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RESEARCH CONTRIBUTION

On the Geometry of the Set of Symmetric Matrices with Repeated Eigenvalues

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Abstract
We investigate some geometric properties of the real algebraic variety Δ of symmetric matrices with repeated eigenvalues. We explicitly compute the volume of its intersection with the sphere and prove a Eckart–Young–Mirsky-type theorem for the distance function from a generic matrix to points in Δ. We exhibit connections of our study to real algebraic geometry (computing the Euclidean distance degree of Δ) and random matrix theory.

Keywords Integral geometry · Random matrices · Euclidean distance degree theory

1 Introduction

In this paper we investigate the geometry of the set Δ (below called *discriminant*) of real symmetric matrices with repeated eigenvalues and of unit Frobenius norm:

$$\Delta = \left\{ Q \in \mathrm{Sym}(n, \mathbb{R}) : \lambda_i(Q) = \lambda_j(Q) \text{ for some } i \neq j \right\} \cap S^{N-1}.$$

Here, $\lambda_1(Q), \ldots, \lambda_n(Q)$ denote the eigenvalues of Q, the dimension of the space of symmetric matrices is $N := \frac{n(n+1)}{2}$ and S^{N-1} denotes the unit sphere in $\mathrm{Sym}(n, \mathbb{R})$ endowed with the Frobenius norm $\|Q\| := \sqrt{\mathrm{tr}(Q^2)}$.

This discriminant is a fundamental object and it appears in several areas of mathematics, from mathematical physics to real algebraic geometry, see for instance (Arnold

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1972, 1995, 2003, 2011; Teytel 1999; Agrachev 2011; Agrachev and Lerario 2012; Vassiliev 2003). We discover some new properties of this object (Theorems 1.1, 1.4) and exhibit connections and applications of these properties to random matrix theory (Sect. 1.4) and real algebraic geometry (Sect. 1.3).

The set Δ is an algebraic subset of S^{N-1}. It is defined by the *discriminant polynomial*:

$$\operatorname{disc}(Q) := \prod_{i<j} \left(\lambda_i(Q) - \lambda_j(Q)\right)^2,$$

which is a non-negative homogeneous polynomial of degree $\deg(\operatorname{disc}) = n(n-1)$ in the entries of Q. Moreover, it is a sum of squares of real polynomials (Ilyushechkin 2005; Parlett 2002) and Δ is of codimension two. The set Δ_{sm} of smooth points of Δ is the set of real points of the smooth part of the Zariski closure of Δ in $\operatorname{Sym}(n, \mathbb{C})$ and it consists of matrices with *exactly* two repeated eigenvalues. In fact, Δ is stratified according to the multiplicity sequence of the eigenvalues; see (1.3).

1.1 The Volume of the Set of Symmetric Matrices with Repeated Eigenvalues

Our first main result concerns the computation of the *volume* $|\Delta|$ of the discriminant, which is defined to be the Riemannian volume of the smooth manifold Δ_{sm} endowed with the Riemannian metric induced by the inclusion $\Delta_{\mathrm{sm}} \subset S^{N-1}$.

Theorem 1.1 (The volume of the discriminant).

$$\frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}.$$

Remark 1 Results of this type (the computation of the volume of some relevant algebraic subsets of the space of matrices) have started appearing in the literature since the 90's (Edelman and Kostlan 1995; Edelman et al. 1994), with a particular emphasis on asymptotic studies and complexity theory, and have been crucial for the theoretical advance of numerical algebraic geometry, especially for what concerns the estimation of the so called *condition number* of linear problems (Demmel 1988). The very first result gives the volume of the set $\Sigma \subset \mathbb{R}^{n^2}$ of square matrices with zero determinant and Frobenius norm one; this was computed in Edelman and Kostlan (1995) and Edelman et al. (1994):

$$\frac{|\Sigma|}{|S^{n^2-1}|} = \sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \sim \sqrt{\frac{\pi}{2}}n^{1/2}.$$

For example, this result is used in Edelman and Kostlan (1995, Theorem 6.1) to compute the average number of zeroes of the determinant of a matrix of linear forms. Subsequently this computation was extended to include the volume of the set of $n \times m$ matrices of given corank in Beltrán (2011) and the volume of the set of *symmetric*

matrices with determinant zero in Lerario and Lundberg (2016), with similar expressions. Recently, in Beltrán and Kozhasov (2018) the above formula and Lerario and Lundberg (2016, Thm. 3) were used to compute the expected condition number of *the polynomial eigenvalue problem* whose input matrices are taken to be random.

In a related paper Breiding et al. (2017) we use Theorem 1.1 for counting the average number of singularities of a random spectrahedron. Moreover, the proof of Theorem 1.1 requires the evaluation of the expectation of the square of the characteristic polynomial of a GOE(n) matrix (Theorem 1.6 below), which constitutes a result of independent interest.

Theorem 1.1 combined with Poincaré's kinematic formula Howard (1993, p. 17) allows to compute the *average* number of symmetric matrices with repeated eigenvalues in a uniformly distributed projective two-plane $L \subset \mathrm{PSym}(n, \mathbb{R}) \simeq \mathbb{R}\mathrm{P}^{N-1}$:

$$\mathbb{E}\,\#(L \cap \mathrm{P}\Delta) = \frac{|\mathrm{P}\Delta|}{|\mathbb{R}\mathrm{P}^{N-3}|} = \frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}, \tag{1.1}$$

where by $\mathrm{P}\Delta \subset \mathrm{PSym}(n, \mathbb{R}) \simeq \mathbb{R}\mathrm{P}^{N-1}$ we denote the projectivization of the discriminant. The following optimal bound on the number $\#(L \cap \mathrm{P}\Delta)$ of symmetric matrices with repeated eigenvalues in a generic projective two-plane $L \simeq \mathbb{R}\mathrm{P}^2 \subset \mathbb{R}\mathrm{P}^{N-1}$ was found in Sanyal et al. (2013, Corollary 15):

$$\#(L \cap \mathrm{P}\Delta) \leq \binom{n+1}{3}. \tag{1.2}$$

Remark 2 Consequence (1.1) combined with (1.2) "violates" a frequent phenomenon in random algebraic geometry, which goes under the name of *square root law*: for a large class of models of random systems, often related to the so called Edelman–Kostlan–Shub–Smale models (Edelman and Kostlan 1995; Shub and Smale 1993b, a, c; Edelman et al. 1994; Kostlan 2002), the average number of solutions equals (or is comparable to) the square root of the maximum number; here this is not the case. We also observe that, surprisingly enough, the *average cut* of the discriminant is an integer number (there is no reason to even expect that it should be a rational number!).

More generally one can ask about the expected number of matrices with a multiple eigenvalue in a "random" compact 2-dimensional family. We prove the following.

Theorem 1.2 (Multiplicities in a random family). *Let $F : \Omega \to \mathrm{Sym}(n, \mathbb{R})$ be a random Gaussian field $F = (f_1, \ldots, f_N)$ with i.i.d. components and denote by $\pi : \mathrm{Sym}(n, \mathbb{R})\backslash\{0\} \to S^{N-1}$ the projection map. Assume that:*

1. *with probability one the map $\pi \circ F$ is an embedding and*
2. *the expected number of solutions of the random system $\{f_1 = f_2 = 0\}$ is finite.*

Then:

$$\mathbb{E}\#F^{-1}(\mathscr{C}(\Delta)) = \binom{n}{2}\mathbb{E}\,\#\{f_1 = f_2 = 0\},$$

where $\mathscr{C}(\Delta) \subset Sym(n, \mathbb{R})$ is the cone over Δ.

Example 1 When each f_i is a Kostlan polynomial of degree d, then the hypotheses of Theorem 1.2 are verified and $\mathbb{E}\#\{f_1 = f_2 = 0\} = 2d|\Omega|/|S^2|$; when each f_i is a degree-one Kostlan polynomial and $\Omega = S^2$, then $\mathbb{E}\#\{f_1 = f_2 = 0\} = 2$ and we recover (1.1).

1.2 An Eckart–Young–Mirsky-Type Theorem

The classical Eckart–Young–Mirsky theorem allows to find a best low rank approximation to a given matrix.

