Canonical Surfaces and Hypersurfaces in Abelian Varieties

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Candidate

Luca Cesarano

Supervisors

Prof. Dr. Fabrizio Catanese
Prof. Dr. Barbara Fantechi

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Scuola Internazionale Superiore di Studi Avanzati - SISSA

Trieste

Abstract

The present work deals with the canonical map of smooth, compact complex surfaces of general type, which induce a polarization of type $(1, 2, 2)$ on an abelian threefold. The aim of the present study is to provide a geometric description of the canonical map of a smooth surface $\mathcal{S}$ of type $(1, 2, 2)$ in an abelian threefold A in some special situations, and to prove that, when A and $\mathcal{S}$ are sufficiently general, the canonical system of $\mathcal{S}$ is very ample. It follows, in particular, a proof of the existence of canonical irregular surfaces in $\mathbb{P}^5$ with numerical invariants $p_g = 6$, $q = 3$ and $K^2 = 24$.

This thesis is organized as follows: The first chapter deals with the basic theoretical results concerning ample divisors on abelian varieties and their canonical map, which can be analytically represented in terms of theta functions (see proposition 1.1.1). In this context, the example of surfaces in a polarization of type $(1, 1, 2)$ on an Abelian threefold, which has been studied in [14], is of particular importance: the behavior of the canonical map of the pullback of a principal polarization by a degree 2 isogeny has been described in [14] by investigating the canonical image and its defining projective equations by means of homological methods. In the last section of the first chapter, we treat in detail these results, as well as the connection with the analytical representation of the canonical map presented at the beginning of the same chapter.

The polarization types $(1, 2, 2)$ and $(1, 1, 4)$ cannot be distinguished by considering only the numeric invariants of the ample surfaces in the respective linear systems. In the second chapter, we study the unramified bidouble covers of a smooth non-hyperelliptic curve of genus 3, and we characterize the unramified bidouble covers of a general Jacobian 3-folds, which carry a polarization of type $(1, 2, 2)$.

In the third and last chapter of this thesis we investigate the behaviour of the canonical map of a general smooth surface in a polarization of type $(1, 2, 2)$ on an abelian threefold A, which is an étale quotient of a product of a $(2, 2)$-polarized abelian surface with a (2)-polarized elliptic curve. With this analysis and with some monodromy arguments, we prove the main result of this thesis, which states that the canonical system of a general smooth surface $\mathcal{S}$ of type $(1, 2, 2)$ in a general abelian threefold A yields a holomorphic embedding in $\mathbb{P}^5$.

Contents

Introduction

The motivational background of this thesis relies on the study of two independent problems, the first consisting in the classification of canonically embedded surfaces in $\mathbb{P}^5$, and the second in the study of the canonical map of ample divisors in abelian varieties. More precisely, the first problem can be formulated as follows:

Question 0.1. *For which values of K^2 can one find canonically embedded smooth surfaces of general type $\mathcal{S}$ in $\mathbb{P}^5$ with $p_g = 6$ and $K_{\mathcal{S}}^2 = K^2$?*

This problem traces its roots back to the mathematical work of F. Enriques, who raised the general question to describe the canonical models of surfaces of general type whose canonical map is at least birational onto its image. As it can easily be observed, the lowest value of the geometric genus for which the canonical map can be birational is $p_g = 4$, and the smooth quintics provide the first example of canonical surfaces $\mathcal{S}$ in $\mathbb{P}^3$ in this setting.
A first answer to the existence question of regular canonical surfaces with $p_g = 4$, at least for small values of the degree K^2, was given first by Enriques and, later, by Ciliberto (see [17]), who provides a practical construction, for every n in the range $5 \leq n \leq 10$, of an algebraic family $\mathcal{K}(n)$ of canonical minimal surfaces in $\mathbb{P}^3$, with $K^2 = n$ and with ordinary singularities.

The construction of examples of regular minimal canonical surfaces in $\mathbb{P}^3$ of higher degrees as determinantal varieties, presented by Ciliberto in his cited work [17], owes to the ideas of Arbarello and Sernesi of representing a projective plane curve with a determinantal equation (see [4]). These ideas also appear in a work of Catanese (see [10]) which improves, in the case of regular surfaces, the description of the known examples of low degree and proves a structure theorem for the equations of the canonical projections in $\mathbb{P}^3$. We recall that, if $\mathcal{S}$ is a minimal surface of general type, a canonical projection is the image of a morphism $\psi\colon \mathcal{S} \longrightarrow \mathbb{P}^N$ defined by $N + 1$ independent global sections of the canonical bundle $\omega_{\mathcal{S}}$. The morphism ψ is called a good birational canonical projection if, moreover, ψ is birational onto its image. This structure theorem for the canonical projections in $\mathbb{P}^3$ has been generalized even to the case of irregular surfaces ([14]), and it states that a good canonical projection $\psi\colon \mathcal{S} \longrightarrow \mathbb{P}^3$, whose image we denote by Y, determines (and is completely determined by) a symmetric map of vector bundles

$$(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{E})^\vee(-5) \xrightarrow{\alpha} (\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{E}) \quad ,$$

where $\mathcal{E}$ in the previous expression is the vector bundle

$$\mathcal{E} = (K^2 - q + p_g - 9)\mathcal{O}_{\mathbb{P}^3}(-2) \oplus q\Omega^1_{\mathbb{P}^3} \oplus (p_g - 4)\Omega^2_{\mathbb{P}^3}$$

such that Y is defined by the determinantal equation $det(\alpha) = 0$. Furthermore, denoting the two blocks of the matrix of α by (α_1, α'), the ideal sheaf generated by the minors of order r of α coincides with the ideal sheaf generated by the minors of order r of α', where $r = rank(\mathcal{E})$.

It is worthwhile to observe here that, in the case $K^2 = 6$, $q = 0$ and $p_g = 4$ we obtain a matrix of polynomials α of the form

$$\alpha = \begin{bmatrix} p_5 & p_3 \\ p_3 & p_1 \end{bmatrix} \quad , \tag{1}$$

where p_i denotes a homogeneous polynomial of degree i. In particular, p_1 represents an adjoint plane cutting Y precisely in a double curve Γ, an irreducible nodal curve which is precisely the non-normal locus of Y (see also [14], remark 2.10). In conclusion, this determinantal structure of the equations perfectly fits with the original construction of Enriques in this case, which turns out to be very important in the analysis of the canonical map of a surface yielding a polarization of type $(1, 1, 2)$ on an abelian 3-fold.

What about the situation in $\mathbb{P}^4$? The first relevant remark about this question is that, in general, the research interest about the surfaces in $\mathbb{P}^4$ was focused more on the classification of the smooth surfaces in $\mathbb{P}^4$, and less on the problem of classification of the canonical ones. A possible reason which motivates this interest relies upon the fact that every smooth projective surface can be embedded in $\mathbb{P}^5$, but the same does not hold true for $\mathbb{P}^4$. The Severi double point formula (see [38]) represents, indeed, a kind of constraint between the numerical invariants of a surface $\mathcal{S}$ embedded in $\mathbb{P}^4$ and the number of its double points, which is an analogue of the genus-degree formula for plane curves:

$$\delta = \frac{d(d-5)}{2} - 5(g-1) + 6\chi - K^2 \tag{2}$$

where d the degree of the surface of $\mathcal{S}$, which has exactly δ double points as singularities in which two smooth branches of $\mathcal{S}$ intersect transversally, and g denotes the genus of a hyperplane section of $\mathcal{S}$. Concerning surfaces of non-general type embedded in $\mathbb{P}^4$, Ellingsrud and Peskine (see [24]) showed that there exists a positive integer d_0 such that every smooth surface $\mathcal{S}$ of degree d greater that d_0 contained in $\mathbb{P}^4$ is necessarily of general type. We don't go deep into this topic, but is known that the choice of $d_0 = 52$ works (see [23]).
Other interesting examples of smooth surfaces which have been the subject of research were the smooth subcanonical surfaces, i.e., surfaces $\mathcal{S}$ in $\mathbb{P}^4$ with the property that $\omega_{\mathcal{S}} \cong \mathcal{O}_{\mathcal{S}}(l)$ for some integer l. The importance of these surfaces is that they are closely related, thanks to the Serre correspondence, to the problem of constructing new examples of holomorphic rank two vector bundles on $\mathbb{P}^4$. More precisely, considering $\mathcal{S}$ a smooth surface in $\mathbb{P}^4$, there exists an integer l such that $\omega_{\mathcal{S}} \cong \mathcal{O}_{\mathcal{S}}(l)$ if and only if there exists a rank two vector bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = l+5$, $c_2(\mathcal{E}) = d$, and a holomorphic global section $\sigma \in H^0(\mathbb{P}^4, \mathcal{E})$ such that $\mathcal{S}$ is exactly its zero locus, and we have the short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{S}}(l+5) \longrightarrow 0 \quad . \tag{3}$$

The cases with $l = 0$ correspond to smooth surfaces in $\mathbb{P}^4$ with trivial canonical bundle. Those are known to be abelian surfaces embedded by a polarization of type $(1,5)$, and the corresponding rank two holomorphic vector bundle $\mathcal{E}$ in the sequence (3), necessarily indecomposable, is called the Horrocks-Mumford bundle (see [26]). It is maybe interesting to notice that this vector bundle is the only known example of a rank 2 indecomposable holomorphic vector bundle over $\mathbb{P}^4$, and that, indeed, even the half-canonical surfaces (i.e.: subcanonical with $l = 2$) which are not complete intersections are proved to be the zero locus of a holomorphic section of $\mathcal{E}(1)$, where $\mathcal{E}$ denotes now the Horrocks-Mumford bundle (see [22]).
We observe moreover that, in general, the cohomology of $\mathcal{I}_{\mathcal{S}}$, and hence the geometry of $\mathcal{S}$, is completely determined by the cohomology of $\mathcal{E}$, and hence by $\mathcal{E}$ itself. However, the complete intersections are precisely the situations in which $\mathcal{E}$ splits as the direct sum of two line bundles. This situation occurrs when $l = -1$, which is the case of a complete intersection of type $(2,2)$, and in the case of canonical surfaces. In this important subcase of the canonical surfaces, the Severi double point formula (2) gives us an integral equation which has only three solutions: the first two correspond to regular surfaces of degree $d = 8$ or $d = 9$ which are complete intersections (see [18]). The last possible case would be the one of a canonically embedded surface of degree $d = 12$ and irregularity $q = 1$. It has been excluded by means of two different methods:

Ballico and Chiantini (see [6]) proved that every smooth canonical surface $\mathcal{S}$ in $\mathbb{P}^4$ is a complete intersection by proving that there are no semistable rank two vector bundles $\mathcal{E}$ on $\mathbb{P}^4$ with Chern classes $c_1 = 0$ and $c_2 = 3$. Catanese observed that the claimed surface would be fibered over an elliptic curve and with fibers of genus 2; hence its canonical map would have degree at least 2, because it would factor through the hyperelliptic involution on each fiber (see for instance [12] pp.38-39). His result has been stated as follows ([11]):

Theorem 0.2. *Assume that $\mathcal{S}$ is the minimal model of a surface of general type with $p_g = 5$, and assume that the canonical map $\phi_{\mathcal{S}}$ embeds $\mathcal{S}$ in $\mathbb{P}^4$.*
Then $\mathcal{S}$ is a complete intersection with $\omega_{\mathcal{S}} = \mathcal{O}_{\mathcal{S}}(1)$, i.e., $\mathcal{S}$ is a complete intersection in $\mathbb{P}^4$ of type $(2,4)$ or $(3,3)$.
Moreover, if $\phi_{\mathcal{S}}$ is birational, and $K^2_{\mathcal{S}} = 8{,}9$, then $\phi_{\mathcal{S}}$ yields an embedding of the canonical model of $\mathcal{S}$ as a complete intersection in $\mathbb{P}^4$ of type $(2,4)$ or $(3,3)$.

As we already observed, every smooth projective surface can be embedded in $\mathbb{P}^5$ and, as a consequence of this fact, it makes sense to restrict our attention first to the smooth canonical surfaces in $\mathbb{P}^5$. The research interest in them revolves also around the possibility to describe their equation in a very special determinantal form. More precisely, for every such surface $\mathcal{S}$ there is a resolution of the form (see [39])

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-7) \longrightarrow \mathcal{E}^{\vee}(-7) \xrightarrow{\alpha} \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{S}} \longrightarrow 0 \quad , \tag{4}$$

where $\mathcal{E}$ is a vector bundle on $\mathbb{P}^5$ of odd rank $2k+1$, α is an antisymmetric map such that $\mathcal{S}$ is defined by the Pfaffians of order $2k$ of α (see [39], [12]).

It is well known, however, that the condition for the canonical map to be birational onto its image leads to some numerical constraints. From the inequality of Bogomolov-Miyaoka-Yau $K^2_{\mathcal{S}} \leq 9\chi$ and the Debarre version of Castelnuovo's inequality $K^2_{\mathcal{S}} \geq 3p_g(\mathcal{S}) + q - 7$, it follows that, if the canonical map of an algebraic surface $\mathcal{S}$ with $p_g = 6$ is birational, then

$$11 + q \leq K^2_{\mathcal{S}} \leq 9(7-q) \quad .$$

By considering $\mathcal{E}$ a sum of line bundles and a suitable antisymmetric vector bundle map α like in the diagram (4), Catanese exhibited examples of canonical regular surfaces in $\mathbb{P}^5$ with $11 \leq K^2 \leq 17$ (see [12])

Another interesting example of a canonical surface in $\mathbb{P}^5$ arises from the technique of ramified bidouble covers (see [13]): given three branch curves D_1, D_2, and D_3 in $Q = \mathbb{P}^1 \times \mathbb{P}^1$ which pairwise intersect transversally, one can construct a bidouble cover Y (possibly singular) of Q branched over $D = D_1 \cup D_2 \cup D_3$.

The resolution of singularities of Y provides examples of surfaces of general type with birational canonical map. Among the examples which have been exhibited in the article [13], Catanese considers the family of surfaces which arise as bidouble covers of Q branched on three smooth curves D_1, D_2, D_3 of bidegree respectively $(2,3)$, $(2,3)$, $(1,4)$, and he obtains a family of surfaces of general type with $K_S^2 = 24$, $p_g = 6$. Moreover, he proved in a recent work (see [16]) that in this case the canonical system is base point free and yields an embedding in $\mathbb{P}^5$.

It is interesting to notice, however, that all the methods we have considered allow us to construct several examples of regular surfaces, but have left open the question 0.1 for irregular ones. Examples of irregular algebraic projective varieties can be constructed using methods which are generally called transcendental methods. If one consideres the general problem of finding "nice" projective models of irregular projective varieties, one focuses in a first moment his attention only on their underlying analytic structure as compact polarized Kähler manifolds and their linear systems. In a second step, an analysis of their very ample linear systems allows to gain more information on their algebraic structure, when we see them as projective varieties.
Abelian varieties provide nice concrete examples of these methods: from the analytic point of view they can be easily described as complex tori with an ample line bundle, even though the structure of their projective equations has always been a very intricate problem (see [33]). Furthermore, examples of irregular varieties of general type arise in a pretty natural way just by considering their smooth ample divisors. In the first chapter of this thesis, we recall, indeed, that the smooth ample divisors of an abelian variety are varieties of general type. This leads naturally to the following question, which is the second general problem we have mentioned at the beginning of this introduction:

Question 0.3. *Let us fix $d = (d_1, \cdots, d_g)$ a polarization type. Let us consider a general d-polarized abelian variety $(A, \mathcal{L})$ of dimension g, and a general smooth divisor $\mathcal{D}$ in the linear system $|\mathcal{L}|$. Is the canonical map $\phi_{\mathcal{D}}$ a holomorphic embedding?*

It is immediately seen, in consequence of the adjunction formula, that the canonical system $|\omega_{\mathcal{D}}|$ of a smooth divisor $\mathcal{D}$ of $|\mathcal{L}|$ contains the sublinear system $|\mathcal{L}| \cap \mathcal{D}$. In the first chapter we see, furthermore, that a suitable set of Theta functions provide a basis for the vector space of the global holomorphic sections of the canonical bundle. This leads to the following result, which is a purely analytical expression of the canonical map:

Proposition 0.4. *Let $A = \mathbb{C}^g/_{\Lambda}$ be an abelian variety and $\mathcal{L}$ an ample line bundle. Let $\mathcal{D}$ be a smooth divisor defined as the zero locus of a holomorphic section θ_0 of $\mathcal{L}$. Moreover, let us suppose $\theta_0, \cdots, \theta_n$ is a basis for the vector space $H^0(A, \mathcal{L})$. Then $\theta_1, \cdots, \theta_n, \frac{\partial \theta_0}{\partial z_1}, \cdots \frac{\partial \theta_0}{\partial z_g}$, where $z_1, \cdots, z_g$ are the flat uniformizing coordinates of $\mathbb{C}^g$, is a basis for $H^0(\mathcal{D}, \omega_{\mathcal{D}})$.*

In the third and most important chapter of this thesis, we study different examples of surfaces in a polarizations of type $(1,2,2)$ on abelian 3-folds, and we give an affirmative answer to the question 0.3 for this polarization type by proving the following theorem.

Theorem 0.5. *Let be $(A, \mathcal{L})$ a general $(1,2,2)$-polarized abelian 3-fold and let be $\mathcal{S}$ a general surface in the linear system $|\mathcal{L}|$. Then the canonical map of $\mathcal{S}$ is a holomorphic embedding.*

Our results provide us, in particular, a new example of family of irregular canonical surfaces of $\mathbb{P}^5$ of dimension 9 with $K_{\mathcal{S}}^2 = 24$, $p_g = 6$ and $q = 3$.

We have not gone into the general problem 0.3 in this work, also because it turned out to be remarkably involved even in our case. Nevertheless, we hope that the methods developed in this thesis lead to results for the general case.

It can be easily seen, for instance by proposition 0.4, that many properties which hold true for $|\mathcal{L}|$ (very ampleness, $\phi_{|\mathcal{L}|}$ finite, $\phi_{|\mathcal{L}|}$ birational...) immediately hold for $|\omega_{\mathcal{D}}|$. However, as it is well known by a theorem of Lefschetz, the linear system $|\mathcal{L}|$ is always very ample if $d_1 \geq 3$, and thus the same holds true for $|\omega_{\mathcal{D}}|$ for every smooth divisor of $|\mathcal{L}|$. Moreover, a result of Ohbuchi states that $\phi_{\mathcal{L}}$ is a morphism if $d_1 \geq 2$ and that $|\mathcal{L}|$ embeds the Kummer variety $A \big/ \langle -1 \rangle$ for the general $(2, \cdots, 2)$ polarized abelian variety. In particular, an easy application of 0.4 implies that question 0.3 has an affirmative answer if $d_1 \geq 2$.
Concerning the case $d_1 = 1$, very little seems to be known about the geometry of abelian varieties which carry such non-principal polarizations. In a remarkable work [34], Nagaraj and Ramanan considered special examples of $(1, 2, \ldots, 2)$-polarized abelian varieties $(A, \mathcal{L})$ of dimension g, with $g \geq 4$. The aim of their research was the study of the behavior of the linear system for the general polarized abelian variety as above. One can see that, considered a general $(1, 2, \ldots, 2)$-polarized abelian variety $(A, \mathcal{L})$ of dimension g, the base locus of $|\mathcal{L}|$ consists precisely of $2^{2(g-1)}$ points and the rational map $\phi_{|\mathcal{L}|} : A \dashrightarrow \mathbb{P}^N$ extends to $\widehat{A}$, the blow-up of A in the points of the base locus of $|\mathcal{L}|$. Moreover, $|\mathcal{L}|$ induces a map $\psi \colon \widehat{A} \big/ (-1) \longrightarrow \mathbb{P}^{N-1}$, $N = 2^{g-1}$. Nagaraj and Ramanan

proved that ψ is birational but there could be a codimension 2 subvariety on which the restriction of ψ is a morphism of degree 2.
Another important work of Debarre (see [21]) deals with the case of a general $(1, \ldots, 1, d)$-polarized abelian variety $(A, \mathcal{L})$, and by making use of degeneration methods it has been proved that $\mathcal{L}$ is very ample if $d > 2^g$. In the case of abelian surfaces, it is indeed well known, by a theorem of Reider, that a polarization of type $(1, d)$ is very ample if $d > 4$, and that this does not hold true for $d = 4$ for topological reasons (see [8] cap. 10). Also in the case of abelian 3-folds the result stated in this article [21] is the best possible: if $d = 8$, then $\phi_{|\mathcal{L}|}$ is a morphism but, according to a result of Iyer (see [27]), it fails to be injective because it identifies a finite number of points. In this case, however, $\phi_{|\mathcal{L}|}$ is unramified (see [21] Remark 26), so we can easily give a positive answer to question 0.3 in the case of a polarization of type $(1, 1, 8)$.

Another important aspect which we have to consider in dealing with question 0.3 is that the behavior of the canonical system of two smooth ample divisors may be different even if they belong to the same linear system. Indeed, it is well known that in the pencil $|\mathcal{L}|$ of a general $(1, 2)$-polarized abelian surface $(A, \mathcal{L})$ the general element is a smooth genus 3 non-hyperelliptic curve, but there exist always hyperelliptic elements in the linear system $|\mathcal{L}|$.
This change of behavior of the linear system seems to become a recurring theme in higher dimension, especially in the case of low degree polarizations. The case of a polarization of type $(1, 1, 2)$ on a general abelian 3-fold was studied in [14]. We return to this important case in the last section of the first chapter but, for the sake of exposition, we recall here the result in this case: what one obtains is a family of surfaces with $p_g = 4$, $K_S^2 = 12$ and $q = 3$, whose general member has birational canonical map. However, it turned out the following fact: for those surfaces in the linear system which are pullback of a principal polarization via an isogeny of degree 2, the canonical map is a two-to-one covering onto a regular canonical surface in $\mathbb{P}^3$ of degree 6, which is one of the surfaces of Enriques we discussed at the beginning of this introduction (see (1)).

A last important remark about problem 0.3 in our case is that the smooth surfaces in a polarization class of type $(1, 2, 2)$ and those in a polarization class of type $(1, 1, 4)$ have the same invariants. In the second chapter, we characterize in a more geometrical way the two polarizations. We consider indeed an abelian threefold A with an isogeny p with kernel isomorphic to $\mathbb{Z}_2^2$ onto a general Jacobian 3-fold $(\mathcal{J}(\mathcal{D}), \Theta)$, and we pull back the curve $\mathcal{D}$ along p on order to obtain a smooth genus 9 curve $\mathcal{C}$ in A. It turns out that the gonality of $\mathcal{C}$ characterises the polarization type of A, and this allows us, in the case of a polarization of type $(1,2,2)$, to give a completely algebro-geometric description of the behavior of the canonical map in the case of the pullback divisor $p^*\Theta$, which has moreover the advantage to be independent of the method of canonical projections used to describe the polarizations of type $(1,1,2)$.

The context of the pullback divisor is one of the few examples in which we can describe the canonical map in a purely algebraic setting, indipendently of its analytical expression in 0.4. Indeed, we are able to deal with this expression of the canonical map in terms of Theta functions only when we consider surfaces yielding a polarization of type $(1,2,2)$ on an abelian 3-fold A, which is isogenous to a polarized product of a $(1,1)$-polarized surface and an elliptic curve. These examples are explained in detail in the last chapter of this work. One important step of the proof of our result is to reduce the study of the differential of the canonical map to the case in which the considered $(1,2,2)$-polarized abelian 3-fold $(A,\mathcal{L})$ is isogenous to a principally polarized product of three elliptic curves E_1, E_2, E_3. In this case, making use of the Legendre normal form for the elliptic curves (see [15]) and their expression as quotient of Theta functions, we see that the equation of a smooth general surface $\mathcal{S}$ in the linear system $|\mathcal{L}|$ can be locally expressed with a very nice polynomial expression, and the same holds for its canonical map. By making use of this algebraic description of the canonical map, we prove that the general surface yielding a $(1,2,2)$ polarization on an abelian 3-fold A has everywhere injective differential, and by means of monodromy arguments, we further conclude that it is injective.

We conclude with the remark that none of the strategies, which we applied in our analysis of the canonical map in the case of a polarization of type $(1,2,2)$, seem to find application to the case of a polarization of type $(1,1,4)$. We only know from case $(1,1,2)$, indeed, that in this case the canonical map of the general member is birational. Nevertheless, question 0.3 is still open in this and many others interesting cases, and remains conseguently a possible stimulating perspective and motivation for future research works in this field.

Chapter 1

Divisors in Abelian Varieties

Throughout this work, a polarized abelian variety will be a couple $(A, \mathcal{L})$, where $\mathcal{L}$ is an ample line bundle on a complex torus A, and we denote by $|\mathcal{L}|$ the linear system of effective ample divisors which are zero loci of global holomorphic sections of $\mathcal{L}$.

The first Chern Class $c_1(\mathcal{L}) \in H^2(A, \mathbb{Z})$ is a integral valued alternating bilinear form on the lattice $H_1(A, \mathbb{Z})$. Applying the elementary divisors theorem, we obtain that there exists a basis $\lambda_1, \cdots, \lambda_g, \mu_1, \cdots, \mu_g$ of Λ with respect to which $c_1(\mathcal{L})$ is given by a matrix of the form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} ,$$

where D is a diagonal matrix $diag(d_1, \cdots, d_g)$ of positive integers with the property that every integer in the sequence divides the next. We call the sequence of integers $d = (d_1, \cdots, d_g)$ the type of the polarization on A induced by $\mathcal{L}$.

Moreover, we will say that an ample divisor $\mathcal{D}$ in an abelian variety A yields a polarization of type $d = (d_1, \cdots, d_g)$ on A, or simply that $\mathcal{D}$ is of type d on A, if the type of the polarization of $\mathcal{L}(\mathcal{D})$ is d.

In this chapter we introduce some notations and we consider the problem of describing analitically the canonical map of a smooth ample divisor $\mathcal{D}$ on an abelian variety A. In a joint work (see [14]) F.Catanese and F.O. Schreyer studied the canonical maps of the smooth ample divisors which yield a polarization of type $(1, 1, 2)$ on an abelian 3-fold. We give an account of the method they applied in the last section of this chapter.

1.1 On the canonical map of a smooth ample divisor on an abelian variety

Let us consider a polarized abelian variety $(A, \mathcal{L})$ of dimension g, where $A := \mathbb{C}^g / _{\Lambda}$, and Λ denote a sublattice in $\mathbb{C}^g$, and $\mathcal{D}$ a smooth ample divisor in the linear system $|\mathcal{L}|$. Denoted by $[\{\phi_\lambda\}_\lambda] \in H^1(\Lambda, H^0(\mathcal{O}_V^*)$ the factor of automorphy corresponding to the ample line bundle $\mathcal{L}$ according to the Appell-Humbert theorem (see [8], p. 32), the vector space $H^0(A, \mathcal{L})$ is isomorphic to the vector space of the holomorphic functions θ on $\mathbb{C}^g$ which satisfy for every λ in Λ the functional equation

$$\theta(z + \lambda) = \phi_\lambda(z)\theta(z) \quad .$$

Let us consider a smooth divisor $\mathcal{D}$ in A which is the zero locus of a holomorphic global section θ_0 of $\mathcal{L}$, which from now on we denote by $\mathcal{O}_A(\mathcal{D})$. By the adjunction formula, we have clearly that

$$\omega_{\mathcal{D}} = (\mathcal{O}_A(\mathcal{D}) \otimes \omega_A)|_{\mathcal{D}} = \mathcal{O}_{\mathcal{D}}(\mathcal{D}) \quad , \tag{1.1}$$

and we can see that the derivative $\frac{\partial \theta_0}{\partial z_j}$ is a global holomorphic sections of $\mathcal{O}_{\mathcal{D}}(\mathcal{D})$ for every $j = 1, \cdots, g$. Indeed, for every λ in Λ and for every z on $\mathcal{D}$ we have

$$\frac{\partial \theta_0}{\partial z_j}(z + \lambda) = \phi_\lambda(z)\frac{\partial \theta_0}{\partial z_j}(z) + \frac{\partial \phi_\lambda}{\partial z_j}(z)\theta_0(z) = \phi_\lambda(z)\frac{\partial \theta_0}{\partial z_j}(z) \quad .$$

This leads naturally to a description of the canonical map of a smooth ample divisor in an abelian variety only in terms of theta functions.

Proposition 1.1.1. *Let $A = \mathbb{C}^g / _{\Lambda}$ be an abelian variety and $\mathcal{L}$ an ample line bundle. Let $\mathcal{D}$ be a smooth divisor defined as the zero locus of a holomorphic section θ_0 of $\mathcal{L}$. Moreover, let us suppose $\theta_0, \cdots, \theta_n$ is a basis for the vector space $H^0(A, \mathcal{L})$. Then $\theta_1, \cdots, \theta_n, \frac{\partial \theta_0}{\partial z_1}, \cdots \frac{\partial \theta_0}{\partial z_g}$, where $z_1, \cdots, z_g$ are the flat uniformizing coordinates of $\mathbb{C}^g$, is a basis for $H^0(\mathcal{D}, \omega_{\mathcal{D}})$.*

Proof. From now on let us consider $\mathcal{L}$ to be $\mathcal{O}_A(\mathcal{D})$. We observe first that, for instance, by the Kodaira vanishing theorem, all the cohomology groups $H^i(A, \mathcal{O}_A(\mathcal{D}))$ vanish, so this implies that $H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D}))$ has the desired dimension $n + g$. In order to prove the assertion of the theorem, it is then enough to prove that the connecting homomorphism $\delta^0 \colon H^0(\mathcal{D}, \mathcal{O}_A(\mathcal{D})) \longrightarrow H^1(A, \mathcal{O}_A)$ maps $\frac{\partial \theta_0}{\partial z_1}, \cdots \frac{\partial \theta_0}{\partial z_g}$ to g linearly independent elements.

