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Spectral triples on the Jiang-Su algebra

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We construct spectral triples on a class of particular inductive limits of matrix-valued function algebras. In the special case of the Jiang-Su algebra, we employ a particular AF-embedding. *Published by AIP Publishing.* https://doi.org/10.1063/1.5026311

I. INTRODUCTION

According to the noncommutative differential geometry program,[2,3] both the topological and the metric information on a noncommutative space can be fully encoded as a *spectral triple* on the noncommutative algebra of coordinates on that space. Nowadays several noncommutative spectral triples have been constructed, with only a partial unifying scheme emerging behind some families of examples, e.g., quantum groups and their homogeneous spaces, like quantum spheres and quantum projective spaces (see, e.g., Refs. 4, 5, and 8). Also some preservation properties with respect to the product, inductive limits, or extensions of algebras have been investigated.

Most of these constructions are still awaiting, however, a proper analysis of properties such as smoothness, dimension (summability), and other conditions selected by Connes. As a testing ground for these and related matters as large as possible, a class of examples should be investigated, including some important new algebras.

In Ref. 10, a general way to construct a spectral triple on arbitrary quasidiagonal C^*-algebras was exhibited. However, in that case, one cannot expect summability. Instead, summability was obtained in Ref. 1 for a certain inductive family of coverings, and p-summability with arbitrary p was obtained for any AF-algebra through the construction in Ref. 6.

In the present paper, we elaborate a construction that extends the latter mentioned approach to a wider class of particular inductive limits of matrix-valued function algebras whose connecting morphisms have a certain peculiar form. In particular, this construction applies to the Jiang-Su algebra $\mathcal{Z}$ (cf. Ref. 7), which was originally constructed in terms of an explicit particular inductive limit of dimension drop algebras. The aim therein was to obtain an example of an infinite-dimensional stably finite nuclear simple unital C^*-algebras with exactly one tracial state and with the same K-theory of the complex numbers. The importance of the Jiang-Su algebra $\mathcal{Z}$ stems from the fact that under some other hypothesis $\mathcal{Z}$-stability entails classification in terms of the Elliott invariant, as proved in Ref. 11.

The organization of the paper is the following: In Sec. II, we recall the definition of the Jiang-Su algebra and construct a particular AF-embedding for it. In Secs. III and IV, we compute the image of elements belonging to a dense subalgebra of the Jiang-Su algebra under the representation obtained by composing the aforementioned AF-embedding with the representation appearing in Ref. 6. In Sec. V, we use the above results to check that some of the Dirac operators considered in Ref. 6 give rise to a spectral triple for the Jiang-Su algebra.

II. SPECTRAL TRIPLE ON THE JIANG-SU ALGEBRA

Let $B = \lim(B_i, \phi_{i,j})$ be a C^*-inductive limit of C^*-algebras, with $B_0 = \mathbb{C}$ and where, for $i > 0$, every B_i is a unital C^*-subalgebra of the algebra of continuous functions on the interval [0, 1] with

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values in M_{n_i} for some natural numbers n_i such that n_i divides n_{i+1}. We assume that every B_i contains a dense $*$-subalgebra of Lipschitz functions. Given $i, l \in \mathbb{N}$, the connecting morphism $\phi_{i,i+l}$ takes the form

$$\phi_{i,i+l}(f) = u_{i,i+l} \begin{pmatrix} f \circ \xi^{i+l}_{i,1} \otimes 1_{N^{i+l}_{i,1}} & & 0 \\ & \ddots & \\ 0 & & f \circ \xi^{i+l}_{i,k^{i+l}_i} \otimes 1_{N^{i+l}_{i,k^{i+l}_i}} \end{pmatrix} u^*_{i,i+l} \tag{1}$$

for some natural numbers $k^{i+l}_i, N^{i+l}_{i,1}, \ldots, N^{i+l}_{i,k^{i+l}_i}$, a unitary $u_{i,i+l} \in C([0,1], M_{n_{i+l}})$, and some continuous paths on the interval (i.e., continuous maps from [0, 1] to itself) $\xi^{i+l}_{i,1}, \ldots, \xi^{i+l}_{i,k^{i+l}_i}$ satisfying

$$|\xi^{i+l}_{i,r}(x) - \xi^{i+l}_{i,r}(y)| \le \frac{1}{2^l}, \qquad \text{for } 1 \le r \le k^{i+l}_i,\ x, y \in [0,1] \tag{2}$$

and such that the resulting $*$-homomorphism $\phi_{i,i+l}$: $B_i \to B_{i+l}$ is injective.