For $r \leq m \leq n$ let us denote by Σ_r the set of $m \times n$ complex matrices of rank r. Then for a given $m \times n$ real or complex matrix A a rank r matrix $\tilde{A} \in \Sigma_r$ which is a global minimizer of the distance function

$$\operatorname{dist}_A : \Sigma_r \to \mathbb{R}, \quad B \mapsto \|A - B\| := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij} - b_{ij}|^2}$$

is called *a best rank r approximation to A*. The Eckart–Young–Mirsky theorem states that if $A = U^* S V$ is the singular value decomposition of A, i.e., U is an $m \times m$ real or complex unitary matrix, S is an $m \times n$ rectangular diagonal matrix with non-negative diagonal entries $s_1 \geq \cdots \geq s_m \geq 0$ and V is an $n \times n$ real or complex unitary matrix, then $\tilde{A} = U^* \tilde{S} V$ is a best rank r approximation to A, where $\tilde{S}$ denotes the rectangular diagonal matrix with $\tilde{S}_{ii} = s_i$ for $i = 1, \ldots, r$ and $\tilde{S}_{jj} = 0$ for $j = r+1, \ldots, m$. Moreover, a best rank r approximation to a sufficiently generic matrix is actually unique. More generally, one can show that any critical point of the distance function $\operatorname{dist}_A : \Sigma_r \to \mathbb{R}$ is of the form $U^* \tilde{S}^I V$, where $I \subset \{1, 2, \ldots, m\}$ is a subset of size r and $\tilde{S}^I$ is the rectangular diagonal matrix with $\tilde{S}^I_{ii} = s_i$ for $i \in I$ and $\tilde{S}^I_{jj} = 0$ for $j \notin I$. In particular, the number of critical points of dist_A for a generic matrix A is $\binom{n}{r}$. In Draisma et al. (2016) the authors call this count the *Euclidean distance degree* of Σ_r; see also Sect. 1.3 below.

In the case of real symmetric matrices similar results are obtained by replacing singular values $\sigma_1 \geq \cdots \geq \sigma_n$ with absolute values of eigenvalues $|\lambda_1| > \cdots > |\lambda_n|$ and singular value decomposition $U \Sigma V^*$ with spectral decomposition $C^T \Lambda C$; see Helmke and Shayman (1995, Thm. 2.2) and Lerario and Lundberg (2016, Sec. 2).

For the distance function from a symmetric matrix to the cone over Δ we also have an Eckart–Young–Mirsky-type theorem. We prove this theorem in Sect. 2.

Theorem 1.3 (Eckart–Young–Mirsky-type theorem). *Let $A \in Sym(n, \mathbb{R})$ be a generic real symmetric matrix and let $A = C^T \Lambda C$ be its spectral decomposition with $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. Any critical point of the distance function*

$$d_A : \mathscr{C}(\Delta_{sm}) \backslash \{0\} \to \mathbb{R}$$

is of the form $C^T \Lambda_{i,j} C$, *where*

$$\Lambda_{i,j} = diag\left(\lambda_1, \ldots, \underset{i}{\frac{\lambda_i + \lambda_j}{2}}, \ldots, \underset{j}{\frac{\lambda_i + \lambda_j}{2}}, \ldots, \lambda_n\right), \quad 1 \le i < j \le n.$$

Moreover, the function $d_A : \mathscr{C}(\Delta) \to \mathbb{R}$ *attains its global minimum at exactly one of the critical points* $C^T \Lambda_{i,j} C \in \mathscr{C}(\Delta_{sm}) \backslash \{0\}$ *and the value of the minimum of* d_A *equals:*

$$\min_{B \in \mathscr{C}(\Delta)} \|A - B\| = \min_{1 \le i < j \le n} \frac{|\lambda_i - \lambda_j|}{\sqrt{2}}.$$

Remark 3 Since $\mathscr{C}(\Delta) \subset \mathrm{Sym}(n, \mathbb{R})$ is the homogeneous cone over $\Delta \subset S^{N-1}$ the above theorem readily implies an analogous result for the spherical distance function from $A \in S^{N-1}$ to Δ. The critical points are $(1 - \frac{(\lambda_i - \lambda_j)^2}{2})^{-1/2} C^T \Lambda_{i,j} C$ and the global minimum of the spherical distance function d^S is $\min_{B \in \Delta} \mathrm{d}^S(A, B) = \min_{1 \le i < j \le n} \arcsin\left(\frac{|\lambda_i - \lambda_j|}{\sqrt{2}}\right)$.

The theorem is a special case of Theorem 1.4 below, that concerns the critical points of the distance function to a fixed *stratum* of $\mathscr{C}(\Delta)$. These strata are in bijection with vectors of natural numbers $w = (w_1, w_2, \ldots, w_n) \in \mathbb{N}^n$ such that $\sum_{i=1}^n i\, w_i = n$ as follows: let us denote by $\mathscr{C}(\Delta)^w$ the smooth semialgebraic submanifold of $\mathrm{Sym}(n, \mathbb{R})$ consisting of symmetric matrices that for each $i \ge 1$ have exactly w_i eigenvalues of multiplicity i. Then, by Shapiro and Vainshtein (1995, Lemma 1), the semialgebraic sets $\mathscr{C}(\Delta)^w$ with $w_1 < n$ form a stratification of $\mathscr{C}(\Delta)$:

$$\mathscr{C}(\Delta) = \bigsqcup_{w\,:\,w_1 < n} \mathscr{C}(\Delta)^w \tag{1.3}$$

In this notation, the complement of $\mathscr{C}(\Delta)$ can be written $\mathscr{C}(\Delta)^{(n,0,\ldots,0)} = \mathrm{Sym}(n, \mathbb{R}) \backslash \mathscr{C}(\Delta)$. By Arnold (1972, Lemma 1.1), the codimension of $\mathscr{C}(\Delta)^w$ in the space $\mathrm{Sym}(n, \mathbb{R})$ equals

$$\mathrm{codim}(\mathscr{C}(\Delta)^w) = \sum_{i=1}^n \frac{(i-1)(i+2)}{2} w_i$$

Let us denote by $\mathrm{Diag}(n, \mathbb{R})^w := \mathrm{Diag}(n, \mathbb{R}) \cap \mathscr{C}(\Delta)^w$ the set of diagonal matrices in $\mathscr{C}(\Delta)^w$ and its Euclidean closure by $\overline{\mathrm{Diag}(n, \mathbb{R})^w}$. This closure is an arrangement of $\frac{n!}{1!^{w_1} 2!^{w_2} 3!^{w_3} \ldots}$ many $(\sum_{i=1}^n w_i)$-dimensional planes. Furthermore, for a sufficiently generic diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ the distance function

$$\mathrm{d}_\Lambda : \mathrm{Diag}(n, \mathbb{R})^w \to \mathbb{R}, \quad \tilde{\Lambda} = \mathrm{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \mapsto \sqrt{\sum_{i=1}^n (\lambda_i - \tilde{\lambda}_i)^2}$$

has $\frac{n!}{1!^{w_1} 2!^{w_2} 3!^{w_3} \ldots}$ critical points each of which is the orthogonal projection of Λ on one of the planes in the arrangement $\overline{\mathrm{Diag}(n, \mathbb{R})^w}$ and the distance d_Λ attains its unique

global minimum at one of these critical points. We will show that an analogous result holds for

$$\mathrm{d}_A : \mathscr{C}(\Delta)^w \to \mathbb{R}, \quad \tilde{A} \mapsto \|A - \tilde{A}\| = \sqrt{\mathrm{tr}((A-\tilde{A})^2)},$$

the distance function from a general symmetric matrix $A \in \mathrm{Sym}(n, \mathbb{R})$ to the smooth semialgebraic set $\mathscr{C}(\Delta)^w$. The proof for the following theorem is given in Sect. 2.

Theorem 1.4 (Eckart–Young–Mirsky-type theorem for the strata). *Let $A \in Sym(n, \mathbb{R})$ be a generic real symmetric matrix and let $A = C^T \Lambda C$ be its spectral decomposition. Then:*

1. *Any critical point of the distance function $d_A : \mathscr{C}(\Delta)^w \to \mathbb{R}$ is of the form $C^T \tilde{\Lambda} C$, where $\tilde{\Lambda} \in Diag(n, \mathbb{R})^w$ is the orthogonal projection of Λ onto one of the planes in $\overline{Diag(n, \mathbb{R})^w}$.*
2. *The distance function $d_A : \mathscr{C}(\Delta)^w \to \mathbb{R}$ has exactly $\frac{n!}{1!^{w_1} 2!^{w_2} 3!^{w_3} \ldots}$ critical points, one of which is the unique global minimum of d_A.*

Remark 4 Note that the manifold $\mathscr{C}(\Delta)^w$ is not compact and therefore the function $\mathrm{d}_A : \mathscr{C}(\Delta)^w \to \mathbb{R}$ might not a priori have a minimum.

