_W^{(0)}$ is called a weak primitive form if $\{z\nabla_i\zeta\}$ form a basis of a section $\mathcal{B}_\zeta$ and the following conditions hold true:

1. Flatness (holonomicity) condition

$$\begin{aligned} z^2 \nabla_i \nabla_j \zeta &\in \mathcal{B}_\zeta \oplus z\mathcal{B}_\zeta, \\ z^2 \nabla_z \nabla_i \zeta &\in z^{-1}\mathcal{B}_\zeta \oplus \mathcal{B}_\zeta. \end{aligned} \tag{2.91}$$

2. Primitive direction flatness
$$z\partial_1\zeta = \zeta, \tag{2.92}$$
where $\Phi_1 = 1$.

3. Homogeneity. There exists a constant $r \in \mathbb{C}$ such that
$$\nabla_{z\partial_z + E}\zeta = r\zeta, \tag{2.93}$$
where E is the Euler vector field which is equal to the class of W in the chiral ring. In the case where W is weighted homogeneous, the last condition is equivalent to the fact that ζ is weighted homogeneous.

These conditions imply, that
$$z^2\nabla_i\nabla_j\zeta = zC_{ij}^k\nabla_k\zeta + z^2\nabla_{\nabla\!\!\!/_i\partial_j}\zeta \tag{2.94}$$
for a flat connection $\nabla\!\!\!/_i$ on TS.

Construction of weak primitive forms We show how to construct weak primitive forms for a given universal unfolding W in more details. To simplify the discussion we consider the weighted homogeneous case. It is convenient to construct weak primitive forms in the formal setting. Namely, we consider a power series completion $\check{\mathcal{H}}_W = \mathcal{H}_W\hat{\otimes}\mathbb{C}[[t^1,\ldots,t^\mu]]$ of $\mathcal{H}_W$ and work in the formal neighbourhood of $0 \in S$.

First we need to pick a good section and a primitive form for $W_0(x)$. In the weighted homogeneous case the good section needs to be weighted homogeneous as well, that is it has to have a weighted homogeneous basis. The weight if z is equal to 1 so that the exponential has weight 0.

We take $\{\Phi_i\}_{i\leq\mu}$ to be an ordered weighted homogeneous basis of the Milnor ring $\mathcal{R}_0$ of $W_0(x)$. A good section projects to Ω_{W_0} isomorphically, so the preimage $\Phi_i^z\,\mathrm{d}^n x$ of the basis $\Phi\,\mathrm{d}^n x$ of Ω_{W_0} is a basis of $\mathcal{B}_0$:
$$\{\Phi_i^z\,\mathrm{d}^n x := (\Phi_i + \sum_{j\leq i} m_{ij}\, z^{wt(i)-wt(j)}\Phi_j)\,\mathrm{d}^n x\}_{i\leq\mu}, \tag{2.95}$$
where in the sum above only terms with $wt(i) - wt(j) \in \mathbb{Z}_{>0}$ are nonvanishing. m_{ij} is a constant matrix with complex coefficients which is responsible for the moduli of the primitive form.

Consider a formal power series expansion [5] of the oscillatory integral
$$\int_{\Gamma_z} e^{W(x,t)/z}\omega = \int_{\Gamma_z} e^{W_0(x)/z+\sum_{s\leq\mu}t^s\Phi_s/z}\omega = \sum_{k\geq 0}\frac{1}{z^k k!}\int e^{W_0(x)}\left(\sum_{s\leq\mu}t^s\Phi_s\right)^k\omega. \tag{2.96}$$

[5] This expansion depends on the projection $Z \to X$, that is on how we split the coordinates x and t.

One can think of $e^{\sum_{s\le\mu} t^s\Phi_s/z}$ as an operator from $\mathcal{H}_{W_0}$ to to $\check{\mathcal{H}}_{W_0} = \mathbb{C}[[t^1,\dots,t^\mu]]\hat{\otimes}\mathcal{H}_{W_0}$: which is defined as

$$A := e^{\sum_{s\le\mu} t^s\Phi_s/z}\,\omega := \left[\sum_{k\ge 0}\frac{(\sum_{s\le\mu} t^s\Phi_s)^k}{z^k k!}\,\omega\right]. \tag{2.97}$$

This operator intertwines the Gauss-Manin connection with the trivial connection:

$$\frac{\partial}{\partial t^i}\left(e^{\sum_{s\le\mu} t^s\Phi_s/z}\,\omega\right) = e^{\sum_{s\le\mu} t^s\Phi_s/z}\,\nabla_i\omega \tag{2.98}$$

Consider its inverse

$$A^{-1} := e^{-\sum_{s\le\mu} t^s\Phi_s/z} = \sum_{k\ge 0}\frac{(-\sum_{s\le\mu} t^s\Phi_s)^k}{z^k k!}. \tag{2.99}$$

One can check that both A and A^{-1} converge in the topology of formal power series in t^i with coefficients at Laurent polynomials in z. We note, that as power of t grows, the negative power of z is unbounded which reflects the fact that $e^{W/z}$ has an essential singularity at $z=0$.

The constant differential forms $\mathcal{H}_{W_0} \subset \mathbb{C}[[t^1,\dots,t^\mu]]\hat{\otimes}\mathcal{H}_{W_0}$ are killed by ∂_i. Therefore, by the intertwining property, if $\alpha \in \mathcal{H}_{W_0}$, then $e^{-\sum_{s\le\mu} t^s\Phi_s/z}\,\alpha$ is a Gauss-Manin flat section.

The section $\mathcal{B}_0$ gives a splitting

$$\mathcal{H}_{W_0} = \mathcal{H}_{W_0}^{(0)} \oplus \mathcal{L}_0, \tag{2.100}$$

where $\mathcal{L}_0 := z^{-1}\mathcal{B}_0[z^{-1}]$. The operator $A^{-1} : \mathcal{H}_{W_0} \to \check{\mathcal{H}}_W$ maps $\mathcal{H}_{W_0}^{(0)}$ to $\check{\mathcal{H}}_W^{(0)}$ and defines a splitting

$$\check{\mathcal{H}}_W = \check{\mathcal{H}}_W^{(0)} \oplus \check{\mathcal{L}}, \tag{2.101}$$

where $\check{\mathcal{L}} = \mathbb{C}[[t^1,\dots,t^\mu]]\hat{\otimes}A^{-1}(\mathcal{L})$.

The weak primitive form belongs to $\check{\mathcal{H}}_W^{(0)}$. The primitive form built from a good section $\mathcal{B}_0$ at $t=0$ is a positive part of the Gauss-Manin flat section of $\mathrm{d}^n x$ in $\check{\mathcal{H}}_W^{(0)}$.

Proposition 2.3.1 (Construction of weak primitive forms)**.** *In the notations of this section let*

$$e^{-\sum_{s\le\mu} t^s\Phi_s/z}\,\mathrm{d}^n x = \zeta_- + \zeta_+, \tag{2.102}$$

where $\zeta_- \in \check{\mathcal{L}}$ and $\zeta \in \mathcal{H}_{W_0}^{(0)}$. Then ζ_+ is a (formal) weak primitive form.

Idea of proof. The form $\zeta_+ = \mathrm{d}^n x + O(t)$ by definition and therefore satisfies the primitivity property. It is weighted homogeneous since $\mathrm{d}^n x$ and A is homogeneous and homogeneity is respected by the splitting.

The holonomicity is essentially the formulas (2.83), (2.75) and (2.90) in the formal setting.

□

The formula (2.73) defines a flat connection $\nabla\!\!\!/$ on TS with flat coordinates s^i, that is ∂_{s^i} are the flat sections of $\nabla\!\!\!/$. The structure constants are integrable and have the form

$$C_{ij}^k = \partial_{s_i}\partial_{s_j}V^k \tag{2.103}$$

for a locally defined vector potential V^k. Such a structure on S is called an F-manifold [87] or a Saito structure without metric [88].

Remark 2.3.1. *In fact, the correspondence $\mathcal{B}_0 \to \zeta_+$ is bijective (in weighted homogeneous case) since given a (weak) primitive form, the restriction of its Gauss-Manin derivatives to $\bar{t} = 0$ spans a good section Bc_0.*

Higher residue pairing and Frobenius manifolds In the construction above there was no nice compatible metric on TS to turn S into a Frobenius manifold. Let $\alpha, \beta \in \mathcal{H}_W^{(0)}$ and an involution $* : \mathcal{H}_W \to \mathcal{H}_W$ acts by sending $z \to -z$. Let us pick a set of dual cycles $\Gamma^i_{\pm z} \in H_n(\mathbb{C}^n, \mathrm{Re}(\pm W/z) \ll 0)$ dual with respect to the intersection pairing [6].

Then the following "two-point function" pairing on $\mathcal{H}_W^{(0)}$

$$K_W(\alpha,\beta) := \sum_i \int_{\Gamma^i_z} e^{W(x,t)/z}\alpha \int_{\Gamma^i_{-z}} e^{-W(x,t)/z}\beta^* = \sum_{k\geq n} z^n K_W^n(\alpha,\beta) \tag{2.104}$$

is called the *higher residue pairing* [27, 101], $K_W^k(\alpha,\beta) \in \mathbb{C}[[\bar{t}]]$. In particular, the lowest degree pairing K^n coincides with the ordinary residue pairing on $\Omega_W \simeq \mathcal{H}_W^{(0)}/\mathcal{H}_W^{(-1)}$. The higher residue pairing behaves well with respect to the Gauss-Manin connection:

$$\partial_{t^i} K_W(\alpha,\beta) = K_W(\nabla_i\alpha,\beta) + K_W(\alpha,\nabla_i\beta). \tag{2.105}$$

Consider the good section $\mathcal{B}_0$ such that

$$K_{W_0}(\mathcal{B}_0,\mathcal{B}_0) \in z^n\mathbb{C}. \tag{2.106}$$

We call (2.106) a metric condition on a good section $\mathcal{B}_0$. The higher residue pairing is invariant with respect to the Gauss-Manin connection and $\check{\mathcal{B}}$ is obtained using the map $e^{\sum_{s\leq\mu} t^s\Phi_s}$ which is a parallel transport with respect to the GM connection, therefore if $\mathcal{B}_0$ satisfies the metric condition, then

$$K_W(\check{\mathcal{B}},\check{\mathcal{B}}) \in z^n\mathbb{C}[[\bar{t}]]. \tag{2.107}$$

Since $\mathcal{B}$ is identified with TS via the primitive form $\partial_i \to z\nabla_i\zeta_+$, the formula (2.107) defines a metric on TS.

$$\eta_{ij} = \eta(\partial_i,\partial_j) = z^{-n}K_W(z\nabla_i\zeta_+, z\nabla_j\zeta_+). \tag{2.108}$$

[6]In this definition we consider z to be a comlex number and $\Gamma^i_{\pm z}$ are families of cycles.

Due to the flatness condition of the higher residue pairing (2.105) the metric η is flat.

The structure constants with lowered indices are integrable

$$C_{ijk} := C_{ij}^{l}\eta_{lk} = \nabla\!\!\!/_i \nabla\!\!\!/_j \nabla\!\!\!/_k F \tag{2.109}$$

for a locally defined function F on S which is called a prepotential. The flat metric η_{ij} on TS together with the multiplication structure $\partial_i \cdot \partial_j = C_{ij}^k \partial_k$ in TS and vector fields ∂_1 and $E = [W]$ define a Frobenius manifold structure on S.

2.3.2 Examples

A_n singularities and Gelfand-Dickey hierarchy Let us give a couple of simple examples of computation of primitive forms. The easiest case is simple singularities [28]. Consider a universal unfolding of an A_n singularity:

$$W(x,\phi) = x^{n+1} + \sum_{s=1}^{n} t^s x^{s-1}. \tag{2.110}$$

This case corresponds to the Frobenius manifolds considered in the first chapter of the current paper in connection with Minimal Liouville Gravity. The polynomial $W_0(x)$ is homogeneous and so must be the primitive form ζ_+. At $\bar{t} = 0$ we have $\zeta_0 = (\zeta_+)|_{\bar{t}=0}\mathrm{d}x$. Since all t^i and x have positive weights it follows that $\zeta_+ = \mathrm{d}x$ by the homogeneity.

The deformed flat coordinates

$$\theta_a(z) := \int_{\Gamma_a} e^{W(x,t)/z}\mathrm{d}x \tag{2.111}$$

are Hamiltonian densities of the A_n Gelfand-Dickey hierarchies (see, e.g. an Appendix in [21] for details. Our variable z should be identified with $-z^{-1}$ from [21]).

Conformal case and Calabi-Yau varieties The next important class of examples considers conformal deformations of $W_0(x) : \mathbb{C}^n \to \mathbb{C}$, that is only the deformations with parameters of weight 0 or *marginal deformations*

$$W_0(x) + \sum_s t^s \Phi_s, \quad wt(\Phi_s) = 1,\ wt(t^s) = 0. \tag{2.112}$$

The bases of such deformation families do not have Frobenius manifold structures but are certain submanifolds in Frobenius manifolds corresponding to the universal unfolding of $W_0(x)$.

In such examples it is still possitble to compute primitive forms explicitly. Since all t_i have weight 0 and all x_i have positive weight, primitive forms of the unfoldings restricted to the marginal subspace (2.112) have the form [7]

$$\zeta_+ = \frac{\mathrm{d}^n x}{f(\bar{t})}. \tag{2.113}$$

Let us compute the coefficient $f(\bar{t})$.

By the formula (2.89)

$$e^{\sum_s t^s \Phi_s / z} \zeta_+ = f(\bar{t})^{-1} e^{\sum_s t^s \Phi_s / z} \mathrm{d}^n x = \mathrm{d}^n x + O(z^{-1}). \tag{2.114}$$

Pick a homogeneous basis of a good section $\mathcal{B}_0$ (2.95) $\{\Phi_i^z = Phi_i + \sum_{j \le \mu} m_{ij} z^{wt(i)-wt(j)} \Phi_j\}_{i \le \mu}$, where Φ_i for a basis of the Milnor ring of W_0. We introduce the inverse matrix $\Phi_i = \sum_j z^{wt(i)-wt(j)} m_{ij}^{-1} \Phi_j^z$ which is also triangular. Let us compute $f(t)$.

$$e^{\sum_s t^s \Phi_s / z} \mathrm{d}^n x = \sum_{i \le \mu} [\Phi_i \, \mathrm{d}^n x] \int_{\Gamma_z^i} e^{W/z} \, \mathrm{d}^n x = \sum_{i \le \mu} z^{-wt(i)} [\Phi_i \, \mathrm{d}^n x] \int_{\Gamma_z^i} e^W \, \mathrm{d}^n x, \tag{2.115}$$

where we used homogeneity property in the last equation. Expanding Φ_i through Φ_j^z we obtain

$$\sum_{i \le \mu} \sum_{j \le \mu} m_{ij}^{-1} z^{-wt(j)} [\Phi_j^z \, \mathrm{d}^n x] \int_{\Gamma_z^i} e^W \, \mathrm{d}^n x = \sum_{i \le \mu} m_{i1}^{-1} [\mathrm{d}^n x] \int_{\Gamma_z^i} e^W \, \mathrm{d}^n x. \tag{2.116}$$

We define $\Gamma_{\mathcal{B}_0} = \sum_i \Gamma_z^i m_{i1}^{-1} \in H_n(\mathbb{C}^n, \mathrm{Re}(W) \ll 0; \mathbb{C})$. Then from (2.116) and (2.114) it follows that

$$f(\bar{t}) = \int_{\mathcal{B}_0} e^{W(x,t)} \mathrm{d}^n x. \tag{2.117}$$

In the case where $W(x,t)$ defines a Calabi-Yau hypersurface $\mathcal{X}_t$ in a weighted projective space, the integral (2.117) is equal to a period integral of a holomorphic volume form on $\mathcal{X}_t$. In particular, if $W_0(x) = x_1^3 + x_2^3 + x_3^3$, the family $W(x,t)$ defines a family of elliptic curves in $\mathbb{P}^2$ and if $W_0(x) = \sum_{i=1}^5 x_i^5$, the corresponding variety is a quintic threefold in $\mathbb{P}^4$. In the Calabi-Yau case the marginal flat coordinates coincide with the mirror map. We will discuss this particular example from another perspective below.

2.3.3 Conclusion

In this section we briefly described a construction of Frobenius manifolds and Frobenius manifolds without metric structure on the unfolding spaces of singularities. The construction of Frobenius manifolds follows closely the one in [29] but has a different

[7] In fact, this is true in the more general case where (2.112) has not only marginal, but also relevant deformations: $wt(t_s) \ge 0$.

focus. In particular, it appears as a specification of a construction of a Frobenius manifold without metric. The details of this construction will be given in a joint paper with Kyoji Saito [30].

In the weighted homogeneous case the construction starts with a good section $\mathcal{B}_0$ which is an embedding of the chiral ring into the space of oscillating integrands $\mathcal{H}_W^{(0)}$:

$$\mathcal{B}_0 := \mathrm{Span}((\Phi_i + \sum_{j \leq i} m_{ij}\, z^{wt(i)-wt(j)} \Phi_j)\, \mathrm{d}^n x)_{i \leq \mu}. \tag{2.118}$$

Such spaces are parametrized by complex matrices $\{m_{ij}\}_{i,j\leq\mu}$ whose entities are nonzero only if $wt(\Phi_i) - wt(\Phi_j) \in \mathbb{N}$. The space of good sections $\mathcal{M}^{weak}$ has the dimension is a number of so-called resonant pairs

$$\dim_{\mathbb{C}} \mathcal{M}^{weak} = \#\{(i,j) \mid wt(\Phi_i) - wt(\Phi_j) \in \mathbb{Z}_{>0}\}. \tag{2.119}$$

Each good section defines a weak primitive form using the exponential map $e^{-\sum_s t^s \Phi_s}$ and splitting of $\mathcal{H}_W$ into positive and negative subspaces defined by the section $\mathcal{B}_0$. The weak primitive form defines a Frobenius manifolds without a metric. Its flat coordinates are computed from the first nontrivial expansion coefficient of a J-function. Saito structures without metric appear in interesting mathematical problems such as unitary reflection groups [102] and we expect to find application to out construction as well.

If the good section $\mathcal{B}_0$ satisfies a metric condition as well, then S turns to an actual Frobenius manifold where the metric coincides with the higher residue pairing evaluated on the section. The J-function expansion coefficients are Hamiltonian densities of a dispersionless integrable hierarchy of the hydrodynamic type.

The metric condition on $\mathcal{B}_0$ is a quadratic condition on the matrix $\{m_{ij}\}$. The dimension of the space $\mathcal{M}$ of good sections satisfying the metric condition is

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{M} =& \#\{(i,j) \mid wt(\Phi_i) - wt(\Phi_j) \in \mathbb{Z}_{>0} \;\&\; i+j < \mu+1\} + \\ &+ \#\{(i,j) \mid wt(\Phi_i) - wt(\Phi_j) \in (2\mathbb{Z}_{>0} - 1) \;\&\; i+j = \mu+1\}. \end{aligned} \tag{2.120}$$

Different choices of good sections are expected to appear in non-unitary Landau-Ginzburg theories. Namely, if the antichiral superpotential does not coincide with the complex conjugate of the chiral superpotential, that is $\bar{U} \neq \bar{W}$, then the Schroedinger equation for the LG theory depends on U. Therefore, even in the holomorphic limit there is a non-trivial dependence of correlation functions on the antichiral superpotential U which leads to different choices of the good section $\mathcal{B}_0$. The question about connection of weak primitive forms or primitive forms without metric with Landau-Ginzburg theories is left to the future research.

2.4 Calabi-Yau moduli spaces

In the beginning of this chapter we discussed geometry of singulary theory which appears in studying Landau-Ginzburg theories. In particular, physics of vacua for genus zero Riemann surface is governed by Frobenius manifold structure with tt^* geometry. We also explained how in a certain class this geometry gives rise to (local) special Kähler geometry on marginal subspaces of Frobenius manifolds. Special Kähler geometry is the geometry of coupling constants of 4-dimensional N=2 supergravity. It was predicted long ago (e.g. [103]that such Landau-Ginzburg orbifolds are equivalent to certain non-linear sigma models to Calabi-Yau manifolds. The superpotential of a Landau-Ginzburg equation becomes a defining equation for a Calabi-Yau manifold in homogeneous coordinates. The idea of Landau-Ginzburg Calabi-Yau correspondence was studied both in physical and mathematical literature [104, 105]. We use the correspondence to compute special Kähler geometry appearing in superstring compactifications on Calabi-Yau manifolds.

2.4.1 Physical preliminaries

In the classical worldsheet approach to superstring theory it is formulated as a superconformal theory on a worldsheet of a Riemann surface. For simplicity we restrict ourselves to type II superstring theories. Then one can construct such a theory starting from an N=(2,2) supersymmetric sigma model action

$$S_{Pol} = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2 z \left(\sqrt{h}\, \partial^\mu Y^i \partial_\mu \overline{Y^j} g_{i\bar{j}}(X) + \mathrm{d}Y^i \wedge \overline{\mathrm{d}Y^j} b_{i\bar{j}}(X) + fermi. \right), \qquad (2.121)$$

where $\{Y^j(z)\}_{j=1}^{10}$ is a map from a Riemann surface to a target space $\mathbb{R}^{1,3} \times X$ with a metric g_{ij} which factorizes into a product of a flat Minkowski metric on $\mathbb{R}^{1,3}$ and a curved metric on X. Such a model is $N = (2,2)$ superconformal only if X has a covariantly constant spinor and therefore is a Calabi-Yau manifold and g_{ij} is a Ricci flat metric.

Scalar fields $Y^i(z)$ become holomorphic coordinates on X and $\mathbb{R}^{1,3} \simeq \mathbb{C}^2$. They have superpartners $\psi^i(z)$ which are worldsheet fermions. On-shell supersymmetry is extended to $N = 2$ in holomorphic and $N = 2$ in antiholomorphic sectors. These fermions can have periodic or antiperiodic boundary conditions on a circle. One says that the corresponding fermion belongs to Ramond or Neveu-Schwarz sector if it has periodic or antiperiodic boundary conditions (this depends on representation of a string worldsheet as a cylinder or as a punctured disk). In string theory with compact worldsheet one takes into account both holomorphic and anti-holomorphic sectors and fermions can be in one of the four sectors (NS, NS), (NS, R), (R, NS), (R, R).

N=2 superconformal theories have important symmetry which is called a spectral flow. (NS, R) and (R, NS) sectors correspond to twisted sectors of the spectral flow symmetry. Orbifolding this symmetry is called GSO projection and gives rise to the massless spectrum of the string theory.

In two-dimensional conformal field theories one can (almost) separate holomorphic and anti-holomorphic sectors. The massless spectrum of the holomorphic sector after GSO projection consists of 10-dimensional gauge field and its fermion superpartner. Both of them being massless transform under vector (NS-sector) and spinor (R-sector) representations of the small 10-dimensional Lorentz group which in the string thory case is $\mathfrak{so}(8)$. Dynkin diagram of $\mathfrak{so}(8)$ is $\mathbb{Z}_3$-symmetric under rotation by $2\pi/3$. In particular, both spinorial representations $8_s, 8_{s'}$ and vector representation 8_v are 8-dimensional. Their highest weights are external vertices of the Dynkin diagram.

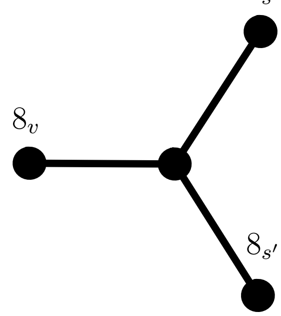

Figure 2.4.1: Dynkin diagram of $so(8)$

Supersymmetry operator is built of the spectral flow which in turn is constructed using a space-time fermion from either 8_s or $8_{s'}$. Then GSO projection leaves only $8_{s'}$ or 8_s massless fermionic states correspondingly which garantees mutual locality of vertex operators.

Massless sector of the closed string theory is obtaied by tensoring massless states from holomorphic and anti-holomorphic sectors on the worldsheet.

Let $8_a, 8_b$ denote a pair of representations of $\mathfrak{so}(8)$, where $a, b \in \{v, s, s'\}$ are different elements. Then there is the following formula for tensor product of representations (due to the symmetry of the Dynkin diagram)

$$\begin{aligned} 8_a \times 8_b &= 56 + 8, \\ 8_a \times 8_a &= 1 + 28 + 35, \end{aligned} \tag{2.122}$$

where the numbers on the right hand side denote dimensions of irreducible representations. In particular,

$$8_v \times 8_v = 1_\Phi + 28_b + 35_g \tag{2.123}$$

is a decomposition of matrices into symmetric traceless 35_g, antisymmetric 28_b and the trace 1_Φ parts.

For spinorial representations $8_b,\ b \in \{s, s'\}$ we have

$$8_v \times 8_b = 8_\lambda + 56_\psi, \tag{2.124}$$

where 56_ψ is a space of spin 3/2 Rarita-Schwinger fields of the same chirality as b and 8_λ is a Weyl fermion of the oposite chirality. Finally, tensor products of spinorial representations decompose into a sum of anti-symmetric tensor representations with the use of gamma-matrices:

$$8_s \times 8_{s'} = 1_{c_0} + 28_{c_2} + 35_{c_4^+} \tag{2.125}$$

and

$$8_s \times 8_s = 8_{c_1} + 56_{c_3}, \tag{2.126}$$

where representations on the right hand sides are antisymmetric forms and c_4^+ is a self-dual antisymmetric form.

There are two non-equivalent choices to perform GSO reduction when pairing both holomorphic and anti-holomorphic sectors. They correspond to the cases where massless fermion vertex operators belong to different spinorial representations of $\mathfrak{so}(8)$ or to the same representation.

Type IIA string theory In the first case the supersymmetry operator is a Dirac space-time fermion and is non-chiral. Such a theory is called superstring theory of type IIA or type $N = (1, 1)$. Its massless sector consists of space-time bosons (NS, NS)+(R,R) sectors and fermions from (NS,R)+(R, NS) sectors:

$$(G_{MN}, B_{MN}, \Phi, C_1, C_3) + (\Psi_{M,\alpha}, \tilde{\Psi}^\alpha_M, \lambda^\alpha, \tilde{\lambda}_\alpha). \tag{2.127}$$

In the formula above M, N are ten-dimensional Minkowski indices and α are 8-dimensional Weyl fermion indices. Fields G, B, Φ are th metric, B-field and dilaton, C_1, C_3 are (R,R) antisymmetric forms, $\Psi, \tilde{\Psi}$ are gravitinos and $\lambda, \tilde{\lambda}$ are Weyl fermions. This is precisely the field content of ten dimensional $N = (1, 1)$ or IIA supergravity theory. In fact, it can be shown that the low-energy limit of string theory indeed reproduces supergravity.

Type IIA supergravity can be obtained by dimensional reduction from eleven-dimensional supergravity. It is believed that full type IIA superstring theory similarly can be obtained from eleven-dimensional M-theory which reduces to supergravity in the low energy limit. Superstring theory backgrounds might include branes which are extended objects which serve as sources for the fields and boundary conditions for open strings. Sources for (R,R) fields are Dp branes, which are $p+1$ dimensional submanfolds in the space-time. In type IIA theory D0 and D2 branes serve as electrical charge sources for C_1 and C_3 whereas D4 and D6 branes are magnetic charge sources for the same fields. Similarly, M-theory should have M2 and M5 branes of dimensions 3 and 6.

Type IIB string theory In the second case the chiralities of fermions in holomorphic and antiholomorphic sectors are the same (a particular choice of one of two is just a convention). Such a theory is chiral. Its massless sector consists of the following fields:

$$(G_{MN}, B_{MN}, \Phi, C_0, C_2, C_4^+) + (\Psi_{M,\alpha}, \tilde{\Psi}_{M,\alpha}, \lambda_\alpha, \tilde{\lambda}_\alpha). \tag{2.128}$$

Apart from different chiralities in the fermion (NS,R)+(R,NS) sector the (R,R) bosonic forms are different. In particular, C_4^+ is a self-dual field strength and in the Lagrangian formulation self-duality should be put as a constraint. There are Dp branes which also play a role of sources of the fields. D1, D3 and D5 branes are electric and magnetic charge sources for C_2 and C_4^+.

The theory reduces to $N = (2,0)$ or IIB supergravity in the low energy limit. It cannot be obtained by dimensional reduction since it is chiral. However, it enjoys a wonderful $\mathrm{SL}(2,\mathbb{Z})$ symmetry which mixes (NS, NS) and (R,R) fields and is called S-duality.

Namely, the complex field

$$\tau := C_0 + ie^{-\Phi} \tag{2.129}$$

transforms as a modular parameter under $\mathrm{SL}(2,\mathbb{Z})$:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1. \tag{2.130}$$

S duality also mixes B with C_2 and D5 brane with the so-called NS5 brane which is magnetically charged under the B-field.

In the F-theory complexified dilaton τ can depend on the position in the spacetime and is interpreted as a modular parameter of a torus fibration over 10-dimensional spacetime.

Superstring dimensional reductions and 4-dimensional N=2 supergravity As explained above, superstring theory reduces to 10-dimensional supergravity in the low energy limit. It is only natural to consider nontrivial vacua configurations of superstring which correspond to supergravity in the curved space-time. Originally, the most interest was attracted to backgrounds of the form $\mathbb{R}^{1,3} \times X$, where X is a compact manifold.