Let us consider then the projection π of $\mathbb{C}^g$ onto A and let us denote by $\widehat{\mathcal{D}}$ the divisor $\pi^*(\mathcal{D})$. We have then clearly the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^g} \longrightarrow \mathcal{O}_{\mathbb{C}^g}(\widehat{\mathcal{D}}) \longrightarrow \mathcal{O}_{\widehat{\mathcal{D}}}(\widehat{\mathcal{D}}) \longrightarrow 0 \quad ,$$

and we can denote the respective cohomology groups by:

$$\begin{aligned} M &:= H^0(\mathbb{C}^g, \mathcal{O}_{\mathbb{C}^g}) = H^0(\mathbb{C}^g, \pi^*\mathcal{O}_A) \\ N &:= H^0(\mathbb{C}^g, \mathcal{O}_{\mathbb{C}^g}(\widehat{\mathcal{D}})) = H^0(\mathbb{C}^g, \pi^*\mathcal{O}_A(\mathcal{D})) \\ P &:= H^0(\widehat{\mathcal{D}}, \mathcal{O}_{\widehat{\mathcal{D}}}(\widehat{\mathcal{D}})) = H^0(\widehat{\mathcal{D}}, \pi^*\mathcal{O}_{\mathcal{D}}(\mathcal{D})) \quad . \end{aligned} \tag{1.2}$$

The cohomology groups in 1.2 are Λ-modules with respect to the following actions: for every element λ of Λ and every elements s, t, u respectively in M, N, and P, the action of Λ is defines as follows:

$$\begin{aligned} \lambda.s(z) &:= s(z+\lambda) \\ \lambda.t(z) &:= t(z+\lambda)\phi_\lambda(z)^{-1} \\ \lambda.u(z) &:= u(z+\lambda)\phi_\lambda(z)^{-1} \quad . \end{aligned} \tag{1.3}$$

According to Mumford ([32], Appendix 2) there exists a natural homomorphism $\psi_\bullet$ from the cohomology groups sequences $H^p(\Lambda, \cdot)$ and $H^p(A, \cdot)$:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^i(\Lambda, M) & \longrightarrow & H^i(\Lambda, N) & \longrightarrow & H^i(\Lambda, P) & \longrightarrow & H^{i+1}(\Lambda, M) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \psi_i^M} & & \downarrow{\scriptstyle \psi_i^N} & & \downarrow{\scriptstyle \psi_i^P} & & \downarrow{\scriptstyle \psi_{i+1}^M} & \\ \cdots \longrightarrow & H^i(A, \mathcal{O}_A) & \longrightarrow & H^i(A, \mathcal{O}_A(\mathcal{D})) & \longrightarrow & H^i(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D})) & \xrightarrow{\delta^i} & H^{i+1}(A, \mathcal{O}_A) & \longrightarrow \cdots \end{array}$$

The homomorphism $\psi_\bullet$ is actually an isomorphism (this means that all the vertical arrows are isomorphisms), because the cohomology groups $H^i(\mathbb{C}^g, \mathcal{O}_{\mathbb{C}^g}(\widehat{\mathcal{D}}))$ vanish for every $i > 0$, being $\mathbb{C}^g$ a Stein manifold. Another possible method to prove that the cohomology sequences $H^p(\Lambda, \cdot)$ and $H^p(A, \cdot)$ are isomorphic, is to use the following result: if X is a variety, G is a group acting freely on X and $\mathcal{F}$ is a G-linearized sheaf, then there is a spectral sequence with E_1 term equal to $H^p(G, H^q(X, \mathcal{F}))$ converging to $H^{p+q}(Y, \mathcal{F})^G$.
The natural identification of these cohomology group sequences allows us to compute $\delta^0\left(\frac{\partial\theta_0}{\partial z_j}\right)$ using the following explicit expression of the connecting homomorphism $H^0(\Lambda, P) \longrightarrow H^1(\Lambda, M)$: given an element s of P^Λ, there exists an element t in N such that $t|_{\widehat{\mathcal{D}}} = s$. Then, by definition of $d\colon N \longrightarrow \mathcal{C}^1(\Lambda; N)$, we have

$$(dt)_\lambda = \lambda.t - t \quad .$$

where $\lambda.t$ is defined according to 1.3. Now, from the invariance of s under the action of Λ, we get

$$(\lambda.t - t)|_{\widehat{\mathcal{D}}} = \lambda.s - s = 0 \quad .$$

Hence, for every λ there exists a constant $c_\lambda \in \mathbb{C}$ such that $\lambda.t - t = c_\lambda \theta_0$, and it follows that

$$\delta^0(s)_\lambda = c_\lambda = \frac{\lambda.t - t}{\theta_0} \quad . \tag{1.4}$$

If we apply now 1.4 to the elements $\frac{\partial \theta_0}{\partial z_j}$, we obtain

$$\begin{aligned}
\delta^0\left(\frac{\partial\theta_0}{\partial z_j}\right)_\lambda &= \frac{\lambda.\frac{\partial\theta_0}{\partial z_j}(z) - \frac{\partial\theta_0}{\partial z_j}(z)}{\theta_0(z)} \\
&= \frac{\frac{\partial\theta_0}{\partial z_j}(z+\lambda)\phi_\lambda^{-1}(z) - \frac{\partial\theta_0}{\partial z_j}(z)}{\theta_0(z)} \\
&= \frac{\left[\phi_\lambda(z)\frac{\partial\theta_0}{\partial z_j}(z) + \frac{\partial\phi_\lambda}{\partial z_j}(z)\theta_0(z)\right]\phi_\lambda^{-1}(z) - \frac{\partial\theta_0}{\partial z_j}(z)}{\theta_0(z)} \\
&= \frac{\theta_0(z)\frac{\partial\phi_\lambda}{\partial z_j}(z)\phi_\lambda^{-1}(z)}{\theta_0(z)} \\
&= \frac{\partial\phi_\lambda}{\partial z_j}(z)\phi_\lambda^{-1}(z) \\
&= \partial_{z_j} log(\phi_\lambda)(z) \\
&= -\partial_{z_j}\pi H(z,\lambda) \\
&= -\pi H(e_j,\lambda) \quad ,
\end{aligned}$$

where $\{\phi_\lambda\}_\lambda$ is the factor of automorphy and H is the positive definite hermitian form on $\mathbb{C}^g$, both corresponding to the ample line bundle $\mathcal{O}_A(\mathcal{D})$ by applying the Appell-Humbert theorem. We can then conclude that:

$$\delta^0\left(\frac{\partial\theta_0}{\partial z_j}\right) = \left[(\pi H(e_j,\lambda))_\lambda\right] \in H^1(\Lambda; M) \quad .$$

We prove now that these images are linearly independent in $H^1(\Lambda; M)$. Let us consider $a_1, \cdots, a_g \in \mathbb{C}$ such that:

$$\left[(a_1 H(e_1,\lambda) + \cdots + a_g H(e_g,\lambda))_\lambda\right] = 0 \quad .$$

This means that there exists $f \in C^0(\Lambda, M)$ such that, for every $\lambda \in \Lambda$, we have:

$$a_1 H(e_1,\lambda) + \cdots + a_g H(e_g,\lambda) = \lambda.f(z) - f(z) = f(z+\lambda) - f(z) \quad .$$

For such f, the differential df is a holomorphic Λ-invariant 1-form. Hence, for some complex constant c and a certain $\mathbb{C}$-linear form L, we can write

$$f(z) = L(z) + c \quad .$$

Hence, for every $\lambda \in \Lambda$ the following holds

$$H(a_1 e_1 + \cdots + a_g e_g, \lambda) = L(\lambda) \quad ,$$

so the same holds for every $z \in \mathbb{C}^g$. This allows us to conclude that $L = 0$, L being both complex linear and complex antilinear. But the form H is non-degenerate, so we conclude that $a_1 = \cdots = a_g = 0$. The proposition is proved. $\square$

By applying the previous proposition, we can easily compute the invariants of an ample divisor $\mathcal{D}$ on an abelian variety A of dimension g.

Proposition 1.1.2. *Let $\mathcal{D}$ a smooth divisor in a polarization of type $(d_1, \cdots, d_g)$ on an abelian variety A. Then the invariants of $\mathcal{D}$ are the following:*

$$\begin{aligned} p_g &= \prod_{j=1}^{g} d_j + g - 1 \\ q &= g \\ K_{\mathcal{D}}^{g-1} &= g! \prod_{j=1}^{g} d_j \quad . \end{aligned}$$

Proof. First of all we recall that, for every $0 \leq j \leq g$, we have

$$h^j(A, \mathcal{O}_A) = \binom{g}{j} \quad .$$

However, $h^0(A, \mathcal{O}_A(\mathcal{D})) = \prod_{j=1}^{g} d_j$, and $h^j(A, \mathcal{O}_A(\mathcal{D}))$ vanishes for every $j > 0$, by the Kodaira vanishing theorem. By considering the usual exact sequence

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_A(\mathcal{D}) \longrightarrow \omega_{\mathcal{D}} \longrightarrow 0$$

we reach the desired conclusion:

$$\begin{aligned} p_g &= h^0(A, \mathcal{O}_A(\mathcal{D})) - h^0(A, \mathcal{O}_A) + h^1(A, \mathcal{O}_A) = \prod_{j=1}^{g} d_j - 1 + g \\ q &= h^{g-2}(\mathcal{D}, \omega_{\mathcal{D}}) = h^{g-1}(A, \mathcal{O}_A) = g \\ K_{\mathcal{D}}^{g-1} &= \mathcal{D}^g = h^0(A, \mathcal{O}_A(\mathcal{D})) \cdot g! = g! \prod_{j=1}^{g} d_j \quad . \end{aligned}$$

$\square$

1.2 The Gauss map

Even though the canonical map of a smooth divisor $\mathcal{D}$ in the linear system $|\mathcal{L}|$ of a (polarized) abelian variety $(A, \mathcal{L})$ can be explicitly expressed in terms of theta functions (as we saw in proposition 1.1.1), its image is not always easy to describe. If we allow $\mathcal{D}$ to be any divisor, not necessarily reduced and irreducible, the same proposition 1.1.1 provides for us a basis for the space of holomorphic sections of $\mathcal{L}|_{\mathcal{D}}$. It makes sense, however, to consider the map defined as follows:

$$\begin{aligned} G : \mathcal{D} &\dashrightarrow \mathbb{P}(V)^\vee \\ x &\mapsto \mathbb{P}(T_x\mathcal{D}) \quad . \end{aligned}$$

This map is called the **Gauss Map**, and it is clearly defined on the smooth part of the support of $\mathcal{D}$. In the particular case in which $\mathcal{D}$ is defined as the zero locus of a holomorphic non-zero section $\theta \in H^0(A, \mathcal{L})$, the map $G : \mathcal{D} \dashrightarrow \mathbb{P}(V)^\vee \cong \mathbb{P}^{g-1}$ is defined by the linear subsystem of $|\mathcal{L}|_{\mathcal{D}}|$ generated by $\frac{\partial\theta}{\partial z_1}, \cdots, \frac{\partial\theta}{\partial z_g}$.

Example 1.2.1. Let us consider the well-known case of a principal polarization Θ of the Jacobian $\mathcal{J}$ of a smooth curve $\mathcal{C}$ of genus g. In this case, the Gauss Map coincides with the map defined by the complete linear system $|\mathcal{J}(\Theta)|_\Theta|$ and it can be geometrically described as follows: the Abel-Jacobi theorem induces an isomorphism $\mathcal{J} \cong Pic^{g-1}(\mathcal{C})$, so Θ can be viewed, after a suitable translation, as a divisor of $Pic^{g-1}(\mathcal{C})$. The Riemann Singularity Theorem states (see. [3]):

$$mult_L\Theta = h^0(\mathcal{C}, L) \quad .$$

By a geometrical interpretation of the Riemann-Roch theorem for algebraic curves, it follows that a point L on the Theta divisor represented by the divisor $D = \sum_{j=1}^{g} P_j$ is smooth precisely when the linear span $\langle \phi(P_1), \cdots, \phi(P_g))\rangle$ in $\mathbb{P}(H^0(C, \omega_{\mathcal{C}}))$ is a hyperplane, where $\phi : \mathcal{C} \longrightarrow \mathbb{P}(H^0(\mathcal{C}, \omega_{\mathcal{C}}))^\vee$ denotes the canonical map of $\mathcal{C}$. Viewing now $\mathcal{J}$ as the quotient of $H^0(\mathcal{C}, \omega_{\mathcal{C}}^\vee)$ by the lattice $H_1(\mathcal{C}, \mathbb{Z})$, the Gauss Map associates to L the tangent space $\mathbb{P}(T_L\Theta)$, which is a hyperplane of $\mathbb{P}(T_L\mathcal{J}) = \mathbb{P}H^0(\mathcal{C}, \omega_{\mathcal{C}})^\vee$ defined as follows:

$$\begin{aligned} G \colon \Theta &\dashrightarrow \mathbb{P}H^0(\mathcal{C}, \omega_{\mathcal{C}})^\vee \\ \sum_{j=1}^{g} P_j &\mapsto \langle \phi(P_1), \cdots, \phi(P_g))\rangle \quad . \end{aligned}$$

It is then easy to conclude that, in this case, the Gauss map is dominant and generically finite, with degree $\binom{2g-2}{g-1}$.

Furthermore, in the particular case in which $\mathcal{C}$ is a genus 3 non-hyperelliptic curve, which we assume to be embedded in $\mathbb{P}^2$ via the canonical map, Θ is smooth and the Gauss map is then nothing but the map which associates, to a divisor $P+Q$, the line in $\mathbb{P}^2$ spanned by P and Q if $P \neq Q$ and $T_P(\mathcal{C})$ if $P = Q$. In particular, the Gauss map G is a covering of degree 6 of $\mathbb{P}^2$ branched on $\mathcal{C}^\vee$, the dual curve of $\mathcal{C}$, which has 28 nodes corresponding to the bitangent lines of $\mathcal{C}$, and 24 cusps corresponding to the tangent lines passing through a Weierstrass points of $\mathcal{C}$.

We will see in a moment, however, that this good behavior of the Gauss map arises in more general situations. There is furthermore a close connection between the property for the Gauss map of a reduced and irreducible divisor $\mathcal{D}$ of being dominant, and the property for $\mathcal{D}$ of being ample and of general type (see even [29]). It is known that a divisor $\mathcal{D}$ on an abelian variety is of general type if and only if there is no non-trivial abelian subvariety whose action on A by translation stabilizes $\mathcal{D}$. Indeed, the following theorem holds (see [29], theorem 4):

Theorem 1.2.2. *(Ueno) Let V a subvariety of an abelian variety A. Then there exist an abelian subvariety B of A and an algebraic variety W which is a subvariety of an abelian variety such that*

- *V is an analytic fiber bundle over W whose fiber is B,*
- *$\kappa(W) = dim W = \kappa(B)$.*

B is characterized as the maximal connected subgroup of A such that $B + V \subseteq V$.

Remark 1.2.3. We can conclude that for a reduced and irreducible divisor $\mathcal{D}$ on an abelian variety A the following are equivalent:

1) The Gauss map of $\mathcal{D}$ is dominant and hence generically finite.

2) $\mathcal{D}$ is an algebraic variety of general type.

3) $\mathcal{D}$ is an ample divisor.

Indeed, we recall that a divisor $\mathcal{D}$ on an abelian variety is ample if and only if it is not translation invariant under the action of any non-trivial abelian subvariety of A. (see [32], p 60). The equivalence of 1) and 3) follows by ([8] 4.4.2), and the idea is that if the Gauss map is not dominant, then $\mathcal{D}$ is not ample because it would be invariant under the action of a non-trivial abelian subvariety. The equivalence of 2) and 3) follows now easily by applying the previous theorem of Ueno 1.2.2.

Remark 1.2.4. If $\mathcal{D}$ is a smooth, ample divisor, then the Gauss map of $\mathcal{D}$ is a finite morphism. Indeed, if G contracted a curve $\mathcal{C}$, without loss of generality we could suppose that $\frac{\partial\theta}{\partial z_j}$ is identically 0 on $\mathcal{C}$ for every $j = 1 \cdots g-1$, and that $\frac{\partial\theta}{\partial z_g}$ has no zeros on $\mathcal{C}$. It would follows that $\omega_{\mathcal{D}}|_{\mathcal{C}} \cong \mathcal{O}_{\mathcal{C}}$, which would contradict the fact that $\omega_{\mathcal{D}}$ is ample on $\mathcal{D}$, $\omega_{\mathcal{D}}$ being the restriction of the ample line bundle $\mathcal{O}_A(\mathcal{D})$ to $\mathcal{D}$.
In particular, an ample smooth surface in an abelian 3-fold A is a minimal surface of general type.

1.3 Moduli

In this section we introduce some notations about moduli and theta functions. When we consider a polarized abelian variety $(A, \mathcal{L})$, we may choose a particular symplectic basis for the alternating form $E := \Im m\ H$ where $H = c_1(\mathcal{L})$. However, a choice of a symplectic basis of Λ for $\mathcal{L}$ naturally induces a decomposition of the vector space V into a sum of two $\mathbb{R}$-vector space of dimension g and a decomposition of the lattice $\Lambda(\mathcal{L})$ into a direct sum of two isotropic free $\mathbb{Z}$-modules of rank g

$$\Lambda(\mathcal{L}) = \Lambda(\mathcal{L})_1 \oplus \Lambda(\mathcal{L})_2 \quad .$$

This decomposition induces a natural decomposition of $K(\mathcal{L}) := {}^{\Lambda(\mathcal{L})}/_{\Lambda}$ into a direct sum of two isomorphic abelian groups K_1 and K_2, both of order $\prod_j d_j$. Moreover, there exists an isogeny $p : A \longrightarrow \mathcal{J}$ with kernel isomorphic to K_2, where $(\mathcal{J}, \mathcal{M})$ denotes a principally polarized abelian variety such that $\mathcal{L} = p^*\mathcal{M}$. Indeed, $\mathcal{J}$ is exactly ${}^{V}/_{\Gamma}$, where

$$\Gamma = \Lambda(\mathcal{L})_1 \oplus \Lambda_2 \quad ,$$

and this decomposition of Γ is similarly symplectic for $\mathcal{M}$. The projection p maps the group K_1 isomorphically onto a finite subgroup $\mathcal{G}$ of $\mathcal{J}$. Once we have chosen a factor of automorphy $[\{\phi_\gamma\}_{\gamma\in\Gamma}]$ representing $\mathcal{M}$ in $H^1(\Gamma, H^0(\mathcal{O}_V^*))$, we denote by θ_0 a generator of $H^0(\mathcal{J}, \mathcal{M})$ satisfying, for every z in V and for every $\gamma \in \Gamma$:

$$\theta_0(z+\gamma) = \phi_\gamma(z)\theta_0(z) \quad .$$

in conclusion, we can easily see, (see [8] p. 55) that a basis for $H^0(A, \mathcal{L})$ is given by $\{\theta_s\}_{s\in\mathcal{G}}$ where:

$$\theta_s(z) := \phi_s(z)^{-1}\theta_0(z+s) \quad . \tag{1.5}$$

Definition 1.3.1. Given a D-polarized abelian variety $(A, \mathcal{L})$, a choice of the symplectic basis defines an isomorphism of A with $A_{(\tau,D)} := \mathbb{C}^g \big/ \Pi$, where

$$\Pi = \tau\mathbb{Z}^g \oplus D\mathbb{Z}^g$$

for a certain τ in $\mathcal{H}_g$. For this reason, we will consider $\mathcal{H}_g$ as a moduli space of D-polarized abelian varieties with symplectic basis, and we will denote by $\mathcal{A}_D = \mathcal{H}_D \big/ \Gamma_D$ the moduli space of D-polarized abelian varieties, which is a normal complex analytic spaces of dimension $\frac{1}{2}g(g+1)$ (see [8], chapter 8).

Definition 1.3.2. For a fixed $\tau \in \mathcal{H}_g$ and D a polarization type, we recall that a global holomorphic section of the polarization on $A_{(\tau,D)}$ is the Riemann theta function defined as follows:

$$\theta_0(z,\tau) := \sum_{n \in \mathbb{Z}^g} e^{\pi i \cdot {}^t n \tau n + 2\pi i \cdot {}^t n z} \quad .$$

The Riemann theta function satisfies, for every λ in $\mathbb{Z}^g$, the functional equation

$$\theta_0(z+\lambda,\tau) = \phi_\lambda(z)\theta_0(z,\tau) \quad ,$$

where

$$\phi_\lambda(z) = e^{-\pi^t \lambda \tau \lambda - 2\pi i^t \lambda z} \quad .$$

Let us consider now $g_1, \cdots, g_n$ a set of representatives of $D^{-1}\mathbb{Z}^g$ in $\mathbb{Z}^g$. A basis for the vector space of the holomorphic sections of the polarization on $A_{(\tau,D)}$ is given by the set of Theta functions $\theta_{g_1}(z), \ldots, \theta_{g_n}(z)$, where

$$\theta_g(z) := \phi_g(z)^{-1}\theta_0(z+g\tau,\tau) \quad .$$

When we have a smooth ample divisor $\mathcal{D}$ in an abelian variety A, we can prove that the family over $\mathcal{A}_D$ with fibers the linear systems $|\mathcal{D}|$ is a Kuranishi family (see [7], Lemma 4.2). We observe moreover that $\mathcal{D}$ moves in a smooth family of dimension $\frac{1}{2}g(g+1) + dim|\mathcal{D}|$. Hence, in order to prove that the Kuranishi family of $\mathcal{D}$ is smooth, is enough to prove that the Kodaira-Spencer map is surjective.

Proposition 1.3.3. *Let A be an Abelian variety of dimension g and let $\mathcal{D}$ be a smooth ample divisor on A. Then:*

$$dim\ Ext^1_{\mathcal{O}_\mathcal{D}}(\Omega^1_\mathcal{D}, \mathcal{O}_\mathcal{D}) = \frac{1}{2}g(g+1) + dim|\mathcal{D}| \quad .$$

Proof. The sheaf $\Omega^1_{\mathcal{D}}$ is locally free, $\mathcal{D}$ being smooth, and we have that

$$Ext^1_{\mathcal{O}_{\mathcal{D}}}(\Omega^1_{\mathcal{D}}, \mathcal{O}_{\mathcal{D}}) \cong H^1(\mathcal{D}, \mathcal{T}_{\mathcal{D}}) \quad .$$

From the tangent bundle sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{D}} \longrightarrow \mathcal{O}^g_{\mathcal{D}} \longrightarrow \mathcal{O}_{\mathcal{D}}(\mathcal{D}) \longrightarrow 0 \quad . \tag{1.6}$$

We see that it is enough to prove the surjectivity of the map

$$H^1(\mathcal{D}, \mathcal{T}_A|_{\mathcal{D}}) = H^1(\mathcal{D}, \mathcal{O}^g_{\mathcal{D}}) \longrightarrow H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D})) \tag{1.7}$$

in the long cohomology exact sequence of 1.6. Indeed, once proved that the map in 1.7 is surjective, we have

$$h^1(\mathcal{D}, \mathcal{T}_{\mathcal{D}}) = h^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D})) + h^1(\mathcal{D}, \mathcal{O}^g_{\mathcal{D}}) - h^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D})) + h^0(\mathcal{D}, \mathcal{O}^g_{\mathcal{D}}) \quad ,$$

and with same computations used to prove proposition 1.1.2, we obtain

$$\begin{aligned} h^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D})) &= dim|\mathcal{D}| + g \\ h^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D})) &= h^2(A, \mathcal{O}_A) = \binom{g}{2} \\ h^0(\mathcal{D}, \mathcal{O}^g_{\mathcal{D}}) &= g \\ h^1(\mathcal{D}, \mathcal{O}^g_{\mathcal{D}}) &= g \cdot h^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}) = g \cdot h^{g-2}(\mathcal{D}, \omega_{\mathcal{D}}) = g \cdot h^{g-1}(A, \mathcal{O}_A) = g^2 \quad . \end{aligned} \tag{1.8}$$

The claim of the proposition follows now easily.

Let us prove that the map in 1.7 is surjective. This map can be described as the cup product

$$H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}) \otimes H^0(A, \mathcal{T}_A) \longrightarrow H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D})) \tag{1.9}$$

this map being nothing but the map which associates to a couple $([\omega], \partial)$ the class $[\omega \cdot \partial\theta]$, where $\mathcal{D} = (\theta)_0$. The polarization on A yields a morphism $\phi : A \longrightarrow Pic^0(A)$, whose differential in 0 is a natural isomorphism $H^0(A, \mathcal{T}_A) \cong T_0(A) \longrightarrow H^1(A, \mathcal{O}_A)$. Hence we conclude that the bilinear map in 1.9 can be seen as the map given by the cup product of $H^1(A, \mathcal{O}_A) \otimes H^1(A, \mathcal{O}_A)$ into $H^2(A, \mathcal{O}_A)$, which is surjective.

□

1.4 Surfaces in $(1,1,2)$-polarized abelian threefolds

In this section, we consider surfaces in the polarization of a general $(1,1,2)$-polarized Abelian 3-fold $(A,\mathcal{L})$. The invariants of such surfaces are (c.f. proposition 1.1.2) $p_g = 4$, $q = 3$ and $K^2 = 12$, and we have a smooth Kuranishi family of dimension 7 (c.f. proposition 1.3.3).
Let $p : A \longrightarrow \mathcal{J}$ denote an isogeny of degree 2, where $\mathcal{J}$ is the Jacobian variety of a smooth quartic plane curve $\mathcal{D}$. We can write in particular $\mathcal{J}$ as a quotient $\mathcal{J} \cong \mathbb{C}^3 / \Lambda$, where Λ is the lattice in $\mathbb{C}^3$ defined as

$$\Lambda := \Lambda_1 \oplus \Lambda_2 \quad , \tag{1.10}$$

where $\Lambda_1 := \tau\mathbb{Z}^3$, $\Lambda_2 := \mathbb{Z}^3$ and τ a general point in the Siegel upper half-space $\mathcal{H}_3$. Without loss of generality, we can suppose that

$$(A,\mathcal{L}) = \left(\mathbb{C}^3 / \Lambda^\eta, p^*\mathcal{O}_{\mathcal{J}}(\Theta)\right) \quad ,$$

where $\eta \in \mathcal{J}[2]$, Θ is the Theta Divisor of $\mathcal{J}$, and

$$\Lambda^\eta := \{\lambda \in \Lambda \mid E(\lambda,\eta) \in \mathbb{Z}\} \quad .$$

The decomposition 1.10 induces a decomposition of $\mathbb{C}^3$ into direct sum of two real subvector spaces

$$\mathbb{C}^3 = V_1 \oplus V_2 \quad . \tag{1.11}$$

Considered this latter decomposition 1.11, we can decompose the lattices

$$\Lambda^\eta \subseteq \Lambda \subseteq \Lambda_\eta := \Lambda + \eta\mathbb{Z}$$

by considering their intersection with the respective real subvector spaces of the decomposition 1.11:

$$\begin{aligned} \Lambda^\eta_j &:= \Lambda^\eta \cap V_j \\ \Lambda_{\eta,j} &:= \Lambda_\eta \cap V_j \quad . \end{aligned} \tag{1.12}$$

We obtain in particular, with $j = 1,2$

$$\Lambda_{\eta,j} \Big/ \Lambda^\eta_j \cong \mathbb{Z}_2 \quad .$$

Once we have chosen representatives of the respective non-zero classes γ in $\Lambda_{\eta,1}$ and δ in $\Lambda_{\eta,2}$, we have that θ_0, θ_γ is a basis for the space $H^0(A,\mathcal{L})$ (cf. 1.3.2), while δ represents the non-trivial element of $Ker(p)$.
Under the previous setup, we can state the following theorem (see [14], Theorem 6.4)

Theorem 1.4.1. *Let $\mathcal{S}$ be a smooth divisor yielding a polarization of type $(1,1,2)$ on an Abelian 3-fold. Then the canonical map of $\mathcal{S}$ is, in general, a birational morphism onto a surface Σ of degree 12.*
In the special case where $\mathcal{S}$ is the inverse image of the theta divisor in a principally polarized Abelian threefold, the canonical map is a degree 2 morphism onto a sextic surface Σ in $\mathbb{P}^3$. In this case the singularities of Σ are in general: a plane cubic Γ which is a double curve of nodal type for Σ and, moreover, a strictly even set of 32 nodes for Σ (according to [9] definition 2.5). Also, in this case, the normalization of Σ is in fact the quotient of $\mathcal{S}$ by an involution i on A having only isolated fixed points (on A), of which exactly 32 lie on $\mathcal{S}$.

To clarify the claim of the theorem and to explain the methods used to prove it, we now quickly recall some definitions.

Definition 1.4.2. (Strictly even set, cf. [9], definition 2.5) Let $\mathcal{S}$ be a surface, $N = \{P_1, \cdots P_t\}$ be a set of nodes, $\pi : \widetilde{\mathcal{S}} \longrightarrow \mathcal{S}$ the blow-up of $\mathcal{S}$ along N, and $E_i := \pi^{-1}(P_i), (1 \leq i \leq t)$. The set N is said to be **strictly even** if the divisor $\sum_i E_i$ is divisible by 2 in $Pic(\widetilde{\mathcal{S}})$.