1.3 Euclidean Distance Degree

Let $X \subset \mathbb{R}^m$ be a real algebraic variety and let $X^{\mathbb{C}} \subset \mathbb{C}^m$ denote its Zariski closure. The number $\#\{x \in X_{\mathrm{sm}} : u - x \perp T_x X_{\mathrm{sm}}\}$ of critical points of the distance to the smooth locus X_{sm} of X from a generic point $u \in \mathbb{R}^m$ can be estimated by the number $\mathrm{EDdeg}(X) := \#\{x \in X^{\mathbb{C}}_{\mathrm{sm}} : u - x \perp T_x X^{\mathbb{C}}_{\mathrm{sm}}\}$ of "complex critical points". Here, $v \perp w$ is orthogonality with respect to the bilinear form $(v, w) \mapsto v^T w$. The quantity $\mathrm{EDdeg}(X)$ does not depend on the choice of the generic point $u \in \mathbb{R}^m$ and it's called *the Euclidean distance degree* of X (Draisma et al. 2016). Also, solutions $x \in X^{\mathbb{C}}_{\mathrm{sm}}$ to $u - x \perp T_x X^{\mathbb{C}}_{\mathrm{sm}}$ are called *ED critical points of u with respect to X* (Drusvyatskiy et al. 2017). In the following theorem we compute the Euclidean distance degree of the variety $\mathscr{C}(\Delta) \subset \mathrm{Sym}(n, \mathbb{R})$ and show that all ED critical points are actually real (this result is an analogue of Drusvyatskiy et al. (2017, Cor. 5.1) for the space of symmetric matrices and the variety $\mathscr{C}(\Delta)$).

Theorem 1.5 *Let $A \in Sym(n, \mathbb{R})$ be a sufficiently generic symmetric matrix. Then the $\binom{n}{2}$ real critical points of $d_A : \mathscr{C}(\Delta_{sm}) \to \mathbb{R}$ from Theorem 1.3 are the only ED critical points of A with respect to $\mathscr{C}(\Delta)$ and the Euclidean distance degree of $\mathscr{C}(\Delta)$ equals $EDdeg(\mathscr{C}(\Delta)) = \binom{n}{2}$.*

Remark 5 An analogous result holds for the closure of any other stratum of $\mathscr{C}(\Delta)^w$. Namely, $\mathrm{EDdeg}(\overline{\mathscr{C}(\Delta)^w}) = \frac{n!}{1!^{w_1} 2!^{w_2} 3!^{w_3} \ldots}$ and for a generic real symmetric matrix $A \in \mathrm{Sym}(n, \mathbb{R})$ ED critical points are real and given in Theorem 1.4.

1.4 Random Matrix Theory

The proof of Theorem 1.1 eventually arrives at Eq. (3.7), which reduces our study to the evaluation of a special integral over the *Gaussian Orthogonal Ensemble* (GOE) (Mehta

2004; Tao 2012). The connection between the volume of Δ and random symmetric matrices comes from the fact that, in a sense, the geometry in the Euclidean space of symmetric matrices with the Frobenius norm and the random GOE matrix model can be seen as the same object under two different points of view.

The integral in (3.7) is the second moment of the characteristic polynomial of a GOE matrix. In Mehta (2004) Mehta gives a general formula for all moments of the characteristic polynomial of a GOE matrix. However, we were unable to locate an exact evaluation of the formula for the second moment in the literature. For this reason we added Proposition 4.2, in which we compute the second moment, to this article. We use it in Sect. 4 to prove the following theorem.

Theorem 1.6 *For a fixed positive integer k we have*

$$\int_{u\in\mathbb{R}} \mathop{\mathbb{E}}_{Q\sim \mathrm{GOE}(k)} [\det(Q-u\mathbb{1})^2]\, e^{-u^2}\, \mathrm{d}u = \sqrt{\pi}\, \frac{(k+2)!}{2^{k+1}}.$$

An interesting remark in this direction is that some geometric properties of Δ can be stated using the language of random matrix theory. For instance, the estimate on the volume of a tube around Δ allows to estimate the probability that two eigenvalues of a GOE(n) matrix are close: for $\epsilon > 0$ small enough

$$\mathbb{P}\{\min_{i\neq j} |\lambda_i(Q)-\lambda_j(Q)| \leq \epsilon\} \leq \frac{1}{4}\binom{n}{2}\epsilon^2 + O(\epsilon^3). \tag{1.4}$$

The interest of this estimate is that it provides a *non-asymptotic* (as opposed to studies in the limit $n\to\infty$, Ben Arous and Bourgade 2013; Nguyen et al. 2017) result in random matrix theory. It would be interesting to provide an estimate of the implied constant in (1.4), however this might be difficult using our approach as it probably involves estimating higher curvature integrals of Δ.

2 Critical Points of the Distance to the Discriminant

In this section we prove Theorems 1.3, 1.4 and 1.5. Since Theorem 1.3 is a special case of Theorem 1.4, we start by proving the latter.

2.1 Proof of Theorem 1.4

Let us denote by $\overline{\mathscr{C}(\Delta)^w} \subset \mathrm{Sym}(n,\mathbb{R})$ the Euclidean closure of $\mathscr{C}(\Delta)^w$. Note that $\overline{\mathscr{C}(\Delta)^w}$ is a (real) algebraic variety, the smooth locus of $\overline{\mathscr{C}(\Delta)^w}$ is $\mathscr{C}(\Delta)^w$ and the boundary $\overline{\mathscr{C}(\Delta)^w}\backslash\mathscr{C}(\Delta)^w$ is a union of some strata $\mathscr{C}(\Delta)^{w'}$ of greater codimension.

The following result is an adaptation of Bik and Draisma (2017, Thm. 3) to the space $\mathrm{Sym}(n,\mathbb{R})$ of real symmetric matrices, its subspace $\mathrm{Diag}(n,\mathbb{R})$ of diagonal matrices and the action $C\in O(n)$, $A\in\mathrm{Sym}(n,\mathbb{R}) \mapsto C^T AC \in \mathrm{Sym}(n,\mathbb{R})$. Note that the assumptions of Bik and Draisma (2017, Thm. 3) are satisfied in this case as shown in Bik and Draisma (2017, Sec. 3).

Lemma 2.1 *Let $X \subset Sym(n, \mathbb{R})$ be a $O(n)$-invariant subvariety. Then for a sufficiently generic $\Lambda \in Diag(n, \mathbb{R})$ the set of critical points of the distance $d_\Lambda : X_{sm} \to \mathbb{R}$ is contained in $X \cap Diag(n, \mathbb{R})$.*

Let now $X = \overline{\mathscr{C}(\Delta)^w}$, let $A \in \mathrm{Sym}(n, \mathbb{R})$ be a sufficiently generic symmetric matrix and fix its spectral decomposition $A = C^T \Lambda C$. By Lemma 2.1 critical points of $\mathrm{d}_\Lambda : \mathscr{C}(\Delta)^w \to \mathbb{R}$ are all diagonal. Since the intersection $X \cap \mathrm{Diag}(n, \mathbb{R}) = \overline{\mathrm{Diag}(n, \mathbb{R})^w}$ is an arrangement of $\frac{n!}{1!^{w_1} 2!^{w_2} 3!^{w_3} \ldots}$ planes, critical points of d_Λ are the orthogonal projections of Λ on each of the components of the plane arrangement. Moreover, one of these points is the (unique) closest point on $\overline{\mathrm{Diag}(n, \mathbb{R})^w}$ to Λ. The critical points of the distance $\mathrm{d}_A : \mathscr{C}(\Delta)^w \to \mathbb{R}$ from $A = C^T \Lambda C$ are obtained via conjugation of critical points of d_Λ by $C \in O(n)$. Both claims follow. □

