Scuola Internazionale Superiore di Studi Avanzati - Trieste

Area of Mathematics

Ph.D. in Mathematical Analysis, Modelling and Application

Thesis

Switching in time-optimal problem

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Academic year 2016-2017

Il presente lavoro costituisce la tesi presentata da Carolina Biolo, sotto la direzione del Prof. Andrei Agrachev, al fine di ottenere l'attestato di ricerca post-universitaria Doctor Philosophiae presso la SISSA, Curriculum in Analisi Matematica, Modelli e Applicazioni, Area di Matematica. Ai sensi dell'art. 1, comma 4, dello Statuto della SISSA pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di Dottore di Ricerca in Matematica.

Trieste, Anno Accademico 2016–2017.

"È un paradosso: la matematica è umana perché è di accesso difficile.
È impossibile, nella matematica è impossibile essere uno scienziato a metà
o al 50%."

L. Lafforgue

"Oggi c'è una pillola per ogni difficoltà.
Ma è così bella la difficoltà, beata...
È una benedizione del cielo non saper come fare,
perché lì diventi uomo e scopri il mondo, la vita, scopri che sei vivo.
Se prendi una pasticca per eliminare questo, è desertificare l'emozione,
non sei più vivo."

R. Benigni

Abstract

The central object of this thesis is time-optimal problem on an affine control system of type

$$\dot{q} = f_0(q) + u_1 f_1(q) + \ldots + u_k f_k(q), \quad q \in M \tag{Σ}$$

where $f_0, \ldots, f_k$ are $k+1$ vector fields defined on the manifold M. We assume that $f_0, \ldots, f_k$ are smooth ($\mathcal{C}^\infty(M)$) and $u = (u_1, \ldots, u_k)$ are L^∞ admissible controls taking value in the k-dimensional closed unitary ball.

We analyse the local regularity of system (Σ), with the classical methods of the optimal control theory: the Pontryagin maximum principle, the second order optimality conditions and other methods based on the relations between geometric local properties of (Σ) and algebraic structure, as the configurations on the Lie brackets of the system.

We are interested in finding generic conditions on the vector fields of the system (Σ) in a point $\bar{q}$ in M, such that each time-optimal trajectory of (Σ) close to $\bar{q}$ is piece-wise smooth with a finite number of smooth components, called *arcs*. More precisely, we look for generic conditions that guaranties the absence of *chattering phemonema*, i.e. the existence of a convergent series of smooth arcs in finite time.

We found this conditions in chapters 3 and 4. In particular, we show that in the case of $k = n-1$ there are sufficient conditions in terms of Lie bracket relations for all optimal controls to be smooth or to have only isolated jump discontinuities; and we characterized the flow of *Pontryagin's extremals*.

In Chapter 5 we analyse the global number of singularities, called *switching*s, considering (Σ) as a linear system: with linear *drift* f_0, and constant controllable vector fields. We show that there will appear a unique or an infinity number of switchings at regular time intervals.

Finally, in Chapter 6 we present the cases for which we were able to prove the optimality of the broken *extremal trajectory*, we found in the previous chapters.

Here we list all the works collected in this thesis:
Chapter 3: A. A. Agrachev, C. Biolo, *Switching in time-optimal problem with control in a ball*, arXiv:1610.06755 (2016), to appear on SIAM J. Control Optim.
Chapter 4: A. A. Agrachev, C. Biolo, *Switching in time-optimal problem: the 3-D case with 2-D control*, J Dyn Control Syst, DOI 10.1007/s10883-016-9342-7, 2016.
Chapter 6: A. A. Agrachev, C. Biolo, *Optimality of a broken extremal*, preprint 2017.

Contents

Chapter 1

Introduction

This thesis is a one more step towards the understanding of the structure of time-optimal controls and trajectories for control affine systems of the form:

$$\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M, \tag{1}$$

where M is a n-dimensional manifold, $u = (u_1, \ldots, u_k)$ are L^∞ admissible controls taking value in $U = \left\{(u_1, \ldots, u_k) : \sum_{i=1}^{k} u_i^2 \leq 1\right\}$ the k-dimensional ball, and $f_0, f_1, \ldots, f_k$ are smooth vector fields. We also assume that $f_1, \ldots, f_k$ are linearly independent in the domain under consideration.

These control systems always attracted a special attention among non linear control theorists, since they represent a sort of laboratory, where nonlinear features appear in great purity. In particular, if the control is scalar ($k = 1$).

Moreover, systems in the form (1) are, at least in first approximation, a convenient modelling of many "real" control systems.

Our attention is devoted to the study of local regularity properties of time-optimal trajectories of (1), which minimize the time needed to join two given points.
Our point of view is local in the following sense: given $\bar{q} \in M$, we study whether there exists a neighbourhood $O_{\bar{q}}$ of $\bar{q}$ such that any optimal trajectory is smooth or is a concatenation of a finite number of smooth arcs.

A piece of trajectory where the control is smooth is called an *arc*. *Bang arc* correspond to a continuous control that lies in the boundary of U, on the other hand an arc which is not bang is called *singular*. If two bang arcs are concatenated, we call switching time the instance at which the control has a discontinuity. A finite concatenation of bang arcs is called a *bang-bang trajectory*.

In order to restrict the family of candidate of time-optimal trajectories we used the Pontryagin maximum principle, it states that if $q(t) : [0, t_1] \to M$ is a time-optimal trajectory of (1) with control $\tilde{u}(\cdot)$, then there exists a non trivial lift $\lambda(t)$ of $q(t)$ in the cotangent bundle T^*M such that, denoted the family of Hamiltonians with respect u

$$h_u(\lambda) = \left\langle \lambda, f_0(q) + \sum_{i=1}^{k} u_i f_i(q) \right\rangle ,$$

$\lambda(t)$ satisfies

$$\dot{\lambda}(t) = \overrightarrow{h}_{\tilde{u}}(\lambda(t))$$

and

$$h_{\tilde{u}}(\lambda(t)) = \max_{u \in U} h_u(\lambda(t)) \geq 0.$$

We call *extremal trajectory* any solution of (1) which satisfies the Pontryagin maximum principle.

The case $k = n$ is the Zermelo navigation problem: optimal controls are smooth in this case (see Remark 2.2.9). In more general situations, discontinuous controls are unavoidable and, in principle, any measurable function can be an optimal control (see [21]).

We call *chattering* a boundary control whose switching times form a monotone sequence that converges in finite time.
Fuller first presented in [10] a control problem, which admits a formulation of type (1), with a time optimal solution corresponding to a chattering control function.

In order to understand whether this phenomenon is stable with respect to small perturbations of the system or not, it is reasonable to focus on generic ensembles of vector fields $f_0, f_1, \dots, f_k$.

If $k = 1$, $n = 2$, then for a generic pair of vector fields f_0, f_1 any optimal control is piecewise smooth; moreover, any point in M has a neighbourhood such that all optimal trajectories contained in are concatenation of at most two smooth arcs, namely they have at most one switching (see [9] and [25]).

The complexity of optimal control grows fast with n.

For $k = 1$, $n = 3$ the generic situation is only partially studied (see [20], [26] and [7]): we know that any point out of a 1-dimensional Whitney-stratified subset of "bad point" has a small neighbourhood that contains only optimal trajectories with at most three switchings. We still do not know if there is any bound on the number of switchings in the points of the "bad" 1-dimensional subset.

We know however that the chattering phenomenon is unavoidable for $k = 1$ and sufficiently big n (see [14] and [28]).

Beside these interesting theoretical challenges, a finite bound on the number of arcs of time-optimal trajectories has a clear role in applications.

Transversality theory gives us adequate instruments for investigating generic properties of the system.
Let $h \geq 0$ and denote by $J^{k+1,h}M$ the vector bundle on M of all h-jets of $k+1$-uple of vector fields $(f_0, f_1, \dots, f_k)$ on M. Let A be a stratified subset of $J^{k+1,h}M$ such that $J_q^{k+1,h}M \cap A$ is of codimension larger than a value m in $J_q^{k+1,h}M$, for every $q \in M$. Then, generically, the set of points $q \in M$, such that the h-jet of $(f_0, \dots, f_k)$ at q, $J_q^{k+1,h}(f_0, \dots, f_k)$ belongs to A, has codimension larger than m in M.

In particular, if we find A such that m is the dimension on M, then generically $J_q^{k+1,h}(f_0,\dots,f_k)$ does not belong to A, for every $q \in M$.
Therefore, if we find A such that, for every $f_0,\dots,f_k$ and q it holds $J_q^{k+1,h}(f_0,\dots,f_k) \notin A$, all short time-optimal trajectories near q are finite concatenations of arcs, and if m is the dimension on M, then, generically, chattering does not occur.
On the other hand, for any m less or equal to the dimension of M, if we can show that no chattering appears near points for which $J_q^{k+1,h}(f_0,\dots,f_k) \in A$, then we actually give a bound on the dimension of the set of points near which chattering cannot be excluded.

Subsets of $J^{k+1,h}M$ are defined in terms of conditions on the configurations of iterated Lie brackets between $f_0, f_1, \dots, f_k$, indeed Lie brackets preserve intrinsic relations between the derivatives of the jets.

When we started to face this topic, we started studying the case $k = 2$ $n = 3$, in particular, for a generic triple (f_0, f_1, f_2). We obtain that any point out a discrete subset of "bad points" in M has a neighbourhood such that any optimal trajectory contained in this neighbourhood has at most one switching. That means that it is possible to avoid generically the chattering trajectories in this setting.
Then we saw that the techniques developed were efficient also in case $k = n-1$ with an arbitrary n and that complexity of the switchings depends much more on $n-k$ than on n.
Let us present the structure of the thesis, presenting a panoramic view of the original contributions that we tried to give in this field.

Structure of the thesis

Chapter 2: Preliminary Theorems. In this Chapter we recall some definitions and preliminary theorems in geometric control theory. We give the definition of optimal control problem, present its reduction to study the attainable set of an extended system and discuss the existence of optimal trajectories.
Moreover we introduce the first and second order optimality conditions: Pontryagin maximum principle and Goh condition.

In the second part of this chapter we focus on the study of Pontryagin extremals and extremal trajectories of the affine control system (1). We define in T^*M the singular locus

$$\Lambda = \{\lambda \in T^*M \,:\, h_1(\lambda) = \ldots = h_k(\lambda) = 0\},$$

where $h_i(\lambda) = \langle \lambda, f_i(q) \rangle$ with $i \in \{0, 1, \dots, k\}$, state the first fact on the local regularity of extremal trajectories: an arc of a time-optimal trajectory, whose extremal is out of the singular locus, is a bang arc (see Corollary 2.2.8).

As a consequence we are interested in what happen if an extremal touches, enters or goes though the singular locus. Extremal trajectories may be not always smooth. For this reason, our attention will focus on the behaviour of the extremal flow close to the singular locus Λ.

In the third section of this chapter we present some other preliminaries for Chapter 6: theory of orbits and the Frobenius theorem. And, finally we give a deeper digression regarding the chattering phenomenon and generic conditions.

Chapter 3: Switching with control in a ball. In this chapter we present the contribute that we wrote in [4]. We analyse local regularity of time-optimal controls and trajectories for an n-dimensional affine control system (1), for any n.

In the case of $k = n-1$, we give generic sufficient conditions in terms of Lie bracket relations for all optimal controls to be smooth or to have only isolated jump discontinuities. Indeed, given $f_0, f_1, \dots, f_{n-1}$ a generic germ of n-uple of vector fields at q, then the germs of extremal trajectories at q may have only these less degenerate singularities (see Theorem 3.1.1).
More precisely, let us define a vector $a \in \mathbb{R}^{n-1}$ and a matrix $A \in \mathrm{so}(n-1)$ by the formulas:

$$a(q) = \{\det(f_1(q), \dots, f_{n-1}(q), [f_0, f_i](q))\}_{i=1}^{n-1}$$

$$A(q) = \{\det(f_1(q), \dots, f_{n-1}(q), [f_i, f_j](q))\}_{i,j=1}^{n-1}.$$

Thus, if

$$a(\bar{q}) \notin A(\bar{q}) S^{n-2}, \tag{2}$$

where $S^{n-2} = \{u \in \mathbb{R}^{n-1} : |u| = 1\}$, then there exists a neighbourhood $O_{\bar{q}}$ of $\bar{q}$ in M such that any time-optimal trajectory contained in $O_{\bar{q}}$ is piecewise smooth with no more than one non smoothness point.

In the general case if $k < n$, we study the flow of extremals in the cotangent bundle T^*M in a neighbourhood $O_{\bar{\lambda}}$ of a singular point $\bar{\lambda} \in \Lambda$ such that $\bar{q} = \pi(\bar{\lambda})$. By the preliminaries, we already know that out of Λ extremals and extremal trajectories are smooth.
Let us define vector $H_{0I} \in \mathbb{R}^k$ and matrix $H_{IJ} \in \mathrm{so}(k)$, univocally given by the system and $\bar{\lambda}$, such that[1]:

$$\begin{aligned} H_{0I} &= \left(h_{0i}(\bar{\lambda})\right)_{i \in \{1,\dots,k\}} \\ H_{IJ} &= \left(h_{ij}(\bar{\lambda})\right)_{i,j \in \{1,\dots,k\}}. \end{aligned}$$

Then by the following generic condition

$$H_{0I} \notin H_{IJ} S^{k-1} \tag{3}$$

with $S^{k-1} = \{u \in \mathbb{R}^k : |u| = 1\}$, we proved that there are no optimal extremals in $O_{\bar{\lambda}}$ that lie in the singular locus Λ for a time interval (see Proposition 3.2.3).

Moreover, we give the complete characterization of the flow of extremals in $O_{\bar{\lambda}}$, explaining in which cases time-optional trajectories are smooth or have at most one isolated singularity (see Theorem 3.2.4).
More precisely, if it holds

$$H_{0I} \in H_{IJ} B^k, \tag{4}$$

[1] see Notation 2.2.4

where $B^k = \{u \in \mathbb{R}^k : |u| < 1\}$, then there exists a neighbourhood $O_{\bar{\lambda}} \subset T^*M$ such that no optimal extremal intersects singular locus in $O_{\bar{\lambda}}$ small enough.
On the other hand, if it holds

$$H_{0I} \notin H_{IJ}\overline{B^k}, \tag{5}$$

then there exists a neighborhood $O_{\bar{\lambda}} \subset T^*M$ such that for any $z \in O_{\bar{\lambda}}$ and $\hat{t} > 0$ there exists a unique contained in $O_{\bar{\lambda}}$ extremal $t \mapsto \lambda(t,z)$ with the condition $\lambda(\hat{t},z) = z$. Moreover, $\lambda(t,z)$ continuously depends on (t,z).
In particular, every extremal in $O_{\bar{\lambda}}$ that passes through the singular locus is piecewise smooth with only one switching.
Besides that, if u is the control corresponding to the extremal through $\bar{\lambda}$, and $\bar{t}$ is its switching time, we have:

$$u(\bar{t} \pm 0) = [\pm d \,\mathrm{Id} + H_{IJ}]^{-1} H_{0I},$$

with $d > 0$ unique, univocally defined by the system and $\bar{\lambda}$, such that

$$\left\langle [d^2 \,\mathrm{Id} - H_{IJ}^2]^{-1} H_{0I}, H_{0I} \right\rangle = 1.$$

This theory is based on the blow-up techniques and the structure of partially hyperbolic equilibria.

Therefore, we observed that in general, the flow of extremals is not locally Lipschitz with respect to the initial value.
Since the Pontryagin maximum principle is a necessary but not sufficient condition of optimality, we cannot guaranty that the broken extremals, that we found, are all optimal. It is possible to say that they are certainly optimal only if the system is a linear system with an equilibrium target. We will present the general case at Chapter 6.

Chapter 4: Switching with 2D control. In this chapter we present a deeper result on local regularity of time-optimal trajectories of system (1) in a 3-dimensional manifold M with control in a 2-dimensional disk U (see [3]).

If $n = 3$ and $k = 2$, the condition (2) that we gave in Chapter 3 reads

$$\begin{aligned}\det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_1](\bar{q})\right) + \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_2](\bar{q})\right) \neq \\ \neq \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_1, f_2](\bar{q})\right).\end{aligned} \tag{6}$$

In this chapter, we study the regularity of optimal trajectories that lies in the neighbourhood $O_{\bar{q}}$ of $\bar{q}$ if

$$\begin{aligned}\det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_1](\bar{q})\right) + \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_2](\bar{q})\right) = \\ = \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_1, f_2](\bar{q})\right),\end{aligned} \tag{7}$$

assuming the weaker condition

$$\operatorname{rank}\{f_1(\bar{q}), f_2(\bar{q}), f_{01}(\bar{q}), f_{02}(\bar{q}), f_{12}(\bar{q})\} = 3. \tag{8}$$

We proved that if (8) and (7) hold in $\bar{q}$ there exists a neighbourhood $O_{\bar{q}}$ of $\bar{q}$ such that any time-optimal trajectory that contains $\bar{q}$ and is contained in the neighbourhood is a bang arc. The correspondent extremal either remains out of the singular locus Λ, or lies in

$$\Lambda \cap \left\{\lambda \in T^*M \,|\, h_{01}^2(\lambda) + h_{02}^2(\lambda) = h_{12}^2(\lambda)\right\}.$$

Anyway, the correspondent optimal control will be smooth without any switching, taking values on the boundary of U, in both cases.

Chapter 5: Linear control system. In this chapter we are going to consider the global regularity of the time-optimal problem if the control system (1) is a linear control system in $\mathbb{R}^n$, with coordinate $x \in \mathbb{R}^n$, and $k = n - 1$. It means that the drift is linear $f_0(x) = Ax$, denoted by a $n \times n$ real matrix A, and the controllable vector fields are constant $f_i(x) = b_i \in \mathbb{R}^n$, with $i \in \{1, \ldots, n-1\}$.
For simplicity we rewrite the system as follows

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$$

where $B = (b_1, \ldots, b_{n-1})$ is a $n \times n - 1$ real matrix, and u the admissible control taking values in U closed unitary ball, as before.
In this system we assume the *Kalman's criterion*

$$\operatorname{rank}\{B, AB, \ldots, A^{n-1}B\} = n. \tag{9}$$

Given any $\bar{x} \in \mathbb{R}^n$ it is always possible to construct an extremal trajectory which has a switching in $\bar{x}$ (see Claim 5.3.1).
We prove that along the global extremal trajectory the switching could be unique or there will be an infinity number of switching in regular intervals of time. It depends on the configurations of (A, B), satisfying (9).
In particular, assuming that A has simple complex and real eigenvalues $\alpha_1 + i\beta_1$, $\alpha_1 - i\beta_1, \ldots, \alpha_j + i\beta_j$, $\alpha_j - i\beta_j$, $\lambda_{2j+1}, \ldots, \lambda_n$, such that $\beta_i \neq 0$ for all $i \in \{1, \ldots, j\}$ and $0 < j \leq \lfloor \frac{n}{2} \rfloor$, given the corresponding eigenvectors that form a basis $\mathcal{B}$ of $\mathbb{R}^n$, and $\bar{p} \in \mathbb{R}^n$, such that $\bar{p}^T B = 0$, described with coordinates $(\bar{p}_i)_{i=1,\ldots n}$ in $\mathcal{B}$. There will be infinite switching if we assume that

- there exists $\beta_A \in \mathbb{R} \setminus \{0\}$ and $K_i \in \mathbb{Q}$, $\forall i \in \{1, \ldots, j\}$, such that $\beta_i = K_i \beta_A$
- $\alpha = \alpha_i$ for all $i \in \{1, \ldots, j\}$ where $\bar{p}_{\alpha_i} \neq 0$, and $\alpha = \lambda_i$ for all $i \in \{2j + 1, \ldots, n\}$ such that $\bar{p}_i \neq 0$.

(see Theorems 5.3.5 and 5.3.6)

Chapter 6: Sufficient optimality condition. Here, we are going to discuss the optimality of the projections of the non smooth extremals detected in the previous chapters, given a non linear affine control system (1), (see [5]).
Let us recall that if condition (5) is satisfied at $\bar{\lambda} \in \Lambda$, there exist a broken extremal that passes through Λ at $\bar{\lambda}$, and the flow of extremals is not locally lipschitz.

Denoting $\bar{q} = \pi(\bar{\lambda})$, and $\mathcal{F} = \{f_1, \ldots, f_k\}$, we prove the sufficient optimality of the normal broken extremal, passing through $\bar{\lambda} \in \Lambda$, if

$$\bar{\lambda} \perp \operatorname{Lie}_{\bar{q}}\mathcal{F}, \quad h_0(\bar{\lambda}) > 0$$

and either $\operatorname{rank}\{\mathrm{Lie}_{\bar q}\mathcal{F}\} = n-1$, or $\operatorname{rank}\{\mathrm{Lie}_q\mathcal{F}\} = \operatorname{rank}\{\mathrm{Lie}_{\bar q}\mathcal{F}\} < n-1$ for all q from a neighbourhood of $\bar q$ in M (see Theorem 6.2.6). Moreover, if $n = 3$ $k = 2$ we prove the optimality for a normal broken extremal if f_1, f_2 form a contact distribution in a neighbourhood of $\bar q$ (see Theorem 6.2.9).
We use a method described by Agrachev and Sachkov in their book [6]. It is a geometrical elaboration of the classical fields of extremals theory, it proves optimality only for normal extremals, assuming the Hamiltonian smooth. We extended this method in the Lipschitzian submanifold, with constructions *ad hoc*.

We also prove optimality of normal (or abnormal) broken extremals for $n > 2$ $k = 2$ and

$$\bar\lambda \perp \operatorname{span}\{f_1(\bar q), f_2(\bar q), [f_1, f_2](\bar q)\} \tag{1.0.1}$$

in just that point (see Theorem 6.3.5). This result is given by direct estimates with time-rescaling.

In this chapter, there is a last section "Open problem", that presents the computations of this method with direct estimates in the general (possible abnormal) case, if (1.0.1) does not hold. It may be useful to answer further questions.

Aknowledgements. I would like to thank my supervisor Andrei Agrachev for having suggested these problems and for having introduced me to the research; his enthusiasm and love for mathematics have fascinated me since the beginning. He taught me that there is no trivial way to overcome a problem, and that very often you just have to look at it with a different perspective; more important, he taught me that, even though a computation or a new statement seems useless, it isn't, because everything can surprise us and help us to find a solution for a problem; this is a lesson that I will use forever in my life.

Chapter 2

Preliminary Theorems

2.1 Optimal control problem

In this section we recall some definitions and preliminary theorems in Geometric control theory.

Definition 2.1.1. Given a n-dimensional manifold M, we call $\mathrm{Vec}(M)$ the *set of* $\mathcal{C}^2$ *vector fields* on M: $f \in \mathrm{Vec}(M)$ if and only if f is a smooth map with respect to $q \in M$ taking value in the tangent bundle,

$$f : M \longrightarrow TM,$$

such that if $q \in M$ then $f(q) \in T_qM$.
Each vector field defines a *dynamical system*

$$\dot{q} = f(q),$$

i. e. for each initial point $q_0 \in M$ it admits a solution $q(t, q_0)$ on an opportune time interval I, such that $q(0, q_0) = q_0$ and

$$\frac{d}{dt}q(t) = f(q(t)), \quad \text{a. e.}\, t \in I.$$

Definition 2.1.2. $f \in \mathrm{Vec}(M)$ is a *complete vector field* if , for each initial point $q_0 \in M$, the solution $q(t, q_0)$ of the *dynamical system* $\dot{q} = f(q)$ is defined for every $t \in \mathbb{R}$. If $f \in \mathrm{Vec}(M)$ has a compact support, it is a complete vector field.

In our local study, we may assume without lack of generality that all vector fields under consideration are complete.

Definition 2.1.3. A *control system* in M is a family of dynamical systems

$$\dot{q} = f_u(q), \quad \text{with } q \in M,\ \{f_u\}_{u \in U} \subseteq \mathrm{Vec}(M),$$

parametrized by $u \in U \subseteq \mathbb{R}^k$, called *space of control parameters.*
Instead of constant values $u \in U$, we are going to consider L^∞ time depending functions taking values in U. Thus, we call

$$\mathcal{U} = \{u : I \to U,\ u \in L^\infty\}$$

the *set of admissible controls* and study the following control system

$$\dot{q} = f_u(q), \quad \text{with } q \in M,\, u \in \mathcal{U}. \tag{2.1.1}$$

In particular, let us present the control system that we are going to study during this Thesis: the affine control system.

Definition 2.1.4. An *affine control system* is a control system of the following form

$$\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M \tag{2.1.2}$$

where $f_0, \dots, f_k \in \mathrm{Vec}(M)$ and $(u_1, \dots, u_k) \in \mathcal{U}$, taking values in the set $U \subseteq \mathbb{R}^k$. We assume that $f_1, \dots, f_k$ are linearly independent in the domain under consideration. The uncontrollable term f_0 is called *drift*.

With the following theorem we want to show that, choosing an admissible control, it is guaranteed the locally existence and uniqueness of the solution of a control system for every initial point.

Theorem 2.1.5. *Fixed an admissible control $u \in \mathcal{U}$, (2.1.1) is a non-autonomous ordinary differential equation, where the right-hand side is smooth with respect to q, and measurable essentially bounded with respect to t, then, for each $q_0 \in M$, there exists a local unique solution $q_u(t, q_0)$ such that $q_u(0, q_0) = q_0$ and it is lipschitzian with respect to t.*

Definition 2.1.6. We denote

$$A_{q_0} = \{q_u(t, q_0) \,|\, t \geq 0, u \in \mathcal{U}\}$$

the *attainable set* from q_0.
In the same way, one can consider the attainable sets for time $t > 0$ from q_0

$$A_{q_0}(t) = \{q_u(t, q_0) \mid u \in \mathcal{U}\},$$

and the attainable sets for time not greater that t:

$$A^t_{(0,q_0)} = \bigcup_{0 \leq \tau < t} A_{q_0}(\tau).$$

We will write $q_u(t) = q_u(t, q_0)$ if we do not need to stress that the initial position is q_0.

Definition 2.1.7. In order to compare admissible controls on a time-interval $[0, t_1]$, we introduce a *cost functional*:

$$J(u) = \int_0^{t_1} \phi(q_u(t), u(t))dt$$

with an integrand

$$\phi : M \times U \to \mathbb{R}$$

smooth with respect to $q \in M$ and continuous with respect to the couple (q, u), $q \in M$ and $u \in \overline{U}$.

Definition 2.1.8. (**Optimal control problem**.)
Given any $q_0 \in M$ and $q_1 \in A_{q_0}$, we define the *optimal control problem* as follows:

Optimal control problem is the minimization problem for $J(u)$ with constrains on u given by control system $\dot{q} = f_u(q)$ and the fixed initial and end points $q(0) = q_0$ and $q(t_1) = q_1$.

We can write it also in the following synthetic way:

$$\begin{cases} \dot{q} = f_u(q), \quad q \in M, \quad u \in U \subset \mathbb{R}^m, \\ q(0) = q_0, \quad q(t_1) = q_1, \\ J(u) = \int_0^{t_1} \phi(q_u(t), u(t))dt \;\to\; \min. \end{cases} \tag{2.1.3}$$

There are two types of problem, *with fixed terminal time* t_1 and *free terminal time.*

We call a solution u of this problem an *optimal control*, and the corresponding curve *optimal trajectory.*

Reduction to study of attainable sets

Let us present how optimal problems are studied.
It turns out that an optimal control problem on the state space M can be essentially reduced to the study of the attainable set of the control system

$$\frac{d\hat{q}}{dt} = \hat{f}_u(\hat{q}), \quad \hat{q} \in \hat{M}, u \in U, \tag{2.1.4}$$

on the extended state space

$$\hat{M} = \mathbb{R} \times M = \{\hat{q} = (y, q) \,|\, y \in \mathbb{R}, q \in M\},$$

such that

$$\hat{f}_u(\hat{q}) = \begin{pmatrix} \phi(q,u) \\ f_u(q) \end{pmatrix}, \quad q \in M,\ u \in U,$$

where $\phi(q, u)$ is the integrand of the cost functional $J(u)$.

In particular, we are interested in $\hat{q}_u(t)$ the solution of the extended system (2.1.4) with initial condition

$$\hat{q}_u(0) = \begin{pmatrix} y(0) \\ q(0) \end{pmatrix} = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \tag{2.1.5}$$

with initial cost 0.
For optimal problem with fixed terminal time t_1, we have the following propositions:

Proposition 2.1.9. *Let $q_{\tilde{u}}(t)$, $t \in [0, t_1]$, be an optimal trajectory in the problem (2.1.3) with the fixed terminal time t_1. Then the corresponding trajectory $\hat{q}_{\tilde{u}}(t)$ of the extended system (2.1.4) comes to the boundary of the attainable set of this system:*

$$\hat{q}_{\tilde{u}}(t_1) \in \partial \hat{A}_{(0,q_0)}(t_1), \tag{2.1.6}$$

where

$$\hat{A}_{(0,q_0)}(t_1) = \left\{ \hat{q}_u(t) \;\middle|\; \hat{q}_u(0) = \begin{pmatrix} 0 \\ q_0 \end{pmatrix},\ t \in [0, t_1],\ u \in \mathcal{U} \right\}.$$

As we can see from the picture if $q_{\tilde{u}}(t)$ is optimal then (y_1, q_1) has to be in the lowest part of $\hat{A}_{(0,q_0)}(t_1)$.

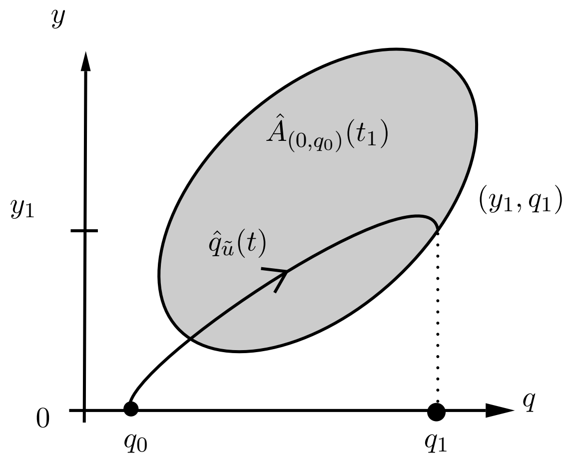

Figure 2.1: Optimal trajectory in the extended space state $\hat{M}$.