In (1), we have identified, as is of common use, $M_{n_{i+l}}$ with $M_{n_i} \otimes M_{n_{i+l}/n_i}$ and for $m \in \mathbb{N}$, we denote by 1_m the $m \times m$-identity matrix; accordingly, for $1 \le r \le k^{i+l}_i$, we denote by $f \circ \xi^{i+l}_{i,r} \otimes 1_{N^{i+l}_{i,r}}$ the $(N_{i,r} \cdot n_i)$-dimensional matrix whose n_i-dimensional diagonal entries are $f \circ \xi^{i+l}_{i,r}$ and the n_i-dimensional off-diagonal entries are 0.

Following a standard notation (see for example Ref. 7 Definition 2.1), we say that, given two nonzero natural numbers p and q, the corresponding dimension drop algebra is the C^*-algebra,

$$Z(p,q) = \{f \in C([0,1], M_p \otimes M_q) : f(0) \in M_p \otimes 1_q,\ f(1) \in 1_p \otimes M_q\}, \tag{3}$$

and that it is called a prime dimension drop algebra if p and q are coprime.

The Jiang-Su algebra $\mathcal{Z}$ is an inductive limit of prime dimension drop algebras Z_i satisfying a certain universal property. We will use the original construction appearing in Ref. 7, where it was proven that given $p_i, q_i, n_i = p_i q_i$ defining the prime dimension drop algebra $Z_i := Z(p_i, q_i)$, there are numbers $N^{i+1}_{i,1}$, $N^{i+1}_{i,2}$, and $N^{i+1}_{i,3}$ such that $n_{i+1} = (N^{i+1}_{i,1} + N^{i+1}_{i,2} + N^{i+1}_{i,3}) n_i$ is equal to $n_{i+1} = p_{i+1} q_{i+1}$ for some coprime numbers p_{i+1} and q_{i+1} and that there is a unitary $u_{i,i+1} \in C([0,1], M_{n_{i+1}})$ such that the map

$$\phi_{i,i+1} : f \mapsto u_{i,i+1} \begin{pmatrix} f \circ \xi^{i+1}_{i,1} \otimes 1_{N^{i+1}_{i,1}} & & 0 \\ & f \circ \xi^{i+1}_{i,2} \otimes 1_{N^{i+1}_{i,2}} & \\ 0 & & f \circ \xi^{i+1}_{i,3} \otimes 1_{N^{i+1}_{i,3}} \end{pmatrix} u^*_{i,i+1} \tag{4}$$

from $C([0,1], M_{n_i})$ to $C([0,1], M_{n_{i+1}})$ restricts to a $*$-homomorphism from Z_i to Z_{i+1} for the paths $\xi^{i+1}_{i,1}(x) = x/2$, $\xi^{i+1}_{i,2}(x) = 1/2$, and $\xi^{i+1}_{i,3}(x) = (x+1)/2$. The Jiang-Su algebra is constructed as the inductive limit of such prime dimension drop algebras, where the connecting morphisms are given by composition of the morphisms just described.

As a consequence, given a natural number l, the connecting morphism $Z_i \to Z_{i+l}$ has the form (1) for some natural numbers $k^{i+l}_i, N^{i+l}_{i,1}, \ldots, N^{i+l}_{i,k^{i+l}_i}$, a unitary $u_{i,i+l} \in C([0,1], M_{n_{i+l}})$, and some paths $\xi^{i+l}_{i,1}, \ldots, \xi^{i+l}_{i,k^{i+l}_i}$ given by

$$\xi^{i+l}_i(x) = \frac{x+r}{2^l}, \qquad \text{for} \quad 0 \le r \le 2^l - 1 \tag{5}$$

or

$$\xi^{i+l}_i(x) = \frac{s}{2^l}, \qquad \text{for} \quad 1 \le s \le 2^l - 1. \tag{6}$$

It follows that the paths appearing in the connecting morphism $\phi_{i,i+l}$ satisfy Eq. (2), and $\mathcal{Z}$ belongs to the class of inductive limit C^*-algebras we want to consider.