2.2 Proof of Theorem 1.3

Let $w = (n-2, 1, 0, \ldots, 0)$ and for a given symmetric matrix $A \in \mathrm{Sym}(n, \mathbb{R})$ let us fix a spectral decomposition $A = C^T \Lambda C$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. From Theorem 1.4 we know that the critical points of the distance function $\mathrm{d}_A : \mathscr{C}(\Delta_{\mathrm{sm}}) \backslash \{0\} \to \mathbb{R}$ are of the form $C^T \Lambda_{i,j} C$, $1 \le i < j \le n$, where $\Lambda_{i,j}$ is the orthogonal projection of Λ onto the hyperplane $\{\lambda_i = \lambda_j\} \subset \overline{\mathrm{Diag}(n, \mathbb{R})^w}$. It is straightforward to check that

$$\Lambda_{i,j} = \mathrm{diag}\left(\lambda_1, \ldots, \underset{i}{\frac{\lambda_i + \lambda_j}{2}}, \ldots, \underset{j}{\frac{\lambda_i + \lambda_j}{2}}, \ldots, \lambda_n\right).$$

From this, it is immediate that the distance between Λ and $\Lambda_{i,j}$ equals

$$\|\Lambda - \Lambda_{i,j}\| = \sqrt{\mathrm{tr}\left(\left(\Lambda - \Lambda_{i,j}\right)^2\right)} = \frac{|\lambda_i - \lambda_j|}{\sqrt{2}}$$

This finishes the proof. □

2.3 Proof of Theorem 1.5

In the proof of Theorem 1.3 we showed that there are $\binom{n}{2}$ real ED critical points of the distance function from a general real symmetric matrix A to $\mathscr{C}(\Delta)$. In this subsection we in particular argue that there are no other (complex) ED critical points in this case. The argument is based on Main Theorem from Bik and Draisma (2017) which is stated first.

Theorem 2.2 (Main Theorem from Bik and Draisma (2017)). *Let V be a finite-dimensional complex vector space equipped with a non-degenerate symmetric bilinear form, let $G^{\mathbb{C}}$ be a complex algebraic group and let $G^{\mathbb{C}} \to O(V)$ be an orthogonal representation. Suppose that $V_0 \subset V$ is a linear subspace such that, for sufficiently generic $v_0 \in V_0$, the space V is the orthogonal direct sum of V_0 and the tangent space $T_{v_0} G^{\mathbb{C}} v_0$ at v_0 to its $G^{\mathbb{C}}$-orbit. Let $X^{\mathbb{C}}$ be a $G^{\mathbb{C}}$-invariant closed subvariety of V. Set*

$X_0^{\mathbb{C}} = X^{\mathbb{C}} \cap V_0$ and suppose that $G^{\mathbb{C}} X_0^{\mathbb{C}}$ is dense in $X^{\mathbb{C}}$. Then the ED degree of $X^{\mathbb{C}}$ in V equals the ED degree of $X_0^{\mathbb{C}}$ in V_0.

We will apply this theorem to the space of complex symmetric matrices $V = \mathrm{Sym}(n, \mathbb{C})$ endowed with the complexified Frobenius inner product, the subspace of complex diagonal matrices $V_0 = \mathrm{Diag}(n, \mathbb{C})$, the complex orthogonal group $G^{\mathbb{C}} = \{C \in M(n, \mathbb{C}) : C^T C = \mathbb{1}\}$ acting on V via conjugation and the Zariski closure $X^{\mathbb{C}} \subset \mathrm{Sym}(n, \mathbb{C})$ of $\mathscr{C}(\Delta) \subset \mathrm{Sym}(n, \mathbb{R})$.

Let us denote by $G = O(n)$ the orthogonal group. Since $\mathscr{C}(\Delta) \subset \mathrm{Sym}(n, \mathbb{R})$ is G-invariant, by Drusvyatskiy et al. (2017, Lemma 2.1), the complex variety $X^{\mathbb{C}} \subset \mathrm{Sym}(n, \mathbb{C})$ is also G-invariant. Using the same argument as in Drusvyatskiy et al. (2017, Thm. 2.2) we now show that $X^{\mathbb{C}}$ is actually $G^{\mathbb{C}}$-invariant. Indeed, for a fixed point $A \in X^{\mathbb{C}}$ the map

$$\gamma_A : G^{\mathbb{C}} \to \mathrm{Sym}(n, \mathbb{C}), \quad C \mapsto C^T A C$$

is continuous and hence the set $\gamma_A^{-1}(X^{\mathbb{C}}) \subset G^{\mathbb{C}}$ is closed. Since by the above $G \subset \gamma_A^{-1}(X^{\mathbb{C}})$ and since $G^{\mathbb{C}}$ is the Zariski closure of G we must have $\gamma_A^{-1}(X^{\mathbb{C}}) = G^{\mathbb{C}}$.

Now for any diagonal matrix $\Lambda \in \mathrm{Diag}(n, \mathbb{C})$ with pairwise distinct diagonal entries the tangent space at Λ to the orbit $G^{\mathbb{C}}\Lambda = \{C^T \Lambda C : C \in G^{\mathbb{C}}\}$ consists of complex symmetric matrices with zeros on the diagonal:

$$T_\Lambda(G^{\mathbb{C}}\Lambda) = \left\{ v\Lambda - \Lambda v : v^T + v = 0 \right\} = \{A \in \mathrm{Sym}(n, \mathbb{C}) : a_{11} = \cdots = a_{nn} = 0\}$$

In particular,

$$\mathrm{Sym}(n, \mathbb{C}) = \mathrm{Diag}(n, \mathbb{C}) + T_\Lambda(G^{\mathbb{C}}\Lambda)$$

is the direct sum which is orthogonal with respect to the complexified Frobenius inner product $(A_1, A_2) \mapsto \mathrm{tr}(A_1^T A_2)$.

As any real symmetric matrix can be diagonalized by some orthogonal matrix we have

$$\mathscr{C}(\Delta) = G(\mathscr{C}(\Delta) \cap \mathrm{Diag}(n, \mathbb{R})) = \left\{ C^T \Lambda C : C \in O(n), \Lambda \in \mathscr{C}(\Delta) \cap \mathrm{Diag}(n, \mathbb{R}) \right\}.$$

This, together with the inclusion $\mathscr{C}(\Delta) \subset G^{\mathbb{C}}(X^{\mathbb{C}} \cap \mathrm{Diag}(n, \mathbb{C}))$, imply that the set $G^{\mathbb{C}}(X^{\mathbb{C}} \cap \mathrm{Diag}(n, \mathbb{C}))$ is Zariski dense in $X^{\mathbb{C}}$. Applying Theorem (2.2) we obtain that the ED degree of $X^{\mathbb{C}}$ in $\mathrm{Sym}(n, \mathbb{C})$ equals the ED degree of $X^{\mathbb{C}} \cap \mathrm{Diag}(n, \mathbb{C})$ in $\mathrm{Diag}(n, \mathbb{C})$. Since $X^{\mathbb{C}} \cap \mathrm{Diag}(n, \mathbb{C}) = \{\Lambda \in \mathrm{Diag}(n, \mathbb{C}) : \lambda_i = \lambda_j, i \neq j\}$ is the union of $\binom{n}{2}$ hyperplanes the ED critical points of a generic $\Lambda \in \mathrm{Diag}(n, \mathbb{C})$ are orthogonal projections from Λ to each of the hyperplanes (as in the proof of Theorem 1.3). In particular, $\mathrm{EDdeg}(X^{\mathbb{C}}) = \mathrm{EDdeg}(X^{\mathbb{C}} \cap \mathrm{Diag}(n, \mathbb{C})) = \binom{n}{2}$ and if $\Lambda \in \mathrm{Diag}(n, \mathbb{R})$ is a generic real diagonal matrix ED critical points are all real. Finally, for a general symmetric matrix $A = C^T \Lambda C$ all ED critical points are obtained from the ones for $\Lambda \in \mathrm{Diag}(n, \mathbb{R})$ via conjugation by $C \in O(n)$.

The proof of the statement in Remark 5 is similar. Each plane in the plane arrangement $\overline{\mathrm{Diag}(n,\mathbb{R})^w}$ yields one critical point and there are $\frac{n!}{1!^{w_1}2!^{w_2}3!^{w_3}\cdots}$ many such planes. □

3 The Volume of the Discriminant

The goal of this section is to prove Theorems 1.1 and 1.2. As was mentioned in the introduction, we reduce the computation of the volume to an integral over the GOE-ensemble. This is why, before starting the proof, in the next subsection we recall some preliminary concepts and facts from random matrix theory that will be used in the sequel.

3.1 The GOE(n) Model for Random Matrices

The material we present here is from Mehta (2004).

