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HIGGS BUNDLES AND FUNDAMENTAL GROUP SCHEMES

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ABSTRACT. Relying on a notion of “numerical effectiveness” for Higgs bundles, we show that the category of “numerically flat” Higgs vector bundles on a smooth projective variety X is a Tannakian category. We introduce the associated group scheme, that we call the “Higgs fundamental group scheme of X ,” and show that its properties are related to a conjecture about the vanishing of the Chern classes of numerically flat Higgs vector bundles.

1. INTRODUCTION

Given a projective scheme X over a field k , a line bundle L on X is said to be numerically effective (abbreviated as “nef”) if $\deg f^*L \geq 0$ for every morphism $f : C \rightarrow X$, where C is an irreducible smooth projective curve. A notion of numerical effectiveness for a vector bundle E can be given by asking that the relative hyperplane bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the projective bundle $\mathbb{P}(E)$ is nef [12, 15, 16]. More generally, if $\mathrm{rk} E = r$, one can consider for every k , with $0 < k < r$, the Grassmann bundle $\mathrm{Gr}_k(E) \rightarrow X$ that parameterizes the quotients of fibers of E of dimension k . The universal quotient bundle $Q_{k,E}$ of rank k on $\mathrm{Gr}_k(E)$ satisfies the well-known property that for any morphism $g : Y \rightarrow X$ if F is a rank k quotient of g^*E , then there is a morphism $h : Y \rightarrow \mathrm{Gr}_k(E)$ which covers g and satisfies the condition that $F \simeq h^*Q_{k,E}$. It turns out that E is nef if and only if all universal quotients $Q_{k,E}$ are nef.

One can consider vector bundles E such that both E and its dual E^* are nef. These are called *numerically flat* bundles. The numerically flat bundles enjoy very special properties; they have vanishing rational Chern classes [10], and they form a Tannakian category $\mathbf{NF}(X)$. The associated group scheme G defined by the property that $\mathbf{NF}(X)$ is the category of representations of G was introduced in [4, 13].

Building on ideas already contained in [7], in [5] a definition of “Higgs numerical effectiveness” (“H-nef” for short) was given (however the basics of this theory in their final form were presented subsequently in [6]). Given a Higgs vector bundle $\mathfrak{E} = (E, \phi)$, and any $0 < k < r$, the idea is to use the Higgs field ϕ to construct a closed subscheme $\mathfrak{Gr}_k(\mathfrak{E}) \subset \mathrm{Gr}_k(E)$, with the property that a rank k quotient F of E is a Higgs quotient of \mathfrak{E} (i.e., the kernel corresponding to it is ϕ -invariant) if and only if the image of the

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associated section of $\mathrm{Gr}_k(E)$ is contained in $\mathfrak{Gr}_k(\mathfrak{E})$. The universal quotient bundle $Q_{k,E}$ restricts to $\mathfrak{Gr}_k(\mathfrak{E})$ to yield a *universal Higgs quotient bundle* $\mathfrak{Q}_{k,\mathfrak{E}}$. This opens the way to define Higgs-numerically effective Higgs bundles, in terms of a recursive positivity property of the bundles $\mathfrak{Q}_{k,\mathfrak{E}}$ (see Definition 2.3 for a precise statement). Higgs-numerically flat bundles (H-nflat Higgs bundles) are then defined as H-nef Higgs bundles for which the dual Higgs bundle is H-nef as well. It turns out that the H-nflat Higgs bundles on a smooth projective variety X make up a Tannakian category $\mathbf{HNF}(X)$ (see section 4). We denote by $\pi_1^H(X, x)$ the associated group scheme, where $x \in X$ is the base point needed to define the fiber functor, and call it the *Higgs fundamental group scheme of X* .

In Section 4 we study some basic properties of this group. It turns out that this group is related to a conjectured property of Higgs bundles [7, 6]. For vector bundles E on a projective manifold X , the following property is known to be true [17, 7, 3].

Theorem 1.1. *The following conditions are equivalent:*

- for every morphism $f : C \rightarrow X$, where C is a smooth irreducible projective curve, the bundle f^*E is semistable;
- E is semistable with respect to some polarization, and the characteristic class

$$\Delta(E) = c_2(E) - \frac{r-1}{2r}c_1(E)^2 \in H^4(X, \mathbb{Q})$$

vanishes (here $r = \mathrm{rk} E$).

For Higgs bundles, it is known that the second condition implies the first [7, 6], but the fact that the first implies the second is an open conjecture (see [8] for the characterization of a class of varieties for which this conjecture holds). It is equivalent to the fact that H-nflat Higgs bundles have vanishing rational Chern classes (see Corollary 3.2). For future convenience, we explicitly state this conjecture.

Conjecture 1.2. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle on a smooth projective variety X , such that for every morphism $f : C \rightarrow X$, where C is a smooth irreducible projective curve, the Higgs bundle $f^*\mathfrak{E}$ is semistable. Then $\Delta(E) = 0$.*

In [8] a characterization was given of some classes of varieties for which the conjecture holds (basically, varieties with nef tangent bundle).