Supergravity in such backgrounds further reduces to 4-dimensional supergravity with matter via Kaluza-Klein procedure. Let us first remind the idea of Kaluza-Klein reduction on a simple example. Consider a free scalar field in 5 dimensions with the following action

$$S[\phi] = \int_{\mathbb{R}^{1,3}\times S^1} \mathrm{d}^5 w\, \partial_M \phi \partial^M \phi = - \int_{\mathbb{R}^{1,3}\times S^1} \mathrm{d}^5 w\, \phi \Delta \phi, \tag{2.131}$$

where S^1 is a circle of radius R and Δ is a Laplace operator on $\mathbb{R}^{1,3} \times S^1$. Either in classical or in quantum setting one can expand $\phi(w)$ in eigenfunctions of Laplace operator (Fourier modes) on a compact circle. Denoting a coordinate $w = (x, y)$, where $x \in \mathbb{R}^{1,3}$ and $y \in S^1$ we obtain

$$\phi(w) = \phi(x, y) = \sum_{n \in \mathbb{Z}} \phi_n(x) \frac{e^{iny/R}}{\sqrt{2\pi R}}. \tag{2.132}$$

Plugging in this expression back to the Lagrangian and integrating over y we get

$$S[\phi] = -\sum_n \int_{R_{1,3}} dd^4x \left[\phi_n(x) \Delta_x \phi_n(x) - \frac{n^2}{R^2} (\phi_n(x))^2 \right]. \tag{2.133}$$

This action is equivalent to an action of an infinite series of free four-dimensional particles with masses n^2/R^2. These particles form a so-called Kaluza-Klein tower of fields. For small radius R the particles become very massive and decouple in the low energy limit.

Very similar phenomenon occurs in superstring compactifications. For low energies the theory is described by ten-dimensional supergravity in the curved background. The Lagrangian is symbolically written as

$$S_{eff} = \int_{\mathbb{R}^{1,3} \times X} \mathrm{d}^{10}w \, L(\Phi(w)) = \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x \int_X \mathrm{d}^6y [K(\Phi(x, y)) + V(\Phi(x, y))], \tag{2.134}$$

where we separated ten-dimensional coordinates w^M into flat coordinates x^μ in $\mathbb{R}^{1,3}$ and 6 coordinates y^m on X. $\Phi(w)$ denotes a collection of all the fields in supergravity multiplet, $K(\Phi)$ denotes kinetic terms (Laplace or Dirac operators depending on the statistics) and $V(\Phi)$ denotes all other (potential) terms.

One decomposes $\Phi(x, y)$ in the eigenfunctions of corresponding Laplace or Dirac operators on X:

$$\Phi(x, y) = \sum_i \phi_i(x) \otimes f_i(y), \ \Delta_X f_i(y) = \lambda_i f_i(y) \tag{2.135}$$

in order to obtain the four dimensional reduction. Integrating over the compact directions one obtains towers of Kaluza-Klein particles with masses proportional to the inverse size of the compact manifold X. Therefore, considering small enough manifold X that the massive particles decouple in the low energy (but not too small so that the supergravity approximation is still valid) the theory effectively reduces to the theory of massless modes of Laplace and Dirac operators.

The zero modes of Laplace operator are called harmonic tensors and they are in one-to-one correspondence with cohomology groups of X. This is how cohomology of compact manifolds appear in effective descriptions of superstring theories. There can be, of course, corrections to the field theory correlation functions given by instantons but they can be controlled and have beautiful mathematical interpretation themselves such as quantum cohomology.

The condition that $\mathbb{R}^{1,3} \times X$ is a four-dimensional supergravity background puts strong restrictions on X. That is the vacuum state $|0\rangle$ corresponding to $\mathbb{R}^{1,3} \times X$ should be annihilated by at least 4 supercharges which are linear combinations of ten-dimensional supercharges

$$\epsilon_\alpha^A(w) Q_A|0\rangle, \quad 1 \leq \alpha \leq 4, \tag{2.136}$$

where $\epsilon_\alpha^A(w)$ are ten-dimensional spinors and α is a four-dimensional Lorentz group spinor index. In particular, vacuum expectation values of supersymmetric variations of the fields of the theory should vanish which means that the field configurations are supersymmetric, or because

$$\langle 0|\delta_\epsilon \Phi|0\rangle\langle 0|[\epsilon_\alpha^A Q_A, \Phi]|0\rangle = \epsilon_\alpha^A \langle 0|Q_A\, \Phi|0\rangle \mp \epsilon_\alpha^A \langle 0|\Phi\, Q_A|0\rangle = 0. \tag{2.137}$$

If we consider a superstring background where only the metric on $\mathbb{R}^{1,3} \times X$ have non-vanishing vacuum expectation value it turns out that it suffices to consider supersymmetric variations of gravitini

$$\delta_\epsilon \Psi_{M,\alpha} = \nabla_M \epsilon_\alpha + \cdots\,, \tag{2.138}$$

where ∇_M is a spin connection associated to the Levi-Civita connection of the metric and $\cdots$ denote terms with vanishing expectation values. The first term is universal for gravitini because they are (vector-spinor) gauge fields associated to supertransformations.

The equation (2.138) implies the existense of a quadruple of covariantly constant spinors on $\mathbb{R}^{1,3} \times X$ which in turn implies existence of a covariantly constant spinor on X because there are four constant Dirac spinors on $\mathbb{R}^{1,3}$. This condition forces the holonomy group of X to be $\mathfrak{su}(3)$, which is a Calabi-Yau condition. Indeed, the (Lie algebra of the) structure group of the tangent bundle of a real six-dimensional metric manifold X is $so(6) \simeq su(4)$. The Weyl spinor bundles $S_\pm \to X$ are isomorphic to 4 and $\bar{4}$ representations associated to a principle bundle with the structure group $SU(4)$ (which is a cover of a one with the structure group $SO(6)$). The holonomy Lie algebra is a subalgebra of $su(4)$ and naturally acts on the spinor bundles. The existence of covariantly constant spinor locally implies that the holonomy is a subalgebra of $su(3) \subset su(4)$ which has a one-dimensional invariant subspace in 4 and $\bar{4}$. If the holonomy is strictly smaller than $su(3)$, then there exist more covariantly constant spinors and therefore the vacuum $|0\rangle$ is invariant under more supersymmetries. We will restrict to the maximal holonomy case $SU(3)$ which is the most interesting for phenomenology.

Six-dimensional manifolds with holonomy $SU(3)$ are Calabi-Yau manifolds endowed with a Ricci-flat metric, in particular, they are complex and Kähler. The easy way to see that is to use holonomy again.

The tangent bundle TX is in the $3 \oplus \bar{3}$ representation of the holonomy group $su(3)$.

Pick a point $p \in X$. Define an automorphism $J : T_pX \to T_pX$ of the tangent space at the point p such that J is $\sqrt{-1} \cdot \mathrm{id}$ on 3 of $su(3)$ and J is $-\sqrt{-1} \cdot \mathrm{id}$ on $\bar{3}$ of $su(3)$. Such an operator is real by construction and $J^2 = -\mathrm{id}_{T_pX}$.

This operator is explicitly invariant under the holonomy group action and thus can be extended to a covariantly constant operator $J : TX \to TX$, $J^2 = -\mathrm{id}$. Real operator in the tangent bundle of a manifold which squares to -1 is called an almost complex structure. It allows to turn tangent spaces into the complex vector spaces by the formula $(a + ib)v = (a + Jb)v$.

If the J satisfies a certain integrability condition (vanishing of the associated Nijenhuis tensor) then it is called a complex structure and it allows to introduce holomorphic coordinates on X with holomorphic transition functions between patches, that is to turn X into a complex manifold. Moreover, if J is a covariantly constant, then X becomes a Kähler manifold with a Kähler form $\omega = Jg$.

Holonomy of the Kähler manifold is a subgroup of $u(3)$, because complex structure defines a decomposition of the complexification of the tangent bundle into 3 and $\bar{3}$ of $u(3)$ and in the Kähler case this composition is covariant. It implies that only Cristoffel symbols with all holomorphic or all anti-holomorphic indices might not vanish.

On the Kähler manifold X the trace of the holonomy is a Ricci tensor and therefore if the holonomy group is traceless $su(3) \subset u(3)$ then the metric is Ricci-flat, therefore X is Calabi-Yau.

We will need a well-known set of properties of Calabi-Yau manifolds:

Proposition 2.4.1. *Let X be a 6-dimensional smooth closed simply connected real manifold. Thenn the following are equivalent:*

1. *There exist a metric g and the covariantly constant spinor ζ on X.*
2. *There exists a metric of SU(3) holonomy.*
3. *There exists a Kähler Ricci flat metric on X.*
4. *There exists a Kähler metric on X and first Chern class of X is trivial $c_1(X) = 0$.*
5. *There exist a Kähler metric on X and a nowhere vanishing holomorphic 3-form (a holomorphic volume form) Ω.*

As we mentioned above the elements of cohomology groups of X manifold produce (via harmonic forms and Kaluza-Klein reduction) massless particles in the superstring theory on $\mathbb{R}^{1,3} \times X$. Cohomology of Kähler manifolds admit so-called Hodge decomposition which is important in the context of studying the massless particles.

Hodge structures On a complex manifold X complexification of the cotangent bundle TX is decomposed into a direct sum of holomorphic and antiholomorphic parts locally generated by differentials of holomorphic and antiholomorphic coordinates. Therefore, the space of differential forms $\Omega^n_X \simeq \Lambda^n TX$ is further decomposed into a sum of (p,q) forms $\Omega^{p,q}_X$ which are represented as products of p holomorphic differentials and q antiholomorphic differentials in the local charts.

On any complex manifold de Rham differential d naturally decomposes into the holomorphic and antiholomorpic parts as well. They are called Dolbeault differentials $\partial, \bar{\partial}$

$$\Omega^n_X = \bigoplus_{p+q=n} \Omega^{p,q}_X, \quad \mathrm{d} = \partial + \bar{\partial}. \tag{2.139}$$

However, in general, this decomposition does not continue on the cohomology level, because the exact forms (the coboundaries) might not have a definite number of differentials. The abovementioned correspondence of harmonic forms and cohomology elements goes through the Hodge theory. Namely, let (X, g) be a compact Riemann manifold. There is a metric on the appropriately completed space of C^∞ differential forms $\Omega^*(X)$:

$$(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle \sqrt{g} \mathrm{d}^n x = \int_X \alpha \wedge \star\beta, \tag{2.140}$$

where α, β are in principle any L^2 forms on X, $\sqrt{g}\mathrm{d}^n x$ is a volume form on X and $\langle \alpha, \beta \rangle$ denotes pointwise scalar product induced by g.

One can introduce an adjoint de Rham differential:

$$(\alpha, \mathrm{d}\beta) = (\mathrm{d}^*\alpha, \beta), \quad \mathrm{d}^* = \pm \star \mathrm{d} \star. \tag{2.141}$$

In particular, in geodesic coordinates we have $\mathrm{d}^* = \sum_{i=1}^n \iota_{\partial_i} \partial_i$, where $\iota_v \alpha$ is an inner product or contraction of the vector field v with the differential form α. The operators $\psi^i := \partial x^i \wedge$ and $\psi^*_i := \iota_{\partial_i}$ acting on differential forms have canonical anticommutation relations for an n-tuple of fermionic creation and annihilation operators:

$$\begin{aligned} \{\psi^i, \psi^j\} = \{\psi^*_i, \psi^*_j\} = 0, \\ \{\psi^i, \psi^*_j\} = \delta^i_j. \end{aligned} \tag{2.142}$$

This is a manifestation of the fact that Riemannian geometry can be reformulated as $N = 2$ supersymmetric quantum mechanics. The Laplace-Beltrami operator $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ is a Hamiltonian of the theory. The kernel of Δ is $\mathrm{Ker}\Delta = \mathrm{Ker}(\mathrm{d}) \cap \mathrm{Ker}(\mathrm{d}^*)$. Zero eigenforms are called harmonic forms.

The *Hodge decomposition* theorem states that any form from $\Omega^*(X)$ can be decomposed into a sum of a boundary, coboundary and harmonic form:

$$\alpha = h + \mathrm{d}\beta + \mathrm{d}^*\gamma, \tag{2.143}$$

where h is a harmonic form. Moreover, any cohomology element has a unique harminic representative in its class.

On a Hermitian manifold there is decomposition $\mathrm{d} = \partial + \bar{\partial}$. One could define three corresponding Laplace operators:

$$\Delta_{\mathrm{d}} := \Delta, \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}, \quad \Delta_{\partial} = \partial\bar{\partial} + \bar{\partial}\partial. \tag{2.144}$$

On a general Hermitian manifold these operators are different, their kernels do not have a distinct (p,q) type decomposition and the Hodge decomposition of differential forms $\Omega_X^n = \bigoplus_{p+q=n} \Omega_X^{p,q}$ does not continue on the harmonic level and thus on cohomology level.

Let X be a Kähler manifold. It was already discussed that covariant differentiation on Kähler manifolds does not mix holomorphic and antiholomorphic objects due to SU(3) holonomy. In particular, $\{\partial, \bar{\partial}^*\} = 0$ and

$$\frac{1}{2}\Delta = \Delta_{\partial} = \Delta_{\bar{\partial}}. \tag{2.145}$$

The harmonic forms can be chosen to have a definite (p,q) type and the Hodge decomposition continues on the level of cohomology

$$H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X), \tag{2.146}$$

where $H^{p,q}(X) = H^q_{\bar{\partial}}(X, \Omega_X^p)$ consists of classes representable by differential forms of type (p,q). This decomposition is called a *Hodge structure* on the cohomology group $H^n(X)$. In our cas it is more convenient to consider an equivalent definition of a Hodge structure. Consider filtered subspaces $F^pH^n = \bigoplus_{k\geq p} H^k, n-k(X) \subset H^n(X)$. The complex conjugation continues from differential forms to cohomology. In particular we have $F^pH^n \cap \overline{F^qH^n} = H^{p,q}(X)$, $F^pH^n \cap \overline{F^{q+1}H^n}$, where $p+q=n$. Hodge decomposition is equivalent to Hodge filtration and complex conjugation or a *real structure*

To sum up, we use the following

Definition 2.4.1. *Let H be a complex vector space. A real structure M on V is an anti-linear involution, that is $M(\lambda v) = \bar{\lambda} M(v)$, $M\bar{M} = \mathrm{id}$.*

If a complex vector space is a complexification of a real space $V = V_{\mathbb{R}} \otimes \mathbb{C}$, then there is a canonical real structure given by complex conjugation. In general, real structure defines a real subspace inside a complex vector space. An example of non-trivial real structure is a real structure on the relative de Rham cohomology group $H^n_{\mathrm{d}+\mathrm{d}W}(\mathbb{C}^n)$ induced by the real structure on the dual homology group of relative cycles $H_n(\mathbb{C}^n, \mathrm{Re}(W) \ll 0; \mathbb{R})$.

Definition 2.4.2. *Let H be a complex vector space. The Hodge structure on H of weight n consists of a Hodge filtration:*

$$H = F^0 H \supset F^1 H \supset \cdots \supset F^{n+1} H = 0 \tag{2.147}$$

together with an integral lattice $H_{\mathbb{Z}} \subset H$ such that $H \simeq H_{\mathbb{Z}} \otimes \mathbb{C}$.

Remark 2.4.1. *Sometimes we weaken the last condition of integral structure to the existence of a real structure on H.*

If X is a smooth Kähler manifold, then there is a natural Hodge structure on its cohomology groups $H^n(X, \mathbb{Z})$ (we work modulo torsion in cohomology).

Hodge structures on Calabi-Yau threefolds We are interested in superstring compactifications on a Calabi-Yau threefold X. The massless particles of low energy four-dimensional theories correspond to harmonic tensors. We consider the case where the holonomy group of X is SU(3), that is the minimally supersymmetric case. For such manifolds the the cohomology groups have a simple structure. Consider a harmonic form of purely holomorphic type $\alpha \in \Omega^{k,0}(X)$ for $k \leq 3$. harmonicity means that $\Delta\alpha = 0$. We use the Lichnerowicz-Weitzenböck formula connecting Laplace-Beltrami operator with the square of covariant derivative:

$$-\nabla_\mu \nabla^\mu \alpha = \Delta\alpha + R[\alpha], \tag{2.148}$$

where ∇ is a Levi-Civita connection for the metric on X and $R[\alpha]$ is a degree zero operator depending on the curvature. Such an operator vanishes on $\Omega^{k,0}(X)$ which implies that α is not just harmonic but also covariantly constant $\nabla\alpha = 0$. Being a (k,0)-form it transforms according to $\Lambda^k 3_{\mathfrak{su}(3)}$ representation of the holonomy group. Such representations have invariant subspaces only in the trivial cases $k = 0$ and $k = 3$ which means that $\dim H^{k,0}(X) \geq 0$ only for $k = 0, 3$. In the first case cohomology $H^0(X)$ are one-dimensional since we assume it to be simply-connected. Consider $[\Omega] \in H^{3,0}(X)$ and its harmonic representative Ω. Such a representative is a holomorphic covariantly constant (3,0) form. In particular it has a constant norm and thus is nowhere vanishing and is defined by its value at any point. It follows that $H^{3,0}(X)$ is one-dimensional.

Further, there are symmetries among the cohomology groups: complex conjugation $\bar{\cdot} \ : \ H^{p,q}(X) \simeq H^{q,p}(X)$ and Hodge star (Poincaré duality) $\star \ : \ H^{p,q}(X) \simeq H^{n-q,n-p}(X)$. Summing up all these considerations we present a Hodge diamond of a compact connected Calabi-Yau threefold with the maximal holonomy group SU(3). We denote $h^{p,q} := \dim H^{p,q}(X)$ and put in the table at n'th row from the bottom at the q'th position from the left the number $h^{p,q}$, where $p+q=n$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.149}$$

In the table above $h^{2,2} = h^{1,1}$ and $h^{1,2} = h^{2,1}$, so there are only two independent dimensions. The Euler characteristic of such a threefold is equal to $2h^{1,1} - 2h^{2,1}$.

Hodge numbers and metric deformations Some four-dimensional massless particles come from harmonic modes of the metric tensor $g = \{g_{mn}\}$ on the Calabi-Yau threefold X. These massless particles belong to the same $N=2$ supermultiplets as the ones coming from Ramond-Ramond forms. Harmonic modes of the metric tensor correspond to infinitesimal deformations of the metric which do not spoil the Ricci-flatness condition. This condition is expressed as Lichnerowicz equation for deformations of the metric. Let $h = \{h_{mn}\}$ be a symmetric tensor on X. Then in the complex coordinates Lichnerowicz equation reads

$$\begin{aligned} \nabla^2 h_{m\bar{n}} + 2R_{m\ \bar{n}}^{\ p\ \bar{q}} h_{p\bar{q}} = 0, \\ \nabla^2 h_{mn} + 2R_{m\ n}^{\ p\ q} h_{pq} = 0. \end{aligned} \tag{2.150}$$

By Lichnerowicz-Weitzenböck formula it is equivalent to the fact that

$$\begin{aligned} \Delta_{\bar{\partial}} h_{m\bar{n}} = 0, \\ \Delta_{\bar{\partial}} h_{\bar{m}}^{n} = 0, \end{aligned} \tag{2.151}$$

where $h_{\bar{m}}^{n} = g^{n\bar{n}} h_{\bar{m}\bar{n}}$. Harmonic forms are isomorphic to corresponding cohomology groups. It follows that type (1,1) Ricci-flat metric deformations are in one-to-one correspondence with $H^{1,1}(X)$ and type (2,0) or (0,2) deformations are in correspondence with $H^1_{\bar{\partial}}(TX)$.

Elements of $H^{1,1}(X)$ are naturally identified with deformations of Kähler structure on X because they change the metric but do not change the complex structure. On the other hand, elements of $H^1_{\bar{\partial}}(TX)$ are so-called Beltrami differentials. They do not change the Kähler (real symplectic) form on X but mix holomorphic and antiholomorphic coordinates. They are identified with infinitesimal deformations of complex structures. Moreover, all infinitesimal deformations can be integrated to finite deformations of complex structure for Calabi-Yau manifolds.For the mathematical exposition of these questions see, e.g. [106].

2.4.2 Special Kähler geometry

Special Kähler geometry is the geometry of coupling constants of four dimensional N=2 supersymmetric gauge theories. There are two types of special Kähler geometry: global or affine which corresponds to gauge theories with global supersymmetry and local or projective special geometry corresponding to local supersymmetry or supergravity. Originally they appeared in physical papers in the 80th.Mathematically, global special Kähler geometry correpsonds to algebraic integrable systems [107] and naturally appears in the study of Riemann curves. Whereas local special geometry is deeply connected with variation of Hodge structures for compact Calabi-Yau threefolds [108].Let us first give the physical motivation.

N=2 supersymmetry We consider the global case first, that is a theory of N=2 supersymmetric (abelian) vector multiplets without gravity. N=2 extended supersymmetry means that the super-Poincaré group has an odd part isomorphic to a sum of two irreducible spinor representations of the ordinary Lorentz group which consist of supercharges. In particular, there is a SU(2) R-symmetry which rotates the two irreducible collections of supercharges. Global theory with N=2 supersymmetry is a special case of the N=1 supersymmetry. The on-shell N=1 vectormultiplet consists of a real U(1) gauge field A_μ and a Weyl fermion λ_α.

Both of them have two physical degrees of freedom. The N=1 chiral multiplet on-shell consists of a complex scalar ϕ and a Weyl fermion χ_α.

The N=2 vector multiplet decomposes into a sum of a N=1 vector and a N=1 chiral multiplets

$$(A_\mu, \lambda_\alpha, \tilde{\lambda}_\alpha, \phi) = (A_\mu, \lambda_\alpha, \tilde{\lambda}_\alpha, \phi). \tag{2.152}$$

The pair of fermions λ_α and $\tilde{\lambda}_\alpha$ form a fundamental representation of SU(2) R-symmetry.

Consider a theory with n N=2 vector multiplets. From the point of view of N=1 gauged sigma model the scalar target space is an n-dimensional Kähler manifold. The kinetic terms of N=1 chiral multiplets have the form

$$\frac{1}{2} g_{i\bar{j}}(\phi)\, \partial^\mu \phi^i \partial_\mu \overline{\phi^j} + g_{i\bar{j}}(\phi)\, \bar{\tilde{\lambda}}^{\bar{j}} \not{D} \tilde{\lambda}^i, \tag{2.153}$$

where locally $g_{i\bar{j}} = \partial_i \overline{\partial_j} K(\phi, \bar{\phi})$. The kinetic terms of N=1 vector multiplets have the form

$$\frac{1}{8\pi}\left(Im(\tau_{ij})\, F^i_{\mu\nu} F^{j,\mu\nu} - Re(\tau_{ij})\, F^i_{\mu\nu} (\star F)^{j,\mu\nu}\right) - \frac{1}{2\pi} Im(\tau_{ij})\, \bar{\lambda}^{\bar{j}} \not{D} \lambda^i, \tag{2.154}$$

where τ is a holomorphic function of $\phi = \{\phi^i\}_{i=1}^n$. Due to the SU(2) R-symmetry, the fermionic kinetic terms should be equal:

$$\frac{\tau_{ij} - \bar{\tau}_{ij}}{4\pi i} = \frac{\partial^2 K(\phi, \bar{\phi})}{\partial \phi^i \partial \overline{\phi^j}}. \tag{2.155}$$

Differiating the equality above with respect to ϕ^i we obtain

$$\frac{\partial}{\partial \phi^i} \tau^{jk} = \frac{\partial^3 K(\phi, \bar{\phi})}{\partial \phi^i \partial \phi^j \partial \overline{\phi^k}} \tag{2.156}$$

up to a non-significant factor. The last equation is an integrability condition for τ, that is

$$\tau_{ij} = \partial_i \partial_j F(\phi) \tag{2.157}$$

holds locally for some holomorphic fuction $F(\phi)$. Using (2.155) we compute the Kähler potential of the metric $g_{i\bar{j}}$

$$K(\phi, \bar{\phi}) = i(\phi^i \overline{\partial_i F(\phi)} - \overline{\phi^i} \partial_i F(\phi)). \tag{2.158}$$

We get that the Kähler potential on the target space $\mathcal{M}$ of scalars of N=2 vectormultiplets is determined by a holomorphic function $F(\phi)$. For this reason such a structure on $\mathcal{M}$ is called a global special Kähler geometry. ϕ^i are flat coordinates of a certain flat symplectic connection. Functions $F_i := \partial_i F(\phi)$ are called dual coordinates. Let us make the symplectic structure explicit. The formula (2.158) can be rewritten as

$$\partial \bar{\partial} K = -\frac{1}{4}(\mathrm{d}\phi^i + \overline{\mathrm{d}\phi^i}) \wedge (\mathrm{d}F_i + \overline{\mathrm{d}F_i}). \tag{2.159}$$

This is true due to the integrability condition:

$$\mathrm{d}F_i \wedge \mathrm{d}\phi^i = \mathrm{d}(F_i \mathrm{d}\phi^i) = \mathrm{d}\mathrm{d}F = 0. \tag{2.160}$$

Structure of the special Kähler geometry is invariant under symplectic transformations preserving the form $\mathrm{d}F_i \wedge \partial \phi^i$:

$$\begin{pmatrix} \mathrm{d}\tilde{\phi} \\ \mathrm{d}\tilde{F} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathrm{d}\phi \\ \mathrm{d}F \end{pmatrix}, \tag{2.161}$$

where $\mathrm{d}\phi$, $\mathrm{d}F$ are one-form valued vectors with components $\mathrm{d}\phi^i$, $\mathrm{d}F_i$. Indeed, such transformations preserve $\partial\bar{\partial}K$ and $\mathrm{d}(\tilde{F}_i\mathrm{d}\tilde{\phi}^i) = 0$, that is there exists a local holomorphic function $\tilde{F}$ such that $\tilde{F}_i = \partial/\partial\tilde{\phi}^i\tilde{F}$.

These symplectic transformations are induced by electro-magnetic duality transformation which mixes field strengths $F^i_{\mu\nu}$ with their Hodge duals $\star F^j_{\mu\nu}$.

Mathematically, (ϕ^i, F_i) form a section of a flat symplectic bundle V over $\mathcal{M}$ and the Kähler form on $\mathcal{M}$ is given by the second derivative of its norm.

Definition 2.4.3. *A Kähler manifold* $(\mathcal{M}, \omega)$ *is called a special Kähler or endowed with a special Kähler structure if there is a holomorphic flat symplectic bundle* $V \to \mathcal{M}$ *with the structure group Sp(2,* $\mathbb{R}$*) which is isomorphic to the tangent bundle* $V \simeq T\mathcal{M}$ *and there exists a section* $\Omega \in \Gamma(\mathcal{M}, V)$ *such that the Kähler form* ω *on* $\mathcal{M}$ *equals to*

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\langle\Omega, \bar{\Omega}\rangle, \tag{2.162}$$

and the following transversality condition is satisfied

$$\langle\partial_i\Omega, \overline{\partial_j\Omega}\rangle = 0, \tag{2.163}$$

where $\langle\cdot,\cdot\rangle$ *is a symplectic pairing on* V*.*

N=2 supergravity In supergravity theories supercharges become local, that is sections of a principle bundle over the space-time. The metric tensor or the vielbein becomes a dynamic field and obtains its own superpartners which are called gravitini which are Rarita-Schwinger fields, or belong to a tensor product of vector and spinor representations of the Lorentz group. In N=2 supersymmetry case there are two gravitiny $\psi_{\mu.\alpha}, \tilde{\psi}_{\mu,\alpha}$ which serve as currents for the local supertransformations. The physical fields of the N=2 gravity multiplet consist of the vielbein, two gravitini and a graviphoton, which is a $U(1)$ gauge field:

$$(E^M_\mu, \psi_{\mu,\alpha}, \tilde{\psi}_{\mu,\alpha}, A_\mu). \tag{2.164}$$

Being a gauge field A_μ mixes with matter vector multiplet gauge fields. In the conformal approach to d=4 N=2 supergravity one works in the N=1 language and introduces $n+1$ vector multiplets and a Weyl multiplet which consists of the metric, gravitini and auxilary fields. The vector multiplets are subject to $\mathbb{C}^*$ gauge transformations which effectively fix scalar and a pair of gaugini (fermions) from one of the vector multiplets and the remaining gauge fields gets identified with the graviphoton. The geomety of how exactly the graviphoton is embedded into $n+1$ gauge fields is called the central charge geometry [109].

Let us call scalars of $n+1$ vector multiplets $\{X^I\}_{I=1}^{n+1}$. The gauge symmetry is $X^I \to e^{f(\phi)}X^I$ for arbitrary transition functions $e^{f(\phi)}$, that is they behave as homogeneous coordinates of a projective space. It means the that X^I are sections of a line bundle $\mathcal{L} \to \mathcal{M}$, where $\mathcal{M}$ is a target space for n independent scalars of the vector multiplets.

If we forget the gauge transformations, that is consider a $n+1$ dimensional total space of the line bundle $\mathcal{L}$, then the usual formula for the global special geometry works. That is if N_{IJ} (which should remind of $\mathrm{Im}\tau_{ij}$) is a metric on the total space of $\mathcal{L}$ them the following formula holds true:

$$N_{IJ} = i\partial_I\overline{\partial_J}(X^I\overline{F_I} - \overline{X^I}F_I) = i(F_{IJ} - \overline{F_{IJ}}), \tag{2.165}$$

where $F_I(X) = \partial/\partial X^I F(X)$ and the locally defined prepotential $F(X)$ is a homogeneous function of degree two due to the gauge invariance of the metric.

Homogeneity of the preotential imply the following equations:

$$\frac{1}{2}X^I F_I = F,\ \partial_I N_{IJ} X^I = N_{IJ},\ \partial_I N_{IJ} X^I \overline{X^J} = N_{IJ}\overline{X^J}. \tag{2.166}$$

The metric on the target space $\mathcal{M}$ is a reduction the metric on the total space of $\mathcal{L}$ by the $\mathbb{C}^*$ action. To compute the metric we introduce orthogonal projection operators P to the planes orthogonal to the group action, that is if $v \in T\mathcal{L}$ then $(XN)_I(Pv)^I = 0$:

$$(Pv)^I = v^I - \frac{\overline{X^I}(NX)_J v^J}{(XN\overline{X})}. \tag{2.167}$$

Then the metric on $\mathcal{M}$ is computed as

$$G_{IJ}v^I\overline{w^J} = -\frac{N_{IJ}(Pv)^I\overline{(Pw)^J}}{XN\overline{X}}, \tag{2.168}$$

where the sign makes the metric positive definite and the denominator makes it gauge invariant. Using the explicit formula (2.167) we compute the metric to be

$$G_{I\bar{J}} = -\frac{N_{IJ}}{X^I N_{IJ}\overline{X^J}} + \frac{(NX)_I\overline{(NX)_J}}{(X^I N_{IJ}\overline{X^J})^2}, \tag{2.169}$$

which can be expressed through a Kähler potential

$$G_{I\bar{J}} = \partial_I\overline{\partial_J}K, \tag{2.170}$$

where

$$e^{-K(X,\bar{X})} = i(X^I\overline{F_I} - \overline{X^I}F_I). \tag{2.171}$$

In particular, for the simplest potential $F = X^2 = X^I X_I$ the metric $G_{I\bar{J}}$ is a Fubini-Studi metric on $\mathbb{C}P^n$. We note that this metric is indeed independent of the gauge transformations $X^I \to e^{f(X)}X^I$ because it implies $K(\phi,\bar{\phi}) \to K(\phi,\bar{\phi}) + f(X) + \overline{f(X)}$.