The GOE(n) *probability measure* of any Lebesgue measurable subset $U \subset \mathrm{Sym}(n,\mathbb{R})$ is defined as follows:

$$\mathbb{P}\{U\} = \frac{1}{\sqrt{2}^n \sqrt{\pi}^N} \int_U e^{-\frac{\|A\|^2}{2}}\, dA,$$

where $dA = \prod_{1\le i\le j\le n} dA_{ij}$ is the Lebesgue measure on the space of symmetric matrices $\mathrm{Sym}(n,\mathbb{R})$ and, as before, $\|A\| = \sqrt{\mathrm{tr}(A^2)}$ is the Frobenius norm.

By Mehta (2004, Sec. 3.1), the joint density of the eigenvalues of a GOE(n) matrix A is given by the measure $\frac{1}{Z_n}\int_V e^{-\frac{\|\lambda\|^2}{2}}|\Delta(\lambda)|\,d\lambda$, where $d\lambda = \prod_{i=1}^n d\lambda_i$ is the Lebesgue measure on $\mathbb{R}^n$, $V \subset \mathbb{R}^n$ is a measurable subset, $\|\lambda\|^2 = \lambda_1^2 + \cdots + \lambda_n^2$ is the Euclidean norm, $\Delta(\lambda) := \prod_{1\le i<j\le n}(\lambda_j - \lambda_i)$ is the Vandermonde determinant and Z_n is the normalization constant whose value is given by the formula

$$Z_n = \int_{\mathbb{R}^n} e^{-\frac{\|\lambda\|^2}{2}}|\Delta(\lambda)|\,\mathrm{d}\lambda = \sqrt{2\pi}^n \prod_{i=1}^n \frac{\Gamma(1+\frac{i}{2})}{\Gamma(\frac{3}{2})}, \tag{3.1}$$

see Mehta (2004, Eq. (17.6.7)) with $\gamma = a = \frac{1}{2}$. In particular, for an integrable function $f : \mathrm{Sym}(n,\mathbb{R}) \to \mathbb{R}$ that depends only on the eigenvalues of $A \in \mathrm{Sym}(n,\mathbb{R})$, the following identity holds

$$\mathop{\mathbb{E}}_{A\sim \mathrm{GOE}(n)} f(A) = \frac{1}{Z_n}\int_V f(\lambda_1,\ldots,\lambda_n)\, e^{-\frac{\|\lambda\|^2}{2}}|\Delta(\lambda)|\,d\lambda. \tag{3.2}$$

3.2 Proof of Theorem 1.1

In what follows we endow the orthogonal group $O(n)$ with the left-invariant metric defined on the Lie algebra $T_{\mathbb{1}}O(n)$ by

$$\langle u, v\rangle = \frac{1}{2}\mathrm{tr}(u^T v),\ u, v \in T_{\mathbb{1}}O(n)$$

The following formula for the volume of $O(n)$ can be found in Muirhead (1982, Corollary 2.1.16):

$$|O(n)| = \frac{2^n \pi^{\frac{n(n+1)}{4}}}{\prod_{i=1}^{n} \Gamma\left(\frac{i}{2}\right)} \tag{3.3}$$

Recall that by definition the volume of Δ equals the volume of the smooth part $\Delta_{\mathrm{sm}} \subset S^{N-1}$ that consists of symmetric matrices of unit norm with exactly two repeated eigenvalues. Let us denote by $(S^{n-2})_*$ the dense open subset of the $(n-2)$-sphere consisting of points with pairwise distinct coordinates. We consider the following parametrization of $\Delta_{\mathrm{sm}} \subset S^{N-1}$:

$$p : O(n) \times (S^{n-2})_* \to \Delta_{\mathrm{sm}}, \quad (C, \mu) \mapsto C^T \,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\, C,$$

where $\lambda_1, \ldots, \lambda_n$ are defined as

$$\lambda_1 = \mu_1,\ \lambda_2 = \mu_2,\ \ldots,\ \lambda_{n-2} = \mu_{n-2},\ \lambda_{n-1} = \frac{\mu_{n-1}}{\sqrt{2}},\ \lambda_n = \frac{\mu_{n-1}}{\sqrt{2}}. \tag{3.4}$$

In Lemma 3.1 below we show that p is a submersion. Applying to it the smooth coarea formula (see, e.g., Bürgisser and Cucker 2013, Theorem 17.8) we have

$$\int_{A \in \Delta_{\mathrm{sm}}} |p^{-1}(A)|\, \mathrm{d}A = \int_{(C,\mu) \in O(n) \times (S^{n-2})_*} \mathrm{NJ}_{(C,\mu)} p \, \mathrm{d}(C, \mu) \tag{3.5}$$

Here $\mathrm{NJ}_{(C,\mu)} p$ denotes the *normal Jacobian* of p at (C, μ) and we compute its value in the following lemma.

Lemma 3.1 *The parametrization $p : O(n) \times (S^{n-2})_* \to \Delta_{sm}$ is a submersion and its normal Jacobian at $(C, \mu) \in O(n) \times (S^{n-2})_*$ is given by the formula*

$$\mathrm{NJ}_{(C,\mu)} p = \sqrt{2}^{\frac{n(n-1)}{2}-1} \prod_{1 \le i < j \le n-2} |\mu_i - \mu_j| \prod_{i=1}^{n-2} \left|\mu_i - \frac{\mu_{n-1}}{\sqrt{2}}\right|^2.$$

Proof Recall that for a smooth submersion $f : M \to N$ between two Riemannian manifolds the normal Jacobian of f at $x \in M$ is the absolute value of the determinant of the restriction of the differential $D_x f : T_x M \to T_{f(x)} N$ of f at x

to the orthogonal complement of its kernel. We now show that the parametrization $p : O(n) \times (S^{n-2})_* \to \Delta_{\text{sm}}$ is a submersion and compute its normal Jacobian.

Note that p is equivariant with respect to the right action of $O(n)$ on itself and its action on Δ_{sm} via conjugation, i.e., for all $C, \tilde{C} \in O(n)$ and $\mu \in S^{n-2}$ we have $p(C\tilde{C}, \mu) = \tilde{C}^T p(C, \mu)\tilde{C}$. Therefore, $D_{(C,\mu)}p = C^T D_{(\mathbb{1},\mu)}p\, C$ and, consequently, $\text{NJ}_{(C,\mu)}p = \text{NJ}_{(\mathbb{1},\mu)}p$. We compute the latter. The differential of p at $(\mathbb{1}, \mu)$ is the map

$$\begin{aligned} D_{(\mathbb{1},\mu)}p : T_{\mathbb{1}}O(n) \times T_\mu S^{n-2} &\to T_{p(\mathbb{1},\mu)}\Delta_{\text{sm}}, \\ (\dot{C}, \dot{\mu}) &\mapsto \dot{C}^T \operatorname{diag}(\lambda_1, \ldots, \lambda_n) + \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\,\dot{C} + \operatorname{diag}(\dot{\lambda}_1, \ldots, \dot{\lambda}_n), \end{aligned}$$

where $\dot{\lambda}_i = \dot{\mu}_i$ for $1 \le i \le n-2$ and $\dot{\lambda}_{n-1} = \dot{\lambda}_n = \frac{\dot{\mu}_{n-1}}{\sqrt{2}}$. The Lie algebra $T_{\mathbb{1}}O(n)$ consists of skew-symmetric matrices:

$$T_{\mathbb{1}}O(n) = \left\{ \dot{C} \in \mathbb{R}^{n\times n} : \dot{C}^T = -\dot{C} \right\}.$$

Let $E_{i,j}$ be the matrix that has zeros everywhere except for the entry (i, j) where it equals 1. Then $\{E_{i,j} - E_{j,i} : 1 \le i < j \le n\}$ is an orthonormal basis for $T_{\mathbb{1}}O(n)$. One verifies that

$$\begin{aligned} D_{(\mathbb{1},\mu)}p(E_{i,j} - E_{j,i}, 0) &= (\lambda_j - \lambda_i)(E_{i,j} + E_{j,i}), \quad \text{and} \\ D_{(\mathbb{1},\mu)}p(0, \dot{\mu}) &= \operatorname{diag}(\dot{\lambda}_1, \ldots, \dot{\lambda}_n). \end{aligned}$$

This implies that p is a submersion and

$$(\ker D_{(\mathbb{1},\mu)}p)^\perp = \operatorname{span}\{E_{i,j} - E_{j,i} : 1 \le i < j \le n, (i, j) \ne (n-1, n)\} \oplus^\perp T_\mu(S^{n-2})_*.$$

Combining this with the fact that the restriction of $D_{(\mathbb{1},\mu)}p$ to $T_\mu(S^{n-2})_*$ is an isometry we obtain

$$\begin{aligned} \text{NJ}_{(\mathbb{1},\mu)}p &= \sqrt{2}^{\frac{n(n-1)}{2}-1} \prod_{1\le i<j\le n,\ (i,j)\ne(n,n-1)} |\lambda_i - \lambda_j| \\ &= \sqrt{2}^{\frac{n(n-1)}{2}-1} \prod_{1\le i<j\le n-2} |\mu_i - \mu_j| \prod_{i=1}^{n-2} \left|\mu_i - \frac{\mu_{n-1}}{\sqrt{2}}\right|^2, \end{aligned}$$

which finishes the proof. □

We now compute the volume of the fiber $p^{-1}(A)$, $A \in \Delta_{\text{sm}}$ that appears in (3.5).

