Results on the extension of isomonodromy deformations to the case of a resonant irregular singularity

Results on the Extension of Isomonodromy Deformations to the case of a Resonant Irregular Singularity

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Abstract

We explain some results of [16], discussed in our talk [13] in Pisa, February 2017. Consider an $n \times n$ linear system of ODEs with an irregular singularity of Poincaré rank 1 at $z = \infty$ and Fuchsian singularity at $z = 0$, holomorphically depending on parameter t within a polydisc in $\mathbb{C}^n$ centred at $t = 0$. The eigenvalues of the leading matrix at ∞, which is diagonal, coalesce along a coalescence locus Δ contained in the polydisc. Under minimal vanishing conditions on the residue matrix at $z = 0$, we show in [16] that isomonodromic deformations can be extended to the whole polydisc, including Δ, in such a way that the fundamental matrix solutions and the constant monodromy data are well defined in the whole polydisc. These data can be computed just by considering the system at point of Δ, where it simplifies. Conversely, if the t-dependent system is isomonodromic in a small domain contained in the polydisc not intersecting Δ, and if suitable entries of the Stokes matrices vanish, then Δ is not a branching locus for the fundamental matrix solutions. The results have applications to Frobenius manifolds and Painlevé equations.

1. Introduction

In these proceedings, we summarise some results extracted from our paper [16], presented in our talk [13] at the workshop "Asymptotic and Computational Aspects of Complex Differential Equations", at CRM in Pisa, in February 2017.

In [16], we study deformations of a class of linear differential systems[1] with a resonant irregular singularity at $z=\infty$, when the eigenvalues of the leading matrix at $z=\infty$ coalesce along a locus in the space of deformation parameters. The above class contains, in particular, the following $n\times n$ ($n\in\mathbb{N}$) system playing an important role in applications

$$\frac{dY}{dz}=A(z,t)Y, \qquad A(z,t)=\Lambda(t)+\frac{A_1(t)}{z}. \tag{1.1}$$

with singularity of Poincaré rank 1 at $z=\infty$. The matrices $\Lambda(t)$ and $A_1(t)$ are holomorphic functions in $t=(t_1,...,t_n)$ in an open connected domain of $\mathbb{C}^n$. Moreover, $\Lambda(t)$ is diagonal, as follows

$$\Lambda(t):=\operatorname{diag}(u_1(t),...,u_n(t)). \tag{1.2}$$

The deformation theory is well understood when $\Lambda(t)$ has distinct eigenvalues $u_1(t)$, $u_2(t)$, ..., $u_n(t)$ for t in the domain. On the other hand, there are important cases for applications when two or more eigenvalues may coalesce along a certain locus Δ in the t-domain, called the *coalescence locus*, the matrix $\Lambda(t)$ remaining diagonal at Δ. This means that $u_a(t)=u_b(t)$ for some indices $a\neq b\in\{1,...,n\}$ whenever t belongs to Δ, while $u_1(t)$, $u_2(t)$, ..., $u_n(t)$ are pairwise distinct for $t\notin\Delta$. In this case, the point $z=\infty$ for $t\in\Delta$ is called a *resonant irregular singularity*, and the deformation is said to be *non-admissible*, because it does not satisfy some of the assumption of the isomonodromy deformation theory of Jimbo-Miwa-Ueno [38] [25]. Indeed, when t leaves the range of admissibility and varies in a neighbourhood of the coalescence locus, several problems arise with the analytic properties of fundamental matrix solutions (for simplicity, we will just write fundamental solutions) and their asymptotic behaviour within prescribed sectors. To the best of our knowledge, the analysis of fundamental matrix solutions and their monodromy, when the diagonal matrix $\Lambda(t)$ has coalescing eigenvalues at Δ, seems to be missing from the existing literature: it is the problem which we have addressed in [16], both in the non-isomonodromic and isomonodromic cases (inspired by [16], a reformulation in the language of Pfaffian systems has been given in [32], and a reformulation in the language of the geometric theory of differential equations has been given by [58]).

An isomonodromic system as above, with antisymmetric A_1, is at the core of the analytic approach to semisimple Frobenius manifolds [19] [20] [21] (see also [59] [60] [61] [49] [57]). Its monodromy data play the role of local moduli. The system (1.1), with coalescing eigenvalues, gives the isomonodromic description of Frobenius manifolds *remaining semisimple at the locus of coalescent canonical coordinates* [17]. An important example of this coalescence is the quantum cohomology of Grassmannians (see [15] [17] and [18]).

For $n=3$, a special case of system (1.1) gives an isomonodromic description of the general sixth Painlevé equation, according to [50] (see also [33]). This description was given also in [19] [21] for a sixth Painlevé equation associated with Frobenius manifolds. Coalescence occurs at critical points of the Painlevé equation, and $A_1(t)$ is holomorphic when the sixth Painlevé transcendents are *holomorphic at a fixed singularity* (critical point) of the Painlevé equation (see Section 3 below).

A motivation for our paper [16] was then to address in a rigorous way the coalescence phenomena, in view of the above applications, especially the computation of monodromy data of Frobenius manifolds. Below, we only review a simple application to Painlevé equations (see Section 3), while the case of (coalescent) Frobenius manifolds is thoroughly studied in [17] and [18], and summarised in [14].

1.1. **Some Examples.** Before stating some of the results of [16] in Section 2 below, it is worth introducing the problem with simple examples.

[1]For the general theory of Stokes phenomena for differential systems not depending on parameters, see [2] [3] [4] [5] [66]. For the local deformation theory, see [34] [35] [63] [64].

Example 1.1. The example shows the typical singular behaviour of fundamental solutions and formal solutions at Δ. Consider the (non-isomonodromic) system

$$\frac{dY}{dz}=\left[\begin{pmatrix}0&0\\0&t\end{pmatrix}+\frac{1}{z}\begin{pmatrix}1&0\\t&2\end{pmatrix}\right]Y.$$

The coalescence locus is $\Delta=\{0\}$. At fixed $t=0$, the system has a fundamental solution

$$\mathring{Y}(z)=I\cdot\begin{bmatrix}z&0\\0&z^2\end{bmatrix}.$$

where I stands for the identity matrix.[2] For $t\neq 0$, the system can be solved by reduction to a second order ODE in a standard way, obtaining the fundamental matrix

$$Y_{hol}(z,t)=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&0\\zte^{zt}\int_1^z\xi^{-2}e^{-t\xi}d\xi&0\end{pmatrix}\right]\begin{pmatrix}z&0\\0&z^2e^{tz}\end{pmatrix}.$$

$Y_{hol}(z,t)$ is holomorphic at $t=0$ and $Y_{hol}(z,0)=\mathring{Y}(z)$, but *it does not have the canonical asymptotic behaviour, for $z\to\infty$*, of type (1.4) below. Introducing the exponential integral

$$\mathrm{Ei}\,(z):=\int_z^\infty\frac{e^{-\zeta}}{\zeta}d\zeta=\int_1^\infty\frac{e^{-z\xi}}{\xi}d\xi,$$

we can rewrite

$$Y_{hol}(z,t)=\ Y_{can}(z,t)\ C(t),$$

where

$$Y_{can}(z,t):=\mathcal{G}(z,t)\ \begin{pmatrix}z&0\\0&z^2e^{tz}\end{pmatrix},$$

$$\mathcal{G}(z,t):=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&0\\\mathcal{G}_{12}(z,t)&0\end{pmatrix},\qquad \mathcal{G}_{12}(z,t):=t\Big[\ tz\ e^{tz}\mathrm{Ei}(tz)-1\Big],$$

$$C(t):=\begin{pmatrix}1&0\\t\left[e^{-t}-t\mathrm{Ei}(t)\right]&1\end{pmatrix},$$

Notice that $C(t)$ is a connection matrix, not holomorphic at $t=0$, due to the logarithmic branching of $\mathrm{Ei}(t)$. The well-known asymptotic behaviour of the exponential integral [66] yields

$$\mathcal{G}_{12}(z,t)\sim\sum_{n=1}^\infty\frac{(-1)^kk!}{t^{k-1}}\frac{1}{z^k},\quad z\to\infty,\quad -\frac{3\pi}{2}<\arg(zt)<\frac{3\pi}{2}.\tag{1.3}$$

Thus, the fundamental solution $Y_{can}(z,t)$ has the following canonical asymptotic behaviour

$$\mathcal{G}(z,t)\sim I+\sum_{k=1}^\infty F_k(t)z^{-k},\tag{1.4}$$

with matrix-coefficients $F_k(t)$ inherited from (1.3), as $z\to\infty$ in the sector $-\frac{3\pi}{2}<\arg(zt)<\frac{3\pi}{2}$. We point out two important features:

- $\Delta=\{t=0\}$ is a *branching locus* for $Y_{can}(z,t)$, because $\mathrm{Ei}(tz)$ has a logarithmic branching at $t=0$.

[2] There is actually the following one parameter family of solutions with canonical asymptotics

$$\mathring{Y}_a(z)=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{1}{z}\begin{pmatrix}1&a\\0&1\end{pmatrix}\right]\begin{pmatrix}z&0\\0&z^2\end{pmatrix},\qquad a\in\mathbb{C}.$$

- $F_1(t)$ is holomorphic at $t=0$, but the other $F_k(t)$, for any $k\geq 2$, *have a pole* of order t^{-k+1}. Actually, the system violates the necessary and sufficient conditions of analyticity for formal solutions, as established in [16] (with reference to [16], the condition that $(A_1)_{12}$ and $(A_1)_{21}$ vanish at $t=0$ implies that $F_1(t)$ must be holomorphic at $t=0$, but all other conditions for the F_k's are violated).