As we discuss in Section 4, the above conjecture is also related to the following product formula for the Higgs fundamental group scheme: if X, Y are smooth projective varieties over a field k , and x, y are points in X, Y , respectively, then

$$\pi_1^H(X \times_k Y, (x, y)) \simeq \pi_1^H(X, x) \times \pi_1^H(Y, y).$$

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2. NUMERICALLY EFFECTIVE HIGGS BUNDLES

Notation. Unless otherwise stated, X will denote a smooth projective variety of dimension n defined over an algebraically closed field k of characteristic zero. The cotangent bundle of X will be denoted by Ω_X^1 . We shall usually denote by a Gothic letter, such as \mathfrak{E} , a pair (E, ϕ) , where E is a coherent sheaf and ϕ is a Higgs field on E (see Definition 2.1). So a roman letter will denote the underlying coherent sheaf of a Higgs sheaf.

We fix a very ample line bundle on X and denote by H its numerical class. The degree of a torsion-free coherent \mathcal{O}_X -module F is defined as to be

$$\deg F := c_1(F) \cdot H^{n-1},$$

and if $\text{rk } F \neq 0$, one defines the *slope* of F to be

$$\mu(F) := \frac{\deg F}{\text{rk } F}.$$

Definition 2.1. A Higgs sheaf \mathfrak{E} on X is a pair (E, ϕ) , where E is a torsion-free coherent sheaf on X and

$$\phi : E \longrightarrow E \otimes \Omega_X^1$$

is a homomorphism of \mathcal{O}_X -modules such that $\phi \wedge \phi = 0$. A Higgs subsheaf of a Higgs sheaf $\mathfrak{E} = (E, \phi)$ is a pair (G, ϕ') , where G is a subsheaf of E such that $\phi(G) \subset G \otimes \Omega_X^1$, and $\phi' = \phi|_G$. A Higgs bundle is a Higgs sheaf \mathfrak{E} such that E is a locally-free \mathcal{O}_X -module. If $\mathfrak{E} = (E, \phi)$ and $\mathfrak{G} = (G, \psi)$ are Higgs sheaves, a morphism $f : \mathfrak{E} \longrightarrow \mathfrak{G}$ is a homomorphism of \mathcal{O}_X -modules $f : E \longrightarrow G$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & G \\ \phi \downarrow & & \downarrow \psi \\ E \otimes \Omega_X^1 & \xrightarrow{f \otimes \text{id}} & G \otimes \Omega_X^1 \end{array}$$

commutes.

Definition 2.2. A Higgs sheaf $\mathfrak{E} = (E, \phi)$ is semistable (respectively, stable) if $\mu(G) \leq \mu(E)$ (respectively, $\mu(G) < \mu(E)$) for every Higgs subsheaf (G, ϕ') of \mathfrak{E} with $0 < \text{rk } G < \text{rk } E$.

From now on, unless otherwise stated, by semistability of a Higgs bundle we will mean semistability in the Higgs sense (as in the above definition). Let us recall the definition of numerical effective vector bundles on a projective variety X . A line bundle L on X is said to be numerically effective (nef for short) if, for every pair (C, f) , where C is a smooth projective irreducible curve and $f : C \rightarrow X$ is a morphism, the line bundle f^*L on C has nonnegative degree. A vector bundle E is numerically effective if the hyperplane line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the projectivization $\mathbb{P}(E)$ of E is numerically effective. For the main properties of numerically effective vector bundles see e.g. [12, 15, 16].

Let E be a vector bundle of rank r on X , and let $s < r$ be a positive integer. We shall denote by $\text{Gr}_s(E)$ the Grassmann bundle on X parameterizing quotients of fibers of E of dimension s . Let $p_s : \text{Gr}_s(E) \rightarrow X$ be the natural projection. There is a universal short exact sequence

$$0 \longrightarrow S_{r-s,E} \xrightarrow{\psi} p_s^*E \xrightarrow{\eta} Q_{s,E} \longrightarrow 0 \quad (1)$$

of vector bundles on $\text{Gr}_s(E)$, with $S_{r-s,E}$ being a universal subbundle of rank $r-s$ and $Q_{s,E}$ a universal quotient of rank s . Given a Higgs bundle $\mathfrak{E} = (E, \phi)$, we have the closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \text{Gr}_s(E)$ parameterizing rank s locally-free Higgs quotients, i.e., locally-free quotients of E whose corresponding kernels are ϕ -invariant. In other words, $\mathfrak{Gr}_s(\mathfrak{E})$ (the Grassmannian of locally free rank s Higgs quotients of \mathfrak{E}) is the closed subscheme of $\text{Gr}_s(E)$ defined by the vanishing of the composed morphism

$$(\eta \otimes \text{Id}) \circ p_s^*(\phi) \circ \psi : S_{r-s,E} \longrightarrow Q_{s,E} \otimes p_s^*\Omega_X^1. \quad (2)$$

Let $\rho_s := p_s|_{\mathfrak{Gr}_s(\mathfrak{E})} : \mathfrak{Gr}_s(\mathfrak{E}) \rightarrow X$ be the restriction. The restriction of (1) to $\mathfrak{Gr}_s(\mathfrak{E})$ provides the universal exact sequence

$$0 \longrightarrow \mathfrak{S}_{r-s,\mathfrak{E}} \xrightarrow{\psi} \rho_s^*\mathfrak{E} \xrightarrow{\eta} \mathfrak{Q}_{s,\mathfrak{E}} \longrightarrow 0, \quad (3)$$

with $\mathfrak{Q}_{s,\mathfrak{E}} := Q_{s,E}|_{\mathfrak{Gr}_s(\mathfrak{E})}$ being equipped with the quotient Higgs field induced by the Higgs field $\rho_s^*\phi$. The universal property satisfied by $\mathfrak{Gr}_s(\mathfrak{E})$ is that given any morphism of k -varieties $f : T \rightarrow X$, f factors through $\mathfrak{Gr}_s(\mathfrak{E})$ if and only if the pullback $f^*(E)$ admits a Higgs quotient of rank s . In that case the pullback of the above universal sequence on $\mathfrak{Gr}_s(E)$ gives the desired quotient of $f^*(E)$.