Analogously, we have a proposition for optimal problems with free terminal time.

Proposition 2.1.10. *Let $q_{\tilde{u}}(t)$, $t \in [0, t_1]$, be an optimal trajectory in the problem (2.1.3) with the free terminal time. Then the corresponding trajectory $\hat{q}_{\tilde{u}}(t)$ of the extended system (2.1.4) comes to the boundary of the attainable set of this system:*

$$\hat{q}_{\tilde{u}}(t_1) \in \partial \hat{A}^t_{(0,q_0)}, \tag{2.1.7}$$

where

$$\hat{A}^t_{(0,q_0)} = \bigcup_{0 \le \tau < t} \hat{A}_{(0,q_0)}(\tau).$$

Due to the reduction of optimal control problems, existence of optimal solutions is reduced to compactness of attainable sets. Thus, sufficient conditions of compactness of the attainable sets $A_{q_0}(t)$ and $A^t_{q_0}$ are given in the following proposition.

Proposition 2.1.11. (Filippov) *Let the space of control parameters $\mathbb{R}^m \Supset U$ be compact. Let there exist a compact K such that $M \Supset K$ and $f_u(q) = 0$ for $q \notin K$, $u \in U$. Moreover, let the velocity sets*

$$f_U(q) = \{f_u(q) \mid u \in U\} \subset T_q M, \quad q \in M,$$

be convex. Then the attainable sets $A_{q_0}(t)$ and $A^t_{q_0}$ are compact for all $q_0 \in M$, $t > 0$.

Proof. See [6], Section 10.3. □

Relaxation

Consider a control system of the form (2.1.1) with a compact set control parameters U. There is a standard procedure called *relaxation* of control system (2.1.1), which extends the velocity set $f_U(q)$ of this system to its convex hull $\mathrm{conv} f_U(q)$.

The *convex hull* of a subset S of a linear space of the minimal convex set that contain S.

Lemma 2.1.12. (Carathéodory) *For any subset $S \subset \mathbb{R}^n$,its convex hull has the form*

$$\mathrm{conv} S = \left\{ \sum_{i=0}^{n} \alpha_i x_i \,|\, x_i \in S,\ \alpha_i \geq 0,\ \sum_{i=0}^{n} \alpha_i = 1 \right\}.$$

Proof. See [19]. □

Relaxation of control system (2.1.1) is constructed in the following way. Let the set of control parameters of the relaxed system

$$V = \Delta^n \times U \times \ldots \times U,$$

with

$$\Delta^n = \left\{ (\alpha_0, \ldots, \alpha_n) \,|\, \alpha_i \geq 0,\ \sum_{i=0}^{n} \alpha_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

is the standard n-dimensional simplex.

If U is compact V is compact as well. The *relaxed system* is

$$\dot{q} = g_v(q) = \sum_{i=0}^{n} \alpha_i f_{u_i}(q), \quad v = (\alpha, u_0, \ldots, u_n) \in V, \quad q \in M. \tag{2.1.8}$$

By Carathéodory lemma, the velocity set $g_V(q)$ is convex, moreover,

$$g_V(q) = \mathrm{conv} f_U(q).$$

Let us give the following theorem on the convexity property of the exponential map.

Theorem 2.1.13. *Let X_τ, Y_τ, $\tau \in [0, t_1]$, be non autonomous vector fields with a common compact support. Let $0 \leq \alpha(\tau) \leq 1$ be a measurable function. Then there exists a sequence of non autonomous vector fields $Z_\tau^n \in \{X_\tau, Y_\tau\}$, i. e., $Z_\tau^n = X_\tau$ or Y_τ for any τ and n, such that*

$$\overrightarrow{\exp} \int_0^t Z_\tau^n d\tau \to \overrightarrow{\exp} \int_0^t (\alpha(\tau) X_\tau + (1 - \alpha(\tau)) Y_\tau)\, d\tau, \quad n \to \infty,$$

uniformly with respect to $(t, q) \in [0, t_1] \times M$ and uniformly with all derivatives with respect to $q \in M$.

Proof. See [6], Chapter 8. □

Hence, by Theorem 2.1.13, any trajectory of the relaxed system (2.1.8) can be uniformly approximated by families of trajectories of initial system.

Thus, the attainable set of the relaxed system (2.1.8) coincide with the closure of the attainable set of the initial system.

2.1.1 Fist order optimality condition: Pontryagin maximum principle

Now we are going to introduce basic notions about Lie brackets, Hamiltonian systems and Poisson brackets, so that we present the first and second order necessary conditions of optimality: Pontryagin maximum principle, and Goh condition.

Definition 2.1.14. Let $f, g \in \mathrm{Vec}(M)$, we define their *Lie brackets* the following vector field

$$[f,g](q) = \frac{1}{2} \frac{\partial^2}{\partial t^2}\bigg|_{t=0} e^{-tg} \circ e^{-tf} \circ e^{tg} \circ e^{tf}(q), \quad \forall q \in M.$$

where e^{-tf} is the flow defined by $-f$.

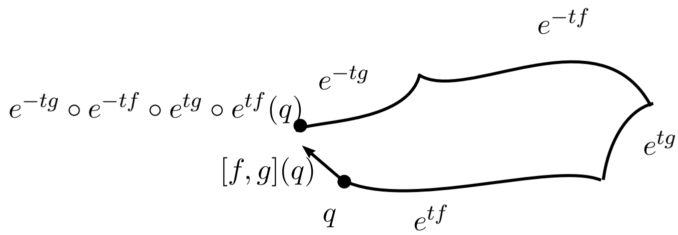

Figure 2.2: Lie Bracket

Definition 2.1.15. An *Hamiltonian* is a smooth function on the cotangent bundle

$$h \in C^\infty(T^*M).$$

The *Hamiltonian vector field* is the vector field associated with h via the canonical symplectic form σ

$$\sigma_\lambda(\cdot, \overrightarrow{h}) = d_\lambda h.$$

We denote

$$\dot{\lambda} = \overrightarrow{h}(\lambda), \quad \lambda \in T^*M,$$

the *Hamiltonian system*, which corresponds to h.
Let $(x_1, \ldots, x_n)$ be local coordinates in M and $(\xi_1, \ldots, \xi_n, x_1, \ldots, x_n)$ induced coordinates in T^*M, $\lambda = \sum_{i=1}^n \xi_i dx_i$. The *symplectic form* has expression $\sigma = \sum_{i=1}^n d\xi_i \wedge dx_i$. Thus, in canonical coordinates, the Hamiltonian vector field has the following form

$$\overrightarrow{h} = \sum_{i=1}^n \left(\frac{\partial h}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial}{\partial \xi_i} \right).$$

Therefore, in canonical coordinates, it is

$$\begin{cases} \dot{x}_i = \frac{\partial h}{\partial \xi_i} \\ \dot{\xi}_i = -\frac{\partial h}{\partial x_i} \end{cases}$$

for $i = 1, \ldots, n$.

Definition 2.1.16. The *Poisson brackets* $\{a,b\} \in \mathcal{C}^\infty(T^*M)$ of two Hamiltonians $a, b \in \mathcal{C}^\infty(T^*M)$ are defined as follows: $\{a,b\} = \sigma(\vec{a}, \vec{b})$; the coordinate expression is:

$$\{a,b\} = \sum_{k=1}^{n} \left(\frac{\partial a}{\partial \xi_k} \frac{\partial b}{\partial x_k} - \frac{\partial a}{\partial x_k} \frac{\partial b}{\partial \xi_k} \right).$$

Remark 2.1.17. Let us recall that, given g_1 and g_2 vector fields in M, considering the Hamiltonians $a_1(\xi,x) = \langle \xi, g_1(x) \rangle$ and $a_2(\xi,x) = \langle \xi, g_2(x) \rangle$, it holds

$$\{a_1, a_2\}(\xi,x) = \langle \xi, [g_1, g_2](x) \rangle .$$

Remark 2.1.18. Given a smooth function Φ in $\mathcal{C}^\infty(T^*M)$, and $\lambda(t)$ solution of the Hamiltonian system $\dot{\lambda} = \overrightarrow{h}(\lambda)$, the derivative of $\Phi(\lambda(t))$ with respect to t is the following

$$\frac{d}{dt}\Phi(\lambda(t)) = \{h, \Phi\}(\lambda(t)).$$

Pontryagin maximum principle

The Pontryagin maximum principle is the fundamental necessary condition of optimality for optimal control problems. The first classical version of PMP was obtained or optimal control problems in $\mathbb{R}^n$ by L. S. Pontryagin [18].

This principle was born studying optimal problems via the reduction to the attainable sets, that we explained previously.

At first let us present the geometric statement of PMP: it gives the necessary conditions for any solution $\tilde{q}(t) = q_{\tilde{u}}(t)$, $t \in [0, t_1]$, of a control system

$$\dot{q} = f_u(q), \quad q \in M, \, u \in U \subset \mathbb{R}^m, \tag{2.1.9}$$

and initial condition

$$q(0) = q_0, \tag{2.1.10}$$

such that

$$\tilde{q}(t_1) \in \partial A_{q_0}(t_1).$$

Define the following family of Hamiltonians:

$$h_u(\lambda) = \langle \lambda, f_u(q) \rangle , \quad \lambda \in T_q^*M, \, q \in M, \, u \in U.$$

The geometric statement of the PMP for fixed terminal time t_1 is the following.

Theorem 2.1.19. (PMP). *Let $\tilde{u}(t)$, $t \in [0, t_1]$, be an admissible control and $\tilde{q}(t) = q_{\tilde{u}}(t)$ the corresponding solution of Cauchy problem (2.1.9) and (2.1.10). If*

$$\tilde{q}(t_1) \in \partial A_{q_0}(t_1),$$

then there exists a Lipschitzian curve in the cotangent bundle

$$\lambda_t \in T^*_{\tilde{q}(t)}M, \quad 0 \le t \le t_1,$$

such that

$$\lambda_t \neq 0 \tag{2.1.11}$$

$$\dot{\lambda}_t = \overrightarrow{h}_{\tilde{u}(t)}(\lambda_t), \tag{2.1.12}$$

$$h_{\tilde{u}(t)}(\lambda_t) = \max_{u\in U} h_u(\lambda_t) \tag{2.1.13}$$

for almost all $t \in [0, t_1]$.

Proof. See [6], Section 12.3. □

On the other hand, we have the geometric statement of the PMP for free terminal time.

Theorem 2.1.20. *Let* $\tilde{u}(\cdot)$ *be an admissible control for control system (2.1.9) such that*

$$\tilde{q}(t_1) \in \partial\left(\bigcup_{|t-t_1|<\varepsilon} A_{q_0}(t)\right),$$

for some $t_1 > 0$ *and* $\varepsilon \in (0, t_1)$. *Then there exists a Lipschitzian curve*

$$\lambda_t \in T^*_{\tilde{q}(t)}M, \quad \lambda_t \neq 0, \quad 0 \leq t \leq t_1,$$

such that

$$\dot{\lambda}_t = \overrightarrow{h}_{\tilde{u}(t)}(\lambda_t),$$

$$h_{\tilde{u}(t)}(\lambda_t) = \max_{u\in U} h_u(\lambda_t)$$

$$h_{\tilde{u}(t)}(\lambda_t) = 0 \tag{2.1.14}$$

for almost all $t \in [0, t_1]$.

Proof. See [6], Section 12.2. □

Hence, thanks to Propositions 2.1.9 and 2.1.10, it is possible to see explicitly the necessary condition of PMP given an admissible control minimizing a cost $J(\cdot)$.

Theorem 2.1.21. *(1) Let* $\tilde{u}$, $t \in [0, t_1]$, *be an optimal control for problem (2.1.3) with fixed terminal time:*

$$J(\tilde{u}) = \min\{J(u) \,|\, q_u(t_1) = q_1\}.$$

Define a Hamiltonian function

$$h^{\nu}_u(\lambda) = \langle \lambda, f_u \rangle + \nu\phi(q, u), \quad \lambda \in T^*_q M, \quad u \in U, \quad \nu \in \mathbb{R}.$$

Then there exist a non trivial pair:

$$(\nu, \lambda_t) \neq 0, \quad \nu \in \mathbb{R}, \quad \lambda_t \in T^*_{\tilde{q}(t)}M,$$

such that the following conditions hold:

$$\begin{aligned}
&\dot{\lambda}_t = \overrightarrow{h}^{\nu}_{\tilde{u}(t)}(\lambda_t),\\
&h^{\nu}_{\tilde{u}(t)}(\lambda_t) = \max\nolimits_{u\in U} h^{\nu}_u(\lambda_t), \quad \text{a.e.}\, t\in[0,t_1],\\
&\nu \leq 0.
\end{aligned} \tag{2.1.15}$$

(2) *If $\tilde{u}$ is an optimal control for the problem (2.1.21) with free terminal time. We define an Hamiltonian function in the same way. Then there exists a non trivial pair $(\nu, \lambda_t) \neq 0$ such that conditions (2.1.15) and*

$$h^{\nu}_{\tilde{u}(t)}(\lambda_t) \equiv 0.$$

hold.

Proof. See [6], Section 12.4. □

Remark 2.1.22. If we have a maximization problem then the inequality for ν should be:

$$\nu \geq 0.$$

Remark 2.1.23. There are two distinct possibilities for the constant parameter ν in Theorem 2.1.21:

(a) If $\nu \neq 0$, then λ_t is a *normal extremal.* Reparametrising (ν, λ_t), in this case $\nu = -1$.

(b) If $\nu = 0$, then λ_t is an *abnormal case.*

2.1.2 Second order optimality condition: Goh condition

Finally, we present the Goh condition, on the singular arcs of the extremal trajectory, in which we do not have information from the maximality condition of the Pontryagin maximum principle. We state the Goh condition only for affine control systems that we denoted in Definition 2.1.4.

Theorem 2.1.24 (*Goh condition*)**.** *Let $\tilde{q}(t)$, $t \in [0, t_1]$ be a time-optimal trajectory corresponding to a control $\tilde{u}$. If $\tilde{u}(t) \in \mathrm{int}U$ for any $t \in (\tau_1, \tau_2)$, then there exist an extremal $\lambda(t) \in T^*_{q(t)}M$ such that*

$$\langle \lambda(t), [f_i, f_j](q(t))\rangle = 0, \qquad t \in (\tau_1, \tau_2),\ i, j = 1, \dots, m. \tag{2.1.16}$$

Proof. See [6] Chapter 20. □

2.2 Time-optimal problem, with control in a ball

Let us present the problem that we are going to discuss in this Thesis.

We are interested in analysing the time-optimal problem of solutions of an affine control system with control in a ball. They are defined as follows.

Definition 2.2.1. An *affine control system* is a control system of the following form

$$\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M \tag{2.2.1}$$

where $f_0, \ldots, f_k \in \mathrm{Vec}(M)$ and $(u_1, \ldots, u_k) \in \mathcal{U}$, taking values in the set $U \subseteq \mathbb{R}^k$. We assume that $f_1, \ldots, f_k$ are linearly independent in the domain under consideration. The uncontrollable term f_0 is called *drift*.
Moreover, we consider

$$U = \overline{B}^k = \{u \in \mathbb{R}^k : ||u|| \leq 1\}$$

the closed unitary ball in $\mathbb{R}^k$.

Definition 2.2.2. Given any control system (2.2.1), $q_0 \in M$ and $q_1 \in A_{q_0}$, the *time-optimal problem* consists in minimizing the time of motion from q_0 to q_1 via admissible trajectories:

$$\begin{cases} \dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad u \in \mathcal{U} \\ q_u(0, q_0) = q_0 \\ q_u(t_1, q_0) = q_1 \\ t_1 \to \min \end{cases} \tag{2.2.2}$$

We call these minimizer trajectories *time-optimal trajectories*, and *time-optimal controls* the corresponding controls.

As we can notice by the Classical Filippov's Theorem 2.1.11, the existence of the time-optimal problem on a affine control system (2.2.1) is guaranteed if U is a convex compact set and q_0 is sufficiently close to q_1.
Moreover, let us present the Pontryagin maximum principle for the time-optimal problem.

Theorem 2.2.3 (Pontryagin maximum principle - time-optimal problem)**.** *Let an admissible control $\tilde{u}$, defined in the interval $t \in [0, \tau_1]$, be time-optimal for the system (2.1.1), and let the Hamiltonian associated with this control system be the action on $f_u(q) \in T_q^*M$ of a covector $\lambda \in T_q^*M$:*

$$h_u(\lambda) = \langle \lambda, f_u(q) \rangle \, .$$

Then there exists $\lambda(t) \in T^*_{q_{\tilde{u}}(t)}M$, for $t \in [0, \tau_1]$, called extremal *never null and lipschitzian, such that for almost all $t \in [0, \tau_1]$ the following conditions hold:*

1. $\dot{\lambda}(t) = \vec{h}_{\tilde{u}}(\lambda(t))$
2. $h_{\tilde{u}}(\lambda(t)) = \max_{u \in U} h_u(\lambda(t))$ *(Maximality condition)*
3. $h_{\tilde{u}}(\lambda(t)) \geq 0$.

Given the canonical projection $\pi : TM \to M$, we denote $q(t) = \pi(\lambda(t))$ the extremal trajectory.

2.2.1 Consequence from the optimality conditions

In this thesis we are going to investigate the local regularity of time-optimal trajectories for the n-dimensional affine control system with a k-dimensional control.

By the Pontryagin maximum principle, every time-optimal trajectory of our system has an extremal in the cotangent bundle T^*M that satisfies a Hamiltonian system, given by the maximized Hamiltonian.

In the following pages we are going to present the first consequences from the optimality condition in the time-optimal case that we are studying.

At first, let us give some notation and define the *singular locus* in T^*M:

Notation 2.2.4. We call $h_i(\lambda) = \langle \lambda, f_i(q) \rangle$, $f_{ij}(q) = [f_i, f_j](q)$, $f_{ijk}(q) = [f_i, [f_j, f_k]](q)$, $h_{ij}(\lambda) = \langle \lambda, f_{ij}(q) \rangle$, and $h_{ijk}(\lambda) = \langle \lambda, f_{ijk}(q) \rangle$, with $\lambda \in T^*_q M$ and $i, j, k \in \{0, 1, \dots, k\}$. Moreover, we denote the following vector $H_{0I}(\lambda) = \{h_{0i}(\lambda)\}_i \in \mathbb{R}^k$ and $k \times k$ matrix $H_{IJ}(\lambda) = \{h_{ij}(\lambda)\}_{ij}$ with respect to $\lambda \in T^*M$.

Definition 2.2.5. The *singular locus* $\Lambda \subseteq T^*M$, is defined as follows:

$$\Lambda = \{\lambda \in T^*M : h_1(\lambda) = \ldots = h_k(\lambda) = 0\}.$$

The following proposition is an immediate Corollary of the Pontryagin maximum principle.

Proposition 2.2.6. *If an extremal $\lambda(t)$, $t \in [0, t_1]$, does not intersect the singular locus Λ, then $\forall t \in [0, t_1]$*

$$\tilde{u}(t) = \begin{pmatrix} \frac{h_1(\lambda(t))}{(h_1^2(\lambda(t)) + \ldots + h_k^2(\lambda(t)))^{1/2}} \\ \vdots \\ \frac{h_k(\lambda(t))}{(h_1^2(\lambda(t)) + \ldots + h_k^2(\lambda(t)))^{1/2}} \end{pmatrix}. \tag{2.2.3}$$

Moreover, this extremal is a solutions of the Hamiltonian system defined by the Hamiltonian $H(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + \ldots + h_k^2(\lambda)}$. Thus, it is smooth.

Definition 2.2.7. We will call *bang arc* any smooth arc of a time-optimal trajectory $q(t)$, whose corresponding time-optimal control $\tilde{u}$ lies in the boundary of the space of control parameters: $\tilde{u}(t) \in \partial U$.

Corollary 2.2.8. *An arc of a time-optimal trajectory, whose extremal is out of the singular locus, is a bang arc.*

From Corollary 2.2.8 we already have an answer about the regularity of time-optimal trajectories: every time-optimal trajectory, whose extremal lies out of the singular locus, is smooth.

Remark 2.2.9. Given an affine control system (4.2.2) with $n = k$, every extremal is smooth, because the singular locus contains only the null covector $\Lambda = \{0\}$.
This case is called *Zermelo Navigation Problem*.

As a consequence we are interested in affine control system where $n > k$.

However, we do not know what happen if an extremal touches the singular locus, optimal controls may be not always smooth.

Definition 2.2.10. A *switching* is a discontinuity of an optimal control.
Given $u(t)$ an optimal control, $\bar{t}$ is a *switching time* if $u(t)$ is discontinuous at $\bar{t}$. Moreover given $q_u(t)$ the admissible trajectory, $\bar{q} = q_u(\bar{t})$ is a *switching point* if $\bar{t}$ is a switching time for $u(t)$.

A concatenation of bang arcs is called *bang-bang trajectory*.
An arc of an optimal trajectory that admits an extremal totally contained in the singular locus Λ, is called *singular arc*.

2.3 Theory of orbits and the Frobenius theorem

In this Subsection we are going to give some preliminaries for Chapter 6: we define and analyse the orbit of a given family $\mathcal{F}$ of vector fields and state die Frobenius Theorem.

Given the control system

$$\dot{q} = f_0(q) + u_1 f_1(q) + \ldots + u_k f_k(q), \quad q \in M \tag{2.3.1}$$

where $f_0, \ldots, f_k$ are smooth vector fields and $(u_1, \ldots, u_k)$ admissible control with value in $\overline{B}^k$, let us denote $\mathcal{F} = \{f_1, \ldots, f_k\} \subset \mathrm{Vec}(M)$, we will call

$$e^{tf_i} : M \to M$$

the exponential map with respect time t and autonomous vector field $f_i \in \mathcal{F}$ and we consider

$$\mathcal{P} = \left\{ e^{t_j f_j} \left(e^{t_{j-1} f_{j-1}} \left(\cdots \left(e^{t_1 f_1} \right) \right) \right) \middle|\ t_i \in \mathbb{R},\ f_i \in \mathcal{F},\ j \in \mathbb{N} \right\} \subset \mathrm{Diff}(M)$$

the group of diffeomorphisms of M generated by flows in $\mathcal{F}$.

Hence, the *orbit of the family* $\mathcal{F}$ through a point $q_0 \in M$ is defined as following:

$$\mathcal{O}_{q_0} = \left\{ e^{t_j f_j} \left(e^{t_{j-1} f_{j-1}} \left(\cdots \left(e^{t_1 f_1}(q_0) \right) \right) \right) \middle|\ t_i \in \mathbb{R},\ f_i \in \mathcal{F},\ j \in \mathbb{N} \right\}.$$

Theorem 2.3.1. (Orbit Theorem, Nagano-Sussmann). *Let $\mathcal{F} \subset \mathrm{Vec}(M)$ and $q_0 \in M$. Then:*

1. *$\mathcal{O}_{q_0}$ is a connected immersed submanifold of M,*
2. *$T_q\mathcal{O}_{q_0} = \mathrm{span}\left\{ P_*^{-1} f(q) \,|\, P \in \mathcal{P},\ f \in \mathcal{F} \right\}$, $q \in \mathcal{O}_{q_0}$.*

Proof. See [6] Chapter 5. ∎

Let us give some corollaries.

Let $\mathcal{O}_{q_0}$ be an orbit of family $\mathcal{F} \subset \mathrm{Vec}(M)$.

If $f \in \mathcal{F}$ then $f(q) \in T_q\mathcal{O}_{q_0}$ for all $q \in \mathcal{O}_{q_0}$. Indeed, the trajectory $t \to e^{tf}(q)$ belongs to orbit $\mathcal{O}_{q_0}$, thus its velocity vector $f(q)$ is in the tangent space $T_q\mathcal{O}_{q_0}$.

Further, if $f_1, f_2 \in \mathcal{F}$, then $[f_1, f_2](q) \in T_q\mathcal{O}_{q_0}$, for all $q \in \mathcal{O}_{q_0}$. This follows since the vector $[f_1, f_2](q)$ is tangent to the trajectory

$$t \longmapsto e^{-tf_2}\left(e^{-tf_1}\left(e^{tf_2}\left(e^{tf_1}(q)\right)\right)\right) \in \mathcal{O}_{q_0}.$$

Given three vector fields $f_1, f_2, f_3 \in \mathcal{F}$, we have $[f_1, [f_2, f_3]](q) \in T_q\mathcal{O}_{q_0}$, $q \in \mathcal{O}_{q_0}$. Indeed, it follows that $[f_2, f_3]$ starting in the immersed submanifold $\mathcal{O}_{q_0}$ do not leave it.

Then we repeat the argument of the previous items.

We can go on and consider Lie brackets of arbitrarily high order

$$[f_1, [\dots [f_{k-1}, f_k] \dots]](q)$$

as tangent vectors to $\mathcal{O}_{q_0}$ if $f_i \in \mathcal{F}$. These considerations can be summarized in terms of the Lie algebra of vector fields generated by $\mathcal{F}$:

$$\mathrm{Lie}\mathcal{F} = \mathrm{span}\{[f_1, [\dots [f_{k-1}, f_k] \dots]] \mid f_i \in \mathcal{F},\, k \in \mathbb{N}\} \subset \mathrm{Vec}(M),$$

and its evaluation at point $q \in M$

$$\mathrm{Lie}_q\mathcal{F} = \{V(q) \mid V \in \mathrm{Lie}\mathcal{F}\} \subset T_qM.$$

We obtain the following statement.

Corollary 2.3.2.

$$\mathrm{Lie}_q\mathcal{F} \subset T_q\mathcal{O}_{q_0} \tag{2.3.2}$$

for all $q \in \mathcal{O}_{q_0}$.

Let us give the definition of *bracket-generating* family $\mathcal{F}$.

Definition 2.3.3. A family $\mathcal{F} \subset \mathrm{Vec}(M)$ that satisfies property

$$\mathrm{Lie}_q\mathcal{F} = T_qM, \quad \forall q \in M, \tag{2.3.3}$$

is called *completely nonholonomic* or *bracket-generating.*

Another important corollary of the Orbit Theorem in the Rashevsky - Chow Theorem.

Theorem 2.3.4. Rashevsky-Chow. *Let M be a connected smooth manifold, and let $\mathcal{F} \subset \mathrm{Vec}(M)$. If the family $\mathcal{F}$ is completely nonholonomic, then*

$$\mathcal{O}_{q_0} = M, \quad \forall q_0 \in M.$$

Proof. By Corollary 2.3.2, equality (2.3.3) means that any orbit $\mathcal{O}_{q_0}$ is an open set in M.

Further, consider the following equivalence relation in M:

$$q_1 \sim q_2 \Leftrightarrow q_2 \in \mathcal{O}_{q_1}, \quad q_1, q_2 \in M.$$

The manifold M is the union of naturally disjoint equivalence classes. Each class is an open subset of M and M is connected. hence there is only one nonempty class. That is, M is single orbit $\mathcal{O}_{q_0}$. □

Analytic case

Let us consider the set $\mathrm{Vec}(M)$ as a module over $\mathcal{C}^\infty(M)$.

Definition 2.3.5. A submodule $\mathcal{V} \subset \mathrm{Vec}(M)$ is called *finitely generated* over $\mathcal{C}^\infty(M)$ if it has a finite global basis of vector fields:

$$\exists V_1, \dots, V_k \in \mathrm{Vec}(M) \text{ s. t. } \mathcal{V} = \left\{ \sum_{i=1}^{k} a_i V_i \mid a_i \in \mathcal{C}^\infty(M) \right\}.$$

Definition 2.3.6. A submodule $\mathcal{V} \subset \mathrm{Vec}(M)$ is called *locally finitely generated* over $\mathcal{C}^\infty(M)$ if any point $q \in M$ has a neighbourhood $O \subset M$ in which the restriction $\mathcal{F}_{|O}$ is finitely generated over $\mathcal{C}^\infty(O)$.

Theorem 2.3.7. *Let $\mathcal{F} \subset \mathrm{Vec}(M)$. Suppose that the module* $\mathrm{Lie}\mathcal{F}$ *is locally finitely generated over $\mathcal{C}^\infty(M)$. Then*

$$T_q\mathcal{O}_{q_0} = \mathrm{Lie}_q\mathcal{F}, \quad q \in \mathcal{O}_{q_0} \tag{2.3.4}$$

for any orbit $\mathcal{O}_{q_0}$, $q_0 \in M$, of the family $\mathcal{F}$.

Proof. See [6] Chapter 5. □

Corollary 2.3.8. *If M and $\mathcal{F}$ are real analytic, then equality (2.3.4) holds.*

Proof. In the analytic case, $\mathrm{Lie}\mathcal{F}$ is locally finitely generated. Indeed, any module generated by analytic vector fields is locally finitely generated. This is Nötherian property of the ring of germs of analytic functions. □

Frobenius Theorem

Definition 2.3.9. A *distribution* $\Delta \subset TM$ on a smooth manifold M is a family of linear subspaces $\Delta_q \subset T_qM$ smoothly depending on a point $q \in M$. Dimension of the subspaces Δ_q, $\forall q \in M$ is assumed constant.

Definition 2.3.10. A distribution Δ on a manifold M is called *integrable* if for any point $q \in M$ there exists an immersed submanifold $N_q \subset M$, $q \in N_q$, such that

$$T_{q'}N_q = \Delta_{q'}, \quad \forall q' \in N_q.$$

The submanifold is called an *integral manifold* of the distribution Δ through the point q.

In other words, integrability of a distribution $\Delta \subset TM$ means that through any point $q \in M$ we can draw a submanifold N_q whose tangent spaces are elements of the distribution Δ.