This special example also has some non-typical features. First, $Y_{can}(z,t)$ is multi valued around $t=0$, but it has limit

$$Y_{can}(z,t)\longrightarrow \mathring{Y}(z),$$

for $t\to 0$ in any sector of finite opening angle and vertex $t=0$ in the universal covering of the punctured t-plane. Moreover, the t-dependent Stokes matrices are holomorphic also at $t=0$. Indeed, consider three successive canonical sectors $\mathcal{S}_{-1}:=S(-3\pi/2,\pi/2)$, $\mathcal{S}_0:=S(-\pi/2,3\pi/2)$ and $\mathcal{S}_1:=S(\pi/2,5\pi/2)$. Their intersection does not contain Stokes rays $\Re(tz)=0$ (namely, $\arg(tz)=\frac{\pi}{2}+m\pi$, $m\in\mathbb{Z}$). There are three fundamental solutions with the canonical asymptotics (1.3)-(1.4) in these sectors. They are respectively $Y_{-1}(z,t):=Y_{can}(z,t)$, $Y_0(z,t):=Y_{can}(z,t)$ and $Y_1(z,t)=Y_{can}(ze^{-2\pi i},t)$. Recalling that $\mathrm{Ei}(z)=\mathrm{Ei}(ze^{-2\pi i})-2\pi i$, we find that the Stokes matrices $\mathbb{S}_{-1}$ and $\mathbb{S}_0$, defined by

$$Y_0=\ Y_{-1}\,\mathbb{S}_{-1},\qquad Y_1=\ Y_0\,\mathbb{S}_0,$$

are respectively

$$S_{-1}=I,\qquad S_0(t)=\begin{pmatrix}1 & 0\\ 2\pi i t^2 & 1\end{pmatrix}.$$

Thus, the second non-typical feature is that, when $t\to 0$, then $S_0(t)\longrightarrow I$, the trivial Stokes matrix of $\mathring{Y}(z)$. $\Box$

Example 1.2 (*Whittaker Isomonodromic System*)**.** Consider an isomonodromic 2×2 system

$$\frac{dY}{dz}=\left[\begin{pmatrix}u_1 & 0\\ 0 & u_2\end{pmatrix}+\frac{A_1(u)}{z}\right]Y. \tag{1.5}$$

Here $\Delta=\{u_1=u_2\}$. This example shows three important facts holding for an isomonodromic system:

- the coalescence locus Δ is a *branching locus* for both $A_1(t)$ and the fundamental solutions (this is well known, see [51]);
- if the entries of the Stokes matrices, with indices corresponding to the coalescing eigenvalues, vanish a Δ, then Δ is not a branching locus. This fact will be the content of Propositions 1.1 and 1.2 below, which anticipate the general result of Theorem 2.2 in Section 2;
- if $A_1(t)$ is holomorphic in a domain containing Δ, and if its entries, with indices corresponding to the coalescing eigenvalues, vanish at Δ, then the fundamental solutions are holomorphic also at Δ and the entries (as above) of the Stokes matrices vanish. See Proposition 1.3 below, which anticipates the general result of Theorem 2.1 in Section 2.

The Jimbo-Miwa-Ueno [38] isomonodromy deformation equations can be written in a small open domain where (u_1,u_2) is an admissible deformation (see Definitions 2.1 and 2.2 below). Explicitly, they are

$$\frac{\partial A_1}{\partial u_a}=[[F_1,E_k],A_1]\,,$$

where $(E_k)_{ab} = \delta_{ka}\delta_{kb}$, $(F_1)_{ab} = (A_1)_{ab}/(u_b - u_a)$, $a \neq b$, and $(F_1)_{aa} = -\sum_{b\neq a}(A_1)_{ab}(F_1)_{ba}$ (here $a, b \in \{1, 2\}$). More explicitly, we find

$$\frac{\partial (A_1)_{11}}{\partial u_1} = \frac{\partial (A_1)_{11}}{\partial u_2} = \frac{\partial (A_1)_{22}}{\partial u_1} = \frac{\partial (A_1)_{22}}{\partial u_2} = 0$$

which implies that $(A_1)_{11}$ and $(A_1)_{22}$ are constant. Let us define constants a and b by $a := (A_1)_{11}$, $a - b := (A_1)_{22}$. Then, the remaining isomonodromy deformation equations are

$$\frac{\partial (A_1)_{12}}{\partial u_1} = \frac{b\ (A_1)_{12}}{u_2 - u_1}, \quad \frac{\partial (A_1)_{12}}{\partial u_2} = \frac{b\ (A_1)_{12}}{u_1 - u_2},$$
$$\frac{\partial (A_1)_{21}}{\partial u_1} = \frac{b\ (A_1)_{12}}{u_1 - u_2}, \quad \frac{\partial (A_1)_{21}}{\partial u_2} = \frac{b\ (A_1)_{12}}{u_2 - u_1}$$

which imply that

$$(A_1)_{12} = c(u_1 - u_2)^{-b}, \qquad (A_1)_{21} = d(u_1 - u_2)^{b}, \qquad c,\ d = \text{ constants.}$$

We conclude that the system (1.5) is isomonodromic if and only if

$$A_1(u) = \begin{pmatrix} a & c(u_1 - u_2)^{-b} \\ d(u_1 - u_2)^b & a - b \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}. \tag{1.6}$$

Therefore, the following holds.

Fact: for generic values of the parameters, A_1 is *singular* (for c and $d \neq 0$) and *has a branching locus at* $\Delta = \{u_1 = u_2\}$ (for $b \notin \mathbb{Z}$).

We compute the fundamental solutions and the Stokes matrices of system (1.5) with residue matrix (1.6). For $u_1 \neq u_2$, there exists the unique "canonical" formal solution

$$Y_F(z, u) = \left(I + \frac{F_1}{z} + \frac{F_2}{z^2} + \cdots\right) z^{\mathrm{diag}(A_1)} \begin{pmatrix} e^{u_1 z} & 0 \\ 0 & e^{u_2 z} \end{pmatrix},$$

with matrix coefficients $F_k = F_k(u)$, uniquely determined by the equation. Actual solutions can be easily computed solving the system (1.5) for a vector solution $\begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}$. This gives

$$\frac{dy_1}{dz} = \left(u_1 + \frac{a}{z}\right) y_1 + \frac{c}{(u_1 - u_2)^b} \frac{y_2}{z}, \qquad \frac{dy_2}{dz} = \left(u_2 + \frac{a - b}{z}\right) y_2 + d(u_1 - u_2)^b \frac{y_1}{z} \tag{1.7}$$

By elimination of y_2, we obtain a second order ODE in y_1. With the substitutions

$$y_1(z) = e^{\frac{1}{2}(u_1 + u_2)z} z^{a - \frac{b+1}{2}} y(z), \ y(z) = w(x), \ x = z(u_1 - u_2),$$

the equation for y_1 becomes the following Whittaker equation:

$$\frac{d^2 w}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right) w = 0,$$
$$\mu^2 := \frac{b^2 + 4cd}{4}, \ \kappa := -\frac{1 + b}{2}.$$

For later convenience, let us set

$$\sigma_+ := \frac{1}{2} + \kappa + \mu, \ \sigma_- := \frac{1}{2} + \kappa - \mu.$$

The u-independent eigenvalues of A_1 are $a + \sigma_\pm$. We are ready to obtain fundamental solutions $Y(z, u)$ admitting $Y_F(z, u)$ as asymptotic representation for $z \to \infty$, namely satisfying the asymptotic condition

$$Y(z, u) = \left(I + \frac{F_1}{z} + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^a e^{u_1 z} & 0 \\ 0 & z^{a-b} e^{u_2 z} \end{pmatrix}, \quad z \to \infty, \tag{1.8}$$

in sector of angular opening greater than π. The result is contained in the following

Lemma 1.1. *Let $u_1-u_2\neq 0$. There are three fundamental matrix solution $Y_{-1}(z,u)$, $Y_0(z,u)$, $Y_1(z,u)$ with the asymptotic behaviour (1.8) respectively for $z(u_1-u_2)$ lying in the successive overlapping sectors*

$$\mathcal{S}_{-1}:=S\Big(-\frac{5\pi}{2},-\frac{\pi}{2}\Big),\quad \mathcal{S}_0:=S\Big(-\frac{3\pi}{2},\frac{\pi}{2}\Big),\quad \mathcal{S}_1:=S\Big(-\frac{\pi}{2},\frac{3\pi}{2}\Big).$$

They are connected by Stokes matrices $\mathbb{S}_{-1}$, $\mathbb{S}_0$ *defined by*

$$Y_0(z)=Y_{-1}(z)\mathbb{S}_{-1},\quad Y_1(z)=Y_0(z)\mathbb{S}_0,\qquad \mathbb{S}_{-1}=\begin{pmatrix}1 & s_{-1}\\ 0 & 1\end{pmatrix},\quad \mathbb{S}_0=\begin{pmatrix}1 & 0\\ s_0 & 1\end{pmatrix},$$

where

$$\begin{aligned} s_0 &= \frac{2\pi i}{c\;\Gamma\left(\frac{1}{2}+\kappa+\mu\right)\Gamma\left(\frac{1}{2}+\kappa-\mu\right)}=\frac{2\pi i}{c\;\Gamma\left(\frac{\sqrt{b^2+4cd}}{2}-\frac{b}{2}\right)\;\Gamma\left(-\frac{\sqrt{b^2+4cd}}{2}-\frac{b}{2}\right)}\\ &=\frac{2\pi i}{c\;\Gamma\left(\sigma_+\right)\Gamma\left(\sigma_-\right)}\end{aligned}$$

$$\begin{aligned} s_{-1} &= \frac{2\pi i\;c\;e^{-2\pi i\kappa}}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}=\frac{-2\pi i\;c\;e^{i\pi b}}{\Gamma\left(\frac{\sqrt{b^2+4cd}}{2}+1+\frac{b}{2}\right)\;\Gamma\left(-\frac{\sqrt{b^2+4cd}}{2}+1+\frac{b}{2}\right)}\\ &=\frac{2\pi i\;c\;e^{-2\pi i\kappa}}{\Gamma\left(1-\sigma_+\right)\Gamma\left(1-\sigma_-\right)}.\end{aligned}$$

Proof: Taking into account that $(F_1)_{12}=(A_1)_{12}/(u_2-u_1)=-c(u_1-u_2)^{-b-1}$, we must have

$$\text{first row of } Y(z,u)=\left[z^a e^{u_1 z}\left(1+O\left(\frac{1}{z}\right)\right),\;-\frac{c}{(u_1-u_2)^{b+1}}z^{a-b-1}e^{u_2 z}\left(1+O\left(\frac{1}{z}\right)\right)\right].\tag{1.9}$$