Note that, given B as above, after reindexing the sequence B_i, for example, sending $i \mapsto i^2$, we can always suppose that the paths appearing in the connecting morphisms satisfy

$$|\xi_{i,r}^{i+1}(x) - \xi_{i,r}^{i+1}(y)| \leq \frac{1}{2^i} \tag{7}$$

for any $1 \leq r \leq k_i^{i+1}$. This relation will be used for the Proof of Lemma 2.1.

Fix a sequence of natural numbers n_i as above and consider the inductive limit $A = \lim(A_i, \phi_{i,j}^{\circ})$, where $A_i = C([0,1], M_{n_i})$ and the connecting morphisms ϕ_i° are constructed in the same way as above, but they are considered as unital $*$-homomorphisms between the A_i's. For any $i, l \in \mathbb{N}$ denoted by $\tilde{\phi}_{i,i+l}^{\circ}: A_i \to A_{i+l}$, the $*$-homomorphism

$$\tilde{\phi}_{i,i+l}^{\circ}(f) = \begin{pmatrix} f \circ \xi_{i,1}^{i+l} \otimes 1_{N_{i,1}^{i+l}} & & 0 \\ & \ddots & \\ 0 & & f \circ \xi_{i,k_i^{i+l}}^{i+l} \otimes 1_{N_{i,k_i^{i+l}}^{i+l}} \end{pmatrix}. \tag{8}$$

Let u_i be the unitary corresponding to the connecting morphism $A_1 \to A_i$ (or $B_1 \to B_i$). For any $f \in A_i$ (or B_i), there is a unique $\tilde{f} \in A_i$ such that $f = u_i \tilde{f} u_i^*$. In this way, the connecting morphisms take the form

$$\phi_{i,i+l}^{\circ}(f) = u_{i,i+l}\tilde{\phi}_{i,i+l}^{\circ}(f)u_{i,i+l}^* = u_{i+l}\tilde{\phi}_{i,i+l}^{\circ}(\tilde{f})u_{i+l}^*. \tag{9}$$

Let now $M = \lim(M_{n_i}, \psi_{i,j})$, where $\psi_{i,i+1}(a) = a \otimes 1_{n_{i+1}/n_i}$.

Lemma 2.1. There is a $*$-isomorphism

$$\alpha : A \to M. \tag{10}$$

Let $\gamma \in (1, 2)$. A Lipschitz function $f \in A_i$ with a Lipschitz constant $L_f < \gamma^i$ is sent to

$$\alpha(f) = \lim_{m\to\infty} \psi_m^{\infty}(\tilde{\phi}_{i,m}^{\circ}(\tilde{f})(0)), \tag{11}$$

where $\psi_m^{\infty}: M_{n_m} \to M$ denotes the natural embedding.

Proof. Define $*$-homomorphisms

$$\alpha_i : A_i \to M_{n_{i+1}}, \tag{12}$$

$$f \mapsto \tilde{\phi}_i^{\circ}(\tilde{f})(0) \tag{13}$$

and

$$\beta_i : M_{n_i} \to A_i, \tag{14}$$

$$a \mapsto u_{i+1}\bar{a}u_{i+1}^*, \tag{15}$$

where $\bar{a} \in A_i$ is the constant matrix-valued function taking value $a \in M_{n_i}$. Let now $\gamma \in (1, 2)$ and take finite sets $F_i \subset A_i$ consisting of Lipschitz matrix-valued functions with a Lipschitz constant less than γ^i and such that their union $\bigcup_i F_i$ is dense in A. For any $f \in F_i$ and $a \in M_{n_i}$, we have

$$\alpha_i \circ \beta_i(a) = \psi_i(a), \tag{16}$$

$$\|\beta_{i+1} \circ \alpha_i(f) - \phi_{i,i+1}^{\circ}(f)\| < \frac{\gamma^i}{2^i}. \tag{17}$$

Hence the result follows by Ref. 9 Proposition 2.3.2. □

III. THE ORTHOGONAL DECOMPOSITION

Let $\mathfrak{H}$ be the Hilbert space considered by Christensen and Antonescu in Ref. 6 corresponding to the Gel'fand–Naimark–Segal (GNS)-representation induced by the unique trace τ on M. This trace is given on the finite-dimensional approximants relative to the inductive limit construction by the normalized trace on matrices. Following Ref. 6, we want to write $\mathfrak{H}$ as an infinite direct sum of the finite dimensional Hilbert spaces on which the M_{n_i}'s are represented.