All the couplings of the vector multiplets are expressed through the holomorphic prepotential. As an important example, the appropriately normalized Pauli couplings (couplings of two gaugini and a photon field strength) are equal to third derivatives of the prepotential $F_{IJK}(X)$.

Mathematically the structure of local special geometry can be described as follows. Local special geomery on $\mathcal{M}$ is a gauge invariant global special Kähler geometry on a total space of a line bundle $\mathcal{L} \to \mathcal{M}$.

Definition 2.4.4. *Let $(\mathcal{M}, \omega)$ be a Kähler manifold. A local special Kähler structure on MM consists of the following data: A flat holomorphic Sp(2n+2, $\mathbb{R}$) bundle $V \to \mathcal{M}$, a linear bundle $\mathcal{L} \to \mathcal{M}$ and a section of their tensor product $\Omega \in \Gamma(\mathcal{M}, V \otimes \mathcal{L})$ subject to the conditions*

1. *The Kähler form on $\mathcal{M}$ is given by the formula:*

$$\omega = -\frac{i}{2\pi}\partial\bar{\partial}\log\langle\Omega, \overline{\Omega}\rangle, \tag{2.172}$$

   in particular, $c_1(\mathcal{L}) = [\omega]$ and thus $\mathcal{M}$ is a Hodge manifold.

2. *Transversality condition:*

$$\langle\Omega, D_i\Omega\rangle = \langle\Omega, \partial_I\Omega - \partial_I K\,\Omega\rangle = \langle\Omega, \partial_I\Omega\rangle = 0. \tag{2.173}$$

2.4.3 Variations of Hodge structures on Calabi-Yau manifolds

Local Special geometry can be thought of as reformulation of variations of Hodge structures. Hodge structure on cohomology of a compact complex manifold X is decomposition of de Rham cohomology groups of X as a real manifold with respect to the complex structure on X. It is therefore natural that the Hodge structure can be used to measure complex structure deformations on X. The mathematical object which is adjusted to it is a variation of Hodge structures or VHS for short [VHSDeligne]. Consider a family $\mathcal{X} \xrightarrow{p} \mathcal{M}$ of complex manifolds, where $\mathcal{M}$ and π are holomorphic and a generic fiber $X_\phi := p^{-1}(\phi)$ is a smooth complex manifold. Then the set $\mathcal{M}^{sing}$ of points whose fibers are singular has complex codimension at least 1. Then all manifolds in the family are diffeomorphic as real manifolds because they can be connected by continuous deformations. In particular, they have the same integral cohomology groups which are canonically identified along any path in the base space using continuous deformations of the cycles. Therefore, there is a vector bundle $\mathcal{H}^* \xrightarrow{\pi} \mathcal{M}\backslash\mathcal{M}^{sing}$ whose fibers are cohomology groups of the fiber manifolds $\pi^{-1}(\phi) = H^*(X_\phi)$ and there is a flat connection ∇^{GM} called Gauss-Manin connection which is essentially a differentiation of differential forms along the fibers such that locally constant singular cocycles are flat.

When the nonsingular fibers X_ϕ are Kähler, then each cohomology group $H^*(X_\phi)$ has a Hodge structure. Consider a family of differential forms $\omega_\phi \in \Gamma(\mathcal{M}, \mathcal{H})$ which is a section of $\mathcal{H}$. Let us fix $\phi_0 \in \mathcal{M}\backslash\mathcal{M}^{sing}$ and pick a representative ω_{ϕ_0} to be of a (p,q) type. To reduce amount of indices consider the case where ω_{ϕ_0} is an (1,1)-form. In the local chart on H^* the form ω_{ϕ_0} can be decomposed as

$$\omega_\phi = \omega^{i,\bar{j}} \mathrm{d}z^i \overline{\mathrm{d}z^j}. \tag{2.174}$$

In the formula above $\mathrm{d}z^i$ are sections of $\mathcal{H}^1$ themselves, their restrictions to the point ϕ_0 belong to $H^{1,0}(X_{\phi_0})$. By the Leibniz rule, Gauss-Manin connection acts on ω_ϕ as

$$\nabla^{GM}_a \omega_\phi = \partial_{\phi^a} \omega^{i,\bar{j}} \mathrm{d}z^i \overline{\mathrm{d}z^j} + \omega^{i,\bar{j}} \nabla^{GM}(\mathrm{d}z^i) \overline{\mathrm{d}z^j} + \omega^{i,\bar{j}} \mathrm{d}z^i \nabla^{GM}(\overline{\mathrm{d}z^j}). \tag{2.175}$$

As a consequence of this formula we have the following Griffiths transversality (or Kodaira lemma):

$$\nabla^{GM} \ : \ F^p H^*(X_\phi) \to F^{p-1} H^*(X_\phi). \tag{2.176}$$

In other words, a derivative with respect to the Gauss-Manin connection can reduce a number of holomorphic differentials only by one.

Basially, Gauss-Manin connection ∇^{GM} acts on representatives as derivatives with respect to local coordinates $\{\phi^a\}_{a=1}^h$ on $\mathcal{M}$.

Let us give a mathematcal definition of variations of Hodge structures.

Definition 2.4.5. *Let $V \stackrel{\pi}{\to} \mathcal{M}$ be a holomorphic vector bundle over an algebraic manifold $\mathcal{M}$. Variation of Hodge structures of weight n on V consists of*

1. *A collection of holomorphic subbundles $F^pV \subset V$, $0 \le p \le n$ such that they restrict to Hodge filtrations on each fiber.*

2. *A holomorphic connection $\nabla \ : \ V \to \Omega^1(\mathcal{M}) \otimes V$ which is called a Gauss-Manin connection and satisfies the Griffiths transversality condition*

$$\nabla \ : \ F^pV \to F^{p-1}V. \tag{2.177}$$

3. *An integral structure $V_{\mathbb{Z}} \subset V$ whose elements are flat with respect to the Gauss-Manin connection.*

Now it is easy to connect variations of Hodge structures with the special Kähler geometry.

Global special Kähler geometry is connected with the variations of Hodge structures on specific families of Riemann curves or, more conceptually from the string theoretic point of view, from variations of Hodge structure on local or non-compact Calabi-Yau threefolds. In this case gravity decouples from the gauge theory with the result of N=2 super Yang-Mills theory.

Local special Kähler geometry appears on variations of Hodge structures of compact Calabi-Yau threefolds as it should: superstring theory compactified on a Calabi-Yau threefold in the low-energy limit is described by N=2 supergravity.

Proposition 2.4.2. *Let $(\mathcal{H}^3 \xrightarrow{\pi} \mathcal{M}, F^pV, \nabla, \mathcal{H}^3_{\mathbb{Z}})$ be a variation of Hodge structures on third cohomology of a family of Calabi-Yau threefolds such that at a generic point $\phi \in \mathcal{M}$ there is a natural isomorphism $T_\phi\mathcal{M} \simeq H^{2,1}(X_\phi)$ given by Gauss-Manin connection. Then it defines a special Kähler structure on $\mathcal{M}$* [8].

Let us give an idea of proof of this proposition. At each point ϕ where X_ϕ is smooth there is a Hodge decomposition of the fiber $\pi^{-1}(\phi) = H^3(X_\phi, \mathbb{C}) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$, where $h^{3,0} = h^{0,3} = 1,\ h^{2,1} = h^{1,2}$. Furthermore, there is a Poincaré pairing on $H^3(X_\phi)$ which is skew-symmetric, integral, non-degenerate and is block antidiagonal with respect to the Hodge decomposition. Therefore, the bundle $\mathcal{H}^3$ is a $\mathrm{Sp}(2h^{2,1}+2, \mathbb{R})$-bundle. This bundle has a flat Gauss-Manin connection. In addition there is a linear holomorphic subbundle with fibers $\mathcal{L} := H^{3,0}(X_\phi) = F^3H^3(X_\phi, \mathbb{C})$. Let us pick any of its local sections $\Omega \in \Gamma(\mathcal{M}, \mathcal{L})$ and define a symplectic form by the formula

$$\omega := \partial\bar{\partial} \log \int_{X_\phi} \Omega_\phi \wedge \overline{\Omega_\phi}. \tag{2.178}$$

This form is independent of the choice of Ω and is defined globally on $\mathcal{M}$. In addition, it has type (1,1) with respect to the complex structure of $\mathcal{M}$. Therefore, it transforms $\mathcal{M}$ into a Hodge Kähler manifold. Finally, the condition $\int \Omega \wedge \partial_I \Omega = 0$ is satisfied due to the Griffiths transversality condition.

Period mapping From the mathematical point of view a variation of Hodge structures on $\mathcal{H}^* \xrightarrow{\pi} \mathcal{M}$ is constructed from a family $\mathcal{X} \xrightarrow{p} \mathcal{M}$ using the period mapping. Since under our assumptions all smooth fibers of p are diffeomorphic, so are their cohomology groups and Gauss-Manin connection identifies them locally. In particular, if we consider an open disk $U \subset \mathcal{M}\backslash M^{sing}$, then there is no monodromy and the bundle $\mathcal{H}^* \to \mathcal{M}$ can be trivialized to $\mathcal{M} \times H^*(X_0, \mathbb{C})$, where $X_0 := p^{-1}(0)$. Then the period mapping P sends a point in U into a flag variety $F(H^*(X_0, \mathbb{C})) = F(H^1(X_0, \mathbb{C}) \times \cdots \times H^n(X_0, \mathbb{C}))$ assigning a flag $0 = F^{k+1}H^k \subset \cdots \subset F^0H^k = H^k(X_\phi, \mathbb{C})$ to a point $\phi \in U$.

In the case of a Calabi-Yau threefold the period map on the middle cohomology is completely determined by the period integrals of a holomorphic volume form. Pick any nonzero section of $\Omega \in \Gamma(U, F^3H^3)$. The coordinates on $H^3(X, \mathbb{C})$ can be chosen by integration over cycles $\{\gamma_i\}_{i=1}^{b_3} \subset H_3(X, \mathbb{Z})$ forming a basis of the latter group. Then

[8] In fact, we will also work with the case where $T_\phi\mathcal{M} \subset H^{2,1}(X_\phi)$, and the special geometry is defined using a subbundle of the cohomology bundle

in these coordinates the map $F^3H^3 \subset \mathcal{H}^3$ is given by the formula

$$\left(\int_{\gamma_1} \Omega_\phi : \ldots : \int_{\gamma_{b_3}} \Omega_\phi\right). \tag{2.179}$$

Different choices of Ω lead to overall scalings of the homogeneous coordinates and do not change the point in the flag variety. The next step of the flag $F^2H^3 \subset \mathcal{H}^3$ is given by the periods of derivatives of Ω with respect to the Gauss-Manin connection. Those are just derivatives of the period integrals with respect to $\partial/\partial\phi^a$ because the cycles of integration are Gauss-Manin flat. Moreover, we use the assumption [9] that $T_\phi\mathcal{M} \simeq H^{2,1}(X_\phi)$. The last nontrivial step in the Hodge filtration F^1H can be obtained either by derivatives of F^2H with respect to the Gauss-Manin connection, or by complex conjugation and Poincaré pairing.

2.4.4 Special Kähler geometry on complex and Kähler moduli of Calabi-Yau threefolds

Complex structures moduli space There are several ways to describe complex structure deformations of Calabi-Yau manifolds. In the previous subsection we described it from the point of view of variations of Hodge structure, where deformations of complex structures are understood as algebraic (or holomorphic) families of manifolds.

Another approach is more differential geometric. On a Kähler manifold X there are three connected structures: a Kähler metric $g_{\mu\nu}$, a symplectic form $\omega_{\mu\nu}$ and a compatible orthogonal complex structure tensor $J^\mu_\nu = \omega_{\nu\lambda}g^{\lambda\mu}$ such that $g_{\mu\nu}J^\mu_\lambda J^\nu_\sigma = g_{\lambda\sigma}$ and $J^2 = -\mathrm{id}$. Deformations of complex structures on a Calabi-Yau manifold can be thought of as Ricci-flat deformations of the metric $g_{\mu\nu}$ which do not change the symplectic form $\omega_{\mu\nu}$ and such that J^μ_ν for the deformed metric also satisfy complex structure conditions.

Consider a Calabi-Yau manifolds X_0 with complex structure J and Ricci-flat metric g. Then Ricci-flat deformations of g up to first order are given by the equations (2.150). Consider a deformation $h_{\bar{i}\bar{j}}$ of the type (0,2). The Ricci-flatness is equivalent to the harmonicity condition on the vector-valued $(0,1)$-form $h^i_{\bar{j}} := g^{i\bar{l}}h_{\bar{l}\bar{j}}$. This form is called a Beltrami differential and it defines a first order deformation of the complex structure by the rule $\mathrm{d}z^i \to \mathrm{d}z^i + h^i_{\bar{j}}\overline{\mathrm{d}z^j}$. The closedness condition implies that the deformed differentials are integrable. Each harminic form $h^i_{\bar{j}}$ determines a cohomology class in $H^1_{\bar{\partial}}(TX)$ which we denote by the same character. This is the well-known fact that first order deformations of complex structure are in one-to-one correspondence with

[9] This assumption is not necessary and will not be satisfied in many of our examples where $T_\phi\mathcal{M} \subset H^{2,1}(X_\phi)$. In these cases the map (2.179) is a projection of the period map on a certain subspace.

elements of $H^1_{\bar{\partial}}(TX)$ cohomology group. In general, they might not lead to finite (all order) deformations, that is there might be obstructions to deformations in general. Deformation theory of Calabi-Yau manifolds is unobstructedand every infinitesimal deformation $h^i_{\bar{j}}$ can be integrated.

Let us describe the metric on the Calabi-Yau complex structures moduli space. We can start from the natural Polyakov/DeWitt metric on the space of deformation of metrics. Let $\partial_a g_{ij}, \partial_b g_{ij}$ be deformations of metric with holomorphic indices (∂_a denotes a derivative with respect to deformation parameters in the finite-dimensional space of Ricci-flat metric deformations). Then the Polyakov metric is defined by the formula

$$G_{a\bar{b}} := G(h^a_{ij}, h^b_{ij}) := \frac{1}{\mathrm{Vol}(X)} \int_X \mathrm{d}^6 x \, g^{1/2} \, g^{i\bar{k}} g^{j\bar{l}} \partial_a g_{ij} \overline{\partial_b g_{kl}}. \tag{2.180}$$

We can rewrite Ricci-flat metric deformations through the corresponding cohomology elements:

$$(h_a)_{ij\bar{k}} := \frac{1}{2} \partial_a g_{\bar{k}\bar{l}} g^{\bar{l}s} \Omega_{sij}, \tag{2.181}$$

where Ω_{sij} is a holomorphic volume form on X which is defined up to a constant. Then the metric (2.180) can be rewritten as

$$G_{a\bar{b}} = \frac{\int_K h_a \wedge \overline{h_b}}{\int_K \Omega \wedge \overline{\Omega}} = \partial_a \overline{\partial_b} \log \int_K \Omega \wedge \overline{\Omega}. \tag{2.182}$$

To get the first equality we write

$$\int_K h_a \wedge \overline{h_b} = \int_K \partial_a g_{\bar{k}\bar{l}} \partial_{\bar{b}} g_{kl} g^{\bar{l}s} g^{l\bar{s}} \Omega_{sij} \bar{\Omega}_{\bar{s}\bar{i}\bar{j}} \mathrm{d}^6 y \sim$$

$$\sim ||\Omega||^2 \int_K \partial_a g_{\bar{k}\bar{l}} \partial_{\bar{b}} g_{kl} g^{\bar{l}s} g^{l\bar{s}} \varepsilon_{sij} \varepsilon_{kij} \varepsilon_{\bar{s}\bar{i}\bar{j}} \varepsilon_{\bar{k}\bar{i}\bar{j}} \mathrm{d}^6 y. \tag{2.183}$$

The second equality in (2.182) we use Griffiths transversality of the Gauss-Manin connection. Let us show it from the point of view of differential geometry.Consider a complex structure deformation on X_0 induced by the metric deformation $\partial_a g_{\bar{i}\bar{j}}$. Pick new holomorphic coordinates on $(X_0, g + \partial_a g, \omega)$

$$x^i \to x^i + m^i(x, \bar{x}), \; \mathrm{d}x^i \to \mathrm{d}x^i + \bar{\partial} m^i + hol.. \tag{2.184}$$

In new coordinates the purely antiholomorphic part of the metric vanishes which implies

$$\partial_a g_{\bar{i}\bar{j}} = -\left(\frac{\partial m^r}{\partial \overline{x^j}} g_{r\bar{i}} + \frac{\partial m^r}{\partial \overline{x^i}} g_{r\bar{j}} \right). \tag{2.185}$$

We can pick m^i in a way that $pd_a g^j_{\bar{i}} = -1/2 \, \partial_{\bar{i}} m^j$. Then under the coordinate change (2.184) the holomorphic volume form Ω on X_0 transforms as

$$\partial_a \Omega = \partial_a \Omega_{123} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + 3\Omega_{123} \, (\partial_a \mathrm{d}x^1) \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{2.186}$$

The (3,0) component in the equality above is closed and is, therefore, proportional to Ω itself at least up to an exact form. It follows that

$$\partial_a \Omega = \kappa_a \Omega + h_a, \tag{2.187}$$

where h_a is a (2,1)-form corresponding to the metric deformation $\partial_a g_{\bar{i}\bar{j}}$ and κ_a is some constant (a function of deformation parameters only).

The second equality in (2.182) is obtained by differentiation of the Kähler potential with the use of (2.187). As a consequence, we showed that the Polyakov metric on a space of Ricci-flat deformations $\mathcal{M}$ preserving the Kähler form is Kähler itself with the potential

$$e^{-K} = \int_X \Omega \wedge \overline{\Omega}. \tag{2.188}$$

We note that $\overline{\Omega \wedge \overline{\Omega}} = -\Omega \wedge \overline{\Omega}$, and the expression above is sometimes conveniently multiplied by $\pm i$ to make it real. However, this leads to an addition of a constant to the Kähler potential and does not change the metric.

Let us explicitly derive existence of the prepotential for the metric, that is find the special geometry structure. As a real manifold X has de Rham cohomology group with integral coefficients $H^3(X, \mathbb{Z})$. Nonsingular variations of metric on X do not change the cohomology of the manifold. Consider a contractible disc $U \subset \mathcal{M}$ and a trivial bundle $\mathcal{H}^3 \to U = H^3(X, \mathbb{C}) \times U$ over the space of metric deformations. It is naturally endowed with a trivial flat connection ∇^{GM} which is a Gauss-Manin connection. If we cover $\mathcal{M}$ or a subset of it with contractible sets and glue them together we obtain a flat bundle $\mathcal{H}^3 \to \mathcal{M}$ with the Gauss-Manin connection. However, the bundle is not necessarily trivial anymore: if $\mathcal{M}$ is not simply connected which is almost always true due to singularities, there might be nontrivial monodromy of the Gauss-Manin connection along non-contractible loops.

Locally we can choose a pair of dual symplectic bases $\{A^a\}_{a \le b_3/2}, \{B_a\}_{a \le b_3/2} \subset H_3(X, \mathbb{Z})$ and $\{\alpha_a\}_{a \le b_3/2}, \{\beta^a\}_{a \le b_3/2} \subset H^3(X, \mathbb{Z})$ such that the intersection form is a symplectic unit that is in the Darboux normal form:

$$\begin{aligned} \int_{A^b} \alpha_a = \int_K \alpha_a \wedge \beta^b = \delta_a^b, \\ \int_{B_a} \beta^b = \int_K \beta^b \wedge \alpha_a = -\delta_a^b, \\ \int_K \alpha_a \wedge \alpha_b = \int_K \beta^a \wedge \beta^b = 0. \end{aligned} \tag{2.189}$$

In these notations A^a are Poincaré dual to β^a and B_a are Poincaré dual to α_a. Special coordinates are periods of the holomorphic volume form Ω:

$$z^a := \int_{A^a} \Omega, \; F_b := -\int_{B_b} \Omega. \tag{2.190}$$

We can decompose the holomorphic volume form in the basis (α_a, β^b).

$$\Omega = z^a \alpha_a - F_b \beta^b. \tag{2.191}$$

Periods z^a are defined up to multiplication by a common constant which is a normalization of Ω and are independent, so they can be used as local coordinates on $\mathcal{M}$. To derive the special geometry relations we compute using (2.187)

$$0 = \int_K \Omega \wedge \partial_c \Omega = \int_k (z^a \alpha_a - F_b \beta^b) \wedge (\alpha_c - \partial_c F_d \beta^d) = \\ = -z^a \partial_c F_a + F_c = 0. \tag{2.192}$$

The last equality implies

$$F_a = \partial_a F(z), \; F(z) = \frac{1}{2} z^a F_a. \tag{2.193}$$

$F(z)$ is a holomorphic prepotential which is homogeneous of degree 2 in $\{z^a\}$. Every intrinsic quantity on the special Kähler manifold $\mathcal{M}$ is encoded in $F(z)$. In particular, the Kähler potential is

$$e^{-K} = \int_k \Omega \wedge \bar{\Omega} = \overline{z^a} \partial_a F(z) - z^a \overline{\partial_a F(z)}. \tag{2.194}$$

It is convenient to rewrite the Kähler potential using the period vector

$$\Pi = (z^a, F_b)^t. \tag{2.195}$$

The Kähler potential becomes

$$e^{-K} = \Pi^t \Sigma \overline{\Pi}, \tag{2.196}$$

where $\Sigma_{ij} = \delta_{i,j-h_{1,1}+1} - \delta_{i-h_{1,1}+1,j}$ is a symplectic unit which coincides with (inverse) intersection matrix of cycles (A^a, B_b). Then in any basis of cycles $\{Q_a\}_{a \leq b_3} \subset H_3(X, \mathbb{Z})$ the Kähler potential reads

$$e^{-K} = \omega^t \, C \, \overline{\omega} = \omega_a(\phi) \, C^{ab} \, \overline{\omega_b(\phi)}, \tag{2.197}$$

where we introduced periods over cycles $\{Q_a\}$

$$\omega_a = \omega_a(\phi) := \int_{Q^a} \Omega \tag{2.198}$$

and $(C^{-1})^{ab} = Q^a \cap Q^b$ is the intersection matrix of cycles.

Three point functions are proportional to what is called Yukawa constants. They are Yukawa constants in heterotic string theory. In type IIB theory they become Pauli constants, that is coupling constants of two fermions and a field strength in vector multiplets [109]. They are computed to be

$$\kappa_{abc} = \int_X \Omega \wedge (h_a \wedge h_b \wedge h_c \Omega) = \int_X \Omega \wedge \partial_a \partial_b \partial_c \Omega, \tag{2.199}$$

where $h_a \wedge h_b \wedge h_c \Omega$ denotes a (0,3)-form $(h_a)^i_{\bar{l}}\,(h_a)^j_{\bar{m}}\,(h_a)^k_{\bar{n}}\,\Omega_{ijk}$. Using the expression for Ω in terms of periods we compute:

$$\kappa_{abc} = \int_X (z^d \alpha_d - F_e \beta^e) \wedge F_{abcf} \beta^f = z^d F_{abcd} = F_{abc} = \partial_a \partial_b \partial_c F. \tag{2.200}$$

In arbitrary coordinates Yukawa constants quite often can be easily computed in terms of combinatorial data or multiplication in local rings of singularities. Special geometry metric and Yukawa constants are enough to find genus zero string contributions to effective four-dimensional theory. These couplings are protected by the supersymmetry and are exact.

Comparing the formulae above with the definition of special Kähler geometry 2.4.4 we see that $\mathcal{L}$ is $H^{3,0}(X) \to \mathcal{M}$.

Each period Π_i is a section of $\mathcal{L}$ that is transforms in the same way as Ω. The bundle $\mathcal{L} \otimes \mathcal{H}$ has a natural connection

$$\mathcal{D}_a \Omega = \partial_a \Omega + \partial_a K\, \Omega = h_a \tag{2.201}$$

Covariant derivative $\mathcal{D} : \mathcal{L} \to \mathcal{L} \otimes T\mathcal{M}^*$ sends $H^{3,0}$ to $H^{2,1}$ so that $H^{2,1} \simeq \mathcal{L} \otimes T^*\mathcal{M}$. Using complex conjugation and Poincaré pairing one obtains the following formulae for (2,1) and (1,2)-forms:

$$\begin{aligned} &\langle h_a, \overline{h_b} \rangle = -e^{-K} G_{a\bar{b}}, \\ &\mathcal{D}_a h_b = e^K \kappa_{abc}\, G^{c\bar{c}}\, \overline{h_c}, \\ &\mathcal{D}_a \overline{h_b} = G_{a\bar{b}}\, \overline{\Omega}. \end{aligned} \tag{2.202}$$

2.4.5 Variation of Hodge structures on Landau-Ginzburg orbifolds

In the first sections of this chapter we discussed N=(2,2) supersymmetric Landau-Ginzburg models and related structures of Frobenius manifold and tt^* geometry. In the case of particular orbifolds of conformal Landau-Ginzburg theories of central charge 9 these structures simplify significantly and define special Kähler geometry and variation of Hodge structures on a conformal deformation space. This is related to the fact that such Landau-Ginzburg orbifolds can be used to define superstring compactifications to four dimensions, where orbifolding plays a role of GSO projection. In this subsection we study special geometries which arise in this way. Certain aspects of this construction can be found, for example, in [110].

Let us remind that a Landau-Ginzburg theory on $\mathbb{C}^n$ is defined by a holomorphic superpotential $W_0(x) : \mathbb{C}^n \to \mathbb{C}$ and a D-term which we disregard as nonuniversal (in particular, ground states couplings do not depend on it). We consider superpotentials which define an isolated singular point at the origin of the target space as before and which are weighted homogeneous. That is there exists a set of rational numbers $0 < q_1, \ldots, q_n < 1$ which are called weights or degrees such that

$$W(\lambda^{q_i} x_i) = \lambda W(x) \tag{2.203}$$

for all $\lambda \in \mathbb{C}^*$. As explained above it implies that W_0 defines a conformal theory with a finite number of vacua.

The chiral ring of the Landau-Ginzburg theory coincides with the Jacobi ideal of the singularity

$$\mathcal{R}_0 \simeq \frac{\mathbb{C}[x_1, \ldots, x_n]}{(\partial_1 W_0, \ldots, \partial_n W_0)}. \tag{2.204}$$

The chiral ring is identified with the ground states space which is a space of differential forms modulo the Jacobi ideal in the topological theory:

$$\mathcal{R}_0 \simeq \frac{\Omega^n_{\mathbb{C}^n}}{\mathrm{d}W_0 \wedge \Omega^{n-1}_{\mathbb{C}^n}}. \tag{2.205}$$

The natural isomorphism between these two descriptions is given by $[f] \to [f\, \mathrm{d}x_1 \cdots \mathrm{d}x_n]$, where $f \in \mathbb{C}[x_1, \ldots, x_5]$. This isomorphism is defined uniquely up to a constant since in the weighted homogeneous case the form $\mathrm{d}^n x := \mathrm{d}x_1 \cdots \mathrm{d}x_n$ is defined as a form of the least weight. The situation becomes more involved in the general case: one has to specify a primitive form which defines an isomorphism (2.205). Each of the descriptions has its own advantages. In particular, there is a ring structure on (2.204) which corresponds to the ring structure on the chiral ring and there is a Frobenius pairing on (2.205) which is a Grothendieck residue pairing:

$$\eta([f\, \mathrm{d}^n x], [g\, \mathrm{d}^n x]) := \mathrm{Res} \frac{f\, g\, \mathrm{d}^n x}{\partial_1 W_0 \cdots \partial_n W_0}. \tag{2.206}$$

The residue is a contour integral around zeros of $\partial_i W_0$ and the pairing is independent on the choice of the representative. This pairing coincides with the topological two-point function of the Landau-Ginzburg theory. The topological three-point function is given by multiplication

$$C(f, g, h) = \eta(\eta([f\, g\, \mathrm{d}^n x], [h\, \mathrm{d}^n x])) = \mathrm{Res} \frac{f\, g\, h\, \mathrm{d}^n x}{\partial_1 W_0 \cdots \partial_n W_0}, \tag{2.207}$$

and is symmetric. The chiral ring of a Landau-Ginzburg theory is naturally a Frobenius algebra. There is also a Hermitian metric g on $\mathcal{R}_0$ which is a tt^* metric for W_0. Together they define a real structure on $\mathcal{R}_0$ by the formula $M = \eta^{-1}g$. This real structure has a simple description for Landau-Ginzburg theories coming from reality structure of Schrodinger equation for wave-forms. We will focus on the real structure a bit later.