Since the Whittaker functions have the asymptotic behaviour

$$W_{\kappa,\mu}(x)=x^{\kappa}e^{-x/2}\left(1+O\left(\frac{1}{x}\right)\right),\quad -\frac{3\pi}{2}<\arg x<\frac{3\pi}{2},\tag{1.10}$$

$$W_{-\kappa,\mu}(-x)=(-x)^{-\kappa}e^{x/2}\left(1+O\left(\frac{1}{x}\right)\right),\quad -\frac{3\pi}{2}<\arg(-x)<\frac{3\pi}{2},\tag{1.11}$$

it follows that we can choose the fundamental solutions $Y_{-1}(z,u)$, $Y_0(z,u)$, $Y_1(z,u)$ as follows:

$$\text{first row of } Y_{-1}(z)=\left[f(z,u)e^{-i\pi\frac{b+1}{2}}W_{-\kappa,\mu}(e^{i\pi}x),\;-cf(z,u)e^{i\pi(b+1)}W_{\kappa,\mu}(e^{2\pi i}x)\right],$$

$$\text{first row of } Y_{0}(z)=\left[f(z,u)e^{-i\pi\frac{b+1}{2}}W_{-\kappa,\mu}(e^{i\pi}x),\;-cf(z,u)W_{\kappa,\mu}(x)\right],$$

$$\text{first row of } Y_{1}(z)=\left[f(z,u)e^{i\pi\frac{b+1}{2}}W_{-\kappa,\mu}(e^{-i\pi}x),\;-cf(z,u)W_{\kappa,\mu}(x)\right],$$

where

$$f(z,u):=\frac{e^{\frac{1}{2}(u_1+u_2)z}z^{a-\frac{b+1}{2}}}{(u_1-u_2)^{\frac{b+1}{2}}}\equiv e^{\frac{1}{2}(u_1+u_2)z}\;z^a\;\Big(z(u_1-u_2)\Big)^{\kappa}.\tag{1.12}$$

Then, $Y_{-1}(z,u)$, $Y_0(z,u)$, $Y_1(z,u)$ have the canonical asymptotic behaviours (1.9), for $z\to\infty$ and $u_1-u_2\neq 0$, when $z(u_1-u_2)$ is respectively in the successive sectors $\mathcal{S}_{-1}$, $\mathcal{S}_0$ and $\mathcal{S}_1$. Notice that the intersections $\mathcal{S}_{-1}\cap\mathcal{S}_0$ and $\mathcal{S}_0\cap\mathcal{S}_1$ do not contain the Stokes rays $\Re(z(u_1-u_2))=0$.

The non-trivial entries s_{-1} and s_0 of the Stokes matrices $\mathbb{S}_{-1}$ and $\mathbb{S}_0$, as defined in the statement of the Lemma (see also Remark 2.1 below), can be computed keeping into account the explicit form of Y_1 and Y_0. We find

$$W_{\kappa,\mu}(x)=\frac{1}{cs_0}\Big(e^{i\pi\kappa}W_{-\kappa,\mu}(e^{i\pi}x)-e^{-i\pi\kappa}W_{-\kappa,\mu}(e^{-i\pi}x)\Big).$$

Now, the Whittaker functions satisfy the cyclic relation

$$W_{\kappa,\mu}(x)=\frac{\Gamma\left(\frac{1}{2}+\mu+\kappa\right)\Gamma\left(\frac{1}{2}-\mu+\kappa\right)}{2\pi i}\Big(e^{i\pi\kappa}W_{-\kappa,\mu}(e^{i\pi}x)-e^{-i\pi\kappa}W_{-\kappa,\mu}(e^{-i\pi}x)\Big).\tag{1.13}$$

This yields s_0 as stated in the Lemma. Keeping into account the explicit form of Y_0 and Y_{-1}, we find

$$W_{-\kappa,\mu}(e^{i\pi}x)=\frac{ce^{-2\pi i\kappa}}{s_{-1}}\Big(e^{-i\pi\kappa}W_{\kappa,\mu}(e^{2\pi i}x)-e^{i\pi\kappa}W_{\kappa,\mu}(x)\Big).$$

The above must be compared with the analogue of (1.13), namely

$$W_{-\kappa,\mu}(e^{i\pi}x)=\frac{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}{2\pi i}\Big(e^{-i\pi\kappa}W_{\kappa,\mu}(e^{2\pi i}x)-e^{i\pi\kappa}W_{\kappa,\mu}(x)\Big).$$

This yields s_{-1} as in the statement of the Lemma. □

Simple computations with the explicit expressions of s_{-1} and s_0 yield the following

Lemma 1.2. *Let $n\geq 1$ and $m\geq 1$ be integers. The Stokes matrices $\mathbb{S}_0$ and $\mathbb{S}_1$ are trivial, namely $s_0=s_{-1}=0$, if and only if one of the following conditions is satisfied*

1) $c=d=0$ *and* $b\in\mathbb{C}$,
2) $cd=mn,\ b=n-m$,
3) *either* $d=0$ *and* $b=-m$, *or* $c=0$ *and* $b=n$.

Now, we look back at the expression (1.6) for A_1 and immediately conclude from Lemma 1.2 that the the following proposition holds.

Proposition 1.1. *If $s_{-1}=s_0=0$, then $A_1(u)$ as in (1.6) is single valued for a loop $(u_1-u_2)\mapsto(u_1-u_2)e^{2\pi i}$ around the coalescence locus $u_1=u_2$. Namely, Δ is not a branching locus for A_1.*

The same statement holds for the fundamental solutions as well.

Proposition 1.2. *If $s_{-1}=s_0=0$, then the fundamental solutions $Y_r(z,u)$, $r=-1,0,1$ are single valued for a loop $(u_1-u_2)\mapsto(u_1-u_2)e^{2\pi i}$ around the coalescence locus $u_1=u_2$. Namely, Δ is not a branching locus.*

It is important to notice that the condition $s_0=s_{-1}=0$ alone does not imply that the fundamental matrix solutions are holomorphic at $u_1-u_2=0$. For example, consider the case $cd=2$, $b=1$ in Lemma 1.2. The fundamental solution, having canonical asymptotic behaviour for every value of $\arg z$ (indeed $\mathbb{S}_0=\mathbb{S}_{-1}=I$) is

$$Y(z)=\left[I+\begin{pmatrix}-\frac{2}{x}+\frac{2}{x^2} & -\frac{2}{d\,(u_1-u_2)^2}\frac{1}{z}\\[2ex] \frac{d}{z}-\frac{2d}{(u_1-u_2)z^2} & \frac{2}{x}\end{pmatrix}\right]\begin{pmatrix}z^a e^{u_1 z} & 0\\[2ex] 0 & z^{a-1}e^{u_2 z}\end{pmatrix}$$

$$=\begin{pmatrix}z^a & 0\\[2ex] 0 & z^{a-1}\end{pmatrix}\cdot\begin{pmatrix}1-\frac{2}{x}+\frac{2}{x^2} & -\frac{2}{d\,x^2}\\[2ex] d-\frac{2d}{x} & 1+\frac{2}{x}\end{pmatrix}\begin{pmatrix}e^{u_1 z} & 0\\[2ex] 0 & e^{u_2 z}\end{pmatrix},\quad x=(u_1-u_2)z,$$

which has a pole at $\Delta=\{u_1-u_2=0\}$.

Moreover, the following holds:

Proposition 1.3. *If $A_1(u)$ is holomorphic at Δ and both $(A_1)_{12}$ and $(A_1)_{21}$ vanish as $u_1 - u_2 \to 0$, then the $Y_r(z,u)$'s are single valued in $u_1 - u_2$ and holomorphic at Δ. The Stokes matrices have entries $s_{-1} = s_0 = 0$.*

The remaining part of the example is devoted to the proof of the above propositions.

Proof of Proposition 1.2: In case 1) of Lemma 1.2 we find $Y_{-1} = Y_0 = Y_1 = \mathrm{diag}\left(z^a e^{u_1 z}, z^{b-a} e^{u_2 z}\right)$, so the proposition is proved. In case 3), the system is integrable by quadratures and variation of parameters, so that it is easy to verify that the fundamental solution $Y_{-1} = Y_0 = Y_1$ with canonical form (1.8) is holomorphic[3] in the whole plane (u_1, u_2). So, we only need to consider case 2). This case can be written as follows:

$$\left\{ \begin{array}{c} \sigma_+ = -n \\ \sigma_- = m \end{array} \right. , \quad \text{or} \quad \left\{ \begin{array}{c} \sigma_+ = m \\ \sigma_- = -n \end{array} \right. . \tag{1.14}$$

Thus, we have

$$\left\{ \begin{array}{c} \mu = -\frac{1}{2}(m+n) \\ \\ \kappa = \frac{1}{2}(m-n) - \frac{1}{2} \end{array} \right. , \left\{ \begin{array}{c} \mu = \frac{1}{2}(m+n) \\ \\ \kappa = \frac{1}{2}(m-n) - \frac{1}{2} \end{array} \right. \implies \mu \neq 0.$$

In order to study the multi-valuedness of the fundamental solutions, we recall that the Whittaker equation is transformed into the confluent hypergeometric equation

$$x\frac{d^2v}{dx^2} + (\gamma - x)\frac{dv}{dx} - \alpha v = 0, \qquad \alpha = \frac{1}{2} + \mu - \kappa \equiv 1 - \sigma_-, \quad \gamma = 1 + 2\mu.$$

by the transformation

$$w = e^{-\frac{x}{2}} x^{\frac{\gamma}{2}} \; v = \; e^{-\frac{x}{2}} x^{\frac{1}{2}+\mu} \; v.$$