Definition 2.3. A Higgs bundle \mathfrak{E} of rank one is said to be Higgs-numerically effective (H-nef for short) if it is numerically effective in the usual sense. If $\text{rk } \mathfrak{E} \geq 2$, we inductively define H-nefness by requiring that

- (1) all Higgs bundles $\mathfrak{Q}_{s,\mathfrak{E}}$ are Higgs-nef (see (3)) for all s , and

(2) *the determinant line bundle $\det(E)$ is nef.*

If both \mathfrak{E} and \mathfrak{E}^* are Higgs-numerically effective, \mathfrak{E} is said to be Higgs-numerically flat (H-nflat).

Definition 2.3 immediately implies that the first Chern class of an H-numerically flat Higgs bundle is numerically equivalent to zero. Note that if $\mathfrak{E} = (E, \phi)$, with E nef in the usual sense, then \mathfrak{E} is H-nef. Moreover, if $\phi = 0$, the Higgs bundle $\mathfrak{E} = (E, 0)$ is H-nef if and only if E is nef in the usual sense (as in this case the Higgs Grassmannian coincides with the usual Grassmannian bundle, and the respective universal bundles coincide).

We recall that in the case of ordinary vector bundles, nefness is defined using only the hyperplane bundle. Let us motivate why one should consider the behavior of the universal Higgs quotients of all ranks, and therefore introduce Higgs Grassmannians corresponding to quotients of all ranks. In the case of ordinary bundles, if the hyperplane bundle is nef, then the universal quotients of all ranks are nef as well; indeed, if a vector bundle E is nef, its pullback to the Grassmannian $\text{Gr}_s(E)$ is nef, and the quotient $Q_{s,E}$ (see equation (1)) is nef too. This is not the case for Higgs bundles, as the following example shows.

Let $\mathfrak{E} = (E, \phi)$ be a rank three nilpotent Higgs bundle on a smooth projective curve C , having the form $E = L_1 \oplus L_2 \oplus L_3$, where each L_i is a line bundle, and $\phi(L_1) \subset L_2 \otimes \Omega_C^1$, $\phi(L_2) \subset L_3 \otimes \Omega_C^1$, $\phi(L_3) = 0$. Denote by d_i the degree of L_i , and assume that $d_1 + d_2 + d_3 = 0$. The computations in Section 3.4 of [7] show that the hyperplane bundle of \mathfrak{E} , restricted to the Higgs Grassmannian $\mathfrak{Gr}_1(\mathfrak{E})$, is nef if $2d_1 - d_2 - d_3 \geq 0$, while the rank two universal quotient on $\mathfrak{Gr}_2(\mathfrak{E})$ is nef if and only if $d_1 + d_2 - 2d_3 \geq 0$. There exist values of the degrees for which the first inequality holds and the second does not. For instance, if C has genus 3, one can take $d_1 = d_3 = 1$, $d_2 = -1$. Note that by Riemann-Roch theorem $h^0(C, K_C) > 0$ and hence an effective divisor exists in the linear system $|K_C|$ (of degree 4). To ensure that there exists a nonzero Higgs morphism, write $K_C = (x_1 + x_2 + x_3 + x_4)$, with x_i points in C , and take $L_1 = (x_1)$, $L_2 = -(x_2 + x_3)$, $L_3 = (x_4)$.

Moreover, one includes the condition that $\det(E)$ is nef in Definition 2.3 to prevent the existence of H-nef Higgs bundles of negative degree. One such example is provided by a Higgs bundle $\mathfrak{E} = (E, \phi)$ on a smooth projective curve, with $E = L_1 \oplus L_2$ (where L_1, L_2 are line bundles), and $\phi : L_1 \rightarrow L_2 \otimes \Omega_X^1$, $\phi(L_2) = 0$. As shown in [7], \mathfrak{E} has only two Higgs quotients, i.e., L_1 and

$$Q = \text{coker}(\phi \otimes \text{id}) : E \otimes T_X \rightarrow E$$

modulo torsion; the latter one will be denoted by \overline{Q} . Note that $\deg(\overline{Q}) \geq \deg(L_1)$. If the genus of X is at least 2, one can for instance take $\deg(L_1) = 0$ and $\deg(L_2) = -2$. Then

\mathfrak{E} satisfies all the conditions in the definition of H-nefness except the one which says that $\det(E)$ is nef.

3. PROPERTIES OF H-NEF HIGGS BUNDLES

We give a few properties of H-nef Higgs bundles.

Proposition 3.1.