Let us pick a basis in $\mathcal{R}_0$ for convenience of the exposition. More precisely, let us pick a set of weighted homogeneous polynomials $\{e_i\}_{i=1}^{\mu} \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\{[e_i\, \mathrm{d}^5 x]\}_{i=1}^{\mu}$ form a basis of $\mathcal{R}_0$ and e_i have the least possible degree. The number μ is the dimension of the chiral ring and is called a Milnor number. Recall that the central charge of a conformal Landau-Ginzburg theory is given by the formula

$$c = 3\sum_{i=1}^{n}(1 - 2q_i), \tag{2.208}$$

where $q_i = wt(x_i)$ with the normalization $wt(W) = 1$. The one third of the central charge is equal to the weight of the Hessian $\det \partial_i\partial_j W_0$ which is a unique element of $\mathcal{R}_0$ of maximal weight.

Let us define a group of *quantum* symmetries Q [111, 112] Since q_i are rational numbers, we can multiply them by their common denominator d and define integers $k_i = q_i$. In the simplest cases $Q \simeq \mathbb{Z}_d$ is a subgroup of weighted homogeneous scalings of W_0 which leaves it invariant. That is $a \in \mathbb{Z}_d$ acts on x_i diagonally by $x_i \to e^{2\pi i\, k_i a/d} x_i$. To define a pure Hodge structure we need to consider a Landau-Ginzburg orbifold on $\mathbb{C}^n/Q$. In this paper we consider such a Landau-Ginzburg orbifold naively, ignoring the so-called twisted sectors.

Let us explain what we mean by an orbifold Landau-Ginzburg theory. Action of Q extends on the ring $\mathbb{C}[x_1, \ldots, x_n]$ and on $\mathcal{R}_0$ if we define the weight of d to be 0 since $\mathrm{d}W$ has weight 1. Chiral ring $\mathcal{R}_0^Q$ of Landau-Ginzburg orbifold is then defined as an invariant part of the chiral ring of the theory without orbifolding.

Important observation is that an element f in $\mathcal{R}_0$ is $\mathbb{Z}_d$-invariant if and only if it has integral weight. Since we restrict to the theories of central charge 9, the maximal weight element is a class of the Hessian $\partial_i\partial_j W_0$ and is equal to 3. Therefore the chiral ring is naturally split into graded components:

$$\mathcal{R}_0^Q = (\mathcal{R}_0^Q)^0 \oplus (\mathcal{R}_0^Q)^1 \oplus (\mathcal{R}_0^Q)^2 \oplus (\mathcal{R}_0^Q)^3. \tag{2.209}$$

This splitting looks like a Hodge decomposition on a Calabi-Yau threefold. However, this structure is somewhat misleading. It turns out that to compare with Hodge structures on Calabi-Yau manifolds we need to consider only the weight filtration (which is not to be mixed with weight filtration in a sence of mixed Hodge structures).

$$\mathcal{R}_0^Q = F^0\mathcal{R}_0^Q \supset F^1\mathcal{R}_0^Q \supset F^2\mathcal{R}_0^Q \supset F^3\mathcal{R}_0^Q = \langle 1\rangle. \tag{2.210}$$

Where $F^i\mathcal{R}_o^Q$ consists of elements of weight not greater than $3-i$. The graded pieces of the filtration are isomorphic to (2.209), but the physical decomposition is generically more complicated. We will discuss the real and integral structures later.

Variations of Hodge structures and 2d gravity To obtain variations of Hodge structures for Landau-Ginzburg orbifolds we need to consider conformal deformations of Landau-Ginzburg theories or Landau-Ginzburg theories coupled to topological gravity. First, let us discuss deformations of Landau-Ginzburg theories. If we disregard D-terms, then the deformations of a Landau-Ginzburg theory are given by deformations of the defining superpotential. Up to coordinate changes such a deformation can be written as $W(x,t) = W_0(x) + \sum_{i=1}^{\mu} t^i e_i$, where $[e_i]$ are basis elements of the chiral ring or chiral fields of the theory. The F-term of the deformed Lagrangian can be written as

$$\int \mathrm{d}^2\theta \left[W_0(X_i) + \sum_{i=1}^{\mu} t^i e_i(X) \right], \tag{2.211}$$

where X_i are chiral superfields and $\mathrm{d}^2\theta$ is a holomorphic volume element on the odd part of the superspace. The deformed theory partition function acquires dependence on the deformation parameters $\{t^i\}$. Expansion coefficients of correlation functions in the variables $\{t^i\}$ are correlation functions of the undeformed theory with insertions of chiral fields integrated over the worldsheet Riemann surface Σ (which is a punctured sphere in this paper):

$$\begin{aligned} &\langle \Phi_1(X)\Phi_2(X)\Phi_3(X) e^{\sum_{i=1}^{\mu} t^i \int_\Sigma \tilde{e}_i(X)} \rangle = \\ &\qquad = \sum_{\bar{m}\geq 0} \frac{t_1^{m_1}\cdots t_\mu^{m_\mu}}{m_1!\cdots m_\mu!} \langle \Phi_1(X)\Phi_2(X)\Phi_3(X) \prod_{k=1}^{\mu} \left(\int_\Sigma \tilde{e}_i(X) \right)^{m_k} \rangle, \end{aligned} \tag{2.212}$$

where $\tilde{e}_i(X)$ is an appropriately dressed $e_i(X)$ which can be integrated over Σ. Its integral is an observable in the Landau-Ginzburg theory coupled to topological gravity.

Space of observables for the latter one is obtained using topological twisting. In the topologically twisted theory one of the supersymmetry operators Q_- is treated as a BRST operator. The energy-momentum tensor becomes exact due to N=2 superalgebra relations $T = \{Q_-, Q_+\}$, and the correlation functions of the theory do not depend on the metric. In particular, the theory is well-defined on any Riemann surface.

The topologically twisted field theory correlation numbers can be obtained from holomorphic limit of $N=2$ theory correlators as discussed in the section 2.2.

In the holomorphic limit the state space of topological gravity is identified with the relative de Rham cohomology group of the twisted de Rham differential $D_z = z\mathrm{d} + \mathrm{d}W\wedge$

$$\mathcal{H}_W^{(0)} = \Omega^n[[z]]/D_z\Omega^{n-1}[[z]]. \tag{2.213}$$

The holomorphic limit map (2.52) $\omega \to \omega^h$ sends a Landau-Ginzburg state with a wave-function ω to a gravitationally dressed state ω^h. Gravitational descendents of ω^h are given by the classes of $z^n \omega^h$ in $\mathcal{H}_W^{(0)}$.

If we fix a value of z, we call the space $\mathcal{H}_W^{(0)}|_{z\in\mathbb{C}^*}$

$$H^n_{D_z}(\mathbb{C}^n) = \Omega^n(\mathbb{C}^n)/D_z\Omega^{n-1}(\mathbb{C}^n). \tag{2.214}$$

This space is isomorphic to $\mathcal{R}$ via the map $\omega \to \omega^h$.

Using the isomorphism with $H^n_{D_z}(\mathbb{C}^n)$ is easy to describe the real structure on $\mathcal{R}$. The homology group $\mathcal{H}_n = H_n(\mathbb{C}^n, \mathrm{Re}(W/z) \ll 0)$ has a natural integral structure given by linear combination of actual cycles. The group $\mathcal{H}_W^{(0)}$ is dual to $\mathcal{H}_n$ and $\omega \in \mathcal{H}^{(0)}$ is integral/real if and only if $\int_\gamma e^{-W/z}\omega$ is an integer/a real number for any integral cycle γ.

Special geometry appears on deformations of conformal Landau-Ginzburg orbifolds of central charge 9. Consider a family of weighted homogeneous polynomials

$$W(x,\phi) = W_0(x) + \sum_{s=1}^{h} \phi_s e_s, \tag{2.215}$$

where $e_s \in \mathcal{R}_0$ of weight 1 and $\phi \in \tilde{\mathcal{M}} \subset \mathbb{C}^h$. For any value $\{\phi_s\}$ there is a Hodge structure on the Q-invariant subspace of the space of oscillating integrands

$$\mathcal{H}^n := H^n_{D_z}(\mathbb{C}^n)^Q = \Omega^n(\mathbb{C}^n)^Q/D_z\Omega^{n-1}(\mathbb{C}^n)^Q. \tag{2.216}$$

For any nonzero z this space is isomorphic to the invariant Milnor ring $\mathcal{R}^Q$. The isomorphism is given by the formula

$$e_i \to e_i\, \mathrm{d}^n x \tag{2.217}$$

and is not canonical, in particular, the form $\mathrm{d}^n x$ is defined up to a constant. The constant can be specified by a choice of the primitive form. The space $\mathcal{H}^n$ consists of elements of weights $1, 2, 3, 4$, if we put $wt(\mathrm{d}) = 0$. There is a corresponding Hodge filtration

$$\mathcal{H}^n = F^0\mathcal{H}^n \supset F^1\mathcal{H}^n \supset F^2\mathcal{H}^n \supset F^3\mathcal{H}^n = \langle \mathrm{d}^n x\rangle. \tag{2.218}$$

The Hodge filtration defined as $F^i\mathcal{H}^n$ consists of elements of weight less or equal to $4-i$. With a slight abuse of notations we also denote $\mathcal{H}^n \xrightarrow{p} \tilde{\mathcal{M}}$ to be a vector bundle with the fiber $p^{-1}(\phi) = H^n_{D_z}(\mathbb{C}^n)^Q$. The Hodge filters are its subbundles $F^i\mathcal{H}^n$ with fibers $F^iH^n_{D_z}(\mathbb{C}^5)^Q$. A set of polynomials $\{e_s\}$ from (2.215) form a basis for almost all $\phi \in \tilde{\mathcal{M}}$ and, therefore, define a trivialization of $\mathcal{R}^Q$ over an open subset of $\tilde{\mathcal{M}}$. In addition, in this trivialization $F^i\mathcal{H}^n$ are trivialized as well, in particular, they are holomorphic vector bundles over $\tilde{\mathcal{M}}$. The Gauss-Manin connection on $\mathcal{H}^n$ comes from the one on $\mathcal{H}_W^{(0)}$ (2.65) and is defined by the formula

$$\nabla_s^{GM}\phi\, \mathrm{d}^n x = \frac{\partial}{\partial t_s}\phi\, \mathrm{d}^n x + z^{-1} e_s\, \phi\, \mathrm{d}^n x, \tag{2.219}$$

where we consider equivalence classes of elements above and fix a nonzero number $z \in \mathbb{C}^*$.

Finally, we define an integral structure on $\mathcal{H}^n$ by duality from the one on $H_n(\mathbb{C}^n, \mathrm{Re}(W/z) \ll 0; \mathbb{Z})$. Integral cycles are continuous families of cycles over the parameter space $\tilde{\mathcal{M}}$. They are flat sections of the Gauss-Manin connection, the fact that parallel transport of a cycle can be locally achieved by a continuous deformation. The formula (2.219) means that:

$$\partial_i \langle \Gamma_z, \omega \rangle := \partial_i \int_{\Gamma_z} e^{W/z}\, \omega = \int_{\Gamma_z} e^{W/z}\, \nabla_i^{GM} \omega, \tag{2.220}$$

$\omega \in \mathcal{H}^n$, $\Gamma_z \in H_n(\mathbb{C}^n, \mathrm{Re}(W/z) \ll 0; \mathbb{Z})$. In pratice, the cycle Γ_z can be locally constant: for small conformal deformations of the superpotential the integral remains convergent over the same cycle.

Corollary 2.4.1. *Let $W(x, \phi)$ be as in* (2.215)*. Then the bundle $\mathcal{H}^n \to \tilde{\mathcal{M}}$ has a natural variation of Hodge structure of weight 3 where the filtration given by homogeneity weights, the Gauss-Manin connection* (2.219) *and the integral structure comes from $H_n(\mathbb{C}^n, \mathrm{Re}(W/z) \ll 0; \mathbb{Z})$.*

The period map of this variation of Hodge structures is defined by exponential periods or oscillating integrals. $F^3\mathcal{H}^n$ is spanned by a class of $\mathrm{d}^n x$ in each chiral ring. Pick a basis $\{\Gamma_z^i\}_{i=1}^h$ in $H_n(\mathbb{C}^n, \mathrm{Re}(W/z) \ll 0; \mathbb{Z})^Q$. Then the embedding $F^3\mathcal{H}^n \subset \mathcal{H}^n$ in the coordinates is given by the formula

$$\left(\int_{\Gamma_z^1} e^{W/z} \mathrm{d}^n x : \ldots : \int_{\Gamma_z^h} e^{W/z} \mathrm{d}^n x \right). \tag{2.221}$$

$F^2\mathcal{H}^n$ is spanned by $\mathrm{d}^n x$ and $\{e_s\, \mathrm{d}^n x = \nabla_s^{GM} \mathrm{d}^n x\}_{s \leq h}$. Similarly, $F^1\mathcal{H}^n$ is spanned by second derivatives of $\mathrm{d}^n x$. It follows that the Hodge filtration on $\mathcal{H}^n$ can be restored from the period integrals (2.221).

2.5 Computation of special Kähler geometries for nonlinear sigma models

2.5.1 Quintic threefold and its mirror

In this section we recall the classical descrition of the moduli space of complex structure deformations on a quintic threefold mirror following [113, 114].Computation of the special geometry on the one-dimensional complex structures moduli space of the mirror quintic was done in [113] almost 30 years ago and was a great success of mirror symmetry as it produced conjectural Gromov-Witten invariants which were not availible by other means at that time. On the contrary, the computation of the special geometry on the 101-dimensional complex structures moduli space of the quintic itself was done by the author and Alexander Belavin in [33].

In this subsection we explain the classical approach to computation of special geometry on the relatively simple example.

Consider a four-dimenional projective space $(\mathbb{C}^5\backslash\{0\})/\mathbb{C}^*$:

$$\mathbb{P}^4 = \{x_1, \cdots, x_5 \in \mathbb{C}^5 \mid \prod_{i\leq 5} x_i \neq 0,\ x_i \sim \lambda x_i,\ \lambda \neq 0\} \tag{2.222}$$

and a complex hypersurface $\mathcal{Q}_0$ in $\mathbb{P}^4$ defined by the equation $W_0(x) = 0$,

$$W_0(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5. \tag{2.223}$$

The polynomial $W_0(x)$ is homoeneous of fifth degree and therefore its zero set is well-defined in $\mathbb{P}^4$. The hypersurface $\mathcal{Q}_0$ is smooth because $\mathrm{d}W_0 = 0$ only at $x = 0$. As a submanifold of a projective space $\mathcal{Q}_0$ is Kähler. We note that the induced Kahler metric on $\mathcal{Q}_0$ is not Ricci-flat. Because of the Lefschetz theorem $H^i(\mathcal{Q}_0) \simeq H^i(\mathbb{P}^4)$ for $i \leq 2$, in particular, $h^{1,1}(\mathcal{Q}_0) = 1$.

The quintic $\mathcal{Q}_0$ is a Calabi-Yau manifold. It can be checked using an explicit holomorphic volume form

$$\Omega_0 := \epsilon^{ijklm} \frac{x_i \mathrm{d}x_j \mathrm{d}x_k \mathrm{d}x_l}{\partial W_0/\partial x_m}, \tag{2.224}$$

where ϵ is an elementary antisymmetric tensor. Ω_0 has a homogeneity degree 0, so it is a well-defined differential form on $\mathcal{Q}_0$. Let us check that it is nonsingular and it does not have zeros. For concreteness pick a coordinate chart $x_5 \neq 0$ on $\mathbb{P}^4$. Wherever $\partial W_0/\partial x_4 \neq 0$ the functions (x_1, x_2, x_3) serve as a coordinate system on $\mathcal{Q}_0$. In such a coodinate system

$$\Omega_0 = \frac{x_5 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{\partial W_0/\partial x_4}. \tag{2.225}$$

In the points where $\partial W_0/\partial x_4$ has a zero, the volume elemet $\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$ has a zero as well cancelling the apparent pole of Ω_0.

Another way to see that Ω_0 is a volume form on $\mathcal{Q}_0$ uses the fact that Ω_0 is a residue at $W_0 = 0$ of a 4-form on $\mathbb{P}^4$.

$$\oint_{W=0} \epsilon^{ijklm} \frac{x_i \mathrm{d}x_j \mathrm{d}x_k \mathrm{d}x_l \mathrm{d}x_m}{W_0(x)} = \oint_{W=0} \frac{\mathrm{d}W_0}{W_0} \epsilon^{ijklm} \frac{x_i \mathrm{d}x_j \mathrm{d}x_k \mathrm{d}x_l}{\mathrm{d}W_0(x)/\mathrm{d}x_m} = \Omega_0. \tag{2.226}$$

The equality above means that an integral of Ω_0 over a cycle γ is equal to an integral of the 4-form on the left hand side over a tubular neighbourhood $T(\gamma)$ of γ in the two-dimensional real plane orthogonal to $W = 0$. One can further rewrite the 4-form (2.226) as a residue of a 5-form in $\mathbb{C}^5$. Indeed, due to homogeneity the integral (2.226) does not depend on x_5. Then we insert a unit $\oint \mathrm{d}x_5/x_5$ inside (in what follows we ignore the $2\pi i$ factors from the residues as they contribute to normalization which is not important in our discussion)

$$\int_{T(\gamma)} \frac{x_5 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4}{W_0(x)} = \int_\Gamma \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5}{W_0(x)}, \tag{2.227}$$

where $\Gamma \subset \mathbb{C}^5$ is a 5-cycle which is a union of fibers of the Hopf fibration $\mathbb{C}^5 \supset S^9 \to \mathbb{P}^4$ over the cycle $T(\gamma)$. Each point on $T(\gamma)$ corresponds to a circle $|x_5| = 1$ in Γ.

One can check that the quintic is a Calabi-Yau manifold by computing Chern classes. At each point of $\mathcal{Q}_0$ tangent space to $\mathbb{P}^4$ decomposes as a direct sum of a tangent space to $\mathcal{Q}_0$ and a normal space to $\mathcal{Q}_0$. In the normal bundle value of W_0 is a local coordinate, therefore the normal bundle is a bundle of polynomials of the fifth degree $\mathcal{O}(5) \to \mathbb{P}^4$. Decomposition of the tangent space to $\mathbb{P}^4$ is expressed by the following short exact sequence of vector bundles:

$$0 \to T\mathcal{Q}_0 \to T\mathbb{P}^4|_{\mathcal{Q}_0} \to \mathcal{O}(5)|_{\mathcal{Q}_0} \to 0 \tag{2.228}$$

In particular, the full Chern class of the quintic $\mathcal{Q}_0$ is a ratio of the full Chern class of the projective space and the full Chern class of $\mathcal{O}(5)$. Let us denote H as a hyperplane section. Then $c(\mathbb{P}^4) = (1+H)^5|_{H^4=0}$ and $c(\mathcal{O}(5)) = 1 + 5H$. The easy computation gives the total Chern class of the quintic $\mathcal{Q}_0$:

$$\begin{aligned} c(T\mathcal{Q}_0) &= 1 + c_1\,H + c_2\,H^2 + c_3\,H^3, \quad H^4|_{Q_0} = 0, \\ c(T\mathcal{Q}_0) &= \frac{(1+H)^5}{1+5H} = 1 + 10H^2 - 40H^3. \end{aligned} \tag{2.229}$$

First Chern class of $\mathcal{Q}_0$ is equal to 0 as it should. The Euler characteristic is computed to be

$$\chi(\mathcal{Q}_0) = \int_{\mathcal{Q}_0} c_3(\mathcal{Q}_0) = \int_{\mathbb{P}^4} c_3(\mathcal{Q}_0)\,(5H) = -\int_{\mathbb{P}^4} 200H^4 = -200. \tag{2.230}$$

For simply connected Calabi-Yau threefolds $\chi = 2h^{1,1} - 2h^{2,1}$ from which it follows that $h^{2,1}(\mathcal{Q}_0) = 101$.

All complex structure deformations can be obtained from polynomial deformations of the quintic. Consider a general homogeneous polynomial deformation of $W_0(x)$:

$$W_0(x) + \sum_{s_1+\cdots+s_5=5} \phi_{s_1\cdots s_5} x_1^{s_1} \cdots x_5^{s_5}. \tag{2.231}$$

Almost all such polynomials define Calabi-Yau threefolds $\mathcal{Q}_\phi$ diffeomeorphic to $\mathcal{Q}_0$ because they can be connected by smooth deformations. However, different manifolds from the family can have different complex structures. Let us see when $\mathcal{Q}_{\phi_1} \simeq \mathcal{Q}_{\phi_2}$ as a complex manifold. In total, there are 126 homogeneous monomials of degree 5:

$$\begin{aligned} x_i^5 - 5,\ x_i^4 x_j - 20,\ x_i^3 x_j^2 - 20,\ x_i^3 x_j x_k - 30,\ x_i^2 x_j^2 x_k - 30, \\ x_i^2 x_j x_k x_l - 20,\ x_1 x_2 x_3 x_4 x_5 - 1. \end{aligned} \tag{2.232}$$

$\mathbb{P}^4$ has an automorphism group PGL(5). Combined with projective dilations it forms a group GL(5). It is clear that any two hypersurfaces connected by a GL(5) transformations are isomorphic. It turns out that these are the only continuous families of isomophisms of $\mathcal{Q}_\phi$. There are in total 126-25=101 continuous independent deformation parameters which coincide with $h^{2,1}$.

Let us compute infinitesimal deformations around $W_0(x)$.

$$W_0(x + \delta x) = W_0(x) + \sum_i \partial_i W_0(x) \delta x_i + O(\delta^2 x). \tag{2.233}$$

It follows that elements from the Jacobi ideal $(\partial_1 W_0, \ldots, \partial_5 W_0)$ are coordinate transformations in the first order. The deformation space is naturally isomorphic to the invariant chiral ring $\mathcal{R}_0^Q$ of $W_0(x)$ with the quantum symmetry group $Q = \mathbb{Z}_5$. The latter group is generated by an element g which acts on x_i as $g(x_i) = \alpha x_i,\ \alpha^5 = 1$.

Mirror quintic The mirror manifold to the quintic was constructed in [111] and is a factor of the quitic itself with respect to a finite group $\mathbb{Z}_5^3$. The group acts on $\mathbb{P}^4$ and the action sends $\mathcal{Q}_0$ to itself and. The generators of the action can be chosen as $g_1 : x_1 \to \alpha x_1,\ x_5 \to \alpha^4 x_5,\ g_2 : x_2 \to \alpha x_3,\ x_5 \to \alpha^4 x_5,\ g_3 : x_3 \to \alpha x_3,\ x_5 \to \alpha^4 x_5,$, where $\alpha^5 = 1$ as above. Out of all the family (2.231) the group $\mathbb{Z}_5^3$ leaves invariant only a one-dimensional subfamily

$$W(x, \phi_1) := \sum_{i \leq 5} x_i^5 + \phi_1 \prod_{i \leq 5} x_i = 0, \tag{2.234}$$

which is called a Dwork family with the fibers $\mathcal{Q}_{\phi_1}$. The group $\mathbb{Z}_5^3$ does not act on $\mathcal{Q}_{\phi_1}$ freely. The action has invariant curves which intersect in invariant points. The factor-variety $\check{\mathcal{Q}}_{\phi_1} := \mathcal{Q}_{\phi_1}/\mathbb{Z}_5^3$ has orbifold singularities. In order to get a smooth mirror quintic one can perform blowups of the singular loci. When blowing up there will be additional cycles in $H^{2,2}$ and $H^{1,1}$. The Euler characteristic of the blown up manifold is 200 [113], therefore $h^{1,1}(\check{\mathcal{Q}}_{\phi_1}) = 101$ as it should be for a mirror manifold.

The orbifold singularities do not affect complex structure deformations of the mirror quintic. Instead of working with orbifold singularities or blow ups we can consider a $\mathbb{Z}_5^3$ invariant part of the variation of Hodge structures over the Dwork family (2.234). This is because the finite global quotient is similar to gauge theory with the finite gauge group.

The holomorphic volume form on $\mathcal{Q}_{\phi_1}$ is equal to the double residue of the following form in $\mathbb{C}^5$

$$\frac{\mathrm{d}x_1 \cdots \mathrm{d}x_5}{W(x, \phi_1)}. \tag{2.235}$$

This form is preserved by $\mathbb{Z}_5^3$ and thus its cohomology class belongs to a $\mathbb{Z}_5^3$-invariant part of $H^3(\mathcal{Q}_{\phi_1})$. The family (2.234) is one-dimensional and is relatively easy to analyze. One thing to notice is that $\mathcal{Q}_{\phi_1} \simeq \mathcal{Q}_{\alpha\phi_1}$ where the isomorphism is given, for example, by $x_1 \to \alpha x_1$. The correct coordinate on the moduli space of complex structures on the mirror quintic is $-z^{-1} := \phi_1^5$ (the sign and power are due to historical reasons). The compactified moduli space of mirror quintics is identified with a plane of coordinate ϕ_1 together with infinity and modulo $\mathbb{Z}_5$ action. Such a space $\mathbb{P}^1_{(1:5)} = \mathbb{P}^1/\mathbb{Z}_5$ is called a weighed projective line.

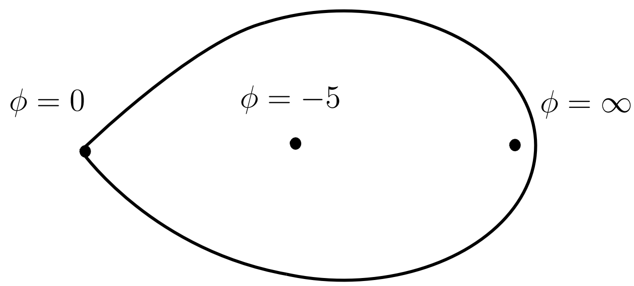

Figure 2.5.1: Mirror quintic complex structures moduli space $\mathbb{P}^1_{(1,5)}$

There are three special points on this moduli space: $z = 0, 5^{-5}, \infty$. The first point $z = 0, \phi_1 = \infty$ corresponds to a degenerate complex structure where the manifold becomes a union of 5 hyperplanes. Such a point is called a maximal unipotent monodomy point since period integrals of the holomorphic volume form have maximal unipotent monodromy around this point. It is this point which under the mirror symmetry corresponds to the large volume point on the Kähler moduli space of the quintic itself around which instanton contributions are exponentially small. Such points in the moduli spaces are interesting due to their connections with Gromov-Witten theories of mirror manifolds.

The point $z = 5^{-5}, \phi_1 = -5$ is called a conifold point since $\mathcal{Q}_{-5}$ has a conifold singularity at $x_1 = 1, \ldots, x_5 = 1$ (125 singularities which are identified by $\mathbb{Z}_5^3$. Indeed, the critical point equation is

$$5x_i^4 + \phi_1 \prod_{j \neq i} x_j = 0 \tag{2.236}$$

and has the solution $x_1 = \cdots = x_5 = \mathit{const}$ at $\phi_1 = -5$. The Hessian matrix at this point is nondegenerate so that the point is really a conifold point of the form $xy = zw$

in $\mathbb{C}^4$ and loocls like a cone over a nontrivial fibration over S^3 with S^2 fibers [115].Such a singularity can be resolved in two natural ways: either using a deformation or a blowup. String theory gives a transition between different resolutions. Studying of conifold singularities is important for phenomenology. In particular, in the brane world scenario some of the background D-branes are placed at conifold singularities [116, 117].Conifold singularities can be interesting from purely mathematical viewpoint as well. In particular, performing small resolutions of all 125 singularities of the mirror quintic one obtains the so-called Schoen quintic [118][?] which does not have any complex structure deformations. Period integrals for such manifold are connected with periods of certain modular forms $f_4(25)$ whose appearance is not directly connected with monodromy and is not very well understood yet.

Finally, the point $z = \infty, \phi_1 = 0$ is called the orbifold point. The manifold $\mathcal{Q}_0$ is smooth but have an extended symmetry group $\mathbb{Z}_5^5$ which independently scales all the coordinates by fifth roots of unity $x_i \to \alpha x_i$. It coincides with an orbifold point of the weighted projective line $\mathbb{P}^1_{(1:5)}$ and reflects the common phenomenon of the moduli problems: objects with enhanced symmetry correspond to orbifold points in moduli spaces, where the orbifolding group is equal to the jump in the symmetry at the orbifold point. Mirror versions of orbifold points are FJRW theories.

Mirror quintic periods Let us proceed to special geometry on the one-dimensional moduli space of mirror quintics. As explained above, it has only one complex structure deformation, that is $h^{2,1} = 1$ and $b^3 = 4$. In other words $\mathbb{Z}_5^3$-invariant part of $H^3(\mathcal{Q}_{\phi_1})$ is four-dimensional and contains $H^{3,0}(\mathcal{Q}_{\phi_1})$. To compute variations of Hodge structure we compute a period map which is reconstructed from four independent period integrals of Ω_{ϕ_1}. We more or less follow the exposition of [113]. First we compute the periods around the maximal unipotent monodromy point $\phi_1 = \infty$. Periods around this point can be computed using power series expansion in $z = -\phi_1^{-5}$:

$$\omega_0(z) := \frac{-z^{-1/5}}{(2\pi i)^5} \int_{T_5} \frac{\mathrm{d}^5 x}{\sum_i x_i^5 - z^{-1/5} \prod x_i}, \tag{2.237}$$

where the normalization $z^{-1/5}/(2\pi i)^5$ is chosen for convenience and the cycle T_5 is a five-dimensional torus in $\mathbb{C}^5$ surrounding the union of coordinate hyperplanes. It is a double tubular neighbourhood of a certain 3-dimensional cycle in $H_3(\mathcal{Q}_{\phi_1})$. This choice of the contour is due to the fact that for small z the integral (2.237) has the form

$$\omega_0(z) \sim \frac{1}{(2\pi i)^5} \int_{T_5} \frac{\mathrm{d}^5 x}{\prod x_i} \tag{2.238}$$

and is easily computed through the product of five residues Expanding (2.237) in power series in z we compute

$$\omega_0(z) = \frac{1}{(2\pi i)^5} \sum_{n=0}^{\infty} \int_{T_5} \frac{\mathrm{d}^5 x}{\prod x_i} \left(z^{1/5} \frac{\sum x_i^5}{\prod x_i} \right)^n . \tag{2.239}$$

After expanding the expression further it becomes a sum of integrals of Laurent polynomials. Only constant terms of such polynomials have nonvanishing residues. These summands are of the type

$$z^n \frac{(5n)!}{(n!)^5}. \tag{2.240}$$

Plugging this expression into the integral (2.239) we get the formula for the period integral

$$\omega_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n. \tag{2.241}$$

To get another useful representation for the period we use the multiplication formula for the gamma function:

$$\Gamma(nx) = (2\pi)^{\frac{n-1}{2}} n^{1/2-nx} \Gamma(x) \Gamma\left(x + \frac{1}{n}\right) \cdots \Gamma\left(x + \frac{n-1}{n}\right) \tag{2.242}$$

with the result

$$\omega_0(z) = (2\pi)^2 5^{1/2} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{5}\right) \Gamma\left(n+\frac{2}{5}\right) \Gamma\left(n+\frac{3}{5}\right) \Gamma\left(n+\frac{4}{5}\right)}{\Gamma(n+1)^3 n!} (z/5^5)^n. \tag{2.243}$$

In this representation it is clear that $\omega_0(z)$ is a generalized hypergeometric function

$$\omega_0(z) = (2\pi)^2 5^{1/2} {}_4F_3 \left(\begin{matrix} \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \\ 1, 1, 1 \end{matrix} ; z/5^5 \right). \tag{2.244}$$

This function is a balanced hypergeometric function, that is the sum of the arguments in the numerator is equal to the sum of the arguments in the denominator + 1. It satisfies a hypergeometric equation of the fourth degree:

$$\left[\theta^4 - \frac{z}{5^5} \left(\theta - \frac{1}{5}\right) \left(\theta - \frac{2}{5}\right) \left(\theta - \frac{3}{5}\right) \left(\theta - \frac{4}{5}\right) \right] \omega_0(z) = 0, \tag{2.245}$$

where $\theta = z\mathrm{d}/\mathrm{d}z$. Three other periods are also solutions of the same differential equation. This reflets a general story for period integrals of Calabi-Yau manifolds. Periods of holomorphic and meromorphic forms on families of algebraic varieties satisfy differential equations of very specific kind which are called Picard-Fuchs equations [119].