Let $\mathfrak{H}_i = \overline{M_{n_i}}^{\tau}$ and let $v \in \mathfrak{H}_i$. We can consider v as a matrix of dimension n_i, and for any $j < i$, we can write v as a block matrix of the form

$$v = \begin{pmatrix} v^{j,i}_{1,1} & \cdots & v^{j,i}_{1,l^i_j} \\ \vdots & \ddots & \vdots \\ v^{j,i}_{l^i_j} & \cdots & v^{j,i}_{l^i_j,l^i_j} \end{pmatrix}, \tag{18}$$

where $l^i_j = n_i/n_j$ is the multiplicity of M_{n_j} in M_{n_i} and the $v^{j,i}_{k,l}$ are the matrices in M_{n_j}; in particular, we can apply the same procedure to these matrices by iteration. With this notation, the projection $P_{i,j}$ from $\mathfrak{H}_i$ to $\mathfrak{H}_j$ reads

$$P_{i,j}(v) = \frac{1}{l^i_j} \sum_{k=1}^{l^i_j} v^{j,i}_{k,k} \quad \in M_{n_j}. \tag{19}$$

If $j > 1$, the projection R_j from $\mathfrak{H}_j$ to the orthogonal complement of $\mathfrak{H}_{j-1}$ in $\mathfrak{H}_j$ reads for $w \in \mathfrak{H}_j$,

$$R_j(w) = \tag{20}$$

$$\begin{pmatrix} w^{j-1,j}_{1,1} - \frac{1}{l^j_{j-1}} \sum_{k=1}^{l^i_{i-1}} w^{j-1,j}_{k,k} & w^{j-1,j}_{1,2} & \cdots & w^{j-1,j}_{1,l^j_{j-1}} \\ w^{j-1,j}_{2,1} & w^{j-1,j}_{2,2} - \frac{1}{l^j_{j-1}} \sum_{k=1}^{l^i_{i-1}} w^{j-1,j}_{k,k} & & w^{j-1,j}_{2,l^j_{j-1}} \\ \vdots & & \ddots & \vdots \\ w^{j-1,j}_{l^j_{j-1},1} & \cdots & & w^{j-1,j}_{l^j_{j-1},l^j_{j-1}} - \frac{1}{l^j_{j-1}} \sum_{k=1}^{l^i_{i-1}} w^{j-1,j}_{k,k} \end{pmatrix}.$$

Hence, if we denote by $\mathfrak{K}_j = \mathfrak{H}_j \ominus \mathfrak{H}_{j-1}$ the orthogonal complement of $\mathfrak{H}_{j-1}$ in $\mathfrak{H}_j$, the projection $Q_j : \mathfrak{H} \to \mathfrak{K}_j$, when applied to an element $v \in \mathfrak{H}_i$, takes the form, for $1 \le s, t \le l^j_{j-1}$,

$$(Q_j(v))^{j-1,j}_{s,t} = \begin{cases} \frac{1}{l^i_j} \sum_{k=1}^{l^i_j} (v^{j,i}_{k,k})^{j-1,j}_{s,s} - \frac{1}{l^i_{j-1}} \sum_{t=1}^{l^j_{j-1}} \sum_{k=1}^{l^i_j} (v^{j,i}_{k,k})^{j-1,j}_{t,t}, & \text{for } s = t, \\ \frac{1}{l^i_j} \sum_{k=1}^{l^i_j} (v^{j,i}_{k,k})^{j-1,j}_{s,t}, & \text{for } s \neq t, \end{cases} \tag{21}$$

where, with a slight abuse of notation, we identify the spaces $\mathfrak{H}_i$ with their images in $\mathfrak{H}$ and correspondingly consider the projections Q_j as operators from $\mathfrak{H}_i$ to $\mathfrak{K}_j$.

IV. THE COMMUTATORS

Taking $i < n < m$ and $v \in \mathfrak{H}_m$, $f \in A_i$. We want to find an explicit form for the elements $Q_n(\tilde{\phi}^\circ_{i,m}(\tilde{f})(0)v)$ and $\tilde{\phi}^\circ_{i,n}(\tilde{f})(0)Q_n v$. To this end, we want to write $\tilde{\phi}^\circ_{i,m}(\tilde{f})$ as the composition $\tilde{\phi}^\circ_{n,m} \circ \tilde{\phi}^\circ_{n-1,n} \circ \tilde{\phi}^\circ_{i,n-1}(\tilde{f})$.