A distribution Δ may be nonintegrable.

Theorem 2.3.11. Frobenius Theorem *A distribution $\Delta \subset TM$ is integrable if and only if in a neighbourhood of any point $q_0 \in M$ any base of Δ is closed with respect to the Lie brackets.*

Proof. Assume that the distribution Δ is integrable. Any vector field in Δ is tangent to the integral manifold N, thus the orbit $\mathcal{O}_q$ of the family of vector fields Δ, restricted to a small enough neighbourhood of $q \in N$, is contained in the integral manifold N.
Moreover, since $\dim\mathcal{O}_q \geq \dim\Delta_q = \dim N$, then locally $\mathcal{O}_q = N$: we can go in N in any direction along vector fields of the family Δ. By the Orbit Theorem, $T_q\mathcal{O}_q \supset \mathrm{Lie}_q\Delta$, that is why

$$\mathrm{Lie}_q\Delta = \Delta_q.$$

it means that in a neighbourhood of any point $q \in M$ any base of Δ is closed with respect to the Lie brackets.

Viceversa, if in a neighbourhood of any point $q_0 \in M$ any base of Δ is closed with respect to the Lie brackets, then $\mathrm{Lie}(\Delta) = \Delta$. By Theorem 2.3.7,

$$T_q\mathcal{O}_{q_0} = \mathrm{Lie}(\Delta), \quad q \in \mathcal{O}_{q_0},$$

thus,

$$T_q\mathcal{O}_{q_0} = \Delta_q, \quad q \in \mathcal{O}_{q_0},$$

i.e. the orbit $\mathcal{O}_{q_0}$ is an integral manifold of Δ through q_0. □

2.4 Main purpose of the thesis

2.4.1 Chattering phenomenon

With the term "chattering phenomenon" we mean optimal control with a convergent series of switchings in a finite time interval.

A typical behaviour of chattering trajectories in a two-dimensional phase space is given by the Fuller's Problem. Let us present it.

Fuller's Problem: Given the system in $\mathbb{R}^2$ with coordinates (x_1, x_2)

$$\dot{x_1} = x_2, \ \dot{x}_2 = u, \ u \in [-1, 1]$$

where the control u is an admissible control, $u(\cdot) \in L_\infty[0, \infty)$, minimize

$$\int_0^{t_1} |x_1|^2 dt \to \min,$$

with initial and final conditions

$$(x_1(0), x_2(0)) = (x_1^0, x_2^0), \ (x_1(t_1), x_2(t_1)) = (0, 0).$$

We want present this problem, because there exist a unique optimal solution $(x_1(t), x_2(t))$ and finite terminal time t_1, with a control $\tilde{u}(t)$ containing a countable set of switches. Let us show briefly why. For a more detailed introduction, see [28].

Considering the cotangent space with coordinates $((p_1, p_2), (x_1, x_2))$, the Hamiltonian parametrized by u is

$$H = p_1 x_2 + p_2 u + \nu x_1^2.$$

If we are going to study normal or abnormal extremals, then $\nu = -1$ or $\nu = 0$.
By maximality condition the optimal control must be $\tilde{u}(t) = \mathrm{sgn}(p_2(t))$ if $p_2(t) \neq 0$.
Moreover, $(p_1(t), p_2(t))$ satisfy

$$\begin{cases} \dot{p}_1 = -2\nu x_1 \\ \dot{p}_2 = -p_1. \end{cases}$$

One can see that there is no singular curve. Indeed, along the singular curve we must have $p_2(t) \equiv 0$, then p_1 x_1 and x_2 must be null too.
It follows that all optimal controls are bang-bang, with switching occurring when $p_2(t) = 0$.
It turns out that the optimal solution has the following properties:

- Optimal controls are bang-bang with infinitely many switchings
- Switchings takes place on the curve $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 + \gamma |x_2| x_2 = 0 \right\}$ where $\gamma \approx 0.445$
- Time intervals between consecutive switchings decrease in geometric progression.

The last property is consistent with the fact that the final time must be finite.
The occurrence of a switching pattern in which switching times form an infinite sequence accumulating near the final time is known as *Fuller's phenomenon*, or *Zeno behaviour*.
The following Figure shows the switching curve and a sketch of an optimal state trajectory.

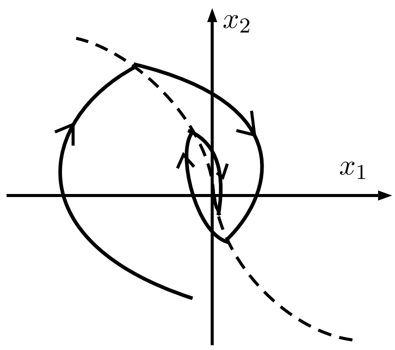

Figure 2.3: Fullers Problem

An other example is the *the Markov-Dubins problem with angular acceleration control*. It is a modifies version of the Markov-Dubins problem, in which the control is angular acceleration rather than angular velocity

$$\begin{cases} \dot{x} = \cos(z) \\ \dot{y} = \sin(z) \\ \dot{z} = w \\ \dot{w} = u. \end{cases}$$

It is proved in [27] that an optimal trajectory can not contain a junction of a bang-bang and singular piece, and there are Pontryagin extremals involving infinite switchings.

2.4.2 Generic conditions

Let us give a digression on the Whitney topology in $\mathcal{C}^\infty$, transversality Theorem, and generic condition.

Definition 2.4.1. Assume, $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$. We define the $\mathcal{C}^\infty$ *Whitney topology* in $\mathcal{C}^\infty(M,N)$ by defining the set of neighbourhoods $U_f^{l,\varepsilon}$ of a given map $f = (f_1, \dots, f_k) : M \to N$, which are parametrized by arbitrary $l \geq 0$ and arbitrary positive defined functions

$$\varepsilon : M \to (0, \infty).$$

They are defined by

$$U_f^{l,\varepsilon} = \left\{ g \in \mathcal{C}^\infty(M,N) : \left| \frac{\partial^{|I|}(g_i - f_i)}{\partial x^I}(x) \right| < \varepsilon(x),\, \forall x \in M,\, i = \{1, \dots, k\},\, |I| \leq l \right\}.$$

If we restrict the possible l to those satisfying $l \leq r$, then we obtain the *Whitney $\mathcal{C}^r$ topology.*

If M N are manifolds, the same definition of a neighbourhoods of a given f can be given, by restricting the domain of x to an open chart $\Phi : U \to M$ in $\mathbb{R}^n$ such that its image under f lies in a given chart $\Psi : V \to N$ in $\mathbb{R}^k$, then the open neighbourhoods of f are also parametrized by such charts Φ and Ψ and form a family $U_f^{l,\varepsilon,\Phi,\Psi}$ defined by the set of $g \in \mathcal{C}^\infty(M,N)$ such that

$$\left| \frac{\partial^{|I|}((\Psi^1 \circ g \circ \Phi)_i - (\Psi^1 \circ f \circ \Phi)_i)}{\partial x^I}(x) \right| < \varepsilon(x), \quad \forall x \in M,\, 1 \leq i \leq k,\, |I| \leq l.$$

If the functions ε were taken constant, we would obtain the usual $\mathcal{C}^\infty$ topology in $\mathcal{C}^\infty(M,N)$. Since the functions ε are taken arbitrary, in particular decaying fast when x tends to the boundary of M, this topology is much stronger then the usual $\mathcal{C}^\infty$ topology.
We have (see [13]):

1. The space $\mathcal{C}^\infty(M,N)$ with Whitney topology is not metrizable.
2. This space has the Baire property: any countable intersection of open dense subsets is dense in $\mathcal{C}^\infty(M,N)$.

By the second property above, we define a residual set in the space $\mathcal{C}^\infty(M,N)$.

Definition 2.4.2. A *residual set* is a countable intersection of some open dense subset in $\mathcal{C}^\infty(M,N)$.

Definition 2.4.3. A *generic property* P of maps in $\mathcal{C}^\infty(M,N)$ is a property which is satisfied by a maps from some residual subset in $\mathcal{C}^\infty(M,N)$.

Definition 2.4.4. We say that f is *transversal to* $S \subset N$ at x if the image of the tangent map f_{*x} is transversal to the tangent space to S at $y = f(x)$, i.e.,

$$f_{*x}(T_x M) + T_y S = T_y M.$$

We call f *transversal to* S, if it is transversal to S to each $x \in M$.

Remark 2.4.5. Given $\text{codim}S = \text{N} - \dim S$, we note that if $\text{codim}S > \dim M$, then

$$\dim f_{*x}(T_x M) + \dim T_y S < \dim N$$

then transversality of f to S means that

$$f(M) \cap S = \emptyset$$

holds.

Proposition 2.4.6. *The set of maps $f : M \to N$ transversal to a given submanifold $S \subset N$ is residual in the Whitney $\mathcal{C}^1$ topology. It is also open and dense, if S is a closed subset of N.*

Definition 2.4.7. Given $O \subset \mathbb{R}^n$ an open set, let us recall that the *h-jet* $j^h\phi(x)$ of a $\mathcal{C}^\infty$ function $\phi : M \to \mathbb{R}$ is the collection of all coefficients of its Taylor expansion of order h at x.
If $f = (f_1, \ldots, f_k) : M \to N \subset \mathbb{R}^k$, then we define $j^h f(x) = (j_1^h f(x), \ldots, j_k^h f(x))$. All such jets fold a space $J^h(M, N)$.

If M and N are differential manifolds, we can define the jets in coordinate charts. We denote by $\approx_{x,h}$, the equivalence relation of kth order tangency of graphs of $f : M \to N$ and $g : M \to N$ at x; in coordinates,

$$f \approx_{x,h} g$$

means the coincidence of their Taylor series up to order h at x.

Definition 2.4.8. The *h-th jet of a map* $f : M \to N$ at x is the equivalence class of f under h-th order tangency at x, namely,

$$j^h f(x) = [f]_{x,h}.$$

Remark 2.4.9. The space $J^h(M, N)$ of all h-jets of smooth maps $f : M \to N$ forms a differentiable manifold, with the coordinate charts induced from the coordinate charts on M and N by "computing the Taylor coefficients up to order h in coordinate charts".

In order to state the Thom transversality theorem recall that a given $\mathcal{C}^\infty$ map $f : N \to M$ has the *h-jet extension*

$$j^h f : M \to J^k(M, N)$$

given by

$$x \mapsto j^h f(x)$$

which assigns to each point x the coefficients of the h-th order Taylor expansion of f at x.

Theorem 2.4.10. Thom Transversality Theorem. *If $S_1, \ldots, S_l$ are submanifolds of $J^h(M, N)$, then the set of maps $f \in \mathcal{C}^\infty(M, N)$ such that $j^h f$ is transversal to each of S_i, $\forall i \in \{1, \ldots, l\}$, is residual in the Whitney $\mathcal{C}^\infty$ topology. Moreover, it is also open if $S_1, \ldots, S_l$ are closed subsets of $J^h(M, N)$.*

Now, let us see why transversality theory gives us adequate instruments for investigating generic properties of the affine control system we are studying.
Instead of $f : M \to N$ we are going to consider $k+1$-uple $(f_0, f_1, \dots, f_k)$ of vector fields on M, taking value in $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$ $k+1$ times.

Let us call $J^{k+1,h}M$ the vector bundle on M of all h-jets of $k{+}1$-uple $(f_0, f_1, \dots, f_k)$ of vector fields on M, i.e. the image of the map $j^h(f_0, f_1, \dots, f_k)$.

Let S be a stratified subset of $J^{k+1,h}M$ such that $J_q^{k+1,h}M \cap S$ is of codimension larger than the value m in $J_q^{k+1,h}M$, for every $q \in M$. Then, generically, the set of points $q \in M$, such that the h-jet of $(f_0, \dots, f_k)$ at q, $J_q^{k+1,h}(f_0, \dots, f_k)$ belongs to S, has codimension larger than m in M.

In particular, if we find S such that m is the dimension on M, then generically $J_q^{k+1,h}(f_0, \dots, f_k)$ does not belong to S, for every $q \in M$.

Therefore, if we find A such that, for every $f_1, \dots, f_k$ and q such that $J_q^{k+1,h}(f_0, \dots, f_k) \notin S$, all short time-optimal trajectories near q are finite concatenations of arcs, and if m is the dimension on M, then, generically, chattering does not occur.

In particular, for any m less equal to the dimension of M, if we can show that no chattering appears near points for which $J_q^{k+1,h}(f_0, \dots, f_k) \in S$, then we actually give a bound on the dimension of the set of points near which chattering cannot be excluded.

Main purpose

We are interested in studying the local regularity of time-optimal trajectories defined by the affine control systems

$$\dot{q} = f_0(q) + u_1 f_1(q) + \ldots + u_k f_k(q), \quad q \in M,$$

where u takes value in the closed unitary ball.
In general situations discontinuous control are unavoidable, and there could be chattering trajectories.
The main purpose of the thesis is to understand how it is possible to avoid chattering phenomenon given generic conditions on the vector fields $f_0, \dots, f_k$. Namely if it possible to obtain more regularity with a perturbation of the system.
Thanks to the previous digression, we saw why we look for generic condition, and which kind of conditions we need to consider in order to complete the theory.

Chapter 3

Switching in time-optimal problem, with control in a ball

3.1 Introduction

In this Chapter we are going to present the most general result that we reach during this years [4]: we analyse local regularity of time-optimal controls and trajectories for an n-dimensional affine control system with a control parameter, taking values in a k-dimensional closed ball.

We study singularities of the extremals of the time-optimal problem for the affine control system of the form:

$$\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M, \ (u_1, \ldots, u_k) \in \mathcal{U}, \tag{3.1.1}$$

where M is a smooth n-dimensional manifold, f_0, f_1, $\ldots$, f_k are smooth vector fields and $(u_1, \ldots, u_k)$ are admissible controls taking values in $U = \{u \in \mathbb{R}^k : |u| \leq 1\}$ the k-dimensional ball. We also assume that $f_1(q), \ldots, f_k(q)$ are linearly independent in the domain under consideration.

In the case of $k = n - 1$, we give sufficient conditions in terms of Lie bracket relations for all optimal controls to be smooth or to have only isolated jump discontinuities.

Indeed, given $f_0, f_1, \ldots, f_{n-1}$ a generic germ of n-tuple of vector fields at q, then the germs of extremal trajectories at q may have only these less degenerate singularities. More precisely, let us define a vector $a \in \mathbb{R}^{n-1}$ and a matrix $A \in$ so$(n-1)$ by the formulas:

$$a(q) = \{\det(f_1(q), \ldots, f_{n-1}(q), [f_0, f_i](q))\}_{i=1}^{n-1},$$

$$A(q) = \{\det(f_1(q), \ldots, f_{n-1}(q), [f_i, f_j](q))\}_{i,j=1}^{n-1},$$

where $[\cdot, \cdot]$ is a Lie bracket. We have the following:

Theorem 3.1.1. *If*

$$a(\bar{q}) \notin A(\bar{q}) S^{n-2}, \tag{3.1.2}$$

then there exists a neighbourhood $O_{\bar{q}}$ of $\bar{q}$ in M such that any time-optimal trajectory contained in $O_{\bar{q}}$ is piecewise smooth with no more than 1 non smoothness point.

Here $S^{n-2} = \{u \in \mathbb{R}^{n-1} : |u| = 1\}$ is the unit sphere.
If $n = 3,\ k = 2$, the inequality (3.1.2) reads:

$$\begin{aligned} \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_1](\bar{q})\right) + \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_2](\bar{q})\right) \neq \\ \neq \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_1, f_2](\bar{q})\right). \end{aligned} \tag{3.1.3}$$

In this case, we will see in Chapter 4 that the result of Theorem 3.1.1 follows from Theorem 4.2.1, and the cited result is a bit stronger than this. Indeed, assumption (3.1.3) is more restrictive than the assumption used in Theorem 4.2.1:

$$\operatorname{rank}\{f_1(\bar{q}), f_2(\bar{q}), [f_0, f_1](\bar{q}), [f_0, f_2](\bar{q}), [f_1, f_2](\bar{q})\} = 3.$$

This theory is based on the blow-up techniques and the structure of partially hyperbolic equilibria.

3.2 Statement of the result

Let us assume that $\dim M = n$ and study the time-optimal problem for the following system

$$\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M,\ u \in \mathcal{U}, \tag{3.2.1}$$

where $k < n,\ f_0, f_1, \ldots, f_k$ are smooth vector fields, and $U = \{u \in \mathbb{R}^k : |u| \leq 1\}$; we also assume that $f_1, \ldots, f_k$ are linearly independent in the domain under consideration, and $f_{ij} = [f_i, f_j]$ with $i, j \in \{0, 1, \ldots, k\}$.

Notation 3.2.1. Recalling Notation 2.2.4, let us introduce the following abbreviated notation: $H_{0I} := H_{0I}(\bar{\lambda}),\ H_{IJ} := H_{IJ}(\bar{\lambda})$, chosen an opportune $\bar{\lambda} \in \Lambda|_{\bar{q}}$.

In order to prove Theorem 3.1.1, we are going to study extremals for any control system of the form (3.2.1) with $k < n$ in a neighbourhood of $\bar{\lambda} \in \Lambda_{\bar{q}} \subseteq T^*_{\bar{q}}M$ such that

$$H_{0I} \notin H_{IJ}\, S^{k-1}, \tag{3.2.2}$$

where $S^{k-1} = \{u \in \mathbb{R}^k : |u| = 1\}$ is the unit sphere.

Remark 3.2.2. If $k = n-1$, we should choose $\bar{\lambda} = f_1(\bar{q}) \wedge \ldots \wedge f_{n-1}(\bar{q})$. One can notice that conditions (3.1.2) and (3.2.2) are equivalent.

From Corollary 2.2.8 we already know that every arc of a time-optimal trajectory, whose extremal lies out of Λ, is bang, and so smooth.
Thus, we are interested to study arcs of a time-optimal trajectories, whose extremals passes through Λ or lies in Λ.
The fist step is to investigate if our system admits singular arcs.

Proposition 3.2.3. *Assuming (3.2.2), there are no optimal extremals in $O_{\bar{\lambda}}$ that lie in the singular locus Λ for a time interval.*

Thanks to Proposition 3.2.3, if it holds (3.2.2), the description of optimal extremals in a neighbourhood of $\bar{\lambda}$ is essentially reduced to the study of the solutions of the Hamiltonian system with a discontinuous right-hand side, defined by the Hamiltonian $H(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + \ldots + h_k^2(\lambda)}$.

Theorem 3.2.4. *Assume that condition (3.2.2) is satisfied.*
If it holds

$$H_{0I} \notin H_{IJ}\overline{B^k}, \tag{3.2.3}$$

where $B^k = \{u \in \mathbb{R}^k : |u| < 1\}$, then there exists a neighbourhood $O_{\bar{\lambda}} \subset T^*M$ such that for any $z \in O_{\bar{\lambda}}$ and $\hat{t} > 0$ there exists a unique contained in $O_{\bar{\lambda}}$ extremal $t \mapsto \lambda(t, z)$ with the condition $\lambda(\hat{t}, z) = z$. Moreover, $\lambda(t, z)$ continuously depends on (t, z) and every extremal in $O_{\bar{\lambda}}$ that passes through the singular locus is piece-wise smooth with only one switching.
Besides that, if u is the control corresponding to the extremal that passes through $\bar{\lambda}$, and $\bar{t}$ is its switching time, we have:

$$u(\bar{t} \pm 0) = [\pm d\,\mathrm{Id} + H_{IJ}]^{-1} H_{0I}, \tag{3.2.4}$$

with $d > 0$ unique, uni vocally defined by the system and $\bar{\lambda}$, such that

$$\left\langle [d^2\,\mathrm{Id} - H_{IJ}^2]^{-1} H_{0I}, H_{0I} \right\rangle = 1. \tag{3.2.5}$$

If it holds

$$H_{0I} \in H_{IJ} B^k, \tag{3.2.6}$$

then there exists a neighbourhood $O_{\bar{\lambda}} \subset T^*M$ such that no one optimal extremal intersects singular locus in $O_{\bar{\lambda}}$.

Note that $H_{IJ}\overline{B^k} = H_{IJ}S^{k-1}$ if the matrix H_{IJ} is degenerate, and that this matrix is always degenerate for odd k. Hence, assuming (3.2.2), we have the following possibilities:

> It holds (3.2.3) if it is verified one of the following scenarios:
>
> (A) k is odd
>
> (B) k is even and H_{IJ} is degenerate
>
> (C') k is even, H_{IJ} is non-degenerate and $H_{0I} \notin H_{IJ}\overline{B^k}$.

> It holds (3.2.6) if it is verified the following scenario:
>
> (C'') k is even, H_{IJ} is non-degenerate and $H_{0I} \in H_{IJ}B^k$.

Remark 3.2.5. In general, the flow of switching extremals from Theorem 3.2.4 is not locally Lipschitz with respect to the initial value. Indeed, we found a simple counterexample for $n = 3$ $k = 2$

$$\dot{x} = \begin{pmatrix} 0 \\ 0 \\ \alpha x_1 \end{pmatrix} + u_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}$$

that can be easily generalized to any $k < n$.

Since the Pontryagin maximum principle is a necessary but not sufficient condition of optimality, even if we have found extremals that passes through the singular locus, we cannot guaranty that they are all optimal, namely that their projections in M are time-optimal trajectory. In some cases they are certainly optimal, in particular, for linear system with an equilibrium target, where to be an extremal is sufficient for optimality. We will study general case at Chapter 6.

3.3 Proof

In this Section we are going to present at first the proof of Theorem 3.2.4, secondly we are going to prove Proposition 3.2.3. All together, these statements contain Theorem 3.1.1.

3.3.1 Proof of Theorem 3.2.4

Let us present the Blow-up technique, in order to analyse the discontinuous right-hand side Hamiltonian system, defined by

$$H(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + \cdots + h_k^2(\lambda)}, \tag{3.3.1}$$

in a neighbourhood $O_{\bar{\lambda}}$ of $\bar{\lambda}$.

Blow-up technique

In view of the fact that this is a local problem in $O_{\bar{\lambda}} \subseteq T^*M$, it is very natural consider directly its local coordinates $(\xi, x) \in \mathbb{R}^{n*} \times \mathbb{R}^n$, such that $\bar{\lambda}$ corresponds to $(\bar{\xi}, \bar{x})$ with $\bar{x} = 0$. Hence,

$$H(\xi, x) = h_0(\xi, x) + \sqrt{h_1^2(\xi, x) + \ldots + h_k^2(\xi, x)}. \tag{3.3.2}$$

Since $f_1, \ldots, f_k$ are linearly independent everywhere, we can define $n - k$ never null vector fields $f_{k+1}, \ldots, f_n$, such that $\{f_1, \ldots, f_n\}$ form a basis at any $q \in M$, then we will have the corresponding $h_j(\xi, x) = \langle \xi, f_j(x) \rangle$, with $j = k+1, \ldots, n$. Therefore, we are allowed to consider the following smooth change of variables

$$\Phi : (\xi, x) \longrightarrow ((h_1, \ldots, h_n), x),$$

so the singular locus becomes the subspace

$$\Lambda = \{((h_1, \ldots, h_n), x) \,:\, h_1 = \ldots = h_k = 0\}.$$

Notation 3.3.1. In order not to do notations even more complicated, we call λ any point defined with respect to the new coordinates $((h_1, \ldots, h_n), x)$, and $\bar{\lambda}$ what corresponds to the singular point.

Thus, let us define the blow-up technique.

Definition 3.3.2. The *blow-up* technique is defined in the following way:
We make a change of variables: $(h_1, \ldots, h_k) = (\rho u_1, \ldots, \rho u_k)$ with $\rho \in \mathbb{R}^+$ and $(u_1, \ldots, u_k) \in S^{k-1}$. Instead of considering the components $h_1, \ldots, h_k$ of the singular point $\bar{\lambda}$ in Λ, as the point $(0, \ldots, 0)$ in the k-dimensional euclidean space, we will consider it as a sphere S^{k-1}, where $\{\rho = 0\}$.

$$((h_1, \ldots, h_k, h_{k+1}, \ldots, h_n), x) \longrightarrow ((\rho, u_1, \ldots, u_k, h_{k+1}, \ldots, h_n), x)$$

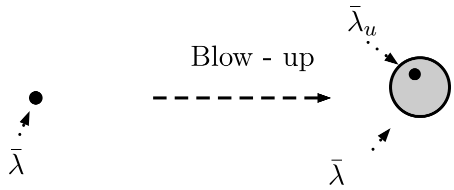

Figure 3.1: Blow-up technique

Let us notice that it is good to denote $u := (u_1, \ldots, u_k)$ the S^{k-1}-coordinates. As it is already know from Proposition 2.2.6, every optimal control $\tilde{u}$, that corresponds to an extremal $\lambda(t)$ out of Λ, satisfies formula (2.2.3): therefore $\tilde{u}$ lies on $\partial U = S^{k-1}$, and it is the normalization of the vector $(h_1(\lambda(t)), \ldots, h_k(\lambda(t)))$.
It is useful denote

$$f_u(x) = u_1 f_1(x) + \ldots + u_k f_k(x)$$

and $h_u(\lambda) = \langle \xi, f_u(x) \rangle$; and finally we can see that

$$h_u(\lambda) = \sqrt{h_1^2 + \ldots + h_k^2},$$

namely $h_u(\lambda) = \rho$, because $h_u(\lambda) = u_1 h_1 + \ldots + u_k h_k$, and $u_i = \frac{h_i}{\sqrt{h_1^2 + \ldots + h_k^2}}$ for all $i \in \{1, \ldots, k\}$.
Hence, with this new formulation the maximized Hamiltonian becomes

$$H(\lambda) = h_0(\lambda) + h_u(\lambda). \tag{3.3.3}$$

Thanks to Notation 2.2.4, Remarks 2.1.18 and 2.1.17, the Hamiltonian system has the following form:

$$\begin{cases} \dot{x} = f_0(x) + f_u(x) \\ \dot{\rho} = \langle H_{0I}(\lambda), u \rangle \\ \dot{u} = \frac{1}{\rho} \left(H_{0I}(\lambda) - \langle H_{0I}(\lambda), u \rangle \, u - H_{IJ}(\lambda) u \right) \\ \dot{h}_j = h_{0j}(\lambda) + h_{uj}(\lambda), \quad j \in \{k+1, \ldots, n\}. \end{cases} \tag{3.3.4}$$

Claim 3.3.3. *If assumption (3.2.3) is satisfied at the singular point* $\bar{\lambda}$*, then in* S^{k-1}

$$u \longmapsto H_{0I} - \langle H_{0I}, u\rangle\, u - H_{IJ}\, u \tag{3.3.5}$$

has two zeros u_+ *and* u_- *defined in the following way:*

$$u_\pm = [\pm d\,\mathrm{Id} + H_{IJ}]^{-1} H_{0I}, \tag{3.3.6}$$

with $d > 0$ *such that*

$$\left\langle [d^2\,\mathrm{Id} - H_{IJ}^2]^{-1} H_{0I}, H_{0I}\right\rangle = 1. \tag{3.3.7}$$

The function (3.3.5) has no zero if it holds assumption (3.2.6).

Proof. Denoting $Z := \langle H_{0I}, u\rangle$, we are looking for $u \in S^{k-1}$ and $Z \in \mathbb{R}$ such that

$$H_{0I} = (Z\,\mathrm{Id} + H_{IJ})u.$$

We already know that, if $Z = 0$, then there is no $u \in S^{k-1}$ such that $H_{0I} = H_{IJ}u$, by assumption (3.2.2). Moreover, since H_{IJ} is a skew-symmetric matrix, if $Z \neq 0$ then $(Z\,\mathrm{Id} + H_{IJ})$ is invertible, and

$$u = (Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}.$$

Let us consider the function

$$Z \longmapsto ||(Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2 \tag{3.3.8}$$

that will be continuous even and monotone in the domains $(-\infty, 0)$ and $(0, +\infty)$, because

$$||(Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2 = \left\langle [Z^2\,\mathrm{Id} - H_{IJ}^2]^{-1} H_{0I}, H_{0I}\right\rangle,$$

and its derivation with respect to Z^2 is negative

$$\frac{\mathrm{d}}{\mathrm{d}(Z^2)} \left\langle [Z^2\,\mathrm{Id} - H_{IJ}^2]^{-1} H_{0I}, H_{0I}\right\rangle < 0.$$

Indeed, it holds

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}(Z^2)} \left\langle [Z^2\,\mathrm{Id} - H_{IJ}^2]^{-1} H_{0I}, H_{0I}\right\rangle &= -\left\langle [Z^2\,\mathrm{Id} - H_{IJ}^2]^{-2} H_{0I}, H_{0I}\right\rangle \\
&= -||[Z^2\mathrm{Id} - H_{IJ}^2]^{-1} H_{0I}||^2.
\end{aligned}$$

We are going to verify if and in which cases the function (3.3.8) takes value 1 two or zero times. Thus, let us compute the limits of $||(Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2$ as $Z \to \pm\infty$ or $Z \to 0^\pm$.

At first one can observe that,

$$\lim_{Z \to \pm\infty} ||(Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2 = 0^+.$$

In order to compute $\lim_{Z \to 0^\pm} ||(Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2$, let us assume that H_{IJ} is in the canonical Jordan form, without loss of generality: it is defined by $j \leq \lfloor \frac{n}{2} \rfloor$ 2×2 skew symmetric blocks with the following form

$$J_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix} \quad i \in \{1, \ldots, j\},$$

and the rest of the matrix is null.