In particular, if we introduce the Tricomi function Ψ (see page 53 of [52]), which has the asymptotic behaviour,

$$\Psi(\alpha, \gamma; x) = x^{-\alpha}\left(1 + O\left(\frac{1}{x}\right)\right) = x^{\kappa - \mu - \frac{1}{2}}\left(1 + O\left(\frac{1}{x}\right)\right), \; x \to \infty, \; x \in S(-3\pi/2, 3\pi/2),$$

then the hypergeometric equation has two independent solutions

$$\Psi(\alpha, \gamma; x), \qquad e^x \Psi(\gamma - \alpha, \gamma; -x).$$

Keeping into account the asymptotic expansions, we see that

$$W_{\kappa,\mu}(x) = e^{-\frac{x}{2}} \; x^{\frac{1}{2}+\mu} \; \Psi(\alpha, \gamma; x),$$
$$W_{-\kappa,\mu}(-x) = e^{\frac{x}{2}} \; (-x)^{\frac{1}{2}+\mu} \; \Psi(\gamma - \alpha, \gamma; -x).$$

In order to study the multi-valuedness in $(u_1 - u_2)$, we express the Tricomi functions on a basis of fundamental solutions at $x = 0$. In the non resonant case

$$\gamma \notin \mathbb{Z} \iff 2\mu \notin \mathbb{Z},$$

which occur when Lemma 1.2 does not apply, we can express them as a linear combinations of

$$\Phi(\alpha, \gamma; x) := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!(\gamma)_n} z^n, \qquad x^{1-\gamma}\Phi(\alpha - \gamma + 1, 2 - \gamma; x),$$

[3]Actually, the factor $\left(I + \frac{F_1}{z} + \cdots\right)$ of (1.8) is a polynomial in u_1 and u_2.

where $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$. Explicitly, we find

$$W_{\kappa,\mu}(x) = e^{-\frac{x}{2}} x^{\frac{1}{2}+\mu}\Psi(\alpha,\gamma;x)$$

$$= e^{-\frac{x}{2}} x^{\frac{1}{2}+\mu}\left[\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\Phi(\alpha,\gamma;x)+\frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}x^{1-\gamma}\Phi\left(\alpha-\gamma+1,2-\gamma;x\right)\right]$$

$$= e^{-\frac{x}{2}}\left[x^{\frac{1}{2}+\mu}\left(\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\kappa-\mu\right)}\Phi\left(\frac{1}{2}-\kappa+\mu,1+2\mu;x\right)\right)+\right.$$

$$\left.+x^{\frac{1}{2}-\mu}\left(\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}-\kappa+\mu\right)}\Phi\left(\frac{1}{2}-\kappa-\mu,1-2\mu;x\right)\right)\right].$$

and

$$W_{-\kappa,\mu}(-x) = e^{\frac{x}{2}}(-x)^{\frac{1}{2}+\mu}\Psi(\gamma-\alpha,\gamma;-x)$$

$$= e^{\frac{x}{2}}(-x)^{\frac{1}{2}+\mu}\left[\frac{\Gamma(1-\gamma)}{\Gamma(1-\alpha)}\Phi(\gamma-\alpha,\gamma;-x)+\frac{\Gamma(\gamma-1)}{\Gamma(\gamma-\alpha)}(-x)^{1-\gamma}\Phi\left(1-\alpha,2-\gamma;-x\right)\right]$$

$$= e^{\frac{x}{2}}\left[(-x)^{\frac{1}{2}+\mu}\left(\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}+\kappa-\mu\right)}\Phi\left(\frac{1}{2}+\kappa+\mu,1+2\mu;-x\right)\right)+\right.$$

$$\left.+(-x)^{\frac{1}{2}-\mu}\left(\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\kappa+\mu\right)}\Phi\left(\frac{1}{2}+\kappa-\mu,1-2\mu;x\right)\right)\right].$$

Keeping (1.12) into account, we see that the monodromy of $Y_r(z,u)$, $r=-1,0,1$ at $z=0$ and $z(u_1-u_2)=0$ depends on the factors

$$z^a\Big(z(u_1-u_2)\Big)^{\frac{1}{2}+\kappa\pm\mu} = z^a\Big(z(u_1-u_2)\Big)^{\sigma_\pm}.$$

The case 2) of Lemma 1.2 corresponds to the resonant cases $2\mu\in\mathbb{Z}$, namely $\gamma\in\mathbb{Z}$ (but $\mu\neq 0$, $\gamma\neq 1$). The two systems (1.14) are respectively

$$\left\{\begin{array}{c}\gamma = 1-(n+m)\\ \alpha=-m+1\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{c}\gamma = 1+(n+m)\\ \alpha=n+1\end{array}\right.,$$

or equivalently

$$\left\{\begin{array}{c}\gamma = 0,-1,-2,-3,....\in\mathbb{Z}_{\leq 0}\\ \alpha=\gamma,\gamma+1,...,-1,0\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{c}\gamma = 2,3,4,....\in\mathbb{Z}_{\geq 2}\\ \alpha=1,2,...,\gamma-1\end{array}\right. \tag{1.15}$$

Let

$$\varphi(\alpha,\gamma;x) := \sum_{s=0}^{-\alpha}\frac{(\alpha)_s\ x^s}{s!(\gamma)_s}.$$

In the first case of (1.15) above, corresponding to case 1° at page 49 in [52], the hypergemetric equation has independent solutions

$$\varphi(\alpha,\gamma;x),\qquad x^{1-\gamma}\Phi(\alpha-\gamma+1,2-\gamma;x).$$

In the second case, corresponding to case 2° at page 49 in [52], independent solutions are

$$\Phi(\alpha,\gamma;x),\qquad x^{1-\gamma}\varphi(\alpha-\gamma+1,2-\gamma;x).$$

There are suitable constants $C_1(\alpha,\gamma)$, $C_2(\alpha,\gamma)$ (see [52]) such that, in the first case we have

$$
\begin{aligned}
W_{\kappa,\mu}(x) \quad &= e^{-\frac{x}{2}}x^{\frac{1}{2}+\mu}\Psi(\alpha,\gamma;x)\\
&= e^{-\frac{x}{2}}x^{\frac{1}{2}+\mu}\left[C_1\ \varphi(\alpha,\gamma;x)+C_1\ x^{1-\gamma}\Phi(\alpha-\gamma+1,2-\gamma;x)\right]\\
&= e^{-\frac{x}{2}}\left[C_1\ x^{\frac{1}{2}+\mu}\varphi(\alpha,\gamma;x)+C_2\ x^{\frac{1}{2}-\mu}\Phi(\alpha-\gamma+1,2-\gamma;x)\right].
\end{aligned}
$$

In the second case we have

$$
W_{\kappa,\mu}(x)=e^{-\frac{x}{2}}\left[C_1\ x^{\frac{1}{2}+\mu}\Phi(\alpha,\gamma;x)+C_2\ x^{\frac{1}{2}-\mu}\varphi(\alpha-\gamma+1,2-\gamma;x)\right].
$$

Similar expressions hold for

$$
W_{-\kappa,\mu}(-x)=e^{\frac{x}{2}}(-x)^{\frac{1}{2}+\mu}\Psi(\gamma-\alpha,\gamma;-x).
$$

The monodromy of the above Whittaker functions at $x=0$ depends on $x^{\frac{1}{2}\pm\mu}$. Keeping (1.12) into account, we see again that the monodromy of $Y_r(z,u)$, $r=-1,0,1$ at $z=0$ and $z(u_1-u_2)=0$ depends on the factors

$$
z^a\Big(z(u_1-u_2)\Big)^{\frac{1}{2}+\kappa\pm\mu}=z^a\Big(z(u_1-u_2)\Big)^{\sigma_\pm}.
$$

Since $\sigma_\pm$ are integers, the fundamental matrix solutions $Y_{-1}(z,u)$ $Y_0(z,u)$, $Y_1(z,u)$ with canonical asymptotics are *single valued* around $u_1-u_2=0$. A similar computation holds for the second row of $Y_r(z,u)$ (just eliminate $y_1(z)$ in system (1.7) and proceed as above for $y_2(z)$). $\square$

Proof of Proposition 1.3: We observe that a necessary condition for $A_1(u)$ to be holomoprhic at $u_1-u_2=0$ is that b is integer. The condition that both $(A_1)_{12}$ and $(A_1)_{21}$ vanish as $u_1-u_2\to 0$ exactly corresponds to case 3) of Lemma 1.2. Thus, (σ_+,σ_-) is either $(0,m)$ or $(m,0)$, $m\geq 1$. From the expressions used in the proof of Proposition 1.2 for the behaviour at $x=0$ of the fundamental solutions, and the the explicit formulae of s_0 and s_{-1}, we conclude the proof. $\square$

2. Some of the Main Results of [16]

For the sake of simplicity, we will not describe all the results of [16]. Rather, we will restrict to the system (1.1), mainly in the isomonodromic case. In this framework, it is known [38] that we can take the eigenvalues of $\Lambda(t)$ to be the deformation parameters. Hence, we can assume that the eigenvalues $u_1(t),...,u_n(t)$ are linear in t.

In order to perform a local analysis in a neighbourhood of a coalescence locus, we restrict to a polydisk containing Δ, defined by

$$
\mathcal{U}_{\epsilon_0}(0):=\big\{t\in\mathbb{C}^m\ \text{ such that }\ |t|\leq\epsilon_0\big\},\qquad |t|:=\max_{1\leq i\leq m}|t_i|
$$

for suitable $\epsilon_0>0$, being $t=0$ a point of Δ. Therefore,

$$
\Lambda(t)=\Lambda(0)+\operatorname{diag}(t_1,...,t_n),
$$

where $\Lambda(0)$ has distinct eigenvalues λ_1, ..., λ_s, $s<n$, of multiplicities p_1, ..., p_s respectively, so that $p_1+\cdots+p_s=n$. Explicitly,

$$
\Lambda(0)=\lambda_1 I_{p_1}\oplus\cdots\oplus\lambda_s I_{p_s},\qquad \lambda_i\neq\lambda_j,\quad 1\leq i\neq j\leq s<n, \tag{2.1}
$$

where I_{p_j} stands for the $p_j\times p_j$ identity matrix. Another way to write is

$$
u_a(t)=u_a(0)+t_a,\quad 1\leq a\leq n. \tag{2.2}
$$

$A_1(t)$ is assumed to be holomorphic in $\mathcal{U}_{\epsilon_0}(0)$.