Observation 1.4.3. In other words, a strictly even set of nodes is precisely a set of nodes such that there exists a double cover $p\colon X \longrightarrow \mathcal{S}$ ramified exactly at the nodes of N. Indeed, if one takes such a set N on a surface $\mathcal{S}$ and considers $\widetilde{\mathcal{S}}$ the blow-up of $\mathcal{S}$ along N, then there exists $\eta \in Pic(\widetilde{\mathcal{S}})$ such that $2\eta = \mathcal{O}_{\widetilde{\mathcal{S}}}(E)$, where $E = \sum_i E_i$ and the divisors E_i are (-2)-curves. There exists then a double cover $\widetilde{p}\colon \widetilde{X} \longrightarrow \widetilde{\mathcal{S}}$ whose branch locus is equal to E. It is now easy to show that the preimages of all exceptional divisors E_i are (-1)-curves on $\widetilde{X}$, and therefore they can be contracted in order to obtain the desired surface X with a double cover $X \longrightarrow \mathcal{S}$ branched only on the set N. We have in particular that the maps p and $\widetilde{p}$ fit into a diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{p}"'] \arrow[d, "\pi_X"] & \widetilde{\mathcal{S}} \arrow[d, "\pi_{\mathcal{S}}"] \\
X \arrow[r, "p"'] & \mathcal{S}
\end{tikzcd}$$

Hence, we have that $\omega_{\widetilde{\mathcal{S}}} = \pi_{\mathcal{S}}^* \omega_{\mathcal{S}}(E)$ and $\omega_{\widetilde{X}} = \widetilde{p}\pi_X^* \omega_{\widetilde{\mathcal{S}}}(A) = \widetilde{p}^*(\omega_{\widetilde{\mathcal{S}}})(A)$, where A is the pullback of the exceptional divisor E in $\widetilde{\mathcal{S}}$. Following the other direction in the diagram, we see that

$$\pi_X^* p^* \omega_{\mathcal{S}} = \widetilde{p}^* \pi_{\mathcal{S}}^* \omega_{\mathcal{S}} = \widetilde{p}^*(\omega_{\widetilde{\mathcal{S}}}(-E)) = \omega_{\widetilde{X}}(-2A) \quad ,$$

and we can conclude that $p^*\omega_{\mathcal{S}} = \omega_X$. If we assume that the surface $\mathcal{S}$ is smooth outside the set N, then X is smooth and, applying the Riemann Roch formula we obtain:

$$\begin{aligned}\chi(X) = \chi(\widetilde{X}) &= \chi(\mathcal{O}_{\widetilde{X}}) \\ &= \chi(\widetilde{p}_*\mathcal{O}_{\widetilde{X}}) \\ &= \chi(\mathcal{O}_{\widetilde{\mathcal{S}}} \oplus -\eta) \\ &= 2\chi(\mathcal{O}_{\mathcal{S}}) + \frac{1}{2}(-\eta).(-\omega_{\widetilde{\mathcal{S}}} - \eta) \quad .\end{aligned}$$

Because we blow up nodes, $E^2 = -2t$, and then $E.K_{\widetilde{\mathcal{S}}} = 0$. We get, in conclusion,

$$\chi(X) = 2\chi(\mathcal{S}) - \frac{t}{4} \quad . \tag{1.13}$$

We can go now deeper into the details of the proof of the theorem 1.4.1. We can consider first the case in which $\mathcal{S}$ is defined as the zero locus of θ_0 in A. It is easy to see that θ_0 and θ_γ are even functions, and the derivatives $\frac{\partial\theta_0}{\partial z_j}$ are then odd functions. This implies that the involution $z \mapsto z + \delta$ on A changes the sign only to θ_γ. Therefore every holomorphic section of the canonical bundle of $\mathcal{S}$ factors through the involution

$$\iota\colon z \mapsto -z + \delta \quad , \tag{1.14}$$

and we conclude that the canonical map of $\mathcal{S}$ factors through the quotient $Z := \mathcal{S}/_{\iota}$ and it cannot be, in particular, birational.
Let us consider now the commutative diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d] & Z \arrow[d] \\ \Theta \arrow[r] & Y := \Theta/\langle -1\rangle \end{tikzcd} \tag{1.15}$$

Multiplication by -1 on $\mathcal{J}$ corresponds to the Serre involution $\mathcal{L} \mapsto \omega_{\mathcal{D}} \otimes \mathcal{L}^\vee$ on $Pic^2(\mathcal{D})$, which can be expressed on Θ as the involution which associates, to the divisor $P+Q$ on $\mathcal{D}$, the unique divisor $R+S$ such that $P+Q+R+S$ is a canonical divisor on $\mathcal{D}$.

It can be now easily seen that the projection of Θ onto Y in diagram 1.15 is a covering branched on 28 points corresponding to the 28 bitangents of $\mathcal{D}$, hence Y has exactly 28 nodes. This set corresponds to the set of odd 2-torsion points of the Jacobian, on which the group $Sp(\mathcal{J}[2], \mathbb{Z}_2)$ acts transitively:

$$\begin{aligned}\mathcal{J}[2]^-(\Theta) &= \{z \in \mathcal{J}[2] \mid mult_z\Theta \text{ is even}\} \\ &= \{z \in \mathcal{J}[2] \mid E(2z_1, 2z_2) = 1 \pmod 2\} \quad .\end{aligned}$$

This set can be viewed as the union of the two sets $p(\mathcal{F}ix(\iota)) \cap \mathcal{J}[2]^-(\Theta)$ and its complementary set:

$$\begin{aligned}p(\mathcal{F}ix(\iota)) \cap \mathcal{J}[2]^-(\Theta) &= \{[v] \in \mathcal{J}[2]^-(\Theta) \mid [2v] = 0 \quad \text{in } A\} \\ &= \{z \in \mathcal{J}[2]^-(\Theta) \mid E(2z, 2\eta) = 0 \pmod 2\} \quad .\end{aligned}$$

This latter set has cardinality equal to 16 and thus the involution ι has exactly 32 fixed points on $\mathcal{S}$. In conclusion, the map $Z \longrightarrow Y$ in diagram 1.15 is a double cover, unramified except over the remaining 12 nodes of Y. We have however that $p_g(Y) = 3$, $p_g(Z) = 4$, $K_Y^2 = 3$, $K_Z^2 = 6$ so according to 1.13, $q(Z) = q(Y) = 0$.

Lemma 1.4.4. *The canonical map of Z is birational. Hence, the canonical map of $\mathcal{S}$ is of degree 2, and its image is a surface Σ of degree 6 in $\mathbb{P}^3$.*

Proof. We recall that the canonical map of $\mathcal{S}$ factors through the involution ι in 1.14 and the canonical map of Z, which we denote by:

$$\phi_Z : Z \longrightarrow \Sigma \subseteq \mathbb{P}^3 \quad .$$

In particular, ϕ_Z is defined by the theta functions $[\theta_\gamma, \frac{\partial\theta_0}{\partial z_1}, \frac{\partial\theta_0}{\partial z_2}, \frac{\partial\theta_0}{\partial z_3}]$. Clearly, the degree of the canonical map of Z is at most 3. Indeed, the Gauss map of $\mathcal{S}$ factors through Z via a map, which we still denote by $G : Z \longrightarrow \mathbb{P}^2$, which is of degree 6 and invariant with respect to the involution $(-1)_Z$ induced by the multiplication by (-1) of S, while the canonical map is not. Moreover, G is ramified on a locus of degree 24 which represents the dual curve $\mathcal{D}^\vee$ of $\mathcal{D}$ (of degree 12) counted with multiplicity 2. If we denote by $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ the projection which forgets the first coordinate, its restriction to Σ must have a ramification locus R whose degree is divisible by 12, and we have:

$$deg\Sigma = degK_\Sigma = -3deg(\pi|_\Sigma) + degR \quad .$$

However, $deg(\pi|_\Sigma) = deg\Sigma$, and hence $4deg\Sigma = degR$. In particular, the canonical map of Z can have degree 1 or 2.
Let us suppose by absurd that ϕ_Z has degree 2. Then ϕ_Z would be invariant respect to an involution j on Z of degree 2, and the map G would be invariant respect to the group $\mathcal{G}$ generated by $(-1)_Z$ and j. Hence, this group has a

natural faithful representation $\mathcal{G} \longrightarrow \mathcal{S}_3$, the symmetric group of degree 3, so $\mathcal{G} \cong \mathcal{S}_3$
The ramification locus in Y of ϕ_Y has two components:

$$\mathcal{T}_1 := \{[2p] \mid p \in \mathcal{D}\}$$
$$\mathcal{T}_2 := \{[p+q] \mid \langle p,q\rangle . \mathcal{D} = 2p+q+r \text{ for some } q,r \text{ on } \mathcal{D}\} \quad .$$

The component $\mathcal{T}_2$ has to be counted twice on Y, while $\mathcal{T}_1$ has multiplicity 1. The components of the counterimage of $\mathcal{T}_1$ in Z are both of multiplicity 1 and the group $\mathcal{G}$ acts on them. Clearly, the involution $(-1)_Z$ exchanges them, while both components must be pointwise fixed under the action of j. But the product $(-1)_Z \cdot j$, has order 3 (because $\mathcal{G} \cong \mathcal{S}_3$), hence must fix both components, and we reach a contradiction.

□

The canonical models of surfaces with invariants $p_g = 4$, $q = 0$, $K^2 = 6$ and birational canonical map have been studied extensively in [10]. In this case, there is a symmetric homomorphism of sheaves:

$$\alpha = \begin{bmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{01} & \alpha_{11} \end{bmatrix} : (\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2))^\vee(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \quad ,$$

where α_{00} is contained in the ideal I generated by α_{01} and α_{11}. The canonical model Y is defined by $det(\alpha)$ and the closed curve Γ defined by I is a cubic contained in the projective plane $\alpha_{11} = 0$.

Remark 1.4.5. In our situation, the curve Γ is nothing but the image of the canonical curve $\mathcal{K}$ defined as the zero locus of θ_γ in $\mathcal{S}$. Indeed, let us denote by $\bar{\mathcal{K}}$ the image of $\mathcal{K}$ in Z. The curve $\mathcal{K}$ does not contain fixed points of ι, so $\mathcal{K}$ and $\bar{\mathcal{K}}$ are isomorphic. Moreover, the locus $\bar{\mathcal{K}}$ is stable under the involution (-1) on $\mathcal{S}$, and the canonical map of Z is of degree 2 on $\bar{\mathcal{K}}$. Then, the image of $\bar{\mathcal{K}}$ in Σ is a curve of nodal type, and then it must be exactly Γ. Moreover, the set $\mathcal{P}$ of pinch points on Γ is exactly the image of the set of the 2-torsion points of A which lie on the canonical curve $\mathcal{K}$, which is in bijection with the set

$$\left\{(x,y) \in \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \;\mid x_2y_2 + x_3y_3 = 1 \right\} \quad .$$

The latter set consists precisely of 24 points, and $\mathcal{P}$ is then nothing else but the image of the branch locus of the map $\bar{\mathcal{K}} \longrightarrow \Gamma$, which has degree 2 and factors with respect to the involution $(-1)_Z$.

Chapter 2

Polarizations of type $(1, 2, 2)$ and $(1, 1, 4)$

From the results of the first chapter (see for instance 1.1.2), we see easily that smooth ample divisors of type $(1, 2, 2)$ and smooth ample divisors of type $(1, 1, 4)$ on an abelian 3-fold have the same invariants. In this section, we want to characterize more geometrically the two polarization types. More precisely, considered the Jacobian variety $(\mathcal{J}, \Theta)$ of a non-hyperelliptic curve $\mathcal{D}$ of genus 3, we can associate to a couple of distinct 2-torsion elements η_1 and η_2 in $\mathcal{J}$ the datum of an abelian variety A together with an isogeny $p : A \longrightarrow \mathcal{J}$. We can then consider the smooth genus 9 curve $\mathcal{C}$ in A obtained by pulling back $\mathcal{D}$ along p. We prove in the last section of this chapter that the gonality of $\mathcal{C}$ is 4 or 6, and the first case occurrs precisely when the polarization $|p^*\Theta|$ on A is of type $(1, 2, 2)$.

2.1 The linear system

In the last section of the first chapter we went deep into the study of the behavior of canonical map of an unramified double cover of the Theta divisor of a Jacobian of $\mathcal{D}$, where $\mathcal{D}$ is a non-hyperelliptic quartic plane curve.

We want now to follow the same strategy to study those special elements in the polarization of an abelian 3-fold A which are bidouble covers of the Theta divisor in a Jacobian 3-fold $\mathcal{J}$. In particular, we are interested in the surfaces

$\mathcal{S}$ such that there exists a cartesian diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook] \arrow[d, "p"] & A \arrow[d, "p"] \\ \Theta \arrow[r, hook] & \mathcal{J} \end{tikzcd} \tag{2.1}$$

where p is an isogeny with kernel isomorphic to $\mathbb{Z}_2^2$. We use the same notation in 1.10, and we write the following decomposition of the lattice $\Lambda = \Lambda_1 \oplus \Lambda_2$ where $\Lambda_1 := \tau\mathbb{Z}^3$, $\Lambda_2 := \mathbb{Z}^3$ and $\tau \in \mathcal{H}_3$. We consider the abelian variety

$$A = \mathbb{C}^3 \big/ \Gamma \quad ,$$

where $\eta_1, \eta_2 \in \mathcal{J}[2]$ and $\Gamma := \{\lambda \in \Lambda \mid E(\lambda, \eta_j) \in \mathbb{Z}, \quad j = 1,2\}$. With $\mathcal{L} := p^*\mathcal{O}_{\mathcal{J}}(\Theta)$, the decomposition of Λ induces a decomposition of the following sublattices (for definitions see also 1.11 and 1.12):

$$\Gamma \subseteq \Lambda \subseteq \Gamma(\mathcal{L}) := \Lambda + \eta_1\mathbb{Z} + \eta_2\mathbb{Z} \quad ,$$

where $2\eta_1, 2\eta_2 \in \Lambda$. Thus, so we can write, with $j = 1,2$,

$$\Gamma_j \subseteq \Lambda_j \subseteq \Gamma(\mathcal{L})_j \quad .$$

We distinguish now two different cases:

$$K(\mathcal{L}) := \Gamma(\mathcal{L}) \big/ \Gamma \cong \begin{cases} \mathbb{Z}_2^4 & \text{if the polarization is of type } (1,2,2) \\ \mathbb{Z}_4^2 & \text{if the polarization is of type } (1,1,4) \end{cases} \tag{2.2}$$

That means that the polarization is of type $(1,1,4)$ if and only if the order of η_1 and η_2 in $K(\mathcal{L})$ is 4. But this happens precisely when $\lambda(\eta_1, \eta_2) = 1$, where λ denotes the natural symplectic pairing induced by E

$$\lambda : \mathcal{J}[2] \times \mathcal{J}[2] \longrightarrow \tfrac{1}{4}\mathbb{Z} \big/ \tfrac{1}{2}\mathbb{Z} \cong \mathbb{Z}_2 \quad . \tag{2.3}$$

Definition 2.1.1. In the case in which the polarization is of type $(1,2,2)$, we will denote by α and β a set of generators of $\Gamma(\mathcal{L})_1 \big/ \Gamma_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and by a and b a set of generators of $\Gamma(\mathcal{L})_2 \big/ \Gamma_2 = Ker(p)$ such that $\lambda(\alpha, a) = \lambda(\beta, b) = 0$ and $\lambda(\alpha, b) = \lambda(\beta, a) = 1$.
The theta functions θ_0, θ_α, θ_β, $\theta_{\alpha+\beta}$ give a basis for $H^0(A, \mathcal{L})$, and for every $\gamma \in \Gamma(\mathcal{L})_1 \big/ \Gamma_1$ and every g in $\mathcal{G} := Ker(p)$ holds

$$\theta_\gamma(z + g) = e^{\pi i \lambda(\gamma, g)}\theta_\gamma(z) \quad .$$

Definition 2.1.2. In the case in which the polarization is of type $(1,1,4)$, we can consider a generator α of $\Gamma(\mathcal{L})_1 \big/ \Gamma_1 \cong \mathbb{Z}_4$ and a generator a of $G := \Gamma(\mathcal{L})_2 \big/ \Gamma_2$ such that $E(\alpha, a) = \frac{1}{4}$.
The theta functions θ_0, θ_α, $\theta_{2\alpha}$, $\theta_{3\alpha}$ give a basis for $H^0(A, \mathcal{L})$, and it holds

$$\theta_\alpha(z+a) = i\theta_\alpha(z) \quad .$$

Remark 2.1.3. It can be easily proved that the linear system $|\mathcal{L}|$ is base point free if the polarization is of type $(1,1,4)$, while in the case in which the polarization is $(1,2,2)$ the base locus is a translated of $K(\mathcal{L})$, and in particular it is a set of cardinality 16. Indeed, if we suppose $\mathcal{L}$ of characteristic 0 as in diagram 2.1 and in definition 2.1.1, then each holomorphic section of $\mathcal{L}$ is even, and thus:

$$\mathcal{B}(\mathcal{L}) = A[2]^-(S) = \{x \in A[2] \mid mult_x S \text{ is odd}\} \quad .$$

2.2 The Weil pairing

We have seen that with the pairing λ defined in 2.2 we can distinguish the polarizations types of the bidouble covers of a principally polarized abelian variety $\mathcal{J}$. When $\mathcal{J}$ is the Jacobian variety of a smooth projective $\mathcal{D}$ of genus 3, we can efficiently compute this pairing in terms of divisors. Indeed, η_1 and η_2 can be considered as elements $Pic^0(\mathcal{D})$ and hence as a couple of 2 torsion line bundles on $\mathcal{D}$ to which corresponds a bidouble unramified cover $p : \mathcal{C} \longrightarrow \mathcal{D}$ and a cartesian diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "p"] & A \arrow[d, "p"] \\ \mathcal{D} \arrow[r, hook] & \mathcal{J} \end{tikzcd} \tag{2.4}$$

Conversely, given a smooth projective curve of $\mathcal{D}$ and 2-torsion line bundles η_1 and η_2 on $\mathcal{D}$, we can construct a cartesian diagram like in 2.4.

Definition 2.2.1. Let f be a meromorphic function on an algebraic curve $\mathcal{C}$ and $D = \sum_{P \in \mathcal{C}} n_P P$ a divisor on $\mathcal{C}$. We denote by $m_P(D) := n_P$ the multiplicity of D at P. If the support of the divisor D does not contain zeros or poles of f, it makes sense to consider:

$$f(D) := \prod_{P \in \mathcal{C}} f(P)^{m_P(D)} \in \mathbb{C}^* \quad . \tag{2.5}$$

Suppose now we have f and g two meromorphic functions, and R, S two divisors on $\mathcal{C}$ such that, for every point P on $\mathcal{C}$, we have:

$$m_P(R)ord_P(g) - m_P(S)ord_P(f) = 0 \quad . \tag{2.6}$$

Then we can extend the previous definition 2.5 to define

$$\frac{f(S)}{g(R)} = \frac{\prod_P f(P)^{m_P(S)}}{\prod_Q g(Q)^{m_Q(R)}} \in \mathbb{C}^* \quad ,$$

where both products are considered on the set of the points of $\mathcal{C}$ which are not zeros or poles of f or g.

With the same procedure used to prove Abel's theorem, one can prove the following proposition.

Proposition 2.2.2. *Let $\eta_1 = \mathcal{O}_{\mathcal{C}}(R)$ and $\eta_2 = \mathcal{O}_{\mathcal{C}}(S)$ be two torsion bundle on a curve $\mathcal{C}$ of genus $g \geq 1$. Then, with $nR = div(f)$, $nS = div(g)$, where f and g are some meromorphic functions on $\mathcal{C}$, and $n \geq 1$ a natural number, we have:*

$$\frac{f(S)}{g(R)} = e^{\frac{2\pi i}{n} E(\eta_1, \eta_2)} \quad .$$

A consequence of proposition 2.2.2 is that the pairing λ defined in on the set of 2-torsion points of a Jacobian is exactly the pairing defined by Weil. (see even J. Harris, [25]):

Definition 2.2.3. (Weil pairing) We have a well-defined pairing:

$$\lambda : \mathcal{J}(\mathcal{C})[2] \times \mathcal{J}(\mathcal{C})[2] \longrightarrow \mathbb{Z}_2$$
$$\lambda(\eta_1, \eta_2) := \frac{1}{\pi i} log \left(\frac{f(S)}{g(R)} \right) \in \mathbb{Z}_2 \quad ,$$

where $\eta_1 = \mathcal{O}_{\mathcal{C}}(R)$, $\eta_2 = \mathcal{O}_{\mathcal{C}}(S)$, $2R = div(f)$ and $2S = div(g)$.

Corollary 2.2.4. *(Reciprocity) Let f and g be two meromorphic functions on a curve $\mathcal{C}$. Then:*

$$f(div(g)) = g(div(f)) \quad .$$

Proof. If the genus g is 0, this is clear, because the meromorphic functions on $\mathbb{P}^1$ are simply rational functions. If $g \geq 1$, then it is enough to apply the previous proposition with $n = 1$. □

2.3 Gonality of the unramified bidouble covers of a smooth quartic curve

In this section, we show that the gonality of pull-back curve $\mathcal{C}$ in diagram 2.4 determines the polarization type of A. In particular, we prove that only two possibilities occur: either $\mathcal{C}$ is a tetragonal curve which belongs to $\overline{\mathcal{M}_{9,4}(2)}$ (see definition 2.3.3), or $\mathcal{C}$ has maximal gonality 6. We prove, moreover, that the first case arises precisely when the polarization of A is of type $(1,2,2)$, while in the second case the polarization type is $(1,1,4)$.

We recall that the gonality of a smooth algebraic curve $\mathcal{C}$, which we assume to be defined over an algebraically closed field, is the smallest possible degree of a nonconstant dominant rational map onto the projective line $\mathbb{P}^1(k)$. As a consequence of the Brill-Noether theory (see [3], Chapter V), the gonality of an algebraic curve of genus g is at most $\lfloor \frac{g+3}{2} \rfloor$, and the equality holds for a general algebraic curve of genus g. It is known that every algebraic curve of genus g of gonality k with $2 \leq k < \lfloor \frac{g+3}{2} \rfloor$ has a unique g^1_k ([2]). However, a general problem is to count the g^1_k's, up to a multiplicity. The following proposition suggests which g^1_k's have to be counted one (see Appendix to the article [20]):

Proposition 2.3.1. *Let $|\mathcal{L}|$ be a g^1_k on an algebraic curve $\mathcal{C}$. Then $h^0(\mathcal{C},\mathcal{L}^2) > 3$ if and only if it is the limit of two different g^1_k's in a family of curves.*

Definition 2.3.2. Let $|\mathcal{L}|$ be a g^1_k on an algebraic curve $\mathcal{C}$, where $\mathcal{L}$ is a line bundle on $\mathcal{C}$. We say that $\mathcal{L}$ **is of type** I if $h^0(\mathcal{C},\mathcal{L}^2) = 3$.

Moreover, the g^1_k's we consider must be independent in the sense of the following definition:

Definition 2.3.3. Let $|\mathcal{L}|$ and $|\mathcal{M}|$ be two g^1_k's on an algebraic curve $\mathcal{C}$. The linear systems $|\mathcal{L}|$ and $|\mathcal{M}|$ are called **dependent** if there is a non-trivial morphism $p : \mathcal{C} \longrightarrow \mathcal{C}'$ and two linear systems $|\mathcal{L}'|$ and $|\mathcal{M}'|$ on $\mathcal{C}'$ such that:

$$\begin{aligned} \mathcal{L} &= p^*\mathcal{L}' \\ \mathcal{M} &= p^*\mathcal{M}' \quad . \end{aligned}$$

Denoted by $\mathcal{M}_g$ the coarse moduli space of algebraic curves of genus g, and $\mathcal{M}_{g,k}$ the locus of k-gonal curves, we define (see [19]):

$$\mathcal{M}_{g,k}(m) = \{\mathcal{C} \in \mathcal{M}_{g,k} \ : \mathcal{C} \text{ has exactly } m \ g^1_k\text{'s pairwise independent and each of type I.}\} \quad .$$

We denote now by $\mathcal{C}$ an algebraic curve of genus 9, and by $p : \mathcal{C} \longrightarrow \mathcal{D}$ an unramified bidouble cover onto $\mathcal{D}$, where $\mathcal{D}$ denotes a non-hyperelliptic algebraic curve of genus 3. We will denote by $\mathcal{G}$ the group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting on $\mathcal{C}$ freely with quotient $\mathcal{D}$. Our goal is to investigate the possible canonical models of $\mathcal{C}$ in $\mathbb{P}^8$.

Definition 2.3.4. We remark that, if $|\mathcal{L}|$ is a g^1_d, we can define a morphism:

$$\begin{aligned} &\psi : \mathbb{P}^1 = \mathbb{P}(H^0(\mathcal{C},\mathcal{L})) \longrightarrow \mathcal{C}^{(d)} \\ &\psi(\big\langle s \big\rangle) := div(s) \quad . \end{aligned}$$

A divisor which belongs to the pencil $|\mathcal{L}|$ can be seen as a divisor on the canonical model of C. Then, called $\phi_{\omega_\mathcal{C}} : C \longrightarrow \mathbb{P}^{g-1}$ the canonical map, the scroll X associated to $\mathcal{L}$ is defined by:

$$X := \bigcup_{P\in\mathbb{P}^1} \overline{\phi_{\omega_\mathcal{C}}(div(s))} \quad .$$

The type of the scroll X is completely determined by the cohomology of $\mathcal{L}$, (see [37], Theorem 2.5). Indeed, the type of X is a list of integers $(e_1,\dots,e_d)$ satisfying

$$\begin{cases} e_1 & \geq \cdots \geq e_d \geq 0 \\ f & := e_1 + \cdots + e_d = g - d + 1 \quad , \end{cases} \tag{2.7}$$

which can be determined in the following way: first we write the partition of $g = h^0(\mathcal{C},\omega_\mathcal{C})$ defined as follows:

$$\begin{cases} d_0 & := h^0(\mathcal{C},\omega_\mathcal{C}) - h^0(\mathcal{C},\omega_\mathcal{C}\otimes\mathcal{L}^\vee) \\ d_1 & := h^0(\mathcal{C},\omega_\mathcal{C}\otimes\mathcal{L}^\vee) - h^0(\mathcal{C},\omega_\mathcal{C}\otimes\mathcal{L}^{2\vee}) \\ \vdots & := \vdots \\ d_j & := h^0(\mathcal{C},\omega_\mathcal{C}\otimes\mathcal{L}^{j\vee}) - h^0(\mathcal{C},\omega_\mathcal{C}\otimes\mathcal{L}^{2j\vee}) \\ \vdots & := \vdots \end{cases} \tag{2.8}$$

The indices of the type of X are given exactly by the following partition of g, which is dual to the partition $\{d_j\}_j$ in 2.8:

$$e_i = \#\{j \mid d_j \geq i\} - 1 \quad . \tag{2.9}$$

With the definitions in 2.3.4, it can be easily seen that $\mathcal{C}$ cannot be hyperelliptic. We see now that the case in which the gonality of $\mathcal{C}$ is 3 or 5 can also be excluded.

Lemma 2.3.5. *Let $\mathcal{C}$ be an algebraic curve of genus 9 with $p : \mathcal{C} \longrightarrow \mathcal{D}$ an unramified bidouble cover of a non-hyperelliptic algebraic curve $\mathcal{D}$ of genus 3. Then the gonality of $\mathcal{C}$ can be neither 3 nor 5.*

Proof. Let us suppose by absurd that the gonality of $\mathcal{C}$ is 3 and let us denote by $|\mathcal{L}|$ the unique g^1_3 on $\mathcal{C}$. Then $|\mathcal{L}|$ is $\mathcal{G}$-invariant, i.e., we have that, for every $g \in \mathcal{G}$ it holds that $g^*\mathcal{L} \cong \mathcal{L}$. Therefore, there is a well-defined action of $\mathcal{G}$ on $\mathbb{P}(H^0(\mathcal{C}, \mathcal{L}))$. Hence, the rational normal scroll X in $\mathbb{P}^{g-1} = \mathbb{P}(H^0(\mathcal{C}, \omega_\mathcal{C}))$ associated to $|\mathcal{L}|$ according to definition 2.3.4 is also $\mathcal{G}$-invariant respect to the natural action on $\mathbb{P}^{g-1}$. Because $|\mathcal{L}|$ is indecomposable, we can suppose that $\mathcal{G}$ is generated by two elements a and b whose action on $\mathbb{P}^1 = \mathbb{P}(H^0(C, \mathcal{L}))$ is represented respect to projective coordinates $[s, t]$ on $\mathbb{P}^1$ by the following matrices:

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad . \tag{2.10}$$

The scroll X in $\mathbb{P}^8$ is, according to 2.7, of type (e_1, e_2) with

$$\frac{2g(\mathcal{C}) - 2}{3} = \frac{16}{3} \geq e_1 \geq e_2 \geq \frac{5}{3} = \frac{g(\mathcal{C}) - 4}{3}$$
$$f = e_1 + e_2 = 7 \quad ,$$

and we may consider its normalization $\pi : \mathbb{P}(\mathcal{E}) \longrightarrow X$, with $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(e_2)$. Let us denote, furthermore, by ϕ_j the element in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H - e_jR))$ which corresponds the inclusion of the j-th summand ($j = 1, 2$):

$$\mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E}(-e_j) \cong \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(H - e_jR) \quad .$$

Because $e_1 \neq e_2$, the group $\mathcal{G}$ acts trivially on the basic sections ϕ_1 and ϕ_2. Hence, we can consider the basis for $H^0(\mathcal{C}, \omega_\mathcal{C})$ given by

$$\phi_1 s^{e_1}, \phi_1 s^{e_1-1}t, \ldots, \phi_1 st^{e_1-1}, \phi_1 t^{e_1}, \phi_2 s^{e_2}, \phi_2 s^{e_2-1}t, \ldots, \phi_2 st^{e_2-1}, \phi_2 t^{e_2} \quad .$$

Because one of the two indices e_i is odd, the action of $\mathcal{G}$ on $\mathbb{P}(H^0(\mathcal{C}, \omega_\mathcal{C}))$ does lift to a linear representation of $\mathcal{G}$ on $H^0(\mathcal{C}, \omega_\mathcal{C})$. This means that $\omega_\mathcal{C}$ is not $\mathcal{G}$-linearizable, which contradicts the assumption that the action of $\mathcal{G}$ on the curve $\mathcal{C}$ is free.

Let us now suppose that the gonality of $\mathcal{C}$ is 5. We may consider all possible configurations of g^1_5's over $\mathcal{C}$. Determining the possible values of m for which the locus $\mathcal{M}_{9,5}(m)$ is non-empty is considerably more involved. However, by applying 2.3.6 to our situation, We can conclude that $\mathcal{M}_{9,5}(m)$ is empty if $m > 6$. Concerning the other cases, one an prove (see [20]) that $\mathcal{M}_{9,5}(m)$ is non-empty precisely when $m \in \{1, 2, 3, 6\}$. We analyze now every possible situation.