Let H_{IJ} be a degenerate matrix. If H_{0I} does not belong to its image, namely $H_{0I} \notin H_{IJ}\mathbb{R}^k$, it holds

$$\lim_{Z\to 0^\pm} ||(Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2 = +\infty.$$

On the other hand, let us show that if $H_{0I} \in H_{IJ}\mathbb{R}^k$ the limit $\lim_{Z\to 0^\pm} ||(Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2$ is finite strictly grater that 1.
Since H_{IJ} is degenerate, it holds $H_{IJ}\overline{B}^k = H_{IJ}S^{k-1}$, then by condition (3.2.2) we have $H_{0I} \notin H_{IJ}\overline{B}^k$. Thus, given condition $H_{0I} \in H_{IJ}\mathbb{R}^k$ we have that for all X, such that $H_{0I} = H_{IJ}X$, it has norm strictly grater than 1.
Finally, let us define

$$X = \begin{pmatrix} J_1^{-1} & & & \\ & \ddots & & \\ & & J_j^{-1} & \\ & & & 0_{(n-2j)\times(n-2j)} \end{pmatrix} H_{0I},$$

and see, by construction, that

$$\lim_{Z\to 0^\pm} ||(Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2 = ||X||^2 > 1.$$

Hence, if H_{IJ} is degenerate, by monotonicity and continuity of (3.3.8), there will be a value $Z = d > 0$ such that

$$||(\pm d\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2 = 1.$$

It means that there exist u_+ and u_- zeros of the function (3.3.5) such that $|\langle H_{0I}, u_\pm\rangle| = d$. We will assume $\langle H_{0I}, u_+\rangle > 0$ and $\langle H_{0I}, u_-\rangle < 0$.
These facts happen in scenarios (A) and (B) of condition (3.2.3).

If H_{IJ} is a non-degenerate matrix, then (3.3.8) is a continuous function for all $Z \in \mathbb{R}$ and

$$\lim_{Z\to 0} ||(Z\,\mathrm{Id} + H_{IJ})^{-1} H_{0I}||^2 = ||H_{IJ}^{-1} H_{0I}||^2.$$

Thus, in this case the function (3.3.5) will have two or no zeros if and only if $||H_{IJ}^{-1}H_{0I}|| > 1$ or $||H_{IJ}^{-1}H_{0I}|| < 1$, namely $H_{0I} \notin H_{IJ}\overline{B^k}$ or $H_{0I} \in H_{IJ}B^k$.
These are, indeed, scenarios (C') and (C''). □

Case $H_{0I} \in H_{IJ}B^k$

Once we have seen that (3.3.5) have no zero in this case, let us present the following Lemma in order to prove Theorem 4.2.5 if $H_{0I} \in H_{IJ}B^k$.

Lemma 3.3.4. *Let us assume (3.2.2), (3.2.6) and give a neighbourhood $O_{\bar{\lambda}}$ small enough such that*

$$H_{0I}(\lambda) - \langle H_{0I}(\lambda), u\rangle\, u - H_{IJ}(\lambda)u \neq 0, \quad \forall \lambda_u \in \overline{O_{\bar{\lambda}}}.$$

Then there exist two constants $c > 0$ and $\alpha > 0$ such that every optimal extremal that lies for a time interval $I \subseteq [0, +\infty)$ in $O_{\bar{\lambda}}$ satisfies the following inequality: $\rho(t) \geq ce^{-t\alpha}\rho(0)$, for $t \in I$.

Proof. Let us call

$$v(\lambda) = H_{0I}(\lambda) - \langle H_{0I}(\lambda), u\rangle\, u - H_{IJ}(\lambda)u, \tag{3.3.9}$$

by construction, we can assume that for all $\lambda \in \overline{O_{\bar{\lambda}}}$ it holds

$$||v(\lambda)|| > 0.$$

Since in the compact set $\overline{O_{\bar{\lambda}}}$ the map $\lambda \to v(\lambda)$ is continuous and not null, then there exist constants $c_1 > 0$ and $c_2 > 0$ such that, for all $\lambda \in O_{\bar{\lambda}}$,

$$c_1 \geq ||v(\lambda)|| \geq c_2 > 0.$$

Given the extremal $\lambda(t)$ in $O_{\bar{\lambda}}$, we can observe that

$$\frac{d}{dt}\rho(t)||v(\lambda(t))|| = \rho(t)\frac{\langle v(\lambda(t)), A(\lambda(t))\rangle}{||v(\lambda(t))||} = \rho(t)\tilde{A}(\lambda(t))$$

where

$$A(\lambda(t)) = \dot{H}_{0I}(\lambda(t)) - \left\langle \dot{H}_{0I}(\lambda(t)), u(t)\right\rangle u(t) - \dot{H}_{IJ}(\lambda(t))u(t).$$

Let us notice that for any Hamiltonian $h(\lambda)$ its time-derivative along $\lambda(t)$ is

$$\begin{aligned}\dot{h}(\lambda(t)) = \{h_0 + \rho, h\}(\lambda(t)) &= \{h_0, h\}(\lambda(t)) + \{\rho, h\}(\lambda(t)) \\ &= \{h_0, h\}(\lambda(t)) + \tfrac{1}{\rho}\textstyle\sum_{i=1}^k h_i(\lambda(t))\{h_i, h\}(\lambda(t)) \\ &= \{h_0, h\}(\lambda(t)) + \textstyle\sum_{i=1}^k u_i(t)\{h_i, h\}(\lambda(t))\end{aligned}$$

and it is bounded.
As a consequence each component of $A(\lambda(t))$ is bounded too, and $\tilde{A}_{|O}$ is bounded from below by a negative constant C

$$\tilde{A}_{|O} \geq C.$$

Finally, we can see that

$$\frac{d}{dt}\left[\frac{\rho(t)||v(\lambda(t))||}{\exp\left(\int_0^t C\,[||v(\lambda(s))||]^{-1}\,ds\right)}\right] \geq 0,$$

hence, for each $t \geq \tau_1$, by the monotonicity:

$$\begin{aligned}\rho(t) &\geq \rho(\tau_1)\,\tfrac{||v(\lambda(\tau_1))||}{||v(\lambda(t))||}\,\exp\left(\textstyle\int_{\tau_1}^t C\,[||v(\lambda(s))||]^{-1}\,ds\right) \\ &\geq \rho(\tau_1)\,\tfrac{c_2}{c_1}\,\exp\left(\tfrac{C}{c_2}(t-\tau_1)\right).\end{aligned}$$

Denoting $c := \frac{c_2}{c_1}$ and $\alpha := -\frac{C}{c_2}$, the thesis follows. □

This Lemma proves Theorem 4.2.5 if $H_{0I} \in H_{IJ}B^k$, because it shows that, given those conditions, every optimal extremal in $O_{\bar{\lambda}}$ does not intersect the singular locus in finite time, and forms a smooth local flow.

Case $H_{0I} \notin H_{IJ}\overline{B^k}$

Proposition 3.3.5. *Given condition (3.2.2) and assumption (3.2.3), there exists a unique extremal that passes through* $\bar{\lambda}$ *in finite time.*

Proof. Let us prove that there is a unique solution of the system (3.3.4) passing through its point of discontinuity $\bar{\lambda}$ in finite time.
In order to detect solutions that go through $\bar{\lambda}$, we rescale the time considering the time $t(s)$ such that $\frac{d}{ds}t(s) = \rho(s)$ and we obtain the following system

$$\begin{cases} x' = \rho\left(f_0(x) + f_u(x)\right) \\ \rho' = \rho\left\langle H_{0I}(\lambda), u\right\rangle \\ u' = H_{0I}(\lambda) - \left\langle H_{0I}(\lambda), u\right\rangle u - H_{IJ}(\lambda)u \\ h'_j = \rho\left(h_{0j}(\lambda) + h_{uj}(\lambda)\right), \quad j \in \{k+1, \ldots, n\}. \end{cases} \tag{3.3.10}$$

with a smooth right-hand side.
This system has an invariant subset $\{\rho = 0\}$ in which only the u-component is moving. Moreover, as we saw from Claim 3.3.3, at $\bar{\lambda} \in \{\rho = 0\}$ there are two equilibria $\bar{\lambda}_{u_-}$ and $\bar{\lambda}_{u_+}$, such that $\langle H_{0I}, u_+\rangle > 0$ and $\langle H_{0I}, u_-\rangle < 0$.

Let us present the Shoshitaishvili's Theorem [22] that explain how is the behaviour of the solutions in $O_{\bar{\lambda}_{u_-}}$ and $O_{\bar{\lambda}_{u_+}}$ neighbourhoods of the equilibria $\bar{\lambda}_{u_-}$ and $\bar{\lambda}_{u_+}$ in T^*M.

Theorem 3.3.6 (Shoshitaishvili's Theorem)**.** *In a n-dimensional manifold N with $\lambda \in N$, let*

$$\dot{\lambda} = f(\lambda) \tag{3.3.11}$$

a dynamical system in N, where $f \in \mathcal{C}^k(N)$, $2 \le k < \infty$. Given $\bar{\lambda} \in N$ there exists an opportune neighbourhood $O_{\bar{\lambda}}$ such that, via the coordinate chart, (3.3.11) is described by the following system in $\mathbb{R}^n$

$$\dot{z} = Bz + r(z), \quad z \in \mathbb{R}^n, \tag{3.3.12}$$

where $r \in C^k(\mathbb{R}^n)$, $r(0) = 0$, $\partial_z r_{|0} = 0$, and $B : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator whose eigenvalues are divided into three groups:

$$\begin{array}{rcl} \mathrm{I} & = & \{\mu_i, 1 \le i \le k^0 \,|\, \mathrm{Re}\mu_i = 0\} \\ \mathrm{II} & = & \{\mu_i, k^0 + 1 \le i \le k^0 + k^- \,|\, \mathrm{Re}\mu_i < 0\} \\ \mathrm{III} & = & \{\mu_i, k^0 + k^- + 1 \le i \le k^0 + k^- + k^+ \,|\, \mathrm{Re}\mu_i > 0\} \end{array}$$

$$k^0 + k^- + k^+ = n.$$

Let the subspaces of $\mathbb{R}^n$, which are invariant with respect B and which correspond to these groups be denoted by X, Y^- and Y^+ respectively, and let $Y^- \times Y^+$ be denoted by Y.
Then the following assertions are true:

1. *There exists a $\mathcal{C}^{k-1}$ manifold γ^0 that is invariant with respect to (3.3.11), may be given by the graph of mapping $\gamma^0 : X \to Y$, $y = \gamma^0(x)$, and satisfies $\gamma^0(0) = 0$ and $\partial_x \gamma^0(0) = 0$.*

2. *The system (3.3.11) in $O_{\bar{\lambda}}$ is homeomorphic to the product of the multidimensional saddle $\dot{y}^+ = y^+$, $\dot{y}^- = -y^-$, and*

$$\dot{x} = \hat{B}x + r_1(x)$$

where $r_1(x)$ is the x-component of the vector $r(z)$, $z = (x, \gamma^0(x))$, i.e. (3.3.11)in $O_{\bar{\lambda}}$ is homeomorphic to the system

$$\begin{cases} \dot{y}^+ = y^+, \quad \dot{y}^- = -y^- \\ \dot{x} = \hat{B}x + r_1(x). \end{cases}$$

Due to the fact that $\bar{\lambda}_{u_-}$ and $\bar{\lambda}_{u_+}$ belong to the invariant subset $\{\rho = 0\}$, where the components ρ, h_j with $j \in \{k+1, \dots, n\}$ and x are fixed, we can observe that Jacobian matrix of (3.3.10) have the following eigenvalues: $\langle H_{0I}, u_\pm \rangle$ that corresponds to the ρ-coordinate, the eigenvalues of the matrix $\partial_u v(\bar{\lambda}_u)_{|\bar{\lambda}_{u_\pm}}$, recalling notation (3.3.9), that correspond to the u-coordinate, and $2n-k$ 0-eigenvalues corresponding to the other coordinates.

Thus, let us study $\partial_u v(\bar{\lambda}_u)_{|\bar{\lambda}_{u_\pm}}$ that has the following form

$$\partial_u v(\bar{\lambda}_u)_{|\bar{\lambda}_{u_\pm}} = -\left[\langle H_{0I}, u_\pm \rangle \, \mathrm{Id} + H_{IJ} + u_\pm \, H_{0I}^T\right] \tag{3.3.13}$$

where H_{0I}^T is the row vector.
Let us prove that the real part of its eigenvalues is equal $-\langle H_{0I}, u_\pm \rangle$.

Let $\alpha + i\beta$ be an eigenvalue of $\partial_u v(\bar{\lambda}_u)_{|\bar{\lambda}_{u_\pm}}$ with $w_R + iw_I \neq 0$ eigenvector, as a consequence we can claim that

$$\begin{cases} \partial_u v(\bar{\lambda}_u)_{|\bar{\lambda}_{u_\pm}} w_R = \alpha w_R - \beta w_I \\ \partial_u v(\bar{\lambda}_u)_{|\bar{\lambda}_{u_\pm}} w_I = \alpha w_I + \beta w_R. \end{cases}$$

Thus, it holds $\left\langle \partial_u v(\bar{\lambda}_u)_{|\bar{\lambda}_{u_\pm}} w_R, w_R \right\rangle + \left\langle \partial_u v(\bar{\lambda}_u)_{|\bar{\lambda}_{u_\pm}} w_I, w_I \right\rangle = \alpha(|w_R|^2 + |w_I|^2)$, and it implies

$$-\langle H_{0I}, u_\pm \rangle (|w_R|^2 + |w_I|^2) = \alpha(|w_R|^2 + |w_I|^2),$$

because w_R and w_I are orthogonal to $u_\pm$. Since $w_R + iw_I \neq 0$, it holds

$$\alpha = -\langle H_{0I}, u_\pm \rangle .$$

By Claim 3.3.3, we know that $\langle H_{0I}, u_- \rangle$ and $\langle H_{0I}, u_+ \rangle$ are not null with opposite sign. Hence, assuming $\langle H_{0I}, u_- \rangle < 0$, we can conclude that in a neighbourhood of $\bar{\lambda}_{u_-}$ there is a stable 1-dimensional submanifold with respect to ρ and an unstable submanifold with respect to u. Analogously in a neighbourhood of $\bar{\lambda}_{u_+}$, we can notice the unstable 1-dimensional submanifold with respect to ρ and the stable one with respect to u.

Central manifolds γ^0 of Theorem 3.3.6 applied to the equilibria $\bar{\lambda}_{u_\pm}$ are $(2n-k)$-dimensional submanifolds defined by the equations $\rho = 0,\ u = u_\pm$. The dynamics on the central manifold is trivial: all points are equilibria.

Hence, according to the Shoshitaishvili theorem, there is a trajectory from the one-dimensional asymptotically stable invariant submanifold that tends to the equilibrium point $\bar{\lambda}_{u_-}$ as $s \to +\infty$, and analogously there is a trajectory from the one-dimensional asymptotically unstable invariant submanifold that escapes from the equilibrium point $\bar{\lambda}_{u_+}$ as $s \to -\infty$.

In order to obtain that exactly one solution of (3.3.10) enters submanifold $\rho = 0$ at $\bar{\lambda}_{u_-}$ and exactly one goes out of this submanifold at $\bar{\lambda}_{u_+}$, let us present together with Shoshitaishvili theorem the following Proposition 3.3.7, that shows the behaviour of solutions with rescaled time s, in the subset $\{\rho = 0\}$ where only the u-component is moving with respect to the equation

$$u' = H_{0I} - \langle H_{0I}, u\rangle\, u - H_{IJ}u. \tag{3.3.14}$$

Then it is completely described the whole phase portrait of the system (3.3.10).

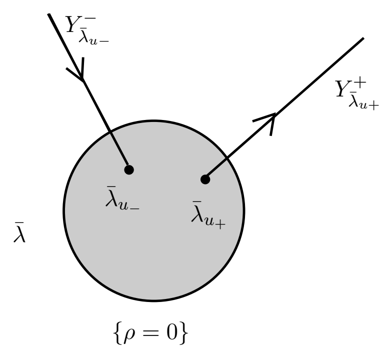

Figure 3.2: Solution of (3.3.10) that passes through $\bar{\lambda} \in \Lambda$.

Proposition 3.3.7. *Let $u(s)$, $s \in \mathbb{R}$, be a solution of system (3.3.14) that is not an equilibrium. Then $u(s) \to u_\pm$ as $s \to \pm\infty$.*

Proof. Let $y(t)$ be a solution of the system $\dot{y} = |y|H_{0I} - H_{IJ}y,\ y \in \mathbb{R}^k$, then $u(t) = \frac{1}{|y(t)|}y(t)$ satisfies system (3.3.14). Consider a linear $(k+1)$-dimensional system

$$\dot{x} = \langle H_{0I}, y\rangle, \quad \dot{y} = xH_{0I} - H_{IJ}y. \tag{3.3.15}$$

Its solutions preserve the Lorentz form $Q(x, y) = x^2 - |y|^2$ and, in particular, the cone

$$C = \{(x, y) \in \mathbb{R}^{k+1} : x^2 = |y|^2\}.$$

We obtain that $s \mapsto y(s)$ is a solution of system $\dot{y} = |y|H_{0I} - H_{IJ}y,\ y \in \mathbb{R}^k$ if and only if $s \mapsto (|y(s)|, y(s))$ is a solution of (3.3.15).

System (3.3.15) has a form $\dot{z} = Bz$, where $z = (x, y)$ and B is a $(k+1) \times (k+1)$-matrix. Moreover, vectors $(1, u_\pm)$ are eigenvectors of the matrix B with eigenvalues $\langle H_{0I}, u_\pm \rangle$. System $\dot{z} = Bz$ preserves any invariant subspace of B and in particular hyperplanes $T_{(1,u_\pm)}C$. Note that the projectivization of C is a strictly convex cone, hence $C \cap T_{(1,u_\pm)}C = \mathrm{span}\{(1, u_\pm)\}$.

We obtain that a co-dimension two subspace $E = T_{(1,u_\pm)}C \cap T_{(1,u_\pm)}C$ has zero intersection with C. It follows that quadratic form Q is sign-definite on the subspace E. Hence all solutions of system $\dot{z} = Bz$ that belong to the invariant subspace E are bounded for both positive and negative time. Any solution of system $\dot{z} = Bz$ has a form:

$$s \mapsto c_+ e^{s\langle H_{0I}, u_+ \rangle}(1, u_+) + c_- e^{s\langle H_{0I}, u_- \rangle}(1, u_-) + e(s),$$

where $e(s) \in E$. Recall that $\langle H_{0I}, u_+ \rangle$ is positive and $\langle H_{0I}, u_- \rangle$ is negative. Collecting now all the information we obtain that any nonzero solution of system $\dot{z} = Bz$ that belong to the invariant cone C asymptotically tends to the line $\mathrm{span}\{(1, u_\pm)\}$ as $s \to \pm\infty$.

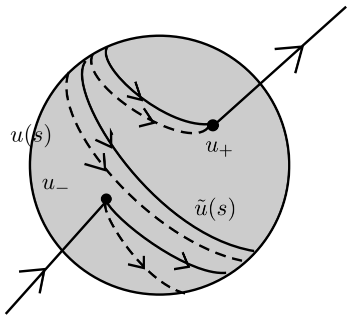

Figure 3.3: Two distinct solution $u(s)$ and $\tilde{u}(s)$ of (3.3.14).

□

Once we have study the system (3.3.10) with rescaled time s, we are going to show that the trajectory that we found, which enters in $\bar{\lambda}_{u^-}$ and goes out from $\bar{\lambda}_{u^+}$, is an extremal of the system (3.3.4) that passes through $\bar{\lambda}$ in finite time.

Thus, let us estimate the time Δt that this extremal needs to reach $\bar{\lambda}$.

Due to the facts that $\langle H_{0I}, u_- \rangle < 0$ and $\langle H_{0I}(\lambda), u \rangle$ at $\bar{\lambda}_{u_-}$ is continuous with respect to λ_u, there exist a neighbourhood $O_{\bar{\lambda}_{u-}}$ of $\bar{\lambda}_{u_-}$, in which $\langle H_{0I}(\lambda), u \rangle$ is bounded from above by a negative constant $c_1 < 0$, namely $\langle H_{0I}(\lambda), u \rangle_{|O_{\bar{\lambda}_{u-}}} < c_1 < 0$.
Hence, in $O_{\bar{\lambda}_{u-}}$ we have the following estimate of the derivative ρ'

$$\rho' = \rho \, \langle H_{0I}(\lambda), u \rangle < \rho \, c_1,$$

consequently until $\rho(s) > 0$, it holds

$$\int_{s_0}^{s} \frac{\rho'}{\rho} ds < \int_{s_0}^{s} c_1 ds,$$

then this inequality implies $\log(\rho(s)) < c_1(s - s_0) + \log(\rho(s_0))$, and so

$$\rho(s) < \rho(s_0)e^{c_1(s-s_0)}.$$

Since $\frac{d}{ds}t(s) = \rho(s)$, the amount of time that we want to estimate is the following

$$\Delta t = \lim_{s\to\infty} t(s) - t(s_0) = \int_{s_0}^{\infty} \rho(s)ds,$$

therefore,

$$\Delta t = \int_{s_0}^{\infty} \rho(s)ds < \rho(s_0)\int_{s_0}^{\infty} e^{c_1(s-s_0)}ds = \frac{\rho(s_0)}{-c_1} < \infty.$$

The amount of time in which this extremal goes out from $\bar{\lambda}$ may be estimate in an analogous way. □

By the previous proposition and the fact that every extremal out of Λ is smooth, it is proven that there exist a neighbourhood $O_{\bar{\lambda}} \subset T^*M$ such that for any $z \in O_{\bar{\lambda}}$ and $\hat{t} > 0$ there exists a unique extremal $t \mapsto \lambda(t, z)$ contained in $O_{\bar{\lambda}} \subset T^*M$ with condition $\lambda(\hat{t}, z) = z$.

Let us conclude the proof with the following proposition.

Proposition 3.3.8. *The map $(t, z) \to \lambda(t, z)$ continuously depends on $(t, z) \in I \times O_{\lambda}$.*

Proof. At first let us observe that for all singular point $\lambda \in O_{\bar{\lambda}}$ the phase portrait in the rescaled time after blow up have the same structure. Moreover, the splitting of the phase space on the hyperbolic and central part continuously depend on λ. This follows from basic facts on invariant submanifold, see [12] for details.

To guarantee continuity of the map $(t, z) \mapsto \lambda(t, z)$ it remains to prove that for each $\varepsilon > 0$ there exists a neighbourhood $O^{\varepsilon}_{\bar{\lambda}}$ such that the maximum time interval of the extremals in this neighbourhood $\Delta_{O^{\varepsilon}_{\bar{\lambda}}} t$ is less than ε.

As we saw previously, the solution of (3.3.10) through $\bar{\lambda}$ arrives and goes out at u_- and u_+. Let us fix two neighbourhoods $O_{\bar{\lambda}_{u_+}}$ of $\bar{\lambda}_{u_+}$ and $O_{\bar{\lambda}_{u_-}}$ of $\bar{\lambda}_{u_-}$, we can distinguish three parts of any trajectory close to $\bar{\lambda}$: the parts in $O_{\bar{\lambda}_{u_-}}$ and in $O_{\bar{\lambda}_{u_+}}$, and the part between those neighbourhoods.
In this last region, since each ρ-component is close to 0 and the corresponding time interval with time s is uniformly bounded, as we saw in Proposition 3.3.7, then Δt is arbitrarily small with respect to $O_{\bar{\lambda}}$.
Hence, in $O_{\bar{\lambda}_{u_-}}$ we are going to show that there exists a sequence of neighbourhoods of $\bar{\lambda}_{u_-}$

$$\left(O^R_{u_-}\right)_R,$$

such that

$$\lim_{R\to 0^+} \Delta_{O^R_{u_-}} t = 0.$$

For simplicity, we are going to prove this fact in $O_{\bar{\lambda}_{u_-}}$, because the situations in $O_{\bar{\lambda}_{u_+}}$ is equivalent.

Let us denote $O^R_{u_-}$ a neighbourhood of $\bar{\lambda}_{u_-}$ such that $O^R_{u_-} \subseteq O_{\bar{\lambda}_{u_-}}$, for each $\lambda \in O^R_{u_-}$ $\rho < R$ and $||u - u_-|| < R$. Therefore, we can define

$$M_R = \sup_{\lambda \in O^R_{u_-}} \langle H_{0I}(\lambda), u \rangle \,,$$

and assume that it is strictly negative and finite, due to the fact that we can choose $O_{\bar{\lambda}_{u_-}}$ in which $\langle H_{0I}(\lambda), u \rangle$ is strictly negative and finite.
Hence, for every $\lambda(t(s))$ in $O^R_{u_-}$, until its ρ-component is different that zero, it holds

$$\frac{\dot{\rho}(s)}{\rho(s)} < M_R,$$

then

$$\rho(s) < \rho(s_0) e^{M_R(s-s_0)},$$

for every $s > s_0$.
Consequently, $\Delta_{O^R_{u_-}} t$ can be estimated in the following way:

$$\Delta_{O^R_{u_-}} t < \int_{s_0}^{\infty} \rho(s_0) e^{M_R(s-s_0)} ds = \frac{\rho(s_0)}{-M_R} < \frac{R}{-M_R}.$$

Due to the fact that $\lim_{R \to 0^+} \frac{R}{-M_R} = 0$, we have proved that for each $\varepsilon > 0$ there exists $O^R_{u_-}$ such that $\Delta_{O^R_{u_-}} t < \varepsilon$. □

3.3.2 Proof of Proposition 3.2.3

Let us assume that there exist a time-optimal control $\tilde{u}$, and an interval (τ_1, τ_2) such that $\tilde{u}$ corresponds to an extremal $\lambda(t)$ in $O_{\bar{\lambda}}$, and $\lambda(t) \in \Lambda$, $\forall t \in (\tau_1, \tau_2)$. By construction, for $t \in (\tau_1, \tau_2)$ it holds

$$\begin{cases} \frac{d}{dt} h_1(\lambda(t)) = 0 \\ \vdots \\ \frac{d}{dt} h_k(\lambda(t)) = 0. \end{cases} \tag{3.3.16}$$

Since the maximized Hamiltonian associated with $\tilde{u}$ is

$$H_{\tilde{u}}(\lambda) = h_0(\lambda) + \tilde{u}_1 h_1(\lambda) + \ldots + \tilde{u}_k h_k(\lambda),$$

by Remark 2.1.18, (3.3.16) implies

$$H_{0I}(\lambda(t)) - H_{IJ}(\lambda(t))\tilde{u} = 0.$$

Moreover, due to condition (3.2.2),we can claim that, choosing $O_{\bar{\lambda}}$ small enough, $H_{0I}(\lambda(t)) \notin H_{IJ}(\lambda(t))\overline{B^k}$ or $H_{0I}(\lambda(t)) \in H_{IJ}(\lambda(t))B^k$, for all $t \in (\tau_1, \tau_2)$.
If $H_{0I}(\lambda(t)) \notin H_{IJ}(\lambda(t))\overline{B^k}$, we arrive to a contradiction, because in this case $||\tilde{u}|| > 1$ but the norm of admissible controls is less equal than 1. On the other hand, if $H_{0I}(\lambda(t)) \in H_{IJ}(\lambda(t))B^k$, such extremals might exist, but they are not optimal by the Goh Condition, presented at Subsection 2.1.2.

Chapter 4

Switching in time-optimal problem, the 3D case with 2D control

4.1 Introduction

In this chapter we are going to present a deeper result on local regularity of time-optimal problem for the affine control system in a 3-dimensional manifold M with control in a 2-dimensional disk U:

$$\dot{q} = f_0(q) + u_1 f_1(q) + u_2 f_2(q), \quad q \in M, \tag{4.1.1}$$

where f_0, f_1 and f_2 are smooth vector fields in M, (u_1, u_2) admissible controls taking values in $U = \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1\}$; we also assume that f_1 and f_2 are linearly independent, in the domain under consideration.

In Chapter 3, we proved that if $k = n - 1$ it is possible to avoid chattering trajectories in a neighbourhood $O_{\bar{q}}$ of $\bar{q}$, if we assume condition (3.1.2) at $\bar{q} \in M$.

If $n = 3$ and $k = 2$ (3.1.2) reads

$$\begin{aligned} \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_1](\bar{q})\right) + \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_2](\bar{q})\right) \neq \\ \neq \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_1, f_2](\bar{q})\right). \end{aligned} \tag{4.1.2}$$

In this chapter we are going to present a more complete result in this dimension, assuming the weaker condition

$$\operatorname{rank}\{f_1(\bar{q}), f_2(\bar{q}), f_{01}(\bar{q}), f_{02}(\bar{q}), f_{12}(\bar{q})\} = 3. \tag{4.1.3}$$

Since in Chapter 3 we have already showed the behaviour of all possible extremals in the neighbourhood $O_{\bar{\lambda}}$ of $\bar{\lambda} \in \Lambda$ such that $\pi(\bar{\lambda}) = \bar{q}$, if it holds (4.1.2), we will focus our attention in the case in which

$$\begin{aligned} \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_1](\bar{q})\right) + \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_0, f_2](\bar{q})\right) = \\ = \det{}^2\left(f_1(\bar{q}), f_2(\bar{q}), [f_1, f_2](\bar{q})\right) \end{aligned} \tag{4.1.4}$$

that is included in condition (4.1.3).

4.2 Statement

Theorem 4.2.1. *Let $\bar{q} \in M$; if it holds*

$$\operatorname{rank}\{f_1(\bar{q}), f_2(\bar{q}), f_{01}(\bar{q}), f_{02}(\bar{q}), f_{12}(\bar{q})\} = 3, \tag{4.2.1}$$

then there exists a neighbourhood $O_{\bar{q}}$ of $\bar{q}$ in M such that any time-optimal trajectory contained in $O_{\bar{q}}$ is bang-bang, with no more than one switching.

Let us give the following notation.

Notation 4.2.2. Let $\bar{\lambda} = f_1(\bar{q}) \times f_2(\bar{q}) \in \Lambda_{\bar{q}}$ and introduce the following abbreviated notations: $h_{ij} := h_{ij}(\bar{\lambda})$, $\forall i, j \in \{0, 1, 2\}$, and $r := (h_{01}^2 + h_{02}^2)^{1/2}$.