Let k^i_j be the amount of different paths appearing in the connecting morphism $\phi_{j,i}$. If $1 \le j \le k^n_{n-1}$, we denote by $\tilde{f} \circ [\xi^{n-1}_i] \circ \xi^n_{n-1,j} = \tilde{\phi}^\circ_{i,n-1}(\tilde{f}) \circ \xi^n_{n-1,j}$ the matrix-valued function

$$\begin{pmatrix} \tilde{f} \circ \xi^{n-1}_{i,1} \circ \xi^n_{n-1,j} \otimes 1_{N^{n-1}_{i,1}} & & 0 \\ & \ddots & \\ 0 & & \tilde{f} \circ \xi^{n-1}_{i,k^{n-1}_i} \circ \xi^n_{n-1,j} \otimes 1_{N^{n-1}_{i,k^{n-1}_i}} \end{pmatrix}, \tag{22}$$

then we can write

$$\tilde{\phi}^\circ_{i,n}(\tilde{f}) = \tilde{\phi}^\circ_{n,n-1} \circ \tilde{\phi}^\circ_{i,n-1}(\tilde{f}) = \tag{23}$$

$$\begin{pmatrix} \tilde{f} \circ [\xi^{n-1}_i] \circ \xi^n_{n-1,1} \otimes 1_{N^n_{n-1,1}} & & 0 \\ & \ddots & \\ 0 & & \tilde{f} \circ [\xi^{n-1}_i] \circ \xi^n_{n-1,k^n_{n-1}} \otimes 1_{N^n_{n-1,k^n_{n-1}}} \end{pmatrix}. \tag{24}$$

For $1 \leq s, \leq l^n_{n-1}$, we denote by $\bar{\xi}^n_{n-1,s}$ the path

$$\bar{\xi}^n_{n-1,s} = \begin{cases} \xi^n_{n-1,1} & \text{for } 1 \leq s \leq N^n_{n-1,1} \\ \xi^n_{n-1,2} & \text{for } N^n_{n-1,1} < s \leq N^n_{n-1,1} + N^n_{n-1,2} \\ \vdots & \\ \xi^n_{n-1,k^n_{n-1}} & \text{for } \sum_{k=1}^{k^n_{n-1}-1} N^n_{n-1,k} < s \leq l^n_{n-1} \end{cases}. \tag{25}$$

Thus we obtain for $1 \leq s, t \leq l^n_{n-1}$,

$$(\tilde{\phi}^\circ_{n-1,n} \circ \tilde{\phi}^\circ_{i,n-1}(\tilde{f})(0)Q_n v)^{n-1,n}_{s,t} = \tag{26}$$

$$\frac{1}{l^m_n}\sum_{j=1}^{l^m_n}\left(\tilde{f} \circ \left[\xi^{n-1}_i\right] \circ \bar{\xi}^n_{n-1,s}\right)(0)\left(v^{n,m}_{j,j}\right)^{n-1,n}_{s,t}, \qquad \text{for } s \neq t \tag{27}$$

and

$$\frac{1}{l^m_n}\sum_{j=1}^{l^m_n}\left(f \circ \left[\xi^{n-1}_i\right] \circ \bar{\xi}^n_{n-1,s}\right)(0)\left(\left(v^{n,m}_{j,j}\right)^{n-1,n}_{s,s} - \frac{1}{l^n_{n-1}}\sum_{k=1}^{l^n_{n-1}}\left(v^{n,m}_{j,j}\right)^{n-1,n}_{k,k}\right), \quad \text{for } s = t. \tag{28}$$

In the same way, for $1 \leq j \leq l^m_n$, we can define paths

$$\bar{\xi}_{n,j} = \begin{cases} \xi^m_{n,1} & \text{for } 1 \leq j \leq N^m_{n,1} \\ \xi^m_{n,2} & \text{for } N^m_{n,1} < j \leq N^m_{n,1} + N^m_{n,2} \\ \vdots & \\ \xi^m_{n,k^m_n} & \text{for } \sum_{k=1}^{k^m_n-1} N^m_{n,k} < j \leq l^m_n \end{cases} \tag{29}$$