The simplest example is an ordinary hypergeometric equation with the parameters $\pm 1/2$ which is satisfied by the periods of elliptic curves or one-dimensional Calabi-Yau manifolds.

Let us compute the other three period integrals. It is convenient to use the Frobenius method for this purpose. Indices of all four solutions of the equation (2.245) are equal to zero at $z=0$, that is the solutions are of the following type:

$$\begin{aligned} &\omega_0(z) = 1 + \cdots\,,\ \omega_1(z) = \omega_0(z)\log z + \cdots\,, \\ &\omega_2(z) = \omega_0(z)\log^2 z + \cdots\,,\ \omega_3(z) = \omega_0(z)\log^3 z + \cdots\,. \end{aligned} \tag{2.246}$$

Point $z=0$ for the equation (2.245) is called a maximal unipotent monodromy point because the monodromy matrix of solutions (2.246) is in the conjugacy class of a unique Jordan block with eigenvalue equal to 1.

$$\omega_i(z) \to \omega_i(z) + \omega_{i-1}(z) + \cdots\,. \tag{2.247}$$

According to the Frobenius method the periods can be evaluated from the following generating function which itself is constructed from $\omega_0(z)$:

$$\begin{aligned} \omega(z,H) := \sum_{n=0}^{\infty} \frac{\Gamma(5n+5H+1)}{\Gamma(n+H+1)^5} z^{n+H}|_{H^4=0} = \\ = \omega_0(z) + \tilde{\omega}_1(z)H + \frac{1}{2}\tilde{\omega}_2(z)H^2 + \frac{1}{6}\tilde{\omega}_3(z)H^3. \end{aligned} \tag{2.248}$$

Periods $\tilde{\omega}_i(z)$ are computed using the Frobenius method and therefore they are not integrals over actual cycles but are complex linear combinations of such integrals. The function $\omega(z,H)$ is closely related with the Givental J-function of the quintic $\mathcal{Q}$ where H should be understood as a Kähler class of the quintic.

We introduce

$$\tilde{\omega}_k(z) = \sum_{i=0}^{k} \binom{k}{i} f_i(z) \log^{k-i} z, \tag{2.249}$$

where

$$f_i(z) = \sum_{n\geq 0} z^n \frac{\partial^i}{\partial H^i}\left(\frac{\Gamma(5n+5H+1)}{\Gamma(n+H+1)^5}|_{H=0}\right). \tag{2.250}$$

Derivatives of the gamma functions are expressed using multigamma functions $\psi^{(0)}(z) = \Gamma'(z)/\Gamma(z)$, $\psi^{(m)}(z) = \partial^m/\partial z^m \psi(z)$ with the result

$$\begin{aligned} f_1(z) &= \sum_n 5\Phi(0)\frac{\Gamma(5n+1)}{\Gamma(n+1)^5} z^n, \\ f_2(z) &= \sum_n (5\Phi'(0) + 25\Phi(0)^2)\frac{\Gamma(5n+1)}{\Gamma(n+1)^5} z^n, \\ f_3(z) &= \sum_n (5\Phi''(0) + 75\Phi(0)'\Phi(0) + 125\Phi(0)^3)\frac{\Gamma(5n+1)}{\Gamma(n+1)^5} z^n, \end{aligned} \tag{2.251}$$

where we introduced another function

$$\begin{aligned}\Phi(H) = \Phi_n(H) =& 5(\psi(5n+5H+1) - \psi(n+H+1)) - \\ & - 5(\psi(5H+1) - \psi(H+1)) = \\ & = 5\sum_{m=1}^{5n} \frac{1}{m+5H} - 5\sum_{m=1}^{n} \frac{1}{m+H}.\end{aligned} \tag{2.252}$$

This data turns out to be enough to compute unnormalized Yukawa couplings and the mirror map. However, in order to compute the special Kähler geometry on the moduli space one has to compute periods in a symplectic basis of cycles or, at least, to compute it in a real basis of cycles and the intersection matrix of the cycles.

The classical approach to compute periods in a symplectic basis of cycles uses analytic continuation of the period $\omega_0(z)$ and monodromy of the periods around special points of the moduli space. Careful examination of the monodromy allows to have enough constraints to compute a symplectic basis of cycles in some cases. Analytic continuation of $\omega_0(z)$ is done with the help of the Mellin-Barnes type representation of hypergeometric functions:

$$\begin{aligned}\omega_0(z) = \sum_{n=0}^{\infty} \frac{\Gamma(5n+1)}{\Gamma(n+1)^5} z^n = \oint \frac{\mathrm{d}s}{e^{2\pi i s}-1} \frac{\Gamma(5s+1)}{\Gamma(s+1)^5} z^s = \\ = \frac{1}{2i} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{5}\right)^5}{n!} z^{-\frac{n+1}{5}} \sin^4 \pi \frac{n+1}{5} e^{\pi i \frac{n+1}{5}},\end{aligned} \tag{2.253}$$

where in the top line the contour is closed to the right hand side picking up the residues at $s = n$, whereas in the bottom line the contour is closed to the left and the residues are at the points $s = -(n+1)/5$. The first series converges where $|z| < 5^{-5}$ and the second where $|z| > 5^{-5}$. The boundary of convergence is a conifold point $z = 5^{-5}$ which illustrates another common point of moduli spaces: periods have different series expansions in different regions. The radius of convergence of such a series is equal to the distance to the closest special point in the moduli space. Different series expansions correspond to different *phases* on the mirror side.

In the last equality in (2.253) the expansion is over fractional powers of z which is consistent with the fact that $z = \infty$ is an orbifold point in the moduli space. This can be used to compute other three periods over integral cycles. The period $\omega_0(z)$ has nontrivial monodromy around $z = \infty,\ z \to e^{2k\pi i} z = \alpha^k z$. Applying this three times we compute four period integrals $\omega_k(z) = \omega_0(\alpha^k z),\ 0 \le k \le 3$ around $z = \infty$:

$$\omega_k(\phi) := \frac{1}{2i} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{5}\right)^5}{n!} z^{-\frac{n+1}{5}} \sin^4 \pi \frac{n+1}{5} \alpha^{(n+1)(k+1/2)}. \tag{2.254}$$

The sum over $k \in \mathbb{Z}_5$ of these periods is equal to zero, therefore there are only four independent periods as it should be.

This method of computation of the special geometry is quite complicated and depends on the parameters of the concrete Calabi-Yau manifolds, in particular, it depends on the monodromy properties in the moduli space. In the next sections we describe our method of computation of the special Kähler geometry which is much simpler in many cases and allows to compute special geometry for many examples which were not known before.

2.5.2 Landau-Ginzburg Calabi-Yau correspondence

In this subsection we explain the main idea behind our method of computation of special Kähler geometry on the moduli space of complex structures. Above we explained how variations of Hodge structures of special sort and special geometry appear on families of Calabi-Yau manifolds and on Landau-Ginzburg orbifolds of central charge $3(n-2)$. In fact, these classes intersect: at least ground states correlation functions coincide with the ones for Calabi-Yau manifolds. This correspondence can be established purely in mathematical terms. In many examples it allows to simplify drastically special geometry computations for Calabi-Yau manifolds.

Remark 2.5.1. *In what follows we consider mostly the case $n = 5$ both for the sake of simplicity and keeping in mind main applications to Calabi-Yau threefolds. Even though most of the formulas hold true in general case with appropriate modifications. In particular, the Hodge structures which appear on the state spaces are of weight $n-2$ insted of 3 and the intersection (topological) pairing is symmetric in the even dimension.*

Landau-Ginzburg Let $W(x,\phi) = W_0(x) + \sum_{s=1}^{h} \phi_s e_s$ be a family of weighted homogeneous polynomials in $\mathbb{C}^n$ as in subsection 2.4.5. In particular, W_0 has an isolated singularity at the origin, $\{e_s\}_{s=1}^{h}$ form a basis of the Milnor ring of $W_0(x)$, and the central charge of $W(x,\phi)$ is equal to $3(n-2)$. We put $\{k_i\}_{i=1}^{n}$ to be minimal integral weights of $\{x_i\}$ and d is a weight of $W(x,\phi)$.

Each such polynomial defines a pair of physical theories: a Landau-Ginzburg orbifold and a nonlinear sigma model. The Landau-Ginzburg orbifold has a superpotential $W(x,\phi)$ and has a target space $\mathbb{C}^n/\mathbb{Z}_d$, where $\alpha \in \mathbb{Z}_d$, $\alpha^d = 1$ acts as $x_i \to \alpha^{k_i} x_i$. We call $\tilde{\mathcal{M}}$ an open subset in the affine space with coordinates $\{\phi_s\}$, that is the deformation space. The tilda stands for the fact that $\tilde{\mathcal{M}}$ is a finite cover of (a subset in) the actual moduli space.

The Landau-Ginzburg theory defines a bundle of oscillating integrals $\mathcal{H}^n \xrightarrow{p} \tilde{\mathcal{M}}$ where a fiber is $p^{-1}(\phi) = H^n_{D_z}(\mathbb{C}^n)^Q = (\mathcal{H}^{(0)}_z)^Q|_{z=const\in\mathbb{C}^*}$ and is isomorphic to the chiral ring $\mathcal{R}_\phi^Q$. We explained in subsection 2.4.5 that there exists a natural variation of Hodge structures of weight $n-2$ on the bundle $\mathcal{H}^n \to \tilde{\mathcal{M}}$, where both the Gauss-Manin connection and the integral structure come from the connection with complex oscillatory integrals, and Hodge filtration coincides with the weight filtration on $H^n_{D_z}(\mathbb{C}^n)^Q$.

Calabi-Yau Second physical theory is a Calabi-Yau non-linear sigma model. The fact that the polynomial $W(x,\phi)$ is weighted homogeneous for all ϕ means that there is a $\mathbb{C}^*$ action on $\mathbb{C}^n$ which leaves zeros of this polynomial invariant. Thus the zero locus $W(x,\phi)=0$ is well-defined on a factor space

$$\mathbb{P}^{n-1}_{\bar{k}} = \mathbb{P}^{n-1}_{(k_1:\ldots:k_n)} := (\mathbb{C}^n \backslash 0)/\mathbb{C}^*, \quad x_i \sim \lambda^{k_i} x_i. \tag{2.255}$$

If the polynomial W_0 is just homogeneous, that is $k_i = 1, d = n$, then the factor space $\mathbb{P}^{n-1}_{(k_1:\ldots:k_n)} = \mathbb{P}^{n-1}$ is an ordinary projective space. In general $\mathbb{P}^{n-1}_{(k_1:\ldots:k_n)}$ is called a weighted projective space [120] and has orbifold singularities.

The equation $W(x,\phi)=0$ defines an orbifold Calabi-Yau hypersurface $\mathcal{X}_\phi$ inside $\mathbb{P}^4_{\bar{k}}$ under our assumptions on $W(x,\phi)$ for almost all $\phi \in \tilde{\mathcal{M}}$. Such an orbifold has only mild singularities and, in particular, there is a pure Hodge structure on the middle cohomology group $H^{n-2}(\mathcal{X}_\phi)$. In fact, there is a subgroup $H^{n-2}_{poly}(\mathcal{X}_\phi)$ which is generated by the holomorphic volume form.

In the case where $n=5$ the middle cohomology is $H^3(\mathcal{X}_\phi)$. The subgroup $HH^3_{poly}(\mathcal{X}_\phi)$ is generated by $H^{3,0}(\mathcal{X}_\phi)$, (2,1)-forms corresponding to polynomial deformations of $W(x,\phi)$ and their complex conjugates.

Unions of cohomology fibers over the deformation space $\tilde{\mathcal{M}}$ form the cohomology bundles $\mathcal{H} \xrightarrow{q} \tilde{\mathcal{M}}$ and $\mathcal{H}_{poly} \xrightarrow{\tilde{q}} \tilde{\mathcal{M}}$ over the deformation space. As in the subsection 2.4.3 these exist natural variations of Hodge structures of weight $n-2$ on $\mathcal{H}$ and on its subbundle $\mathcal{H}_{poly}$ since it is compatible with the integral structure and is invariant with respect to the Gauss-Manin connection. For more details on variations of Hodge structures for orbifolds see [114] or appendix 2.A.

The main result of this section is

Proposition 2.5.1. *Let $W(x,\phi)$ be as above. Then the variations of Hodge structures on the deformations of the singularity $\mathcal{H}^n \to \tilde{\mathcal{M}}$ and on deformations of the Calabi-Yau manifold $\mathcal{H}_{poly} \to \tilde{\mathcal{M}}$ are isomorphic, that is there is a vector bundle isomorphism $\mathcal{H}^n \overset{\iota}{\simeq} \mathcal{H}_{poly}$ which commutes with integral structures and Gauss-Manin connections.*

Proof. Let us explain why this is true. A holomorphic volume form on a threefold $\mathcal{X}_\phi$ is

$$\Omega_\phi := \epsilon^{ijklm} \frac{x_i \mathrm{d}x_j \mathrm{d}x_k \mathrm{d}x_l}{\partial W_0/\partial x_m} \tag{2.256}$$

We can rewrite it as a double residue using the same argument as was used in the case of the quintic threefold.

$$\oint_{W=0} \epsilon^{ijklm} \frac{x_i \mathrm{d}x_j \mathrm{d}x_k \mathrm{d}x_l \mathrm{d}x_m}{W_0(x)} = \oint_{W=0} \epsilon^{ijklm} \frac{\mathrm{d}W_0}{W_0} \frac{x_i \mathrm{d}x_j \mathrm{d}x_k \mathrm{d}x_l}{\mathrm{d}W_0(x)/\mathrm{d}x_m} = \Omega_0. \tag{2.257}$$

The 4-form above is expressed as a residue of the followng form in $\mathbb{C}^5$

$$\int_\Gamma \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5}{W_0(x)}. \tag{2.258}$$

$\Omega \in \Gamma(\tilde{\mathcal{M}}, \mathcal{H}_{poly})$. We define an isomorphism ι as $\iota(\Omega_\phi) \to \mathrm{d}^5x \in H^5_{D_z}(\mathbb{C}^5)^Q$. As was explained in subsection 2.4.3, variations of Hodge structures of the type considered here can be completely reconstructed from the period integrals of a holomorphic volume form which is a section of $\Gamma(\tilde{\mathcal{M}}, F^3\mathcal{H}_{poly})$.

To prove the proposition we prove that the period integrals of Ω_ϕ and $\mathrm{d}^n x$ coincide.

Lemma 2.5.1 (Key lemma)**.** *Let a cycle* $\gamma \in H^3_{poly}(\mathcal{X}_\phi)^*$. *Then there exist* $\Gamma_z \in H_5(\mathbb{C}^5, \mathrm{Re}(W/z) \ll 0)$ *such that*

$$\int_\gamma \Omega_\phi = z \int_{\Gamma_z} e^{-W(x,\phi)/z} \mathrm{d}^5 x. \tag{2.259}$$

If γ *is an integral cycle then so is* Γ.

Proof. This lemma is a variation of well-known results in the singularity theory about the connection between complex oscillating integrals and periods [121]. First of all we use the definition of Ω_ϕ to rewrite the period in the left hand side as a residue integral (2.259)

$$\int_\gamma \Omega_\phi = \int_{T(\gamma)} \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5}{W_0(x)}, \tag{2.260}$$

where $T(\gamma) \in H^5(\mathbb{C}^5 \backslash \{W = 0\})$. This cycle surrounds union of Hopf fibers over γ in the direction normal to $W = 0$.

Consider a nearby Milnor fiber $\{W(x, \phi) = w\} \subset \mathbb{C}^5$. Outside of a ball around the critical point 0 it is a small perturbation of the central Milnor fiber $\{W = 0\}$. Since the cycle $T(\gamma)$ can be deformed away of the origin, it defines an element of $H^5(\mathbb{C}^5 \backslash \{W = w\})$ for small w as well. We denote a class of such a cycle $T(\gamma_w)$ because it surrounds a 4-dimensional cycle γ_w inside (we note that γ is three-dimensional and γ_0 would be a union of Hopf fibers over γ). We compute

$$\int_{T(\gamma_w)} \frac{\mathrm{d}^5 x}{W(x,\phi_1) - w} = \int_\gamma \frac{\mathrm{d}^5 x}{W(x,\phi_1)} \sum_{n=0}^{\infty} \left(\frac{w}{W(x,\phi_1)} \right) = \int_\gamma \frac{\mathrm{d}^5 x}{W(x,\phi_1)}. \tag{2.261}$$

due to weighted homogeneity. Using this and inserting the 1 as $z\int_{w\geq 0} e^{-w/z}\mathrm{d}w$ we have

$$\int_{T(\gamma)} \frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\mathrm{d}x_5}{W(x,\phi_1)} = \int_{T(\gamma_w)} \frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\mathrm{d}x_5}{W(x,\phi_1)-w} =$$
$$= z\int_{w>0} e^{-w/z}\left(\int_{T(\gamma_w)} \frac{\mathrm{d}^5x}{W(x,\phi_1)-w}\right)\mathrm{d}w, \quad (2.262)$$

Now we take a residue at $W=w$ in the inner integral in (2.262)

$$z\int_{w>0} e^{-w/z}\left(\int_{T(\gamma_w)} \frac{\mathrm{d}^5x}{W(x,\phi_1)-w}\right)\mathrm{d}w = z\int_{\Gamma_z:=\cup_w\gamma_w} e^{-w/z}\frac{\mathrm{d}^4x\mathrm{d}w}{\partial W(x,\phi_1)/\partial x_5}. \quad (2.263)$$

The five-dimensional cycle Γ_z is a union of all cycles γ_w for $w\in z\mathbb{R}_+$.

For any $x\in\Gamma_z$ we have $W(x,\phi)=w$. The last step is a coordinate change $x_1,\dots,x_4,w\to x_1,\dots,x_5$:

$$z\int_{\Gamma_z} e^{-w/z}\frac{\mathrm{d}^4x\mathrm{d}w}{\partial W(x,\phi_1)/\partial x_5} = z\int_{\Gamma_z} e^{-W(x,\phi_1)/z}\mathrm{d}^5x. \quad (2.264)$$

□

With this lemma it is easy to construct the map ι. Elements of the form

$$\nabla^{GM}_{i_1}\cdots\nabla^{GM}_{i_1}\mathrm{d}^5x = \partial_{\phi_{i_1}}W\cdots\partial_{\phi_{i_1}}W\,\mathrm{d}^5x = e_{i_1}\cdots e_{i_s}\,\mathrm{d}^5x \quad (2.265)$$

generate all $H^5_{D_z}(\mathbb{C}^5)^Q$ because 1 and $\partial_{\phi_k}W$ form a multiplicative basis in $\mathcal{R}^Q_\phi$ by construction. Since ι commutes with the Gauss-Manin connection we should have

$$\iota(e_{i_1}\cdots e_{i_s}\,\mathrm{d}^5x) = \frac{\partial}{\partial\phi_{i_1}}\cdots\frac{\partial}{\partial\phi_{i_1}}\Omega_\phi. \quad (2.266)$$

□

In the physical language the proposition (2.5.1) establish the equality between deformed topologically twisted disk one-point functions for Landau-Ginzburg orbifolds and for corresponding non-linear Calabi-Yau sigma models. Complex oscillatory integrals or exponential periods being disk one-point functions in Landau-Ginzburg theory and periods of the holomorphic forms are disk one-point functions for sigma models. The map ι is an isomorphism between topologically B-twisted observables and A-twisted branes or boundary conditions [100]. The isomorphism between the branes is $\iota^*(\gamma)\to\Gamma_z$. The A-twisted branes are special Lagrangian representatives in $H_3(\mathcal{X}_\phi)$ and $H_5(\mathbb{C}^5,\mathrm{Re}(W/z)\ll 0)$ whose pairing with B-twisted observables is given by the (exponential) periods. In order to prove equivalence of the special geometries we need to prove that the pairings of the states or branes coincide.

Proposition 2.5.2. *In the setting of the proposition* (2.5.1) *the intersection matrix of cycles* $C_{ij}^{-1} := \gamma_i \cap \gamma_j$ *coincides with the intersection matrix* $C_{ij}^{-1} = \Gamma_z^i \cap \Gamma_{-z}^j$ *up to overall normalization.*

Proof. We present the proof for the cocycles. The intersection pairing on $H^5_{D_z}(\mathbb{C}^5)^Q$ is the higher residue pairing

$$K(e_i\,\mathrm{d}^5x, e_j\,\mathrm{d}^5x) = C^{ab}\int_{\Gamma_+^a} e^{W/z} e_i\,\mathrm{d}^5x \int_{\Gamma_-^a} e^{-W/z} e_j\,\mathrm{d}^5x = z^5 \mathrm{Res}\frac{e_i\,e_j\mathrm{d}^5x}{\partial_1 W\cdots\partial_5 W} + O(z), \tag{2.267}$$

where the $O(z)$ part vanishes if $wt(e_i) + wr(e_j) < wt(\mathrm{Hess}W)$ by the saddle point approximation and homogeneity.

Let us denote $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ and $\partial_I = \partial_{s^{i_1}} cdots \partial_{i^p}$. It is known (e.g. [113]) that

$$\int_{\mathcal{X}} \partial_I\Omega \wedge \partial_J\Omega = \mathrm{Res}\frac{(\prod_{s\in I\sqcup J} e_s)\,\mathrm{d}^5x}{\partial_1 W\cdots\partial_5 W}, \tag{2.268}$$

if $|I|+|J| = 3$ and is equal to zero if $|I|+|J| < 3$. Via the map ι we see that the pairings of the cocycles coincide on $F^p\mathcal{H}_{poly} \otimes F^q\mathcal{H}_{poly}$ and $F^p\mathcal{H}^n \otimes F^q\mathcal{H}^n$ for $p+q \leq 3$. Since the both pairings are real, they coincide also on $\overline{F^p\mathcal{H}_{poly}} \otimes \overline{F^q\mathcal{H}_{poly}}$ and $\overline{F^p\mathcal{H}^n} \otimes \overline{F^q\mathcal{H}^n}$ and, therefore, on the full spaces as well.

□

Remark 2.5.2. *The propositions above extend to the case of general* n. *The spaces* $H^{n-2}_{poly}(\mathcal{X}_\phi) \subset H^{n-2}(\mathcal{X}_\phi)$ *are generated by the map* ι (2.266) *Gauss-Manin derivatives of the holomorphic volume form* Ω *with respect to the polynomial deformations of* $\mathcal{X}_\phi$.

Remark 2.5.3. *The proposition 2.5.2 says that the Hodge structures on* $\mathcal{H}_{D_z}$ *and on* $\mathcal{H}_{poly}$ *are isomorphic as polarized Hodge structures, where polarization is given by the Poincaré pairing which is dual to intersection of cycles.*

Corollary 2.5.1. *As a consequence of two propositions above special Kähler geomeries on Landau-Ginzburg orbifold and Calabi-Yau complex structures deformation spaces are isomorphic.*

Special geomety for Calabi-Yau manifolds connected with Landau-Ginzburg orbifolds We use the propositions above to write a formula for special Kähler metric on the complex structure deformation space of a Calabi-Yau purely in Landau-Ginzburg terms.

Let us explain the idea. Special geometry metric is a pairing between holomorphic and anti-holomorphic disk one-point functions, in particular, its Kähler potential is a logarithm of a disc one-point functions of an identity operator with its complex conjugates (2.197). On a Calabi-Yau manifold it is a pairing of the period integrals. On the Landau-Ginzburg side it is a pairing of complex oscillatory integrals and the intersection is the intersection pairing of relative cycles in $\mathbb{C}^n$ instead of cycles in $\mathcal{X}_\phi$.

Consider the formula (2.197):

$$e^{-K} = \omega_i(\phi) C^{ij} \overline{\omega_k(\phi)}. \tag{2.269}$$

Using the proposition 2.5.1 we can write

$$\omega_i(\phi) = \int_{Q^i_\pm} e^{\pm W(x,\phi)} \mathrm{d}^n x. \tag{2.270}$$

where $Q^i_\pm \in H_n(\mathbb{C}^n, \mathrm{Re}(\pm W) \ll 0; \mathbb{Z})$. The proposition 2.5.2 implies that

$$(C^{-1})^{ij} \sim Q^i_+ \cap Q^j_-, \tag{2.271}$$

where $\sim$ means that they are equal up to a nonsignificant common factor. To simplify the expression we introduce a basis $\{e_i \, \mathrm{d}^n x\}_{i=1}^{2h+2}$ of $H^n_{D_z}(\mathbb{C}^n)^Q$ and a dual basis of cycles $\{\Gamma^j_z\}_{j=1}^{2h+2}$ in $H_n(\mathbb{C}^n, \mathrm{Re}(W/z) \ll 0)^Q \otimes \mathbb{C}$ which are Gauss-Manin flat and

$$\int_{\Gamma^j_z} e_i \, e^{W_0(x)/z} \mathrm{d}^n x = \delta^j_i. \tag{2.272}$$

We also define $\Gamma^j_+ := \Gamma^j_1$ and Γ^j_- such that

$$\int_{\Gamma^i_-} e^{-W_0(x) - \sum_{s \le h} \phi_s e_s(x)} \mathrm{d}^5 x = \int_{\Gamma^i_+} e^{W_0(x) + \sum_{s \le h} \phi_s e_s(x)} \mathrm{d}^5 x. \tag{2.273}$$

The oscillating integrals over Γ^j_z can be computed in purely algebraical terms using power series expansion of the exponent in the deformation parameters and adding D_z-exact terms to integrand to decompose them as sums of basis elements $e_i \, \mathrm{d}^n x$. We show how this works in examples below.

The remaining question is how to compute the intersection matrix $\Gamma^i_z \cap \Gamma^j_z$. Since these cycles have complex coefficients, there are two notions of intersection matrices for them. On is holomorphic continuation from the integral cycles, another is Hermitian. These are connected with holomorpic and tt^* metrics of Frobenius manifolds. We split this question into two: first we compute the holomorphic pairing of cycles and then the real structure which connects it with the Hermitian pairing.

Holomorphic pairing, Hermitian pairing and Milnor ring We introduce the matrix η^{ij} of inverse holomorphic intersection pairing

$$(\eta^{-1})^{ij} := \Gamma^i_z \cap^{hol} \Gamma^j_{-z}, \tag{2.274}$$

which is a biholomorphic continuation of the pairing from integral cycles. We can compute it with the help of the following two propositions. First of them express cycles intersection matrix through intersection (higher residue pairing) of cocycles.

Proposition 2.5.3. *In the notations of this section the following formula holds up to a constant:*

$$\eta^{ik}\int_{\Gamma^i_-} e^{-W_0(x)}e_j(x)\,\mathrm{d}^5x\int_{\Gamma^k_+} e^{W_0(x)}e_l(x)\,\mathrm{d}^5x=(-1)^{|l|}\eta_{jl}, \tag{2.275}$$

where $|l|$ is a weighted degree of e_l in $\mathcal{R}_0^Q$.

Proof. The integrals in (2.278) can be obtained from (2.273) by differentiation at $\phi=0$ due to (2.265). The basis $\{e_i\}_{i\le 2+2h}$ can be always chosen such that $e_i=e_{s_1}\cdots e_{s_k}$ such that e_{s_k} have weighted degree 1. Then

$$\int_{\Gamma^i_\mp} e_i\,e^{\mp W_0(x)}\mathrm{d}^5x=(\mp1)^k\left(\partial_{s_1}\cdots\partial_{s_k}\int_{\Gamma^i_\mp} e^{\mp W(x,\phi)}\mathrm{d}^5x\right)|_{\phi=0}. \tag{2.276}$$

As a consequence we obtain

$$\int_{\Gamma^i_-} e_j\,e^{-W_0(x)}\mathrm{d}^5x=(-1)^{|i|}\int_{\Gamma^i_+} e_j\,e^{W_0(x)}\mathrm{d}^5x=(-1)^{|l|}\delta^i_j. \tag{2.277}$$

Plugging this to the left hand side of (2.278) we compute

$$\eta^{ik}\int_{\Gamma^i_-} e^{-W_0(x)}e_j(x)\,\mathrm{d}^5x\int_{\Gamma^k_+} e^{W_0(x)}e_l(x)\,\mathrm{d}^5x=\eta_{ik}\delta^i_j(-1)^{|l|}\delta^k_l=(-1)^{|l|}\eta_{jl}, \tag{2.278}$$

□

The second proposition is due to [101, 122] and connects the oscillating integral pairing (higher residue pairing) in (2.278) with the ordinary residue.