As we did in Chapter 3, we are going to study directly the behaviour of extremals in the cotangent bundle in the neighbourhood of $\bar{\lambda}$, that is any lift of $\bar{q}$ in $\Lambda_{\bar{q}} \subseteq T_{\bar{q}}^*M$, not null.
Let us give an equivalent condition to (4.2.1) at the point $\bar{\lambda}$.

Claim 4.2.3. *Given $\bar{\lambda} \in \Lambda_{\bar{q}} \subseteq T_{\bar{q}}^*M$, $\bar{\lambda} \neq 0$, equation (4.2.1) is equivalent to*

$$h_{01}^2 + h_{02}^2 + h_{12}^2 \neq 0,$$

namely,

$$r^2 + h_{12}^2 \neq 0. \tag{4.2.2}$$

Moreover, analogously it is possible to rewrite condition (4.1.2) at $\bar{\lambda}$, it is

$$r^2 \neq h_{12}^2. \tag{4.2.3}$$

Due to the homogeneity of any h_{ij} with respect to λ, inequalities (4.2.2) and (4.2.3) dos not depend on the choice of $\bar{\lambda} \in \Lambda_{\bar{q}}$.

Proof. Since by construction $\bar{\lambda}$ is orthogonal to $f_1(\bar{q})$ and $f_2(\bar{q})$, (4.2.1) will be true if and only if the valuers $h_{01}(\bar{\lambda})$ $h_{02}(\bar{\lambda})$ and $h_{12}(\bar{\lambda})$ can not be all null. On the other hand inequality (4.2.3) comes from definition of $\bar{\lambda}$. □

Let us recall with this simplified notation what was claimed by Proposition 3.2.3.

Proposition 4.2.4. *If $r^2 \neq h_{12}^2$ there are no optimal extremals in $O_{\bar{\lambda}}$ that lie in the singular locus Λ for a time interval. On the other hand, if $r^2 = h_{12}^2$ there might be arcs of optimal extremal contained in Λ.*

In the same way we have Theorem 3.2.4:

Theorem 4.2.5. *Suppose that $r^2 \neq h_{12}^2$.*
If

$$r^2 > h_{12}^2, \tag{4.2.4}$$

then there exists a neighbourhood $O_{\bar{\lambda}} \subset T^*M$ such that for any $z \in O_{\bar{\lambda}}$ and $\hat{t} > 0$ there exists a unique contained in $O_{\bar{\lambda}}$ extremal $t \mapsto \lambda(t, z)$ with the condition $\lambda(\hat{t}, z) = z$. Moreover, $\lambda(t, z)$ continuously depends on (t, z) and every extremal in $O_{\bar{\lambda}}$ that *passes through the singular locus is piece-wise smooth with only one switching. Besides that, we have:*

$$u(\bar{t} \pm 0) = \frac{1}{r^2}\left(-h_{02}h_{12} \pm h_{01}(r^2 - h_{12}^2)^{\frac{1}{2}}, h_{01}h_{12} \pm h_{02}(r^2 - h_{12}^2)^{\frac{1}{2}}\right), \tag{4.2.5}$$

where u is the control correspondent to the extremal that passes through $\bar{\lambda}$, and $\bar{t}$ is its switching time. If

$$r^2 < h_{12}^2, \tag{4.2.6}$$

then there exists a neighbourhood $O_{\bar{\lambda}} \subset T^*M$ such that every optimal extremal does not intersect the singular locus in $O_{\bar{\lambda}}$; all the optimal trajectories which are close to $\bar{q}$ are smooth bang arcs.

Remark 4.2.6. We would like to stress the fact that formula (4.2.5) explicitly describes the jump of the time-optimal control at the switching point in terms of Lie brackets relations.
If the value h_{12} equals zero at the jump point, then the control reaches the antipodal point of the boundary of the disk. This happen at points where f_1 f_2 and f_{12} are linearly dependent.
Moreover, if the inequality $r^2 > h_{12}^2$ is close to being an equality the jump will be smaller and smaller.

In the limit case $r^2 = h_{12}^2$ we have the following result:

Proposition 4.2.7. *If*

$$r^2 = h_{12}^2, \tag{4.2.7}$$

there exists a neighbourhood of $\bar{q}$ such that any time-optimal trajectory that contains $\bar{q}$ and is contained in the neighbourhood is a bang arc. The correspondent extremal either remains out of the singular locus Λ, or lies in

$$\Lambda \cap \left\{\lambda \in T^*M \mid h_{01}^2(\lambda) + h_{02}^2(\lambda) = h_{12}^2(\lambda)\right\}. \tag{4.2.8}$$

Anyway, the correspondent optimal control will be smooth without any switching, taking values on the boundary of U, in both cases.

Remark 4.2.8. One can notice that the case, in which an extremal $\lambda(t)$ lies in (4.2.8) for a time interval, is very rare. Indeed, necessarily along the curve the following conditions (P_k) on (f_0, f_1, f_2) hold, i.e. tag the equalities as (P_k)

$$\frac{d^k}{dt^k}\left(h_{01}^2(\lambda(t)) + h_{02}^2(\lambda(t)) - h_{12}^2(\lambda(t))\right) = 0, \quad k \in \mathbb{N},$$

and it is easy to see that at least conditions (P_0) (P_1) and (P_2) are distinct and independent.

Theorem 4.2.1 is given by Proposition 4.2.4, Theorem 4.2.5 and Proposition 4.2.7. Since Proposition 4.2.4 and Theorem 4.2.5 are included in Proposition 3.2.3 and Theorem 3.2.4, hence it remains to prove Proposition 4.2.7.

4.3 Proof of Proposition 4.2.7

At first, let us remember that we have already proved by Proposition 4.2.4 that there could exist an optimal extremal $\lambda(t)$ contained in the singular locus Λ, in particular in (4.2.8) and the correspondent controls take values on the boundary of the disk U, with equation

$$\begin{cases} u_1(t) = -\frac{h_{02}(\lambda(t))}{h_{12}(\lambda(t))} \\ u_2(t) = \frac{h_{01}(\lambda(t))}{h_{12}(\lambda(t))} \end{cases}$$

in the corresponding time interval I.

Now, let us study the extremal out side the singular locus and analyse the Hamiltonian system if $n = 3$ and $k = 2$, it can be written in the following way

$$\begin{cases} \dot{x} = f_0(x) + f_\theta(x) \\ \dot{\rho} = \cos(\theta)h_{01}(\lambda) + \sin(\theta)h_{02}(\lambda) \\ \dot{\theta} = \frac{1}{\rho}\left[h_{12}(\lambda) + (-\sin(\theta)h_{01}(\lambda) + \cos(\theta)h_{02}(\lambda))\right] \\ \dot{h}_3 = h_{03} + h_{\theta 3}. \end{cases} \tag{4.3.1}$$

We are going to show that, given a time-optimal trajectory through $\bar{q}$, whose extremal has a point out of the singular locus, then it does not attain Λ in finite time.

At first, let us consider the neighbourhood $O_{\bar{\lambda}_{\bar{\theta}}}$ of $\bar{\lambda}_{\bar{\theta}}$ included in $O_{\bar{\lambda}}$, such that $\bar{\theta}$ is the unique angle such that $h_{12} + \cos(\bar{\theta})h_{02} - \sin(\bar{\theta})h_{01} = 0$. In this neighbourhood we are going to see that in $O_{\bar{\lambda}_{\bar{\theta}}}$ the ρ component of the extremal .
Without loss of generality, we assume that $\bar{\theta} = 0$.
We omit some routine details and focus on the essential part of the estimate. First we freeze slow coordinates x, h_3 and study the system (4.3.1) with only two variables ρ, θ. In the worst scenario we get the following system:

$$\begin{cases} \dot{\rho} = -\sin(\theta) - \rho \\ \dot{\theta} = \frac{1}{\rho}(1 - \cos(\theta)) + 1. \end{cases}$$

Consequently, the behaviour of ρ-component with respect the θ-component is described by the following equation:

$$\rho'(\theta) = \frac{-\rho(\sin(\theta) + \rho)}{1 - \cos(\theta) + \rho}. \tag{4.3.2}$$

With the next Lemma 4.3.1 we analyse (4.3.2) and prove that, on the θ-axis there exists an interval I containing 0, on which ρ has a positive increment for any sufficiently small initial condition $\rho(0) = \rho_0 > 0$.

Lemma 4.3.1. *Given $O_{\bar{\lambda}_{\bar{\theta}}}$, there exist $\eta > 0$ small enough and $\theta_1 > 0$, such that for every initial values $(\rho(0), \theta(0)) = (\rho_0, 0)$ with $\rho_0 \neq 0$, the solution of system (4.3.2) satisfies the following implication: if $\theta > \theta_1$ then*

$$\rho(-\theta) < \rho(\eta\,\theta).$$

Proof. Given any $\eta > 0$ and any solution of (4.3.2) $\rho(\theta)$, we are going to compare the behaviour of $\tilde{\rho}(\theta) = \rho(-\theta)$ and $\hat{\rho}(\theta) = \rho(\eta\,\theta)$ for $\theta > 0$.
They will be solutions for $\theta > 0$ of the following two systems

$$\tilde{\rho}'(\theta) = \frac{\tilde{\rho}(\tilde{\rho} - \sin(\theta))}{1 - \cos(\theta) + \tilde{\rho}}$$

and

$$\hat{\rho}'(\theta) = -\eta\frac{\hat{\rho}(\hat{\rho} + \sin(\eta\,\theta))}{1 - \cos(\eta\,\theta) + \hat{\rho}}.$$

We can see that $\tilde{\rho}'(0) > \hat{\rho}'(0)$, thus if θ is very small it holds $\tilde{\rho}(\theta) > \hat{\rho}(\theta)$.
On the other hand, let us notice that choosing $\eta > 0$ small there exists $\nu > 1$ such that if $\theta > \nu\,\rho$ then $\hat{\rho}'(\theta) > \tilde{\rho}'(\theta)$. By the classical theory of dynamical system, this implies that in the domain

$$\{(\rho, \theta)\,|\,\theta > \nu\rho\}$$

if $\hat{\rho}(\theta) > \tilde{\rho}(\theta)$ at a certain $\theta > 0$, then the inequality remains true for every bigger value.
In order to compare the behaviour of $\tilde{\rho}(\theta)$ and $\hat{\rho}(\theta)$ when ρ_0 tends to zero, we consider the following re-scaling:

$$\begin{cases} \theta = st \\ \tilde{\rho} = s + s^2 x(t) \\ \hat{\rho} = s + s^2 y(t) \end{cases}$$

where s is the initial value ρ_0 and $x(0) = y(0) = 0$.
One can easily notice that if s tends to 0 then

$$\begin{cases} x'(t) = 1 - t + O(s) \\ y'(t) = \eta(-1 - \eta t) + O(s), \end{cases}$$

hence, it holds

$$\begin{cases} x_0(t) = t - \frac{1}{2}t^2 + O(s) \\ y_0(t) = -\eta t - \frac{\eta^2}{2}t^2 + O(s), \end{cases}$$

and

$$x_0(t) - y_0(t) = t\left((1+\eta) - \frac{(1-\eta^2)}{2}t\right) + O(s).$$

Hence, there exist $T > 2\frac{1+\eta}{1-\eta^2} > 2$, such that, denoting ρ_0^{MAX} the maximum among the initial values ρ_0 in $O_{\bar{\lambda}_{\bar{\theta}}}$, and calling $\theta_1 = \rho_0^{\mathrm{MAX}} T$, it holds that if $\theta > \theta_1$ then $\tilde{\rho}(\theta) < \hat{\rho}(\theta)$, namely

$$\rho(-\theta) < \rho(\eta\,\theta).$$

□

Hence, given $\lambda(t)$ extremal that runs close to $\bar{\lambda}$, we can denote $J \subseteq [0, \infty)$ the time interval that the extremal needs to arrive in $\bar{\lambda}$.
We can assume that, until $\lambda(t)$ attains $\bar{\lambda}$, the θ-component is monotone $\dot{\theta} > 0$, and that $\lambda(0) \in O_{\bar{\lambda}} \setminus \overline{O_{\bar{\lambda}_{\bar{\theta}}}}$.

Let us denote $[t_i, t_{i+1}]$, with $i \in \{0, 2, 4, \ldots\}$ even, the sub intervals of J in which $\lambda(t) \in O_{\bar{\lambda}_{\bar{\theta}}}$ and the ρ component increases, as we saw in the previous lemma. Thus, J is the union of a infinite number of separated intervals.

Denoting $\rho(t)$ and $\theta(t)$ the ρ and θ-components of $\lambda(t)$, let us define $\hat{\lambda}(t)$ with components $\hat{\rho}(t)$ $\hat{\theta}(t)$ such that

$$\hat{\lambda}(0) = \lambda(0)$$

$$\hat{\rho}(t_i) = \hat{\rho}(t_{i+1}), \ \forall i \text{ even}$$

$$\hat{\theta}(t_j) = \theta(t_j), \ \forall j \in \mathbb{N}$$

and $\hat{\lambda}(t)$ satisfies the Hamiltonian system (4.3.1) in $O_{\bar{\lambda}} \setminus \overline{O_{\bar{\lambda}_{\bar{\theta}}}}$.

Now, we redefine $\hat{\lambda}(t)$ gluing together its arcs in $O_{\bar{\lambda}} \setminus \overline{O_{\bar{\lambda}_{\bar{\theta}}}}$, it will be a curve in $O_{\bar{\lambda}} \setminus \overline{O_{\bar{\lambda}_{\bar{\theta}}}}$ discontinuous in all its components except for the $\hat{\rho}$ component. We redefine $\hat{\lambda}(t)$ in this time interval $\widehat{J}$ denoted as follows: it is the interval

$$J \setminus \cup_{i \,\text{even}} [t_i, t_{i+1}),$$

gluing together all its parts.

By construction we have that

$$\rho(t) \geq \hat{\rho}(t), \quad \forall t \in \hat{J}.$$

With the following claim let us estimate $\hat{\rho}(t)$ with respect to $\hat{\rho}(0)$, that is equal to $\rho(0)$, in order to give a final estimate of $\rho(t)$ with respect to $\rho(0)$.

Claim 4.3.2. *There exist two constants $c > 0$ and $\alpha > 0$ such that $\hat{\rho}(t) \geq c\, e^{-t\alpha} \hat{\rho}(0)$ for all $t > 0$.*

Proof. Let us call

$$v(\lambda) = h_{12}(\lambda) + (-\sin(\theta) h_{01}(\lambda) + \cos(\theta) h_{02}(\lambda)).$$

By construction, $\forall \lambda \in O_{\bar{\lambda}} \setminus \overline{O_{\bar{\lambda}_{\bar{\theta}}}}$

$$v(\lambda) \neq 0,$$

moreover in $\overline{O_{\bar{\lambda}} \setminus \overline{O_{\bar{\lambda}_{\bar{\theta}}}}}$ the map $\lambda \to v(\lambda)$ is continuous and not null, then it is bounded and there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \geq v(\lambda) \geq c_2 > 0.$$

Given the $\hat{\lambda}(t)$ we can observe that

$$\frac{d}{dt} \hat{\rho}(t) v(\hat{\lambda}(t)) = \rho(t) A(\hat{\lambda}(t))$$

where

$$A(\hat{\lambda}(t)) = \dot{h}_{12}(\hat{\lambda}(t)) + \cos(\hat{\theta}(t)) \dot{h}_{02}(\hat{\lambda}(t)) - \sin(\hat{\theta}(t)) \dot{h}_{01}(\hat{\lambda}(t)).$$

Moreover, we can claim that $A(\hat{\lambda}(t))$ is bounded from below by a negative constant C

$$A(\hat{\lambda}(t)) \geq C.$$

Finally, we can see that

$$\frac{d}{dt}\left[\frac{\hat{\rho}(t)v(\hat{\lambda}(t))}{\exp\left(\int_0^t Cv(\hat{\lambda}(s))^{-1}ds\right)}\right] \geq 0,$$

hence, for each $t \geq 0$, by the monotonicity:

$$\begin{array}{rcl} \hat{\rho}(t) & \geq & \hat{\rho}(0)\,\frac{v(\hat{\lambda}(0))}{v(\hat{\lambda}(t))}\,\exp\left(\int_0^t C\left[\hat{\lambda}(s)\right]^{-1} ds\right) \\ & \geq & \hat{\rho}(0)\,\frac{c_2}{c_1}\,\exp\left(\frac{C}{c_2}t\right). \end{array}$$

Denoting $c := \frac{c_2}{c_1}$ and $\alpha := -\frac{C}{c_2}$, the thesis follows. $\square$

Thanks to Claim 4.3.2 and this construction we conclude that there exist $c > 0$ and $\alpha > 0$ such that

$$\rho(t) \geq \rho(0)\; ce^{-t\alpha}, \quad \forall t \in J$$

and prove that the extremal $\lambda(t)$ can not attain in finite time the singular point $\bar{\lambda}$.

Chapter 5

Switching in time-optimal problem: Linear control system

5.1 Introduction

In this Chapter we are going to consider the global regularity of the time-optimal problem for a linear control system in $\mathbb{R}^n$, that is a particular case of affine control system.

5.1.1 Linear control systems

Let us start with some Definition

Definition 5.1.1. A *linear control system* is a family of dynamical system

$$\dot{x} = Ax + c + Bu, \quad x \in \mathbb{R}^n \tag{5.1.1}$$

parametrized by admissible control u taking values in $U \subseteq \mathbb{R}^k$, where A and B are a real constant matrices $n \times n$ and $n \times k$, $c \in \mathbb{R}^n$ is constant too.
Let us assume B with the maximum rank: $\text{rank}(B) = k$. During this chapter, we will consider $B = (b_1, \dots, b_k)$ matrix formed by k linearly independent vector in $\mathbb{R}^n$.

Due to the fact that admissible controls are locally integrable, given an initial point $x_0 \in \mathbb{R}^n$ let us directly compute the corresponding solution $x(t, u, x_0)$:

$$\begin{array}{rcl} x(t,u,x_0) & = & e^{tA}\left(x_0 + \int_0^t e^{-\tau A}\left(Bu(\tau) + c\right) d\tau\right) \\ & = & e^{tA}x_0 + \frac{e^{tA} - Id}{A}c + e^{tA}\int_0^t e^{-\tau A} Bu(\tau) d\tau \end{array} \tag{5.1.2}$$

Let us analyse the attainable sets of this system considering $U = \mathbb{R}^k$ the whole k-dimensional space as the space of control parameters.

Definition 5.1.2. The system (5.1.1) on $\mathbb{R}^n$ is called *completely controllable* for all $t > 0$ if

$$A_{x_0}(t) = \mathbb{R}^n, \quad \forall x_0 \in \mathbb{R}^n.$$

This means that for any pair of point x_0 and x_1 in M there exists an admissible control $u(\cdot)$ such that $x(\cdot, u, x_0)$ of the control system goes from x_0 to x_1 in t time

$$x(0, u, x_0) = x_0, \quad x(t, u, x_0) = x_1.$$

The completely controllability of linear systems can be explicitly studied by the following observation. The affine mapping (5.1.2) is surjective if and only if its linear part

$$u \to e^{tA} \int_0^t e^{-\tau A} B u(\tau) d\tau \tag{5.1.3}$$

is onto. Moreover, it is surjective if and only if the map

$$u \to \int_0^t e^{-\tau A} B u(\tau) d\tau \tag{5.1.4}$$

is onto. Let us present the following Theorem that give an equivalent condition on the matrices A and B of the system that is equivalent to the completely controllability for a time $t > 0$. It is called Kalman's Criterion.

Theorem 5.1.3. Kalman's Criterion *The linear control system (5.1.1) is completely controllable for time $t \geq 0$ if and only if.*

$$\operatorname{rank}\{B, AB, \dots, A^{n-1}B\} = n, \tag{5.1.5}$$

where $\{B, AB, \dots, A^{n-1}\}$ is the Kalman's controllability matrix $n \times nk$ *formed by the columns of all those matrices.*

Proof. Let us assume by contradiction that (5.1.5) does not hold. Then, there exists $p \in \mathbb{R}^n$ not null such that

$$p^T A^j B = 0 \in \mathbb{R}^k, \tag{5.1.6}$$

for all $j \in \{0, \dots, n-1\}$. By the Cayley-Hamilton Theorem[1] we can claim that A^n is a linear combination of $A, \dots, A^{n-1}$, thus for all $\tilde{n} \geq n$ also $A^{\tilde{n}}$ is equal to a linear combination of $A, \dots, A^{n-1}$.
Hence, the assumption (5.1.6) implies

$$p^T A^i B = 0 \in \mathbb{R}^k$$

for all $i \in \mathbb{N}$.
Since the exponential matrix is defined in the following way

$$e^{-tA} = Id - tA + \frac{t^2}{2!} A^n + \dots + (-1)^n \frac{t^n}{n!} A^n + \dots,$$

[1]The characteristic polynomial of A is a monic polynomial and its degree is n, such that $p_A(\lambda) = \det(\lambda I - A)$. The Cayley-Hamilton Theorem proves that $p_A(A) = 0$, hence with opportune real α_j it holds

$$A^n = \sum_{j=1}^{n-1} \alpha_j A^j.$$

we have that

$$p^T e^{-tA} B = 0 \in \mathbb{R}^k.$$

As a result, there exists $p \in \mathbb{R}^n$ not null such that

$$p^T \left(\int_0^t e^{-\tau A} B u(\tau) d\tau \right) = 0,$$

for all admissible control $u \in \mathcal{U}$, and it gives the contradiction: the function (5.1.4) is not surjective in $\mathbb{R}^n$.

Vice-versa, if we assume by contradiction that the map (5.1.4) is not surjective, then there exists $p \in \mathbb{R}^n$ not null such that

$$p^T \left(\int_0^t e^{-\tau A} B u(\tau) d\tau \right) = 0 \tag{5.1.7}$$

for all admissible control $u \in \mathcal{U}$. In particular, let us chose the following $u \in \mathcal{U}$ not null only in the i-th component:

$$u(t) = (0, ..., 0, v_s(\tau), 0, ..., 0)$$

where

$$v_s(\tau) = \begin{cases} 1 & 0 \le \tau \le s \\ 0 & \tau > s \end{cases} .$$

Then the equation (5.1.7) becomes:

$$p^T \left(\int_0^s e^{-\tau A} b_i d\tau \right) = 0$$

with $s \in \mathbb{R}$ and b_i the i-th column of B.
Since it holds for all $s \in [0, t_1]$ and $i \in \{1, \dots, k\}$ we have

$$p^T e^{-tA} B = 0 \in \mathbb{R}^k, \quad \forall t \in [0, t_1]. \tag{5.1.8}$$

Let us, hence, derive this equality $n-1$ times at $t = 0$: we obtain that

$$p^T A^j B = 0, \quad \forall j \in \{0, \dots, n-1\}.$$

It gives a contradiction of the condition (5.1.5). □

5.2 Linear Time-optimal Problem, with control in a ball

Now, we are going to study an optimal problem for a linear control system, in particular let us consider the *linear time-optimal problem.*

The linear time-optimal problem for such a control system is the following

$$\begin{cases} \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathcal{U}, \\ x(0) = x_0, \quad x(t_1) = x_1 \in A_{x_0}(t_1), \quad x_0, x_1 \in \mathbb{R}^n \text{ fixed}, \\ t_1 \to \min, \end{cases} \tag{5.2.1}$$

where A B are constant matrices of order $n \times n$ and $n \times k$ respectively and u is an admissible control taking value in $U \subset \mathbb{R}^k$ strictly contained in $\mathbb{R}^k$.
In particular we are going to assume $U = \overline{B}^k$ the k-dimensional closed unitary ball.

Let us spend few words about this assumption.

As we saw in Chapter 2, we reduce the problem of optimality to the study of attainable sets. In order to have the existence of the optimal trajectories we have to guaranteed the compactness of the extended attainable sets that we denoted in Proposition 2.1.9 and Proposition 2.1.10.
Since we are considering time-optimal problem, it is enough study directly the attainable set of the system.
Thanks to the Filippov's Theorem, in order to have a compact attainable set A_{x_0} for any $x_0 \in \mathbb{R}^n$ initial point, we need to consider U compact and $f_U(q)$ convex.
From Chapter 2, we saw that there is a standard procedure, called Relaxation, which extends the velocity set $f_U(q)$ to its convex hull.
For the linear control system $\dot{x} = Ax + Bu$ with A B constant, that we are considering, we extend the velocity set $f_U(q)$ to its convex hull extending U to its convex hull of U.
An example of compact and convex set $\mathbb{R}^k$ is the polytope, that is the convex hull of a finite number of point $a_1, \dots, a_m$ in $\mathbb{R}^k$. Indeed, the classical linear time-optimal problem, that was studied and explained in the book [18], written by Pontryagin Boltyanskij Granmkrelidze and Mishchenko, consider the space of control parameters U as a polytope in $\mathbb{R}^k$.

In this Thesis, we are going to study directly the time-optimal problem with $U = \overline{B}^k$. Anyway let us notice that many of the results that we are going to show holds if U is compact and strictly convex with analytic boundary.

Let us recall the Pontryagin maximum principle in this setting, then we will present its consequences.

Theorem 5.2.1. (Pontryagin maximum principle) *Let $\tilde{u}(t)$ be, for $t \in [0, \infty)$, a time-optimal control, and $x_{\tilde{u}}(t)$ the corresponding trajectory.*
We define the Hamiltonian

$$h_u(x, p) = p^T(Ax + c + Bu)$$

with $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$, $u \in U$.
Then there exists $p(t) \in \mathbb{R}^n$ Lipschitzian curve never null, such that for almost every $t \in [0, \infty)$ it holds:

1.
$$\begin{cases} \dot{x} = Ax + c + Bu \\ \dot{p}^T = -p^T A \end{cases}$$

2. $h_{\tilde{u}(t)}(x(t), p(t)) = \max_{u \in U} h_u(x(t), p(t))$, *namely*
$$p(t)^T B\tilde{u}(t) = \max_{u \in U} p(t)^T Bu$$

3. $h_{\tilde{u}(t)}(p(t)) \geq 0$.

Let us stress the fact that the component $p(t)$ of the extremal $(x(t), p(t))$ is defined only by the matrix A and an initial value p_0:

$$p(t)^T = p_0^T e^{-tA}.$$

Remark 5.2.2. We can see that given the linear control system (5.2.1), by the maximality condition of the Pontryagin maximum principle if, for a time interval $\tilde{I}$, it holds

$$p(t)^T B \neq 0 \in \mathbb{R}^k, \quad \forall t \in \tilde{I}.$$

then the corresponding time-optimal control $\tilde{u}(t)$ is explicitly defined by the equation

$$\tilde{u}(t) = \frac{B^T p(t)}{||B^T p(t)||} \tag{5.2.2}$$

at $\tilde{I}$ and it will be analytic.

Regarding the problem to avoid the chattering phenomenon for the linear control system (5.2.1), we are going to see in this case the classical Bang-Bang Theorem: if the system satisfies the condition of completely controllability (5.1.5) then optimal controls $u(t)$ are piece-wise analytic.

Theorem 5.2.3. (Bang-Bang Theorem) *Given a linear control system (5.2.1) that satisfies the condition*

$$\text{rank}\{B, AB, \ldots, A^{n-1}B\} = n, \tag{5.2.3}$$

any of its time-optimal control $u(t)$ is piece-wise analytic.

Proof. We are going to show that if the system (5.2.1) satisfies (5.2.3) then there exists no accumulating series of time $(t_n)_n \subseteq [0, t_1]$ such that it holds

$$p(t_n)^T B = 0 \in \mathbb{R}^k, \quad \forall n \in \mathbb{N}. \tag{5.2.4}$$

By Remark 5.2.2, it implies the thesis.

Let us assume, by contradiction, that there exists an accumulating series of time $(t_n)_n$ such that it holds (5.2.4). Since it holds $p(t)^T = p_0^T e^{-tA}$, each component of the vector $p(t)^T B$ is analytic, then, as a consequence from (5.2.4), we have that

$$p(t)^T B = p_0^T e^{-tA} B \equiv 0, \quad \forall t \in [0, t_1].$$

Let us, hence, derive this equality $n-1$ times at $t = 0$: we obtain that

$$p_0^T A^j B = 0, \quad \forall j \in \{0, \ldots, n-1\},$$

and it gives a contradiction to the hypothesis (5.2.3), since $p_0 \neq 0$. □

Moreover, if $\bar{t}$ is a switching time, the corresponding extremal $(x(t), p(x))$ satisfies this equation at $\bar{t}$

$$p(\bar{t})^T B = 0 \in \mathbb{R}^k.$$

Therefore, let us show that the Bang-Bang Theorem implies the uniqueness of the time-optimal solution from any $x_0 \in \mathbb{R}^n$ and any $x_1 \in A_{x_0}$

Theorem 5.2.4. (Uniqueness of time-optimal solution) *Let $x_0 \in \mathbb{R}^n$ and $x_1 \in A_{x_0}$ via admissible trajectories of the linear control system (5.2.1) with condition (5.2.3), then the solution of the time-optimal problem is unique.*

Proof. We assume by contradiction that $u_1(t)$ e $u_2(t)$ are two time-optimal control such that $x_1 = x_{u_1}(t_1) = x_{u_2}(t_1)$, we have that

$$\begin{aligned} x_1 &= e^{t_1 A}\left(x_0 + \int_0^{t_1} e^{-tA} B u_1(t) dt\right) \\ &= e^{t_1 A}\left(x_0 + \int_0^{t_1} e^{-tA} B u_2(t) dt\right), \end{aligned}$$

thus,

$$\int_0^{t_1} e^{-tA} B u_1(t) dt = \int_0^{t_1} e^{-tA} B u_2(t) dt.$$

Let $p_1(t)^T = p_1(0)^T e^{-tA}$ be the curve defined by $u_1(t)$ via the Pontryagin maximum principle. Applying to the last equation $p_1(0)^T$ we will have

$$\int_0^{t_1} p_1(t)^T B u_1(t) dt = \int_0^{t_1} p_1(t)^T B u_2(t) dt. \tag{5.2.5}$$

From the maximality condition it holds

$$p_1(t)^T B u_1(t) = \max_{u \in U} p_1(t)^T B u$$

then

$$p_1(t)^T B u_1(t) \geq p_1(t)^T B u_2(t).$$

Due to (5.2.5) equality for almost every $t \geq 0$ we have

$$p_1(t)^T B u_1(t) = p_1(t)^T B u_2(t).$$

Hence, the analyticity of $p(t)$ implies $u_1(t) = u_2(t)$ for almost every $t \geq 0$. □

Now, we are going to show how is the jump of any optimal control (5.2.2) at the switching time: as we can imagine by what we studied in Chapter 3, the control jumps in the antipodal point of the ball.