and compute for $1 \leq s, t \leq l^n_{n-1}$,

$$(Q_n\tilde{\phi}^\circ_{i,m}(\tilde{f})(0)v)^{n-1,n}_{s,t} = (Q_n(\tilde{\phi}^\circ_{n,m} \circ \tilde{\phi}^\circ_{n-1,n} \circ \tilde{\phi}^\circ_{i,n-1})(\tilde{f})(0)v)^{n-1,n}_{s,t} = \tag{30}$$

$$\frac{1}{l^m_n}\sum_{j=1}^{l^m_n}(\tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,s} \circ \bar{\xi}^m_{n,j})(0)(v^{n,m}_{j,j})^{n-1,n}_{s,t}, \quad \text{for } s \neq t \tag{31}$$

and

$$\frac{1}{l^m_n}\sum_{j=1}^{l^m_n}[(\tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,s} \circ \bar{\xi}^m_{n,j})(0)(v^{n,m}_{j,j})^{n-1,n}_{s,s} \tag{32}$$

$$-\frac{1}{l^n_{n-1}}\sum_{k=1}^{l^n_{n-1}}(\tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,k} \circ \bar{\xi}^m_{n,j})(0)(v^{n,m}_{j,j})^{n-1,n}_{k,k})], \quad \text{for } s = t. \tag{33}$$

Thus we can write the commutators

$$(Q_n(\tilde{\phi}^\circ_{i,m}(\tilde{f})(0)v) - \tilde{\phi}^\circ_{i,n}(\tilde{f})(0)Q_n v)^{n-1,n}_{s,t} = \tag{34}$$

$$\frac{1}{l^m_n}\sum_{j=1}^{l^m_n}(\tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,s} \circ \bar{\xi}^m_{n,j} - \tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,s})(0)(v^{n,m}_{j,j})^{n-1,n}_{s,t}, \quad \text{for } s \neq t \tag{35}$$

and

$$\frac{1}{l^m_n}\sum_{j=1}^{l^m_n}[(\tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,s} \circ \bar{\xi}^m_{n,j} - \tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,s})(0)(v^{n,m}_{j,j})^{n-1,n}_{s,t} + \tag{36}$$

$$\frac{1}{l^n_{n-1}}\sum_{k=1}^{l^n_{n-1}}(\tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,s} - \tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,k} \circ \xi^m_{n,j})(0)(v^{n,m}_{j,j})^{n-1,n}_{k,k} \circ \bar{\xi}^m_{n,j}], \quad \text{for } s = t. \tag{37}$$

Lemma 4.1. Let $i < l < m \leq k$ be natural numbers and let ξ_i^l, ξ_l^m, ξ_l^k be paths on the interval $[0, 1]$ *such that*

$$|\xi_i^l(x) - \xi_i^l(y)| \leq \frac{1}{2^{l-i}}, \qquad \text{for any } x, y \in [0, 1]. \tag{38}$$

Then, given any $n > 0$ and any Lipschitz function in $C([0, 1], M_n)$ with Lipschitz constant L_f, we have

$$\|(f \circ \xi_i^l \circ \xi_l^m)(0) - (f \circ \xi_i^l \circ \xi_l^k)(0)\| \leq \frac{2^i L_f}{2^l}. \tag{39}$$

Proof. This is a consequence of the fact that $|\xi_i^l(x) - \xi_i^l(y)| \leq \frac{1}{2^{l-i}}$ for every $x, y \in [0, 1]$. □

V. THE SPECTRAL TRIPLE

Note that if $D = \sum_n \alpha_n Q_n$ for a certain sequence of real numbers $\{\alpha_n\}$, then the domain of D, $\mathrm{dom}(D) = \{v \in \mathfrak{H} : \{\|\alpha_n Q_n v\|\} \in l^2(\mathbb{N})\}$, is left invariant under the action of any $f \in A$; thus, in particular, for every $f \in B$, it makes sense to consider the (in general unbounded) operator $[D, f]$.

Moreover, it follows from the Hahn-Banach extension theorem that if T is an unbounded operator on $\mathfrak{H}$ whose domain contains the algebraic direct sum $\oplus_{alg}\mathfrak{K}_i$ and $\|TP_n\|$ is uniformly bounded on n, then T extends (uniquely) to a bounded operator on the whole Hilbert space $\mathfrak{H}$.

Hence, to obtain boundedness of $[D, f]$, we want to compute estimates for $\|[D, f]P_n\|$ for every n.