- (i) *An H-numerically flat Higgs bundle is semistable.*
- (ii) *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle whose first Chern class is numerically equivalent to zero. Assume that for all morphisms $f : C \rightarrow X$, where C is a smooth irreducible projective curve, the pullback $f^*\mathfrak{E}$ is semistable. Then \mathfrak{E} is H-nflat.*
- (iii) *Let*

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{G} \longrightarrow 0$$

be a short exact sequence of Higgs bundles. If \mathfrak{F} and \mathfrak{G} are H-nflat, so is \mathfrak{E} .

- (iv) *If \mathfrak{E} and \mathfrak{G} are H-nflat Higgs bundles, then the tensor product $\mathfrak{E} \otimes \mathfrak{G}$ is H-nflat.*

Proof. (i) and (ii) are Proposition 8.8 and Lemma 8.7 in [6], respectively.

(iii) Let C be a smooth irreducible projective curve, and $f : C \rightarrow X$ a morphism. Then the sequence

$$0 \longrightarrow f^*\mathfrak{F} \longrightarrow f^*\mathfrak{E} \longrightarrow f^*\mathfrak{G} \longrightarrow 0$$

is exact. As $f^*\mathfrak{F}$ and $f^*\mathfrak{G}$ are H-nflat, their first Chern classes are numerically equivalent to zero and they are semistable by part (i). It follows that $f^*\mathfrak{E}$ is semistable as well. Hence \mathfrak{E} is H-nflat.

(iv) Again, let C be a smooth irreducible projective curve, and $f : C \rightarrow X$ a morphism. By the same argument as in part (iii), $f^*\mathfrak{E}$ and $f^*\mathfrak{G}$ are semistable. Then, as shown in [1], $f^*\mathfrak{E} \otimes f^*\mathfrak{G} \simeq f^*(\mathfrak{E} \otimes \mathfrak{G})$ is semistable as well. Moreover,

$$c_1(E \otimes G) = \text{rk } E \cdot c_1(G) + \text{rk } G \cdot c_1(E) \equiv 0.$$

So by part (ii), $\mathfrak{E} \otimes \mathfrak{G}$ is H-nflat. □

A corollary to Proposition 3.1 is that Conjecture 1.2 is equivalent to the property that all H-nflat Higgs bundles have vanishing rational Chern classes (the analogous fact for vector bundles was proved in [7]).

Corollary 3.2. *The following facts are equivalent.*

- (i) *If $\mathfrak{E} = (E, \phi)$ is a Higgs bundle on X , and for all morphisms $f : C \rightarrow X$, where C is a smooth irreducible projective curve, the pullback $f^*\mathfrak{E}$ is semistable, then $\Delta(E) = 0$.*

(ii) *All rational Chern classes of an H-nflat Higgs bundle vanish.*

Proof. Assume that (i) holds, and let $\mathfrak{E} = (E, \phi)$ be H-nflat. Then it follows easily that $f^*(\mathfrak{E})$ is also H-nflat. By Proposition 3.1 (i), every pullback $f^*\mathfrak{E}$ is semistable. Since (i) holds, we have $\Delta(E) = c_2(E) = 0$. By Theorem 2 in [19], \mathfrak{E} has a filtration whose quotients are flat, so that all Chern classes of \mathfrak{E} vanish.

Conversely, assume that (ii) holds, and let \mathfrak{E} be a Higgs bundle such that all pullbacks $f^*\mathfrak{E}$ are semistable. We may assume that \mathfrak{E} has vanishing first Chern class by replacing it with its endomorphism bundle. By Proposition 3.1 (ii), \mathfrak{E} is H-nflat. Since (ii) holds, we have in particular $\Delta(E) = 0$. \square

Actually Proposition 3.1 (iv) can be generalized to H-nef Higgs bundles. We shall use the Harder-Narasimhan filtration for Higgs bundles on curves [11]. Given a Higgs bundle \mathfrak{E} on a smooth, projective curve Y defined over k , there exists a unique filtration of \mathfrak{E} by Higgs subsheaves $0 \subset \mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \cdots \subset \mathfrak{E}_r = \mathfrak{E}$ such that the successive quotients $\mathfrak{E}_i/\mathfrak{E}_{i-1}$ are semistable as Higgs sheaves with their slopes satisfying the inequalities $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ for all i . Set $\mu_{\max}(\mathfrak{E}) = \mu(E_1)$ and $\mu_{\min}(\mathfrak{E}) = \mu(E_r/E_{r-1})$.

In the rest of the paper, by the Harder-Narasimhan filtration of a Higgs bundle we will mean a filtration as above.

The Harder-Narasimhan filtration for Higgs bundles on a curve has the following basic properties, analogous to those of the Harder-Narasimhan filtration for torsion-free sheaves (see [1]):

1. If \mathfrak{E} and \mathfrak{F} are two Higgs bundles, $\mu_{\max}(\mathfrak{E} \otimes \mathfrak{F}) = \mu_{\max}(\mathfrak{E}) + \mu_{\max}(\mathfrak{F})$ and $\mu_{\min}(\mathfrak{E} \otimes \mathfrak{F}) = \mu_{\min}(\mathfrak{E}) + \mu_{\min}(\mathfrak{F})$.