When Δ is not empty, the dependence on t of fundamental solutions of (1.1) near $z=\infty$ is quite delicate. If $t \notin \Delta$, then there is a unique formal solution (see [35]),

$$Y_F(z,t) := \Big(I + \sum_{k=1}^{\infty} F_k(t) z^{-k}\Big) z^{B_1(t)} e^{\Lambda(t)z}, \qquad B_1(t) := \mathrm{diag}(A_1(t)), \tag{2.3}$$

where the matrices $F_k(t)$ are uniquely determined by the equation and are holomorphic on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$.

In order to find actual solutions, we need the concept of admissible deformation. Let $t=0$ and consider the *Stokes rays* associated with the matrix $\Lambda(0)$, namely rays in the universal covering of the z-punctured plane $\mathbb{C}\backslash\{0\}$, denoted by $\mathcal{R}$, defined by the condition that

$$\Re e[(u_a(0)-u_b(0))z] = 0,$$

with $u_a(0) \neq u_b(0)$ $(1 \leq a \neq b \leq n)$. Then, we consider an *admissible ray*, namely a ray in $\mathcal{R}$, with a certain direction $\widetilde{\tau}$, that does not contain any of the Stokes rays above. Finally, let us take the Stokes rays

$$\{z \in \mathcal{R} \mid \Re e[(u_a(t)-u_b(t))z] = 0\},$$

associated with $\Lambda(t)$, $t \notin \Delta$. As t varies in the polydisc, these Stokes rays rotate, and for some values of t, actually for t along a certain locus that we call $X(\widetilde{\tau})$, they come to cross the admissible ray $\arg z = \widetilde{\tau}$. If we look at the polydisc, we see that as long as t does not lie on Δ or on $X(\widetilde{\tau})$, then Stokes rays may rotate with varying t, but behave nicely, namely they neither cross the admissible ray nor they do disappear (as it may happen for $t \in \Delta$). Thus, the good place to stay within the polydisc is $\mathcal{U}_{\epsilon_0}(0)\backslash\,(\Delta \cup X(\widetilde{\tau}))$. With some careful study (see [16]) we can prove that $\Delta \cup X(\widetilde{\tau})$ is a union of real hyperplanes which disconnect $\mathcal{U}_{\epsilon_0}(0)$. Every connected component is actually simply connected, and homeomorphic to a ball in $\mathbb{R}^{2n}$. Thus, it is a cell in the topological precise sense, so in [16] we have called it *a $\widetilde{\tau}$-cell*.

Definition 2.1. *The deformation of the linear system (1.1) is called an* **admissible deformation in** B *if t varies in a domain B contained in the polydisc $\mathcal{U}_{\epsilon_0}(0)$, enjoying the following property: B is a subset of a $\widetilde{\tau}$-cell and also its closure $\overline{B}$ is properly contained in the cell.*[4] *For simplicity, we will just say that t is an admissible deformation.*

By definition, an admissible deformation means that as long as t varies within B, then *no Stokes rays of $\Lambda(t)$ cross the admissible ray* of direction $\widetilde{\tau}$.

If t belongs to a domain B as above, then we prove in [16] that there is a family of actual fundamental solutions $Y_r(z,t)$, labelled by $r \in \mathbb{Z}$, uniquely determined by the canonical asymptotic representation

$$Y_r(z,t) \sim Y_F(z,t),$$

for $z \to \infty$ in suitable sectors $\mathcal{S}_r(B)$ of the universal covering $\mathcal{R}$ of $\mathbb{C}\backslash\{0\}$. Each $Y_r(z,t)$ is holomorphic within $\mathcal{R}$ for large $|z|$, and in $t \in B$. The asymptotic series $I + \sum_{k=1}^{\infty} F_k(t) z^{-k}$ is uniform in $\overline{B}$. The sectors $\mathcal{S}_r(B)$ will be constructed in the technical Appendix. Here it suffices to say that they have angular central opening strictly greater than π, that $\mathcal{S}_r(B)$ and $\mathcal{S}_{r+1}(B)$ overlap, but their intersection does not contain any of the Stokes rays associated with $\Lambda(t)$, $t \in \overline{B}$.

In [16], we extend the deformation theory to non-admissible deformations beyond B, to the whole $\mathcal{U}_{\epsilon_0}(0)$. We prove – for general deformations that are not necessarily isomonodromic – that the t-analytic continuation of $Y_r(z,t)$ exists at least for t in the $\widetilde{\tau}$-cell containing B. Now, if we assume that the t-analytic continuation extends beyond the $\widetilde{\tau}$-cell, then the delicate points – some already pointed out in Examples 1.1 and 1.2 – emerge, as follows.

[4] The definition of *admissible deformation* of a linear system is in accordance with the definition given in [25].

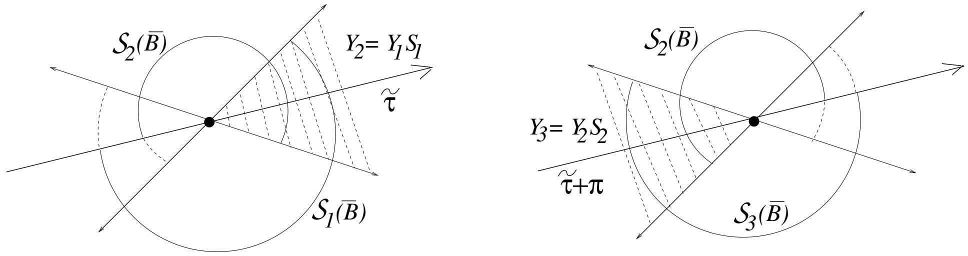

FIGURE 1. Stokes phenomenon of formula (2.4). In the left figure is represented the sheet of the universal covering $\widetilde{\tau}-\pi < \arg z < \widetilde{\tau}+\pi$ containing $\mathcal{S}_1(B)\cap\mathcal{S}_2(B)$, and in the right figure the sheet $\widetilde{\tau} < \arg z < \widetilde{\tau}+2\pi$ containing $\mathcal{S}_2(B)\cap\mathcal{S}_3(B)$. The rays $\arg z = \widetilde{\tau}$ and $\widetilde{\tau}+\pi$ (and then $\widetilde{\tau}+k\pi$ for any $k\in\mathbb{Z}$) are *admissible rays*, namely such $\Re e\Big[(u_a(0)-u_b(0))z\Big]\neq 0$ along these rays, for any $u_a(0)\neq u_b(0)$.

- While the expressions $\Re e\,[(u_a(t)-u_b(t))z]$, $1\leq a\neq b\leq n$, have constant sign in B (by definition, because B is in a cell!), they may vanish for values of t sufficiently far away from B, precisely when t crosses $X(\widetilde{\tau})$ and leaves the cell of B, which means that some Stokes ray $\Re e\,[(u_a(t)-u_b(t))z]=0$, associated with $\Lambda(t)$, cross the admissible direction $\widetilde{\tau}$. Hence, the asymptotic representation $Y_r(z,t)\sim Y_F(z,t)$ for $z\to\infty$ in $\mathcal{S}_r(B)$ does no longer hold for t outside the $\widetilde{\tau}$-cell containing B.
- The locus Δ is expected to be a locus of singularities for the coefficients $F_k(t)$ and for the $Y_r(z,t)$'s.
- The *Stokes matrices* $\mathbb{S}_r(t)$, defined for $t\in B$ by the following relations (see Figure 1)
$$Y_{r+1}(z,t)=Y_r(z,t)\;\mathbb{S}_r(t), \tag{2.4}$$
are expected to be singular as t approaches Δ.

Remark 2.1. In order to completely describe the Stokes phenomenon, it suffices to consider only three fundamental solutions, for example $Y_r(z,t)$ for $r=1,2,3$, and $\mathbb{S}_1(t)$, $\mathbb{S}_2(t)$ (this has been done in Examples 1.1 and 1.2, with $r=-1,0,1$ and $\mathbb{S}_{-1}$ and $\mathbb{S}_0$).

In [16], we give necessary and sufficient conditions for the coefficients $F_k(t)$ of the formal solution (2.3) to be holomorphic also at Δ.[5] Then, we give sufficient conditions such that fundamental solutions $Y_r(z,t)$, $r\in\mathbb{Z}$, as above, defined in B, together with their Stokes matrices $\mathbb{S}_r(t)$, are actually holomorphic also at Δ and on the whole $\mathcal{U}_{\epsilon_0}(0)$, where the asymptotic representation $Y_r(z,t)\sim Y_F(z,t)$ continues to hold. In this case, the limits
$$\lim_{t\to t_\Delta}\mathbb{S}_r(t),\qquad t_\Delta\in\Delta. \tag{2.5}$$
exist and are finite. They give the Stokes matrices for the system (1.1) with matrix coefficient $A(z,t_\Delta)$.

We now turn to isomonodromic deformations. Fist of all, we will define the mondromy data. For given t, a matrix $G(t)$ puts $A_1(t)$ in Jordan form
$$J(t):=G^{-1}(t)\;A_1(t)\;G(t).$$

[5]Notice that our result cannot be derived from [1] and [62], where holomorphic confluence for $t\to 0$ of formal solutions is studied, since $\Lambda(t)z$ is in general not "well-behaved" (condition (4.2) of [62] is violated).

Close to the Fuchsian singularity $z=0$, and for a given t, the system (1.1) has a fundamental solution in Levelt form

$$Y^{(0)}(z,t)=G(t)\Big(I+\sum_{l=1}^{\infty}\Psi_l(t)z^l\Big)z^{D(t)}z^{S(t)+R(t)}. \tag{2.6}$$

The matrix coefficients $\Psi_l(t)$ of the convergent expansion are constructed by a recursive procedure. $D(t)=\mathrm{diag}(d_1(t),...,d_n(t))$ is a diagonal matrix of integers, piecewise constant in t, $S(t)$ is a Jordan matrix whose eigenvalues have real part in $[0,1[$, and the nilpotent matrix $R(t)$ has non-vanishing entries only if some eigenvalues of $A_1(t)$ differ by non-zero integers. Then, $J(t)=D(t)+S(t)$. It is important to remark that, under the assumptions of our Theorem 2.1 below, the solution (2.6) turns out to be holomorphic in $t\in\mathcal{U}_{\epsilon_0}(0)$.