Proposition 5.2.5. *Given $\bar{t}$ any switching time of the time-optimal control $\tilde{u}$, it holds*

$$\lim_{\epsilon \to 0^+} \tilde{u}(\bar{t} + \epsilon) = - \lim_{\epsilon \to 0^+} \tilde{u}(\bar{t} - \epsilon).$$

Proof. At first, let us write the Taylor expansion of $p(\bar{t} + \epsilon)$ and $p(\bar{t} - \epsilon)$ at $\epsilon = 0$ until the second order

$$\begin{aligned} p(\bar{t} + \epsilon) = p(\bar{t}) + e^{-\bar{t}A^T}(-A^T) p(0)\epsilon + o(\epsilon) \\ p(\bar{t} + \epsilon) = p(\bar{t}) + e^{-\bar{t}A^T} A^T p(0)\epsilon + o(\epsilon). \end{aligned}$$

We can see that, since $\tilde{u}(t)$ satisfies equation (5.2.2) it holds

$$\tilde{u}(\bar{t} \pm \epsilon)||B^T p(\bar{t} \pm \epsilon)|| = B^T p(\bar{t} \pm \epsilon)$$

and than by the hypothesis on $\bar{t}$ we have

$$\tilde{u}(\bar{t} \pm \epsilon)||B^T p(\bar{t} \pm \epsilon)|||= -B^T e^{-\bar{t}A^T} A^T p(0)\epsilon + o(\epsilon)$$

$$\tilde{u}(\bar{t} \pm \epsilon)||B^T p(\bar{t} \pm \epsilon)|| = B^T e^{-\bar{t}A^T} A^T p(0)\epsilon + o(\epsilon).$$

With the same computation we can notice that $||p(\bar{t}+\epsilon)^T B||$ and $||p(\bar{t}-\epsilon)^T B||$ coincide until the second order, and moreover they are $O(\epsilon)$.

Hence, if ϵ tends to 0^+ the thesis holds. □

5.3 Global number of switchings, if $k = n - 1$

Instead of the non linear case, given a linear control system (5.2.1) with condition (5.2.3) in $\mathbb{R}^n$ it is more clear how to investigate globally any time-optimal trajectory $x(t)$ for $t \in \mathbb{R}$.

In this Section we are going to investigate on the number of switching that an optimal control has if $k = n - 1$, namely, we have such a linear control system with co dimension 1 control.

Claim 5.3.1. *Given any couple $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\bar{p}^T B = 0$, we can give the equation of an extremal trajectory $x(t)$ that at time 0 it is in $\bar{x}$ and has a switching at that point.*

Proof. Let us define

$$p(t)^T = \bar{p}^T e^{-tA}, \quad t \in \mathbb{R}$$

then $\tilde{u}(t)$ and $x(t)$ are defined by (5.2.2) (5.2.1). □

Claim 5.3.2. *Given the extremal defined at Claim 5.3.1, there exists another switching time $\tilde{t} \in \mathbb{R}$, $\tilde{t} \neq 0$, if and only if*

$$e^{-\tilde{t}A^T}\bar{p} = \lambda\bar{p},$$

namely $\bar{p}$ is an eigenvector of $e^{-\tilde{t}A^T}$ with a real eigenvalue.

Proof. Let us observe that there exists another switching time $\tilde{t} \in \mathbb{R}$, $\tilde{t} \neq 0$, if and only if

$$p(\tilde{t})^T B = 0.$$

It means that $\tilde{t}$ is a switching time if and only if $p(t)$ is orthogonal to the subspace generated by the vector columns of the matrix B.

Since we assume that $k = n - 1$, the subspace of $\mathbb{R}^n$ generated by the vector columns of the matrix B has dimension $n-1$ and without loss of generality we claim that its orthogonal complement is defined by $\bar{p}$:

$$\langle\bar{p}\rangle = \{p \in \mathbb{R}^n \,|\, p^T B = 0 \in \mathbb{R}^n\}.$$

As a consequence, it holds that there exists another switching time $\tilde{t} \in \mathbb{R}$, $\tilde{t} \neq 0$, if and only if

$$p(\tilde{t}) \in \langle\bar{p}\rangle\,,$$

it means that there exists a coefficient $\lambda \in \mathbb{R}$ such that

$$\bar{p}^T e^{-\tilde{t}A} = \lambda \bar{p}^T$$

namely,

$$e^{-\tilde{t}A^T} \bar{p} = \lambda \bar{p}$$

$\bar{p}$ is an eigenvector of $e^{-\tilde{t}A^T}$, with $\tilde{t} \neq 0$, with a real eigenvalue. $\square$

Let us see the following Proposition

Proposition 5.3.3. *Given (5.2.1) with $k = n-1$, it does not hold condition (5.2.3) if and only if the vector $\bar{p}$, such that $\bar{p}^T B = 0$, is an eigenvector of A^T with real eigenvalue.*

Proof. Given $\bar{p} \neq 0$ such that $\bar{p}^T B = 0$, it holds condition (5.2.3)

$$\text{rank}\{B, AB, \ldots, A^{n-1}B\} = n,$$

if and only if there exists $j \in \{1, \ldots, n-1\}$ such that $\bar{p}^T A^j B \neq 0$, namely

$$B^T (A^T)^j \bar{p} \neq 0.$$

If, by contradiction, $\bar{p}$ is an eigenvector of A^T with eigenvalue $\mu \in \mathbb{R}$, then

$$(A^T)^j \bar{p} = \mu^j \bar{p}$$

for all $j \in \{1, \ldots, n-1\}$, and there is a contradiction with condition (5.2.3).

Viceversa, if the vector $\bar{p}$, such that $B^T \bar{p} = 0$, is not an eigenvector of A^T with real eigenvalue, then

$$(A)^T \bar{p} \notin \langle \bar{p} \rangle .$$

Hence, it holds $B^T A^T \bar{p} \neq 0$, and so

$$\text{rank}\{B, AB\} = n,$$

that means that it holds condition (5.2.3). $\square$

Remark 5.3.4. Given (5.2.1) with $k = n-1$, satisfying condition (5.2.3), the extremal defined at Claim 5.3.1 has another switching time $\tilde{t} \in \mathbb{R}$, $\tilde{t} \neq 0$, if the vector $\bar{p}$ is not an eigenvector of A^T with real eigenvalue but it is an eigenvector of $e^{-\tilde{t}A^T}$ with a real eigenvalue.

Hence we obtain the following Theorems:

Theorem 5.3.5. *Given (5.2.1) in $\mathbb{R}^n$ with $k = n-1$, satisfying condition (5.2.3),*

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n \tag{5.3.1}$$

it is always possible to give an optimal trajectory with a switching point at any given $\bar{x} \in \mathbb{R}^n$. Moreover, along that time-optimal trajectory this switching is unique, if A^T has only real eigenvalues.

Proof. We are going to directly consider the extremal denoted in Claim 5.3.1.

As we saw by Claim 5.3.2 and Proposition 5.3.3, we are going to see when it is satisfied what we claimed in Remark 5.3.4.

We analyse the problem with respect to the eigenvalues of A^T. Without loss of generality we consider the matrix A^T in canonical Jordan blocks.
Assuming A with only real eigenvalue, A^T has only real eigenvalues too. Let us distinguish and analyse if the eigenvalues are simple or not.

(1) If A^T has simple real eigenvalues.

We are going to see that, if A^T has simple real eigenvalues, then any time-optimal trajectories has at most a unique switching.

Let us consider $\lambda_1, \ldots, \lambda_n$ eigenvalues of A^T with $v_{\lambda_1}, \ldots, v_{\lambda_n}$ corresponding eigenvectors that form a basis of $\mathbb{R}^n$. Given $\bar{p}$ with this basis

$$\bar{p} = \bar{p}_1 v_{\lambda_1} + \ldots + \bar{p}_n v_{\lambda_n}, \quad \bar{p}_i \in \mathbb{R}, \quad i \in \{1, \ldots, n\},$$

we will have

$$e^{tA^T} \bar{p} = \sum_{i=1}^{n} \bar{p}_i e^{t\lambda_i} v_{\lambda_i}.$$

If, by contradiction, there exists $\tilde{t} \neq 0$ such that $e^{\tilde{t}A^T}$ has a real eigenvalue with respect to $\bar{p}$, then $\lambda_i = \lambda$ for all $i \in \{1, \ldots, n\}$ where $\bar{p}_i \neq 0$.

As a consequence, we have that $\bar{p}$ is a real eigenvector of A^T

$$A^T \bar{p} = \lambda \bar{p},$$

that is a contradiction.

(2) If A^T has real eigenvalues and some of them are not simple.

We are going to see that, if A^T has real eigenvalues and some of them are not simple, then any time-optimal trajectories has at most a unique switching.

Let us consider $\lambda_1, \ldots, \lambda_j$ with $j < n$, eigenvalues of A^T with $v_{\lambda_1}, \ldots, v_{\lambda_j}$ corresponding eigenvectors. Let us consider the generalized eigenvalues $v'_{\lambda_1}, \ldots, v'_{\lambda_{j_1}}$ of $v_{\lambda_1}, \ldots, v_{\lambda_{j_1}}$, with $j_1 < j$, moreover we will have $v''_{\lambda_1}, \ldots, v''_{\lambda_{j_2}}$ of $v'_{\lambda_1}, \ldots, v'_{\lambda_{j_2}}$, with $j_2 < j_1$, and so on.

Now, we assume that the basis of $\mathbb{R}^n$ with eigenvector and generalized eigenvector is the following: $v_{\lambda_1}, \ldots, v_{\lambda_j}, v'_{\lambda_1}, \ldots, v'_{\lambda_{j_1}}, v''_{\lambda_1}, \ldots, v''_{\lambda_{j_2}}$.

We can observe that

$$\begin{cases} A^T v_{\lambda_i} = \lambda_i v_{\lambda_i}, \quad i \in \{1, \ldots, j\} \\ \\ A^T v'_{\lambda_i} = \lambda_i v'_{\lambda_i} + v_{\lambda_i}, \quad i \in \{1, \ldots, j_1\} \\ \\ A^T v''_{\lambda_i} = \lambda_i v''_{\lambda_i} + v'_{\lambda_i}, \quad i \in \{1, \ldots, j_2\} \end{cases}$$

and

$$\begin{cases} e^{tA^T} v_{\lambda_i} = e^{t\lambda_i} v_{\lambda_i}, \quad i \in \{1, \ldots, j\} \\ e^{tA^T} v'_{\lambda_i} = e^{t\lambda_i} v'_{\lambda_i} + te^{t\lambda_i} v_{\lambda_i}, \quad i \in \{1, \ldots, j_1\} \\ e^{tA^T} v''_{\lambda_i} = e^{t\lambda_i} v''_{\lambda_i} + te^{t\lambda_i} v'_{\lambda_i} + t^2 e^{t\lambda_i} v_{\lambda_i}, \quad i \in \{1, \ldots, j_2\} \end{cases}$$

Let us give the following vector $\bar{p}$,

$$\bar{p} = \bar{p}_1 v_{\lambda_1} + \ldots + \bar{p}_j v_{\lambda_j} + \bar{p}_{j+1} v'_{\lambda_1} + \ldots + \bar{p}_{j+j_1} v'_{\lambda_{j_1}} + \bar{p}_{j+j_1+1} v''_{\lambda_1} + \ldots + \bar{p}_n v''_{\lambda_{j_2}},$$

with $\bar{p}_i \in \mathbb{R}$ $i \in \{1, \ldots, n\}$, if we assume that it is not an eigenvector of A^T with real eigenvalue, then it means that

$$\sum_{i=j+1}^{n} \bar{p}_i^2 \neq 0. \tag{5.3.2}$$

Hence, we have

$$\begin{aligned} e^{tA^T} \bar{p} = \; & \textstyle\sum_{i=1}^{j} \bar{p}_i e^{t\lambda_i} v_{\lambda_i} + \sum_{i=1}^{j_1} \bar{p}_{j+i} \left(e^{t\lambda_i} v'_{\lambda_i} + te^{t\lambda_i} v_{\lambda_i} \right) + \\ & + \textstyle\sum_{i=1}^{j_2} \bar{p}_{j+j_1+i} \left(e^{t\lambda_i} v''_{\lambda_i} + te^{t\lambda_i} v'_{\lambda_i} + t^2 e^{t\lambda_i} v_{\lambda_i} \right), \end{aligned}$$

thus,

$$\begin{aligned} e^{tA^T} \bar{p} \; = \; & \textstyle\sum_{i=1}^{j} e^{t\lambda_i} \bar{p}_i \, v_{\lambda_i} + \sum_{i=1}^{j_1} e^{t\lambda_i} \bar{p}_{j+i} \, v'_{\lambda_i} + \sum_{i=1}^{j_2} e^{t\lambda_i} \bar{p}_{j+j_1+i} \, v''_{\lambda_i} + \\ & + t \textstyle\sum_{i=1}^{j_1} e^{t\lambda_i} \bar{p}_{j+i} \, v_{\lambda_i} + t \sum_{i=1}^{j_2} e^{t\lambda_i} \bar{p}_{j+j_1+i} \, v'_{\lambda_i} + t^2 \sum_{i=1}^{j_2} e^{t\lambda_i} \bar{p}_{j+j_1+i} \, v_{\lambda_i}, \end{aligned}$$

if, by contradiction, there exists $\tilde{t} \neq 0$ such that $e^{\tilde{t}A^T}$ has a real eigenvalue with respect to $\bar{p}$, then $\lambda_i = \lambda$ for all $i \in \{1, \ldots, j\}$ where $\bar{p}_i \neq 0$, and $\bar{p}_{j+i} = 0$ for all $i \in \{1, \ldots, n-j\}$. And this contradicts inequality (5.3.2). □

Theorem 5.3.6. *Given (5.2.1) in $\mathbb{R}^n$ with $k = n - 1$, satisfying condition (5.2.3),*

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n \tag{5.3.3}$$

it is always possible to give an optimal trajectory with a switching point at any given $\bar{x} \in \mathbb{R}^n$. Assuming that

(3) *A has simple complex and real eigenvalues $\alpha_1 + i\beta_1$, $\alpha_1 - i\beta_1, \ldots, \alpha_j + i\beta_j$, $\alpha_j - i\beta_j$, $\lambda_{2j+1}, \ldots, \lambda_n$, such that $\beta_i \neq 0$ for all $i \in \{1, \ldots, j\}$ and $0 < j \leq \lfloor \frac{n}{2} \rfloor$,*

given the corresponding eigenvectors that form a basis $\mathcal{B}$ of $\mathbb{R}^n$, and $\bar{p}$ described with coordinates $(\bar{p}_i)_{i=1,\ldots n}$ in $\mathcal{B}$. There will be an infinite number of switchings in regular intervals if we assume that

- *there exists $\beta_A \in \mathbb{R} \setminus \{0\}$ and $K_i \in \mathbb{Q}$, $\forall i \in \{1, \ldots, j\}$, such that $\beta_i = K_i \beta_A$*

- $\alpha = \alpha_i$ *for all* $i \in \{1, \dots, j\}$ *where* $\bar{p}_{\alpha_i} \neq 0$, *and* $\alpha = \lambda_i$ *for all* $i \in \{2j+1, \dots, n\}$ *where* $\bar{p}_i \neq 0$.

Therefore, if A *has simple complex and real eigenvalues such that*

- *there exists* $\beta_A \in \mathbb{R} \setminus \{0\}$ *and* $K_i \in \mathbb{Q}$, $\forall i \in \{1, \dots, j\}$, *such that* $\beta_i = K_i \beta_A$
- $\alpha = \alpha_i$ *for all* $i \in \{1, \dots, j\}$, *and* $\alpha = \lambda_i$ *for all* $i \in \{2j+1, \dots, n\}$,

then for all matrix B, *every time-optimal trajectory will have infinite isolated switching, in regular time-interval.*

Proof. As we did in previous Theorem, we are going to directly consider the extremal denoted in Claim 5.3.1, and see when it is satisfied what we claimed in Remark 5.3.4. Without loss of generality we consider the matrix A^T in canonical Jordan blocks.

Let us start assuming that $j = 1$ in (3A), then we will study the general case (3B) with $0 < j \leq \lfloor \frac{n}{2} \rfloor$.

(3A) If A^T has a couple of complex eigenvalues and $n-2$ real simple eigenvalues.

Let us consider $\alpha + i\beta$, $\alpha - i\beta$, with $\beta \neq 0$, and $\lambda_3, \dots, \lambda_n$ eigenvalues of A^T with v_α, v_β and $v_{\lambda_3}, \dots, v_{\lambda_n}$ corresponding eigenvectors that form a basis of $\mathbb{R}^n$.

We can observe that

$$\begin{cases} A^T v_\alpha = \alpha v_\alpha + \beta v_\beta \\ A^T v_\beta = \alpha v_\beta - \beta v_\alpha \qquad i \in \{3, \dots, n\} \\ A^T v_{\lambda_i} = \lambda_i v_{\lambda_i} \end{cases}$$

and

$$\begin{cases} e^{tA^T} v_\alpha = e^{t\alpha} \cos(t\beta) v_\alpha + e^{t\alpha} \sin(t\beta) v_\beta \\ e^{tA^T} v_\beta = e^{t\alpha} \cos(t\beta) v_\beta - e^{t\alpha} \sin(t\beta) v_\alpha \qquad i \in \{3, \dots, n\}. \\ e^{tA^T} v_{\lambda_i} = e^{\lambda_i} v_{\lambda_i} \end{cases}$$

Given the vector $\bar{p}$ with basis $\mathcal{B}$

$$\bar{p} = \bar{p}_\alpha v_\alpha + \bar{p}_\beta v_\beta + \bar{p}_3 v_{\lambda_3} + \ldots + \bar{p}_n v_{\lambda_n}, \quad \bar{p}_\alpha, \bar{p}_\beta, \bar{p}_i \in \mathbb{R}, \quad \forall i \in \{3, \dots, n\},$$

if we say that it is not an eigenvector of A^T with a real eigenvalue, it means that $\bar{p}_\alpha^2 + \bar{p}_\beta^2 \neq 0$.

Hence, we can see that

$$\begin{aligned} e^{tA^T} \bar{p} = \;& \bar{p}_\alpha (e^{t\alpha} \cos(t\beta) v_\alpha + e^{t\alpha} \sin(t\beta) v_\beta) + \\ & + \bar{p}_\beta (e^{t\alpha} \cos(t\beta) v_\beta - e^{t\alpha} \sin(t\beta) v_\alpha) + \\ & + \textstyle\sum_{i=3}^n \bar{p}_i e^{t\lambda_i} v_{\lambda_i} \end{aligned}$$

and there exists $\tilde{t}$ sucht that $\bar{p}$ is an eigenvector with a real eigenvalue of $e^{\tilde{t}A^T}$ if and only if $\tilde{t} = \frac{2K\pi}{\beta}$ with any $K \in \mathbb{Z}$ and $\alpha = \lambda_i$ for all $i \in \{3, \dots, n\}$ where $\bar{p}_i \neq 0$. Therefore, we can see that there will be infinite isolated switching at switching time

$$\tilde{t}_K \beta = 2K\pi \quad K \in \mathbb{Z}.$$

in regular time-intervals.

(3B) If A^T has complex and real simple eigenvalues: $\alpha_1 + i\beta_1$, $\alpha_1 - i\beta_1, \dots, \alpha_j + i\beta_j$, $\alpha_j - i\beta_j$ and $\lambda_{2j+1}, \dots, \lambda_n$, with $\beta_i \neq 0$ for all $i \in \{1, \dots, j\}$ and $j \leq \lfloor \frac{n}{2} \rfloor$.

Let $v_{\alpha_1}, \dots, v_{\alpha_j}$, $v_{\beta_1}, \dots, v_{\beta_j}$ and $v_{\lambda_{2j}}, \dots, v_{\lambda_n}$ be the corresponding eigenvectors that form basis $\mathcal{B}$ of $\mathbb{R}^n$.

We can notice that

$$\begin{cases} A^T v_{\alpha_i} = \alpha_i v_{\alpha_i} + \beta_i v_{\beta_i} & i \in \{1, \dots, j\} \\ A^T v_{\beta_i} = \alpha_i v_{\beta_i} - \beta_i v_{\alpha_i} & \\ A^T v_{\lambda_i} = \lambda_i v_{\lambda_i} & i \in \{2j+1, \dots, n\} \end{cases}$$

and

$$\begin{cases} e^{tA^T} v_{\alpha_i} = e^{t\alpha_i} \cos(t\beta_i) v_{\alpha_i} + e^{t\alpha_i} \sin(t\beta_i) v_{\beta_i} & i \in \{1, \dots, j\} \\ e^{tA^T} v_{\beta_i} = e^{t\alpha_i} \cos(t\beta_i) v_{\beta_i} - e^{t\alpha_i} \sin(t\beta_i) v_{\alpha_i} & \\ e^{tA^T} v_{\lambda_i} = e^{t\lambda_i} v_{\lambda_i} & i \in \{2j+1, \dots, n\} \end{cases}$$

Given $\bar{p}$ with basis $\mathcal{B}$

$$\bar{p} = \bar{p}_{\alpha_1} v_{\alpha_1} + \bar{p}_{\beta_1} v_{\beta_1} + \dots + \bar{p}_{\alpha_j} v_{\alpha_j} + \bar{p}_{\beta_j} v_{\beta_j} + \bar{p}_{2j+1} v_{\lambda_{2j+1}} + \dots + \bar{p}_n v_{\lambda_n}, \quad \bar{p}_{\alpha_i}, \bar{p}_{\beta_i}, \bar{p}_i \in \mathbb{R},$$

we can see that if it is not an eigenvector of A^T with a real eigenvalue, it holds

$$\sum_{i=1}^{j} \left[\bar{p}_{\alpha_i}^2 + \bar{p}_{\beta_i}^2 \right] \neq 0.$$

Hence,

$$\begin{aligned} e^{tA^T} \bar{p} = & \textstyle\sum_{i=1}^{j} \left[\bar{p}_{\alpha_i} (e^{t\alpha_i} \cos(t\beta_i) v_{\alpha_i} + e^{t\alpha_i} \sin(t\beta_i) v_{\beta_i}) + \right. \\ & \left. + \bar{p}_{\beta_i} (e^{t\alpha_i} \cos(t\beta_i) v_{\beta_i} - e^{t\alpha_i} \sin(t\beta_i) v_{\alpha_i}) \right] + \\ & + \textstyle\sum_{i=2j+1}^{n} \bar{p}_i e^{t\lambda_i} v_{\lambda_i} \end{aligned}$$

and there exists $\tilde{t}$ such that $\bar{p}$ is an eigenvector with a real eigenvalue of $e^{\tilde{t}A^T}$ if and only if

- there exists $\beta_A \in \mathbb{R} \setminus \{0\}$ and $K_i \in \mathbb{Q}$, $\forall i \in \{1, \dots, j\}$, such that $\beta_i = K_i \beta_A$
- $\alpha = \alpha_i$ for all $i \in \{1, \dots, j\}$ where $\bar{p}_{\alpha_i} \neq 0$, and $\alpha = \lambda_i$ for all $i \in \{2j+1, \dots, n\}$ where $\bar{p}_i \neq 0$.

Indeed, given $K_i = \frac{m_i}{n_i}$ for $i \in \{1, \dots, j\}$, it is possible to denote the switching time $\tilde{t} = \frac{2\pi}{\beta_A} n_1 \dots n_j$, and for all $i \in \{1, \dots, j\}$

$$\tilde{t}\beta_i = 2\pi \cdot \underbrace{n_1 \dots \widehat{n_i} \cdot \dots \cdot n_j \cdot m_i}_{=K \in \mathbb{Z}}.$$

Finally we can see that there will be infinite isolated switchings at switching times

$$\tilde{t}_K = \frac{2K\pi}{\beta_A} \cdot n_1 \dots \cdot n_j, \quad K \in \mathbb{Z}$$

in regular time-intervals.

In general we can say that if

- there exists $\beta_A \in \mathbb{R} \setminus \{0\}$ and $K_i \in \mathbb{Q}$, $\forall i \in \{1, \ldots, j\}$, such that $\beta_i = K_i \beta_A$
- $\alpha = \alpha_i$ for all $i \in \{1, \ldots, j\}$, and $\alpha = \lambda_i$ for all $i \in \{2j+1, \ldots, n\}$,

then for any matrix B, with the corresponding $\bar{p}$, any optimal trajectory with a switching, actually has an infinite series of switchings at regular time intervals.

$\square$

Chapter 6

Sufficient optimality condition

6.1 Introduction

At Chapters 3, 4 and 5 we proved that with some generic conditions it is possible to avoid chattering phenomenon, moreover we denoted in which cases it is possible to find non smooth optimal trajectories. Actually, at the moment, we can not claim that they exist. Indeed, via the Pontryagin maximum principle, we know that every time-optimal trajectory has a lift, called extremal, in T^*M. But, on the other hand it is not guaranteed that given any extremal its projection on M is time-optimal: even though we have found extremals through the singular locus Λ that projects in piece-wise smooth trajectories, non necessarily those trajectories are time-optimal.

It is guaranteed the optimality of the projection of any extremal, only if we consider a linear control system, satisfying (5.2.3), and put the final point in a equilibrium. It is true due to the fact that the uniqueness of the time-optimal solution and the uniqueness of the extremal hold.

In this Chapter we are going to discuss the optimality of the projections of the non smooth extremals, that we call *broken extremals*, detected in the previous Chapters, given a non linear affine control system.

Let us briefly recall the conditions that we need in a neighbourhood $O_{\bar{\lambda}}$ of $\bar{\lambda} \in \Lambda$ in order to have and study broken extremals.

Given a n-dimensional manifold M, let us consider our system

$$\dot{q} = f_0(q) + u_1 f_1(q) + \ldots + u_k f_k(q), \quad q \in M \tag{6.1.1}$$

where $f_0, \ldots, f_k$ are smooth vector fields and $(u_1, \ldots, u_k)$ admissible control with value in $\overline{B}^k$. From the Pontryagin maximum principle necessarily every time-optimal trajectory of this control system is a projection of an extremal defined in T^*M.

Out of the singular locus $\Lambda = \{\lambda \in T^*M | h_1(\lambda) = \ldots = h_k(\lambda) = 0\}$ extremals satisfy the Hamiltonian system defined by[1]

$$H(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + \ldots + h_k^2(\lambda)}. \tag{6.1.2}$$

[1]See Notation 2.2.4

In Chapter 3 we proved that if, at $\bar{\lambda} \in \Lambda$, it is satisfied the condition

$$H_{0I} \notin H_{IJ}\overline{B}^k \tag{6.1.3}$$

there exist an extremal that passes through Λ at $\bar{\lambda}$. And the flow of extremals is not locally Lipschitz.

We started to investigate this topic from the sufficient optimality that Andrei A. Agrachev and Sachkov present in their book [6]. They considered only normal extremals and assumed the Hamiltonian to be smooth.

In order to present our contribute, we will recall their method for the time-optimal problem. Hence, we are going to show the sufficient optimality of normal extremals in a neighbourhood $O_{\bar{\lambda}}$ of $\bar{\lambda}$, where the Hamiltonian is not smooth and has form (6.1.2).
Denoting $\bar{q} = \pi(\bar{\lambda})$, and $\mathcal{F} = \{f_1, \ldots, f_k\}$, we prove the sufficient optimality of the normal broken extremal, passing through $\bar{\lambda} \in \Lambda$, if

$$\bar{\lambda} \perp \mathrm{Lie}_{\bar{q}}\mathcal{F}, \quad h_0(\bar{\lambda}) > 0$$

and either $\operatorname{rank}\{\mathrm{Lie}_{\bar{q}}\mathcal{F}\} = n-1$, or $\operatorname{rank}\{\mathrm{Lie}_{q}\mathcal{F}\} = \operatorname{rank}\{\mathrm{Lie}_{\bar{q}}\mathcal{F}\} < n-1$ for all q from a neighbourhood of $\bar{q}$ in M (see Theorem 6.2.6). Moreover, if $n = 3$ $k = 2$ we prove the optimality for a normal broken extremal if f_1, f_2 form a contact distribution in a neighbourhood of $\bar{q}$ (see Theorem 6.2.9).
We use a method described by Agrachev and Sachkov in their book [6]. It is a geometrical elaboration of the classical fields of extremals theory, it proves optimality only for normal extremals, assuming the Hamiltonian smooth. We extended this method in the Lipschitzian submanifold, with constructions *ad hoc*.

We also prove optimality of normal (or abnormal) broken extremals for $n > 2$ $k = 2$ and

$$\bar{\lambda} \perp \mathrm{span}\{f_1(\bar{q}), f_2(\bar{q}), [f_1, f_2](\bar{q})\}$$

in just that point (see Theorem 6.3.5). This result is given by direct estimates with time-rescaling.

6.2 Sufficient optimality for normal extremals

In the following subsection we would like to present what is already know regarding sufficient optimality condition for time-optimal control problem with free time. Let us recall what Andrei A. Agrachev and Yuri L. Sachkov observed in their book [6].

6.2.1 With smooth Hamiltonian

The authors considered the following optimal control problem with free time

$$\begin{cases} \dot{q} = f_u(q), \quad q \in M, u \in U \\ q(0) = q_0,\, q(t_1) = q_1, \quad q_0,\, q_1 \text{ fixed} \\ \int_0^{t_1} \phi(q(t), u(t))\mathrm{d}t \to \min, \end{cases}$$

if we assume $\phi(q(t),u(t)) \equiv 1$ we face the time-optimal problem.
Let us present their contribute regarding the sufficient optimality condition of a normal extremal in the time-optimal problem.