For every $i \in \mathbb{N}$, we will denote by LB_i the subset of B_i consisting of Lipschitz functions with the Lipschitz constant smaller than γ^i for some $\gamma \in (1, 2)$. Observe that $\phi_{i,i+1}^\circ|_{LB_i}$ is a linear map sending LB_i into LB_{i+1} and that the algebraic direct limit $\bigcup_i LB_i$ is a dense $*$-subalgebra of B.

Theorem 5.1. *Let $D = \sum_n \alpha_n Q_n$, with $\{\alpha_n\}$, a diverging sequence of real numbers satisfying $\alpha_0 = 0$, $|\alpha_n| \leq \beta^{(n-1)}$ with $\beta < 2$ and $n > 0$. Then $(\bigcup_i LB_i, \mathfrak{H}, D)$ is a spectral triple for B.*

It is p-summable whenever the sequences of numbers $\{\alpha_i\}$, $\{n_i\}$ satisfy

$$\sum_{i \geq 1} (1 + \alpha_i^2)^{-p/2}(n_i^2 - n_{i-1}^2) < \infty \tag{40}$$

for some $p > 0$.

Proof. After reindexing $i \mapsto i^2$, the $*$-isomorphism $\alpha: A \to M$ has the concrete description given in Lemma 2.1, and we will suppose that the index set is already reindexed, if necessary. Thus we can compose it with the GNS representation of M induced by the unique trace τ.

Let $l, m \in \mathbb{N}$ and $v \in \mathfrak{H}_l$. Denote by $\beta_{l,m}^{\mathfrak{H}} : \mathfrak{H}_l \to \mathfrak{H}_m$ and $\beta_{l,\infty}^{\mathfrak{H}} : \mathfrak{H}_l \to \mathfrak{H}$ the connecting isometries. Note that for $i \in \mathbb{N}$ and $f \in LB_i$ the action of f on v reads

$$\lim_{m\to\infty} \beta_{m,\infty}^{\mathfrak{H}} \tilde{\phi}_{i,m}^\circ(\tilde{f})(0)\beta_{l,m}^{\mathfrak{H}} v, \tag{41}$$

where we use the convention that $\beta_{m,\infty}^{\mathfrak{H}} \tilde{\phi}_{i,m}^\circ(\tilde{f})(0)\beta_{l,m}^{\mathfrak{H}}(v) = \beta_{l,\infty}^{\mathfrak{H}} \tilde{\phi}_{i,l}^\circ(\tilde{f})(0)v$ for $\max\{i, m\} \leq l$ and $\beta_{m,\infty}^{\mathfrak{H}} \tilde{\phi}_{i,m}^\circ(\tilde{f})(0)\beta_{l,m}^{\mathfrak{H}} v = \beta_{i,\infty}^{\mathfrak{H}} \tilde{f}(0)\beta_{l,i}^{\mathfrak{H}} v$ for $\max\{m, l\} \leq i$.

Thus we can write

$$\|Q_n f v - f Q_n v\| = \|\beta_n^{\mathfrak{H},\infty} Q_n \lim_{m\to\infty} \tilde{\phi}_{i,m}^\circ(\tilde{f})(0)\beta_{l,m}^{\mathfrak{H}} v - \lim_{m\to\infty} \beta_{m,\infty}^{\mathfrak{H}} \tilde{\phi}_{i,m}^\circ(\tilde{f})(0)\beta_{n,m}^{\mathfrak{H}} Q_n v\|. \tag{42}$$

Since the sequence $\beta_{m,\infty}^{\mathfrak{H}} \tilde{\phi}_{i,m}^\circ(f)(0)\beta_{l,m}^{\mathfrak{H}} v$ converges, there is an M such that

$$\|\beta_{k,\infty}^{\mathfrak{H}} \tilde{\phi}_{i,k}^\circ(\tilde{f})(0)\beta_{l,k}^{\mathfrak{H}} v - \lim_{m\to\infty} \beta_{m,\infty}^{\mathfrak{H}} \tilde{\phi}_{i,m}^\circ(\tilde{f})(0)\beta_{l,m}^{\mathfrak{H}} v\| \leq \frac{1}{2^{(n-1)}}, \tag{43}$$