2. If $\mathfrak{E}^\bullet = \{\mathfrak{E}_0 \subset \mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \cdots \subset \mathfrak{E}_s\}$ is any filtration of \mathfrak{E} such that each filter is preserved by the Higgs field, the Harder-Narasimhan polygon for \mathfrak{E}^\bullet lies under the Harder-Narasimhan polygon for \mathfrak{E} . (See [13] for the definition of the Harder-Narasimhan polygon.)

The following lemma generalizes a criterion holding for numerically effective bundles [2].

Lemma 3.3. *Let \mathfrak{E} be a Higgs bundle on X . Then \mathfrak{E} is H-nef if and only if for any morphism $f: C \rightarrow X$, where C is a smooth projective irreducible curve, one has $\mu_{\min}(f^*\mathfrak{E}) \geq 0$.*

Proof. Suppose \mathfrak{E} is H-nef. Let $f: C \rightarrow X$ be any morphism from a smooth projective irreducible curve C to X . Let

$$0 \subset \mathfrak{E}_1 \subset \mathfrak{E}_2 \subset \cdots \subset \mathfrak{E}_r = f^*\mathfrak{E}$$

be the Harder-Narasimhan filtration of the pullback of \mathcal{E} to C . Since \mathcal{E} is H-nef, it follows that $\deg f^*\mathcal{E} \geq 0$. Let $s = \text{rk}(\mathcal{E}_r/\mathcal{E}_{r-1})$. By the universal property of the Higgs Grassmannian $\mathfrak{Gr}_s(f^*\mathcal{E})$, from the natural quotient morphism $\phi_r : \mathcal{E}_r \rightarrow \mathcal{E}_r/\mathcal{E}_{r-1}$ we get a morphism $C \rightarrow \mathfrak{Gr}_s(f^*\mathcal{E})$ such that the pullback of the universal quotient on $\mathfrak{Gr}_s(f^*\mathcal{E})$ coincides with ϕ_r . By the H-nefness of \mathcal{E} , it follows that $\deg(\mathcal{E}_r/\mathcal{E}_{r-1}) \geq 0$.

Conversely, suppose \mathcal{E} has the property that for any smooth projective irreducible curve C and any morphism $f : C \rightarrow X$, $\mu_{\min}(f^*(\mathcal{E})) \geq 0$. We want to show that \mathcal{E} is an H-nef Higgs bundle on X . The assumption on \mathcal{E} implies that the degree of \mathcal{E} is non-negative on every curve, so that $\det f^*\mathcal{E}$ is nef and hence H-nef.

To prove the other condition in the definition of H-nef bundles, we recall that, as explained in [6], the H-nefness of \mathcal{E} is equivalent to the nefness, in the usual sense, of a collection of line bundles \mathcal{L}_S , each defined on a scheme S equipped with a projection $\pi_S : S \rightarrow X$ (these line bundles are obtained by successively taking the universal Higgs quotient until one reaches the rank one quotient bundles). Let $\psi_S : \pi_S^*\mathcal{E} \rightarrow \mathcal{L}_S$ denote the quotient morphism, and let $g : C \rightarrow S$ be any morphism, where C is a smooth curve. The pullback of ψ_S to C produces a quotient $f^*\mathcal{E} \rightarrow \mathfrak{F}$ on C , where $f = \pi_S \circ g$. Let \mathfrak{F}' denote the kernel of this quotient. By property (1) of the Harder-Narasimhan filtration explained before, the polygon corresponding to the filtration $0 \subset \mathfrak{F}' \subset f^*\mathcal{E}$ lies under the Harder-Narasimhan polygon of $f^*\mathcal{E}$. Since $\mu_{\min}(f^*\mathcal{E}) \geq 0$, this immediately implies that $\deg \mathfrak{F}' \leq \deg f^*\mathcal{E}$ and hence $\deg \mathfrak{F} \geq 0$, so that \mathcal{L}_S is nef. This shows that \mathcal{E} is H-nef, thereby completing the proof of the lemma. \square

Lemma 3.4. *If $f : Y \rightarrow X$ is a surjective morphism of smooth projective varieties, and \mathcal{E} is a Higgs bundle on X , then \mathcal{E} is H-nef if and only if $f^*\mathcal{E}$ is.*

Proof. Suppose \mathcal{E} is H-nef. Consider $f^*(\mathcal{E})$. Let $\psi : C \rightarrow Y$ be any morphism from a smooth projective curve C to Y . Then $\deg(\psi^*f^*(\mathcal{E})) = \deg((f \circ \psi)^*(\mathcal{E})) \geq 0$ by H-nefness of \mathcal{E} . Hence $\det(f^*(\mathcal{E}))$ is nef. Also $\mu_{\min}(\psi^*f^*(\mathcal{E})) = \mu_{\min}((f \circ \psi)^*(\mathcal{E})) \geq 0$ by H-nefness of \mathcal{E} thereby showing that $f^*(\mathcal{E})$ is H-nef. Conversely suppose $f^*(\mathcal{E})$ is H-nef on Y . Let $\phi : C \rightarrow X$ be any morphism from a smooth projective curve to X . Then there exists a surjective morphism from a smooth projective curve $g : \tilde{C} \rightarrow C$ and a morphism $\tilde{\phi} : \tilde{C} \rightarrow Y$ lying over the morphism ϕ . Then $\deg(f \circ \tilde{\phi})^*(\mathcal{E}) \geq 0$ by H-nefness of $f^*(\mathcal{E})$ and hence by the commutativity of the diagram, $\deg(\phi \circ g)^*\mathcal{E} \geq 0$. Since g is a finite morphism this shows that $\deg(\phi^*(\mathcal{E})) \geq 0$ thus proving that $\det(\mathcal{E})$ is nef. Similarly $\mu_{\min}(f \circ \tilde{\phi})^*(\mathcal{E}) \geq 0$ by H-nefness of $f^*(\mathcal{E})$ and hence by the commutativity of the diagram, $\mu_{\min}(\phi \circ g)^*(\mathcal{E}) \geq 0$. Since g is a finite morphism, it follows that $\mu_{\min}(\phi^*(\mathcal{E})) \geq 0$ as well. \square