Chosen a solution $Y^{(0)}(z,t)$ with normal form (2.6), a *central connection matrix* $C^{(0)}$ is defined by the relation

$$Y_1(z,t)=Y^{(0)}(z,t)\ C^{(0)}(t),\qquad z\in\mathcal{S}_1(B). \tag{2.7}$$

In order to completely describe the monodromy of the system (1.1), we need its *essential* monodromy data (the adjective "essential" is inspired by a similar definition in [38]), defined to be

$$\mathbb{S}_1(t),\quad \mathbb{S}_2(t),\quad B_1(t)=\mathrm{diag}(A_1(t)),\quad C^{(0)}(t),\quad J(t),\quad R(t). \tag{2.8}$$

Now, when t tends to a point $t_\Delta\in\Delta$, the limits of the above data may not exist. If the limits exist, they *do not in general give the monodromy data of the system with matrix $A(z,t_\Delta)$*. The latter have in general different nature, as it is clear from the results of [3], and from [16].

Definition 2.2. *If the deformation is admissible in a small domain B, as in Definition 2.1, we say that it is* **isomonodromic in** *B if the essential monodromy data (2.8) do not depend on $t\in B$.*

When this definition holds, the classical theory of Jimbo-Miwa-Ueno [38] applies in B.[6] We are interested in extending the theory to the whole $\mathcal{U}_{\epsilon_0}(0)$, including the coalescence locus Δ. The existing literature (to be reviewed below) does not seem to include this extension, so we have given such an extension in our [16]. Some of our results are summarised in Theorem 2.1 and Corollary 2.1 below.

Before stating these results, we should explain how small ϵ_0 is. Since this is a little bit technical point, we postpone it to the Appendix. We also postpone the construction of new sectors $\widehat{\mathcal{S}}_r(t)$ and $\widehat{\mathcal{S}}_r=\bigcap_{t\in\mathcal{U}_{\epsilon_0}(0)}\widehat{\mathcal{S}}_r(t)$, which appear in the following theorem and are bigger than $\mathcal{S}_r(B)$, namely $\mathcal{S}_r(B)\subset\widehat{\mathcal{S}}_r\subset\widehat{\mathcal{S}}_r(t)$, with $t\in\mathcal{U}_{\epsilon_0}(0)$.

Theorem 2.1. *Consider the system (1.1), with eigenvalues of $\Lambda(t)$ linear in t as in (2.2), and with $A_1(t)$ holomorphic on a closed polydisc $\mathcal{U}_{\epsilon_0}(0)$ centred at $t=0$, with sufficiently small radius ϵ_0 (as specified in the Appendix). Let Δ be the coalescence locus in $\mathcal{U}_{\epsilon_0}(0)$, passing through $t=0$. Let the dependence on t be isomonodromic in $B\subset\mathcal{U}_{\epsilon_0}(0)$ as in Definition 2.2.*

If the matrix entries of $A_1(t)$ satisfy in $\mathcal{U}_{\epsilon_0}(0)$ the vanishing conditions

$$\Big(A_1(t)\Big)_{ab}=\ \textit{holomorphic multiple of } u_a(t)-u_b(t)\to 0,\qquad 1\le a\neq b\le n, \tag{2.9}$$

whenever $u_a(t)$ and $u_b(t)$ coalesce as t tends to a point of Δ, then the following results hold:

[6] Notice that in [38] it is also assumed that $A_1(t)$ is diagonalisable with eigenvalues not differing by integers. We do not make this assumption in [16].

- *The formal solution $Y_F(z,t)$ of (1.1) as given in (2.3) is holomorphic on the whole $\mathcal{U}_{\epsilon_0}(0)$.*
- *The three fundamental matrix solutions $Y_r(z,t)$, $r=1,2,3$, initially defined on B, with asymptotic representation $Y_F(z,t)$ for $z\to\infty$ in the sectors $\mathcal{S}_r(B)$ introduced above, can be t-analytically continued as single-valued holomorphic functions on $\mathcal{U}_{\epsilon_0}(0)$, with asymptotic representation*

$$Y_r(z,t)\sim Y_F(z,t) \quad \text{for } z\to\infty \text{ in the wider sectors } \widehat{\mathcal{S}}_r,$$

for any $t\in\mathcal{U}_{\epsilon_1}(0)$ and any $0<\epsilon_1<\epsilon_0$. In particular, they are defined at any $t_\Delta\in\Delta$ with asymptotic representation $Y_F(z,t_\Delta)$. The fundamental matrix solution $Y^{(0)}(z,t)$ is also t-analytically continued as a single-valued holomorphic function on $\mathcal{U}_{\epsilon_0}(0)$
- *The constant Stokes matrices $\mathbb{S}_1$, $\mathbb{S}_2$, and a central connection matrix $C^{(0)}$, initially defined for $t\in B$, are actually* globally defined on $\mathcal{U}_{\epsilon_0}(0)$. *They coincide with the Stokes and connection matrices $Y_r(z,0)$ and $Y^{(0)}(z,0)$ of the system*

$$\frac{dY}{dz}=A(z,0)Y,\qquad A(z,0)=\Lambda(0)+\frac{A_1(0)}{z}. \tag{2.10}$$

Also the remaining t-independent monodromy data in (2.8) coincide with those of (2.10).
- *The entries (a,b) of the Stokes matrices are characterised by the following vanishing property:*

$$(\mathbb{S}_1)_{ab}=(\mathbb{S}_1)_{ba}=(\mathbb{S}_2)_{ab}=(\mathbb{S}_2)_{ba}=0 \quad \text{whenever } u_a(0)=u_b(0),\quad 1\le a\neq b\le n. \tag{2.11}$$

Theorem 2.1 allows to holomorphically define the fundamental solutions and the monodromy data on the whole $\mathcal{U}_{\epsilon_0}(0)$, under the only condition (2.9). This fact is remarkable, compared to the general fact that Δ is expected to be a branching locus and the $Y_r(z,t)$ are expected to lose their asymptotic representation $Y_r(z,t)\sim Y_F(z,t)$ in $\mathcal{S}_r(B)$.

There is more to say, about the computation of the essential monodromy data. Let the assumptions of Theorem 2.1 hold. Then, the system (2.10) has a formal solution (here we denote objects Y, $\mathbb{S}$ and C referring to the system (2.10) with the symbols $\mathring{Y}$, $\mathring{\mathbb{S}}$ and $\mathring{C}$) with behaviour[7]

$$\mathring{Y}_F(z)=\Big(I+\sum_{k=1}^{\infty}\mathring{F}_k z^{-k}\Big)z^{B_1(0)}e^{\Lambda(0)z},\qquad B_1(0)=\mathrm{diag}(A_1(0)). \tag{2.12}$$

The coefficients $\mathring{F}_k$ can be recursively constructed from the differential system, but *there is not a unique choice for them.*

Actually, there is a family of formal solutions with structure (2.12), depending on a finite number of complex parameters. To each element of the family there correspond unique actual solutions $\mathring{Y}_1(z)$, $\mathring{Y}_2(z)$, $\mathring{Y}_3(z)$ such that $\mathring{Y}_r(z)\sim\mathring{Y}_F(z)$ for $z\to\infty$ in a sector $\mathcal{S}_r\supset\mathcal{S}_r(B)$, $r=1,2,3$, with Stokes matrices defined by

$$\mathring{Y}_{r+1}(z)=\mathring{Y}(z)\,\mathring{\mathbb{S}}_r,\qquad r=1,2.$$

Notice that only one element of the family of formal solutions (2.12) satisfies the condition $\mathring{F}_k=F_k(0)$ for any $k\ge 1$. By Theorem 2.1, the relation $\mathbb{S}_r=\mathring{\mathbb{S}}_r$ holds only for the actual solutions with asymptotic representation given by this element.

To complete the picture, let us also choose a solution $\mathring{Y}^{(0)}(z)$ close to $z=0$ in Levelt form, and define the corresponding central connection matrix $\mathring{C}^{(0)}$ such that

$$\mathring{Y}_1(z)=\mathring{Y}^{(0)}(z)\,\mathring{C}^{(0)}.$$

[7] If the vanishing assumption (2.9) fails, formal solutions are more complicated [3].

The following holds [16]

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold. If the diagonal entries of $A_1(0)$ do not differ by non-zero integers, then there is a unique formal solution (2.12) of the system (2.10), whose coefficients necessarily satisfy the condition*

$$\mathring{F}_k \equiv F_k(0).$$

Hence, (2.10) only has at $z = \infty$ canonical fundamental solutions $\mathring{Y}_1(z)$, $\mathring{Y}_2(z)$, $\mathring{Y}_3(z)$, such that:

$$Y_1(z,0) = \mathring{Y}_1(z), \quad Y_2(z,0) = \mathring{Y}_2(z), \quad Y_3(z,0) = \mathring{Y}_3(z).$$

Moreover, for any $\mathring{Y}^{(0)}(z)$ there exists $Y^{(0)}(z,t)$ such that $Y^{(0)}(z,0) = \mathring{Y}^{(0)}(z)$. The following equalities hold:

$$\mathbb{S}_1 = \mathring{\mathbb{S}}_1, \quad \mathbb{S}_2 = \mathring{\mathbb{S}}_2, \quad C^{(0)} = \mathring{C}^{(0)}.$$

Corollary 2.1 has a practical computational importance: the *constant monodromy data* (2.8) of the system (1.1) on the whole $\mathcal{U}_{\epsilon_0}(0)$ are computable just by considering the system (2.10) at the coalescence point $t = 0$. This is useful for applications. One case is when $A_1(t)$ is known in a whole neighbourhood of a coalescence point, but the computation of monodromy data can be explicitly done (only) at a coalescence point, where (1.1) simplifies due to (2.9). Another case is when $A_1(t)$ is *explicitly known only* at a coalescence point. This may happen in the case of Frobenius manifolds, like the quantum cohomology of Grassmannians [15], [17], [18]. Theorem 2.1 and Corollary 2.1 allow to compute the monodromy data of a semisimple Frobenius manifold just by considering the Frobenius structure at a coalescence point, as explained in our paper [17] (these data can then be extended to the whole manifold by an action of the braid group [21] [17] [18]).