In the case in which $\mathcal{C}$ has 1 or 3 simple g^1_5, arguing as in the case in which the gonality is 3, we can see that there exists a $\mathcal{G}$-invariant g^1_5 on $\mathcal{C}$, which has a $\mathcal{G}$-invariant rational normal scroll X. However, the possible types for X would be, according to 2.7,

$$(5,0,0,0) \quad (4,1,0,0) \quad (3,2,0,0) \quad (3,1,1,0) \quad (2,2,1,0) \quad (2,1,1,1) \quad .$$

Repeating the same procedure as before in the case of gonality 3, we see that a projective representation of $\mathcal{G}$ on $\mathbb{P}(H^0(\mathcal{C},\omega_{\mathcal{C}}))$ does not lift to a linear representation on $H^0(\mathcal{C},\omega_{\mathcal{C}})$. In the case in which we have 6 different g^1_5's, the curve $\mathcal{C}$ has a plane model $\mathcal{C}'$ of degree 7, with 6 double points, and the projections from those double points define the 6 simple g^1_5 (see [20]). The group $\mathcal{G}$ is represented as a group of projective linear transformation acting on this plane, and the action on the double points represents exactly the action on the different g^1_5's. Hence, there exists a non-trivial element γ of $\mathcal{G}$ which fixes two of the double points. Hence, this element γ fixes the line spanned by them, which must contain other 3 points of $\mathcal{C}'$ (counted with multiplicity). Thus, also in this case, the action of γ cannot be free on $\mathcal{C}$, contrary to our hypothesis.
If we suppose that $\mathcal{C}$ has only 2 different simple g^1_5's, then $\mathcal{C}$ has a plane model $\mathcal{C}'$ of degree 8 with 3 double points and 2 triple points, and the projection from the triple points define the 2 simple g^1_5. Then we can find again a non-trivial element γ which fixes the triple points. Blowing up $\mathcal{C}'$ in one of those triple points, we see that γ has to fix one of the 3 points in the preimage, and again we conclude that the action of γ cannot be free on $\mathcal{C}$. □

We analyze now the case in which the gonality of $\mathcal{C}$ is 4. As a first introductory step, we will show that $\mathcal{M}_{9,4}(m)$ is empty if $m \geq 3$ by applying the following result of D.M. Accola which generalized the Castelnuovo inequality:

Proposition 2.3.6. *(A generalized Castelnuovo inequality for more linear series g^1_d, see (Accola, [1])) Let $\mathcal{C}$, an algebraic curve of genus g, admit $s \geq 2$ different linear series g^1_d. Assume all these linear series are simple and independent accordingly to the definition 2.3.3. Let $d = m(s-1)+q$ where q is the residue modulo $(s-1)$ so that $-s+3 \leq q \leq 1$. Then:*

$$2g \leq sm^2(s-1) + 2m(q-1) + (q-2)(q-1) \quad .$$

By 2.3.6, we can easily deduce the following:

Proposition 2.3.7. *Let be $m \geq 3$. Then $\mathcal{M}_{9,4}(m)$ is empty.*

Proof. By applying the previous proposition, it can be easily seen that

$$2g \leq s^2 m^2 \quad .$$

However, because $m = q(s-1) + q$ and $q \leq 1$, we have

$$2g \leq s^2 \left(\frac{d-1}{s-1}\right)^2 = \left(\frac{s}{s-1}\right)^2 (d-1)^2 \quad .$$

The desired conclusion follows. □

We want now to study the canonical models of tetragonal curves of genus 9 which admit an unramified bidouble cover of a non-hyperelliptic algebraic curve of genus 3. We want to prove, in particular, that such curves belong to $\overline{\mathcal{M}_{9,4}(2)}$, and the general one can be realized as a complete intersection of a smooth quadric ad a certain quartic smooth surface. In order to do this, let us consider a genus 9 tetragonal curve $\mathcal{C}$, $p : \mathcal{C} \longrightarrow \mathcal{D}$ an unramified bidouble cover and the rational normal scroll X determined by a g^1_4.
The possible scrolls are then, according to 2.7, of the following types:

$$\text{(a) } (4,2,0) \quad \text{(b) } (4,1,1) \quad \text{(c) } (3,3,0) \quad \text{(d) } (3,2,1) \quad \text{(e) } (2,2,2) \quad . \tag{2.11}$$

Using 2.8, in each of the cases above we can determine $h^0(\mathcal{C}, \omega_{\mathcal{C}} \otimes \mathcal{L}^{j\vee})$ for every $j = 0 \cdots 4$:

$$\begin{gathered} \text{(a) } (9,6,4,2,1) \quad \text{(b) } (9,6,3,2,1) \quad \text{(c) } (9,6,4,2,0) \\ \text{(d) } (9,6,3,1,0) \quad \text{(e) } (9,6,3,0,0) \quad . \end{gathered} \tag{2.12}$$

Definition 2.3.8. (g^1_4's of type II) If $|\mathcal{L}|$ is a g^1_4 on $\mathcal{C}$ then, according to definition 2.3.2, $|\mathcal{L}|$ is said of type I if $h^0(\mathcal{C}, \mathcal{L}^2) = 3$. However, the other possibile value for $h^0(\mathcal{C}, \mathcal{L}^2)$ is 4. In this case we will say that $|\mathcal{L}|$ **is of type II**. From the cohomology of $\omega_{\mathcal{C}} \otimes \mathcal{L}^{j\vee}$ in 2.12 we can easily see that the scroll types a) and c) in 2.11 correspond to linear systems $|\mathcal{L}|$ of type II.

In the case of a tetragonal curve $\mathcal{C}$, the canonical model is always a complete intersection of two divisors D_1 and D_2 inside the scroll defined by a g^1_4 on $\mathcal{C}$ (see [37], Corollary 4.4), of type respectively $2H + b_1 R$ and $2H + b_2 R$ with the conditions that

$$\begin{cases} b_1 \geq 0 \\ b_2 \geq 0 \\ b_1 + b_2 \quad = 4 \quad . \end{cases}$$

Observation 2.3.9. Suppose $\mathcal{C}$ has $|\mathcal{L}|$ and $|\mathcal{M}|$ two distinct but dependent g^1_4's. Then, by definition, there exists a curve $\mathcal{E}$ with a morphism $p : C \longrightarrow \mathcal{E}$ and $|\mathcal{L}'|$, $|\mathcal{M}'|$ two distinct g^1_2's on $\mathcal{E}$ such that:

$$p^*\mathcal{L}' = \mathcal{L}$$
$$p^*\mathcal{M}' = \mathcal{M} \quad .$$

We note moreover that $\mathcal{E}$ is necessary of genus 1 because a g^1_2 on a curve of genus greater then 2 is unique. Thus, we can see that $h^0(C, \mathcal{L}^2) = 4$ and hence, according to definition 2.3.8, $|\mathcal{L}|$ is a g^4_1 of type II and, in particular, unique. In such case (see [37] 6.5), $\mathcal{C}$ is a complete intersection of X of two divisors of type respectively $2H - 4R$ and $2H$.

Observation 2.3.10. Let us suppose $\mathcal{C}$ has a complete g^3_8, which we denote by $|\mathcal{N}|$. Then $|\mathcal{N}|$ is base point free and, moreover, $\omega_{\mathcal{C}} \cong \mathcal{N}^2$. Indeed, by applying the Riemann Roch Theorem, we have that $h^0(C, \mathcal{N}) = h^1(C, \mathcal{N}) = 4$. The mobile part of $|\mathcal{N}|$ is a special g^3_{8-r}, where r denotes the number of fixed points. We have that

$$Cliff(\mathcal{M}) = deg(\mathcal{M}) - 2|\mathcal{M}| = 8 - r - 6 = 2 - r \quad .$$

By the fact that a curve of genus 9 has Clifford index 2 if and only if it is tetragonal, we conclude that $r = 0$.

Observation 2.3.11. We see now which are the possible models for the curve $\mathcal{C}$ we are looking for. We begin with the observation that $\mathcal{C}$ cannot have a plane model $\mathcal{C}'$ of degree 6 in $\mathbb{P}^2$. Indeed, the group $\mathcal{G}$ would act on $\mathcal{C}'$ with projective linear transformations on $\mathbb{P}^2$. The curve $\mathcal{C}'$ has however only one simple node P, which must be fixed under the action of $\mathcal{G}$. Thus, the action of the group $\mathcal{G}$ could be described as the action on $\mathcal{C}$ in the blow-up of $\mathbb{P}^2$ in P, but this is a contradiction, since $\mathcal{G}$ does not act without fixed points on this model of $\mathcal{C}$.
We have the following list of possibilities:

- $\mathcal{C}$ has a unique g^1_4 of type I. In our particular situation this case will be excluded. (See prop. 2.3.12)

- The curve $\mathcal{C}$ has $|\mathcal{L}|$ and $|\mathcal{M}|$ two distinct g^1_4's and $|\mathcal{L} \otimes \mathcal{M}|$ is a very ample linear system g^3_8. In this case, the image of $\mathcal{C}$ is $\mathbb{P}^3$ is a divisor of type $(4,4)$ a non-singular quadric S, which we consider naturally isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The two projections onto $\mathbb{P}^1$ cut out the two g^1_4's on $\mathcal{C}$. The canonical model of $\mathcal{C}$ in $\mathbb{P}^8$ is contained in the rational surface S embedded in $\mathbb{P}^8$ via the complete linear system of the quadrics in $\mathbb{P}^3$, and S is also contained in a rational normal scroll defined by one of the g^1_4's.

- The curve $\mathcal{C}$ has a unique g^1_4 of type II which we denote by $|\mathcal{L}|$, and $|\mathcal{L}^2|$ is very ample. In this case, the image of $\mathcal{C}$ in $\mathbb{P}^3$ is contained in a singular quadric S, which is the image in $\mathbb{P}^3$ of the Hirzebruck surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)\oplus \mathcal{O}_{\mathbb{P}^1})$. Called H the hyperplane section in S, $Pic(S)$ is generated by H and R, the ruling class. Is easy now to see that the adjoint linear series $\mathcal{C}+K_S$ correspond to the linear system given by $2H$, which is very ample on S.

We see now that in all the previous cases, (except the first) the canonical curve $\mathcal{C}$ can be considered as a complete intersection of two divisors of type $(2,-4)$ and $(2,0)$ in a rational normal scroll in $\mathbb{P}^8$. Indeed, S must be one of those surfaces, so let's suppose that S is a divisor of type $(2,b)$ in X. We will prove this claim in the second case, the other cases being similar. Denoted by R_1 and R_2 two divisors respectively in $|\mathcal{L}|$ and $|\mathcal{M}|$, and denoted by H the hyperplane class in $\mathbb{P}^8$, we have that:

$$\begin{aligned}
H|_S &\cong 2R_1 + 2R_2\\
K_S &= (K_X + S)|_S = (-3H + 4R_1 + 2H - bR_1)|_S\\
&= (-H + (4-b)R_1)\,|_S = -2R_1 - 2R_2 + (4-b)R_1 \quad .
\end{aligned}$$

On the other hand, we know that $K_S = -2R_1 - 2R_2$, and we conclude that $b = 4$.

Proposition 2.3.12. *Let $\mathcal{C}$ be a tetragonal algebraic curve of genus* 9 *and $p : \mathcal{C} \longrightarrow \mathcal{D}$ be an unramified bidouble cover of a non-hyperelliptic curve $\mathcal{D}$ of genus* 3. *Then the rational normal scroll in $\mathbb{P}^8$ defined by a g^1_4 of $\mathcal{C}$ is of type* $(2,2,2)$ *or* $(4,2,0)$. *In the first case, $\mathcal{C}$ has precisely* 2 *distinct $\mathcal{G}$-invariant g^1_4's of type I, while in the second it has a unique g^1_4 of type II.*

Proof. Let us suppose, first of all, that $\mathcal{C}$ has a $\mathcal{G}$-invariant g^1_4 of type I, which we denote by $|\mathcal{L}|$. We will show that $\mathcal{C}$ has another $\mathcal{G}$-invariant g^1_4 of type I. Let us consider $\{s,t\}$ a basis for $H^0(\mathcal{C},\mathcal{L})$ and a, b two generators of $\mathcal{G}$ such that their action in the chosen basis is representable in the following form:

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad . \tag{2.13}$$

The associated scroll cannot be of type $(1,2,3)$: if it were the case, we could consider the basic sections $\phi_1 \in H^0(X,\mathcal{O}_X(H-R))$, $\phi_2 \in H^0(X,\mathcal{O}_X(H-2R))$ and $\phi_3 \in H^0(X,\mathcal{O}_X(H-3R))$, and we would have the following basis for $H^0(\mathcal{C},\omega_{\mathcal{C}})$:

$X_0 := s^3\phi_1$	$X_4 := s^2\phi_2$	$X_7 := s\phi_3$
$X_1 := s^2t\phi_1$	$X_5 := st\phi_2$	$X_8 := t\phi_3$
$X_2 := st^2\phi_1$	$X_6 := t^2\phi_2$	
$X_3 := t^3\phi_1$		

Following the same procedure in the proof of proposition 2.3.5, we would deduce that the projective representation on $\mathbb{P}(H^0(\mathcal{C},\omega_{\mathcal{C}}))$ would not lift to a faithful linear representation on $H^0(\mathcal{C},\omega_{\mathcal{C}})$, which contradicts again our hypothesis on $\mathcal{C}$.
So the type of the scroll associated to $|\mathcal{L}|$ must be $(2,2,2)$, and we have in this case

$$H^0(X,\mathcal{O}_X(H-2R)) = \langle \phi_1,\phi_2,\phi_3\rangle \quad .$$

We can suppose, moreover, that the group $\mathcal{G}$ can be represented on $\mathbb{P}^2 = \mathbb{P}(H^0(X,(1,-2)))$ with two matrices of the form

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad .$$

Indeed, if there were a non-trivial element g of $\mathcal{G}$ acting trivially, then the degree 8 morphism $\mathcal{C} \longrightarrow \mathbb{P}^2$ defined by $\mathcal{O}_X(H-2R)|_{\mathcal{C}}$ would factor through $\mathcal{C}/\langle g\rangle$, yielding a g^2_4 on this quotient. By applying Clifford's theorem, we would conclude that $\mathcal{C}/\langle g\rangle$ is hyperelliptic, which is absurd. Recalling that $\mathcal{C}$ is defined as a complete intersection in X of two divisors of type $2H+b_1R$ and $2H+b_2R$ respectively where b_1 and b_2 are both positive.
With this type of scroll, we can then write a basis for $H^0(X,(1,0)) \cong H^0(\mathcal{C},\omega_{\mathcal{C}})$ given by

$+\ +$	$+\ -$	$-\ +$	$-\ -$
$X_0 := s^2\phi_1 + t^2\phi_3$	$X_3 := s^2\phi_1 - t^2\phi_3$	$X_5 := st(\phi_1+\phi_3)$	$X_7 := (s^2-t^2)\phi_2$
$X_1 := st\phi_2$	$X_4 := s^2\phi_3 - t^2\phi_1$	$X_6 := (s^2+t^2)\phi_2$	$X_8 := st(\phi_1-\phi_3)$
$X_2 := s^2\phi_3 + t^2\phi_1$			

where, in the top row, we express the sign of the action of the generators a and b of $\mathcal{G}$ on the coordinates. Moreover, it is easy to see that $H^0(X,\mathcal{O}_X(2H-4R))$ is generated by

$+\ +$	$+\ -$	$-\ +$	$-\ -$
$\phi_1^2+\phi_3^2$	$\phi_1^2-\phi_3^2$	$(\phi_1+\phi_3)\phi_2$	$(\phi_1-\phi_3)\phi_2$
ϕ_2^2			
$\phi_1\phi_3$			

If $\mathcal{C}$ were not contained in a $\mathcal{G}$-invariant divisor of type $2H-4R$ defined by a section ω_1 of the form

$$\omega_1 = a(\phi_1^2+\phi_3^2)+b\phi_2^2+c\phi_1\phi_3 \quad , \tag{2.14}$$

then the map defined by the global $\mathcal{G}$-invariant holomorphic sections of $\mathcal{O}_X(2H-4R)$ would define a non-degenerate map of degree 4 on $\mathcal{D}=\mathcal{C}/\mathcal{G}$ in $\mathbb{P}^2$. This map would be then the canonical map of $\mathcal{D}$, and we would have $p_*(\omega_{\mathcal{C}}^2\otimes\mathcal{L}^{-4})^{\mathcal{G}}\cong\omega_{\mathcal{D}}$ and hence $\omega_{\mathcal{C}}^2\otimes\mathcal{L}^{-4}$. It would follows that $\omega_C\cong\mathcal{L}^4$, which contradicts the assumption that X is of type $(2,2,2)$.
Hence, $\mathcal{C}$ must be contained in a divisor in the linear system $|2H-4R|$ on X, and $\mathcal{C}$ is then a complete intersection of this divisor and another linearly equivalent to $2H$. Let us suppose from now on that $\mathcal{C}$ is contained in $(\omega_1)_0$, where ω_1 is a holomorphic section like at the point 2.3. Thus, the image of the morphism $\psi\colon\mathcal{C}\longrightarrow\mathbb{P}^2$ defined by the sections $[\phi_1,\phi_2,\phi_3]$ is contained in the plane quadric defined by the equation . Hence, ψ factors through a morphism $\mathcal{C}\longrightarrow\mathbb{P}^1$ of degree 4 and the Veronese map of degree 2. There exists then $|\mathcal{M}|$ a g^1_4 on $\mathcal{C}$ such that

$$\mathcal{O}_X(H-2R)|_{\mathcal{C}}=\omega_{\mathcal{C}}\otimes\mathcal{L}^{-2}\cong\mathcal{M}^2 \quad ,$$

and $|\mathcal{M}|$ is clearly $\mathcal{G}$-invariant and of type I.
It remains now to prove that, if $\mathcal{C}$ has $|\mathcal{L}|$ and $|\mathcal{M}|$ two g^1_4's of type I, then both are $\mathcal{G}$-invariant. Indeed, the linear system $|\mathcal{L}\otimes\mathcal{M}|$ is a $\mathcal{G}$-invariant g^3_8. From the previous argument of the proof we know that if $|\mathcal{L}|$ is $\mathcal{G}$-invariant, then there exists another $\mathcal{G}$-invariant g^1_4 of type I, which we denote by $|\mathcal{M}|$. So let us suppose by absurd that there exist a and b generators for $\mathcal{G}$ such that:

$$\begin{aligned} a^*\mathcal{L}=\mathcal{L} \quad & a^*\mathcal{M}=\mathcal{M} \\ b^*\mathcal{L}=\mathcal{M} \quad & b^*\mathcal{M}=\mathcal{L} \quad . \end{aligned}$$

We can now choose two basis $\mathcal{A}=\{s,t\}$ for $H^0(C,\mathcal{L})$ and $\mathcal{B}=\{u,v\}$ for $H^0(C,\mathcal{M})$ respect to which we can represent the action of a with matrices of the following form:

$$[a]^{\mathcal{A}}_{\mathcal{A}}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=[a]^{\mathcal{B}}_{\mathcal{B}} \quad .$$

The only possible actions of b are then represented by $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ respect to the bases $\mathcal{A}$ and $\mathcal{B}$. In the first case we have that $s\otimes u$ and $t\otimes v$ are $\mathcal{G}$-invariant sections which define a map $\mathcal{C}/\mathcal{G}\longrightarrow\mathbb{P}^1$ of degree 2, which is absurd. The second case can be similarly excluded. □

Corollary 2.3.13. *Let $\mathcal{C}$ be as in 2.3.12. Then one of the following cases occurs:*

- *The curve $\mathcal{C}$ has a unique g^1_4 of type II, denoted by $|\mathcal{L}|$, and it holds that $\omega_{\mathcal{C}} \cong \mathcal{L}^4$ and $|\mathcal{L}^2|$ is a very ample g^3_8. The image of $\mathcal{C}$ in $\mathbb{P}^3$ is the intersection of a $\mathcal{G}$-invariant singular cone in $\mathbb{P}^3$ and a quartic projective $\mathcal{G}$-invariant surface.*
- *The curve $\mathcal{C}$ admits $\mathcal{L}$ and $\mathcal{M}$ two distinct $\mathcal{G}$-invariant g^1_4's, both of type I, with $\omega_{\mathcal{C}} \cong (\mathcal{L}\otimes\mathcal{M})^2$. Moreover, $\mathcal{L}\otimes\mathcal{M}$ is a very ample g^3_8, and it holds that $\mathcal{L}^2 \not\cong \mathcal{M}^2$. The image of $\mathcal{C}$ in $\mathbb{P}^3$ is a complete intersection of the following type:*

$$\mathcal{C} : \begin{cases} X^2+Y^2+Z^2+T^2 = 0 \\ q(X^2,Y^2,Z^2,T^2) = XYZT \quad , \end{cases} \tag{2.15}$$

where q is a quadric, and there exist coordinates $[X,Y,Z,T]$ on $\mathbb{P}^3 = \mathbb{P}(H^0(\mathcal{C},\mathcal{L}))$, and two generators a, b of $\mathcal{G}$ such that the projective representation of $\mathcal{G}$ on $\mathbb{P}^3$ is represented by

$$\begin{aligned} a.[X,Y,Z,T] &= [X,Y,-Z,-T] \\ b.[X,Y,Z,T] &= [X,-Y,Z,-T] \quad . \end{aligned} \tag{2.16}$$

Moreover, the covering $p : \mathcal{C} \longrightarrow \mathcal{D}$ can be expressed as the map obtained by restricting to $\mathcal{C}$ the rational map $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by

$$\psi : [X,Y,Z,T] \mapsto [x,y,z,t] := [X^2,Y^2,Z^2,T^2] \quad . \tag{2.17}$$

Proof. We have seen in the proof of proposition 2.3.12 that, if $\mathcal{C}$ has $|\mathcal{L}|$ and $|\mathcal{M}|$ two linear systems g^1_4 of type I, then both are $\mathcal{G}$-invariant, and we can choose a basis $\mathcal{A} = \{s,t\}$ for $H^0(\mathcal{C},\mathcal{L})$ and $\mathcal{B} = \{u,v\}$ for $H^0(\mathcal{C},\mathcal{M})$, respect to which we can represent the actions of two generators a and b of $\mathcal{G}$ in the following form:

$$\begin{aligned} [a]^{\mathcal{A}}_{\mathcal{A}} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [a]^{\mathcal{B}}_{\mathcal{B}} \\ [b]^{\mathcal{A}}_{\mathcal{A}} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [b]^{\mathcal{B}}_{\mathcal{B}} \quad . \end{aligned}$$

We can now easily see that on the system of coordinates $[X,Y,Z,T] := [su+tv, su-tv, sv+tu, sv-tu]$ on $\mathbb{P}(H^0(\mathcal{C},\mathcal{L}\otimes\mathcal{M}))$ the group $\mathcal{G}$ acts exactly as claimed in 2.16.

Under the assumption that $|\mathcal{L} \otimes \mathcal{M}|$ is very ample, the image in $\mathbb{P}^3$ with respect to $\phi_{|\mathcal{L}\otimes\mathcal{M}|}$ is clearly contained in the image of $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}(H^0(C, \mathcal{L})) \times \mathbb{P}(H^0(C, \mathcal{M}))$, which is the quadric Q_2 defined by the equation $X^2+Y^2+Z^2+T^2 = 0$.

Hence, the image of $\mathcal{C}$ in $\mathbb{P}^3$, which we still denote $\mathcal{C}$ by an abuse of notation, is a complete intersection of Q_2 and a smooth quartic curve Q_4 defined by an equation $s_4 = 0$, where s_4 is a homogeneous quartic. We want to prove that s_4 is $\mathcal{G}$-invariant.

The vector space of the quartics in $\mathbb{P}^3$ has the following decomposition, according to the action of the generators a and b of $\mathcal{G}$:

$+\ +$	$+\ -$	$-\ +$	$-\ -$
$X^4 \quad X^2Y^2 \quad Y^2Z^2$	X^3Y	X^3Z	X^3T
$Y^4 \quad X^2Z^2 \quad Y^2T^2$	X^2ZT	X^2YT	X^2YZ
$Z^4 \quad X^2T^2 \quad Z^2T^2$	XY^3	XZ^3	XT^3
T^4	$XYZ^2 \quad XYT^2$	$XY^2Z \quad XZT^2$	$XZ^2T \quad XY^2T$
$XYZT$	$Z^3T \quad ZT^3$	$Y^3T \quad YT^3$	$Y^3Z \quad YZ^3$
	Y^2ZT	YZ^2T	YZT^2

To complete the proof is enough to prove that the quartic s_4 belongs to $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))^{\mathcal{G}}$. If it were not the case, we could suppose (using the equation of Q_2) that the quartic s_4 can be written in the following form

$$XYp^2 = ZTq^2 \quad ,$$

where p and q are quadrics in the vector space generated by X^2, Y^2, Z^2. This means that, considered $\mathbb{P}^3$ with coordinates $[x, y, z, t]$ as in 2.17, we could write the equation of $\mathcal{D}$ in the following form:

$$\mathcal{D} : \begin{cases} x + y + z + t = 0 \\ xyp^2 = ztq^2 \quad , \end{cases}$$

where p and q are two lines. In this case, $\mathcal{D}$ would be singular in the point in which the lines p and q intersect. Hence, $\mathcal{D}$ would be hyperelliptic, which contradicts our assumptions on $\mathcal{D}$. $\square$

We conclude this section by showing the following:

Theorem 2.3.14. *Let A be general polarized abelian 3-fold with a cartesian* *diagram:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "p"] & A \arrow[d, "p"] \\ \mathcal{D} \arrow[r, hook] & \mathcal{J} = \mathcal{J}(\mathcal{D}) \end{tikzcd}$$

where $\mathcal{D}$ is a general algebraic curve of genus 3, p is an unramified bidouble cover defined by two elements η_1 and η_2 belonging to $\mathcal{J}[2]$ with $\lambda(\eta_1,\eta_2) = 0$, where λ is the Weil pairing defined in 2.2. Then the following are equivalent:

(a) $\lambda(\eta_1,\eta_2) = 0$.

(b) *$\mathcal{C}$ is tetragonal.*

(c) *A is $(1,2,2)$-polarized.*

Proof. We have already observed in 2.2 that (a) and (c) are equivalent.
Let us suppose now that $\lambda(\eta_1,\eta_2) = 0$. We show that $\mathcal{C}$ is tetragonal. Let us fix a point $Q_0 \in \mathcal{C}$ whose image $q_0 \in \mathcal{D}$ with respect to p does not lie on a bitangent line of $\mathcal{D}$, and let us denote by $\mathcal{A}$ the Abel map defined respect to a point different from q_0.
We see that there exists ζ on Θ such that the following conditions hold

$$\begin{aligned} \zeta + \eta_1 &\in \Theta \\ \zeta + \eta_2 &\in \Theta \\ \zeta + \eta_1 + \eta_2 &\in \Theta \quad , \end{aligned} \tag{2.18}$$

and such that for every special divisor of degree 3and every $\eta \in K(\mathcal{L})_2$ we have $\zeta \neq \mathcal{A}(q_0) - \kappa - \eta - \mathcal{A}(D)$, where is the vector of Riemann constants (See [35] Thm. 2 p.100).
Indeed, the conditions in 2.18 say that ζ is the image of a 2-torsion point in A belonging to the base locus of the linear system $(1,2,2)$ in A. If however, the fourth condition does not hold, it exists $\eta \in K(\mathcal{L})_2$ and a special divisor D of degree 3 on $\mathcal{D}$ such that:

$$\zeta = \mathcal{A}(q_0) - \kappa - \eta - \mathcal{A}(D) \quad ,$$

and it follows that, in particular (recall that $\mathcal{A}(K) = -2\kappa$, where K is a canonical divisor on $\mathcal{D}$)

$$0 = 2\zeta = \mathcal{A}(2q_0 + K - 2D) - 2\eta = \mathcal{A}(2q_0 + K - 2D) \quad .$$

Hence, by Abel's theorem, the divisor $2(D - q_0)$ is a canonical divisor. But D is supposed to be a special divisor of degree 3, hence linearly equivalent to $K - r$ where r is some point on $\mathcal{D}$. (D is a g^1_3 on $\mathcal{D}$). That means, in particular, that:

$$K \equiv 2(D - q_0) \equiv 2(K - r - q_0) \quad ,$$

and we conclude then that $r + q_0$ is an odd theta-characteristic. But by our assumption on q_0 we can exclude this case.
With these assumptions on ζ, we can define, with $\eta \in K(\mathcal{L})_2$ and $\gamma \in K_1$:

$$s^{q_0}_\gamma := t^*_{\mathcal{A}(q_0)-\zeta}\theta_\gamma|_{\mathcal{C}} \in H^0(\mathcal{C}, \mathcal{T}) \quad ,$$

where $\mathcal{T} := (t^*_{\mathcal{A}(q_0)-\zeta}\mathcal{L})|_{\mathcal{C}}$. Our goal is to show that $|\mathcal{T}|$ is a linear system on $\mathcal{C}$ with 4 base points and of degree 12, and that its mobile part defines a g^3_8 on $\mathcal{C}$.
For, if we consider a point $X \in \mathcal{C}$ and its image x in $\mathcal{D}$ with respect to p, we have that, by definition

$$s^{q_0}_\gamma(X) = \theta_\gamma(\mathcal{A}(x) - \mathcal{A}(q_0) - \zeta) \quad .$$

Hence, $s^{q_0}_\gamma(X)$ vanishes if and only if $\theta_0(\mathcal{A}(x) - \mathcal{A}(q_0) - (\zeta + \gamma)) = 0$. We can then conclude that (see [35] Lemma 2 p. 112):

$$\begin{aligned} div(\theta(\mathcal{A}(x) - \mathcal{A}(q_0) - \zeta - \gamma)) &= x + D_\gamma \\ \mathcal{A}(D_\gamma) &= \zeta + \eta - \kappa \quad , \end{aligned}$$

where D_γ is a divisor of degree 2 independent on q_0. Thus:

$$div(s^{q_0}_\gamma) = \mathcal{G}.Q_0 + p^*(D_\gamma) \quad ,$$

where $\mathcal{G}.Q_0$ is the orbit of Q_0 with respect to the action of $\mathcal{G}$.
This means that $\mathcal{T}$, which is of degree 12, has a fixed part of degree 4, and its mobile part, which we denote by $|\mathcal{M}|$, has degree 8. We can conclude now that

$$H^0(A, \mathcal{I}_{\mathcal{C}} \otimes t^*_{\mathcal{A}(q_0)-\zeta}\mathcal{L}) = \sum_\eta H^0(\mathcal{J}, \mathcal{I}_{\mathcal{D}} \otimes t^*_{\mathcal{A}(q_0)-\zeta-\gamma}\Theta) = 0 \quad .$$

Indeed, if $h^0(\mathcal{J}, \mathcal{I}_{\mathcal{D}} \otimes t^*_{\mathcal{A}(q_0)-\zeta-\gamma}\Theta) = 1$ for some γ then, for every $p \in \mathcal{D}$, we would have:

$$\theta_0(\mathcal{A}(q_0) - \zeta - \gamma - \mathcal{A}(p)) = 0 \quad ,$$

and we would find a special divisor D such that:

$$\mathcal{A}(q_0) - \eta - \zeta - \kappa = \mathcal{A}(D) \quad .$$

But this would contradict the conditions on ζ. We have then the following exact sequence:

$$0 \longrightarrow H^0(A, t^*_{A(q_0)-\zeta}\mathcal{L}) \longrightarrow H^0(\mathcal{C},\mathcal{T}) \longrightarrow H^1(A, \mathcal{I}_{\mathcal{C}} \otimes t^*_{A(q_0)-\zeta}\mathcal{L}) \longrightarrow 0 \quad ,$$

from which we can conclude that $h^0(\mathcal{C},\mathcal{M}) = h^0(\mathcal{C},\mathcal{T}) \geq 4$. We can apply the Riemann Roch theorem to conclude that the linear system $|\mathcal{M}|$ is special on $\mathcal{C}$. Moreover, its Clifford index is:

$$Cliff(\mathcal{M}) = 8 - 2|\mathcal{M}| \leq 2 \quad .$$

But the Clifford index of $\mathcal{M}$ cannot be 0 by the Clifford theorem, so $\mathcal{M}$ is of Clifford index 2. This implies that $\mathcal{C}$ is an algebraic curve of genus 9 with Clifford Index 2, and we conclude that $\mathcal{C}$ is a tetragonal curve.
Suppose not that $\mathcal{C}$ is a tetragonal curve. We may assume by the generality of A that $\mathcal{C}$ can be represented as a smooth curve in $\mathbb{P}^3$ with coordinates $[X,Y,Z,T]$ like in 2.15. The quartic curve $\mathcal{D}$ in $\mathbb{P}^3$ can be expressed in a suitable system of coordinates $[x,y,z,t]$ of $\mathbb{P}^3$ as

$$\begin{cases} h\colon x+y+z+t & = 0 \\ q(x,y,z,t)^2 & = xyzt \quad , \end{cases} \tag{2.19}$$

where q is a quadratic form which defines a $\mathcal{Q}$ in the plane h in 2.19. The bidouble cover p is defined by the 2 torsion line bundles:

$$\begin{aligned} \eta_1 &= \mathcal{O}((x=0)\cap\mathcal{Q}) \\ \eta_2 &= \mathcal{O}((y=0)\cap\mathcal{Q}) \quad . \end{aligned}$$

We can then perform $\lambda(\eta_0,\eta_1)$ by applying the reciprocity 2.2.4, and we reach the desired conclusion

$$\lambda(\eta_0,\eta_1) = \frac{1}{\pi i} log\left(\frac{\frac{u}{v}(div_{\mathcal{Q}}(\frac{u}{w}))}{\frac{u}{w}(div_{\mathcal{Q}}(\frac{v}{w}))}\right) = 0 \quad .$$

$\square$

Chapter 3

The canonical map of the $(1,2,2)$-Theta divisor

In this chapter we start by considering a general abelian 3-fold $(A, \mathcal{L})$ with an isogeny $p : A \longrightarrow \mathcal{J}$ onto a principally polarized abelian 3-fold $(\mathcal{J}, \Theta)$, such that the polarization induced by $p^*\Theta$ on A is of type $(1,2,2)$. By generality of A and by Torelli's theorem, we can assume that $(\mathcal{J}, \Theta)$ is the Jacobian variety of a non-hyperelliptic genus 3 curve $\mathcal{D}$.
The goal of the first two sections is a geometric description of the canonical map of the pullback divisor $p^*\Theta$. The canonical map can be expressed, in this context, by means of bihomogeneous polynomials of bidegree $(2,2)$ on $\mathbb{P}^3 \times \mathbb{P}^3$. This polynomial expression turns out to be closely related to the biquadratic expression of certain addition laws of bidegree $(2,2)$ on very special elliptic curves embedded in $\mathbb{P}^3$.
We introduce, in the first section of this chapter, the general notion of addition law on an embedded elliptic curve presented in the article [31] of Lange and Ruppert. In the second section of this chapter, we give a geometrical description of the canonical map of $p^*\Theta$ by making use of this notion. This description, furthermore, is achieved without making use of the method of the canonical projections.