Lemma 3.2 *The volume of the fiber over $A \in \Delta_{sm}$ equals $|p^{-1}(A)| = 2^n \pi\, (n-2)!$.*

Proof Let $A = p(C, \mu) \in \Delta_{\mathrm{sm}}$. The last coordinate μ_{n-1} is always mapped to the double eigenvalue $\lambda_{n-1} = \lambda_n$ of A, whereas there are $(n-2)!$ possibilities to arrange $\mu_1, \ldots, \mu_{n-2}$. For a fixed choice of μ there are $|O(1)|^{n-2}|O(2)|$ ways to choose $C \in O(n)$. Therefore, by (3.3) we obtain $|p^{-1}(A)| = |O(1)|^{n-2}|O(2)|\,(n-2)! = 2^{n-2} \cdot 2^2\pi \cdot (n-2)! = 2^n\pi(n-2)!$. □

Combining (3.5) with Lemmas 3.1 and 3.2 we write for the normalized volume of Δ:

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{\sqrt{2}^{\frac{n(n-1)}{2}-1-2n}}{\pi(n-2)!|S^{N-3}|} \times \int_{(C,\mu)\in O(n)\times(S^{n-2})_*} |\Delta(\mu_1,\ldots,\mu_{n-2})| \prod_{i=1}^{n-2}\left|\mu_i - \frac{\mu_{n-1}}{\sqrt{2}}\right|^2 \mathrm{d}(C,\mu),$$

where $\Delta(\mu_1, \ldots, \mu_{n-2}) = \prod_{1\le i<j\le n-2}(\mu_i - \mu_j)$.

The function being integrated is independent of $C \in O(n)$. Thus, using Fubini's theorem we can perform the integration over the orthogonal group. Furthermore, the integrand is a homogeneous function of degree $\frac{(n-2)(n+1)}{2}$. Passing from spherical coordinates to spatial coordinates and extending the domain of integration to the measure-zero set of points with repeated coordinates we obtain

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{\sqrt{2}^{\frac{n(n-1)}{2}-1-2n}|O(n)|}{\pi\,(n-2)!\,|S^{N-3}|\,K} \times \int_{\mu\in\mathbb{R}^{n-1}} |\Delta(\mu_1,\ldots,\mu_{n-2})| \prod_{i=1}^{n-2}\left|\mu_i - \frac{\mu_{n-1}}{\sqrt{2}}\right|^2 e^{-\frac{\|\mu\|^2}{2}}\mathrm{d}\mu$$

where

$$K = \int_0^\infty r^{\frac{(n-2)(n+1)}{2}+n-2} e^{-\frac{r^2}{2}}\,\mathrm{d}r = 2^{\frac{n(n+1)}{4}-2}\,\Gamma\left(\frac{n(n+1)}{4} - 1\right).$$

Let us write $u := \frac{\mu_{n-1}}{\sqrt{2}}$ for the double eigenvalue and make a change of variables from μ_{n-1} to u. Considering the eigenvalues $\mu_1, \ldots, \mu_{n-2}$ as the eigenvalues of a symmetric $(n-2)\times(n-2)$ matrix Q, by (3.2) we have

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{\sqrt{2}^{\frac{n(n-1)}{2}-2n}|O(n)|\,Z_{n-2}}{\pi\,(n-2)!\,|S^{N-3}|\,K} \int_{u\in\mathbb{R}} \mathop{\mathbb{E}}_{Q\sim\mathrm{GOE}(n-2)}[\det(Q-u\mathbb{1})^2]\,e^{-u^2}\mathrm{d}u. \tag{3.6}$$

Using formulas (3.1) and (3.3) for Z_{n-2} and $|O(n)|$ respectively we write

$$|O(n)|\cdot Z_{n-2} = \frac{2^n\pi^{\frac{n(n+1)}{4}}}{\prod_{i=1}^n \Gamma(\frac{i}{2})} \cdot \sqrt{2\pi}^{n-2} \prod_{i=1}^{n-2}\frac{\Gamma(1+\frac{i}{2})}{\Gamma(\frac{3}{2})}$$

$$
\begin{aligned}
&= \frac{2^n \pi^{\frac{n(n+1)}{4}}}{\prod_{i=1}^{n} \Gamma(\frac{i}{2})} \cdot \sqrt{2\pi}^{n-2} \frac{\prod_{i=1}^{n-2} \frac{i}{2} \Gamma(\frac{i}{2})}{(\frac{\sqrt{\pi}}{2})^{n-2}} \\
&= \frac{\sqrt{2}^{3n-2} \pi^{\frac{n(n+1)}{4}} (n-2)!}{\Gamma(\frac{n}{2}) \Gamma(\frac{n-1}{2})} \\
&= \sqrt{2}^{5n-6} \pi^{\frac{n(n+1)}{4}-1},
\end{aligned}
$$

where in the last step the duplication formula for Gamma function $\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2}) = 2^{2-n}\sqrt{\pi}\,(n-2)!$ has been used. Let us recall the formula for the volume of the $(N-3)$-dimensional unit sphere: $|S^{N-3}| = 2\pi^{\frac{N-2}{2}} / \Gamma(\frac{N-2}{2})$. Recalling that $N = \frac{n(n+1)}{2}$ we simplify the constant in (3.6)

$$
\frac{\sqrt{2}^{\frac{n(n-1)}{2}-2n}}{\pi\,(n-2)!} \cdot \frac{|O(n)| \cdot Z_{n-2}}{|S^{N-3}| \cdot K} = \frac{\sqrt{2}^{\frac{n(n-1)}{2}-2n}}{\pi\,(n-2)!} \cdot \sqrt{2}^{5n-6} \pi^{\frac{n(n+1)}{4}-1} \cdot \frac{\Gamma\left(\frac{N-2}{2}\right)}{2\pi^{\frac{N-2}{2}}} \cdot \frac{2^{2-\frac{n(n+1)}{4}}}{\Gamma\left(\frac{n(n+1)}{4}-1\right)} = \frac{2^{n-2}}{\sqrt{\pi}\,(n-2)!}.
$$

Plugging this into (3.6) we have

$$
\frac{|\Delta|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\,n!} \binom{n}{2} \int_{u \in \mathbb{R}} \mathop{\mathbb{E}}_{Q \sim \mathrm{GOE}(n-2)} [\det(Q - u\mathbb{1})^2]\, e^{-u^2} \mathrm{d}u. \tag{3.7}
$$

Combining the last formula with Theorem 1.6 whose proof is given in Sect. 4 we finally derive the claim of Theorem 1.1: $\frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}$.

Remark 6 The proof can be generalized to subsets of Δ that are defined by an eigenvalue configuration given by a measurable subset of $(S^{n-2})_*$. Such a configuration only adjusts the domain of integration in (3.7). For instance, consider the subset

$$
(S^{n-2})_1 = \{(\mu_1, \ldots, \mu_{n-1}) \in (S^{n-2})_* \mid \mu_{n-1} < \mu_i \text{ for } 1 \le i \le n-2\}.
$$

It is an open semialgebraic subset of $(S^{n-2})_*$ and $\Delta_1 := p(O(n) \times (S^{n-2})_1)$ is the smooth part of the matrices whose two smallest eigenvalues coincide. Following the proof until (3.7), we get

$$
\frac{|\Delta_1|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\,n!} \binom{n}{2} \int_{u \in \mathbb{R}} \mathop{\mathbb{E}}_{Q \sim \mathrm{GOE}(n-2)} [\det(Q - u\mathbb{1})^2\, \mathbf{1}_{\{Q \succ u\mathbb{1}\}}]\, e^{-u^2} \mathrm{d}u,
$$

where $\mathbf{1}_{\{Q \succ u\mathbb{1}\}}$ is the indicator function of $Q - u\mathbb{1}$ being positive definite.