Proposition 2.5.4. *In the notations of the section*

$$\eta_{ik}\int_{\Gamma^i_+} e^{W_0(x)/z}e_j(x)\,\mathrm{d}^5x\int_{\Gamma^k_-} e^{-W_0(x)/z}e_l(x)\,\mathrm{d}^5x=$$

$$=z^5\left(\mathrm{Res}\frac{e_j(x)e_l(x)\,\mathrm{d}^5x}{\partial_1W_0\cdots\partial_5W_0}+O(z)\right). \tag{2.279}$$

Proof. In order to prove the proposition we perturb W_0 to a Morse function $\tilde{W}_0=W_0+\sum_i c_ix_i$ and use the steepest descent method. Let us denote $\mu=\dim(\mathcal{R}_0)$ to be a Milnor number of W_0. We change the integration cycles to Lefschetz thimbles $\{L^i_\pm\}_{i\le\mu}$ which emanate from Morse singular points $\{p_i\}_{i\le\mu}$ in direction of the gradient of W_0. A period over a cycle of steepest descent (ascent) has the following semi-classical asymptotics:

$$\int_{L^i_\mp} e^{\mp\tilde{W}_0(x)/z}e_k(x)\,\mathrm{d}^5x=z^{5/2}\frac{e^{\mp z\tilde{W}_0(p_i)}e_k(p_i)}{\sqrt{\det\partial_i\partial_j\tilde{W}_0(p_i)}}+O(z^{7/2}). \tag{2.280}$$

We can compute (2.279) using this asymptotics:

$$\sum_{i\le 4}\int_{L^i_-} e^{-\tilde W_0(x)/z} e_j(x)\,\mathrm{d}^5x \int_{L^i_+} e^{\tilde W_0(x)/z} e_l(x)\,\mathrm{d}^5x =$$

$$= z^5\left(\sum_{i\le 4}\frac{e_j(p_i)e_l(p_i)\,\mathrm{d}^5x}{\det\partial_k\partial_p\tilde W_0} + O(z)\right). \quad (2.281)$$

To finish the proof we recall that Grothendieck residue of a Morse function $f\ :\ \mathbb{C}^n\to\mathbb{C}$ has the following expression

$$\mathrm{Res}\frac{\alpha\,\mathrm{d}^5x}{\partial_1 f\cdots\partial_n f} = \sum_{p_i\in Crit(f)}\frac{\alpha(p_i)}{\det\partial_j\partial_k f} \quad (2.282)$$

and take the limit $c_i\to 0$. □

In the favorable cases the $O(z^{-1})$ correction to the residue vanishes and the holomorphic pairing just coincides with the residue pairing. In general case one can pick particular representatives $e_i\,\mathrm{d}^n x + \mathrm{d}W\wedge\alpha$ form equivalence classes such that the $O(z^{-1})$ term vanishes.

Now we turn to the Hermitian pairing of cycles. It is equal to a product of the holomorphic pairing and real structure matrix. namely, if $\overline{\Gamma^i_z} = M^i_j\Gamma^j_z$, then $\overline{e_i(x)\,\mathrm{d}^n x} = M^j_i e_j\,\mathrm{d}^n x$ and the Hermitian pairing is expressed as $\eta^{ik}M^j_k$.

We check this by the following computation:

$$e^{-K} = \eta^{ij}\int_{\Gamma^i_-} e^{-W}\mathrm{d}^5x\int_{\overline{\Gamma^j_+}}\overline{e^W\mathrm{d}^5x} = \eta^{ik}M^j_k\int_{\Gamma^i_-} e^{-W}\mathrm{d}^5x\,\overline{\int_{\Gamma^j_+} e^W\mathrm{d}^5x}. \quad (2.283)$$

Let us introduce a notation for the periods

$$\sigma_i(\phi) := \int_{\Gamma^i_+} e^W\mathrm{d}^n x. \quad (2.284)$$

Then the formula for the Kähler potential can be written compactly as

$$e^{-K} = \sigma_i(\phi)\eta^{ik}M^j_k\overline{\sigma_j(\phi)}, \quad (2.285)$$

which is our main worling formula.

In some cases special Kähler geometry on Landau-Ginzburg orbifold deformations is especially simple to compute. In these cases much of the computation is done with the help of additional symmetry. Below we use the framework of this section and additional symmetry to explicitly compute Kähler potentials of the special geometry for non-linear sigma models.

2.5.3 Phase symmetry

In this subsection we explain the common feature of our examples: additional diagonal symmetry at the special points of moduli spaces. It was mentioned that such points become orbifold points in the moduli spaces. We compute Kähler potentials of special geometry metrics as power series expansions around orbifold points.

Let $W(x,\phi) = W_0(x) + \sum_{s=1}^{h} \phi_s e_s$ be a family of weighted homogeneous polynomials in n variables of central charge $3(n-2)$. We also assume that for $\phi = 0$ the polynomial $W(x,\phi)$ has an isolated singularity at the origin and that e_s form a basis of $\mathcal{R}_0$ of weight 1. There is a $(\mathbb{C}^*)^n$ action on $\mathbb{C}^n$ acting by diagonal rescalings:

$$(\lambda_1, \ldots, \lambda_n) \cdot (x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n). \tag{2.286}$$

Definition 2.5.1. *A phase symmetry group Π_{W_0} is a maximal discrete subgroup of $(\mathbb{C}^*)^n$ acting on $\mathbb{C}^n$ as above and preserving $W_0(x)$, that is $g \in \Pi_{W_0}$ acts as $W_0(g \cdot x) = W_0(x)$.*

Since $W_0(x)$ is weighted homogeneous, there exists a set of integers $\{k_i\}_{i \leq n}, d$ without a common factor such that

$$W(\lambda^{k_i} x_i, \phi) = \lambda^d W(x, \phi). \tag{2.287}$$

It follows that the phase symmetry group is non-empty and includes a subgroup $\mathbb{Z}_d$ acting as in (2.287) with $\lambda^d = 1$. This group is not specific to W_0 and its action is defined for the whole family $W(x,\phi)$. This group acts trivially on $\mathbb{P}^{n-1}_{\bar{k}}$ and therefore is not seen on the level of nonlinear sigma model on $\mathcal{X}_\phi = \{W(x,\phi) = 0\} \subset \mathbb{P}^{n-1}_{\bar{k}}$. Following the old conventions of [111]we call such a group a quantum symmetry group of $\mathcal{X}_\phi$. In general, quantum symmetry group Q_{W_0} is a subgroup of the phase symmetry group which acts trivially on the Calabi-Yau manifold which we associate with the Landau-Ginzburg orbifold. The Landau-Ginzburg orbifold is defined on $\mathbb{C}^n / Q_{W_0}$. In the minimal case the quantum symmetry group coincides with $\mathbb{Z}_d$.

The factor $G_{\mathcal{X}_0} := \Pi_{W_0} / Q_{W_0}$ is called a geometric symmetry group and acts without a kernel on $\mathcal{X}_0$ in favorable cases. A generic neighbouring fiber $\mathcal{X}_\phi$ does not have this symmetry which is implied by the fact that $W(x,\phi)$ is not invariant under $G_{\mathcal{X}_0}$. There is a way to modify this action in such a way that the modified action preserves $W(x,\phi)$. For this purpose we extend the action of $G_{\mathcal{X}_0}$ on $\mathbb{C}^n \times \tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}} \simeq \mathbb{C}^h$ is a deformation space with the coordinats ϕ_s. If $g \in G_{\mathcal{X}_0}$ and $g \cdot e_s = g_s e_s$ we define $g \cdot \phi_s := g_s^{-1} e_s$.

The action of the geometric symmetry group extended in this way preserves $W(x,\phi)$ by construction. Moreover, it defines an isomorphism between $\mathcal{X}_\phi$ and $\mathcal{X}_{g \cdot \phi}$ by the formula $x \to g \cdot x$. It follows that $\tilde{\mathcal{M}}$ is at most a ramified finite cover of actual moduli space with the generic fiber greater or equal to the order of $G_{\mathcal{X}_0}$ modulo stabilizers.

Phase symmetry and special geometry Points with additional symmetry in the moduli space are orbifold points and we can use monodromy considerations to simplify the formulas a lot. We focus on the case where we can pick a basis $\{e_i\}_{2+2h}$ of $\mathcal{R}^Q_\phi$ such that each e_i has a unique weight with respect to the phase symmetry group which is the most symmetric case. We show that in this case both holomorphic pairing η^{ij} and real structure matrix M_i^j are antidiagonal in the natural basis.

Proposition 2.5.5. *1. We can reorder elements of the basis $\{e_i\}$ such that the residue pairing becomes antidiagonal matix at $\phi = 0$.*

2. In the basis $\{e_i\,\mathrm{d}^n x\}_i \subset H^n_{D_z}(\mathbb{C}^n)$ the oscillatory integral pairing (the higher residue pairing) (2.279) *coincides with the ordinary residue pairing.*

3. In the same basis at $\phi = 0$ the real structure matrix M_i^k is antidiagonal as well.

Proof. The first part is almost trivial. The residue pairing on the Milnor ring is non-degenerate and invariant with respect to the phase symmetry group. Indeed, let us perform a coordinate transformation $x \to g \cdot x$ inside the residue pairing. Let $g \cdot e_i\,\mathrm{d}^n x = g_i\,e_i\,\mathrm{d}^n x$. Then after the coordinate change we have

$$\mathrm{Res}\frac{e_a\,e_b\,\mathrm{d}^n x}{\partial_1 W_0 \cdots \partial_n W_0} = \mathrm{Res}\frac{g_a g_b e_a\,e_b\,\mathrm{d}^n x}{\partial_1 W_0 \cdots \partial_n W_0}, \tag{2.288}$$

where weights of group action on $\mathrm{d}^n x$ come from the transformation of the numerator $\mathrm{d}^n x$ and from ∂_i in the denominator. It follows that the residue pairing vanishes unless product of weights of $e_a\,\mathrm{d}^n x$ and $e_j\,\mathrm{d}^n x$ is equal to 1, $g_a g_b = 1$ for all g. This is equivalent to the weight of e_i times the weight of e_j is equal to the weight of $\mathrm{Hess}W_0$. For every $e_s \in \mathcal{R}^Q_0$ of weight less or equal to 1 we define e_{3+2h-s} of weight $3-|s|$ to be the unique element of $\mathcal{R}^Q_0$ of complimentary weight. In this basis the residue pairing is antidiagonal by construction.

To show the second claim we make a coordinate change in (2.279) induced by a phase symmetry transformation $x \to g \cdot x$ similarly to (2.288) we get

$$\begin{aligned}\eta_{ik} \int_{\Gamma^i_+} e^{W_0(x)/z} e_a(x)\,\mathrm{d}^5 x \int_{\Gamma^k_-} e^{-W_0(x)/z} e_b(x)\,\mathrm{d}^5 x = \\ = g_a g_b\,\eta_{ik} \int_{g(\Gamma^i_+)} e^{W_0(x)/z} e_a(x)\,\mathrm{d}^5 x \int_{g(\Gamma^k_-)} e^{-W_0(x)/z} e_b(x)\,\mathrm{d}^5 x.\end{aligned} \tag{2.289}$$

Transformations $\Gamma^\pm_k \to g(\Gamma^\pm_k)$ do not change the intersection matrix and thus the expression above vanishes whenever $g_a g_b \neq 1$ for all $g \in \Pi_{W_0}$. To show that (2.289) is equal to the ordinary residue we note that the oscillatory pairing (2.289) is weighted homogeneous of degree n. Since the corrections to the ordinary residue are $O(z^{n+1})$ they are nonvanishing only if $wt(e_a\,\mathrm{d}^n x) + wt(e_b\,\mathrm{d}^n x) \geq n+1$. But $g_a g_b = 1$ can be true for all g only if $wt(e_a\,\mathrm{d}^n x) + wt(e_b\,\mathrm{d}^n x) = n$, that is all the corrections of order $O(z^{n+1})$ vanish. As a corollary, normalizing e_a if necessary, we obtain a formula for the holomorphic intersection matrix:

$$\eta^{ij} = \delta_{i+j,2h+3}(-1)^{|j|}. \tag{2.290}$$

The final claim of the proposition is almost immediate. If $g \cdot e_a\,\mathrm{d}^n x = g_a e_a\,\mathrm{d}^n x$ then by duality $g \cdot \Gamma_a = g_a^{-1}\Gamma_a$. We use phase symmetry action and complex conjugation

$$\overline{g \cdot \Gamma_a} = \overline{g_a}^{-1} M_a^b \Gamma_b = g_a M_a^b \Gamma_b, \tag{2.291}$$

where the last equality is because g acts as roots of unity, in particular $|g_a| = 1$. Therefore $M_a^b\Gamma_b$ has a definite weight g_a which implies

$$M_a^b = A_b \delta_{a+b,2h+3}. \tag{2.292}$$

Since $M\bar{M} = 1$ we have $A_a \bar{A}_{2h+3-a} = 1$.

□

This proposition tells that the holomorphic inverse intersection matrix of cycles up to signs coincide with the ordinary residue pairing in the Milnor ring and is antidiagonal in a good basis.

We can obtain an important consequence from the proposition above. The corollary is obtained by applying the formulas (2.292) and (2.290) to our main formula for the Kähler potential (2.285)

Corollary 2.5.2. *The exponent of the special Kähler metric around good orbifold points is diagonal in the periods $\sigma_k(\phi)$ and is expressed as:*

$$e^{-K} = \sum_{k=1}^{2h+2} (-1)^{|k|} A_k |\sigma_k(\phi)|^2, \tag{2.293}$$

where $\overline{\Gamma_{2h+3-a}} = A_a \Gamma_a$ *and* $A_{2h+3-a} = \bar{A}_a^{-1}$.

The formula (2.293) will be our main formula for computation in examples.

2.5.4 Periods and real structure: invertible singularities

In this subsection we explain how to compute both periods $\sigma_a(\phi)$ and the real structure matrix $\mathbf{M}_a^b$ which we denote by a bold letter in this section in order to not mix it with another matrix.

We consider deformations of a Calabi-Yau variety which is a zero locus of a weighted homogeneous polynomial $W_0(x)$ in a weighted projective space. Let $W_0(x)$ be an *invertible singularity* that is

$$W_0(x) = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{M_{ij}}, \tag{2.294}$$

where the matrix $M = \{M_{ij}\}_{i,j\leq 5}$ consists of non-negative integers and is invertible [123, 124].Such singularities are also called singularities of Berglund-Hübsch type due to their mirror symmetry construction for such Calabi-Yau manifolds in [125]. We denote the inverse matrix $M^{-1} = \{(M^{-1})_{ij}\}_{i,j\leq n}$. All invertible singularities are weighted homogeneous:

$$W_0(\lambda_k^{(M^{-1})_{jk}} x_j) = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{M_{ij}} \lambda_k^{M_{ij} M_{jk}^{-1}} = \sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n} x_j^{M_{ij}}. \tag{2.295}$$

When $\lambda_1 = \cdots = \lambda_n = \lambda$ the right hand side is $\lambda W_0(x)$. The quasihomogeneity weights are $k_i = d\sum_j (M_{ij}^{-1})$, where d is the least common denominator of $\sum_j (M_{ij}^{-1})$. From the same formula it is cleat that W_0 have a large phase symmetry group Π_{W_0}. An element $g \in \Pi_{W_0}$ acts on $\mathbb{C}^n$ as $g \cdot x_i = e^{2\pi\sqrt{-1}g_i} x_i$, where

$$g_i = (M^{-1})_{ij} n_j,\ n_j \in \mathbb{Z}. \tag{2.296}$$

Zero loci of $W_0(x)$ are well-defined in $\mathbb{P}^{n-1}_{(k_1,\ldots,k_n)}$. Weighted projecctive spaces are orbifolds and so are $\mathcal{X}_0 = \{W_0(x) = 0\} \subset \mathbb{P}^{n-1}_{\bar{k}}$. As we mentioned before these singularities do not spoil Hodge structure on the middle cohomology $H^n(\mathcal{X}_0)$. Two additional condition we require are as always the transversality $\mathrm{d}W_0 = 0 \iff x = 0$ and the Calabi-Yau condition: $d = \sum_i k_i$. These two conditions garantee that $\mathcal{X}_0$ is a quasismooth (orbifold) Calabi-Yau. The Calabi-Yau condition can be written as

$$\sum_{i,j\leq n} (M^{-1})_{ij} = 1. \tag{2.297}$$

Space of polynomial deformations of complex structure is the base of the family

$$W(x,\phi) = W_0(x) + \sum_{s=1}^{h} \phi_s e_s(x), \tag{2.298}$$

where $h = h^{2,1}_{poly}$ is a number of independent polynomial deformations of complex structure or $e_s(x) \in \mathbb{C}[x_1,\ldots,x_5]/(\partial_i W_0)$ is a basis of the homogeneous part of the chiral ring $(\mathcal{R}_0^Q)^1$. such polynomial deformations span a cohomology group $H^{2,1}_{poly}(\mathcal{X}_0)$ consisting of Beltrami differentials corresponding to polynomial deformations. $H^3_{poly}(\mathcal{X}_0)$

is defined as a span of $H^{3,0} \oplus H^{2,1}_{poly}(\mathcal{X}_0)$ and its complex conjugate. In general not all deformations of complex structures have polynomial representatives and there are group embeddings $H^{2,1}_{poly}(\mathcal{X}_0) \subset H^{2,1}(\mathcal{X}_0)$ and $H^{3}_{poly}(\mathcal{X}_0) \subset H^{3}(\mathcal{X}_0)$.

We consider maximally symmetric cases that is only singularities W_0 such that $\mathcal{R}^Q_0$ is decomposed into pairwise differents one-dimensional represntations of the phase symmetry group Π_{W_0}. Let us choose a basis $\{e_a\}_{a=1}^{2=2h}$ of the Milnor ring whose elements are eigenvectors with respect to the phase symmetry group and the residue pairing is

$$\mathrm{Res} \frac{e_a\, e_b\, \mathrm{d}^5 x}{\partial_1 W_0 \cdots \partial_5 W_0} = \delta_{a+b,2h+3}. \tag{2.299}$$

Periods First we find the periods (we perform the computation for general n)

$$\sigma_a(\phi) = \int_{\Gamma_a^-} e^{-W(x,\phi)}\, \mathrm{d}^n x. \tag{2.300}$$

For this purpose we expand $W(x,\phi)$ in ϕ and recursively reduce integrands to the basis forms $e_b\, \mathrm{d}^n x \in H^n_{D_+}(\mathbb{C}^n)$.

$$\sigma_a(\phi) = \sum_{m_1,\dots,m_h \geq 0} \frac{\phi_1^{m_1} \cdots \phi_h^{m_h}}{m_1! \cdots m_h!} \int_{\Gamma^a_-} e^{-W_0(x)} \prod_{i \leq n} x_i^{\sum_{s=1}^h m_s s_i}\, \mathrm{d}^n x. \tag{2.301}$$

Consider an integral

$$\int_{\Gamma^a_-} e^{-W_0(x)} \prod_{i \leq n} x_i^{b_i}\, \mathrm{d}^n x, \tag{2.302}$$

We note that the Jacobi ideal of $W_0(x)$ is generated by

$$\partial_k W_0(x) = \sum_i M_{ik} \prod_j x_j^{M_{ij} - \delta_{jk}}. \tag{2.303}$$

In particular, all the monomials of W_0 belong to the Jacobi ideal themselves

$$\prod_j x_j^{M_{aj}} = \sum_k M^{-1}_{ka} x_k \partial_k W_0(x). \tag{2.304}$$

In general only W_0 itself belongs to its own Jacobi ideal in the weighted homogeneous case [28]. Using the formula (2.304) we can recursively reduce the differential form in (2.302) shifting the exponents of x_i by the rows of the matrix M. For the k-th row we explicitly compute

$$\prod_{i \leq 5} x_i^{M_{ki} + a_i} \mathrm{d}^n x - D_- \left(\sum_b M^{-1}_{bk} \prod_{i \leq n} x_i^{a_i + \delta_{ib}} \mathrm{d}^n x / \mathrm{d} x_b \right) = \\ = \sum_b (a_b + 1) M^{-1}_{bk} \prod_{i \leq n} x_i^{a_i} \mathrm{d}^n x. \tag{2.305}$$

Using induction we obtain the following formula

$$x^{Mv+k}\mathrm{d}^n x = \prod_{i=1}^{n}\left(\sum_j (k_j+1)M^{-1}_{ji}\right)_{v_i}\prod_{i=1}^{n} x^{k_i}\mathrm{d}^n x,\ b = Mv+k, \tag{2.306}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is a Pochhammer symbol.

By the formula for the phase symmetry group action, $x^{Mv+k}\mathrm{d}^n x$ and $x^k \mathrm{d}^n x = \prod_i x_i^{k_i}\,\mathrm{d}^n x$ belong to the same representation of Π_{W_0} that is they transform in the same way. Rows of the matrix M_{ij} are n linearly independent integral vectors in the lattice $\mathbb{Z}^n$. They have a fundamental domain with $\det M$ integral points. Every point $\mathbb{Z}^n$ can be shifted to the fundamental domain by the rows of M. We further assume that elements of the fundamental domain have pairise different weights with respect to Π_{W_0} and that we can pick a basis in the Milnor ring so that for all $a \leq 2+2h$ exponent vectors $(a_1,\cdots,a_n)$ of the basis monomials $e_a = x_1^{a_1}\cdots x_n^{a_n}$ belong to the fundamental domain. This assumption always holds in our examples.

Under the assumption we can use (2.306) to compute the period in (2.301):

$$\begin{aligned}
&\sigma_a(\phi) = \sum_{v_1,\ldots,v_5\geq 0}\prod_{i\leq 5}\left((a_j+1)M^{-1}_{ji}\right)_{v_i}\sum_{\sum_{s=1}^{h} m_s s_i = M_{ij}v_j + a_i}\frac{\phi_1^{m_1}\cdots\phi_h^{m_h}}{m_1!\cdots m_h!},\\
&a = (a_1,\ldots,a_5)\in\mathcal{R}_0,\\
&\sum_{i\leq 5} M_{ij}a_j = 0,\ d,\ 2d,\ 3d.
\end{aligned} \tag{2.307}$$

Let us make a comment on the formula above. It defines a period integral when the Pochhammer symbols are non-vanishing which is certainly true if the arguments $(a_j+1)M^{-1}_{ji}$ are non-integral. Otherwise the actual periods contain logarithmic terms which are not seen in the expansion. We will not discuss such cases in this text.

Real structure In order to compute the real structure on cycles Γ_+^a or on the cohomology elements $e_a\,\mathrm{d}^n x \in H^n_{D_+}(\mathbb{C}^n)$ we compute a set of period integrals over actual integral cycles. We choose the cycles carefully so that period integrals decompose into the products of gamma functions. We switch to the case of general n and non-invariant cohomology groups $H^n_{D_+}(\mathbb{C}^n)$. The real and integral structure on the Q-invariant cohomology is induced from the one considered here.

Consider an integral cycle $\Gamma_+ \in H_n(\mathbb{C}^n, \mathrm{Re}(W)\ll 0;\mathbb{Z})$ and the following integral

$$\int_{\Gamma_+}\prod_{i\leq 5} x_i^{k_i} e^{-\sum_i\prod_j x_j^{M_{ij}}}\mathrm{d}^n x = \int_{L_-} x^k e^{-\sum_i x^{M_i}}\mathrm{d}^n x, \tag{2.308}$$

where we use the notation $x^v = \prod_j x_j^{v_j}$ for any vector v of length n as above. Let us perform a singular coordinate change

$$y_i := x^{M_i} = \prod_j x_j^{M_{ij}}. \tag{2.309}$$

This coordinate change is singular on the union of coordinate hyperplanes and is not injective. The integral (2.308) in the new coordinates becomes

$$\int_{L_-} x^k e^{-\sum_i x^{M_i}} \mathrm{d}^n x = \det M^{-1} \int_{L_-} y^{(k+1)M^{-1}-1} e^{-\sum_i y_i} \mathrm{d}^n y, \tag{2.310}$$

where 1 in the exponent stands for $(1,1,1,1,1)$. In this form the integral decomposes into a product of n gamma-function integrals for an appropriate contour L_-. Since the coordinate change and the integrand is not well-defined on the coordinate planes, we choose L_- to be a product of n Pochhammer contours [126].

To describe L_- more carefully we consider a basic one-dimensional contour C which is defined parametrically in $\mathbb{C}$ with complex coordinate z by the formula $z(s) = \rho(s)e^{i\theta(s)}$ which goes below the real exis from $+\infty$ to zero, encircles zero from the left and then goes a bit above the real axis to $+\infty$.

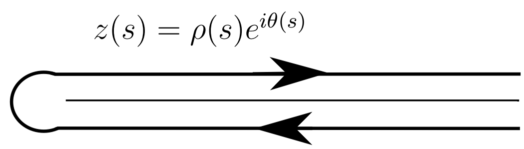

Figure 2.5.2: Contour $C = z(s)$

Using the basic contour C we can define a basis (under our assumptions on the phase symmetry) of integral cycles in $\mathcal{H}_n^- = H_n(\mathbb{C}^n, \mathrm{Re}(W) \gg 0; \mathbb{Z})$ parametrically as an image of $\mathbb{R}^n$ with coordinates $s_1, \cdots s_n$ in $\mathbb{C}^n$:

$$L_-^b := \left\{ x_i(s) = \prod_{j\leq 5} \rho(s_j)^{M_{ij}^{-1}} \exp\left(\sum_{j \leq 5} (\theta(s_j) + 2\pi b_{ij}) M_{ij}^{-1} \right), \quad \text{no summation in } i \right\}, \tag{2.311}$$

where b_{aj} is a matrix with integral coefficients. The integral (2.308) obviously converges over L_-^b. Since the phase symmetry action is defined as $x_j \to \exp(2\pi\sqrt{-1}\sum_a M_{aj}^{-1} b_{aj})$, we can pick enough such cycles to form a basis in $\mathcal{H}_n^-$.

In the coordinates y_i the contours L_-^b decompose into a product of n one-dimensional Pochhammer contours

$$y_i(s) = \rho(s_i)e^{i\theta(s_i)}. \tag{2.312}$$

Since the coordinate change $x \to y$ is non-injective, there are several preimages of this contour in the x-coordinates which corresponds to different choices of roots of unity in the integals (2.308).

Using the formula

$$(e^{2\pi is}-1)\Gamma(s)=\int_C e^{-x}x^{s-1}\mathrm{d}x \tag{2.313}$$

we easily compute the integral (2.308)

$$\begin{aligned}\det M^{-1}\int_{\Gamma_+} y^{M^{-1}(k+1)-1}e^{-\sum_i y_i}\mathrm{d}^n y = \\ = \det M^{-1}\sum_{(s_1,\ldots,s_n)\in\{0,1\}^N}(-1)^{|s|}\exp\left[2\pi\sqrt{-1}\sum_a M^{-1}_{aj}(k_j+1)b_{aj}s_a\right]\times \\ \times\prod_{i\le n}\Gamma\left((k_j+1)M^{-1}_{ji}\right).\end{aligned} \tag{2.314}$$

The integral (2.314) is a complex number. This integral is a pairing of a complex cohomology element $e_k\,\mathrm{d}^n x=\prod_i x_i^{k_i}\,\mathrm{d}^n x$ with an integral cycle L_-^b. The cocycle $e_k\,\mathrm{d}^n x+\overline{e_k\,\mathrm{d}^n x}$ is obviously real and the integral

$$\int_{L_-^b} e^{-W_0}\left(e_k\,\mathrm{d}^n x+\overline{e_k\,\mathrm{d}^n x}\right) \tag{2.315}$$

is real for any k and b. We use this constraint together with the formula (2.314) to compute the real structure matrix $\mathbf{M}$.

The real structure on the cohomology is given by the formula $\overline{e_k\,\mathrm{d}^n x}=\mathbf{M}^p{}_k e_p\,\mathrm{d}^n x=A_p\delta_{k+p,\rho+1}e_p\,\mathrm{d}^n x$, where $e_\rho=\mathrm{Hess}W_0$. For any $e_k=\prod_i x_i^{k_i}$ we introduce a notation for the dual with respect to the residue pairing element $e_{2h+3-k}=\prod_i x_i^{\check{k}_i}$. Then we have the following equation on the coefficients A_p of the real structure matrix:

$$\mathrm{Im}\left[\int_{L_-^b}(e_a(x)+A_{\rho+1-a}e_{\rho+1-a}(x))e^{-W_0(x)}\mathrm{d}^5 x\right]=0. \tag{2.316}$$

We compute the integral with the help of (2.314):

$$\begin{aligned}\mathrm{Im}\left(\sum_{(s_1,\ldots,s_5)\in\{0,1\}^N}(-1)^{|s|}\exp\left[2\pi\sqrt{-1}\sum_a M^{-1}_{ja}(a_j+1)b_{aj}s_a\right]\right)\times \\ \times\left[\prod_{i\le 5}\Gamma((a_j+1)M^{-1}_{ji})-A_{\rho+1-a}\prod_{i\le 5}\Gamma((\rho_j+1-a_j)M^{-1}_{ji})\right],\end{aligned} \tag{2.317}$$

where we used the condition $g_a g_{\rho+1-a} = 1$ for any $g \in \Pi_{W_0}$ to factor out the common factor.