Analysing only normal extremal, the control-dependent Hamiltonian of PMP is

$$h_u(\lambda) = \langle \lambda, f_u(q) \rangle - 1, \quad \lambda \in T^*M,\, q = \pi(\lambda) \in M,\, u \in U,$$

and the maximized Hamiltonian is

$$H(\lambda) = \max_{u \in U} h_u(\lambda).$$

constant equal 0 along the extremal $\lambda(t)$.
Without loss of generality, we redenote the maximized Hamiltonian along $\lambda(t)$ in the following way

$$H(\lambda(t)) = \max_{u \in U} \langle \lambda(t), f_u(q(t)) \rangle = 1.$$

Given any arbitrary smooth function $a \in \mathcal{C}^\infty(M)$, we consider the graph of differential da, that is a n-dimensional smooth submanifold in T^*M

$$\mathcal{L}_0 = \{ \mathrm{d}_q a \,|\, q \in M \} \subset T^*M.$$

The first and second assumptions that the authors gave were the following

(A1) H is defined and smooth in T^*M, and $\overrightarrow{H}$ is complete.

(A2) 1 is a regular value of the restriction $H_{|\mathcal{L}_0}$, i.e. $d_\lambda H_{|T_\lambda \mathcal{L}_0} \neq 0$, for all $\lambda \in \mathcal{L}_0 \cap H^{-1}(1)$.

Moreover, let us denote s the tautological 1-form on T^*M, $s_\lambda = \lambda \circ \pi_*$, and its differential is the canonical symplectic structure in T^*M, $\mathrm{d}s = \sigma$.

Notation 6.2.1. If $F : M \to N$ is a smooth mapping, we denote $F^* : \Lambda^k N \to \Lambda^k M$ the mapping of differential forms, if $\omega \in \Lambda^k N$, $(F^*\omega)_q(v_1, \dots, v_k) = \omega_{F(q)}(F_* v_1, \dots, F_* v_k)$, $q \in M$ $v_i \in T_q M$

Proposition 6.2.2. *Assuming (A2) the map*

$$\Phi : \mathcal{L}_0 \cap H^{-1}(1) \times \mathbb{R} \longrightarrow T^*M, \quad \Phi(\lambda_0, t) = e^{t\overrightarrow{H}}(\lambda_0), \tag{6.2.1}$$

is an immersion and $\Phi^* s$ is an exact form.

Proof.

(1) Assumption (A2) implies that $\mathcal{L}_0 \cap H^{-1}(1)$ is a smooth manifold.

(2) Let us prove that $\Phi^* s$ is an exact form in $\Lambda^1(\mathcal{L}_0 \cap H^{-1}(1) \times \mathbb{R})$.

(2a) At first we are going to prove that it is a closed form.
We can immediately see that

$$d(\Phi^* s)_{|(\lambda_0,t)} = \Phi^* \sigma_{|(\lambda_0,t)} = \sigma_{|e^{t\overrightarrow{H}}(\lambda_0)} = \sigma_{|\lambda_0}$$

for the properties of the exterior derivative and the form σ, moreover by the definition of $\mathcal{L}_0$

$$\sigma_{|\lambda_0} = d\, s_{|\lambda_0} = d(d\, a \circ \pi)_{|\lambda_0} = 0.$$

(2b) Now, let us show that $\Phi^* s$ is an exact form, namely for any closed curve

$$\gamma : \tau \mapsto (\lambda_0(\tau), t(\tau)) \in \mathcal{L}_0 \cap H^{-1}(1) \times \mathbb{R},$$

it holds

$$\int_\gamma \Phi^* s = 0.$$

Indeed,

$$\int_\gamma \Phi^* s = \int_{\Phi(\gamma)} s$$

$\Phi(\gamma)$ is homeomorphic to

$$\tilde{\gamma}_0 : \tau \mapsto \lambda_0(\tau) \in \mathcal{L}_0 \cap H^{-1}(1),$$

then, by the Stokes Theorem and the definition of $\mathcal{L}_0$,

$$\int_{\Phi(\gamma)} s = \int_{\tilde{\gamma}_0} s = \int_{\tilde{\gamma}_0} d(a \circ \pi) = 0.$$

(3) To prove that Φ is an immersion, it is enough to show that the vector

$$\frac{\partial \Phi}{\partial t}(\lambda_0, t) = \overrightarrow{H}(\lambda_t), \quad \lambda_t = \Phi(\lambda_0, t)$$

it is not tangent to the image of $\mathcal{L}_0 \cap H^{-1}(1)$ under the diffeomorphism

$$e^{t\overrightarrow{H}} : T^*M \to T^*M$$

for all $\lambda_0 \in \mathcal{L}_0 \cap H^{-1}(1)$.
Let us notice that $e^{t\overrightarrow{H}}(\mathcal{L}_0 \cap H^{-1}(1)) = \mathcal{L}_t \cap H^{-1}(1)$, where

$$\mathcal{L}_t = e^{t\overrightarrow{H}}(\mathcal{L}_0).$$

Hence, it is enough prove that $\overrightarrow{H}(\lambda_t)$ is not tangent to $\mathcal{L}_t$.
We saw that $\sigma_{|\mathcal{L}_t} = ds_{|\mathcal{L}_t} = 0$, then we show that the form $i_{\overrightarrow{H}}\sigma$ does not vanish at the point λ_t

$$(i_{\overrightarrow{H}}\sigma)_{|\mathcal{L}_t} = \left(e^{t\overrightarrow{H}}\right)^* ((i_{\overrightarrow{H}}\sigma)_{|\mathcal{L}_0}) = -\left(e^{t\overrightarrow{H}}\right)^* (dH_{|\mathcal{L}_0}),$$

since $\left(e^{t\overrightarrow{H}}\right)^*$ is invertible, it is enough claim that

$$dH_{|\mathcal{L}_0} \neq 0$$

does not vanish, and it is true by the assumption (A2).

$\square$

Now given W a domain of $\mathcal{L}_0 \cap H^{-1}(1) \times \mathbb{R}$ let us put an other assumption:

(A3) The map

$$\pi \circ \Phi_{|W} : W \to M \tag{6.2.2}$$

is a map of W onto a domain in M.

Theorem 6.2.3. *Let W be a domain in $\mathcal{L}_0 \cap H^{-1}(1) \times \mathbb{R}$ such that it holds (A3), and let*

$$\tilde{\lambda}_t = e^{t\overrightarrow{H}}(\tilde{\lambda}_0), \quad t \in [0, t_1],$$

be a normal extremal such that $(\tilde{\lambda}_0, t) \in W$ for all $t \in [0, t_1]$. Then the extremal trajectory $\tilde{q}(t) = \pi(\tilde{\lambda}_t)$ (with the corresponding control $\tilde{u}(t)$) realizes a strict minimum of the cost $\int_0^\tau \phi(q(t), u(t))dt$ among all admissible trajectories such that $q(t) \in \pi \circ \Phi(W)$ for all $t \in [0, \tau]$, $q(0) = \tilde{q}(0)$, $q(\tau) = \tilde{q}(t_1)$, $\tau > 0$.

Proof. Let us set $\mathcal{L} = \Phi(W)$, then $\pi : \mathcal{L} \to \pi(\mathcal{L})$ is a diffeomorphism and $s_{|\mathcal{L}}$ is an exact form.
Let $q(t)$, $t \in [0, \tau]$, be an admissible trajectory generated by a control $u(t)$ and contained in $\pi(\mathcal{L})$, with the boundary conditions $q(0) = \tilde{q}(0)$, $q(\tau) = \tilde{q}(t_1)$. Then there exist a smooth curve $t \mapsto \lambda(t)$ in $\mathcal{L}$ such that $\lambda(0) = \tilde{\lambda}_0$, $\lambda(\tau) = \tilde{\lambda}_{t_1}$ and $q(t) = \pi(\lambda(t))$, $0 \geq t \geq \tau$.

Recalling that $\int_{\lambda(\cdot)} s = \int_{\tilde{\lambda}} s$ and $H(\lambda(t)) = \max_{u \in U} \langle \lambda(t), f_u(q(t)) \rangle = 1$ we have

$$\int_{\tilde{\lambda}} s = \int_0^{t_1} \left\langle \tilde{\lambda}_t, \dot{\tilde{q}}(t) \right\rangle dt = \int_0^{t_1} \left\langle \tilde{\lambda}_t, f_{\tilde{u}(t)}(\tilde{q}(t)) \right\rangle dt = t_1.$$

On the other hand,

$$\int_{\lambda(\cdot)} s = \int_0^\tau \langle \lambda(t), \dot{q}(t) \rangle \, dt = \int_0^\tau \langle \lambda(t), f_{u(t)}(q(t)) \rangle \, dt \leq \tau$$

Moreover, the inequality is strict if the curve $t \mapsto \lambda(t)$ is not a solution of the equation $\dot{\lambda} = \overrightarrow{H}(\lambda)$, namely is does not coincide with $\tilde{\lambda}(t)$. $\square$

6.2.2 With non smooth Hamiltonian

In this section we are going to see some cases in which we prove the sufficient optimality of normal extremals through the singular locus.

Here we are going to show a generalization of the previous method, applied to the broken extremal defined by the system

$$\dot{q} = f_0(q) + \sum_{i=1}^k u_i f_i(q), \quad q \in M, u \in \mathcal{U} \tag{6.2.3}$$

where the space of control parameters is the k-dimensional closed unitary ball $U = \{u \in \mathbb{R}^k : ||u|| \leq 1\}$. In this setting extremals satisfies the Hamiltonian system denoted by the non smooth Hamiltonian

$$H(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + \ldots + h_k^2(\lambda)}.$$

We consider only normal extremal that passes through the singular locus Λ. Hence, assuming that every $\lambda(t)$ must remain in the level set $H(\lambda(t)) = 1$, necessarily $H(\bar{\lambda}) = h_0(\bar{\lambda}) = 1$.

Theorem 6.2.4. *Let $\tilde{\lambda}(t)$ be broken extremal passing through $\bar{\lambda} \in \Lambda$. If it is possible to define*

(1) A co-dimension one submanifold N of M such that the curve $\tilde{q}(t) \in \pi(\tilde{\lambda}(t))$ passes transversally through N in both sides with $\tilde{q}(\bar{t}) = \bar{q} = \pi(\bar{\lambda}) \in N$

(2) A section ω of bundle $T^*M_{|N}$ such that

$$\omega : q \in N \to \omega_q \in T^*M,$$

$H(\omega_q) = 1$ and $\langle \omega_q, f_i(q) \rangle = 0$ with $i \in \{1, \ldots, k\}$ for all $q \in N$; moreover $\omega_{|N}$ is a well defined differential 1-form of N and $d\omega_{|N} = 0$.

Then $\tilde{q}(t)$ is time-optimal at $\bar{q}$: there exists an interval $J = (t_1, t_2)$ with $\bar{t} \in J$, such that $\tilde{q}(t)$ with $t \in J$ realizes a strict minimum time among all admissible trajectories $q(t)$ such that $q(t_1) = \tilde{q}(t_1)$ and $q(\tau) = \tilde{q}(t_2)$ with $\tau > t_1$.

Proof. Given $N \subset M$ and $\omega : q \in N \to \omega_q \in T_qM$ with those hypothesis, let us consider

$$\mathcal{N} = \{\omega_q \,:\, q \in N\}$$

that is a submanifold in T^*M such that $\mathcal{N} \subset \Lambda \cap H^{-1}(1)$, in particular $\bar{\lambda} \in \mathcal{N}$.

From what we have proved in chapter 3, if $\bar{\lambda} \in \Lambda$ satisfies condition (3.2.3), given $O_{\bar{\lambda}}$ a small enough neighbourhood of $\bar{\lambda}$, for all $\hat{\lambda} \in O_{\bar{\lambda}} \cap \Lambda$ there exists a unique broken extremal $\lambda_{\hat{\lambda}}(t)$ that passes through the singular locus at $\hat{\lambda}$. We assume $\lambda_{\hat{\lambda}}(\bar{t}) = \hat{\lambda}$. Moreover, let us recall that out of Λ each extremal satisfies the Hamiltonian system $\dot{\lambda} = \overrightarrow{H}(\lambda)$, with $H(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + \ldots + h_k^2(\lambda)}$.

Hence, we restrict $\mathcal{N}$ to those points close to $\bar{\lambda}$ and define the map

$$\Phi : \mathcal{N} \times I \to T^*M$$

where $I \subseteq \mathbb{R}$ is an interval with $\bar{t} \in I$, such that $\Phi(\hat{\lambda}, t) = \lambda_{\hat{\lambda}}(t)$.

From what we have explained in chapter 3: given any $\hat{\lambda} \in \mathcal{N}$ $\lambda_{\hat{\lambda}}(t)$ is piece-wise smooth with respect to t in the two sides where $t < \bar{t}$ or $t > \bar{t}$, and it is globally lipschitzian because the right and left limits of $\dot{\lambda}(t)$ as $t \to \bar{t} \pm 0$ are well defined. Moreover, Theorem 3.2.4 claims that at $O_{\bar{\lambda}}$ it is defined a continuous flow of extremals that is not locally Lipschitz. Nevertheless, considering just broken extremals passing through $\mathcal{N}$, the image of map Φ is a piece-wise smooth manifold composed by two smooth manifolds with boundary $\mathcal{N}$. Globally $\Phi(\mathcal{N} \times I)$ is Lipschitzian, because we have that $\frac{\partial \Phi}{\partial t}(\hat{\lambda}, t)_{|t \neq \bar{t}} = \overrightarrow{H}\left(\lambda_{\hat{\lambda}}(t)\right)$ for all $(\hat{\lambda}, t) \in \mathcal{N} \times \{I \setminus \{\bar{t}\}\}$ and the limits as $t \to \bar{t} \pm 0$ are explicitly defined (see Chapter 3).

Let us stress that, given W a domain in $\mathcal{N} \times I$ such that $(\bar{\lambda}, 0) \in W$, the map

$$\pi \circ \Phi_{|W} : W \to M$$

is a Lipschitzian (even piece-wise smooth) homeomorphism of W into a domain in M, by construction. This is because we assume that $\tilde{q}(t)$ passes transversally through N in both sides.

As a consequence, we have that Φ is a piece-wise smooth immersion since $\pi \circ \Phi_{|W}$ is immersion.

Now, we need to prove a technical fact: $\Phi^* s$ is an exact form.

It is a closed form, because

$$d(\Phi^* s)_{|(\hat{\lambda},t)} = \Phi^* \sigma_{|(\hat{\lambda},t)} = \sigma_{|\Phi(\hat{\lambda},t)} \quad \forall (\hat{\lambda}, t) \in \mathcal{N} \times \mathbb{R}$$

by the properties of the exterior derivative, and

$$\sigma_{|\Phi(\hat{\lambda},t)} = \sigma_{|\hat{\lambda}} = ds_{|\hat{\lambda}} = d(\omega \circ \pi)_{|\hat{\lambda}} = 0 \quad \forall (\hat{\lambda}, t) \in \mathcal{N} \times \mathbb{R}$$

because of the properties of form σ and by definition of $\mathcal{N}$.

On the other hand, it is exact because, given any closed curve

$$\gamma : \tau \mapsto (\lambda_0(\tau), t(\tau)) \in \mathcal{N} \times \mathbb{R},$$

one can see that

$$\int_\gamma \Phi^* s = 0.$$

We have

$$\int_\gamma \Phi^* s = \int_{\Phi(\gamma)} s$$

$\Phi(\gamma)$ is homeomorphic to

$$\gamma_0 : \tau \mapsto \lambda_0(\tau) \in \mathcal{N},$$

then, by the Lipschitzian version of Stokes Theorem [15] and the definition of $\mathcal{N}$

$$\int_{\Phi(\gamma)} s = \int_{\gamma_0} s = \int_{\gamma_0} \omega \circ \pi = 0.$$

Finally, we prove the thesis of the theorem.

Let us call $\mathcal{N}_W = \Phi(W) \subset T^*M$ such that $\pi : \mathcal{N}_W \to \pi(\mathcal{N}_W)$ is a Lipschitzian (even piece-wise smooth) homeomorphism and $s_{|\mathcal{N}_W}$ is an exact form.
Given $\tilde{q}(t) = \pi(\tilde{\lambda}(t))$ with $t \in (t_1, t_2)$ such that $t_1 < 0 < t_2$ and $\tilde{q}(0) = \bar{q} = \pi(\bar{\lambda})$, let us consider $q(t)$ with $t \in (t_1, \tau)$ an admissible trajectory generated by a control $u(t)$ and contained in $\pi(\mathcal{N}_W)$, with the boundary conditions $q(t_1) = \tilde{q}(t_1)$ and $q(\tau) = \tilde{q}(t_2)$. Then, by the map $\pi_{|\mathcal{N}_W}$, there exists a curve $\lambda(\cdot) : t \mapsto \lambda(t)$ in $\mathcal{N}_W$ such that $\lambda(t_1) = \tilde{\lambda}(t_1)$ $\lambda(\tau) = \tilde{\lambda}(t_2)$ and $q(t) = \pi(\lambda(t))$ for all $t \in (t_1, \tau)$.
Since $\int_{\lambda(\cdot)} s = \int_{\tilde{\lambda}(\cdot)} s$ and $H(\lambda(t)) = \max_{u \in U} \left\langle \lambda(t), f_0(q(t)) + \sum_{i=1}^k u_i(t) f_i(q(t)) \right\rangle = 1$ we have

$$\int_{\tilde{\lambda}} s = \int_0^{t_1} \left\langle \tilde{\lambda}_t, \dot{\tilde{q}}(t) \right\rangle dt = \int_0^{t_1} \underbrace{\left\langle \tilde{\lambda}_t, f_0(\tilde{q}(t)) + \sum_{i=1}^k \tilde{u}_i(t) f_i(\tilde{q}(t)) \right\rangle}_{=H(\tilde{\lambda}(t))=1} dt = t_1.$$

On the other hand,

$$\int_{\lambda(\cdot)} s = \int_0^\tau \langle \lambda(t), \dot{q}(t) \rangle \, dt = \int_0^\tau \underbrace{\left\langle \lambda(t), f_0(q(t)) + \sum_{i=1}^k u_i(t) f_i(q(t)) \right\rangle}_{\leq 1} dt \leq \tau$$

Moreover, the inequality is strict if the curve $t \mapsto \lambda(t)$ is not a solution of the equation $\dot{\lambda} = \overrightarrow{H}(\lambda)$, namely is does not coincide with $\tilde{\lambda}(t)$.

Therefore, we have proved that $\tilde{q}(t)$ is locally time-optimal in the switching point $\bar{q}$. Actually, it is globally optimal.
It is optimal with respect to the whole trajectory. Indeed, it will spend strictly more time going out side the neighbourhood.

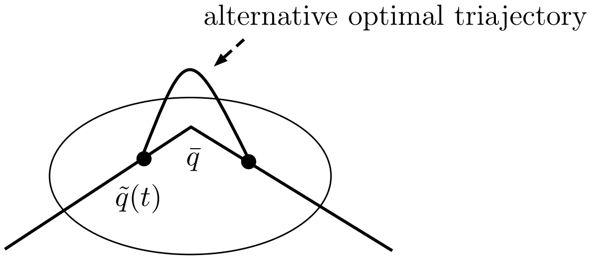

Figure 6.1: Global optimality of $\tilde{q}(t)$.

□

Remark 6.2.5. On the other hand, if we study the problem with a smooth Hamiltonian H and $\overrightarrow{H}$ complete, as before, it is enough give a Lagrangian $\mathcal{N}$ such that

$$\mathcal{N} = \{ \mathrm{d}_q a \,|\, q \in M \},$$

with $a \in \mathcal{C}^\infty(M)$ any arbitrary smooth function. We called it $\mathcal{L}_0$.
As a consequence $\omega = \mathrm{d}a$.

Now, let us present two cases in which we found such a Lagrangian submanifold $\mathcal{N}$.
This method proves the optimality of those extremals that pass through Λ.

Theorem 6.2.6. *Given an affine control system (6.2.3) with $f_1, \ldots, f_k$ analytic fields, let $\bar{\lambda} \in \Lambda$ be a singular point such that it holds (6.1.3) and $H(\bar{\lambda}) = 1$.
In this setting let us consider the normal broken extremal $\tilde{\lambda}(t)$ through $\bar{\lambda}$, such that $\tilde{\lambda}(0) = \bar{\lambda}$.
We denote $\mathcal{F} = \{f_1, \ldots, f_k\}$ the family of controllable vector fields. If*

$$\bar{\lambda} \perp \mathrm{Lie}_{\bar{q}}\mathcal{F}, \quad h_0(\bar{\lambda}) > 0$$

and either $\operatorname{rank}\{\mathrm{Lie}_{\bar{q}}\mathcal{F}\} = n-1$, *or* $\operatorname{rank}\{\mathrm{Lie}_{q}\mathcal{F}\} = \operatorname{rank}\{\mathrm{Lie}_{\bar{q}}\mathcal{F}\} < n-1$ *for all q from a neighbourhood $O_{\bar{q}}$ of $\bar{q}$ in M, then $\tilde{q}(t) = \pi(\tilde{\lambda}(t))$ is locally time-optimal among all admissible trajectory in $O_{\bar{q}}$ with the same boundary conditions.*

Proof. As we discussed previously, it is enough find an opportune Lagrangian submanifold $\mathcal{N}$ with the said conditions.
Let us consider $\mathcal{F}$ and the distribution $\mathrm{Lie}_q\mathcal{F}$, that, by definition, is closed with respect to the Lie brackets.

If $\mathrm{rank}\{\mathrm{Lie}_{\bar{q}}\mathcal{F}\} = n-1$, we will denote N the orbit $\mathcal{O}_{\bar{q}}$ of distribution $\mathrm{Lie}\mathcal{F}$ at point $\bar{q}$ that is a $n-1$ dimensional submanifold of M by Nagano Theorem (see Theorem 2.3.1).

Otherwise, if $\mathrm{rank}\,\{\mathrm{Lie}_q\mathcal{F}\} = \mathrm{rank}\,\{\mathrm{Lie}_{\bar{q}}\mathcal{F}\} = m < n-1$ $\forall q \in O_{\bar{q}}$, then, by Frobenius Theorem (see Theorem 2.3.11), it is defined a fibration in $O_{\bar{q}}$ give by the m dimensional submanifold N' of M and other $n-m$ components. By construction, one can define the codimension 1 submanifold N, such that such that $N' \subset N$, $\mathcal{F} \subseteq T_qN$ $\forall q \in O_{\bar{q}}$ and $f_0(\bar{q}) \notin T_{\bar{q}}N$, and the said curve $\tilde{q}(t)$ will cross transversally N at $\bar{q}$.

Moreover, let us define ω the 1-form that annihilates T_qN such that $\omega(f_0)_{|q} = 1$, for all $q \in N$. By construction, ω satisfies $d\omega_{|N} = 0$, and denoting

$$\mathcal{N} = \{\omega_{|q} \,|\, q \in N\},$$

it holds $\mathcal{N} \subseteq \Lambda \cap H^{-1}(1)$.
All these facts imply the thesis. □

Remark 6.2.7. Let us notice that, in the setting of Theorem 6.2.6, the corresponding smooth function a, denoted in Remark 6.2.5, is such that $a_{|N} = 0$ and $\mathrm{d}a = \omega$.

Before presenting the next result let us define the Reeb vector field.

Definition 6.2.8. In a 3-dimensional manifold M let us consider a *contact 1-form* $\omega \in \Lambda^1(M)$ such that $\omega \wedge d\omega \neq 0$ in never vanishing.
The *Reeb vector field* $\xi \in \mathrm{Vec}(M)$ is the unique element of the (one-dimensional) kernel of $d\omega$ such that $\omega(\xi) = 1$.

Theorem 6.2.9. *Given an affine control system (6.2.3) with $n = 3$ and $k = 2$ and with f_1, f_2 analytic fields, let $\bar{\lambda} \in \Lambda$ be a singular point such that it holds (6.1.3) and $H(\bar{\lambda}) = 1$.*
In this setting let us consider the normal extremal $\tilde{\lambda}(t)$ through $\bar{\lambda}$, such that $\tilde{\lambda}(0) = \bar{\lambda}$. If the distribution

$$\Delta = \mathrm{span}\{f_1, f_2\}$$

is contact in $\bar{q} = \pi(\bar{\lambda})$, then $\tilde{q}(t) = \pi(\tilde{\lambda}(t))$ is locally time-optimal among all admissible trajectory in a neighbourhood $O_{\bar{q}}$ of $\bar{q}$ with the same boundary conditions.

Proof. Since Δ is a contact distribution in a neighbourhood $O_{\bar{q}}$ of $\bar{q}$, there exists $\omega \in \Lambda^1(M)$ a 1-form such that $\Delta = \ker(\omega)$ and $\omega \wedge d\omega \neq 0$, moreover we can assume $\omega_q(f_0(q)) \equiv 1$ $\forall q \in O_{\bar{q}}$.
We can define $\xi \in \mathrm{Vec}(M)$ the Reeb field such that $\langle\xi\rangle = \ker(d\omega)$.
We construct a co-dimension 1 submanifold N in the following way.
Given the control $\tilde{u}$ corresponding to the extremal trajectory $\tilde{q}(t) = \pi(\tilde{\lambda}_t)$, let us denote $f_-(\bar{q})$ and $f_+(\bar{q})$ at point $\bar{q}$

$$\begin{cases} f_-(\bar{q}) = \tilde{u}_1(0^-)f_1(\bar{q}) + \tilde{u}_2(0^-)f_2(\bar{q}) \\ f_+(\bar{q}) = \tilde{u}_1(0^+)f_1(\bar{q}) + \tilde{u}_2(0^+)f_2(\bar{q}). \end{cases}$$

Let us give any integral curve $\hat{\gamma}$ whose velocities belong to the distribution $\mathrm{span}\{f_1(q), f_2(q)\}$, with $q \in O_{\bar{q}}$, as follows such that $f_-(\bar{q})$ and $f_+(\bar{q})$ appear in the same side.

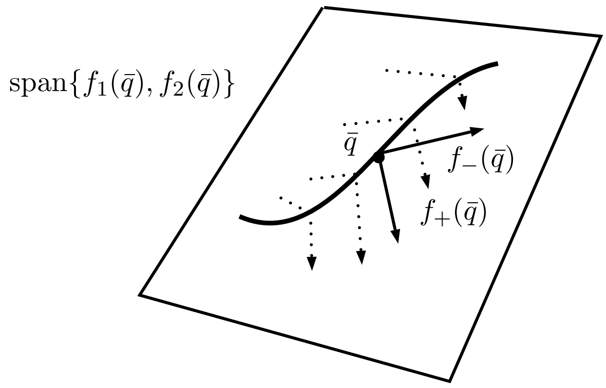

Figure 6.2: Curve in $\mathrm{span}\{f_1, f_2\}$

Then we apply the flow generated by the Reeb field ξ. We denote this surface N. Therefore we denote

$$\mathcal{N} = \{\omega_q \in T^*M \mid q \in N\}.$$

Let us stress the fact that we chose the curve in $\mathrm{span}\{f_1, f_2\}$, as it is described at Figure 2, because we need to assume that $\tilde{q}(t)$ passes transversally through N. This construction implies the thesis. □

Remark 6.2.10. Let us notice that, in the setting of Theorem 6.2.9, the corresponding smooth function a, denoted in Remark 6.2.5, is the time-function along the Reeb curves such that $a(\hat{\gamma}) \equiv 0$.
Indeed, given $\gamma(t)$ a curve along the Reeb flow with $\gamma(0) \in \hat{\gamma}$, we have

$$a(\gamma(t)) = \underbrace{a(\gamma(0))}_{=0} + \int_0^t \frac{d}{dt} a(\gamma(\tau)) d\tau = \int_0^t \underbrace{\langle d_{\gamma(\tau)} a, \dot{\gamma}(\tau) \rangle}_{\langle \omega_{\gamma(\tau)}, \xi(\gamma(\tau)) \rangle = 1} d\tau = t.$$

6.3 Sufficient optimality, with 2-dimensional control

In this new Section we are going to present an alternative method to prove the sufficient optimality of extremals through the singular locus defined by systems of type (6.1.1) with 2-dimensional control.

At first we present how we reduce the problem and then the result that we were able to gave.