Lemma 3.5. *Every quotient Higgs bundle of an H-nef Higgs bundle \mathcal{E} on X is H-nef.*

Proof. Let $\mathfrak{E} \twoheadrightarrow \mathfrak{E}''$ be a non-trivial Higgs quotient. Let \mathfrak{E}' denote the kernel. Let $f : C \rightarrow X$ be a morphism from a smooth curve C to X . By the property of the (Higgs) Harder-Narasimhan filtration mentioned earlier, $\mu_{\min}(f^*(\mathfrak{E})) \geq 0$ and hence $\deg(f^*(\mathfrak{E}')) \leq \deg(f^*(\mathfrak{E}))$. Thus $\deg(f^*(\mathfrak{E}'')) \geq 0$ proving that $\det(\mathfrak{E}'')$ is nef. To prove the second condition, let $f^*(\mathfrak{E}'') \rightarrow \mathfrak{F}$ be a Higgs quotient. Then \mathfrak{F} is also a Higgs quotient of $f^*(\mathfrak{E})$ and hence by H-nefness of \mathfrak{E} , $\deg(\mathfrak{F}) \geq 0$. This shows that $\mu_{\min}(f^*(\mathfrak{E}'')) \geq 0$ thus completing the proof that \mathfrak{E}'' is H-nef as well. \square

The remaining results in this section will be the key to prove that H-nflat Higgs bundles make up a Tannakian category.

Theorem 3.6. *Let X be a smooth projective variety. Let \mathfrak{E} and \mathfrak{F} be two H-nef bundles on X . Then $\mathfrak{E} \otimes \mathfrak{F}$ is also H-nef.*

Proof. Let $f : C \rightarrow X$ be any smooth projective curve mapping to X . Since \mathfrak{E} and \mathfrak{F} are both H-nef, $\mu_{\min}(f^*\mathfrak{E})$ and $\mu_{\min}(f^*\mathfrak{F})$ are both non-negative. By property 2 of the Harder-Narasimhan-filtration explained earlier, $\mu_{\min}(f^*(\mathfrak{E} \otimes \mathfrak{F})) \geq 0$. Hence by Lemma 3.3, the tensor product $\mathfrak{E} \otimes \mathfrak{F}$ is also H-nef. \square

Finally, we have the following property of morphisms between H-nflat Higgs bundles.

Proposition 3.7. *Let $\beta : \mathfrak{E} = (E, \phi) \rightarrow \mathfrak{F} = (F, \psi)$ be a morphism of H-nflat Higgs bundles on a smooth projective variety X . The kernel and cokernel of β are both locally free.*

Proof. The proposition is equivalent to the statement that $\dim \beta(E_x)$ is independent of $x \in X$. Therefore, it suffices to show the following: for every pair (C, f) , where C is a smooth projective curve and $f : C \rightarrow X$ is a morphism, the image $(f^*\beta)(f^*E)$ is a subbundle of f^*F .

From Lemma 3.3 we know that \mathfrak{E} and \mathfrak{F} are Higgs semistable of degree zero. Therefore, it is enough to prove the proposition for smooth projective curves.

So take X to be a smooth projective curve. Take semistable Higgs bundles $\mathfrak{E} = (E, \phi)$ and $\mathfrak{F} = (F, \psi)$ of degree zero on X , and let $\beta : \mathfrak{E} \rightarrow \mathfrak{F}$ be a nonzero homomorphism. Since $\beta(E)$ is a quotient of E (respectively, subsheaf of F), we have $\deg \beta(E) \geq 0$ (respectively, $\deg \beta(E) \leq 0$). Therefore, it follows that

$$\deg \beta(E) = 0. \tag{4}$$

Next, we will show that the quotient $F/\beta(E)$ is torsion-free. Let T be the torsion part of $F/\beta(E)$. Let F' be the inverse image of T in F . We have

$$\deg F' = \deg \beta(E) + \deg T = \deg T.$$

So if $T \neq 0$, then $\deg F' = \deg T > 0$, and hence in this case F' contradict the semistability condition for \mathfrak{F} . Consequently, we have $T = 0$. This implies that $\beta(E)$ is a subbundle of F . \square

Proposition 3.8. *Let $\beta : \mathfrak{E} = (E, \phi) \rightarrow \mathfrak{F} = (F, \psi)$ be a morphism of H -nflat Higgs bundles on a smooth projective variety X . The kernel and cokernel of β are H -nflat Higgs bundles.*