By our assumptions on the phase symmetry group there exists L_-^b such that the prefactor has a nonzero imaginary part, therefore

$$A_{\rho+1-a} = \prod_{i\le 5} \frac{\Gamma\left((a_j+1)M^{-1}_{ji}\right)}{\Gamma\left((\check{a}_j+1)M^{-1}_{ji}\right)}. \tag{2.318}$$

Alternative way to compute the coefficients A_a uses the following trick. Since we choose the cycles Γ_+^a such that

$$\int_{\Gamma_+^a} e_b\, e^{W_0(x)} \mathrm{d}^n x = \delta_{ab}, \tag{2.319}$$

the set of integrals over the Lefschetz thimbles

$$T_b^a := \int_{L_+^a} e_b\, e^{W_0(x)} \mathrm{d}^n x \tag{2.320}$$

defines a transition matrix between the cycles

$$L_+^a = T_b^a \Gamma_+^b, \quad \Gamma_+^a = (T^{-1})_b^a L_+^b. \tag{2.321}$$

The real structure matrix $\mathbf{M}$ is computed from

$$\overline{\Gamma_+^a} = \mathbf{M}^a{}_b \Gamma_+^b = (\mathbf{M}\cdot T^{-1})^a{}_c L_+^c \tag{2.322}$$

and $\overline{\Gamma_+^a} = \overline{(T^{-1})^a{}_b L_+^b} = \overline{T^{-1}}^a{}_b L_+^b$. Comparing this with (2.322) we compute $M = \overline{T^{-1}} \cdot T$.

2.5.5 The quintic threefold

Let us consider the quintic threefold $\mathcal{Q}_\phi \subset \mathbb{P}^4$ again. Its equation is

$$W(x,\phi) = W_0(x) + \sum_{s=1}^{101} \phi_s e_s = \sum_{i\le 5} x_i^5 + \sum_{s=1}^{101} \phi_s x_1^{s_1} \cdots x_5^{s_5} = 0. \tag{2.323}$$

We recall that ϕ without an index means a vector of all $\phi_s,\ 1 \le s \le 101$ for the quintic. An index s means either a number or a set $(s_1,\dots,s_5)$ of exponents of the corresponding deformation monomial $e_s = x_1^{s_1}\cdots x_5^{s_5}$. Another notation is $(s_1,\dots,s_5) := x_1^{s_1}\cdots x_5^{s_5}$. We pick e_s to be a basis of elements of $(\mathcal{R}_0^Q)^1$ represented by the monomials from (2.232) which do not contain x_i^4. These are monomials with exponent vectors $(1,1,1,1,1)$,

$(2,1,1,1,0)$, $(2,2,1,0,0)$, $(3,1,1,0,0)$, $(3,2,0,0,0)$ and their permutations. The quantum symmetry group is $\mathbb{Z}_5$ which acts as $x_i \to \alpha x_i$, where $\alpha^5 = 1$.

The top weight element of $\mathcal{R}_0^Q$ is $e_\rho = (3,3,3,3,3)$ which is proportional to HessW_0 and has the degree 15. The invariant Milnor ring is

$$\mathcal{R}_0^Q = \langle 1 \rangle \oplus \langle e_s \rangle_{s=1}^{101} \oplus \langle e_{\rho+1-s} \rangle_{s=1}^{101} \oplus \langle e_\rho \rangle, \tag{2.324}$$

where $(\mathcal{R}_0^Q)^2$ is spanned by $e_{\rho+1-s} = (3-s_1, 3-s_2, 3-s_3, 3-s_4, 3-s_5)$. The topological residue pairing in this basis is antidiagonal $\eta^{ab} = \eta(e_a, e_b) = \delta_{a+b,\rho+1} = \prod_{i=1}^5 \delta_{a_i+b_i,3}$.

Phase symmetry and special geometry The phase symmetry group is $\mathbb{Z}_5^5$ which scales all coordinates independently by roots of unity: $x_i \to \alpha_i x_i$, $\alpha_i^5 = 1$. Each monomial from the Milnor ring have a unique weight with respect to this group action: $\alpha \cdot e_a = e_a \prod_{i\le 5} \alpha_i^{a_i}$. It follows that we can apply the machinery of the previous sections. The matrix of exponents M for the quintic is $M = \mathrm{diag}\{5,5,5,5,5\}$. We use the formulas (2.293), (2.307) and (2.318) to write the Kähler potential of the special geometry metric:

$$e^{-K} = \sum_{\substack{k_1,\ldots,k_5 \le 3, \\ \sum_i k_i = 0,5,10,15}} (-1)^{\sum_i k_i/5} \prod_{i\le 5} \gamma\left(\frac{k_i+1}{5}\right) |\sigma_{(k_1,\ldots,k_5)}(\phi)|^2, \tag{2.325}$$

where

$$\begin{aligned} \sigma_a(\phi) &= \sum_{n_1,\ldots,n_5 \ge 0} \prod_{i\le 5} \left(\frac{a_i+1}{5}\right)_{n_i} \sum_{\sum_{s=1}^{101} m_s s_i = 5n_i + a_i} \frac{\phi_1^{m_1}\cdots\phi_{101}^{m_{101}}}{m_1!\cdots m_{101}!}, \\ a &= (a_1,\ldots,a_5),\ 0 \le a_i \le 3, \sum_{i\le 5} a_i = 0,\ 5,\ 10,\ 15. \end{aligned} \tag{2.326}$$

This formula was used for computations of distances in the moduli space of complex structures on the quintic around the orbifold point to check the Refined Swampland Distance Conjecture [127].

2.5.6 Fermat hypersurfaces

Fermat hypersurfaces from the point of view of our computation are not very different from the quintic. Consider a weighted homogeneous polynomial

$$W_0(x) = x_1^{d/k_1} + x_2^{d/k_2} + x_3^{d/k_3} + x_4^{d/k_4} + x_5^{d/k_5}, \tag{2.327}$$

where $d = \sum_{i\le 5} k_i$ and all d/k_i are integral. The equation $\{W_0(x) = 0\}$ defines a hypersurface $\mathcal{X}_0$ in the weighted projective space $\mathbb{P}^4_{(k_1,k_2,k_3,k_4,k_5)}$. The quantum symmetry group is $Q = \mathbb{Z}_d$ which acts as $x_i \to \alpha^{k_i} x_i$, $\alpha^d = 1$.

The main difference between general Fermat hypersurfaces and the quintic is that weighted projective spaces and Xc_0 themselves can have orbifold sigularities.

Consider, for example $\mathbb{P}^2_{(1,1,2)}$ with homogeneous coordinates $x_1 x_2, x_3$ and a chart $x_3 = 1$. Then the gauge transformation $x_i \to \lambda^{k_i} x_i$ with $\lambda = -1$ does not change x_3 which implies that $(x_1, x_2) \simeq (-x_1, -x_2)$ in this chart which is isomorphic $\mathbb{C}^2/\mathbb{Z}_2$. This chart can be embedded into $\mathbb{C}^3$ with the coordinates u, v, w using a map $(x_1, x_2) \to (x_1^2, x_1 x_2, x_2^2)$. It follows that $\mathbb{C}^2/\mathbb{Z}_2$ is isomorphic to a quadratic cone $uw = v^2$.

We work with variations of Hodge structures on families of orbifold varieties. There are various ways to work with singular varieties. One of the simplest ones is to *blow up* the singularities. That is if $V \subset X$ is a singular locus, then gluing a projectivization $\mathbb{P}(NV)$ of the normal bundle to V is called a blow up and makes X less singular. Intuitively, the blow up turns each point $x \in V$ to a set of normal directions to V at this point. The blown up variety is called $\tilde{X}$ and has a projection map back to X.

Blowing up introduces new cycles to a variety. We are interested in hypersurfaces $\mathcal{X}_0 \subset \mathbb{P}^4_{\bar{k}}$ in weighted projective spaces which become smooth when we resolve projective space orbifold singularities. When resolving curve singularities in $\mathbb{P}^4_{\bar{k}}$ we might add curves and surfaces into $\tilde{\mathcal{X}}_0$ which is the resolution of $\mathcal{X}_0$. These curves span additional Kähler classes on $\tilde{\mathcal{X}}_0$. Moreover, if $C := \mathcal{X}_0 \cap \mathrm{sing}(\mathbb{P}^4_{\bar{k}})$ contains a curve of positive genus, then it contain one-dimensional cycles. Blowing up C produces additional 3-cycles which may add complex structure deformations to $\mathcal{X}_0$ [114].

Another way to work with orbifolds is using stacks and more involves cohomology theories such as Chen-Ruan cohomology.In this paper we disregard these complications and consider the smallest consistent Hodge structure on the middle cohomology of the singular variety $\mathcal{X}_0$ which coinside with the one on intersection cohomology of $\mathcal{X}_0$.

Our approach is to consider only such $(2,1)$ forms which generate polynomial deformations of $\mathcal{X}_0 \subset \mathbb{P}^4_{\bar{k}}$, that is consider $H^3_{poly}(\mathcal{X}_0)$ and $H^{2,1}_{poly}(\mathcal{X}_0)$ as above. A generic polynomial deformation is

$$W(x, \phi) = \sum_{i \leq 5} x_i^{d/k_i} + \sum_{s=1}^{h} \phi_s x_1^{s_1} \cdots x_5^{s_5} = 0, \tag{2.328}$$

where $h = h^{2,1}_{poly} = \dim(H^{2,1}_{poly}(\mathcal{X}_0))$ is the dimension of $(\mathcal{R}_0^Q)^1$. The latter is spanned by $(s_1, \ldots, s_5)$ such that $\sum_{i \leq 5} k_i s_i = d$ and $s_i < d/k_i - 1$. The Hessian of W_0 is the maximal weight element of $\mathcal{R}_0^Q$ is

$$e_\rho = (d/k_i - 2, \ldots, d/k_5 - 2) = x_1^{d/k_1 - 2} \cdots x_5^{d/k_5 - 2} \tag{2.329}$$

in the notations of the previous subsection. The $(\mathcal{R}_0^Q)^2$ is spanned by $\{e_{\rho - s}\}_{s=1}^h$, where $e_{\rho - s} = (d/k_1 - 2 - s_1, \ldots, d/k_5 - 2 - s_5)$ and e_s belongs to $(\mathcal{R}_0^Q)^1$. In such a basis the holomorphic paring is antidiagonal as well $\eta^{ab} = \eta(e_a, e_b) = \prod_{i=1}^5 \delta_{a_i + b_i, d/k_i - 2}$.

Phase symmetry and special geometry The phase symmetty group $\Pi_{W_0} = \mathbb{Z}_{d/k_1} \times \cdots \times \mathbb{Z}_{d/k_5}$ is a product of groups of roots of unity preserving each monomial x^{d/k_i}. The phase symmety group acts as $\Pi_{W_0} \ni (\alpha_1, \ldots, \alpha_5) \;:\; x_i \to \alpha_i x_i$, where $\alpha_i^{d/k_i} = 1$. Each element of the Milnor ring has a unique weight with respect to this group. The matrix of exponents is $M = \mathrm{diag}\{d/k_1, \ldots, d/k_5\}$. Applcation of the formulas (2.293), (2.307) and (2.318) gives the following expression:

$$e^{-K} = \sum_{\substack{a_i \le d/k_i - 2 \\ \sum_i k_i a_i = 0, d, 2d, 3d}} (-1)^{\sum_i k_i a_i / d} \prod_{i=1}^{5} \gamma\left(\frac{k_i(a_i+1)}{d}\right) |\sigma_a(\phi)|^2, \tag{2.330}$$

where

$$\begin{aligned} &\sigma_a(\phi) = \sum_{n_1,\ldots,n_5 \ge 0} \prod_{i \le 5} \left(\frac{k_i(a_i+1)}{d}\right)_{n_i} \sum_{\sum_{s=1}^{h} m_s s_i = d/k_i n_i + a_i} \frac{\phi_1^{m_1} \cdots \phi_h^{m_h}}{m_1! \cdots m_h!}, \\ &a = (a_1, \ldots, a_5),\ 0 \le a_i \le d/k_i - 2, \sum_{i \le 5} k_i a_i = 0,\ d,\ 2d,\ 3d. \end{aligned} \tag{2.331}$$

2.6 Gauged Linear Sigma Models and special geometry

In this section we discuss an interesting connection between special geometry and partition functions of Gauge Linear Sigma Models (GLSM). GLSM are known to reduce to non-linear sigma models, Landau-Ginzburg orbifolds and other theories in particular limits [104].

We are interested in partition functions Z_{S^2} of special abelian GLSM on S^2 with a round metric. It was shown by [91] and [128] that this partition function can be computed exactly using supersymmetric localization techniques.

Shortly after the original computations it was conjectured in [93] that when vacua manifold $\mathcal{Y}$ of a GLSM is a Calabi-Yau manifold, the sphere partition function is equal to the quantum corrected exponent of a Kähler potential of the special geometry metric on the Kähler class moduli space of $\mathcal{Y}$.

In this section we check the mirror symmetric version of this conjecture for Calabi-Yau threefolds where we can compute the special Kähler metric on the complex structures moduli space using a slightly modified version of Batyrev mirror symmetry construction [129].

2.6.1 GLSM

Let us briefly discuss the main features of GLSM which are important for us following [130].

Linear Gauged Sigma Models are 2d N=(2,2) supersymmetric theories of chiral superfields Φ_i and gauge superfields $V = \theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) + \ldots$, where $v = (v_0, v_1)$ is a gauge field. Second superderivative of V is a twisted chiral field

$$\Sigma := \overline{D_+} D_- V \tag{2.332}$$

which is called the super field-strength.

Consider a theory of k vector U(1) superfields and N chiral superfields with the charge matrix $\{Q_{ia}\}_{i\leq N, a\leq k}$. The Lagrangian of GLSM of our interest is

$$\begin{aligned} L = \int \mathrm{d}^4\theta \left(\sum_{i=1}^N \overline{\Phi_i} e^{Q_{ia}V_a} \Phi_i - \sum_a \frac{1}{2e_a^2} \overline{\Sigma_a} \Sigma_a \right) + \\ + \frac{1}{2} \left(- \int \mathrm{d}^2\tilde{\theta} \sum_{a=1}^k t_a \Sigma_a + \int \mathrm{d}^2\theta W(\Phi) + \text{h.c.} \right), \end{aligned} \tag{2.333}$$

where the superpotential $W(\Phi)$ is gauge invariant. The complex coupling constant $t_a = r_a - i\theta_a$ is a sum of the Fayet-Iliopoulos parameter r_a and the theta angle θ_a. These parameters play a role of complexified Kähler class of the vacua manifold.

The scalars potential can be obtained from the Lagrangian above by expanding superfields in components and integrating out auxilary fields (fields without kinetic terms).

$$U = \sum_{i=1}^N |Q_{ia}\sigma_a|^2 |\phi_i|^2 + \sum_{a=1}^k \left(Q_{ia}|\phi_i|^2 - r_a \right)^2 + \sum_{i=1}^k \left| \frac{\partial W}{\partial \phi_i} \right|^2, \tag{2.334}$$

where σ_a are the scalars from the vector multiplets and ϕ are the scalar components of the chiral multiplets.

The supersymmetric vacua manifold coincides with the zeros of the classical potential

$$\mathcal{Y}_r = \left\{ (\phi_1, \ldots, \phi_N) \in \mathbb{C}^N \;\middle|\; \sum_{a=1}^N Q_{al}|\phi_a|^2 = r_l,\ l = 1, \ldots, k,\ \frac{\partial W}{\partial \phi_a} = 0 \right\} / U(1)^k, \tag{2.335}$$

where $U(1)^k$ acts on ϕ_i with the charge matrix Q_{ia}. In different regions of parameters r_l the manifold $\mathcal{Y}_r$ has different topology. In general, the space $\mathbb{R}^k$ of the Fayet-Iliopoulos parameters is split into chambers by hyperplanes which separate regions with different topology. These different regions are called *phases* of GLSM [104].

Sending all coupling constants e_a to infinity one achieves that all low-energy dynamics is concentrated in the infenitesimal neighbourhood of the vacua manifold $\mathcal{Y}_r$. In the case where this manifold has the maximal dimension, all the massless modes are tangent to $\mathcal{Y}_r$ and the dynamics is that of the non-linear sigma model. On the other extreme case where $\mathcal{Y}_r = 0$ and all the modes are tangent to $\mathcal{Y}_r$ the theory flows to a Landau-Ginzburg theory in the infrared. In general some of the massless modes are tangent and some are normal to $\mathcal{Y}_r$.

2.6.2 Localization and mirror symmetry

Localization In the previous discussion the GLSM was described on a flat Riemann surface. In order to connect to the special geometry one has to consider a supersymmetric background on the round sphere.

Supersymmetric localization was successfully performed in supersymmetric gauge theories in various dimensions [131, 132, 133]: on $S^2, S^3, S^4, S^4, \Omega - \text{deformed}\mathbb{R}^4$ and others. The idea of supersymmetric localization goes back to a generalization of equivariant localization to the case of infinite-dimensional Lie superalgebra actions. In the presence of an odd supersymmetry $\mathbb{Q}$ the functional integral reduces to a semi-classical computation around the saddle points of the supersymmetry. The total partition function (or correlators of symmetric observables) in an integral over fixed point locus a classical action times one-loop determinant (the quadratic saddle point approximation contribution). The fixed loci are often finite dimensional and the integral can be computed.

In practice, one usually adds a large $\mathbb{Q}$-exact term δS to the action which does not change the integral. In the limit where δS goes to infinity, the functional integral localizes around the saddle points of δS. The choice of the deformation term δS is different in different examples.

In the S^2 localization in GLSM there are two choices of this deformation term which lead to localization on the Higgs branch (non-zero vevs of scalars in the chiral multiplets) or on the Coulomb branch (non-zero vevs of scalars from vector multiplets). The coincidence of the two computations is a nontrivial identity.

We use the Coulomb branch localization for a theory of N chiral multiplets Φ_i, k vector U(1) multiplets V_a with the charge matrix Q_{ia} and a set of R-charges q_i of the scalars. The localization computation [91, 92] give the following formula for the S^2 partition function:

$$Z_{S^2} = \sum_m \int \left(\prod_{j \le k} \frac{\mathrm{d}\sigma_j}{2\pi} \right) Z_{class}(\sigma, m) \prod_{i \le N} Z_{\Phi_i}(\sigma, m), \tag{2.336}$$

where the one-loop determinant of the abelian Gauge fields is trivial, the chiral field one-loop determinant is

$$Z_{\Phi_i} = \frac{\Gamma(q_i/2 - i\sum_l (Q_{il}\sigma_l - m_l/2))}{\Gamma(1 - q_i/2 - i\sum_l (Q_{il}\sigma_l + m_l/2))}, \tag{2.337}$$

and the classical action on the localization locus is

$$Z_{class} = e^{-4\pi i r_l \sigma_l - i\theta_l m_l}. \tag{2.338}$$

The integral in (2.336) is over all real values of the scalars σ_l from the vector multiplet and over all integral values of twisted masses m_l. Both σ_l and m_l belong to the Cartan algebra of the gauge group which is $U(1)^k$ in our case. The twisted masses m_l belong to an intrinsically defined integral lattice of the Cartan algebra which is defined as a lattice of weights of all possible representations of $U(1)^k$ as follows from the Dirac quantization condition which is required by the well-definedness of the functional integral.

Let us change the coordinates in the localization formula $\tau_l := -i\sigma_l$ so that

$$Z_{S^2} = \sum_m e^{-i\theta_l m_l} \int_{C_1} \cdots \int_{C_k} \left(\prod_{l \leq k} \frac{\mathrm{d}\tau_;}{2\pi i} \right) e^{4\pi r_l \tau_l} \prod_{i \leq N} \frac{\Gamma\left(q_i/2 + \sum_{l=1}^k Q_{il}(\tau_l - \frac{m_l}{2}\right)}{\Gamma\left(1 - q_i/2 - \sum_{l=1}^k Q_{il}(\tau_l + \frac{m_l}{2}\right)}, \tag{2.339}$$

where the contours C_i go along imaginary axes. The spherical partition function does not depend either on the coupling constants e_l or the explicit form of the superpotential W. In particular, sending them to zero, we see that the integral computes a quantity in the massless sector of the theory.

It was conjectured in [93] that in the geometric phase (non-linear sigma model case) the partition function coincides with the exponent of the Kähler potential of the special geometry on the Kähler structures moduli space on $\mathcal{Y}_r$, where r parametrizes the Kähler parameters. This conjecture was checked for several cases in the original paper [93]. Later there appeared physical proofs of the conjecture [94]. One of the ideas is that the special geometry on the moduli space is a Zamolodchikov metric on the deformation space of conformal field theories in the infrared limit, whereas the partition function computes the conformal anomaly connected with Zamolodchikov metric.

The direct check of the conjecture is complicated due to the fact that the Kähler structures special geometry $K^{\mathcal{Y}_r}$ is relatively difficult to compute. We check the mirror version of this conjecture. Mirror symmetry states that there exists a family of Calabi-Yau varieties $\mathcal{X}_\phi$ such that the complex structure moduli space metric on $\mathcal{X}_\phi$ coincides with the Kähler structures metric on $\mathcal{Y}_r$ after an appropriate mirror map $r = r(\phi)$:

$$K^{\mathcal{Y}_r} \sim K^{\mathcal{X}}, \quad e^{-K^{\mathcal{X}}} = -i \int_{\mathcal{X}} \Omega \wedge \overline{\Omega}. \tag{2.340}$$

We developed an effective method of computation of $e^{-K^{\mathcal{X}}}$ for a class of Calabi-Yau manifolds. Below we check the mirror version of Jockers et al. conjecture for the cases where we computed the complex structures moduli space metric:

$$\int_{\mathcal{X}_\phi} \Omega_\phi \wedge \overline{\Omega_\phi} \sim Z_{S^2}, \tag{2.341}$$

where $\mathcal{Y}_{r(\phi)}$ and $\mathcal{X}_\phi$ form a mirror pair and $\sim$ means an equality up to a Kähler transformation. On the left hand side Kähler transformations correspond to different choices of $\Omega_\phi \to f(\phi)\Omega_\phi$ whereas on the right hand side they depend on the regularization of the partition function.

Batyrev Mirror symmetry We use a version of Batyrev mirror symmetry construction [129] for Calabi-Yau hypersurfaces in toric varieties.

Toric varieties are natural generalizations of complex projective spaces $\mathbb{P}^n$. An n-dimensional variety X is toric if there is an algebraic action of a complex torus $(\mathbb{C}^*)^n$ on it and one of the orbits is open and dense in X. For $\mathbb{P}^n$ one can pick as a torus the diagonal action on homogeneous coordinates and the dense orbit is a set where none of the homogeneous coordinates vanish.

There exist several natural ways to construct toric varieties. The closest one to the GLSM is the GIT quotient construction:

$$X = \mathbb{C}^N//_r(\mathbb{C}^*)^k = (\mathbb{C}^N - Z_r)/(\mathbb{C}^*)^k, \tag{2.342}$$

where $//_r$ stands for a geometric invariant theory quotient with the stability parameter r which controls the Kähler class of the resulting variety, $n = N - k$ and Z_r is a $(\mathbb{C}^*)^k$-invariant subset such that the factor is Hausdorff. There is an obvious torus action $\mathbb{C}^n \simeq (\mathbb{C}^*)^N/(\mathbb{C}^*)^k$ on X. In the case of the projective space

$$\mathbb{P}^n = \mathbb{C}^{n+1}//_r\mathbb{C}^* := (\mathbb{C}^{n+1} - 0)/\mathbb{C}^*, \tag{2.343}$$

and the Kähler form is the Fubini-Studi form times r which is a number.

It is well-known that the GIT quotient is isomorphic to a Hamiltonian reduction of the form

$$X_r = \left\{ (x_1, \ldots, x_N) \in \mathbb{C}^N \;\middle|\; \sum_{a=1}^N Q_{al}|x_a|^2 = r_l,\ l = 1, \ldots, k \right\} /U(1)^k, \tag{2.344}$$

where Q_{al} are the weights of the $(\mathbb{C}^*)^k$ action on $\mathbb{C}^N$ and $U(1)^k \subset (\mathbb{C}^*)^k$ consists of elements of modulus 1.

The vacua manifolds of GLSM are naturally subvarieties in toric varieties via this correspondence.

The classical ways of defining of a toric variety are fans and polytopes. Consider $N_\mathbb{R} := \mathbb{R}^n$ with an integral lattice given by the points with integral coefficients. A (rational, polyhedral) cone is a convex span of a set of integral vectors:

$$\sigma = \{\sum_i a_i v_i \mid v_i = (v_{i1}, \ldots, v_{in}),\ a_i \in \mathbb{R}_+\}. \tag{2.345}$$

We shall consider only strongly convex cones, that is the cones such that $\sigma \cap (-\sigma) = 0$. A fan Σ is a set of cones $\{\sigma_I\}$ which is closed under operation of taking the boundary, that is if $\sigma \in \Sigma$, then the cones of $\partial\sigma$ also belong to Σ. Instead of describing the original construction of a toric variety via fans we reduce this construction to a quotient.

Consider a 1-skeleton $\Sigma(1) \subset \Sigma$ of a fan consisting of 1-dimensional cones of Σ or rays. The 1-skeleton consists of the rays spanned by integral vectors v_i, $1 \leq i \leq N$. We will identify rays with the smallest integral vectors v_i belonging to them.

Consider an integral basis $\{Q_{ia}\}_{a \leq k}$ of all possible linear dependencies between v_i. That is any linear dependence with integral coefficients

$$\sum_{i \leq N} m_i v_i = 0. \; m_i \in \mathbb{Z}. \tag{2.346}$$

can be written as $m_i = \sum_a Q_{ia} c_a$ with integral coefficients c_a.

Mathematically it means that there is a short exact sequence

$$0 \to \mathbb{Z}^k \to \mathbb{Z}^N \to \mathbb{Z}^n \to 0, \tag{2.347}$$

where the first map has a matrix Q_{ia}, and the second has a matrix v_{ij}.

We can define $\mathbb{C}^n$ with coordinates x_i for every vector v_i from $\Sigma(1)$. We also define a $(\mathbb{C}^*)^k$-action on $\mathbb{C}^n$ with the charge matrix Q_{ia}.

The *unstable locus* Z_Σ is defined as a union of hyperplanes $\{x_{i_1}, \ldots, x_{i_p}\}$ such that $v_{i_1}, \ldots, v_{i_p}$ do not form a cone of Σ. Then the toric variety with a fan Σ is defined as

$$\mathbb{P}_\Sigma := (\mathbb{C}^N - Z_\Sigma)/(\mathbb{C}^*)^k. \tag{2.348}$$

Each p-dimensional cone σ of Σ defines a toric-invariant subvariety $(\{x_{i_1} = 0, \ldots, x_{i_p} = 0\} \backslash Z_\Sigma)/(\mathbb{C}^*)^k$ of dimension $n - p$.

Another way of defining toric varieties uses polytopes. A polytope in $M_\mathbb{R} := N^*_\mathbb{R} \simeq \mathbb{R}^n$ is a convex hull of a set of rational points in $M_\mathbb{R}$

$$\Delta := \{\sum_{i \leq h} a_i v_i \mid v_i = (v_{i1}, \ldots, v_{in}), \; 0 \leq a_i \leq 1, \; \sum_i a_i = 1\}. \tag{2.349}$$

A polytope defines a toric variety $\mathbb{P}_\Delta$ together with an embedding into a projective space. Let us denote $\{m_i\}_{i=1}^H$ to be a set if all integral points of Δ. Every integral point in $M_\mathbb{R}$ defines a function on a torus by the rule $t_j \to t_j^{m_{ij}}$, where t_i are coordinates on a torus and $m_i = (m_{i1}, \ldots, m_{in})$. A toric variety $\mathbb{P}_\Delta$ is defined as a closure of an image of a torus $(t^{m_1} : \cdots : t^{m_H})$ inside $\mathbb{P}^{H-1}$. Such a toric variety in good cases coincide with $\mathbb{P}_{\Sigma_\Delta}$, where Σ_Δ is a *normal fan* of Δ. The normal fan of Δ consists of normals to proper faces of Δ. In particular, one-dimensional cones of Σ_Δ are the normals to facets of Δ (in the classical construction there is an additional minus sign). The integral points of a polytope Δ are sectons of a line bundle $\mathcal{L}$ whose pushforward to $\mathbb{P}^{H-1}$ is $\mathcal{O}(1)$. We will be interested in polytops which correspond to anticanonical line bundles $K^{-1} \to \mathbb{P}_\Sigma$.

The Batyrev mirror symmetry holds for Calabi-Yau hypersurfaces in toric varieties defined by *reflexve politopes.*

Definition 2.6.1. *A polytope Δ is called reflexive if*

1. *its facets (faces of maximal dimension) $\sum_{i=1}^n v_i x_i = -1$, where $v_i \in \mathbb{Z}$.*
2. *Δ contains the origin.*

There is a duality property on reflexive polytopes which is called a *polar duality.* Given a polytope Δ the polar dual polytope ∇ is defined as

$$\nabla := \{y \in (\mathbb{R}^4)^* \mid \forall x \in \Delta \ \langle x, y\rangle \geq -1\}. \tag{2.350}$$

If Δ is reflexive then so is ∇. Each p-dimensional face of Δ corresponds to a $n-k-1$-dimensional face of ∇ and vice versa. The polar dual polytope to ∇ is Δ again. There is a nice correspondence between integral points of Δ and 1-dimensional cones of a normal fan of a dual polytope

Σ_∇. Namely, all vectors v_i of Σ_∇ are integral points of Δ and vice versa, that is all vectors $\check{v}_j$ of Σ_Δ are integral points of ∇. Let us define a fan $\tilde{\Sigma}_\nabla$ as a refinement of theirfan Σ_∇ by adding additional 1-dimensional cones which correspond to integral points of the polyope Δ.

The toric variety $\mathbb{P}_{\tilde{\Sigma}_\nabla}$ is a (partial) resolution of singularities of $\mathbb{P}_\nabla$ since additional 1-dimensional cones correspond to additional divisors. Moreover, by construction, the integral points of the dual polytope (which correspond to anticanonical sections on the dual toric variety) are unchanged.