for any $k \geq M$. Moreover, by Lemma 4.1 and the discussion preceding it,

$$\begin{aligned}&\|[\beta_{n,m}^{\mathfrak{H}} \tilde{\phi}_{i,n}^\circ\big(\tilde{f}\big)(0) - \tilde{\phi}_{i,m}^\circ\big(\tilde{f}\big)(0)\beta_{n,m}^{\mathfrak{H}}]Q_n v\| \\ &\qquad = \|(\beta_{n,m}^{\mathfrak{H}} \tilde{f} \circ [\xi_i^n](0) - \tilde{f} \circ [\xi_i^n] \circ [\xi_n^m](0)\beta_{n,m}^{\mathfrak{H}})Q_n v\| \leq \frac{2^i L_f}{2^{(n-1)}}\end{aligned} \tag{44}$$

for $m > n$ and

$$\|Q_n \tilde{\phi}^{\circ}_{i,M}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,M} v - \tilde{\phi}^{\circ}_{i,n}(\tilde{f})(0) Q_n \beta^{\mathfrak{H}}_{l,M} v\| \leq \frac{2^i L_f}{2^{(n-1)}}. \tag{45}$$

We can suppose $M > n$ and obtain

$$\begin{aligned}
&\|\beta^{\mathfrak{H}}_{n,\infty} Q_n \lim_{m\to\infty} \beta^{\mathfrak{H}}_{m,\infty} \tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,m} v - \lim_{m\to\infty} \beta^{\mathfrak{H}}_{m,\infty} \tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{n,m} Q_n v\| \\
&\qquad \leq \|\beta^{\mathfrak{H}}_{n,\infty} Q_n [\beta^{\mathfrak{H}}_{M,\infty} \tilde{\phi}^{\circ}_{i,M}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,M} v - \lim_{m\to\infty} \beta^{\mathfrak{H}}_{m,\infty} \tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,m} v]\| \\
&\qquad + \|Q_n \tilde{\phi}^{\circ}_{i,M}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,M} v - \tilde{\phi}^{\circ}_{i,n}(\tilde{f})(0) Q_n \beta^{\mathfrak{H}}_{l,M} v\| \\
&\qquad + \|\lim_{m\to\infty} \beta^{\mathfrak{H}}_{m,\infty} [\beta^{\mathfrak{H}}_{n,m} \tilde{\phi}^{\circ}_{i,n}(\tilde{f})(0) - \tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{n,m}] Q_n \beta^{\mathfrak{H}}_{l,M} v\| \\
&\qquad \leq \frac{1 + 2^{i+1} L_f}{2^{(n-1)}}.
\end{aligned} \tag{46}$$

Thus we obtain

$$\|[\alpha_n Q_n, f] P_m\| \leq \frac{|\alpha_n|(1 + 2^{i+1} L_f)}{2^{(n-1)}} \leq (1 + 2^{i+1} L_f)(\beta/2)^{(n-1)}. \tag{47}$$

Hence

$$\begin{aligned}
\|[D, f]\| &\leq \|[\sum_{n=1}^{i} \alpha_n Q_n, f\| + \|\sum_{n>i} \alpha_n Q_n, f]\| \\
&\leq 2\|f\| \sum_{n=1}^{i} |\alpha_n| + (1 + 2^{i+1} L_f) \sum_{n>i} (\beta/2)^{(n-1)} < \infty
\end{aligned} \tag{48}$$

and $[D, f]$ extends to a bounded operator.

Moreover D has a compact resolvent since it has a discrete spectrum and its eigenvalues have finite multiplicity. Suppose we have sequences $\{\alpha_i\}$, $\{n_i\}$ and a real number $p > 0$ as in the statement. Then

$$\mathrm{Tr}((1 + D^2)^{-p/2}) = 1 + \sum_{i\geq 1} (1 + \alpha_i^2)^{-p/2} (n_i^2 - n_{i-1}^2) < \infty. \tag{49}$$

□

As the final comment, we observe that by looking at the growth of the dimensions n_i of the matrix algebras appearing in the original construction of the Jiang-Su algebra (cfr. Ref. 7), it is clear that (40) cannot be satisfied for any choice of a sequence $\{\alpha_i\}$ defining a Dirac operator as in Theorem 5.1 and any $p > 0$; hence the spectral triples exhibited above are not p-summable in this case. Also, with the help of Stirling formula, it can be seen that $\mathrm{Tr}\exp(-D^2)$ diverges and thus the θ-summability does not hold either.

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