In [16], we also prove the somehow weaker converse of Theorem 2.1. Assume that the deformation is admissible and isomonodromic as in Definitions 2.1 and 2.2 on a simply connected domain $B \subset \mathcal{U}_{\epsilon_0}(0)$. Note that now *we are not assuming* that $A_1(t)$ is holomorphic in the whole $\mathcal{U}_{\epsilon_0}(0)$, contrary to what has been done so far. As a result of [51], the fundamental solutions $Y_r(z,t)$, $r = 1, 2, 3$, and $A_1(t)$ can be analytically continued as multi-valued functions on $\mathcal{U}_{\epsilon_0}(0) \backslash \Delta$, with movable poles at the Malgrange divisor [53] [46] [47] [48]. Moreover, $Y_r(z,t)$ is no longer asymptotic to $Y_F(z,t)$ in $\mathcal{S}_r(B)$ when t moves sufficiently far away from B. Nevertheless, if the vanishing condition (2.11) on Stokes matrices holds, then we can prove that the fundamental solutions $Y_r(z,t)$ and $A_1(t)$ have *single-valued* meromorphic continuation on $\mathcal{U}_{\epsilon_0}(0) \backslash \Delta$, so that Δ *is not a branching locus*. Moreover, the asymptotic behaviour is preserved, according to the following

Theorem 2.2. *Consider the system (1.1), such that $A_1(t)$ is holomorphic on an open simply connected domain $B \subset \mathcal{U}_{\epsilon_0}(0)$, where the deformation is admissible and isomonodromic as in Definitions 2.1 and 2.2. Let ϵ_0 be sufficiently small (as specified in the Appendix). If the entries of the constant Stokes matrices satisfy the vanishing condition*

$$(\mathbb{S}_1)_{ab} = (\mathbb{S}_1)_{ba} = (\mathbb{S}_2)_{ab} = (\mathbb{S}_2)_{ba} = 0 \qquad \textit{whenever } u_a(0) = u_b(0), \quad 1 \leq a \neq b \leq n,$$

then, the fundamental solutions $Y_r(z,t)$ and $A_1(t)$ admit single-valued analytic continuation on $\mathcal{U}_{\epsilon_0}(0) \backslash \Delta$ as meromorphic functions of t. Moreover, for any $t \in \mathcal{U}_{\epsilon_0}(0) \backslash \Delta$ which is not a pole *of $Y_r(z,\tilde{t})$ (i.e. which is not a point of the Malgrange divisor), we have*

$$Y_r(z,t) \ \sim Y_F(z,t) \ \ \text{for } z\to\infty \text{ in } \widehat{\mathcal{S}}_r(t), \ \ r=1,2,3,$$

and

$$Y_{r+1}(z,t)=Y_r(z,t)\ \mathbb{S}_r, \quad r=1,2.$$

The sectors $\widehat{\mathcal{S}}_r(t)$'s are described in the Appendix.

2.1. **Some Comments on the Literature.** To conclude this Section, we would like to review some of the existing literature, where coalescence of eigenvalues has been discussed, in order compare it to our results. In the literature, sometimes the irregular singular point is taken at $z=0$, which is equivalent to $z=\infty$ by a change $z\mapsto 1/z$. One considers a "folded" system $A(z,0)=z^{-k-1}\sum_{j=0}^{\infty}A_j(0)z^j$, with an irregular singularity of Poincaré rank k at $z=0$ and studies its holomorphic unfolding $A(z,t)=p(z,t)^{-1}\sum_{j=0}^{\infty}A_j(t)z^j$, where $p(z,t)=(z-a_1(t))\cdots(z-a_{k+1}(t))$ is a polynomial. The problem is than to study the limits for $t\to 0$ of solutions and monodromy data of the "unfolded" system and their relation to solutions and data of the "folded" one. Early studies were started by Garnier [26]. The problem was again proposed by V.I. Arnold in 1984 and studied by many authors in the '80's and '90's of the XX century, for example see [55], [23], [9]. Under suitable conditions, some results have been recently established regarding the convergence for $t\to 0$, in sectors or suitable ramified domains, of suitable monodromy data (transition or connection matrices) of the "unfolded" system to the Stokes matrices of the "folded" one [55], [23], [9], [1], [62], [27], [28], [36], [39], [44], [42]. Nevertheless, to our knowledge, the case when $A_0(0)$ is *diagonalisable with coalescing eigenvalues* has not yet been studied. Indeed, either the leading matrix $A_0(0)$ is assumed to have distinct eigenvalues, as in [27] [36] [39], or $A_0(0)$ is a single Jordan block, as in [28] and [42], so that the irregular singularity is ramified for the system at $t=0$, and the fundamental matrices $Y_r(z,t)$ diverge when $t\to 0$, together with the corresponding Stokes matrices. In all the above cases, we cannot find the discussion about the sectors $\widehat{\mathcal{S}}_r$, the cell decomposition, analytic continuation beyond $\widetilde{\tau}$-cells, and sufficient conditions for the existence of the limit (2.5).

Also in the isomonodromic case, to the best of our knowledge, the literature does not seem to contain results analogue to our [16], as exposed in the present proceedings.

In case Δ is empty, a generalisation of the classical theory of [38], consisting in allowing any matrix $A_1(t)$, including the case when the eigenvalues differ by integers, can be easily done when we require that the *monodromy exponents* J, R and the connection matrix $C^{(0)}$ in (2.8) are constant (this is an *isoprincipal deformation*, in the language of [41]). For example, the case when Δ is empty and $A_1(t)$ is skew-symmetric and diagonalisable has been studied in [19], [21], in the context of Frobenius manifolds.[8]

Isomonodromy deformations at irregular singular points with leading matrix admitting a Jordan form J *independent of* t were studied in [6], with some *Lidskii* generic conditions. The system in [6], with singularity is at $z=\infty$, can be written as $A(z,t)=z^{k-1}(J+\sum_{j=1}^{\infty}A_j(t)z^{-j})$, being J a matrix in Jordan form, and the Poincaré rank is $k\geq 1$. Although the eigenvalues of J have algebraic multiplicity greater than 1, J is "rigid", namely $u_1,...,u_n$ are not deformed.

[8] We also recall that in case of Fuchsian singularities only, isomonodromic deformations were completely studied in [11] and [41]. In [11] it is only assumed that the monodromy matrices are constant. This generates non-Schlesinger deformations. On the other hand, an isopricipal deformation always leads to Schlesinger deformations [41].

Other investigations of isomonodromy deformations at irregular singularities can be found in [24] and [7]. Nevertheless, these results do not apply to our coalescence problem. For example, the third admissibility conditions of definition 10 of [7] is not satisfied in our case. In [24] the system with $A(z,t)=z^{r-1}B(z,t)$, $r\in\mathbb{Q}$, is considered, such that $B(\infty,t)$ has distinct eigenvalues; $z=\infty$ satisfying this condition is called a simple irregular singular point. This simplicity condition does not apply in our case.

The results of [42], cited above, are applied in [43] to the 3×3 isomonodromic description of the Painlevé VI equation and its coalescence to Painlevé V. In this case, the limiting system for $t\to 0$ has leading matrix with a 2×2 Jordan block, so that the fundamental matrices $Y_r(z,t)$ diverge.

Isomonodromic deformations of a system such as our (1.1) are also studied in [12]. Nevertheless, the eigenvalues $u_1,...,u_n$ of the matrix Z in [12], which is the analogue of our Λ, are always inside the same coalescence "stratum". Namely, there are $s<n$ deformation parameters λ_1,...,λ_s, ($\lambda_i\neq\lambda_j$ for $i\neq j$) and the eigenvalues are always equal to these parameters, namely

$$u_1=\cdots=u_{p_1}=\lambda_1, \tag{2.13}$$

$$u_{p_1+1}=\cdots=u_{p_1+p_2}=\lambda_2, \tag{2.14}$$

$$\cdots \tag{2.15}$$

$$u_{p_1+\cdots+p_{s-1}+1}=\cdots=u_{p_1+\cdots+p_s}=\lambda_s, \tag{2.16}$$

with $p_1+\cdots+p_s=n$. Thus, no splitting of coalescences occurs. Moreover, the matrix $f=f(Z)$ in [12], which is the analogue of our $A_1=A_1(u)$, satisfies the quite restrictively requirements that the diagonal is zero and $(A_1)_{ab}=0$ whenever $u_a=u_b$, $1\leq a\neq b\leq n$. With these strict requirements, an adaptation of the classical Jimbo-Miwa-Ueno isomonodromy deformation theory can be done in a straightforward way. Thus, the deformation theory in [12] is a very particular and simple sub-case of our [16], where we have studied the *general* isomonodromic deformations of the system (1.1), not only the simple decomposition of the spectrum as in (2.13)-(2.16).