In the second part of the chapter, we study different many other situations in which $(1,2,2)$-polarized abelian 3-folds arise naturally as quotients of a $(2,2,2)$-polarized abelian 3-folds, which is a polarized product of a $(2,2)$-polarized surface with a 2-polarized elliptic curve. In such situations, we see in the fourth section of this chapter that the canonical map of a general surface in the induced polarization has everywhere injective differential, but is never injective, because there exists a canonical curve on which the restriction of the canonical map has degree 2. Nevertheless, the use of monodromy arguments enables us to prove that the canonical map of a general surface yielding a

$(1,2,2)$-polarization on a general abelian 3-fold A is injective and in the end a holomorphic embedding in $\mathbb{P}^5$.

3.1 Addition laws of bidegree $(2,2)$ on elliptic curves in $\mathbb{P}^3$

Throughout this section, we will denote by $(A,\mathcal{L})$ a polarized abelian variety, with $\mathcal{L}$ very ample. We denote by ϕ the holomorphic embedding $\phi_{|\mathcal{L}|}\colon A \longrightarrow \mathbb{P}^N$ defined by the linear system $|\mathcal{L}|$ on A. Moreover, we will denote by $\mu,\delta\colon A\times A \longrightarrow A$ the morphisms respectively defined by the sum and the difference in A, and by π_1 and π_2 the projections of $A\times A$ onto the respective factors. To prevent misunderstandings based on notation, we will refer to μ as the **group law** on A, to distinguish it from the notion of addition law which we are going to introduce in this section. The aim is to provide a description of the group law μ on an open set of $A\times A$ by means of a rational map $\oplus\colon \mathbb{P}^N\times\mathbb{P}^N \dashrightarrow \mathbb{P}^N$ defined by an ordered set of bihomogeneous polynomials $(f_0,\cdots,f_N)$ of a given bidegree (m,n). Such a set of bihomogeneous polynomials is called **addition law** (see [5]).
Assigned an addition law $\oplus$ on A defined by bihomogeneous polynomials $(f_0\cdots f_N)$, we denote by $W(\oplus)$ the sublinear system of $|\mathcal{O}_{\mathbb{P}^N}(m)\boxtimes\mathcal{O}_{\mathbb{P}^N}(n)|$ generated by $f_0,\cdots,f_N$.
An addition law on A can be viewed then as a rational map $\oplus\colon \mathbb{P}^N\times\mathbb{P}^N \dashrightarrow \mathbb{P}^N$ such that the following diagram commutes:

$$\begin{tikzcd} A\times A \arrow[r, "\phi\times\phi"'] \arrow[d, "\mu"] & \mathbb{P}^N\times\mathbb{P}^N \arrow[d, dashed, "\oplus"] \\ A \arrow[r, "\phi"'] & \mathbb{P}^N \end{tikzcd} \tag{3.1}$$

In particular, the rational map $A\times A \dashrightarrow \mathbb{P}^N$, which in diagram 3.1 is given by the composition $\phi\times\phi$ with $\oplus$, is defined by the $N+1$ linearly independent global sections of

$$(\phi\times\phi)^*\left(\mathcal{O}_{\mathbb{P}^N}(m)\boxtimes\mathcal{O}_{\mathbb{P}^N}(n)\right) = \pi_1^*\mathcal{L}^m\otimes\pi_2^*\mathcal{L}^n \quad .$$

The morphism $\phi\circ\mu$ is defined, on the other side, by the complete linear system $|\mu^*\mathcal{L}|$ on $A\times A$. By applying the projection formula and by the fact that μ is a morphism with connected fibers, we have that

$$H^0(A\times A,\mu^*\mathcal{L}) \cong H^0(A,\mathcal{L}) \quad .$$

Hence, a rational map $\oplus : \mathbb{P}^N \times \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ of bidegree (m,n) such that the previous diagram can be expressed then as a global section of

$$\mathcal{M}_{m,n} := \mu^*\mathcal{L}^{-1} \otimes \pi_1^*\mathcal{L}^m \otimes \pi_2^*\mathcal{L}^n \quad .$$

This leads to the following definition:

Definition 3.1.1. (Addition law, [31]) Let $(A, \mathcal{L})$ be a polarized abelian variety, with $\mathcal{L}$ very ample. Let m, n two non-zero natural numbers. An **addition law of bidegree** (m,n) **on** A is a global section of $\mathcal{M}_{m,n}$, where $\mathcal{M}_{m,n} := \mu^*\mathcal{L}^{-1} \otimes \pi_1^*\mathcal{L}^m \otimes \pi_2^*\mathcal{L}^n$.

Let $s \in H^0(A \times A, \mathcal{M}_{m,n})$ be a non-zero addition law of bidegree (m,n). If we consider the rational map $\oplus : \mathbb{P}^N \times \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ defined by s, and I_A the homogeneous ideal defining A in $\mathbb{P}^N$, then the restriction $\oplus$ to $A \times A$ is given by some bihomogeneous polynomials $f_0 \cdots f_N$ of bidegree (m,n) in $k[A] = k[\mathbb{P}^N]/I_A$ which express the group law μ on A away from the base locus Z of $W(\oplus)$. The locus Z, which is the indeterminacy locus of the rational map $\oplus$, will be called **exceptional locus of** s.

Proposition 3.1.2. *Z is a divisor in $A \times A$.*

Proof. The map $\mu \circ \phi$ in diagram 3.1 is the morphism defined by the complete base point free linear system $|\mu^*\mathcal{L}|$. We consider then $\theta_0 \dots \theta_N$ a basis for$H^0(A \times A, \mu^*\mathcal{L})$. We have then that (recall that $s \in H^0(A \times A, \mathcal{M}_{m,n})$):

$$s(\theta_i \circ \mu) = p_i \quad , \tag{3.2}$$

and It follows then that

$$Z = div(s) \quad .$$

□

From now on we do not distinguish anymore an addition law of a given bidegree (m,n) from its expression as a rational map $\oplus : \mathbb{P}^N \times \mathbb{P}^N \dashrightarrow \mathbb{P}^N$. Closely related to the notion of addition law for an embedded abelian variety, there is the notion of complete set of addition laws.

Definition 3.1.3. A set of addition laws $s_1 \cdots s_k$ of bidegree (m,n) is said to be a **complete set of addition laws** if:

$$div(s_1) \cap \cdots \cap div(s_k) = \emptyset \quad .$$

In particular, there exists a complete set of addition laws of bidegree (m,n) if and only if $|\mathcal{M}_{m,n}|$ is base point free.

The problem of determining, whether for a given bidegree (m, n) with $m, n \geq 2$ there exists an addition law (resp. a complete set of addition laws), has been solved by Lange and Ruppert (see [31] p. 610). Their main result is:

Theorem 3.1.4. *Let A be an abelian variety embedded in $\mathbb{P}^N$, and $\mathcal{L} = \mathcal{M}^m$, with $m \geq 3$, a very ample line bundle defining the embedding of A in $\mathbb{P}^N$. Then:*

- *There are complete systems of addition laws on $A \subseteq \mathbb{P}^N$ of bidegree $(2, 3)$ and $(3, 2)$.*
- *There exists a system of addition laws on $A \subseteq \mathbb{P}^N$ of bidegree $(2, 2)$ if and only if $\mathcal{L}$ is symmetric. Furthermore, in this case, there exists a complete system of addition laws.*

We focus our attention now on the case of biquadratic addition laws. Keeping the notations of the previous section, we show as a first step that the space of the biquadratic addition laws for the embedded abelian variety A is naturally isomorphic to the space of global holomorphic sections of $\mathcal{L}$.

Proposition 3.1.5. *(Addition laws of bidegree $(2, 2)$) Let A be an abelian variety and $\mathcal{L}$ be a symmetric ample line bundle. Then:*

$$H^0(A \times A, \mathcal{M}_{(2,2)}) \cong H^0(A, \mathcal{L}) \quad .$$

Proof. Because $\mathcal{L}$ is symmetric, we have that $\mu^*\mathcal{L} \otimes \delta^*\mathcal{L} \cong \pi_1^*\mathcal{L}^2 \otimes \pi_2^*\mathcal{L}^2$. By applying the projection formula, and from the fact that the morphism δ is a proper morphism with connected fibers, we can conclude that

$$H^0(A \times A, \mathcal{M}_{(2,2)}) = H^0(A \times A, \delta^*\mathcal{L}) \cong H^0(A, \mathcal{L}) \quad .$$

□

We see first a model of a smooth elliptic curve in $\mathbb{P}^3$ not contained in any hyperplane:

Definition 3.1.6. (Jacobi's model, see also [5] p.21) Let u, v, w three non-zero complex numbers such $u + v + w = 0$. We define $\mathcal{J}_{u,v}$ the elliptic curve in $\mathbb{P}^3$ with coordinates $X, \cdots, T$ defined by the following quadric equations:

$$\mathcal{J}_{u,v} : \begin{cases} uX^2 + Y^2 & = Z^2 \\ vX^2 + Z^2 & = T^2 \\ wX^2 + T^2 & = Y^2 \quad . \end{cases} \tag{3.3}$$

On $\mathbb{P}^3 \times \mathbb{P}^3$, we denote by $[X_0 \cdots T_0]$ the coordinates for the first factor and by $[X_1 \cdots T_1]$ the coordinates for the second one. An explicit basis of the space of the biquadratic addition laws has been in determined [5]:

Theorem 3.1.7. *The vector space $H^0(J_{u,v} \times J_{u,v}, \mathcal{M}_{(2,2)})$ of the addition laws of bidegree $(2, 2)$ for the elliptic curve $J_{u,v}$ in $\mathbb{P}^3$ defined by the Jacobi quadratic equation is generated by:*

$$\begin{aligned}
\oplus_X :=&[X_0^2Y_1^2 - Y_0^2X_1^2, X_0Y_0Z_1T_1 - Z_0T_0X_1Y_1,\\
&X_0Z_0Y_1T_1 - Y_0T_0X_1Z_1, X_0T_0Y_1Z_1 - Y_0Z_0X_1T_1]\\
\oplus_Y :=&[X_0Z_0Y_1T_1 + Y_0T_0X_1Z_1, -uX_0T_0X_1T_1 + Y_0Z_0Y_1Z_1,\\
&uvX_0^2X_1^2 + Z_0^2Z_1^2, vX_0Y_0X_1Y_1 + Z_0T_0Z_1T_1]\\
\oplus_Z :=&[X_0Y_0Z_1T_1 + Z_0T_0X_1Y_1, uwX_0^2X_1^2 + Y_0^2Y_1^2,\\
&uX_0T_0X_1T_1 + Y_0Z_0Y_1Z_1, -wX_0Z_0X_1Z_1 + Y_0T_0Y_1T_1]\\
\oplus_T :=&[u(X_0T_0Y_1Z_1 + Y_0Z_0X_1T_1), u(wX_0Z_0X_1Z_1 + Y_0T_0Y_1T_1),\\
&u(-vX_0Y_0X_1Y_1 + Z_0T_0Z_1T_1), -vY_0^2Y_1^2 - wZ_0^2Z_1^2] \quad .
\end{aligned}$$

Moreover, for every $H \in \{X, Y, Z, T\}$, the exceptional divisor of $\oplus_H$ is $\delta^*(H)$, where H denotes the corresponding hyperplane divisor $\mathbb{P}^3$.

Proof. See [5], p.22 □

The theorem 3.1.7 states that, in particular, the exceptional divisor of $\oplus_X$ is $\delta^*((X = 0))$. On the other side, the divisor $(X = 0)$ on the elliptic curve $\mathcal{J}_{u,v}$ given by the equations 3.3 is known to be the set of the 2-torsion points of $\mathcal{J}_{u,v}$. We denote this set by Δ_2,

$$\Delta_2 = \{T_0, T_1, T_2, T_3\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \quad ,$$

where:

$$T_0 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \quad T_1 = \begin{bmatrix}0\\1\\-1\\-1\end{bmatrix} \quad T_2 = \begin{bmatrix}0\\-1\\1\\-1\end{bmatrix} \quad T_3 = \begin{bmatrix}0\\-1\\-1\\1\end{bmatrix} \quad . \tag{3.4}$$

Via $\oplus_X$, the 2-torsion points in 3.4 act naturally on $J_{u,v} - \Delta_2)$. Indeed, considered $[X,Y,Z,T] \in J_{u,v} - \Delta_2$, we see immediately that:

$$\begin{aligned} T_0 \oplus_X \begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix} &= \begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix} \quad T_1 \oplus_X \begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix} = \begin{bmatrix} X \\ Y \\ -Z \\ -T \end{bmatrix} \\ T_2 \oplus_X \begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix} &= \begin{bmatrix} X \\ -Y \\ Z \\ -T \end{bmatrix} \quad T_3 \oplus_X \begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix} = \begin{bmatrix} X \\ -Y \\ -Z \\ T \end{bmatrix} \quad . \end{aligned} \tag{3.5}$$

It can be in particular easily seen that the addition law $\oplus_X$ is not defined on the union of four copies of $\mathcal{J}_{u,v}$ in $\mathcal{J}_{u,v} \times \mathcal{J}_{u,v}$ corresponding to the 2-torsion points:

$$Z := div(\oplus_X) = \bigcup_{i=0}^{3} \{(P, \oplus_X(T_i, P) \mid P \in \mathcal{J}_{u,v}\} \subseteq \mathbb{P}^3 \times \mathbb{P}^3 \quad .$$

We denote the bihomogeneous polynomials which define the addition law $\oplus$ by

$$\begin{aligned} \eta_{01} &:= \begin{vmatrix} X_0^2 & X_1^2 \\ Y_0^2 & Y_1^2 \end{vmatrix} \\ \omega_{45} &:= \begin{vmatrix} X_0Y_0 & X_1Y_1 \\ Z_0T_0 & Z_1T_1 \end{vmatrix} \\ \omega_{67} &:= \begin{vmatrix} X_0Z_0 & X_1Z_1 \\ Y_0T_0 & Y_1T_1 \end{vmatrix} \\ \omega_{89} &:= \begin{vmatrix} X_0T_0 & X_1T_1 \\ Y_0Z_0 & Y_1Z_1 \end{vmatrix} \quad . \end{aligned} \tag{3.6}$$

Is worth to notice here that, considered a general $(1,2,2)$-polarized abelian 3-fold $(A, \mathcal{L})$ with an isogeny $p : A \longrightarrow \mathcal{J}$ onto a principally polarized abelian 3-fold $(\mathcal{J}, \Theta)$, the polynomial terms in the expression of the canonical map of the surface $p^*\Theta$ in 3.11 perfectly correspond with those in 3.6.

Example 3.1.8. (A more general model in $\mathbb{P}^3$) Under the hypothesis that a,b,c,d are all distinct complex numbers, the locus $\mathcal{E}$ in $\mathbb{P}^3$ defined by the following couple of quadrics is a smooth elliptic curve:

$$\mathcal{E} := \begin{cases} aX^2 + bY^2 + cZ^2 + dT^2 = 0 \\ X^2 + Y^2 + Z^2 + T^2 = 0 \end{cases} \quad . \tag{3.7}$$

We show here that, up to a choice of signs which represents the action of a 2-torsion point, we can define on this embedded elliptic curve a special addition law, which we denote again by $\oplus_X$, which plays the role of the addition law $\oplus_X$ defined on the Jacobi model in 3.1.7. As a first step, we work out the equations 3.7 in order to obtain a model which is similar to the Jacobi model in 3.3.

$$\mathcal{E} := \begin{cases} \frac{a-d}{b-d}X^2 + Y^2 = \frac{c-d}{b-d}Z^2 \\ -\frac{a-c}{b-c}X^2 + \frac{c-d}{b-c}T^2 = Y^2 \end{cases} . \tag{3.8}$$

With α and β square roots of $\frac{c-d}{b-d}$ and $\frac{c-d}{b-c}$ respectively, we obtain on $\mathbb{P}^3 \times \mathbb{P}^3$ a rational map corresponding to $\oplus_X$, which represent the group law of $\mathcal{E}$:

$$\oplus_X : (P,Q) \dashmapsto \begin{bmatrix} \eta_{01}(P,Q) \\ \alpha\beta\omega_{45}(P,Q) \\ \beta\omega_{67}(P,Q) \\ \alpha\omega_{89}(P,Q) \end{bmatrix} . \tag{3.9}$$

The rational map defined in 3.9, however, is an addition law up to the action of a 2-torsion point, according to 3.5. Indeed, if we choose another branch of the square root used to define α and β the signs in the expression 3.9 change exactly according to the action in 3.5. This means that this rational map $\oplus_X$ represents an operation on $\mathcal{E}$ of the following form:

$$\tilde{\mu}(P,Q) = \mu(T, \mu(P,Q)) = T + P + Q \quad ,$$

where T is a 2-torsion point.

3.2 The canonical map of the $(1,2,2)$ Theta-divisor and its geometry

In this section, we use the notation which we introduced in chapter 2.1. The goal of this section is to achieve an exhaustive description of the geometry of the canonical map of the surface $\mathcal{S}$ in the pullback diagram:

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook] \arrow[d, "p"] & A \arrow[d, "p"] \\ \Theta \arrow[r, hook] & \mathcal{J} \end{tikzcd} \tag{3.10}$$

where A is a general $(1,2,2)$-polarized abelian 3-fold. By applying proposition 2.3.14, we can easily see that the surface $\mathcal{S}$ can be geometrically described as a quotient of the form

$$\mathcal{S} = \mathcal{C} \times \mathcal{C} \Big/ \Delta_{\mathcal{G}} \times \mathbb{Z}_2 \quad ,$$

where:

- $\mathcal{C}$ is a smooth curve of genus 9 in $\mathbb{P}^3$, which is a complete intersection of the form 2.15.

- $\Delta_{\mathcal{G}}$, the diagonal subgroup of $\mathcal{G} \times \mathcal{G}$, acts naturally on $\mathcal{C} \times \mathcal{C}$, while $\mathbb{Z}_2$ acts by switching the two factors of $\mathcal{C} \times \mathcal{C}$.

Whenever necessary, we will denote the points of $\mathcal{S}$ by representatives of the form $[(P,Q)]$, where P and Q are points on $\mathcal{C}$. We denote, moreover, the coordinates on the two factors of $\mathbb{P}^3 \times \mathbb{P}^3$ by $[X_0 \cdots T_0]$ and $[X_1, \cdots T_1]$ respectively. We remark that the action of the group $\mathcal{G}$ on $\mathcal{S}$ is defined by the action of $\mathcal{G}$ on the second component: considered g an element of $\mathcal{G}$ and $[(P,Q)]$ a point on $\mathcal{S}$, we have

$$g.[P,Q] := [P, g.Q] \quad .$$

We are now in position to exhibit a basis for the space $H^0(\mathcal{S}, \omega_{\mathcal{S}})$: we consider first the basis for $H^0(\mathcal{C}, \omega_{\mathcal{C}})$ given by the quadrics on $\mathbb{P}^3$, and we arrange them in a table according to the sign of the action of the generators a and b of $\mathcal{G}$ on the coordinates of $\mathbb{P}^3$ (see 2.16):

$+\ +$	$+\ -$	$-\ +$	$-\ -$
$\eta_1 := X^2\|_{\mathcal{C}}$	$\eta_4 := XY\|_{\mathcal{C}}$	$\eta_6 := XZ\|_{\mathcal{C}}$	$\eta_8 := XT\|_{\mathcal{C}}$
$\eta_2 := Y^2\|_{\mathcal{C}}$	$\eta_5 := ZT\|_{\mathcal{C}}$	$\eta_7 := YT\|_{\mathcal{C}}$	$\eta_9 := YZ\|_{\mathcal{C}}$
$\eta_3 := Z^2\|_{\mathcal{C}}$			

Let us denote now by π_1 and π_2 the projections of $C \times C$ onto the two factors. We have that

$$H^0(\mathcal{S}, \omega_{\mathcal{S}}) = H^0(\mathcal{C} \times \mathcal{C}, \omega_{\mathcal{C}} \boxtimes \omega_{\mathcal{C}})^{\Delta_G \times \mathbb{Z}_2}$$

Thus, the following is easily seen to be a basis for $H^0(\mathcal{S},\omega_{\mathcal{S}})$:

$$
\begin{aligned}
\eta_{01} &:= \begin{vmatrix} \pi_1^*\eta_1 & \pi_2^*\eta_1 \\ \pi_1^*\eta_2 & \pi_2^*\eta_2 \end{vmatrix} = \begin{vmatrix} X_0^2 & X_1^2 \\ Y_0^2 & Y_1^2 \end{vmatrix} \\
\eta_{02} &:= \begin{vmatrix} \pi_1^*\eta_1 & \pi_2^*\eta_1 \\ \pi_1^*\eta_3 & \pi_2^*\eta_3 \end{vmatrix} = \begin{vmatrix} X_0^2 & X_1^2 \\ Z_0^2 & Z_1^2 \end{vmatrix} \\
\eta_{12} &:= \begin{vmatrix} \pi_1^*\eta_2 & \pi_2^*\eta_2 \\ \pi_1^*\eta_3 & \pi_2^*\eta_3 \end{vmatrix} = \begin{vmatrix} Y_0^2 & Y_1^2 \\ Z_0^2 & Z_1^2 \end{vmatrix} \\
\omega_{45} &:= \begin{vmatrix} \pi_1^*\eta_4 & \pi_2^*\eta_4 \\ \pi_1^*\eta_5 & \pi_2^*\eta_5 \end{vmatrix} = \begin{vmatrix} X_0Y_0 & X_1Y_1 \\ Z_0T_0 & Z_1T_1 \end{vmatrix} \\
\omega_{67} &:= \begin{vmatrix} \pi_1^*\eta_6 & \pi_2^*\eta_6 \\ \pi_1^*\eta_7 & \pi_2^*\eta_7 \end{vmatrix} = \begin{vmatrix} X_0Z_0 & X_1Z_1 \\ Y_0T_0 & Y_1T_1 \end{vmatrix} \\
\omega_{89} &:= \begin{vmatrix} \pi_1^*\eta_8 & \pi_2^*\eta_8 \\ \pi_1^*\eta_9 & \pi_2^*\eta_9 \end{vmatrix} = \begin{vmatrix} X_0T_0 & X_1T_1 \\ Y_0Z_0 & Y_1Z_1 \end{vmatrix} \quad .
\end{aligned}
\tag{3.11}
$$

The $\mathcal{G}$-invariant subspace generated by η_{01}, η_{02} and η_{12} is the sublinear system of $|\omega_{\mathcal{S}}|$ which defines the Gauss map $G : \mathcal{S} \longrightarrow \mathbb{P}^{2\vee}$. This map factors clearly through the isogeny p and the Gauss map of Θ, whose geometrical interpretation has been discussed in 1.2.1.
We give now a geometrical interpretation of the information carried by the three holomorphic sections ω_{45}, ω_{67} and ω_{89}.
To a point $U := [(P,Q)] \in \mathcal{S}$, we can associate the line in $\mathbb{P}^2$ with coordinates $[x,y,z]$

$$r := G(U) = \{ax + by + cz = 0\} \quad ,$$

where

$$[a,b,c] = [\eta_{12}(U), -\eta_{02}(U), \eta_{01}(U)] \in \mathbb{P}^2 \quad . \tag{3.12}$$

We pullback now this line to $\mathbb{P}^3$ through $\psi : [X,Y,Z,T] \dashrightarrow [x,y,z,t] := [X^2,Y^2,Z^2,T^2]$ (see 2.17) to obtain the quadric

$$\mathcal{R}_U : aX^2 + bY^2 + cZ^2 = 0 \quad .$$

Finally, we denote bt $\mathcal{E}_U$ the locus defined by the intersection of $\mathcal{R}_U$ with the $\mathcal{G}$-invariant quadric of $\mathbb{P}^3$ containing $\mathcal{C}$ (see equation 2.15):

$$\mathcal{E}_U := \begin{cases} aX^2 + bY^2 + cZ^2 = 0 \\ X^2 + Y^2 + Z^2 + T^2 = 0 \end{cases} \quad .$$

The curve $\mathcal{E}_U$ is a smooth curve of genus 1 if and only if a, b and c are non zero and all distinct. In this case, (c.f. 3.1.8) there exist two constants α_U and

β_U, which depend only on a, b, c, and a biquadratic addition law $\oplus_X^U$ on $\mathcal{E}_U$, which is defined as follows

$$\oplus_X^U : (X,Y) \dashmapsto \begin{bmatrix} \eta_{01}(X,Y) \\ \alpha_U\beta_U\omega_{45}(X,Y) \\ \beta_U\omega_{67}(X,Y) \\ \alpha_U\omega_{89}(X,Y) \end{bmatrix} \quad .$$

It follows by construction that, if for two points $U=[P,Q]$ and $V=[R,S]$ it holds that $\phi_{\mathcal{S}}(U)=\phi_{\mathcal{S}}(V)$, then U and V define the same locus $\mathcal{E}_U$. Actually, a closer relationship between the group law $\mathcal{E}_U$ and the canonical group of $\mathcal{S}$ holds:

Lemma 3.2.1. *Let be $U=[P,Q]$ and $V=[R,S]$ two points of $\mathcal{S}$ such that $\mathcal{E}_U$ and $\mathcal{E}_V$ are smooth. If $\phi_{K_S}(U)=\phi_{K_S}(V)$, then $\mathcal{E}_U=\mathcal{E}_V$ and $\mu_U(P,Q)=\mu_U(R,S)$ holds, where μ_U is the group law in $\mathcal{E}_U$.*

Proof. Let's consider the addition law $\oplus_X^U$ from 3.1.8. For every point $W=[A,B]$ in a suitable neighborhood $\mathcal{U}$ of U in $\mathcal{S}$, the locus $\mathcal{E}_W$ is still a smooth elliptic curve, and we can then denote by τ_W a corresponding element in $\mathcal{H}_1$. Moreover, is well-defined $\mu_W(W)$, where μ_W is the group law in $\mathcal{E}_W$:

$$\mu_W(W) := \mu_W(A,B) \quad .$$

We see indeed that this definition does not depend on the choice of the representative of W. For, let us consider g an element of the group $\mathcal{G}$ and $(g.A, g.B)$ the corresponding representative of W. Then, according to 3.5, there exists a 2-torsion point T on $\mathcal{E}_W$ such that:

$$\begin{aligned} \mu_W(A,T) &= g.A \\ \mu_W(B,T) &= g.B \quad . \end{aligned}$$

Hence, we can easily conclude that $\mu_W(g.A, g.B)=\mu_W(A,B)$, T being a 2-torsion point.
We denote now by $\theta_0(z,\tau_W), \theta_1(z,\tau_W), \theta_2(z,\tau_W), \theta_3(z,\tau_W)$ the four theta functions defining the embedding of $\mathcal{E}_W$ in $\mathbb{P}^3$, and by Ψ the holomorphic map $\Psi : \mathcal{U} \longrightarrow \mathbb{P}^3$ defined as follows:

$$\begin{pmatrix} 1 & & & \\ & \alpha & & \\ & & \beta & \\ & & & \alpha\beta \end{pmatrix} \circ \pi \circ \phi_{\omega_S} \quad ,$$

where π is the following projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^3$:

$$[\eta_{01}, \eta_{02}, \eta_{12}, \omega_{45}, \omega_{67}, \omega_{89}] \dashmapsto [\eta_{01}, \omega_{45}, \omega_{67}, \omega_{89}] \quad ,$$

and α and β square roots of $-\frac{\eta_{01}}{\eta_{02}}$ and $-\frac{\eta_{01}}{\eta_{02}+\eta_{01}}$ respectively, which are defined according to definitions 3.12 and 3.1.8. The map Ψ is defined everywhere on $\mathcal{U}$ because, on every point of $\mathcal{U}$, we have that $\eta_{01} \neq 0$ and $\eta_{01} \neq -\eta_{02}$ by definition of $\mathcal{U}$, and in particular α and β can be considered simply as holomorphic functions defined on $\mathcal{U}$ as well and with values in $\mathbb{C}^*$. We remark, furthermore, that the choice of the branch of the square root used to define α and β is not important because another choice leads to a sign-change of the coordinates to the function Ψ accordingly to the action of the group $\mathcal{G}$ on the coordinates of $\mathbb{P}^3$ (see 3.1.8). The map Ψ is then:

$$\begin{aligned}\Psi(W) &= [\eta_{01}(W), \alpha\beta\omega_{45}(W), \beta\omega_{67}(W), \alpha\omega_{89}(W)] = \oplus_X^W(W) \\ &= [\theta_0 \circ \mu_W(W), \theta_1 \circ \mu_W(W), \theta_2 \circ \mu_W(W), \theta_3 \circ \mu_W(W)] \quad .\end{aligned}$$

Where the last line follows by 3.2 in proposition 3.1.2. Under this setting, if $\phi_{\mathcal{S}}(U) = \phi_{\mathcal{S}}(V)$ and $\mathcal{E}_U = \mathcal{E}_V$ are smooth elliptic curves, then $\Psi(U) = \Psi(V)$ and it must then exist $\zeta \in \mathbb{C}^*$ a constant such that, for every $j = 0, \ldots 3$,

$$\theta_j \circ \mu_U(U) = \zeta \cdot \theta_j \circ \mu_U(V) \quad .$$

On the other hand, the sections θ_j on $\mathcal{E}_U$, with $j = 0, \ldots 3$, embed $\mathcal{E}_U$ in $\mathbb{P}^3$, so we can conclude that $\mu_U(U) = \mu_U(V)$. $\square$

Observation 3.2.2. If U is a point of $\mathcal{S}$ such that $\phi_{\mathcal{S}}(U) = \phi_{\mathcal{S}}(U+g)$ where g is a non-trivial element of $\mathcal{G}$, then U belongs to the intersection of two others translated of $\mathcal{S}$ in A corresponding to the zero-locus of two others theta functions. Using the notation of 2.1.1, we denote by $\mathcal{S}_\gamma$ the zero locus in A of the theta function θ_γ. The theta function θ_α corresponds in particular to the a-invariant global holomorphic differential ω_{45}, the theta function θ_β to ω_{67} and $\theta_{\alpha+\beta}$ to ω_{89}.