6.3.1 How we reduce the problem

Let us consider a control system of type (6.1.1) in the n-dimensional manifold M with $k = 2$. We assume $\bar{\lambda} \in \Lambda$ satisfying the condition

$$H_{0I} \notin H_{IJ} \overline{B^k},$$

namely there exists an extremal $\lambda(t)$ that passes through $\bar{\lambda}$, going through the singular locus. Let us consider the perturbation of a trajectory $q(t) = \pi(\lambda(t))$ that is the projection of the extremal.
With opportune rotation of the system, we may assume that $\bar{\lambda} = \lambda(0)$ and the trajectory $q(t)$ satisfies the following system with constant piece-wise control:

$$\begin{cases} \dot{q} = f_-(q) := f_0(q) + \cos(\hat{\theta}) f_1(q) - \sin(\hat{\theta}) f_2(q), & t < 0 \\ \dot{q} = f_+(q) := f_0(q) + \cos(\hat{\theta}) f_1(q) + \sin(\hat{\theta}) f_2(q), & t > 0 \\ q(0) = \bar{q}, \end{cases}$$

with $\hat{\theta} \in (0, \frac{\pi}{2})$.
As we saw in Chapter 4, it is possible to give explicitly the jump of the control at the switching by equation (4.2.5).
In this setting we will have at $\bar{\lambda}$ $h_{01} = 0$, $h_{02} > 0$ and $h_{12} \leq 0$, and calling $\alpha := \frac{|h_{12}|}{|h_{02}|}$ we have

$$(\cos(\hat{\theta}), \sin(\hat{\theta})) = \left(\alpha, \sqrt{1-\alpha^2}\right).$$

In order to perturb the control with admissible controls, we denote

$$g_v(q) := v_1 f_1(q) + v_2 f_2(q),$$

where v_1 and v_2 are time depending function such that

$$\begin{cases} \left|\left|\left(\alpha + v_1(t), -\sqrt{1-\alpha^2} + v_2(t)\right)\right|\right| \leq 1, & t < 0 \\ \\ \left|\left|\left(\alpha + v_1(t), \sqrt{1-\alpha^2} + v_2(t)\right)\right|\right| \leq 1, & t > 0. \end{cases} \tag{6.3.1}$$

Defining

$$a := \begin{pmatrix} \alpha \\ \sqrt{1-\alpha^2} \end{pmatrix}$$

the condition (6.3.1) becomes

$$\begin{cases} \left|\begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}\right|^2 \leq -2 \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} \cdot a, & t < 0 \\ \left|\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right|^2 \leq -2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot a, & t > 0. \end{cases} \tag{6.3.2}$$

Now, let us study the behaviour of the following path in the neighbourhood $O_{\bar{q}}$ of $\bar{q}$ at time $t \in [-\varepsilon, \varepsilon]$, with $\varepsilon > 0$ small, using the chronological calculus described in [6] Chapter 2,

$$\bar{q} \circ F_\varepsilon(v(t)) = \bar{q} \circ e^{(-\varepsilon - 0) f_-} \circ \overrightarrow{\exp} \int_{-\varepsilon}^{0} f_- + g_v \, \mathrm{dt} \circ \overrightarrow{\exp} \int_0^{\varepsilon} f_+ + g_v \, \mathrm{dt} \circ e^{(0-\varepsilon) f_+}.$$

Claim 6.3.1. *In order to prove the optimality of the switched curve among all perturbations, it is enough to prove the following:*
Statement*:*
There exists $\bar{\varepsilon} > 0$ such that $\forall v \neq 0$ and $\forall \varepsilon < \bar{\varepsilon}$ the functional $F_\varepsilon(v) \neq \mathrm{Id}$.

Now, let us study deeply this functional $F_\varepsilon(v)$.
Thanks to the variational formula we simplify it in such a way

$\bar{q} \circ F_\varepsilon(v(t)) = \bar{q} \circ e^{-\varepsilon f_-} \circ e^{\varepsilon f_-} \circ \overrightarrow{\exp} \int_{-\varepsilon}^{0} e^{t\,\mathrm{ad} f_-} g_v\,\mathrm{d}t \circ \overrightarrow{\exp} \int_0^\varepsilon e^{t\mathrm{ad} f_+} g_v\,\mathrm{d}t \circ e^{\varepsilon f_+} \circ e^{-\varepsilon f_+}$

$= \bar{q} \circ \overrightarrow{\exp} \int_{-\varepsilon}^{0} e^{t\,\mathrm{ad} f_-} g_v\,\mathrm{d}t \circ \overrightarrow{\exp} \int_0^\varepsilon e^{t\mathrm{ad} f_+} g_v\,\mathrm{d}t$

Rescaling the time in the integrals we have

$$\bar{q} \circ F_\varepsilon(v(t)) = \bar{q} \circ \overrightarrow{\exp} \int_{-1}^{0} \varepsilon e^{\varepsilon\, t\mathrm{ad} f_-} g_v\,\mathrm{d}t \circ \overrightarrow{\exp} \int_0^1 \varepsilon e^{\varepsilon t\,\mathrm{ad} f_+} g_v\,\mathrm{d}t.$$

Hence, we can rewrite it as follows

$$F_\varepsilon(v(t)) = \overrightarrow{\exp} \int_{-1}^{1} V_t(\varepsilon)\,\mathrm{d}t$$

where

$$\overrightarrow{\exp} \int_{-1}^{1} V_t(\varepsilon)\,\mathrm{d}t = \overrightarrow{\exp} \int_{-1}^{0} \varepsilon\, g^-_{\varepsilon t}(v)\,\mathrm{d}t \circ \overrightarrow{\exp} \int_0^1 \varepsilon g^+_{\varepsilon t}(v)\,\mathrm{d}t$$

and

$$g^-_{\varepsilon t}(v) = e^{\varepsilon t\,\mathrm{ad}\, f_-} g_v$$
$$g^+_{\varepsilon t}(v) = e^{\varepsilon t\,\mathrm{ad}\, f_+} g_v.$$

Notation 6.3.2. We will use the following notation

$$F_\varepsilon(v)_{|[t\to 1]} = \overrightarrow{\exp} \int_t^1 V_\tau(\varepsilon)\,\mathrm{d}\tau$$

and

$$F_\varepsilon(v)_{|[1\to t]} = \overrightarrow{\exp} \int_1^t V_\tau(\varepsilon)\,\mathrm{d}\tau.$$

In order to verify what we state in Claim 6.3.1, we are going to study the Taylor expansion of $F_\varepsilon(v)$

$$F_\varepsilon(v) = \mathrm{Id} + \partial_\varepsilon F_\varepsilon(v)_{|\varepsilon=0}\ \varepsilon + \frac{1}{2}\partial^2_\varepsilon F_\varepsilon(v)_{|\varepsilon=0}\ \varepsilon^2 + O(\varepsilon^3)$$

then the first derivative is

$$\partial_\varepsilon F_\varepsilon(v) = F_\varepsilon(v) \circ \int_{-1}^{1} F_\varepsilon(v)_{|[t\to 1]} \circ \partial_\varepsilon V_t(\varepsilon) \circ F_\varepsilon(v)_{|[1\to t]}\mathrm{d}t$$

and the second

$$\begin{aligned} \partial^2_\varepsilon F_\varepsilon(v) \quad = \quad & \partial_\varepsilon F_\varepsilon(v) \circ \textstyle\int_{-1}^{1} F_\varepsilon(v)_{|[t\to 1]} \circ \partial_\varepsilon V_t(\varepsilon) \circ F_\varepsilon(v)_{|[1\to t]}\mathrm{d}t + \\ & + F_\varepsilon(v) \circ \textstyle\int_{-1}^{1} \partial_\varepsilon F_\varepsilon(v)_{|[t\to 1]} \circ \partial_\varepsilon V_t(\varepsilon) \circ F_\varepsilon(v)_{|[1\to t]}\mathrm{d}t + \\ & + F_\varepsilon(v) \circ \textstyle\int_{-1}^{1} F_\varepsilon(v)_{|[t\to 1]} \circ \partial^2_\varepsilon V_t(\varepsilon) \circ F_\varepsilon(v)_{|[1\to t]}\mathrm{d}t + \\ & + F_\varepsilon(v) \circ \textstyle\int_{-1}^{1} F_\varepsilon(v)_{|[t\to 1]} \circ \partial_\varepsilon V_t(\varepsilon) \circ \partial_\varepsilon F_\varepsilon(v)_{|[1\to t]}\mathrm{d}t. \end{aligned}$$

Since by construction

$$F_\varepsilon(v)_{|\varepsilon=0} = \mathrm{Id} \qquad \partial_\varepsilon V_t(\varepsilon)_{|\varepsilon=0} = g_v$$

and

$$\int_{-1}^{1} \partial_\varepsilon^2 V_t(\varepsilon)_{|\varepsilon=0} = \int_{-1}^{0} 2t[f_-, g_v]\mathrm{d}t + \int_{0}^{1} 2t[f_+, g_v]\mathrm{d}t$$

it holds

$$\partial_\varepsilon F_\varepsilon(v)_{|\epsilon=0} = \int_{-1}^{1} g_v \mathrm{d}t,$$

and

$$\begin{aligned}\partial_\varepsilon^2 F_\varepsilon(v)_{|\varepsilon=0} = &\left[\textstyle\int_{-1}^{0} 2t[f_-, g_v]\mathrm{d}t + \int_0^1 2t[f_+, g_v]\mathrm{d}t\right] + \\ &+ \textstyle\int_{-1}^{1}\int_t^1 [g_{v(\theta)}, g_v]\mathrm{d}\theta\mathrm{d}t + \int_{-1}^{1} g_v \circ \int_{-1}^{1} g_v\end{aligned}$$

Remark 6.3.3. Given the Taylor expansion

$$F_\varepsilon(v) = \mathrm{Id} + \partial_\varepsilon F_\varepsilon(v)_{|\varepsilon=0}\ \varepsilon + \frac{1}{2}\partial_\varepsilon^2 F_\varepsilon(v)_{|\varepsilon=0}\ \varepsilon^2 + O(\varepsilon^3),$$

if $\partial_\varepsilon F_\varepsilon(v)_{|\varepsilon=0}\ + \frac{1}{2}\varepsilon\, \partial_\varepsilon^2 F_\varepsilon(v)_{|\varepsilon=0} \neq 0$ then the statement of Claim 6.3.1 is proved.

Thus, we are interested in proving if the statement of Claim 6.3.1 can be proved even in the worst case. So, let us assume that

$$\partial_\varepsilon F_\varepsilon(v)_{|\varepsilon=0}\ + \frac{1}{2}\varepsilon\, \partial_\varepsilon^2 F_\varepsilon(v)_{|\varepsilon=0} = 0,$$

then

$$\partial_\varepsilon F_\varepsilon(v)_{|\varepsilon=0} = \int_{-1}^{1} g_{v(t)}\mathrm{d}t \in O(\varepsilon),$$

and

$$\int_{-1}^{1} g_v \circ \int_{-1}^{1} g_v \in O(\varepsilon^2),$$

and finally we rewrite the functional in the following way

$$\begin{aligned}\tfrac{1}{\varepsilon}\left(F_\varepsilon(v) - \mathrm{Id}\right) \quad = \quad & \textstyle\int_{-1}^{1} g_{v(t)}\mathrm{d}t + \frac{1}{2}\varepsilon\Big(\int_{-1}^{0} 2t[f_-, g_v]\mathrm{d}t + \int_0^1 2t[f_+, g_v]\mathrm{d}t \\ & + \textstyle\int_{-1}^{1}\int_t^1 [g_{v(\tau)}, g_{v(t)}]\mathrm{d}\tau\,\mathrm{d}t\Big) + O(\varepsilon^2).\end{aligned}$$

At this point we calculate and study the scalar product $\left\langle\bar\lambda, \frac{1}{\varepsilon}\left(F_\varepsilon(v) - \mathrm{Id}\right)\right\rangle$, with $\bar\lambda \in \Lambda$, because if we show that it is strictly negative, the statement is proven and the projection of the extremal that we are analysing is optimal.

Thus, we have

$$\begin{aligned}\left\langle\bar\lambda, \tfrac{1}{\varepsilon^2}\left(F_\varepsilon(v) - \mathrm{Id}\right)\right\rangle \quad = \quad & \tfrac{1}{2}\Big(\textstyle\int_{-1}^{0} 2t\left\langle\bar\lambda, [f_-, g_v]\right\rangle \mathrm{d}t + \int_0^1 2t\left\langle\bar\lambda, [f_+, g_v]\right\rangle \mathrm{d}t + \\ & + \textstyle\int_{-1}^{1}\int_t^1 \left\langle\bar\lambda, [g_{v(\tau)}, g_{v(t)}]\right\rangle \mathrm{d}\tau\,\mathrm{d}t\Big) + O(\varepsilon).\end{aligned} \tag{6.3.3}$$

One can give the following estimate for $O(\varepsilon)$

$$O(\varepsilon) \le \varepsilon \text{ const} \int_{-1}^{1} |t||v|^2 dt.$$

Hence let us give the following Claim

Claim 6.3.4. *In order to prove the optimality of the switched curve among all perturbations, denoting*

$$\begin{aligned} J(v) &= \textstyle\int_{-1}^{0} 2t \left\langle \bar{\lambda}, [f_-, g_v] \right\rangle \mathrm{d}t + \int_0^1 2t \left\langle \bar{\lambda}, [f_+, g_v] \right\rangle \mathrm{d}t + \\ &+ \textstyle\int_{-1}^{1} \int_t^1 \left\langle \bar{\lambda}, [g_{v(\tau)}, g_{v(t)}] \right\rangle \mathrm{d}\tau \, \mathrm{d}t + \varepsilon \text{ const} \int_{-1}^{1} |t||v|^2 dt. \end{aligned} \tag{6.3.4}$$

it is enough to prove the following:
Statement*:*
There exists $\bar{\varepsilon} > 0$ such that $\forall v \neq 0$ and $\forall \varepsilon < \bar{\varepsilon}$ the following inequality holds

$$J(v) < 0.$$

6.3.2 Result

The reduction of the problem that we explained in the previous subsection, permits to show the following result.

Theorem 6.3.5. *Given an affine control system of type (6.1.1) with any n and $k = 2$, and $\bar{\lambda} \in \Lambda$ that satisfies (6.1.3), if*

$$[f_1, f_2](\bar{q}) \in \operatorname{span}\{f_1(\bar{q}), f_2(\bar{q})\},$$

then the projection on M of the extremal through $\bar{\lambda}$ is time-optimal.

Proof. Let us compute explicitly equation (6.3.4), in particular it holds

$$\left\langle \bar{\lambda}, [f_-, g_v] \right\rangle = -|h_{02}|\sqrt{1-\alpha^2} \left[\begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} \cdot a \right]$$

$$\left\langle \bar{\lambda}, [f_+, g_v] \right\rangle = |h_{02}|\sqrt{1-\alpha^2} \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot a \right]$$

$$\left\langle \bar{\lambda}, [g_{v(\tau)}, g_{v(t)}] \right\rangle = -|h_{12}| \left(v_1(\tau) v_2(t) - v_1(t) v_2(\tau) \right)$$

For simplicity let us denote

$$\begin{cases} V^- := \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} & t < 0 \\ \\ V^+ := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} & t > 0, \end{cases}$$

and (6.3.4) becomes

$$J(V^-,V^+) = 2|h_{02}|\,\sqrt{1-\alpha^2}\left(\int_{-1}^{0} -t\ V^- \cdot a\,\mathrm{d}t + \int_0^1 t\ V^+ \cdot a\,\mathrm{d}t\right) +$$

$$+\int_{-1}^{1}\int_t^1 \left\langle \bar\lambda, [g_{v(\tau)}, g_{v(t)}]\right\rangle \mathrm{d}\tau\,\mathrm{d}t + \varepsilon\ \mathrm{const}\ \left(\int_0^1 |t||V^+|^2 dt + \int_{-1}^{0} |t||V^-|^2 dt\right).$$

Moreover, we can consider for $t > 0$

$$\begin{cases} \tilde V^-(t) := V^-(-t) \\ \tilde V^+(t) := V^+(t), \end{cases}$$

then we have

$$\begin{aligned} J(\tilde V^+,\tilde V^-) &= 2|h_{02}|\,\sqrt{1-\alpha^2}\left(\int_0^1 t\ \tilde V^- \cdot a\,\mathrm{d}t + \int_0^1 t\ \tilde V^+ \cdot a\,\mathrm{d}t\right) + \\ &+\int_{-1}^{1}\int_t^1 \left\langle \bar\lambda, [g_{v(\tau)}, g_{v(t)}]\right\rangle \mathrm{d}\tau\,\mathrm{d}t + \varepsilon\ \mathrm{const}\ \left(\int_0^1 t|\tilde V^+|^2 dt + \int_0^1 t|\tilde V^-|^2 dt\right). \end{aligned} \tag{6.3.5}$$

From equation (6.3.2) we have $\left|\tilde V^\pm\right|^2 \le -2\tilde V^\pm \cdot a$, thus, assuming const $= |h_{02}|$ it holds

$$J(\tilde V^+,\tilde V^-) \le 2|h_{02}|\ \left(\sqrt{1-\alpha^2} - \varepsilon\right)\left(\int_0^1 t\ |\tilde V^-|^2\,\mathrm{d}t + \int_0^1 t\ |\tilde V^+|^2\,\mathrm{d}t\right) +$$

$$+\int_{-1}^{1}\int_t^1 \left\langle \bar\lambda, [g_{v(\tau)}, g_{v(t)}]\right\rangle \mathrm{d}\tau\,\mathrm{d}t.$$

Thus, we prove that it holds the statement of Claim 6.3.4, if $[f_1,f_2](\bar q) \in \mathrm{span}\{f_1(\bar q), f_2(\bar q)\}$, indeed we will have $h_{12} = 0,\ \alpha = 0$ and

$$J(\tilde V^+,\tilde V^-) \le -|h_{02}|\ \left(\sqrt{1-\alpha^2} - \varepsilon\right)\left(\int_0^1 t\ |\tilde V^-|^2\,\mathrm{d}t + \int_0^1 t\ |\tilde V^+|^2\,\mathrm{d}t\right) \tag{6.3.6}$$

is strictly negative if the perturbation v is not null. □

6.4 Open problem

Let us describe what we have already seen if $[f_1,f_2](\bar q) \notin \mathrm{span}\{f_1(\bar q), f_2(\bar q)\}$.

At first let us calculate

$$\begin{aligned} \int_{-1}^{1}\int_t^1 \left\langle \bar\lambda, [g_{v(\tau)}, g_{v(t)}]\right\rangle dt \quad &= \quad |h_{12}|\left(\int_0^1\int_t^1 -\tilde V^+(t)\cdot A\tilde V^+(\theta)d\theta\,dt + \right. \\ &\quad + \int_0^1\int_0^1 \tilde V^-(t)\cdot B\tilde V^+(\theta)d\theta\,dt + \\ &\quad \left. + \int_0^1\int_0^t \tilde V^-(t)\cdot A\tilde V^-(\theta)d\theta\,dt\right) \end{aligned}$$

with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we can rewrite (6.3.5) as follows

$$J(\tilde V^+,\tilde V^-) = |h_{02}|\,\sqrt{1-\alpha^2}\left(\int_0^1 2t\ \tilde V^-\cdot a\,\mathrm{d}t + \int_0^1 2t\ \tilde V^+\cdot a\,\mathrm{d}t\right)+$$

$$+|h_{12}|\left(\int_0^1\int_t^1 -\tilde V^+(t)\cdot A\tilde V^+(\theta)d\theta\,dt+ \int_0^1\int_0^1 \tilde V^-(t)\cdot B\tilde V^+(\theta)d\theta\,dt+\right.$$

$$\left.+\int_0^1\int_0^t \tilde V^-(t)\cdot A\tilde V^-(\theta)d\theta\,dt\right)+\varepsilon\text{ const}\left(\int_0^1 t|\tilde V^+|^2dt+\int_0^1 t|\tilde V^-|^2dt\right).$$

If we define

$$\dot X^\pm = \tilde V^\pm$$

and multiply the expression by $\frac{1}{|h_{02}|}$, it holds

$$\tfrac{1}{|h_{02}|}J(\tilde V^+,\tilde V^-) = \sqrt{1-\alpha^2}\left(\int_0^1 2t\ \tilde V^-\cdot a\,\mathrm{d}t + \int_0^1 2t\ \tilde V^+\cdot a\,\mathrm{d}t\right)+$$

$$\alpha\left(\int_0^1 -\tilde V^+(t)\cdot A\left[X_1^+ - X^+(t),\right]\,dt+\left[X_1^- - X_0^-\right]\cdot B\left[X_1^+ - X_0^+\right]+\right.$$

$$\left.+\int_0^1 \tilde V^-(t)\cdot A\left[X_1^- - X_0^-\right]\,dt - \int_0^1 \tilde V^-(t)\cdot A\left[X_1^- - X^-(t)\right]\,dt\right)+$$

$$+\varepsilon\text{ const}\int_0^1 t|\tilde V^+|^2dt+\int_0^1 t|\tilde V^-|^2dt,$$

namely

$$\tfrac{1}{|h_{02}|}J(\tilde V^+,\tilde V^-) = \sqrt{1-\alpha^2}\left(\int_0^1 2t\ \tilde V^-\cdot a\,\mathrm{d}t + \int_0^1 2t\ \tilde V^+\cdot a\,\mathrm{d}t\right)+$$

$$+\alpha\int_0^1 -\tilde V^+(t)\cdot A\left[X_1^+ - X^+(t),\right]\,dt+\alpha\int_0^1 -\tilde V^-(t)\cdot A\left[X_1^- - X^-(t)\right]\,dt+$$

$$+\alpha\left[X_1^- - X_0^-\right]\cdot B\left[X_1^+ - X_0^+\right]+\varepsilon\text{ const}\left(\int_0^1 t|\tilde V^+|^2dt+\int_0^1 t|\tilde V^-|^2dt\right).$$

Now, assuming

$$\begin{cases} v^\pm = -V^\pm \\ x^\pm(t) = X_1^\pm - X^\pm(t), \end{cases}$$

the constraint become

$$||v^\pm - a|| \le 1,$$

hence we are going to study the following control system

$$\dot x^\pm = v^\pm$$

and we want to minimize

$$\begin{aligned} I(v^+,v^-) = -J(v^+,v^-) \quad &= \quad \int_0^1 2t\sqrt{1-\alpha^2}(v^-+v^+)\cdot a\ dt \\ &- \int_0^1 \alpha v^+\cdot Ax^+dt - \int_0^1 \alpha v^-\cdot Ax^-dt \\ &-\alpha x_0^-\cdot Bx_0^+ - \varepsilon\text{ const}\int_0^1 t(|v^-|^2+|v^+|^2)dt, \end{aligned}$$

and see that it remains greater that 0.
From equation (6.3.2) we have $|v^{\pm}|^2 \leq 2v^{\pm} \cdot a$, then $-|v^{\pm}|^2 \geq -2v^{\pm} \cdot a$, and assuming const $= 1$ we have

$$I(v^+, v^-) \geq \int_0^1 2t \left(\sqrt{1-\alpha^2} - \varepsilon\right)(v^- + v^+) \cdot a \; dt$$

$$- \int_0^1 \alpha v^+ \cdot Ax^+ dt - \int_0^1 \alpha v^- \cdot Ax^- dt - \alpha x_0^- \cdot Bx_0^+.$$

Calling $\hat{\beta} = \sqrt{1-\alpha^2} - \varepsilon$, let us redenote

$$I(v^+, v^-) := \int_0^1 2t\hat{\beta}(v^- + v^+) \cdot a \; dt$$

$$- \int_0^1 \alpha v^+ \cdot Ax^+ dt - \int_0^1 \alpha v^- \cdot Ax^- dt - \alpha x_0^- \cdot Bx_0^+.$$

Now, we rewrite it in complex coordinates instead of $\mathbb{R}^2$ coordinates

$$I(v^+, v^-) = \int_0^1 t\hat{\beta} \langle v^- + v^+, a \rangle \; dt$$

$$+ \int_0^1 \alpha \langle v^+, ix^+ \rangle dt - \int_0^1 \alpha \{v^-, ix^- \rangle dt - \alpha \left\langle \overline{x}_0^-, ix_0^+ \right\rangle,$$

then we obtain

$$I(v^+, v^-) = \underbrace{\int_0^1 \left\langle v^+, 2\hat{\beta} t\, a + \alpha\, i\, x^+ \right\rangle dt}_{I^+} + \underbrace{\int_0^1 \left\langle v^-, 2\hat{\beta} t\, a + \alpha\, i\, x^- \right\rangle dt}_{I^-} + \langle \bar{x}_0^-, ix_0^+ \rangle .$$

Remark 6.4.1. If we consider the system

$$\dot{q} = f(q, u)$$

and we minimize

$$a(q_0) + \int_0^t \phi(q(t))dt,$$

we want to investigate if at time 0 at q_0 the covector is $\lambda_0 = +dq_0 a$.
Let us add a further variable y

$$\begin{cases} \dot{y} = 0 \\ \dot{q} = f(q, u) \end{cases}$$

and assume $y_0 = a(q_0)$, and apply the Pontryagin maximum principle with the transversality condition, then $\dot{\nu} = 0$, $y_0 - a(q_0) = 0$ and

$$(\nu, \lambda_0) = \mathrm{const}(-1, d_{q_0} a);$$

since $\nu < 0$ then

$$(\nu, \lambda_0) = (-1, d_{q_0} a)$$

and we have the thesis.

Hence we can consider directly the system

$$\begin{cases} \dot{x}^+ = v^+ \\ x_1^+ = 0 \\ I^+ = \int_0^1 \left\langle v^+, 2\hat{\beta}t\, a + \alpha\, i\, x^+ \right\rangle dt \to \min \end{cases}$$

or

$$\begin{cases} \dot{x}^- = v^- \\ x_1^- = 0 \\ I^- = \int_0^1 \left\langle v^-, 2\hat{\beta}t\, a + \alpha\, i\, x^- \right\rangle dt \to \min \end{cases}$$

and prove that the minimum of $I^\pm$ is strictly positive.

Let us consider directly the system

$$\begin{cases} \dot{x} = v \\ x_1 = 0 \\ ||v - a|| \leq 1 \\ I = \int_0^1 \left\langle v, 2\hat{\beta}t\, a + \alpha\, i\, x \right\rangle dt \to \min \end{cases}$$

and apply the Pontryagin maximum principle.
The family of Hamiltonian function parametrized by the control v is

$$h_v(p, x) = \langle p, v \rangle$$

if we analyse abnormal extremal, and

$$h_v(p, x) = \langle p, v \rangle - \left\langle v, 2\hat{\beta}t\, a + \alpha\, i\, x \right\rangle,$$

namely,

$$h_v(p, x) = \left\langle v, p - 2\hat{\beta}t\, a - \alpha\, i\, x \right\rangle$$

if we consider normal extremal.
We are going to study normal extremals.
Denoting

$$y = p - 2\hat{\beta}t\, a - \alpha\, i\, x,$$

since $||v - a|| \leq 1$, the maximized Hamiltonian is

$$H = \langle a, y \rangle + ||y||$$

with

$$v_{\max} = a + \frac{y}{||y||}.$$

Moreover, the normal extremals, thus, in T^*M they satisfy the following Hamiltonian system

$$\begin{cases} \dot{x} = a + \frac{y}{||y||} \\ \dot{p} = i\,\alpha \left(a + \frac{y}{||y||} \right), \end{cases}$$

and

$$\dot{y} = -2\alpha\, i \left(a + \frac{y}{||y||} \right) - 2\hat{\beta}a. \tag{6.4.1}$$

Claim 6.4.2. *If we denote*

$$a = e^{i\phi}$$
$$y = re^{i\theta}$$

we can see that the phase portrait of any curve $y(t)$ *is well define.*

Proof. By equation (6.4.1) we can see that

$$\left(\dot{r} + i\,r\,\dot{\theta}\right)e^{i\theta} = -2\hat{\beta}e^{i\phi} - i\left[2\alpha\left(e^{i\phi} + e^{i\theta}\right)\right]$$

then

$$\dot{r} + i\,r\,\dot{\theta} = -2\hat{\beta}e^{i(\phi-\theta)} - i\left[2\alpha\left(e^{i(\phi-\theta)} + 1\right)\right].$$

In order to give a smarter notation let us define $\bar{r}e^{i\bar{\phi}} = \alpha + i\hat{\beta}$, then

$$\dot{r} + i\,r\,\dot{\theta} = -2\bar{r}\sin(\bar{\phi})e^{i(\phi-\theta)} - i\left[2\bar{r}\cos(\bar{\phi})\left(e^{i(\phi-\theta)} + 1\right)\right]$$

namely

$$\dot{r} + i\,r\,\dot{\theta} = -2\bar{r}e^{i((\phi-\bar{\phi})-\theta)} - i2\bar{r}\cos(\bar{\phi})$$

thus,

$$\begin{cases} \dot{r} = -2\bar{r}\cos\left((\phi-\bar{\phi})-\theta\right) \\ \\ \dot{\theta} = -\frac{2\bar{r}}{r}\left[\cos(\bar{\phi}) + \sin\left((\phi-\bar{\phi})-\theta\right)\right]. \end{cases}$$

We can see that

$$r\left[\cos(\bar{\phi}) + \sin\left((\phi-\bar{\phi})+\theta\right)\right] = \text{const}$$

is constant, indeed,

$$\dot{r}\left[\cos(\bar{\phi}) + \sin\left((\phi-\bar{\phi})-\theta\right)\right] + r\tfrac{d}{dt}\left[\cos(\bar{\phi}) + \sin\left((\phi-\bar{\phi})-\theta\right)\right] =$$

$$= -2\bar{r}\cos\left((\phi-\bar{\phi})-\theta\right)\left[\cos(\bar{\phi}) + \sin\left((\phi-\bar{\phi})-\theta\right)\right] +$$

$$+\cos\left((\phi-\bar{\phi})-\theta\right)\left(2\bar{r}\left[\cos(\bar{\phi}) + \sin\left((\phi-\bar{\phi})-\theta\right)\right]\right) = 0$$

□

6.5 Sufficient optimality condition with n=3 and k=2

Finally, let us summarise sufficient optimality results for a system (6.2.3) when $n = 3$ and $k = 2$.

We proved the optimality of broken extremals, that passes through $\bar{\lambda} \in \Lambda$ such that $\bar{q} \in \pi(\bar{\lambda})$, if

- $f_0 \wedge f_1 \wedge f_2 \neq 0$ at $\bar{q}$, namely f_0, f_1, f_2 are linearly independent at point $\bar{q}$,

or

- $f_1 \wedge f_2 \wedge [f_1, f_2] = 0$ at $\bar{q}$, namely those fields are linearly dependent at point $\bar{q}$.

It means that each normal extremal that projects in $O_{\bar{q}}$, a neighbourhood of $\bar{q}$ small enough, is optimal. On the other hand, if at point $\bar{q}$ the distribution $\mathrm{span}\{f_1(q), f_2(q)\}$ is not contact, any broken extremal (even abnormal) passing through $\bar{\lambda}$ is optimal.

Among all settings, it remains the case in which

- $f_0 \wedge f_1 \wedge f_2 = 0$ and $f_1 \wedge f_2 \wedge [f_1, f_2] \neq 0$ at point $\bar{q}$,

namely, we have a broken abnormal extremal passing through $\bar{\lambda}$ and the fields f_1 and f_2 generate a contact distribution at $\bar{q}$.

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