Proof. From Proposition 3.7 we know that both kernel and cokernel of β are locally free. As in the proof of Proposition 3.7, take X to be a smooth projective curve. Then by proposition 3.1, \mathfrak{E} and \mathfrak{F} are semistable of degree 0. Since $\deg E = 0$, from eq. (4) it follows immediately that $\deg(\ker \beta) = 0$ if $\beta \neq 0$. Similarly, since $\deg F = 0$, from eq. (4) it follows immediately that $\deg(\operatorname{coker} \beta) = 0$ if $\beta \neq 0$. Since E and F are Higgs-semistable of degree zero, and $\ker \beta$ and $\operatorname{coker} \beta$ are of degree 0, it follows that $\ker \beta$ and $\operatorname{coker} \beta$ are also Higgs-semistable of degree zero. Since the pullbacks of $\ker \beta$ and $\operatorname{coker} \beta$ to any smooth curve are Higgs semistable of degree 0, by Proposition 3.1, it follows that both kernel and cokernel of β are H -nflat Higgs bundles. \square

4. CATEGORIES OF NUMERICALLY FLAT BUNDLES

Definition 4.1. *Given a smooth projective variety X over a field k of characteristic zero, we consider the following categories.*

- (1) *The category $\mathbf{NF}(X)$ whose objects are numerically flat vector bundles on X , and morphisms are morphisms of \mathcal{O}_X -modules;*
- (2) *The category $\mathbf{HNF}(X)$ whose objects are H -numerically flat Higgs bundles on X , and morphisms are morphisms of Higgs sheaves.*

Note that by Proposition 3.7 kernels and cokernels in these categories are locally free. $\mathbf{NF}(X)$ and $\mathbf{HNF}(X)$ are Abelian categories (the case of $\mathbf{HNF}(X)$ follows as a consequence of Proposition 3.8), and $\mathbf{NF}(X)$ is a proper subcategory of $\mathbf{HNF}(X)$. Both are tensor categories (cf. in particular Proposition 3.1 (iv)). Moreover, they are rigid in the sense of [9], Definition 1.7.

We remind the reader that a *neutral Tannakian category over a field k* is a rigid Abelian k -linear tensor category \mathbf{C} together with a faithful k -linear tensor functor $\omega : \mathbf{C} \rightarrow \mathbf{Vect}_k$. Here \mathbf{Vect}_k is the category of k -vector spaces, and ω is called the *fiber functor*. Then, there exists an affine group scheme G over k such that \mathbf{C} is equivalent to the category $\mathbf{Rep}_k(G)$ of k -linear representations of G (see [9]).

$\mathbf{HNF}(X)$ is indeed a neutral Tannakian category (with ω the functor that associates to an H-flat Higgs bundle $\mathfrak{E} = (E, \phi)$ its fiber E_x at a fixed point $x \in X$), so that the following definition makes sense.

Definition 4.2. *Let $x \in X$. The Higgs fundamental group scheme $\pi_1^H(X, x)$ is the affine group scheme representing the category $\mathbf{HNF}(X)$ with the fiber functor $\mathfrak{E} \mapsto E_x$.*

If $\pi_1^S(X, x)$ is the fundamental group scheme associated with the category $\mathbf{NF}(X)$ [13], the inclusion $\mathbf{NF}(X) \hookrightarrow \mathbf{HNF}(X)$ induces a faithfully flat homomorphism of group schemes $\pi_1^H(X, x) \rightarrow \pi_1^S(X, x)$.

We conclude this paper by giving a few properties of the Higgs fundamental group scheme. A more thorough study of this group will form the object of a future paper.

Proposition 4.3. *Let $f : X' \rightarrow X$ be a surjective, flat morphism of projective varieties over k . If $f_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ and $f(x') = x$, then the induced morphism $\pi_1^H(X', x') \rightarrow \pi_1^H(X, x)$ is a surjective faithfully flat morphism.*

Proof. By [9, Prop. 2.21(a)], it suffices to show that if E is an H-numerically flat bundle on X and $F' \subset f^*E = E'$ (say) is an H-numerically flat subbundle of E' on X' , then there exists an H-numerically flat subbundle $F \subset E$ on X such that $f^*F = F'$. Fix $y \in X$. Let E_y (respectively, E'_y, F'_y) denote the restrictions of E (respectively, E', F') to y (respectively, X'_y). Consider the surjection $E'_y \twoheadrightarrow F'_y$ corresponding to the inclusion $F'_y \subseteq E'_y$. Since E'_y is trivial and hence globally generated, it follows that F'_y is globally generated as well. But since $c_1(F'_y)$ is numerically equivalent to zero, it follows that any section of F'_y has no zero's and hence F'_y and therefore F' is trivial on the fibers of f . Since by flatness of f , $h^0(F'|_{X'_y})$ is independent of $y \in X$, by Grauert's theorem it follows that f_*F' is locally free. This and the given condition that $f_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ together imply that the natural map $f^*f_*F' \rightarrow F'$ is an isomorphism of bundles. Taking F to be f_*F' thereby produces a subbundle $F \subseteq E$ such that f^*F is isomorphic to F' . It is easy to see that F is invariant under the Higgs field on E . The vector bundle F equipped with the induced Higgs field is also H-nflat by Proposition 3.4 since its pullback under f is H-nflat, thereby completing the proof of the proposition. \square

We also mention the following facts.