3.3 Multiplicities in a Random Family

In this subsection we prove Theorem 1.2.

The proof is based on the integral geometry formula from Howard (1993, p. 17). We state it here for the case of submanifolds of the sphere S^{N-1}. If $A, B \subset S^{N-1}$ are smooth submanifolds of dimensions a and b respectively and $a + b \ge N - 1$, then

$$\frac{1}{|O(N)|}\int_{O(N)} |A \cap gB|\, dg = |S^{a+b-N+1}|\frac{|A|}{|S^a|}\frac{|B|}{|S^b|} \tag{3.8}$$

Note that the intersection $A \cap gB$ is transverse for a full measure set of $g \in O(N)$ and hence its volume is almost surely defined.

Denote $\widehat{F} = \pi \circ F : \Omega \to S^{N-1}$. Then, by assumption, with probability one we have:

$$\#F^{-1}(\mathscr{C}(\Delta)) = \#\widehat{F}^{-1}(\Delta) = \#\widehat{F}(\Omega) \cap \Delta.$$

Observe also that, since the list $(f_1, \ldots, f_N)$ consists of i.i.d. random Gaussian fields, then for every $g \in O(N)$ the random maps $\widehat{F}$ and $g \circ \widehat{F}$ have the same distribution and

$$\begin{aligned}
\mathbb{E}\,\#F^{-1}(\mathscr{C}(\Delta)) &= \mathbb{E}\,\#\widehat{F}(\Omega) \cap \Delta = \mathbb{E}\,\#g(\widehat{F}(\Omega)) \cap \Delta \\
&= \frac{1}{|O(N)|}\int_{O(N)} \mathbb{E}\,\#(g(\widehat{F}(\Omega)) \cap \Delta)\, dg \\
&= \mathbb{E}\,\frac{1}{|O(N)|}\int_{O(N)} \#(g(\widehat{F}(\Omega)) \cap \Delta)\, dg \\
&= \mathbb{E}\,\frac{1}{|O(N)|}\int_{O(N)} \#(g(\widehat{F}(\Omega)) \cap \Delta_{\mathrm{sm}})\, dg \\
&= \mathbb{E}\, 2\frac{|\widehat{F}(\Omega)|}{|S^2|}\frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}\mathbb{E}\, 2\frac{|\widehat{F}(\Omega)|}{|S^2|}.
\end{aligned}$$

In this derivation we have applied (3.8) to Δ_{sm}, $\widehat{F}(\Omega) \subset S^{N-1}$ (for almost any $g \in O(N)$ the embedded surface $g(\widehat{F}(\Omega))$ intersects Δ only along Δ_{sm}). Let us denote by $L = \{x_1 = x_2 = 0\}$ the codimension-two subspace of $\mathrm{Sym}(n, \mathbb{R})$ given by the vanishing of the first two coordinates (in fact: any two coordinates). The conclusion follows by applying (3.8) again:

$$\mathbb{E}\, 2\frac{|\widehat{F}(\Omega)|}{|S^2|} = \mathbb{E}\,\frac{1}{|O(N)|}\int_{O(N)} \#(g(\widehat{F}(\Omega)) \cap L)dg = \mathbb{E}\#F^{-1}(L) = \mathbb{E}\,\#\{f_1 = f_2 = 0\}.$$

This finishes the proof. □

4 The Second Moment of the Characteristic Polynomial of a GOE Matrix

In this section we give a proof of Theorem 1.6. Let us first recall some ingredients and prove some auxiliary results.

Lemma 4.1 *Let* $P_m = 2^{1-m^2}\sqrt{\pi}^m \prod_{i=0}^m (2i)!$ *and let* Z_{2m} *be the normalization constant from* (3.1)*. Then* $P_m = 2^{1-2m} Z_{2m}$.

Proof The formula (3.1) for Z_{2m} reads

$$Z_{2m} = \sqrt{2\pi}^{2m} \prod_{i=1}^{2m} \frac{\Gamma\left(\frac{i}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)} = (2\pi)^m \prod_{i=1}^{m} \frac{\Gamma\left(\frac{2i-1}{2}+1\right)\Gamma\left(\frac{2i}{2}+1\right)}{\left(\frac{\sqrt{\pi}}{2}\right)^2}.$$

Using the formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}2^{1-2z}\Gamma(2z)$ (Spanier et al. 2000, 43:5:7) with $z = i + 1/2$ we obtain

$$\begin{aligned} Z_{2m} &= 2^{3m} \prod_{i=1}^{m} \sqrt{\pi} 2^{1-2(i+1/2)} \Gamma(2(i+1/2)) \\ &= 2^{2m-m^2} \sqrt{\pi}^m \prod_{i=1}^{m} (2i)! = 2^{2m-1} P_m. \end{aligned}$$

This proves the claim. □

Recall now that the *(physicist's) Hermite polynomials* $H_i(x)$, $i = 0, 1, 2, \ldots$ form a family of orthogonal polynomials on the real line with respect to the measure $e^{-x^2}dx$. They are defined by

$$H_i(x) = (-1)^i e^{x^2} \frac{\mathrm{d}^i}{\mathrm{d}x^i} e^{-x^2}, \quad i \geq 0$$

and satisfy

$$\int_{u\in\mathbb{R}} H_i(u) H_j(u) e^{-u^2} \,\mathrm{d}u = \begin{cases} 2^i i! \sqrt{\pi}, & \text{if } i = j \\ 0, & \text{else.} \end{cases} \tag{4.1}$$

A Hermite polynomial is either odd (if the degree is odd) or even (if the degree is even) function:

$$H_i(-x) = (-1)^i H_i(x); \tag{4.2}$$

and its derivative satisfies

$$H_i'(x) = 2i\, H_{i-1}(x); \tag{4.3}$$

see Spanier et al. (2000, (24:5:1)) and Gradshteyn and Ryzhik (2015, (8.952.1)) for these properties.

The following proposition is crucial for the proof of Theorem 1.6.

Proposition 4.2 (Second moment of the characteristic polynomial). *For a fixed positive integer k and a fixed $u \in \mathbb{R}$ the following holds.*

1. *If $k = 2m$ is even, then*

$$\mathop{\mathbb{E}}_{Q\sim \mathrm{GOE}(k)} \det(Q - u\mathbb{1})^2 = \frac{(2m)!}{2^{2m}} \sum_{j=0}^{m} \frac{2^{-2j-1}}{(2j)!} \det X_j(u),$$

where

$$X_j(u) = \begin{pmatrix} H_{2j}(u) & H'_{2j}(u) \\ H_{2j+1}(u) - H'_{2j}(u) & H'_{2j+1}(u) - H''_{2j}(u) \end{pmatrix}.$$

2. *If $k = 2m+1$ is odd, then*

$$\mathop{\mathbb{E}}_{Q\sim \mathrm{GOE}(k)} \det(Q - u\mathbb{1})^2 = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \det Y_j(u),$$

where

$$Y_j(u) = \begin{pmatrix} \frac{(2j)!}{j!} & H_{2j}(u) & H'_{2j}(u) \\ 0 & H_{2j+1}(u) - H'_{2j}(u) & H'_{2j+1}(u) - H''_{2j}(u) \\ \frac{(2m+2)!}{(m+1)!} & H_{2m+2}(u) & H'_{2m+2}(u) \end{pmatrix}.$$

Proof In Section 22 of Mehta (2004) one finds two different formulas for the even $k = 2m$ and odd $k = 2m+1$ cases. We evaluate both separately.