3. A Simple Application of Theorem 2.1 and Corollary 2.1 to Painlevé Equations

Theorem 2.1 and Corollary 2.1 have relevant applications to Frobenius manifolds, as discussed in [17]. Here we explain a simple application to Painlevé equations. They provide an alternative to Jimbo's approach for the computation of the monodromy data associated with Painlevé VI transcendents holomorphic at a critical point. As an example, we consider the A_3-algebraic solution of Dubrovin-Mazzocco [22]. The following Painlevé VI equation, depending on a parameter $\mu\in\mathbb{C}$,

$$\frac{d^2y}{dt^2}=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right]\left(\frac{dy}{dt}\right)^2-\left[\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right]\frac{dy}{dt}+ \\ +\frac{1}{2}\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[(2\mu-1)^2+\frac{t(t-1)}{(y-t)^2}\right],$$

is the isomonodromicity condition for a 3×3 system of type (1.1) (see [21] [50]). Suppose we want to study the coalescence $u_2-u_1\to 0$, with $u_3-u_1\neq 0$. With the substitutions $Y(z)\mapsto e^{u_1 z}Y(z)$, and $z\to(u_3-u_1)z$, (1.1) becomes

$$\frac{dY}{dz}=\left[\begin{pmatrix}0&0&0\\0&t&0\\0&0&1\end{pmatrix}+\frac{A_1(t)}{z}\right]Y,\qquad t=\frac{u_2-u_1}{u_3-u_1}. \tag{3.1}$$

The Painlevé equation is associated with the system with skew-symmetric A_1, namely

$$A_1(t) =: \begin{pmatrix} 0 & \Omega_2 & -\Omega_3 \\ -\Omega_2 & 0 & \Omega_1 \\ \Omega_3 & -\Omega_1 & 0 \end{pmatrix}.$$

The isomonodromy deformation equations are then

$$\frac{d\Omega_1}{dt} = \frac{1}{t}\ \Omega_2\Omega_3, , \qquad \frac{d\Omega_2}{dt} = \frac{1}{1-t}\ \Omega_1\Omega_3, \qquad \frac{d\Omega_3}{dt} = \frac{1}{t(t-1)}\ \Omega_1\Omega_2. \tag{3.2}$$

The eigenvalues of $A_1(t)$ are $\mu, 0, -\mu$. The system (3.2) is equivalent to the above Painlevé equation through the following formulae (see [29] [31] for the formulae)

$$\Omega_1 = i\frac{\sqrt{y-1}\sqrt{y-t}}{\sqrt{t}}\left[\frac{A}{(y-1)(y-t)} + \mu\right], \qquad \Omega_2 = i\frac{\sqrt{y}\sqrt{y-t}}{\sqrt{1-t}}\left[\frac{A}{y(y-t)} + \mu\right],$$

$$\Omega_3 = -\frac{\sqrt{y}\sqrt{y-1}}{\sqrt{t}\sqrt{1-t}}\left[\frac{A}{y(y-1)} + \mu\right], \qquad A := \frac{1}{2}\left[\frac{dy}{dt}t(t-1) - y(y-1)\right].$$

The A_3-algebraic solution of PVI_μ, $\mu = -\frac{1}{4}$, obtained in [22] (there is a misprint in $t(s)$ of [22]), have the parametric representation

$$y(s) = \frac{(1-s)^2\ (1+3s)\ (9s^2-5)^2}{(1+s)\ (243s^6+1539s^4-207s^2+25)}, \qquad t(s) = \frac{(1-s)^3\ (1+3s)}{(1+s)^3\ (1-3s)}, \quad s \in \mathbb{C}. \tag{3.3}$$

As it is shown in [22], the Jimbo's monodromy data [37] of the Jimbo-Miwa-Ueno [38] isomonodromic Fuchsian system associated with algebraic solutions of PVI_μ are $\mathrm{tr}(M_iM_j) = 2 - \mathbb{S}_{ij}^2$, $1 \le i < j \le 3$, where $\mathbb{S}$ is the Stokes matrix (in upper triangular form) of the corresponding Frobenius manifold [19], and $\mathbb{S}+\mathbb{S}^T$ is the Coxeter matrix of the reflection group A_3. Moreover, Jimbo's isomonodromic method [37], as applied in [22] (see also [40], [30] for holomorphic solutions) allows to compute $\mathrm{tr}(M_iM_j)$. Here we apply Theorem 2.1 and obtain S in an alternative, and probably simpler, way, as exposed below.

A holomorphic branch is obtained by letting $s \to -\frac{1}{3}$ in (3.3), which gives the convergent Taylor expansion

$$\Omega_1(t) = i\sqrt{2}\left(\frac{1}{8} - \frac{1}{256}t - \frac{17}{16384}t^2 - \frac{257}{524288}t^3 + O(t^4)\right),$$
$$\Omega_2(t) = -\frac{1}{32}t - \frac{1}{64}t^2 - \frac{173}{16384}t^3 + O(t^4),$$
$$\Omega_3(t) = i\sqrt{2}\left(\frac{1}{8} + \frac{1}{256}t + \frac{47}{16384}t^2 + \frac{1217}{524288}t^3 + O(t^4)\right).$$

Since $\lim_{t\to 0}\Omega_2(t) = 0$, Theorem 2.1 holds. Since $\mathrm{diag}(A_1) = (0,0,0)$, also Corollary 2.1 holds. Accordingly, the Stokes matrices can be computed using (3.1) at $t = 0$, namely:

$$\frac{dY}{dz} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{A_1(0)}{z}\right]Y, \qquad A_1(0) = \begin{pmatrix} 0 & 0 & -i\sqrt{2}/8 \\ 0 & 0 & i\sqrt{2}/8 \\ i\sqrt{2}/8 & -i\sqrt{2}/8 & 0 \end{pmatrix}. \tag{3.4}$$

This system is integrable by reduction to a second order differential equation (and a quadrature), in a standard way. The second order equation is a Bessel equation, so its Stokes matrices can be computed using Hanckel functions. For technical details we refer to [16], and just give the result. With the three sectors

$$\mathcal{S}_1 = S\Big(-\frac{3\pi}{2}, \frac{\pi}{2}\Big),\ \mathcal{S}_2 = S\Big(-\frac{\pi}{2}, \frac{3\pi}{2}\Big),\ \mathcal{S}_3 = S\Big(\frac{\pi}{2}, \frac{5\pi}{2}\Big).$$

we obtain

$$\mathbb{S}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbb{S}_2 = \mathbb{S}_1^{-T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

The result is in accordance with Theorem 2.1, which predicts that the entry $(1,2)$ of $\mathbb{S}_1$ and the entry $(2,1)$ of $\mathbb{S}_2$ must be zero. It is also in accordance with the monodromy data of $y(t)$ obtained in [22].

Remark 3.1. If we choose $A_1(0)$ with different signs, we obtain different signs in $\mathbb{S}_1$. This sign freedom corresponds to the invariance of $U = \operatorname{diag}(u_1, u_2, u_3)$, namely $JUJ \equiv U$, where $J = \operatorname{diag}(\pm 1, \pm 1 \pm 1)$. For example, if $J := \operatorname{diag}(1, -1, 1)$, we take the system with $JA_1(0)J$ and find the Stokes matrices

$$\mathbb{S} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = J\mathbb{S}_1 J, \quad \text{and} \quad \mathbb{S}^{-T}.$$

The above is in accordance with the known result of [19] that the Stokes matrix $\mathbb{S}$ of the A_3 Frobenius manifold (up to permutation, change of signs and action of the braid group) is such that $\mathbb{S} + \mathbb{S}^T$ is the Coxeter matrix of the reflection group A_3.

Appendix

3.1. **Sectors $\mathcal{S}_r(B)$.** The sectors $\mathcal{S}_r(B)$ are constructed as follows: take for example the " half plane" $\Pi_1 := \{z \in \mathcal{R} \mid \widetilde{\tau} - \pi < \arg z < \widetilde{\tau}\}$. We call $\mathcal{S}_1(t)$ the open sector containing Π_1 and extending up to the closest Stokes rays of $\Lambda(t)$ outside Π_1. Then, we define $\mathcal{S}_1(B) := \bigcap_{t\in\overline{B}} \mathcal{S}_1(t)$. Analogously, we consider the "half-planes" $\Pi_r := \{z \in \mathcal{R} \mid \widetilde{\tau}+(r-3)\pi < \arg z < \widetilde{\tau}+(r-1)\pi\}$ and repeat the same construction for $\mathcal{S}_r(B)$. The sectors $\mathcal{S}_r(B)$ have central opening angle greater than π and their successive intersections do not contain Stokes rays $\Re e[(u_a(t) - u_b(t))z] = 0$, $t \in B$. The sectors $\mathcal{S}_r(B)$ for $r = 1, 2, 3$ are represented in Figure 1. An admissible ray $\arg z = \widetilde{\tau}$ in $\mathcal{S}_1(B) \cap \mathcal{S}_2(B)$ is also represented.

3.2. **Size of ϵ_0 and Sectors $\widehat{\mathcal{S}}_r$.** We write $\Lambda(t) = \Lambda_1(t) \oplus \cdots \oplus \Lambda_s(t)$, so that $\Lambda_j(t) \to \lambda_j I_{p_j}$ as in (2.1), when $t \to 0$. In Theorems 2.1 and 2.2 we need ϵ_0 sufficiently small to ensure that $\Lambda_i(t)$ has no eigenvalues in common with $\Lambda_j(t)$, for $i \neq j$. Moreover, the following constraint must hold

$$\epsilon_0 < \min_{1\leq j\neq k\leq s} \delta_{jk}, \tag{3.5}$$

where

$$\delta_{jk} := \frac{1}{2} \min_{\rho\in\mathbb{R}} \Big\{ |\lambda_k - \lambda_j + i\rho \exp\{-i\tilde{\tau}\}| \Big\}$$

(here i is the imaginary unit). This condition has a geometrical reason. If we represent $\lambda_1, ..., \lambda_s$ in the same λ-plane, the distance between the two parallel lines through λ_k and λ_j of angular direction $3\pi/2 - \widetilde{\tau}$ is exactly $2\delta_{jk}$. Let us consider Stokes rays associated with couples $u_a(t)$, $u_b(t)$, $a, b \in \{1, 2, ..., n\}$, as in (2.2), and such that $u_a(0) = \lambda_j$ and $u_b(0) = \lambda_k$, with $1 \leq j \neq k \leq s$. Then, any of these rays never cross the admissible directions $\widetilde{\tau} + k\pi$, $k \in \mathbb{Z}$, when t varies in $\mathcal{U}_{\epsilon_0}(0)$ with ϵ_0 as in (3.5). For a given t, let $\mathfrak{R}(t)$ be the set of all the above rays for all $j \neq k$. We construct a sector $\widehat{\mathcal{S}}_r(t)$ containing the "half-plane" Π_r (defined above), and extending up to the closest Stokes rays of $\mathfrak{R}(t)$ lying outside Π_r. Clearly, $\widehat{\mathcal{S}}_r(t) \supset \mathcal{S}_r(t)$ (the sectors $\mathcal{S}_r(t)$ are introduced above). Then, we define

$$\widehat{\mathcal{S}}_r := \bigcap_{t\in\mathcal{U}_{\epsilon_0}(0)} \widehat{\mathcal{S}}_r(t).$$

By construction, if ϵ_0 is as in (3.5), then this sector has central opening angle greater than π.

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