Lemma 3.2.3. *The set of the base points of the polarization $|\mathcal{S}|$ on A consists of* 16 2*-torsion points, on which the group $\mathcal{G}$ acts with precisely four distinct $\mathcal{G}$-orbits of order* 4*.*
Each $\mathcal{G}$-orbit corresponds to one fundamental bitangent line on $\mathcal{D}$, i.e, the bitangent lines with equations $x = 0$, $y = 0$, $z = 0$, $t = 0$. Moreover, the canonical map of $\mathcal{S}$ sends the points of the same orbit to a unique point.

Proof. We observe first that a base point $U = [P, Q]$ defines a locus $\mathcal{E}_U$ which is not smooth. Indeed, let us consider the addition law $\oplus_X$ defined in 3.1.8. Using the same notation of 3.2.1, we see that if the addition law were defined in U, then we would have

$$\theta_1(\mu(U)) = \theta_2(\mu(U)) = \theta_3(\mu(U)) = 0 \quad .$$

This means that the group law would send the couple of points representing U to the point $[1, 0, 0, 0]$, which doesn't belong to $\mathcal{E}_U$. Hence, the addition law $\oplus_X$ is not defined on U, and this implies that Q belongs to the $\mathcal{G}$-orbit of P. On the other side, the canonical image of two different base points belonging to the same $\mathcal{G}$-orbit is the same (see also 3.2.2), and then the same would be true for the base points $[P, P]$ and $[P, g.P]$. But this would contradict the lemma 3.2.1: the group law μ_U is defined independently of the addition law, and for every $g \in \mathcal{G}$ the internal sum $\mu(P, P)$ and $\mu(P, gP)$ would be the same. We conclude then that $\mathcal{E}_U$ can't be smooth.
A base point in the linear system $|\mathcal{S}|$ in A defines a bitangent line on $\mathcal{D}$: indeed, a base point is an odd 2-torsion point in A whose image in Θ is still an odd 2-torsion point.
It can be now easily seen that two base points which yield the same bitangent, must be in the same $\mathcal{G}$-orbit.
Furthermore, this ensures also that a base point must be of the form $[P, Q]$, with P and Q not belonging to the same $\mathcal{G}$-orbit, provided that $\mathcal{D}$ has no hyperflex, which can be excluded by the generality of $\mathcal{D}$. Again, by the generality of $\mathcal{D}$, we can suppose that, to every bitangent line l to $\mathcal{D}$ different from the lines x,y,z,t, there corresponds a locus $\mathcal{E}_t$ which is smooth, and then l it cuts on $\mathcal{D}$ a divisor of the form $2(p + q)$ such that no point of $\mathcal{S}$ in the preimage of $p + q \in \Theta$ with respect to the isogeny p is a base point. $\square$

Example 3.2.4. With the notation of 2.3.13, we consider the quartic curve $\mathcal{D}$ in $\mathbb{P}^3$ defined by

$$\mathcal{D} : \begin{cases} x + y + z + t & = 0 \\ q(x, y, z, t) & = xyzt \end{cases} .$$

Then we have, for every line $l \in \{x, y, z, t\}$ in the plane $H : x + y + z + t = 0$

$$l.\mathcal{D} = 2(l_1 + l_2) \quad ,$$

and we can select two points L_1 and L_2 in the respective preimages in $\mathcal{C}$ respect to p. Then, by 3.2.3, we see that $\mathcal{G}.[(L_1, L_2)]$ is a $\mathcal{G}$-orbit of base points for $\mathcal{L}$ in A. In particular, the set of the 16 base points in the linear system $|\mathcal{S}|$ is exactly the union of four $\mathcal{G}$-orbits, each corresponding to a fundamental bitangent.

Proposition 3.2.5. *Let U, V be points on $\mathcal{S}$ such that $\phi_{\mathcal{S}}(U) = \phi_{\mathcal{S}}(V)$. Then one of the following cases occurs:*

- $V = U$
- $V = -g.U$ *for some non-trivial element g of $\mathcal{G}$. This case arises precisely when U and V belong to the canonical curve $\mathcal{S} \cap \mathcal{S}_g$.*

- *$V = g.U$ for some non-trivial element g of $\mathcal{G}$. This case arises precisely when U and V belong to the translate $\mathcal{S}_h$, for every $h \in \mathcal{G} - \{g\}$.*

- *U and V are two base points of $|\mathcal{S}|$ which belong to the same $\mathcal{G}$-orbit.*

Proof. Let us consider $U = [P, Q]$ and $V = [R, S]$ two points on $\mathcal{S}$, and let us assume that $\phi_{\mathcal{S}}(U) = \phi_{\mathcal{S}}(V)$. Let p, q, r and s denote, moreover, the corresponding points on $\mathcal{D}$, and $[a, b, c] = [\eta_{12}, -\eta_{02}, \eta_{01}]$ the coefficients of the line $l := G(U) = G(V) \in \mathbb{P}^{2^\vee}$ according to 3.12.
Depending on the coefficients, the locus $\mathcal{E} := \mathcal{E}_U$ will be smooth or not. However, up to exchange a, b, and c, we can assume that we are in one of the following cases:

i) a, b and c are all distinct and non-zero. In this case, $\mathcal{E}$ is a smooth elliptic curve.

ii) $c = 0$, but $b \neq 0 \neq a$ and $a \neq b$. In this case, the locus $\mathcal{E}$ is the union of two irreducible plane conics in $\mathbb{P}^3$ meeting in a point not belonging to the curve $\mathcal{C}$.

iii) $c = 0$ and $b = 0$. In this case, l is the bitangent x, and $\mathcal{E}$ is a double conic contained in the hyperplane $\{X = 0\}$ in $\mathbb{P}^3$. This case occurs precisely when U and V are base points. (c.f. the lemma 3.2.3)

iv) $c = 0$ and $a = b \neq 0$. In this case, the locus $\mathcal{E}$ is the union of four lines, each couple of them lying on a plane and intersecting in a point not belonging to $\mathcal{C}$.

Let us begin with the first case, in which $\mathcal{E}$ is a smooth elliptic curve. Then by lemma 3.2.1, we have that:

$$\mu(P, Q) = \mu(R, S) \tag{3.13}$$

where μ is the group law in $\mathcal{E}$. Suppose that $U \neq V$. By 3.13 we can suppose moreover that $R \neq P$, up to exchange R and S, and also S can be supposed to be different from Q (again by 3.13).
If S belongs to the $\mathcal{G}$-orbit of P, then, acting with $\Delta_{\mathcal{G}}$, we can suppose that $S = P$, and applying (3.13) we can conclude that $R = Q$, and hence that $U = V$. Thus, we can suppose that S does not belong to the $\mathcal{G}$-orbit of P. Symmetrically, we can suppose that R and S both belong neither to the $\mathcal{G}$-orbit of P, nor to the $\mathcal{G}$-orbit of Q. We can then consider the following canonical divisor on $\mathcal{D}$:

$$l.D = p + q + r + s \quad ,$$

and we have that $p \neq r$, $p \neq s$, $q \neq r$ and $q \neq s$. In particular, the divisor $R+S$ on $\mathcal{C}$ is the pullback of the Serre dual of the divisor $p+q$ on $\mathcal{D}$, and then it must exist an element $g \in G$ such that:

$$V = -g.U$$

The element g is not the identity because otherwise U and V were both base points, and in such a case we would reach a contradiction by applying lemma 3.2.2 since $\mathcal{E}$ is supposed to be smooth.

Concerning the remaining cases, we have to treat them independently of lemma 3.2.1, which cannot be applied if $\mathcal{E}$ is not smooth.
Suppose we are in the second case. Then $\mathcal{E}$ is a locus in $\mathbb{P}^3$ defined by the equations:

$$\mathcal{E} := \begin{cases} aX^2 + bY^2 = 0 \\ X^2 + Y^2 + Z^2 + T^2 = 0 \end{cases} ,$$

where:

$$\mathcal{E} = \mathcal{Q}^+ \cup \mathcal{Q}^-$$

$$\mathcal{Q}^\epsilon = \begin{cases} Y = \epsilon i \sqrt{\frac{b}{a}} X \\ X^2 + Y^2 + Z^2 + T^2 = 0 \end{cases}$$

and ϵ denotes a sign. We choose the following parametrizations $f^\epsilon : \mathbb{P}^1 \longrightarrow \mathcal{Q}^\epsilon \subseteq \mathbb{P}^3$ of the quadrics $\mathcal{Q}^\epsilon$:

$$f^\epsilon([u,v]) = \left[\frac{uv}{\sqrt{1-\frac{b}{a}}}, \epsilon i \sqrt{\frac{b}{a}} \frac{uv}{\sqrt{1-\frac{b}{a}}}, \frac{i}{2}(u^2+v^2), \frac{i}{2}(u^2-v^2) \right] \tag{3.14}$$

$$\mathcal{Q}^+ \cap \mathcal{Q}^- = f^*([1,0]) = f^*([0,1]) \notin \mathcal{C} \quad .$$

The choice of the square roots in 3.14 is not important. We notice, furthermore, that the group $\mathcal{G}$ acts in the following form:

$$\begin{aligned} a.f^\epsilon([u,v]) &= f^\epsilon([u,-v]) \\ b.f^\epsilon([u,v]) &= f^{-\epsilon}([v,u]) \quad . \end{aligned}$$

Hence, we can consider, without loss of generality, two points $U := [P,Q]$ and $V = [P,R]$ such that

$$\begin{aligned} P &:= f^1([u,1]) \\ Q &:= f^1([v,1]) \\ R &:= f^\epsilon([w,1]) \\ \phi_{\mathcal{S}}(U) &= \phi_{\mathcal{S}}(V) \quad . \end{aligned}$$

Moreover, without loss of generality we can assume that R does not belong to the $\mathcal{G}$-orbit of P. In this setting, we have to prove that $v=w$ and that $\epsilon=1$. First of all, we have that:

$$\eta_{02}(V)=\eta_{02}([f^1([u,1]),f^1([w,1])])=-\frac{1}{4(1-\frac{a}{b})}(u^2-w^2)(1-u^2w^2)$$
$$\eta_{12}(V)=\eta_{12}([f^1([u,1]),f^1([w,1])])=-\frac{1}{4}\frac{\frac{a}{b}}{(1-\frac{a}{b})}(u^2-w^2)(1-u^2w^2)\quad .$$

Similarly, we can now write down, up to a constant independent from u, v and ϵ, the following expressions of the sections ω_{45}, ω_{67} and ω_{89}:

$$\begin{aligned}
\omega_{45}(V)&=\begin{vmatrix}u^2 & \epsilon w^2\\ u^4-1 & w^4-1\end{vmatrix}=\begin{cases}-(u^2-w^2)(u^2w^2+1) & \text{if } \epsilon=1\\ (u^2+w^2)(u^2w^2-1) & \text{if } \epsilon=-1\end{cases}\\
\omega_{67}(V)&=\begin{vmatrix}u(u^2+1) & \epsilon w(w^2+1)\\ u(u^2-1) & w(w^2-1)\end{vmatrix}=\begin{cases}-2uw(u^2-w^2) & \text{if } \epsilon=1\\ -2uw(u^2w^2-1) & \text{if } \epsilon=-1\end{cases}\\
\omega_{89}(V)&=\begin{vmatrix}u(u^2-1) & \epsilon w(w^2-1)\\ u(u^2+1) & w(w^2+1)\end{vmatrix}=\begin{cases}2uw(u^2-w^2) & \text{if } \epsilon=1\\ -2uw(u^2w^2-1) & \text{if } \epsilon=-1\end{cases}
\end{aligned}\tag{3.15}$$

In particular, if we apply the previous expressions 3.15 to U, we obtain:

$$\phi_{\mathcal{S}}(U)=\begin{bmatrix}\eta_{01}(U)\\ \eta_{02}(U)\\ \eta_{12}(U)\\ \omega_{45}(U)\\ \omega_{67}(U)\\ \omega_{89}(U)\end{bmatrix}=\begin{bmatrix}0\\ -\frac{1}{4(1-\frac{a}{b})}(u^2-v^2)(1-u^2v^2)\\ -\frac{1}{4}\frac{\frac{a}{b}}{(1-\frac{a}{b})}(u^2-v^2)(1-u^2v^2)\\ -(u^2-v^2)(u^2v^2+1)\\ -2uv(u^2-v^2)\\ 2uv(u^2-v^2)\end{bmatrix}=\begin{bmatrix}0\\ \frac{1}{4(1-\frac{a}{b})}(1+u^2v^2)\\ \frac{1}{4}\frac{\frac{a}{b}}{(1-\frac{a}{b})}(1-u^2v^2)\\ u^2v^2+1\\ 2uv\\ -2uv\end{bmatrix}\quad .$$

If we had that $\epsilon=-1$, then we would have:

$$\phi_{\mathcal{S}}(V)=\begin{bmatrix}\eta_{01}(V)\\ \eta_{02}(V)\\ \eta_{12}(V)\\ \omega_{45}(V)\\ \omega_{67}(V)\\ \omega_{89}(V)\end{bmatrix}=\begin{bmatrix}0\\ -\frac{1}{4(1-\frac{a}{b})}(u^2-w^2)(1-u^2w^2)\\ -\frac{1}{4}\frac{\frac{a}{b}}{(1-\frac{a}{b})}(u^2-w'^2)(1-u^2w^2)\\ (u^2+w^2)(u^2w^2-1)\\ -2uw(u^2w^2-1)\\ -2uw(u^2w^2-1)\end{bmatrix}=\begin{bmatrix}0\\ -\frac{1}{4(1-\frac{a}{b})}(u^2-w^2)\\ -\frac{1}{4}\frac{\frac{a}{b}}{(1-\frac{a}{b})}(u^2-w^2)\\ u^2+w^2\\ -2uw\\ -2uw\end{bmatrix}\quad ,$$

which would imply that $\phi_{\mathcal{S}}(U)\neq\phi_{\mathcal{S}}(V)$ since neither u nor w can vanish.

Hence, we can conclude that $\epsilon = \epsilon' = 1$. In this case, we have, as points on $\mathbb{P}^5$:

$$\phi_{\mathcal{S}}(U) = \begin{bmatrix} 0 \\ \frac{1}{4(1-\frac{a}{b})}(1-u^2v^2) \\ \frac{1}{4}\frac{\frac{a}{b}}{(1-\frac{a}{b})}(1-u^2v^2) \\ u^2v^2+1 \\ 2uv \\ -2uv \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4(1-\frac{a}{b})}(1-u^2w^2) \\ \frac{1}{4}\frac{\frac{a}{b}}{(1-\frac{a}{b})}(1-u^2w^2) \\ u^2w^2+1 \\ 2uw \\ -2uw \end{bmatrix} = \phi_{\mathcal{S}}(V) \quad ,$$

and it can be easily seen that $v = w$ holds.
It only remains to consider the fourth case. The locus $\mathcal{E}$ is reducible and it can expressed as union of four lines:

$$\begin{aligned} \mathcal{E} &= r^{1,1} \cup r^{1,-1} \cup r^{-1,1} \cup r^{-1,-1} \\ r^{\gamma,\delta} &= \begin{cases} Y = \gamma i X \\ T = \delta i Z \end{cases} \qquad \gamma, \delta \in \{+1,-1\} \end{aligned} \quad . \tag{3.16}$$

We parametrize the lines in 3.16 as follows:

$$\begin{aligned} g^{\gamma,\delta} &: \mathbb{P}^1 \longrightarrow r^{\gamma,\delta} \\ g^{\gamma,\delta}([u,v]) &= [u, \gamma i u, v, \delta i v] \quad . \end{aligned}$$

For every $u \in \mathbb{C}$ we denote:

$$\begin{aligned} g^{\gamma,\delta}(u) &:= g^{\gamma,\delta}([u,1]) \\ g^{\gamma,\delta}(\infty) &:= g^{\gamma,\delta}([1,0]) \quad . \end{aligned}$$

It can be easily seen that $g^{\gamma,\delta}(0)$ and $g^{\gamma,\delta}(\infty)$ does not belong to $\mathcal{C}$, and that, on the lines defined in 3.16, the group $\mathcal{G}$ acts in the following way:

$$\begin{aligned} a.g^{\gamma,\delta}([u,v]) &= g^{\gamma,\delta}([-u,v]) \\ b.g^{\gamma,\delta}([u,v]) &= g^{-\gamma,-\delta}([u,v]) \quad . \end{aligned}$$

Let us consider now $U := [g^{\gamma,\delta}(u), g^{\gamma',\delta'}(u')]$ and $V := [g^{\gamma,\delta}(u), g^{\gamma'',\delta''}(u'')]$ two points with the same image with respect to the canonical map. By 3.11, the evaluation at U of the canonical map $\phi_{\mathcal{S}}$ can be expressed as follows:

$$\phi_{\mathcal{S}}(U)=\begin{bmatrix}\eta_{01}(U)\\ \eta_{02}(U)\\ \eta_{12}(U)\\ \omega_{45}(U)\\ \omega_{67}(U)\\ \omega_{89}(U)\end{bmatrix}=\begin{bmatrix}0\\ u^2-u'^2\\ u'^2-u^2\\ -\begin{vmatrix}\gamma u^2 & \gamma' u'^2\\ \delta & \delta'\end{vmatrix}\\ \begin{vmatrix}u & u'\\ -\gamma\delta u & -\gamma'\delta' u'\end{vmatrix}\\ \begin{vmatrix}\delta iu & \delta' iu'v\\ \gamma iu & \gamma' iu'\end{vmatrix}\end{bmatrix}=\begin{bmatrix}0\\ u^2-u'^2\\ u'^2-u^2\\ -\begin{vmatrix}\gamma u^2 & \gamma' u'^2\\ \delta & \delta'\end{vmatrix}\\ -\gamma\gamma' uu'\begin{vmatrix}\gamma & \gamma'\\ \delta & \delta'\end{vmatrix}\\ uu'\begin{vmatrix}\gamma & \gamma'\\ \delta & \delta'\end{vmatrix}\end{bmatrix}$$

By the hypothesis that $\phi_{\mathcal{S}}(U)=\phi_{\mathcal{S}}(V)$, it follows that there exists $\lambda\in\mathbb{C}^*$ such that:

$$\begin{cases}u^2-u''^2 & =\lambda(u^2-u'^2)\\ \begin{vmatrix}\gamma u^2 & \gamma'' u''^2\\ \delta & \delta''\end{vmatrix} & =\lambda\begin{vmatrix}\gamma u^2 & \gamma' u'^2\\ \delta & \delta'\end{vmatrix}\\ \gamma'' u''\Delta'' & =\lambda\gamma' u'\Delta'\\ u''\Delta'' & =\lambda u'\Delta'\end{cases}\tag{3.17}$$

where $\Delta':=\begin{vmatrix}\gamma & \gamma'\\ \delta & \delta'\end{vmatrix}$ and $\Delta'':=\begin{vmatrix}\gamma & \gamma''\\ \delta & \delta''\end{vmatrix}$. In consequence of the last two identities in 3.17, we can easily infer that $\gamma'=\gamma''$. In particular, we see that $\delta'=\delta''$ because Δ' vanishes if and only if Δ'' does. Hence, $\Delta'=\Delta''$ and the equations 3.17 can be rewritten in the following form:

$$\begin{cases}u^2-u''^2 & =\lambda(u^2-u'^2)\\ \begin{vmatrix}\gamma u^2 & \gamma' u''^2\\ \delta & \delta'\end{vmatrix} & =\lambda\begin{vmatrix}\gamma u^2 & \gamma' u'^2\\ \delta & \delta'\end{vmatrix}\\ u'' & =\lambda u' \quad ,\end{cases}$$

and we finally obtain the following linear system in the variables u^2, u'^2:

$$\begin{cases}\gamma\delta'(1-\lambda)u^2 & +\lambda\gamma'\delta(1-\lambda)u'^2=0\\ (1-\lambda)u^2 & +(1-\lambda)\lambda u'^2=0 \quad .\end{cases}$$

The determinant of this linear system must vanish because u and u' are supposed to be non-zero. Hence, we have that $\delta\delta'\lambda(1-\lambda)^2\Delta=0$, and we distinguish two cases: if $\lambda=1$ we can conclude that $U=V$. Otherwise, if $\Delta=0$, we have $\omega_{67} = \omega_{89} = 0$. In this case, we have

$$\begin{cases} u'' & = \lambda u' \\ u^2 & = -\lambda u'^2 \\ u^2 - u''^2 & = \lambda(u^2 - u'^2) \end{cases} ,$$

and finally

$$(-\lambda - 1)u'^2 = \lambda(-\lambda u'^2 - u'^2) = -\lambda(\lambda + 1)u'^2 \quad .$$

In conclusion, $\lambda = -1$ and $(\gamma'', \delta'') = \pm(\gamma', \delta')$, and there exists then a non-trivial element g of $\mathcal{G}$ such that $g.U = V$. This completes the proof of the proposition. □

We conclude this section by proving the following proposition.

Proposition 3.2.6. *The differential of the canonical map of $\mathcal{S}$ is everywhere injective.*

Proof. Throughout the proof, we use the notation which has been introduced in section 2.1.1. For every non-trivial element g of $\mathcal{G}$, we denote by A_g the $(1,1,2)$-polarized abelian threefold obtained as the quotient of A by g, by q_g the projection of A onto A_g, and by $\mathcal{T}_g$ the image $q_g(\mathcal{S})$ in A_g. For every non-trivial element g of $\mathcal{G}$, we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[d, "q_g"] & \mathcal{S} \arrow[l, hook'] \arrow[r, "\phi_{\mathcal{S}}"] \arrow[d, "q_g"] & \Sigma \arrow[r, hook] \arrow[d] & \mathbb{P}^5 \arrow[d, dashed] \\ A_g & \mathcal{T}_g \arrow[l, hook'] \arrow[r, "\phi_{\mathcal{T}_g}"] \arrow[dr] & \Sigma_g \arrow[r, hook] & \mathbb{P}^3 \\ & & Z_g \arrow[u] & \end{tikzcd} \tag{3.18}$$

where Σ denotes the image of the canonical map $\phi_{\mathcal{S}}$ of $\mathcal{S}$ in $\mathbb{P}^5$, while Z_g denotes the quotient of $\mathcal{T}_g$ by the involution $z \mapsto -z + h$ in A_g, where $h \in \mathcal{G} - \{1, g\}$. The canonical map $\phi_{\mathcal{T}_g}$ has degree 2, and it is defined by the theta functions $[\theta_\gamma, \frac{\partial\theta_0}{\partial z_1}, \frac{\partial\theta_0}{\partial z_2}, \frac{\partial\theta_0}{\partial z_3}]$, where $\gamma \in \langle\alpha, \beta\rangle$ is the unique non-trivial element such that $\lambda(g, \gamma) = 0$ (see 1.4.4).
Let us consider now a point z on $\mathcal{S}$ such that the differential $d_z\phi_{\mathcal{S}}$ is not injective. From diagram 3.18 we have that the differential at $q_g(z)$ of the canonical map of Z_g is not injective. Consequently (see theorem 1.4.1), the

image in $\mathbb{P}^3$ of $q_g(z)$ with respect to $\phi_{\mathcal{T}_g}$ must be one of the pinch points inside Γ_g in $\mathbb{P}^3$, which are contained in the plane $\theta_\gamma = 0$ by remark 1.4.5. Hence, z must be a base point of the linear system $|\mathcal{O}_A(\mathcal{S})|$ in A.
Thus, it is enough to prove the proposition for the base points of the linear system $|\mathcal{O}_A(\mathcal{S})|$ in $\mathcal{S}$.
Let us consider in particular a base point z_0. We have to prove that, for every tangent vector ν to $\mathcal{S}$ in z_0, there exists a divisor D in the canonical class $|K_{\mathcal{S}}|$ such that D contains z_0, but ν is not tangent to D in z_0. To conclude the proof of the proposition is then enough to prove the following lemma. $\square$

Lemma 3.2.7. *Let b be a base point of the linear system $|\mathcal{O}_A(\mathcal{S})|$. There exist an invertible matrix Ξ and non-zero constants δ, γ, λ, μ in $\mathbb{C}$ such that:*

$$\Xi \begin{pmatrix} \frac{\partial}{\partial z_0}\theta_0(b) & \frac{\partial}{\partial z_0}\theta_\alpha(b) & \frac{\partial}{\partial z_0}\theta_\beta(b) & \frac{\partial}{\partial z_0}\theta_{\alpha+\beta}(b) \\ \frac{\partial}{\partial z_1}\theta_0(b) & \frac{\partial}{\partial z_1}\theta_\alpha(b) & \frac{\partial}{\partial z_1}\theta_\beta(b) & \frac{\partial}{\partial z_1}\theta_{\alpha+\beta}(b) \\ \frac{\partial}{\partial z_2}\theta_0(b) & \frac{\partial}{\partial z_2}\theta_\alpha(b) & \frac{\partial}{\partial z_2}\theta_\beta(b) & \frac{\partial}{\partial z_2}\theta_{\alpha+\beta}(b) \end{pmatrix} = \begin{pmatrix} \delta & 0 & 0 & \mu \\ 0 & \gamma & 0 & \mu \\ 0 & 0 & \lambda & \mu \end{pmatrix} \quad .$$

Proof. Without loss of generality, we can suppose that $\eta_1 = \alpha$ and $\eta_2 = \beta$ belong to Γ_1. We denote by $\mathcal{A} : \mathcal{D} \longrightarrow \mathcal{J}(\mathcal{D})$ the Abel map with respect to a fixed point p_0 on $\mathcal{D}$ and $\tilde{\mathcal{A}} : \mathcal{C} \longrightarrow A$ the induced map in the cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "\tilde{\mathcal{A}}"] \arrow[d, "p"] & A \arrow[d, "p"] \\ \mathcal{D} \arrow[r, hook] & \mathcal{J} = \mathcal{J}(\mathcal{D}) \end{tikzcd}$$

With the notation from example 3.2.4, and denoting by l a bitangent of $\mathcal{D}$ among the fundamental bitangent lines $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$ and $\{t = 0\}$, we consider the base point $b_l = \tilde{\mathcal{A}}(L_1 + L_2) + \kappa$ of the linear system $|\mathcal{S}|$. The unramified bidouble covering $p : \mathcal{C} \longrightarrow \mathcal{D}$ is defined by the 2-torsion points:

$$\begin{aligned} \eta_1 &= \mathcal{O}_{\mathcal{D}}(y_1 + y_2 - x_1 - x_2) \\ \eta_2 &= \mathcal{O}_{\mathcal{D}}(z_1 + z_2 - x_1 - x_2) \\ \eta_1 \otimes \eta_2 &= \mathcal{O}_{\mathcal{D}}(t_1 + t_2 - x_1 - x_2) \quad . \end{aligned}$$

With this notation, the proof follows, since

$$\begin{aligned} \nabla\theta_0(b_y) &= \nabla\theta_0(b_x + \eta_1) = \phi_{\eta_1}(b_x) \cdot \nabla\theta_{\eta_1}(b_x) \\ \nabla\theta_0(b_z) &= \nabla\theta_0(b_x + \eta_2) = \phi_{\eta_2}(b_x) \cdot \nabla\theta_{\eta_2}(b_x) \\ \nabla\theta_0(b_t) &= \nabla\theta_0(b_x + \eta_1 + \eta_2) = \phi_{\eta_1+\eta_2}(b_x) \cdot \nabla\theta_{\eta_1+\eta_2}(b_x) \quad . \end{aligned}$$

$\square$

3.3 Linear systems on $(1, 2)$-polarized abelian surfaces.

In this section, we quickly recall some basic facts about $(1, 2)$-polarized abelian surfaces and their ample linear systems. Let $(A, \mathcal{L})$ be a $(1, 2)$-polarized abelian surface. It easy to prove (see [8]) that, up to translation, the set of base points $\mathcal{B}$ of $|\mathcal{L}|$ consists of 4 elements and it is contained in the set A_2 of the 2-torsion points of A. The rational map $\phi_{\mathcal{L}} : A \dashrightarrow \mathbb{P}^1$, moreover, can be extended to a map $\psi : \widehat{A} \longrightarrow \mathbb{P}^1$, where $\widehat{A}$ denotes the blow-up of A at the points of $\mathcal{B}$. This morphism ψ is clearly a fibration, and its general fiber is a smooth non-hyperelliptic curve of genus 3. By applying the following Zeuthen Segre Formula, we can count the singular fibers of ψ.