- If $\pi_1^H(X, x) = \{e\}$, the category $\mathbf{HNF}(X)$ is equivalent to the category \mathbf{Vect}_k of finite-dimensional vector spaces. As a consequence, all H-nflat Higgs bundles are trivial.
- If the natural morphism $\pi_1^H(X, x) \rightarrow \pi_1^S(X, x)$ is an isomorphism, the categories $\mathbf{HNF}(X)$ and $\mathbf{NF}(X)$ are equivalent. This means that all H-nflat Higgs bundles only have zero Higgs field, which also implies that the Conjecture 1.2 holds true.

Let X, Y be projective varieties over k , and let x, y be points in X, Y , respectively. Given Higgs bundles (E, θ) and (F, ϕ) on X and Y respectively, we have the Higgs bundle $(E \boxtimes F, \theta \otimes \text{Id} + \text{Id} \otimes \phi)$ on $X \times Y$. This construction produces a homomorphism

$$\pi_1^H(X \times_k Y, (x, y)) \longrightarrow \pi_1^H(X, x) \times \pi_1^H(Y, y). \quad (5)$$

At the moment we do not know whether the above homomorphism is an isomorphism. This fact, via Corollary 3.2, is related to the conjecture that Theorem 1.1 also holds for Higgs bundles. If indeed the morphism (5) is an isomorphism, then any numerically flat Higgs bundle on $C_1 \times \dots \times C_d$, where C_i are smooth projective curves, would arise from numerically flat Higgs bundles on the curves C_i . A numerically flat Higgs bundle on a curve is of degree zero. Therefore, all higher Chern classes of a numerically flat Higgs bundle on $C_1 \times \dots \times C_d$ would be numerically equivalent to zero. On the other hand, the numerical vanishing of higher Chern classes of a numerically flat Higgs bundle is the key obstruction if one tries to generalize of proof of the product formula for the usual numerically flat case (no Higgs field), as given in [14], to Higgs bundles.

REFERENCES

- [1] V. BALAJI AND A. J. PARAMESWARAN, *Tensor product theorem for Hitchin pairs—an algebraic approach*, Ann. Inst. Fourier (Grenoble), **61** (2011), pp. 2361–2403.
- [2] C. M. BARTON, *Tensor products of ample vector bundles in characteristic p* , Amer. J. Math., **93** (1971), pp. 429–438.
- [3] I. BISWAS AND U. BRUZZO, *On semistable principal bundles over a complex projective manifold*, Int. Math. Res. Not. IMRN, (2008), pp. Art. ID rnn035, 28.
- [4] I. BISWAS AND Y. I. HOLLA, *Semistability and numerical effectiveness in positive characteristic*, Internat. J. Math. **22** (2011), 25–46.
- [5] U. BRUZZO AND B. GRAÑA OTERO, *Numerically flat Higgs bundles*, Commun. Contemp. Math. **9** (2007), pp. 437–446.
- [6] ———, *Semistable and numerically effective principal (Higgs) bundles*, Adv. Math., **226** (2011), pp. 3655–3676.
- [7] U. BRUZZO AND D. HERNÁNDEZ RUIPÉREZ, *Semistability vs. nefness for (Higgs) vector bundles*, Differential Geom. Appl., **24** (2006), pp. 403–416.
- [8] U. BRUZZO AND A. LO GIUDICE, *Restricting Higgs bundles to curves*, Asian J. Math. **20** (2016), pp. 309–408.
- [9] P. DELIGNE AND J. S. MILNE, *Tannakian categories*, in Hodge cycles, motives, and Shimura varieties, P. Deligne, J. S. Milne, A. Ogus, and K.-y. Shih, eds., vol. 900 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1982, pp. ii+414.
- [10] J.-P. DEMAILLY, T. PETERNELL, AND M. SCHNEIDER, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom., **3** (1994), pp. 295–345.
- [11] A. DEY AND R. PARTHASARATHI, *On Harder-Narasimhan reductions for Higgs principal bundles*, Proc. Indian Acad. Sci. Math. Sci., **115** (2005), pp. 127–146.
- [12] R. HARTSHORNE, *Ample vector bundles*, Inst. Hautes Études Sci. Publ. Math. **29** (1966), pp. 63–94.

- [13] A. LANGER, *On the S -fundamental group scheme*, Ann. Inst. Fourier (Grenoble), **61** (2011), 2077–2119.
- [14] —, *On the S -fundamental group scheme. II*, J. Inst. Math. Jussieu **11** (2012), 835–854.
- [15] R. LAZARSFELD, *Positivity in algebraic geometry. I*, vol. 48 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.
- [16] —, *Positivity in algebraic geometry. II*, vol. 49 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.
- [17] N. NAKAYAMA, *Normalized tautological divisors of semi-stable vector bundles*, Sūrikaiseikikenkyūsho Kōkyūroku, (1999), pp. 167–173. Free resolutions of coordinate rings of projective varieties and related topics (Japanese) (Kyoto, 1998).
- [18] C. T. SIMPSON, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc., **1** (1988), pp. 867–918.
- [19] —, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math., **75** (1992), pp. 5–95.

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