Consider a maximal resolution $\tilde{\mathbb{P}}_\nabla$ which is specified by adding all integral points of Δ to Σ_∇. Such a resolution is not unique on the level of higher dimensional cones. Integral points of ∇ define anticanonical sections in $\tilde{\mathbb{P}}_\nabla$. A zero locus of a generic anticanonical section is Calabi-Yau variety since its canonical class is trivial.

The Batyrev mirror symmetry states that the anticanonical hypersurfaces in $\tilde{\mathbb{P}}_\nabla$ and $\tilde{\mathbb{P}}_\Delta$ form a pair of mirror dual Calabi-Yau varieties.

Below we use a slight modification of this construction in our check of the conjecture of [93] and build explicit mirrors and mirror maps.

2.6.3 Mirror quintic

Consider a generic equation of the quintic threefolds $\mathcal{X}_\phi$ one more time

$$W(x, \phi) = \sum_{i=1}^{5} x_i^5 + \sum_{l=1}^{101} \phi_l e_l(x) = \sum_{i=1}^{106} C_a(\phi) \prod_{j=1}^{5} x_j^{v_{ij}}, \tag{2.351}$$

where we introduced an exponent matrix v_{ij}. Let us build a mirror quintic as a hypersurface in a toric variety. Vectors $v_i = (v_{i,1}, \dots v_{i,5})$ define (not all) integral points of a polytope Δ of a projective space $\mathbb{P}^4$. All these points belong to a $\mathbb{R}^4$ which is defined by an equation $\sum_{j\leq 5} v_{ij} = 5$ in $\mathbb{R}^5$.

We make use of a fact that the points of a polytope Δ can be used in the construction of a toric variety $\tilde{\mathbb{P}}_\nabla$ containing the mirror quintic.

The Batyrev construction suggests taking all integral points of Δ to be 1-dimensional cones of the fan $\tilde{\Sigma}_{nabla}$ of $\tilde{\mathbb{P}}_\nabla$.

We take only the points from the equation (2.351) instead which corresponds to a partial resolution of a singular variety $\mathbb{P}_\nabla$. We consider a fan whose 1-dimensional skeleton is a set of vectors $\{v_i\}_{i=1}^{106}$

$$v_{ij} = \begin{cases} 5\delta_{i,j}, & 1 \leq i \leq 5, \\ s_{i-5,j}, & 6 \leq i \leq 106. \end{cases} \tag{2.352}$$

with nonstandart integral structure (so that v_i are smallest integral vectors spanning their cones) whose explicit form is not important for us.

This fan defines a canonical bundle $K \to \tilde{\mathbb{P}}_\nabla$ which is a five-dimensional non-compact Calabi-Yau variety

In order to build a GLSM we need to pick an integral basis in the linear relations among b_i. It turns out more convenient to pick a rational basis Q_{ai} in the relations instead.

$$\sum_{i\leq 106} Q_{ai} v_i = 0. \tag{2.353}$$

In such a basis the integral lattice in the Cartan algebra of $U(1)^{101}$ is non-standart. The element $m = \{m_a\}_{a=1}^{101}$ from the Cartan algebra is integral if and only if $\sum_{a=1}^{101} m_a Q_{ai}$ is integral for all i.

The convenient rational basis for the mirror quintic is

$$\tilde{Q}_{ai} = \begin{cases} s_{ai}, & 1 \leq i \leq 5, \\ -5\delta_{i-5,a}, & 6 \leq i \leq 106. \end{cases} \tag{2.354}$$

These relations tell that any monomial $x_1^{s_1} \cdots x_5^{s_5}$ is a product of five monomials x_i^5 with some powers which are rationals with the denominator 5.

These relations clearly do not form an integral basis in all relations. Consider a pair of deformations $s_2 = (3, 2, 0, 0, 0)$ and $s_3 = (2, 3, 0, 0, 0)$, which correspond to monomials $x_1^3 x_2^2$ and $x_1^2 x_2^3$. We have $Q_{2i} + Q_{3i} = (5, 5, 0, 0, 0, 0, -5, -5, 0, \dots, 0)$ which is 5 times an integral relation $(1, 1, 0, 0, 0, 0, -1, -1, 0, \dots, 0)$, that is $v_1 + v_2 - v_7 - v_8 = 0$. It is easy to see that the last relation cannot be obtained as an integral combination of Q_{ai}.

From the formula (2.354) it is clear that $\sum_{i=1}^{106} Q_{ai} = 0$ which reflects the fact that $K \to \tilde{\mathbb{P}}_\nabla$ is a Calabi-Yau variety and r_l, θ_l are free parameters on the quantum level.

Let us write down a chiral superpotential of the theory. It will be convenient to split the chiral fields into

$$\Phi_i = \begin{cases} S_i, & 1 \le i \le 5, \\ P_{i-5}, & 6 \le i \le 106. \end{cases} \tag{2.355}$$

A chiral field P_1 corresponds to the vector $v_{6i} = (1,1,1,1,1)$ and is a coordinate in the fibers of the anticanonical bundle. The superpotential is

$$W_Y := P_1\, G(S_1, \dots, S_5; P_2, \dots, P_{101}). \tag{2.356}$$

Being a bit sloppy we assign the R-charges as $q_{P_1} = 2,\ q_{P_l} = 0,\ l > 1$ and $q_{S_i} = 0$. The R-charge of W is 2 as it should to preserve the symmetry.

The scalars potential is

$$U(\boldsymbol{\phi}) = \sum_{l=1}^{101} \frac{e_l^2}{2} \left(\sum_{i=1}^{5} s_{li} |S_a|^2 - 5|P_l|^2 - r_l \right)^2 + \frac{1}{4} |G(S_1, \dots, S_5; P_2, \dots, P_{101})|^2 + \\ + \frac{1}{4} |P_1|^2 \sum_{i=1}^{5} \left| \frac{\partial G}{\partial S_i} \right|^2 + \frac{1}{4} |P_1|^2 \sum_{l=2}^{101} \left| \frac{\partial G}{\partial P_l} \right|^2 . \tag{2.357}$$

Depending on the values of the Fayet-Iliopoulos parameters r_l such a GLSM is in different phases. In the geometric phases ot describes a non-linear sigma model in the vacua manifolds which is one of the mirror quintics

$$\sum_{i=1}^{5} s_{li} |S_a|^2 - 5|P_l|^2 - r_l = 0, \quad G(S_1, \dots, S_5; P_2, \dots, P_{101}) = 0, \ \ P_1 = 0 \tag{2.358}$$

modulo $U(1)^{101}$. The spherical partition function can be written explicitly

$$Z_{S^2} = \sum_{m_l \in V} \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_{101}} \prod_{l=1}^{101} \frac{d\tau_l}{(2\pi i)} \left(z_l^{-\tau_l + \frac{m_l}{2}} \bar{z}_l^{-\tau_l - \frac{m_l}{2}} \right) \times \\ \times \frac{\Gamma\big(1 - 5(\tau_1 - \frac{m_1}{2})\big)}{\Gamma\big(5(\tau_1 + \frac{m_1}{2})\big)} \prod_{a=1}^{5} \frac{\Gamma\big(\sum_l s_{la}(\tau_l - \frac{m_l}{2})\big)}{\Gamma\big(1 - \sum_l s_{la}(\tau_l + \frac{m_l}{2})\big)} \prod_{l=2}^{101} \frac{\Gamma\big(-5(\tau_l - \frac{m_l}{2})\big)}{\Gamma\big(1 + 5(\tau_l + \frac{m_l}{2})\big)}, \tag{2.359}$$

where

$$z_l := e^{-(2\pi r_l + i\theta_l)}. \tag{2.360}$$

Contours C go slightly to the left of the imaginary axes $\tau_l = -\epsilon + it_l$ which can be acheved by assigning small positive R-charges to the chiral fields (except P_1).

The summation set V is defined by the qunatization condition that m is in the integral lattice of the Cartan algebra: $\sum_{a\leq 101} m_a Q_{ai} \in \mathbb{Z}$.

We consider the Landau-Ginzburg phase of this GLSM as it is mirror to our computations. The LG phase corresponds to large values of $|z_l|$ or $r_l \ll 0$. Each of the contours can be closed to the right and picks up the poles at

$$5\left(\tau_l - \frac{m_l}{2}\right) - 1 = p_1,\ 5\left(\tau_l - \frac{m_l}{2}\right) = p_l;$$
$$p_1 = 1, 2, \ldots,\ p_l = 0, 1, \ldots \quad \text{so that} \quad p_l + 5m_l > 0. \quad (2.361)$$

It is convenient to introduce a notation $\bar{p}_l := p_l + 5m_l$. After computing the residues which are all at the first order poles, the partition function becomes

$$Z_{S^2} = \pi^{-5} \sum_{p_1>0, p_l\geq 0} \sum_{\bar{p}_l \in \Sigma_p} \prod_l \frac{(-1)^{p_l}}{p_l! \bar{p}_l!} z_l^{-\frac{p_l}{5}} \bar{z}_l^{-\frac{\bar{p}_l}{5}}$$
$$\prod_{i=1}^{5} \Gamma\left(\frac{1}{5}\sum_{l=1}^{h} s_{li} p_l\right) \Gamma\left(\frac{1}{5}\sum_{l=1}^{h} s_{li} \bar{p}_l\right) \sin\left(\frac{\pi}{5}\sum_{l=1}^{h} s_{li}\bar{p}_l\right), \quad (2.362)$$

where the set Σ_p - is a set of all $\{\bar{p}_l\}$ such that $\sum_a (\bar{p}_a - p_a) Q_{ai}/5 = \sum_a m_a Q_{ai} \in \mathbb{Z}$ as is dictated by the quantization condition. Using the formula (2.354) we rewrite the condition as $\bar{p}_l \in \mathbb{Z}$ and $\sum_a (\bar{p}_a - p_a) s_{ai} \in 5\mathbb{Z}$. Every term such that $\sum_{a=1}^{101} \bar{p}_a s_{ai} = 0 \bmod 5$ in (2.362) vanishes and the sum in (2.362) is over the sets

$$S_{\boldsymbol{a}} = \left\{ p_l, \bar{p}_l \ :\ \sum_{l=1}^{101} s_{li} p_l \equiv \sum_{l=1}^{101} s_{li} \bar{p}_l \equiv a_i \ (\mathrm{mod}\ 5),\ 1 \leq a_i \leq 4 \right\}. \quad (2.363)$$

Finally, we use the following identity

$$\prod_{i=1}^{5} \sin\left(\frac{\pi}{5}\sum_{l=1}^{h} s_{li}\bar{p}_l\right) = (-1)^{|\boldsymbol{a}|} \prod_{i=1}^{5} \sin\left(\frac{\pi a_i}{5}\right) \prod_{l=1}^{h} (-1)^{\bar{p}_l}, \quad (2.364)$$

we find

$$Z_{S^2} = \sum_{\boldsymbol{a}} (-1)^{|\boldsymbol{a}|} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{a_i}{5}\right)}{\Gamma\left(1 - \frac{a_i}{5}\right)} |\sigma_{\boldsymbol{a}}(\boldsymbol{z})|^2, \quad (2.365)$$

where

$$\sigma_{\boldsymbol{a}}(\boldsymbol{z}) = \sum_{n_i \geq 0} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{a_i}{5} + n_i\right)}{\Gamma\left(\frac{a_i}{5}\right)} \sum_{\boldsymbol{p} \in S_{\boldsymbol{a},\boldsymbol{n}}} \prod_{l=1}^{101} \frac{(-1)^{p_l} z_l^{-\frac{p_l}{5}}}{p_l!}. \quad (2.366)$$

The S^2 partition function given by the formulas above coincides with the exponent of the Kähler potential for the complex structures moduli space of the quintic which we computed in (2.325) under a simple mirror map

$$z_a = -\phi_l^{-5}. \tag{2.367}$$

As expected, in the region $r_l \ll 0$ the spherical partition function of the mirror quintic GLSM reproduces the complex structure moduli space geometry of the quintic. It is remarkable how simple the form of the mirror map is as opposed to the "geometric" mirror map at the FJRW point

$$t^s_{LG}(\phi) = \sigma_s(\phi)/\sigma_{(00000)}(\phi_1), \tag{2.368}$$

where $s = (s_1, \ldots, s_5),\ 0 \le s_i \le 3, \sum_i s_i = 5$.

2.6.4 Fermat hypersurfaces

The discussion above easily generalizes to the case fo Fermat hypersurfaces in weighted projective spaces. Since this case is parallel to the quintic, we write down the formulas with the minimum amount of comments.

Consider a family of Fermat hypersurfaces

$$W(x,\phi) = x_1^{d/k_1} + x_2^{d/k_2} + x_3^{d/k_3} + x_4^{d/k_4} + x_5^{d/k_5} + \sum_{s=1}^{h} \phi_s x_1^{s_1} \cdots x_5^{s_5} = 0. \tag{2.369}$$

We define a fan for the mirror variety as in the quintic case by the 1-dimensional skeleton

$$v_{ij} = \begin{cases} d\delta_{i,j}, & 1 \le i \le 5, \\ k_j s_{i-5,j}, & 6 \le i \le 106. \end{cases} \tag{2.370}$$

Instead of discussing the higher dimensional cones we construct a GLSM whose different phases correspond to different fans with the same 1-skeleton.

We pick a simple rational basis Q_{ai} in the linear relations among v_i:

$$Q_{ai} = \begin{cases} k_i s_{ai}, & 1 \le i \le 5, \\ -d\delta_{i-5,a}, & 6 \le i \le h. \end{cases} \tag{2.371}$$

In this basis the quantization condition on twisted masses has the form

$$\sum_a m_a Q_{ai} \in \mathbb{Z} \tag{2.372}$$

instead of just $m_l \in \mathbb{Z}$.

We regroup the chiral fields of the theory as

$$\Phi_i = \begin{cases} S_i, & 1 \leq i \leq 5, \\ P_{i-5}, & 6 \leq i \leq h+5. \end{cases} \tag{2.373}$$

As in the quintic case the chiral field P_1 is somewhat distinguished and corresponds to the vector $v_{6i} = (1,1,1,1,1)$. The weighted homogeneous superpotential is

$$W_Y := P_1\, G(S_1, \ldots, S_5; P_2, \ldots, P_h), \tag{2.374}$$

whose explicit expression is not important for us. The R-charges are assigned according to $q_{P_1} = 2,\ q_{P_l} = 0,\ l > 1$ and $q_{S_i} = 0$.

The partition function of thei GLSM on S^2 reads

$$Z_{S^2} = \sum_{m_l \in V} \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_h} \prod_{l=1}^{h} \frac{d\tau_l}{(2\pi i)} \left(z_l^{-\tau_l + \frac{m_l}{2}} \bar{z}_l^{-\tau_l - \frac{m_l}{2}} \right) \times \\ \times \frac{\Gamma\big(1 - d(\tau_1 - \frac{m_1}{2})\big)}{\Gamma\big(d(\tau_1 + \frac{m_1}{2})\big)} \prod_{a=1}^{5} \frac{\Gamma\big(\sum_l k_a s_{la}(\tau_l - \frac{m_l}{2})\big)}{\Gamma\big(1 - \sum_l k_a s_{la}(\tau_l + \frac{m_l}{2})\big)} \prod_{l=2}^{h} \frac{\Gamma\big(-d(\tau_l - \frac{m_l}{2})\big)}{\Gamma\big(1 + d(\tau_l + \frac{m_l}{2})\big)}, \tag{2.375}$$

where

$$z_l := e^{-(2\pi r_l + i\theta_l)}. \tag{2.376}$$

The contours C go slightly to the left from the imaginary axes $\tau_l = -\epsilon + it_l$. The summation set V is defined by the quantizatin, condition $m \in V \iff \sum_{a \leq h} m_a Q_{ai} \in \mathbb{Z}$.

We perform the computation in the Landau-Ginzburg phase of the GLSM which is at $|z_l| \gg 0,\ r_l \ll 0$. Each of the contours C can be closed to the right and picks up the residues at first order poles at

$$d\left(\tau_l - \frac{m_l}{2}\right) - 1 = p_1,\ d\left(\tau_l - \frac{m_l}{2}\right) = p_l; \\ p_1 = 1, 2, \ldots,\ p_l = 0, 1, \ldots \quad \text{so that} \quad p_l + d m_l > 0. \tag{2.377}$$

The partition function is evaluated to be

$$Z_{S^2} = \pi^{-5} \sum_{p_1 > 0, p_l \geq 0} \sum_{\bar{p}_l \in \Sigma_p} \prod_{l} \frac{(-1)^{p_l}}{p_l! \bar{p}_l!} z_l^{-\frac{p_l}{d}} \bar{z}_l^{-\frac{\bar{p}_l}{d}} \\ \prod_{i=1}^{5} \Gamma\left(\frac{1}{d} \sum_{l=1}^{h} k_i s_{li} p_l\right) \Gamma\left(\frac{1}{d} \sum_{l=1}^{h} k_i s_{li} \bar{p}_l\right) \sin\left(\frac{\pi}{d} \sum_{l=1}^{h} k_i s_{li} \bar{p}_l\right), \tag{2.378}$$

where the set Σ_p is a set of all $\{\bar{p}_l\}$ such that $\sum_a (\bar{p}_a - p_a) Q_{ai}/d = \sum_a m_a Q_{ai} \in \mathbb{Z}$, that is $\bar{p}_l \in \mathbb{Z}$ and $\sum_a (\bar{p}_a - p_a) s_{ai} k_i \in d\mathbb{Z}$.

Terms where $\sum_{a=1}^{h} \bar{p}_a s_{ai} j_i = 0$ mod d vanish and the summation in (2.378) reduces to the sets

$$S_{\boldsymbol{a}} = \left\{ p_l, \bar{p}_l \ : \ \sum_{l=1}^{h} k_i s_{li} p_l \equiv \sum_{l=1}^{h} k_i s_{li} \bar{p}_l \equiv a_i \ (\mathrm{mod}\ d), \ 1 \leq a_i \leq d/k_i - 1 \right\}. \quad (2.379)$$

The product of sines can be simplified using

$$\prod_{i=1}^{5} \sin\left(\frac{\pi}{d} \sum_{l=1}^{h} k_i s_{li} \bar{p}_l \right) = (-1)^{|\boldsymbol{a}|} \prod_{i=1}^{5} \sin\left(\frac{\pi k_i a_i}{d} \right) \prod_{l=1}^{h} (-1)^{\bar{p}_l}. \quad (2.380)$$

The final expression for the partition function is

$$Z_{S^2} = \sum_{\boldsymbol{a}} (-1)^{|\boldsymbol{a}|} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{k_i a_i}{d}\right)}{\Gamma\left(1 - \frac{k_i a_i}{d}\right)} |\sigma_{\boldsymbol{a}}(\boldsymbol{z})|^2, \quad (2.381)$$

where

$$\sigma_{\boldsymbol{a}}(\boldsymbol{z}) = \sum_{n_i \geq 0} \prod_{i=1}^{5} \frac{\Gamma\left(\frac{k_i a_i}{d} + n_i\right)}{\Gamma\left(\frac{k_i a_i}{d}\right)} \sum_{\boldsymbol{p} \in S_{\boldsymbol{a},\boldsymbol{n}}} \prod_{l=1}^{h} \frac{(-1)^{p_l} z_l^{-\frac{p_l}{d}}}{p_l!}. \quad (2.382)$$

These formulas coincide with the special geometry Kähler potential on the moduli space of the Fermat hypersurfaces (2.330) up to a common factor.

The mirror map is given by

$$z_a = -\phi_l^{-d} \quad (2.383)$$

and is as simple as in the case of the (mirror) quintic.

Let us comment on the general case of invertible singularities. The main difference is that in the general case some of the poles will have the second order even in the Landau-Ginzburg case which will lead to an appearance of logarithms in the periods.

2.6.5 Conclusion

In this section we applied the supersymmetric localization methods of [134, 92] and mirror symmetry construction of [129] to check the mirror version of the conjecture stated in [93]. The main result of this section is an explicit construction of a model which is a mirror to a Fermat hypersurface in a weighted projective space, a check of the equality between the sphere partition function and the exponent of the complex structures moduli space metric Kähler potential with the help of an explicit mirror map (2.383)

$$e^{-K} \sim \int_{\mathcal{X}} \Omega_\phi \wedge \overline{\Omega_\phi}. \quad (2.384)$$

In addition to the main results we can use the localization formulas to study complex structures moduli space of Calabi-Yau manifolds. In particular, analytic continuation between different phases of GLSM gives special geometry in different regions of the complex structures moduli space.

Appendix

2.A Variations of Hodge structures for orbifolds

A huge class of Calabi-Yau manifolds is given by hypersurfaces in *weighted projective spaces* [120]. On the Landau-Ginzburg side such hypersurfaces correspond to *weighted homogeneous* superpotentials. The whole story of variations of Hodge structures and LG CY correspondence is pretty much the same as for hypersurfaces in projective spaces with the notable exception that the CY hypersurfaces are singular. General singular variety does not have Hodge structure in its cohomology but instead have mixed Hodge structures. However, the singularities are very mild, they are Gorenstein orbifold singularities for a generic (transverse) hypersurface. Then there are many equivalent languages to define appropriate cohomology theory and variation of Hodge structures on middle cohomology on the Calabi-Yau variety X. For example, one could make a small resolution of singularities of X or consider intersection homology.

Consider a ring of polynomial functions $\mathbb{C}[x_1, \dots, x_N]$ in $\mathbb{C}^N$. A set of coprime positive integers $k_1, \dots, k_N$ defines a weight grading on this ring by the rule $wt(x_i) = k_i$. Any graded component $\mathbb{C}[x_1, \dots, x_N]^w$ of the ring is finite dimensional since all k_i are positive. Graded components of the ring $\mathbb{C}[x_1, \dots, x_N]$ consist of global sections of line bundles on weighted projective spaces.

Definition 2.A.1. *Weighted projective space* $\mathbb{P}^N_{\bar{k}} = \mathbb{P}^{N-1}_{k_1:\cdots:k_N}$ *is an algebraic variety defined as a quotient*

$$\mathbb{P}^N_{\bar{k}} := (\mathbb{C}^N \backslash 0)/\mathbb{C}^* = \mathrm{Proj} \bigoplus_{w \geq 0} \mathbb{C}[x_1, \cdots, x_N]^w. \tag{2.385}$$

The points of $\mathbb{P}^N_{\bar{k}}$ *are denoted as* $(x_1 : \cdots : x_N)$ *which stands for an equivalence class under the relation* $(x_1, \dots, x_N) \sim (\lambda^{k_1} x_1, \dots, \lambda^{k_N} x_N),\ \lambda \in \mathbb{C}^*$.

Weighted projective spaces are covered by singular (global quotient) charts $\mathbb{P}^{N-1}_{\bar{k}} = \cup_{i=1}^N U_i$,

$$U_i = \{(x_1 : \cdots : 1 : \cdots : x_N) \sim (\alpha^{k_1} x_1 : \cdots : 1 : \cdots : \alpha^{k_N} x_N)\}, \tag{2.386}$$

where $\alpha^{k_i} = 1$. We see that $U_i = \mathbb{C}^{N-1}/\Gamma_i = \operatorname{Spec}\mathbb{C}[y_1, \cdots, y_{N-1}]^{\Gamma_i}$, where $\Gamma_i \subset \mathbb{Z}/k_i\mathbb{Z}$.

Detailed description of weighted projective spaces and their properties can be found in the original work [120].

Definition 2.A.2. *[114] A complex variety X is an n-dimensional orbifold if it is locally analytically isomorphic to U/Γ, where Γ is a finite subgroup of $\mathrm{GL}(d, \mathbb{C})$ without nontrivial complex reflections and $U \subset \mathbb{C}^n$ is a Γ-stable neighbourhood of the origin.*

By slight abuse of notations we will call U/Γ an orbifold chart on X

Cohomology of orbifolds over $\mathbb{Q}$ behave pretty much like cohomology of manifolds.

Remark 2.A.1. *Weighted projective space is an orbifold where orbifold charts U/Γ are given by U_i.*

Since we work with variations of Hodge structure using period map given as a set of integrals we need a notion of differential forms on orbifolds.

A differential form (smooth or analytic) on an orbifold chart $U\Gamma$ is a Γ-invariant differential form on U. A differential form on X is naturally a collection of differential forms on every orbifold chart U_i such that they patch on intersections. In particular, sheaves of holomorphic differential p-forms are called Zarski sheaves and will be denoted as $\hat{\Omega}^p_X$. It can be shown [114] that these sheaves are pushforwards of sheaves of holomorphic forms from the smooth locus.

De-Rham and Dolbeault differentials are defined on complexes of orbifold differential forms locally in each chart and allow to define de-Rham $H^*(X, \mathbb{Q})$ and Dolbeault $H^{*,*}(X, \mathbb{Q})$ cohomology groups as in the smooth case. The pairing with homology can be defined in terms of integration of orbifold forms over actual homology cycles.

Let X be an n-dimensional orbifold, ω be a k-dimensional orbifold chart and $C \subset X$ be a singular k-cycle (we consider X just a topological space endowed with analytic topology).

Then the integration of ω over C in a chart (U/Γ) is defined as

$$\int_{C|_{U/\Gamma}} \omega := \frac{1}{|\Gamma_0|} \int_{p^{-1}(C|_{U/\Gamma})} p^*\omega. \tag{2.387}$$

Integration over C inside X is defined using an orbifold atlas and partitions of unity. For our purposed intersections of C with U/Γ will be dense inside C and we will not need to wor with partitions of unity.

The following proposition (see e.g. [114])allows us to work with variation of Hodge structures on orbifolds $\otimes\mathbb{Q}$ as if they were for smooth manifolds.

Proposition 2.A.1. *Let a complex variety X be an orbifold and $X_o \overset{\iota}{\hookrightarrow} X$ be an embedding of the smooth locus. Then the following holds true:*

1. $\hat{\Omega}^p_X = \iota^*(\Omega^p_{X_0})$.

2. $(\hat{\Omega}^p_X, \mathrm{d})$ *is a resolution of a constant sheaf* $\mathbb{C}$.

3. $H^q(X, \hat{\Omega}^p_X) \simeq H^{p,q}(X)$.

4. $H^*(X, \mathbb{Q}) \to IH^*(X, \mathbb{Q})$ *is an isomorphism, where* IH *is intersection homology.*

5. $H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$ *and it has a pure Hodge structure over* $\mathbb{Q}$ *and satisfies the Hard Lefschetz theorem.*

6. $IH^{sing}_*(X, \mathbb{Q}) \simeq H^{sing}_*(X, \mathbb{Q})$ *under the natural identification of representative singular chains.*

We are interested in the case where $X \subset \mathbb{P}^{N-1}_{\bar{k}}$ is a hypersurface in a weighted projective space. By the proposition above its middle cohomology have a pure Hodge structure. We want to describe variations of Hodge structures for families $\mathcal{X} \xrightarrow{\pi} S$ where $X_\phi = \pi^{-1}(\phi)$ is an orbifold hypersurface in $\mathbb{P}^{N-1}_{\bar{k}}$ in terms of period integrals as we did for the smooth case.

Proposition 2.A.2. *Let* $\mathcal{X} \xrightarrow{\pi} S$ *where* S *is a* h*-dimensional smooth complex manifold (or orbifold).*

$$X_\phi = \pi^{-1}(\phi) = \{W(x, \phi) = 0\} \subset \mathbb{P}^{N-1}_{\bar{k}}\} \tag{2.388}$$

is a quasi-smooth orbifold hypersurface in $\mathbb{P}^{N-1}_{\bar{k}}$ *for each* $\phi \in S$.

1. *The holomorphic vector bundle* $\mathcal{H} \xrightarrow{p} S$*, where* $p^{-1}(\phi) = H^{N-2}(X_\phi)$ *with natural Hodge decompositions and Gauss-Manin connections is a variation of Hodge structures.*

2. *Family of differential forms:*

$$\Omega_\phi := \frac{\mathrm{d}x_1 \cdots \widehat{\mathrm{d}x_i} \cdots \mathrm{d}x_{N-1}}{\partial W(x, \phi)/\partial x_N} \tag{2.389}$$

defines a holomorphic volume form, that is a nonvanishing section of K_{X_ϕ} *for generic* ϕ.

3. *The period map is recovered from the following integrals*

$$\int_{\gamma_i} \Omega_\phi = \frac{1}{|\Gamma_i|} \int_{\pi^{-1}(\gamma|_{U_i})} \frac{\mathrm{d}x_1 \cdots \widehat{\mathrm{d}x_i} \cdots \mathrm{d}x_{N-1}}{\partial W/\partial x_N}. \tag{2.390}$$

4. *Let* $\mathcal{L} = H^{3,0}(X_\phi) \subset H^3_{poly}(X_\phi)$ *be a holomorphic linear subbundle of* $\mathcal{H}$*. Then* $(S, \mathcal{H}, \mathcal{L}, \Omega_\phi)$ *is a projective special Kähler manifold with* $\omega_S = \partial\bar{\partial}K$ *and*

$$e^{-K} = i \int_{X_\phi} \Omega \wedge \overline{\Omega}. \tag{2.391}$$

Proof. □

Definition 2.A.3. *As in the smooth case we understand (polynomial) complex structure moduli space* $\mathcal{M}^c_X$ *of* X *as the special Kähler manifold* $(S, \mathcal{H}, \mathcal{L}, \Omega_\phi)$.

Proposition 2.A.3. *The Kähler potential on* $\mathcal{M}^c_X$ *can be written as a Hermitian sum of the period integrals. Let* $\{q_i\} \in H_3(X, \mathbb{Q})$ *be a basis of* $H_3^{poly}(X, \mathbb{Q}) := H^3_{poly}(X, \mathbb{Q})^*$ *and* $\omega_i(\phi) := \int_{q_i} \Omega_\phi$*. Then*

$$e^{-K} = \sum_{i,j} \omega_i(\phi) C^{ij} \overline{\omega_j(\phi)}, \tag{2.392}$$

where inverse of a non-degenerate matrix C *is* $(C^{-1})_{ij} = [q_i] \cap [q_j]$ *an intersection matrix of cycles.*

The proof is analogous to the smooth case.

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