Theorem 3.3.1. *Let S be a smooth projective surface, and let $\{C_\lambda\}_\lambda$ a linear pencil of curves of genus g which meet transversally in δ distinct points. If μ is the number of singular curves in the pencil (counted with multiplicity), then*

$$\mu - \delta - 2(2g - 2) = I + 4 = \chi_{top}(S) \quad ,$$

where the integer I is an algebraic invariant of the algebraic surface S, called the Zeuthen-Segre invariant. Moreover, the Zeuthen-Segre invariant I can be expressed in terms of the surface S blown up in the points of the set $\mathcal{B}$ of the base points of the pencil. It holds, indeed:

$$I + 4 + \delta = \chi_{top}(Blow_{\mathcal{B}}(S)) \quad .$$

By applying 3.3.1, it follows that the number of singular fibers of ψ (counted with multiplicity) is

$$\mu = 8 + \chi_{top}(Blow_{\mathcal{B}}(A)) = 8 + 4 = 12 \quad .$$

Observation 3.3.2. We determine now all possible configurations of singular fibers of ψ. We can suppose that $\mathcal{L}$ is of characteristic 0, and we can consider a basis for $H^0(A, \mathcal{L})$ of even theta functions. Considered $\mathcal{D}$ a symmetric divisor in $|\mathcal{L}|$, it holds that (see [8] p. 97)

$$\begin{aligned} \#A_2^+(\mathcal{D}) &:= \{x \in A_2 \mid mult_x\mathcal{D} \text{ is even}\} = 12 \\ \#A_2^-(\mathcal{D}) &:= \{x \in A_2 \mid mult_x\mathcal{D} \text{ is odd}\} = 4 \quad , \end{aligned}$$

and we have, moreover, that $A_2^-(\mathcal{D}) = \mathcal{B}$. It follows that:

a) No divisor $\mathcal{D}$ of $|\mathcal{L}|$ is singular at a base point.

b) If a divisor $\mathcal{D}$ of $|\mathcal{L}|$ contains a 2-torsion point x which is not contained in $\mathcal{B}$, then $\mathcal{D}$ is singular in x. Moreover, if a divisor $\mathcal{D}$ of $|\mathcal{L}|$ contains two distinct points of $A_2^+(\mathcal{L})$, then $\mathcal{D}$ is reducible, or A is isomorphic to a polarized product of elliptic curves. Indeed, let us suppose x and y are distinct 2-torsion points for which $mult_x(\mathcal{D}) = mult_y(\mathcal{D}) = 2$. If $\mathcal{D}$ were irreducible, then its normalization $\mathcal{C}$ would be a smooth elliptic curve, and we would have a homomorphism from $\mathcal{C}$ to A, which means that A is isomorphic to a polarized product of elliptic curves.

We can now describe all possible configurations of reducible fibers of ψ. Let us consider a reducible fiber $\mathcal{D}$ of ψ, which we write as the union of its reducible components

$$\mathcal{D} = E_1 + \cdots + E_s \quad . \tag{3.19}$$

We have clearly, by adjunction, that

$$2p_a(E_j) - 2 = E_j^2 \quad .$$

On the other side, none of the divisors E_s in the decomposition 3.19 can be ample: the polarization on A, which is of type $(1,2)$, would be in this case the tensor product of two polarizations. Thus, we can conclude that $E_j^2 = 0$ and E_j is a curve of genus 1 for every j. On the other side, we have that

$$\begin{aligned} 3 = p_a(D) &= \sum_j p_a(E_j) + \sum_{i\neq j} E_i.E_j - s + 1 \\ &= \sum_{i\neq j} E_i.E_j + 1 \quad . \end{aligned}$$

Furthermore, because $\mathcal{D}$ is connected, the only possible configurations are the following:

a) $\mathcal{D} = E_1 + E_2 + E_3$ with $E_2.E_1 = E_2.E_3 = 1$ and $E_1.E_3 = 0$.

b) $\mathcal{D} = E_1 + E_2$ and $E_1.E_2 = 2$.

Note that, in both cases, each irreducible component E_i is a smooth elliptic curve, but in the first case we have that (see 10.4.6 [8]):

$$(A,\mathcal{L}) \cong (E_1, \mathcal{O}_{E_1}(O_{E_1})) \boxtimes (E_1, \mathcal{O}_{E_2}(O_{E_2})) \cong (E_3, \mathcal{O}_{E_3}(O_{E_3})) \boxtimes (E_2, \mathcal{O}_{E_2}(O_{E_2})) \quad .$$

Finally, in the case in which $\mathcal{D} = E_1 + E_2$ and $E_1.E_2 = 2$, we can consider the difference morphism $\phi\colon E_1 \times E_2 \longrightarrow A$ defined by $\phi(p,q) := p - q$. This kernel of ϕ consists of the two points in which E_1 and E_2 intersect. Thus, ϕ is an isogeny, and we conclude that

$$\phi^*(\mathcal{L}) = (\mathcal{D}.E_1, \mathcal{D}.E_2) = (2,2) \quad .$$

In conclusion, A is in this case isogenous to a product of (2)-polarized elliptic curves, which carries a natural polarization of type $(2,2)$.

3.4 Degenerations of polarizations of type $(1, 2, 2)$ to quotients of products.

In the previous section, we investigated the behavior of the canonical map of a suitable unramified bidouble cover of a principal polarization of a general Jacobian 3-fold, and it turned out (see proposition 3.2.5) that the canonical map of such a surface is never injective, because it has degree 2 on some special canonical curves. However, this case is quite special. We consider, in this section, other possible degenerate situations, which arise naturally by considering surfaces yielding a polarization of type (1, 2, 2) on abelian 3-fold A, which is isogenous to a polarized product of a $(1, 1)$-polarized surface and a (1)-polarized elliptic curve. In such cases, it is possible to investigate and to determine the behavior of the canonical map of every sufficiently general member of the linear system $|\mathcal{L}|$.

Notation 3.4.1. Let us consider a point $\tau = \begin{bmatrix} \tau_1 & \delta & 0 \\ -\delta & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix}$ in the Siegel upper half-space $\mathcal{H}_3$, and the $(2, 2, 2)$-polarized abelian threefold

$$T := \mathbb{C}^3 \Big/ \Big\langle \begin{bmatrix} \tau I_3 & 2I_3 \end{bmatrix} \Big\rangle_{\mathbb{Z}} \cong B \times E$$

where B and E respectively denote the $(2, 2)$-polarized abelian surface and the 2-polarized elliptic curve defined as follows:

$$B := \mathbb{C}^2 \Big/ \left\langle \begin{bmatrix} \tau_1 & \delta & 2 & 0 \\ -\delta & \tau_2 & 0 & 2 \end{bmatrix} \right\rangle_{\mathbb{Z}}$$
$$E := \mathbb{C} \Big/ \langle \tau_3, 2 \rangle_{\mathbb{Z}} \quad .$$

We denote, moreover, by $\mathcal{N}$ the polarization of type $(2, 2)$ on B, and we respectively denote the holomorphic sections of the polarizations on B and E by:

$$H^0(B, \mathcal{N}) = \langle \theta_{00}, \theta_{01}, \theta_{10}, \theta_{11} \rangle$$
$$H^0(E, \mathcal{O}_E(2O_E)) = \langle \psi_0, \psi_1 \rangle$$

where the theta-functions θ_{ij} and ψ_k are defined according to definition 1.3.1. We can now consider the $(1, 2, 2)$-abelian threefold A obtained as the quotient of $B \times E$ by the translation $e_1 + e_2 + e_3$. Indeed, denoting by $p : T \longrightarrow A$ the isogeny with kernel $e_1 + e_2 + e_3$, there exists an ample line bundle $\mathcal{L}$ on A such that

$$p^*\mathcal{L} \cong \mathcal{N} \boxtimes \mathcal{O}_E(2O_E)$$

It is easily seen, moreover, that the set of theta functions $\theta_{ijk}(z,t) := \theta_{ij}(z)\psi_k(t)$, where $(ijk) \in \{(000, 011, 101, 110\}$, is a basis for $H^0(A, \mathcal{L})$.

Observation 3.4.2. (Degeneration to product of elliptic curves)
In the case in which $\delta = 0$, T is the product $E_1 \times E_2 \times E_3$ of elliptic curves $E_j := \mathbb{C} \big/ \langle 2, \tau_j \rangle$, with $j = 1,2,3$. We consider, for every couple of indices (i,j) the isogeny of degree 2

$$p_{ij} : E_i \times E_j \longrightarrow S_{ij} := E_i \times E_j \big/ \langle e_i + e_j \rangle \quad .$$

On S_{ij} we will distinguish two line bundles in the same algebraic equivalence class: there exists a line bundle $\mathcal{M}_{ij}$ such that $p_{ij}^* \mathcal{M}_{ij} \cong \mathcal{O}_{E_i}(2O_{E_i}) \boxtimes \mathcal{O}_{E_j}(2O_{E_i})$, and we have clearly that

$$p_{ij*}(\mathcal{O}_{E_i}(2O_{E_i}) \boxtimes \mathcal{O}_{E_j}(2O_{E_i})) \cong \mathcal{M}_{ij} \oplus t^*_{\frac{\tau_i}{2}} \mathcal{M}_{ij} \quad .$$

We will respectively denote the space of global holomorphic sections of these line bundles by:

$$H^0(S_{ij}, \mathcal{M}_{ij}) = \left\langle \theta_{00}^{(ij)}, \theta_{11}^{(ij)} \right\rangle$$
$$H^0(S_{ij}, t^*_{\frac{\tau_i}{2}} \mathcal{M}_{ij}) = \left\langle \theta_{10}^{(ij)}, \theta_{01}^{(ij)} \right\rangle$$

where $\theta_{hk}^{(ij)}$ denotes just $\theta_{ih}\theta_{jk}$, and where $H^0(E_j, \mathcal{O}_{E_j}(2OE_j))$ is generated by the functions θ_{j0} and θ_{j1} according to definition 1.3.2.

Definition 3.4.3. (The condition $\natural$) Let us consider $\tau = (\tau_{ij})_{ij}$ an element of the Siegel upper half-space $\mathcal{H}_3$. We say that **the point** $(b,c,d) \in \mathbb{C}^3$ **satisfies the condition** $\natural$ **respect to** τ if, considered the $(1,2,2)$-polarized abelian variety A in 3.4.2 obtained as the quotient of the product $T = E_1 \times E_2 \times E_3$ with $E_j := \mathbb{C} \big/ \langle 2, \tau_{jj} \rangle$ for every $j = 1,2,3$, the following conditions hold:

- The numbers b, c and d are all non-zero.
- The curves in S_{23} defined by $\{\theta_{00}^{(23)} + b\theta_{11}^{(23)} = 0\}$ in the linear system $|\mathcal{M}_{23}|$ and $\{c\theta_{01}^{(23)} + d\theta_{10}^{(23)} = 0\}$ in the linear system $|t^*_{\frac{\tau_2}{2}}\mathcal{M}_{23}|$ are smooth and non-hyperelliptic.
- The curves in S_{13} defined by $\{\theta_{00}^{(13)} + c\theta_{11}^{(13)} = 0\}$ in the linear system $|\mathcal{M}_{13}|$ and $\{b\theta_{01}^{(13)} + d\theta_{10}^{(13)} = 0\}$ in the linear system $|t^*_{\frac{\tau_1}{2}}\mathcal{M}_{13}|$ are smooth and non-hyperelliptic.
- The curves in S_{12} defined by $\{\theta_{00}^{(12)} + d\theta_{11}^{(12)} = 0\}$ in the linear system $|\mathcal{M}_{12}|$ and $\{b\theta_{01}^{(12)} + c\theta_{10}^{(12)} = 0\}$ in the linear system $|t^*_{\frac{\tau_1}{2}}\mathcal{M}_{12}|$ are smooth and non-hyperelliptic.

Notation 3.4.4. Let us consider (b,c,d) a point in $\mathbb{C}^3$, and $\theta := \theta_{000} + b\theta_{011} + c\theta_{101} + d\theta_{110}$ the corresponding theta function. Denoted by $\mathcal{S}$ the surface defined as the zero locus of θ in A, we will denote by ω_j the derivatives $\frac{\partial\theta}{\partial z_j}$ and by W_j the corresponding zero locus in $\mathcal{S}$. We recall that the ω_j's are holomorphic global sections of $\mathcal{O}_{\mathcal{S}}(\mathcal{S})$.

In the case of a product as in 3.4.1, we can determine the behavior of the canonical map of the general member of the linear system when condition $\natural$ is satisfied.

Proposition 3.4.5. *Let $\tau \in \mathcal{H}_3$ be as in 3.4.1 and (b,c,d) be a point in $\mathbb{C}^3$ which satisfies condition $\natural$ respect to τ, according to definition 3.4.3 . Let us consider the surface in A defined by*

$$\mathcal{S} : \theta_{000} + b\theta_{011} + c\theta_{101} + d\theta_{110} = 0 \tag{3.20}$$

Denoting by ι_j the involution on $\mathbb{C}^3$ which changes the sign of the jth coordinate, the canonical map $\phi_{\mathcal{S}}$ of $\mathcal{S}$ is one-to-one except:

- *on the finite set $W_1 \cap W_2$, on which the canonical map is two-to-one and factors through the involution $\iota_1\iota_2 : (z_1, z_2, z_3) \mapsto (-z_1, -z_2, z_3)$.*
- *on the canonical curve W_3, where the canonical map is two-to-one and factors through the involution $\iota_3 : (z_1, z_2, z_3) \mapsto (z_1, z_2, -z_3)$.*

Proof. With the notation which we have introduced in 3.4.4, we consider two points on $\mathcal{S}$

$$\begin{aligned} z &= [x, s] := p(x, s) \\ w &= [y, t] := p(y, t) \end{aligned}$$

such that $\phi_{\mathcal{S}}(z) = \phi_{\mathcal{S}}(w)$. In particular, there exists $\lambda \in \mathbb{C}^*$ such that, for every $(ijk) \in \{(000), (011), (101), (110)\}$, and for every $h = 1, 2, 3$, we have

$$\begin{aligned} \theta_{ij}(x)\psi_k(s) &= \lambda\theta_{ij}(y)\psi_k(t) \\ \omega_h(x, s) &= \lambda\omega_h(y, t) \end{aligned} \tag{3.21}$$

We denote, moreover, by Θ_{ij} the divisor $div(\theta_{ij})$. To prove the proposition is enough to consider the following cases:

a) $\psi_0(s) \neq 0 \neq \psi_1(s)$ and neither x nor y belongs to $(\Theta_{00}\cap\Theta_{11})\cup(\Theta_{01}\cap\Theta_{10})$, where $\Theta_{ij} := div(\theta_{ij})$ for every couple of indices ij.

b) $\psi_0(s) = 0$.

c) $\psi_0(s) \neq 0$ and x belongs to $\Theta_{00} \cap \Theta_{11}$.

We begin now with the first one. Here we have that, on B, $\phi_{\mathcal{N}}(x) = \phi_{\mathcal{N}}(y)$, and thus $x = \pm y$ because $\phi_{\mathcal{N}}$ induces an embedding of the Kummer surface $K := B \big/ \{\pm 1\}$ in $\mathbb{P}^3$. For the same reason, we have that $t = \pm s$, and we infer that $\lambda = 1$.
We obtain then easily, in this case, that $\phi_{\mathcal{S}}([x,s]) = \phi_{\mathcal{S}}([-x,s])$ if and only if (x,s) belongs to $W_1 \cap W_2$, and $\phi_{\mathcal{S}}([x,s]) = \phi_{\mathcal{S}}([x,-s])$ if and only if (x,s) belongs to W_3. This concludes the proof in the case a).

Let us suppose now we are in the case b), with $\psi_0(s) = 0$. In this case, $[x,s]$ and $[y,t]$ belong to:

$$\mathcal{S} \cap \Theta_{000} \cap \Theta_{110} \subseteq \{b\theta_{011} + c\theta_{101} = 0\} \quad . \tag{3.22}$$

We must necessarily have that $\psi_1(t) \neq 0$. If it were not the case, we would also have that $\psi_0(t) \neq 0$, and by 3.21 we could conclude that $\theta_{00}(y) = \theta_{11}(y) = 0$. This would imply, however,

$$\begin{aligned} \theta_{00}(x) = \theta_{11}(x) = 0 \\ b\theta_{01}(x) + c\theta_{10}(x) = 0 \end{aligned}$$

In conclusion, x would represent a singular point on the curve $\mathcal{C}$ in $S_{12} := B \big/ \langle e_1 + e_2 \rangle$ defined by the equation

$$\mathcal{C} : b\theta_{01} + c\theta_{10} = 0 \tag{3.23}$$

Thus, we have $\psi_1(t) \neq 0 \neq \psi_1(s)$, and x and y can be considered as points on the curve $\mathcal{C}$ defined in 3.23 (see also 3.22). We can infer that $\psi_0(t) = 0$. Indeed, arguing again by contradiction, if $\psi_0(t)$ were non zero, by the equation of $\mathcal{S}$ in 3.20 and by 3.21, we would conclude that:

$$\begin{aligned} \theta_{00}(y) = \theta_{11}(y) = 0 \\ b\theta_{01}(y) + c\theta_{10}(y) = 0 \end{aligned} \tag{3.24}$$

which again contradicts point b) in 3.3.2.
Hence, y would represent in S_{12} a base point of $|\mathcal{M}_{12}|$, which is, in particular, a 2-torsion point contained in the curve $\mathcal{C}$, which belongs to the linear system $|t^*_{\frac{\tau_3}{2}} \mathcal{M}_{12}|$. This, however, would contradict again condition $\natural$ on the coefficients, by point b) in 3.3.2.
Hence, we have that $\psi_0(t) = 0$. In particular, because $\psi_0(s) = 0$,

$$t \in \{s, -s\} = \left\{ \frac{1+\tau_3}{2}, \frac{-1+\tau_3}{2} \right\} \quad . \tag{3.25}$$

We recall now that, by proposition 1.1.1, the canonical map of $\mathcal{C}$ is defined on a point z on $\mathcal{C}$ as follows (recall that by condition $\natural$, b and c are non-zero):

$$\phi_{\omega_{\mathcal{C}}} = [\theta_{01}(z), b\partial_{z_1}\theta_{01}(z) + c\partial_{z_1}\theta_{10}(z), b\partial_{z_2}\theta_{01}(z) + c\partial_{z_2}\theta_{10}(z)] \tag{3.26}$$

On the other side, under these conditions, we have that

$$\begin{aligned}\phi_{\omega_{\mathcal{C}}}(x) &= [\theta_{01}(x), b\partial_{z_1}\theta_{01}(x) + c\partial_{z_1}\theta_{10}(x), b\partial_{z_2}\theta_{01}(x) + c\partial_{z_2}\theta_{10}(x)] \\ &= [\theta_{011}(x,s), \omega_1(x,s), \omega_2(x,s)] \\ \phi_{\omega_{\mathcal{C}}}(y) &= [\theta_{01}(y), b\partial_{z_1}\theta_{01}(y) + c\partial_{z_1}\theta_{10}(y), b\partial_{z_2}\theta_{01}(y) + c\partial_{z_2}\theta_{10}(y)] \\ &= [\theta_{011}(y,t), \omega_1(y,t), \omega_2(y,t)]\end{aligned}$$

and by 3.21 we conclude that $\phi_{\omega_{\mathcal{C}}}(x) = \phi_{\omega_{\mathcal{C}}}(y)$. We can, in particular, infer that the points x end y are equal in S_{12} because $\mathcal{C}$ is non-hyperelliptic by condition $\natural$, which by hypothesis holds true. Hence, $y \in \{x, x+e_1+e_2\}$ when we consider x and y as points on B. In conclusion, in virtue of 3.25, is enough to consider the case in which $x = y$ and the claim of the proposition follows now easily: if $s = -t$, then both w and z belong to W_3, and $w = \iota_3(z)$.

Let us assume, finally, that we are in the case c). Up to exchange the role of ψ_0 and ψ_1, we can assume we are in case b). Indeed, if it were not the case, then we would have:

$$\begin{aligned}\psi_0(s) \neq 0 \neq \psi_1(s) \\ \psi_0(t) \neq 0 \neq \psi_1(t)\end{aligned}$$

Hence, by 3.20 and 3.21 that x and y both satisfy 3.24, and this again contradicts condition $\natural$ on the coefficients, by point b) in 3.3.2.

□

In the case of a quotient of a product of three elliptic curves, the behavior of the canonical map of the general member of the linear system is similar, and the same procedure used to prove the proposition 3.4.5 can be used to prove the following proposition:

Proposition 3.4.6. *Let us consider τ as in 3.4.2 and b, c, d be a point in $\mathbb{C}^3$ which satisfies the condition $\natural$ (see 3.4.3) . Let us consider the surface*

$$\mathcal{S} : \theta_{000} + b\theta_{011} + c\theta_{101} + d\theta_{110} = 0 \quad .$$

Then the canonical $\phi_{\mathcal{S}}$ of $\mathcal{S}$ is one-to-one except on the intersections of $\mathcal{S}$ with one of the three canonical divisors W_j, on which the canonical map is two-to-one and factors through the involution ι_j.

3.5 On the differential of the canonical map

In the notations introduced in 3.4.2, we will prove in this section that, considered the $(2,2,2)$-polarized abelian 3-fold $T := E_1 \times E_2 \times E_3$, where E_1, E_2, and E_3 are three general elliptic curves, the canonical map of the surfaces $\mathcal{S}$ in $A := T \big/ \langle e_1 + e_2 + e_3 \rangle$ defined by the equation

$$\{(z_1, z_2, z_3) \in A \ \mid \ f := \theta_{000} + b\theta_{011} + c\theta_{101} + d\theta_{110} = 0\}$$

has everywhere injective differential, provided that (b, c, d) are sufficiently general. We denote by $\widehat{\mathcal{S}}$ the surface in T defined by the same equation.

Observation 3.5.1. We can describe the elliptic curves $E_j = \mathbb{C} \big/ \langle 1, \tau_{jj} \rangle_{\mathbb{Z}}$ as Riemann surfaces defined in a neighborhood $\mathcal{U}_i$ by an affine curve of the form

$$\begin{aligned} p_i(x_i) &:= (x_i^2 - 1)(x_i^2 - \delta_i^2) \\ g_i(x_i, y_i) &:= y_i^2 - (x_i^2 - 1)(x_i^2 - \delta_i^2) \end{aligned} \tag{3.27}$$

where (x_i,y_i) are the coordinates of an affine plane, and δ_i is a parameter depending only on τ_{jj}. The Riemann surface defined by the equation 3.27 has two points at infinity, which we denote by ∞_+ and ∞_-. Around them, the function x_i has a simple pole, so we can consider $v_i := \frac{1}{x_i}$ to be a local parameter around ∞_+ and ∞_-. At infinity, in particular, E_i is defined in a neighborhood $\mathcal{V}_i$ by the following affine curve, defined in an affine plane with coordinates (v_i,w_i):

$$\begin{aligned} q_i(v_i) &:= (1 - v_i^2)(1 - \delta_i^2 v_i^2) \\ h_i(v_i, w_i) &:= w_i^2 - (1 - v_i^2)(1 - \delta_i^2 v_i^2) \end{aligned} \tag{3.28}$$

The change of coordinate charts between $\mathcal{U}_i$ and $\mathcal{V}_i$ is precisely $(v_i, w_i) = (x_i^{-1}, y_i x_i^{-2})$, defined wherever $v_i \neq 0$ and $x_i \neq 0$.
The function x_i defines a ramified cover $x_i : E_i \longrightarrow \mathbb{P}^1$ of degree 2, ramified over the points of the set $\{1, -1, \delta_i, -\delta_i\}$
The affine model in 3.27 is called the Legendre normal form of E_i (see [15]). Moreover, the function x_i is a Legendre function for E_i, according to the following definition.

Definition 3.5.2. (see [7], p. 60) Let us consider an elliptic curve $E = \mathbb{C} \big/ \langle 1, \tau \rangle$. A **Legendre function for** E is a holomorphic function $\mathcal{P} : E \longrightarrow \mathbb{P}^1$ which is a double cover of $\mathbb{P}^1$ branched over the four distinct points ± 1, $\pm\delta \in \mathbb{P}^1 - \{0, \infty\}$, with $\delta \neq \pm 1$.

Observation 3.5.3. Let E be an elliptic curve as in the previous definition 3.5.2. A Legendre function $\mathcal{P} : E \longrightarrow \mathbb{P}^1$ for E is unique and satisfies the following properties (see [28]):

- $\mathcal{P}(z+1) = \mathcal{P}(z+\tau)) = \mathcal{P}(z)$, $\mathcal{P}\left(z+\frac{1}{2}\right) = -\mathcal{P}(z)$, $\mathcal{P}(-z) = \mathcal{P}(z)$ $\mathcal{P}\left(z+\frac{\tau}{2}\right) = \frac{\delta}{\mathcal{P}(z)}$ for every z.
- $\mathcal{P}(\frac{1}{2}) = -1$, $\mathcal{P}(0) = 1$, $\mathcal{P}(\frac{\tau}{2}) = \delta$, $\mathcal{P}\left(\frac{1+\tau}{2}\right) = -\delta$.
- $\mathcal{P}'(z) = 0$ if and only if $z \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$.

Moreover, according to Inoue (see Lemma 3.2 [28]), if we denote by ψ_0 and ψ_1 two non-zero holomorphic sections of $H^0(E, \mathcal{O}_E(2O_E))$ such that ψ_0 is invariant and ψ_1 is anti-invariant with respect to the translation by $\frac{1}{2}$, we have that the Legendre function for E is:

$$\mathcal{P}(z) = \frac{\psi_0(0,\tau)\psi_1(z,\tau)}{\psi_1(0,\tau)\psi_0(z,\tau)} \tag{3.29}$$

Proposition 3.5.4. *Let us consider $T := E_1 \times E_2 \times E_3$ the product of three general (2)-polarized elliptic curves $E_i = \mathbb{C} \big/ \langle 1, \tau_i \rangle$, $i = 1,2,3$. Let us consider a general smooth surface $\mathcal{S}$ yielding the natural $(1,2,2)$-polarization on the abelian 3-fold $A := T \big/ \langle e_1 + e_2 + e_3 \rangle$ induced by T. Then the differential of the canonical map of $\mathcal{S}$ is everywhere injective.*

Proof. Let us assume, as usual, that $\mathcal{S}$ is defined on A by an equation of the form

$$\mathcal{S}\colon\ f := \theta_{000} + b\theta_{011} + c\theta_{101} + d\theta_{110} = 0 \ . \tag{3.30}$$

and let us denote by $\widehat{\mathcal{S}}$ the corresponding surface on T. In the notations of 3.5.1 and 3.29, the function x_i can be reexpressed in terms of theta functions as follows:

$$x_i = \frac{\theta_0(0,\tau_i)\theta_1(z_i,\tau_i)}{\theta_1(0,\tau_i)\theta_0(z_i,\tau_i)} \ . \tag{3.31}$$

Let us moreover denote by $\mathcal{P}\colon T \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ the holomorphic function whose components are the functions x_i in 3.31. The function $\mathcal{P}$ factors through the isogeny $p : T \longrightarrow A$ and induce a holomorphic function

$$\overline{\mathcal{P}}\colon A \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \ .$$

It can be now easily seen that the set of base points $\mathcal{B}(|\mathcal{L}|)$ is the union of four $\mathbb{Z}_2^2$-orbits

$$\mathcal{B}(|\mathcal{L}|) = \mathcal{B}_{000} \cup \mathcal{B}_{011} \cup \mathcal{B}_{101} \cup \mathcal{B}_{110}$$

where

$$\begin{aligned}
\mathcal{B}_{000} &:= \overline{\mathcal{P}}^{-1}\left((\infty,\infty,\infty)\right) = \mathbb{Z}_2^2 \cdot \left(\frac{1+\tau_1}{2}, \frac{1+\tau_2}{2}, \frac{1+\tau_3}{2}\right)\\
\mathcal{B}_{011} &:= \overline{\mathcal{P}}^{-1}\left((\infty,0,0)\right) = \mathbb{Z}_2^2 \cdot \left(\frac{1+\tau_1}{2}, \frac{1}{2}, \frac{1}{2}\right)\\
\mathcal{B}_{101} &:= \overline{\mathcal{P}}^{-1}\left((0,\infty,0)\right) = \mathbb{Z}_2^2 \cdot \left(\frac{1}{2}, \frac{1+\tau_2}{2}, \frac{1}{2}\right)\\
\mathcal{B}_{110} &:= \overline{\mathcal{P}}^{-1}\left((0,0,\infty)\right) = \mathbb{Z}_2^2 \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1+\tau_3}{2}\right) \quad .
\end{aligned} \tag{3.32}$$

We first prove that, if $z = (z_1, z_2, z_3) \in \mathcal{B}(|\mathcal{L}|)$, the differential at z of the canonical map of $\mathcal{S}$ is injective. Without loss of generality, we can assume furthermore that z belongs to $\mathcal{B}_{110}$. In order to prove that the differential at z of the canonical map of $\mathcal{S}$ is injective , is enough to observe that the following matrix has rank 4:

$$\begin{bmatrix}
\theta_{000}(z) & \theta_{011}(z) & \theta_{101}(z) & \theta_{110}(z) & \frac{\partial f}{\partial z_1}(z) & \frac{\partial f}{\partial z_2}(z) & \frac{\partial f}{\partial z_3}(z)\\
\partial_{z_1}\theta_{000}(z) & \partial_{z_1}\theta_{011}(z) & \partial_{z_1}\theta_{101}(z) & \partial_{z_1}\theta_{110}(z) & \frac{\partial^2 f}{\partial z_1^2}(z) & \frac{\partial^2 f}{\partial z_1 \partial z_2}(z) & \frac{\partial^2 f}{\partial z_1 \partial z_3}(z)\\
\partial_{z_2}\theta_{000}(z) & \partial_{z_2}\theta_{011}(z) & \partial_{z_2}\theta_{101}(z) & \partial_{z_2}\theta_{110}(z) & \frac{\partial^2 f}{\partial z_1 \partial z_2}(z) & \frac{\partial^2 f}{\partial z_2^2}(z) & \frac{\partial^2 f}{\partial z_2 \partial z_3}(z)\\
\partial_{z_3}\theta_{000}(z) & \partial_{z_3}\theta_{011}(z) & \partial_{z_3}\theta_{101}(z) & \partial_{z_3}\theta_{110}(z) & \frac{\partial^2 f}{\partial z_1 \partial z_3}(z) & \frac{\partial^2 f}{\partial z_2 \partial z_3}(z) & \frac{\partial^2 f}{\partial z_3^2}(z)
\end{bmatrix}$$
$$= \begin{bmatrix}
0 & 0 & 0 & 0 & c\partial_{z_1}\theta_{101}(z) & b\partial_{z_2}\theta_{011}(z) & \partial_{z_3}\theta_{000}(z)\\
0 & 0 & \partial_{z_1}\theta_{101}(z) & 0 & * & * & *\\
0 & \partial_{z_2}\theta_{011}(z) & 0 & 0 & * & * & *\\
\partial_{z_3}\theta_{000}(z) & 0 & 0 & 0 & * & * & *
\end{bmatrix} \tag{3.33}$$

By assumption, we have that $z_1 \in div(\theta_1(\cdot,\tau_1))$, $z_2 \in div(\theta_1(\cdot,\tau_2))$ and $z_3 \in div(\theta_0(\cdot,\tau_3))$, and the claim follows easily, the terms of the first row in the matrix 3.33 beeing non-zero.