If $k = 2m$, we have by Mehta (2004, (22.2.38)) that

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{(2m)!\,P_m}{Z_{2m}} \sum_{j=0}^{m} \frac{2^{2j-1}}{(2j)!} \det \begin{pmatrix} R_{2j}(u) & R'_{2j}(u) \\ R_{2j+1}(u) & R'_{2j+1}(u) \end{pmatrix},$$

where $P_m = 2^{1-m^2}\sqrt{\pi}^m \prod_{i=0}^{m}(2i)!$ is as in Lemma 4.1, Z_{2m} is the normalization constant (3.1) and where $R_{2j}(u) = 2^{-2j} H_{2j}(u)$ and $R_{2j+1}(u) = 2^{-(2j+1)}(H_{2j+1}(u) - H'_{2j}(u))$. Using the multilinearity of the determinant we get

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{(2m)!\,P_m}{Z_{2m}} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \det X_j(u).$$

By Lemma 4.1 we have $\frac{P_m}{Z_{2m}} = 2^{1-2m}$. Putting everything together yields the first claim.

In the case $k = 2m+1$ we get from Mehta (2004, (22.2.39)) that

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{(2m+1)!\,P_m}{Z_{2m+1}} \sum_{j=0}^{m} \frac{2^{2j-1}}{(2j)!} \det \begin{pmatrix} g_{2j} & R_{2j}(u) & R'_{2j}(u) \\ g_{2j+1} & R_{2j+1}(u) & R'_{2j+1}(u) \\ g_{2m+2} & R_{2m+2}(u) & R'_{2m+2}(u) \end{pmatrix},$$

where P_m, $R_{2j}(u)$, $R_{2j+1}(u)$ are as above and

$$g_i = \int_{u\in\mathbb{R}} R_i(u) \exp(-\tfrac{u^2}{2})\,\mathrm{d}u.$$

By (4.2), the Hermite polynomial $H_{2j+1}(u)$ is an odd function. Hence, we have $g_{2j+1} = 0$. For even indices we use Gradshteyn and Ryzhik (2015, (7.373.2)) to get $g_{2j} = 2^{-2j}\sqrt{2\pi}\frac{(2j)!}{j!}$. By the multilinearity of the determinant:

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{\sqrt{2\pi}(2m+1)!\,P_m}{2^{2m+2}Z_{2m+1}} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \det Y_j(u). \tag{4.4}$$

From (3.1) one obtains $Z_{2m+1} = 2\sqrt{2}\,\Gamma(m+\frac{3}{2})\,Z_{2m}$, which together with Lemma 4.1 implies

$$\frac{P_m}{Z_{2m+1}} = \frac{2^{-2m}}{\sqrt{2}\,\Gamma\left(m+\frac{3}{2}\right)}.$$

Plugging this into (4.4) we conclude that

$$\mathbb{E}\det(Q - u\mathbb{1})^2 = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma\left(m+\frac{3}{2}\right)} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \det Y_j(u). \tag{4.5}$$

□

Everything is now ready for the proof of Theorem 1.6

Proof of Theorem 1.6 Due to the nature of Proposition 4.2 we have to make a case distinction also for this proof.

In the case $k = 2m$ we use the formula from Proposition 4.2 (1) to write

$$\int_{u\in\mathbb{R}} \mathbb{E}\det(Q - u\mathbb{1})^2 e^{-u^2}\,\mathrm{d}u = \frac{(2m)!}{2^{2m}} \sum_{j=0}^{m} \frac{2^{-2j-1}}{(2j)!} \int_{u\in\mathbb{R}} \det X_j(u)\,\mathrm{d}u.$$

By (4.3) we have $H_i'(u) = 2i\,H_{i-1}(u)$. Hence, $X_j(u)$ can be written as

$$\begin{pmatrix} H_{2j}(u) & 4j\,H_{2j-1}(u) \\ H_{2j+1}(u) - 4j\,H_{2j-1}(u) & 2(2j+1)H_{2j}(u) - 8j(2j-1)H_{2j-2}(u) \end{pmatrix}.$$

From (4.1) we can deduce that

$$\begin{aligned} \int_{u\in\mathbb{R}} \det X_j(u)\,\mathrm{d}u &= 2(2j+1)2^{2j}(2j)!\sqrt{\pi} + 16j^2 2^{2j-1}(2j-1)!\sqrt{\pi} \\ &= 2^{2j+1}(2j)!\sqrt{\pi}(4j+1). \end{aligned}$$

From this we see that

$$\sum_{j=0}^{m} \frac{2^{-2j-1}}{(2j)!} \int_{u\in\mathbb{R}} \det X_j(u)\,\mathrm{d}u = \sqrt{\pi}\sum_{j=0}^{m}(4j+1) = \sqrt{\pi}\,(m+1)(2m+1). \tag{4.6}$$

and hence,

$$\int_{u\in\mathbb{R}} \mathbb{E}\det(Q-u\mathbb{1})^2 e^{-u^2}\,\mathrm{d}u = \frac{(2m)!}{2^{2m}}\sqrt{\pi}\,(m+1)(2m+1) = \frac{(2m+2)!}{2^{2m+1}}\sqrt{\pi}.$$

Plugging back in $m=\frac{k}{2}$ finishes the proof of the case $k=2m$.

In the case $k=2m+1$ we use the formula from Proposition 4.2 (2) to see that

$$\int_u \mathbb{E}\det(Q-u\mathbb{1})^2\, e^{-u^2}\,\mathrm{d}u = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma\left(m+\frac{3}{2}\right)}\sum_{j=0}^{m}\frac{2^{-2j-2}}{(2j)!}\int_u \det Y_j(u)\, e^{-u^2}\,\mathrm{d}u.$$

Note that the top right 2×2-submatrix of $Y_j(u)$ is $X_j(u)$, so that $\det Y_j(u)$ is equal to

$$\frac{(2m+2)!}{(m+1)!}\det X_j(u) + \frac{(2j)!}{j!}\det\begin{pmatrix} H_{2j+1}(u)-H'_{2j}(u) & H'_{2j+1}(u)-H''_{2j}(u)\\ H_{2m+2}(u) & H'_{2m+2}(u)\end{pmatrix}. \tag{4.7}$$

Because taking derivatives of Hermite polynomials decreases the index by one (4.3) and because the integral over a product of two Hermite polynomials is non-vanishing only if their indices agree, the integral of the determinant in (4.7) is non-vanishing only for $j=m$, in which case it is equal to

$$\int_{u\in\mathbb{R}} H_{2m+1}(u)H'_{2m+2}(u)\,e^{-u^2}\,\mathrm{d}u = 2(2m+2)2^{2m+1}(2m+1)!\,\sqrt{\pi},$$

by (4.1) and (4.3). Hence,

$$\int_{u\in\mathbb{R}} \det Y_j(u)\,e^{-u^2}\,\mathrm{d}u = \begin{cases} \frac{(2m+2)!}{(m+1)!}\int_{u\in\mathbb{R}}\det X_m(u)\,e^{-u^2}\,\mathrm{d}u + \frac{(2m)!}{m!}\,2^{2m+2}(2m+2)!\sqrt{\pi}, & \text{if } j=m,\\ \frac{(2m+2)!}{(m+1)!}\int_{u\in\mathbb{R}}\det X_j(u)\,e^{-u^2}\,\mathrm{d}u, & \text{else.}\end{cases}$$

We find that

$$\begin{aligned}
&\sum_{j=0}^{m}\frac{2^{-2j-2}}{(2j)!}\int_u \det Y_j(u)\,e^{-u^2}\,\mathrm{d}u\\
&\quad= \frac{(2m+2)!}{m!}\sqrt{\pi} + \frac{(2m+2)!}{(m+1)!}\sum_{j=0}^{m}\frac{2^{-2j-2}}{(2j)!}\int_u \det X_j(u)\,e^{-u^2}\,\mathrm{d}u\\
&\quad= \frac{(2m+2)!}{m!}\sqrt{\pi} + \frac{(2m+2)!}{(m+1)!}\frac{\sqrt{\pi}}{2}(m+1)(2m+1)\\
&\quad= \frac{\sqrt{\pi}}{2}\frac{(2m+3)!}{m!},
\end{aligned}$$

where we used (4.6) in the second step. It follows that

$$\int_{u\in\mathbb{R}} \mathbb{E}\det(Q-u\mathbb{1})^2\, e^{-u^2}\,\mathrm{d}u = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})}\,\frac{\sqrt{\pi}}{2}\,\frac{(2m+3)!}{m!} = \frac{\pi(2m+1)!(2m+3)!}{2^{4m+3}\,\Gamma(m+\frac{3}{2})\,m!}.$$

It is not difficult to verify that the last term is $2^{-2m-2}\sqrt{\pi}\,(2m+3)!$. Substituting $2m+1=k$ shows the assertion in this case. □

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