Let us now consider a point $P = \left[\binom{x_1}{y_1}, \binom{x_2}{y_2}, \binom{x_3}{y_3}\right]$ on $\mathcal{S}$, which is represented by a point in $\widehat{\mathcal{S}}$ lying in the affine open set

$$\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 = T - div(\theta_{000}) \quad . \tag{3.34}$$

If we divide the holomorphic section f which defines $\mathcal{S}$ by θ_{000} (see 3.30) we obtain that the equation of $\widehat{\mathcal{S}}$ can be expressed, in the open set $\mathcal{U}$ in the following form

$$\left\{\left(\begin{pmatrix}x_1\\y_1\end{pmatrix}, \begin{pmatrix}x_2\\y_2\end{pmatrix}, \begin{pmatrix}x_3\\y_3\end{pmatrix}\right) \in \mathcal{U} \ \mid \ f' := 1 + b'x_2x_3 + c'x_1x_3 + d'x_1x_2 = 0)\right\}$$

where

$$b' = b\tfrac{\theta_1(0,\tau_2)}{\theta_0(0,\tau_3)} \quad c' \quad = c\tfrac{\theta_1(0,\tau_1)}{\theta_0(0,\tau_3)} \quad d' \quad = d\tfrac{\theta_1(0,\tau_1)}{\theta_0(0,\tau_2)} \quad . \tag{3.35}$$

By an abuse of notation we will still denote by b, c and d the respective terms in 3.35.
Under the assumption

$$\frac{\partial f'}{\partial x_3} = bx_2 + cx_1 \neq 0 \tag{3.36}$$

we can use x_1 and x_2 as local parameters of $\mathcal{S}$ in P, and we can write then the global sections of the canonical bundle of $\mathcal{S}$ locally in P as holomorphic forms of the type $g(x_1,x_2)dx_1 \wedge dx_2$, where g denotes a holomorphic function defined around P.
For every $(ij) \in \{(12),(13),(23)\}$, the elements $\omega_{ij} := dz_i \wedge dz_j$ can be looked at as non-zero elements of $H^0(\mathcal{S},\omega_{\mathcal{S}})$ when restricted to $\mathcal{S}$. Hence, we can write them in the following form:

$$\begin{aligned}
\omega_{12} &:= \frac{dx_1}{y_1} \wedge \frac{dx_2}{y_2} &&= \frac{1}{y_1y_2}dx_1 \wedge dx_2 \\
\omega_{13} &:= \frac{dx_1}{y_1} \wedge \frac{dx_3}{y_3} &&= -\frac{dx_1 + bx_3}{(bx_2+cx_1)y_1y_3}dx_1 \wedge dx_2 \\
\omega_{23} &:= \frac{dx_2}{y_2} \wedge \frac{dx_3}{y_3} &&= \frac{(cx_3+dx_2)}{(bx_2+cx_1)y_1y_2}dx_1 \wedge dx_2 \quad .
\end{aligned} \tag{3.37}$$

We write down also the global holomorphic differentials which arise by the residue map $H^0(A,\mathcal{O}_A(\mathcal{S})) = H^0(A,\omega_A(\mathcal{S})) \longrightarrow H^0(\mathcal{S},\omega_{\mathcal{S}})$. We denote, with $(ijk) \in \{(000),(011),(101),(110)\}$,

$$\psi_{ijk} := (\theta_{ijk} \cdot dz_1 \wedge dz_2 \wedge dz_3) \neg \left(\frac{\partial x_3}{\theta_{000}\frac{\partial f}{\partial x_3}} \right)$$

where $\neg$ is the contraction operator. We have in conclusion, up to a non-zero constant:

$$\begin{aligned}
\psi_{000} &= \frac{1}{(bx_2+cx_1)y_1y_2y_3}dx_1 \wedge dx_2 \\
\psi_{011} &= \frac{x_2x_3}{(bx_2+cx_1)y_1y_2y_3}dx_1 \wedge dx_2 \\
\psi_{101} &= \frac{x_1x_3}{(bx_2+cx_1)y_1y_2y_3}dx_1 \wedge dx_2 \\
\psi_{110} &= \frac{x_1x_2}{(bx_2+cx_1)y_1y_2y_3}dx_1 \wedge dx_2 \quad .
\end{aligned} \tag{3.38}$$

Once we have multiplied the expressions in 3.37 and 3.38 by $(b'x_2 + c'x_1)y_1y_2y_3$, we obtain the following expression of the canonical map of $\mathcal{S}$, which is defined independently on the assumption 3.36 and every point of the affine space $\mathbb{A}^6$ of coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$:

$$\phi_{\mathcal{S}} = \Big[(bx_2 + cx_1)y_3 \quad (bx_3 + dx_1)y_2 \quad (dx_2 + cx_3)y_1 \quad 1 \quad x_1x_2 \quad x_1x_3 \quad x_2x_3\Big] \quad .$$

We consider finally the affine map $\Phi : \mathbb{A}^6 \longrightarrow \mathbb{A}^9$ defined by

$$\Phi = \Big((bx_2 + cx_1)y_3 \quad (bx_3 + dx_1)y_2 \quad (dx_2 + cx_3)y_1 \quad x_1x_2 \quad x_1x_3 \quad x_2x_3 \quad g_1 \quad g_2 \quad g_3\Big) \tag{3.39}$$

where g_i are defined as 3.27. The differential of the canonical map of $\mathcal{S}$ is injective at a point of P of $p(\mathcal{U})$ if the matrix of the differential of Φ at P has maximal rank. The matrix of this differential is exactly

$$N := \begin{bmatrix} cy_3 & dy_2 & 0 & 0 & x_3 & x_2 & 2x_1(\delta_1^2 - 2x_1^2 + 1) & 0 & 0 \\ by_3 & 0 & dy_1 & x_3 & 0 & x_1 & 0 & 2x_2(\delta_2^2 - 2x_2^2 + 1) & 0 \\ 0 & by_2 & cy_1 & x_2 & x_1 & 0 & 0 & 0 & 2x_3(\delta_3^2 - 2x_3^2 + 1) \\ 0 & 0 & dx_2 + cx_3 & 0 & 0 & 0 & 2y_1 & 0 & 0 \\ 0 & dx_1 + bx_3 & 0 & 0 & 0 & 0 & 0 & 2y_2 & 0 \\ cx_1 + bx_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2y_3 \end{bmatrix} \tag{3.40}$$

Is easily seen now that the injectivity of the differential of Φ fails at the points where $y_3 = 0 = cx_1 + bx_2$, $y_2 = 0 = dx_1 + bx_3$ or $y_1 = 0 = dx_2 + cx_3$. We threat this case now, and we consider a point P of $p(\mathcal{U})$ with $y_1 = 0 = dx_2 + cx_3$. By the generality of the coefficients b, c and d we have that $\frac{\partial f'}{\partial x_3} = bx_2 + cx_1 \neq 0$, so we can use local parameters y_1 and x_2 around the point in $\widehat{\mathcal{S}}$ which represents P. In this case we have, on A:

$$df = (cx_3 + dx_2)\frac{\partial x_1}{\partial y_1}dy_1 + (dx_1 + bx_3)dx_2 + (bx_2 + cx_1)dx_3$$

and hence, on $\mathcal{S}$:

$$\begin{aligned} \omega_{12} &= \frac{1}{p_1y_2}dy_1 \wedge dx_2 \\ \omega_{13} &= \frac{1}{p_1y_3}dy_1 \wedge dx_3 = -\frac{(bx_3 + dx_1)}{p_1y_3(bx_2 + cx_1)}\frac{\partial x_1}{\partial y_1}dy_1 \wedge dx_2 \\ \omega_{23} &= \frac{1}{y_2y_3}dx_2 \wedge dx_3 = \frac{(cx_3 + dx_2)}{y_2y_3(bx_2 + cx_1)}\frac{\partial x_1}{\partial y_1}dy_1 \wedge dx_2 \end{aligned}$$

We have that $\frac{\partial x_i}{\partial y_1}(P) = 0$ for every $i = 1, 2, 3$, and thus we have, in particular, that $\omega_{13}(P) = \omega_{23}(P) = 0$ and $\frac{\partial \omega_{13}}{\partial x_2}(P) = \frac{\partial \omega_{23}}{\partial x_2}(P) = 0$. On the other side, by generality of the coefficients, we can assume that $bx_3 + dx_1$ does not vanish in P. Hence

$$\begin{aligned}\frac{\partial \omega_{13}}{\partial y_1}(P) &= -\frac{(bx_3+dx_1)}{p_1y_3(bx_2+cx_1)}\frac{\partial^2 x_1}{\partial y_1^1} \neq 0\\ \frac{\partial \omega_{23}}{\partial y_1}(P) &= -\frac{(cx_3+dx_2)}{y_2y_3(bx_2+cx_1)}\frac{\partial^2 x_1}{\partial y_1^1} = 0\end{aligned}$$

In conclusion, the matrix of the differential at P of ϕ_S can be written in this case in the following form:

$$\begin{bmatrix} \frac{1}{p_1y_2} & 0 & 0 & 1 & x_2x_3 & x_1x_3 & x_1x_2\\ 0 & *\neq 0 & 0 & 0 & 0 & 0 & 0\\ *\neq 0 & 0 & 0 & 0 & * & * & x_1\neq 0\end{bmatrix}$$

which has maximal rank.

From now on, let us suppose that none of the last three rows of the matrix N in 3.40 vanish.
Given $L \subseteq \{1, \cdots, 10\}$ a list of indeces of colums of N, we denote by N_L the submatrix formed from the colums in L. We have that

$$\begin{aligned} det(N_{1,3,4,5,6,8}) &:= \begin{vmatrix} cy_3 & 0 & 0 & x_3 & x_2 & 0\\ by_3 & dy_1 & x_3 & 0 & x_1 & 2\delta_2^2x_2-4x_2^3+2x_2\\ 0 & cy_1 & x_2 & x_1 & 0 & 0\\ 0 & dx_2+cx_3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 2y_2\\ cx_1+bx_2 & 0 & 0 & 0 & 0 & 0\end{vmatrix}\\ &= 2(cx_1+bx_2)(dx_2+cx_3)y_2\begin{vmatrix}0 & x_3 & x_2\\ x_3 & 0 & x_1\\ x_3 & x_2 & 0\end{vmatrix}\\ &= 4x_1x_2x_3y_2(cx_1+bx_2)(dx_2+cx_3)\end{aligned}$$

and we can compute the following 7×7 minors of N:

$$\begin{aligned} det(N_{1,2,4,5,6,7}) &= 4x_1x_2x_3y_1(cx_1+bx_2)(dx_1+bx_3)\\ det(N_{1,3,4,5,6,8}) &= 4x_1x_2x_3y_2(cx_1+bx_2)(dx_2+cx_3)\\ det(N_{2,3,4,5,6,9}) &= 4x_1x_2x_3y_3(dx_1+bx_3)(dx_2+cx_3)\\ det(N_{1,4,5,6,7,8}) &= -8x_1x_2x_3y_1y_3(cx_1+bx_2)\\ det(N_{2,4,5,6,7,9}) &= 8x_1x_2x_3y_1y_3(dx_1+bx_3)\\ det(N_{3,4,5,6,8,9}) &= -8x_1x_2x_3y_2y_3(dx_2+cx_3)\quad . \end{aligned} \tag{3.41}$$

By the generality of the coefficients and under the assumption that none of the last three rows of the matrix N are zero, we conclude that all the minors listed in 3.41 vanish simultaneously if and only if $x_i = 0$ for some i. If $x_3 = 0$, then without loss of generality we can assume the following

$$\begin{aligned} y_3 &= \delta_3 \\ x_1 \neq 0 &\neq x_2 \end{aligned} \tag{3.42}$$

and the matrix N has the following form

$$N := \begin{bmatrix} c\delta_3 & dy_2 & 0 & 0 & 0 & x_2 & 2x_1(\delta_1^2 - 2x_1^2 + 1) & 0 & 0 \\ b\delta_3 & 0 & dy_1 & 0 & 0 & x_1 & 0 & 2x_2(\delta_2^2 - 2x_2^2 + 1) & 0 \\ 0 & by_2 & cy_1 & x_2 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & dx_2 & 0 & 0 & 0 & 2y_1 & 0 & 0 \\ 0 & dx_1 & 0 & 0 & 0 & 0 & 0 & 2y_2 & 0 \\ cx_1 + bx_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\delta_3 \end{bmatrix}$$

In finally consider, under the hypothesis in 3.42, the following determinants:

$$\begin{aligned} det(N_{1,4,6,7,8,9}) &= -8cy_1^2 y_2(cx_1 - bx_2) \\ det(N_{2,4,6,7,8,9}) &= 8dx_1 y_1 x_2(x_2^2 - \delta_2^4) \\ det(N_{3,4,6,7,8,9}) &= -8dy_2 x_2^2(x_1^2 - \delta_1^4) \quad . \end{aligned}$$

Those determinants do not vanish simultaneously: indeed, if this were the case and all y_i are non-zero, then we would have that $cx_1 - bx_2 = 0$ and $x_1^2 - \delta_1^4 = x_2^2 - \delta_2^4 = 0$. But this situation can be avoided if we suppose the coefficients b, c, d to be sufficiently general. If otherwise $y_1 = 0$, then we have clearly that $det(N_{3,4,6,7,8,9}) \neq 0$, and the conclusion of the theorem follows.

It remains only to consider the case of a point P on $\mathcal{S}$ which is not a base point and such that $x_3(P) = \infty$. More specifically, we assume (see 3.32), that $x_1(P) \neq 0 \neq x_2(P)$, and without loss of generality we can assume that $w_3 = 1$, and that P is not contained in the divisor $div(\Theta_{011})$. In this case, P can be represented by a point in

$$\mathcal{U}_\infty := \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{V}_3 \quad .$$

We follow the same strategy in 3.34 and we divide the holomorphic section f which defines $\mathcal{S}$ by θ_{011} in order to obtain a polynomial equation which expresses $\hat{\mathcal{S}}$ in the affine open set $\mathcal{U}_\infty$:

$$\left\{ \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} v_3 \\ w_3 \end{pmatrix} \right) \in \mathcal{U}_\infty \;\mid\; f' := v_3 + bx_2 + cx_1 + dx_1 x_2 v_3 = 0) \right\} \quad .$$

We repeat the procedure we used in 3.36, and we assume that

$$\frac{\partial f'}{\partial v_3} = 1 + dx_1x_2 \neq 0 \tag{3.43}$$

Under this assumption, we can use local parameters x_1 and x_2 around the point in $\widehat{\mathcal{S}}$ which represents P, and we can write

$$\begin{aligned}
\omega_{12} &:= \frac{dx_1}{y_1} \wedge \frac{dx_2}{y_2} = \frac{1}{y_1y_2} dx_1 \wedge dx_2 \\
\omega_{13} &:= \frac{dx_1}{y_1} \wedge \frac{dv_3}{w_3} = -\frac{dx_1v_3 + b}{(1+dx_1x_2)y_1w_3} dx_1 \wedge dx_2 \\
\omega_{23} &:= \frac{dx_2}{y_2} \wedge \frac{dv_3}{w_3} = \frac{(c + dx_2v_3)}{(1+dx_1x_2)y_2w_3} dx_1 \wedge dx_2 \quad .
\end{aligned} \tag{3.44}$$

We denote, with $(ijk) \in \{(000), (011), (101), (110)\}$,

$$\psi_{ijk} := (\theta_{ijk} \cdot dz_1 \wedge dz_2 \wedge dz_3) \neg \left(\frac{\partial v_3}{\theta_{011} \frac{\partial f'}{\partial v_3}} \right)$$

where $\neg$ denotes as usual the contraction operator. Up to a non-zero constant, we conclude as in 3.38 that

$$\begin{aligned}
\psi_{000} &= \frac{1}{x_2x_3(1+d'x_1x_2)y_1y_2w_3} dx_1 \wedge dx_2 = \frac{v_3}{x_2(1+d'x_1x_2)y_1y_2w_3} dx_1 \wedge dx_2 \\
\psi_{011} &= \frac{1}{(1+d'x_1x_2)y_1y_2w_3} dx_1 \wedge dx_2 \\
\psi_{101} &= \frac{x_1}{x_2(1+d'x_1x_2)y_1y_2w_3} dx_1 \wedge dx_2 \\
\psi_{110} &= \frac{x_1}{x_3(1+d'x_1x_2)y_1y_2w_3} dx_1 \wedge dx_2 = \frac{x_1v_3}{(1+d'x_1x_2)y_1y_2w_3} dx_1 \wedge dx_2 \quad .
\end{aligned} \tag{3.45}$$

Thus, we have that the canonical map of $\mathcal{S}$ can be written in the open set $p(\mathcal{U}_\infty)$, once we have multiplied the right members in 3.44 and 3.45 by $(1+d'x_1x_2)y_1y_2x_2w_3$, in the following form:

$$\phi_{\mathcal{S}} = \Big[(w_3(1+dx_1x_2)x_2 \quad (dx_1v_3+b)y_2x_2 \quad (dx_2v_3+c)y_1x_2 \quad v_3 \quad x_2 \quad x_2 \quad x_1x_2v_3\Big] \quad . \tag{3.46}$$

We repeat the procedure we applied in 3.39, and we observe that the map in 3.46 is defined independently on the assumption 3.43 on every point of the affine space $\mathbb{A}^6$ with coordinates $(x_1, x_2, v_3, y_1, y_2, w_3)$, and we can consider the map $\Phi_\infty : \mathbb{A}^6 \longrightarrow \mathbb{P}^9$ defined as follows:

$$\Phi_\infty = \Big[(w_3(1+dx_1x_2)x_2 \quad (dx_1v_3+b)y_2x_2 \quad (dx_2v_3+c)y_1x_2 \quad v_3 \quad x_2 \quad x_2 \quad x_1x_2v_3 \quad g_1 \quad g_2 \quad h_3\Big]$$

where g_1, g_2 and h_3 are defined as 3.27.
The differential of the canonical map of $\mathcal{S}$ is injective at a point of $p(\mathcal{U}_\infty)$ if the following matrix of the differential at P of Φ_∞ has maximal rank.

$$M := \begin{bmatrix} dx_2(x_1x_2+1) & bx_2y_2 & cy_1x_2 & 0 & x_2 & x_1 & 0 & 0 & 0 & 0 \\ dx_2^2 & 0 & 0 & 0 & 0 & 1 & 0 & 2x_1(\delta_1^2-2x_1^2+1) & 0 & 0 \\ 2dx_1x_2+1 & by_2 & cy_1 & 0 & 1 & 0 & 0 & 0 & 2x_2(\delta_2^2-2x_2^2+1) & 0 \\ 0 & dx_1x_2y_2 & dy_1x_2^2 & 1 & 0 & 0 & x_1x_2 & 0 & 0 & 0 \\ 0 & 0 & cx_2 & 0 & 0 & 0 & 0 & 2y_1 & 0 & 0 \\ 0 & bx_2 & 0 & 0 & 0 & 0 & 0 & 0 & 2y_2 & 0 \\ dx_1x_2(x_2+1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

We conclude that the differential has maximal rank, by considering the following minor:

$$det(M_{1,2,3,5,6,7,10}) = \begin{vmatrix} dx_2(x_1x_2+1) & bx_2y_2 & cy_1x_2 & x_2 & x_1 & 0 & 0 \\ dx_2^2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2dx_1x_2+1 & by_2 & cy_1 & 1 & 0 & 0 & 0 \\ 0 & dx_1x_2y_2 & dy_1x_2^2 & 0 & 0 & x_1x_2 & 0 \\ 0 & 0 & cx_2 & 0 & 0 & 0 & 0 \\ 0 & bx_2 & 0 & 0 & 0 & 0 & 0 \\ dx_2(x_1x_2+1) & 0 & 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$
$$= 2bcx_2^2 \begin{vmatrix} dx_2(x_1x_2+1) & x_2 & x_1 & 0 \\ dx_2^2 & 0 & 1 & 0 \\ 2dx_1x_2+1 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_1x_2 \end{vmatrix}$$
$$= 2bcx_1x_2^3 \begin{vmatrix} dx_2(x_1x_2+1) & x_2 & x_1 \\ dx_2^2 & 0 & 1 \\ 2dx_1x_2+1 & 1 & 0 \end{vmatrix} = -4bcdx_1^2x_2^5 \neq 0$$

□

A more general result follows by applying 3.5.4 together with an easy degeneration argument.

Corollary 3.5.5. *Let us consider $T := B \times E$ the product of a general $(2,2)$-polarized abelian surface B with a general 2-polarized elliptic curve E. Let us consider a general smooth surface $\mathcal{S}$ yielding the natural $(1,2,2)$-polarization on the abelian 3-fold $A := T / \langle e_1 + e_2 + e_3 \rangle$ induced by T. Then the differential of the canonical map of $\mathcal{S}$ is everywhere injective.*

Theorem 3.5.6. *Let be $(A, \mathcal{L})$ a general $(1,2,2)$-polarized abelian 3-fold and let be $\mathcal{S}$ a general surface in the linear system $|\mathcal{L}|$. Then the canonical map of $\mathcal{S}$ is a holomorphic embedding.*

Proof. By propositions 3.2.6 and 3.5.4, it is enough to prove the injectivity of the canonical map of the general smooth surface in the linear system $|\mathcal{L}|$ for the general $(1,2,2)$-polarized abelian 3-fold A.
For every element τ of $\mathcal{H}_3$ we denote the by T_τ the corresponding $(2,2,2)$-polarized abelian 3-fold:

$$T_\tau = \mathbb{C}^3 \Big/ \Big\langle \begin{bmatrix} \tau & | & 2I_3 \end{bmatrix} \Big\rangle_{\mathbb{Z}} \tag{3.47}$$

and, with $(i,j,k) \in \mathbb{Z}_2^3$ we denote by θ_{ijk} the theta function which corresponds, according to definition 1.3.1, to (i,j,k) under the isomorphism $\mathbb{Z}_2^3 \cong \frac{1}{2}\mathbb{Z}^3 / \mathbb{Z}^3$.
We consider, furthermore,

$$\begin{aligned} A_\tau &:= T_\tau \Big/ \langle e_1 + e_2 + e_3 \rangle \\ \mathcal{J}_\tau &:= \mathbb{C}^3 \Big/ \Big\langle \begin{bmatrix} \tau & | & I_3 \end{bmatrix} \Big\rangle_{\mathbb{Z}} \quad . \end{aligned}$$

Then A_τ is clearly $(1,2,2)$-polarized.

To prove the claim of the theorem, we start by considering τ_{11}, τ_{22} and τ_{33} three general points in $\mathcal{H}_1$, and we denote by $\mathcal{H}_{3\Delta}$ the closed subset of $\mathcal{H}_3$ which consists of the matrices whose diagonal entries are the fixed parameters τ_{11}, τ_{22} and τ_{33}.
We choose, moreover, a general point (b,c,d) on $\mathbb{C}^3$ which satisfies condition $\natural$ respect to τ_{11}, τ_{22} and τ_{33}, and such that the claim of proposition 3.5.4 holds true for the corresponding surface. There exists then a suitable neighborhood $\mathcal{U}$ of $\tau_0 := \begin{bmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{bmatrix}$ in $\mathcal{H}_{3\Delta}$ such that the following conditions hold:

- the point (b,c,d) satisfies condition $\natural$ with respect to every τ contained in in the neighborhood $\mathcal{U}$.
- for every couple if indices (ij), the claim of the corollary holds true for every τ in the closed set:

$$\mathcal{H}_{3\Delta}^{(ij)} := \mathcal{H}_{3\Delta} \cap \{\tau_{ik} = \tau_{jk} = 0\} \quad .$$

For every τ in $\mathcal{H}_{3\Delta}$, we denote by $\mathcal{S}_\tau$ the zero locus in A_τ of the theta function $\theta(\tau) := \theta_{000}(\tau) + b\theta_{011}(\tau) + c\theta_{101}(\tau) + d\theta_{110}(\tau)$, and we have then, in particular, a family of surfaces $\mathcal{S} \longrightarrow \mathcal{H}_{3\Delta}$, which we restrict to a family $\mathcal{S}_{\mathcal{U}}$ on the open set $\mathcal{U}$.

Considered (i, j, k) a permutation of $(1, 2, 3)$, we define furthermore

$$\mathcal{U}^{(ij)} := \mathcal{U} \cap \mathcal{H}_{3\Delta}^{(ij)} \quad .$$

We recall that, by the definition of $\mathcal{H}_{3\Delta}$:

$$\{\tau_0\} = \mathcal{U}^{(12)} \cap \mathcal{U}^{(13)} \cap \mathcal{U}^{(23)} \quad . \tag{3.48}$$

Let us suppose by absurd that the claim of the theorem is false. Then, denoted by $\Delta_{\mathcal{U}}$ the diagonal subscheme of $\mathbb{P}^5_{\mathcal{U}} \times_{\mathcal{U}} \mathbb{P}^5_{\mathcal{U}}$, there exists a closed subset $\mathcal{Q}$ of $(\phi_{\mathcal{S}_{\mathcal{U}}} \times_{\mathcal{U}} \phi_{\mathcal{S}_{\mathcal{U}}})^{-1}(\Delta_{\mathcal{U}}) \subseteq \mathcal{S}_{\mathcal{U}} \times_{\mathcal{U}} \mathcal{S}_{\mathcal{U}}$ different from the diagonal and dominant over $\mathcal{U}$.

For every couple of indices (ij), the restriction of $\mathcal{Q}$ on $\mathcal{U}^{(ij)}$ still has an irreducible component which is dominant on $\mathcal{U}^{(ij)}$. We denote this component by $\mathcal{Q}^{(ij)}$. Because by hypothesis the canonical map of $\mathcal{S}_\tau$ has everywhere injective differential for every τ in $\mathcal{U}$, this component $\mathcal{Q}^{(ij)}$ does not intersect the diagonal subscheme of $\mathcal{S}_{\mathcal{U}^{(ij)}} \times_{\mathcal{U}^{(ij)}} \mathcal{S}_{\mathcal{U}^{(ij)}}$. Indeed, the geometric points of such intersection represent infinitely near couples of points on a certain surface in the family $\mathcal{U}$ which have the same image with respect to the canonical map, and in particular they represent points on a certain surface in the family $\mathcal{U}$, at which the differential of the canonical map fails to be injective. However, the existence of such geometric points would contradict the hypothesis on the family $\mathcal{U}$, according to which the statement of corollary 3.5 holds for the surfaces in the family $\mathcal{U}$.
On the other hand, by proposition 3.4.5, we have that

$$\mathcal{Q}^{(ij)} \subseteq \mathcal{X}_{ij}^{(ij)} \cup \mathcal{W}_k^{(ij)} \tag{3.49}$$

where, for every $h = 1, 2, 3$ and for every τ in $\mathcal{U}$, denoted by $\iota_h : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ the involution which exchanges the sign to the h-th coordinate,

$$\mathcal{X}_{\tau,ij} := \{(P, \iota_i \iota_j(P)) \in \mathcal{S}_\tau \times \mathcal{S}_\tau \mid P \in div\left(\frac{\partial \theta}{\partial z_i}\right) \cap div\left(\frac{\partial \theta}{\partial z_j}\right)\}$$
$$\mathcal{W}_{\tau,k} := \{(P, \iota_k(P)) \in \mathcal{S}_\tau \times \mathcal{S}_\tau \mid P \in div\left(\frac{\partial \theta}{\partial z_k}\right)\} \quad .$$

From 3.49 and 3.48, it follows immediately that the following intersection is non-empty:

$$\mathcal{R}_{\tau_0} := \bigcap_{ijk} (\mathcal{X}_{\tau_0,ij} \cup \mathcal{W}_{\tau_0,k}) = \bigcup_{ijk} (\mathcal{X}_{\tau_0,ij} \cap \mathcal{W}_{\tau_0,i} \cap \mathcal{W}_{\tau_0,j})$$

Because the claim of the proposition 3.5.4 holds true, by hypothesis, for the surfaces in the family $\mathcal{U}$, we have that $\mathcal{R}_{\tau_0}$ does not intersect the diagonal

subspace in $\mathcal{S}_{\tau_0} \times \mathcal{S}_{\tau_0}$, which represents, in our context, points on which the differential of the canonical map fails to be injective.
On the other hand, every point of $\mathcal{X}_{\tau_0,ij} \cap \mathcal{W}_{\tau_0,i} \cap \mathcal{W}_{\tau_0,j}$ is of the form (P,Q) such that $Q = \iota_i \iota_j(P)$, $Q = \iota_i(P)$ and $Q = \iota_j(P)$. But this implies that $P = Q$, and we reach a contradiction. The proof of the theorem is complete. $\square$

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