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Structure and Regularity of Solutions to Nonlinear Scalar Conservation Laws

Ph.D. Thesis

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Contents

Introduction

We consider the Cauchy problem for the scalar conservation law in one-space dimension:

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,\cdot) = u_0(\cdot) & \text{in } \mathbb{R}, \end{cases} \tag{1}$$

where the flux function $f : \mathbb{R} \to \mathbb{R}$ is given and supposed to be smooth and $u : \mathbb{R}^+_t \times \mathbb{R}_x \to \mathbb{R}$ is the unknown spatial density of the conserved quantity.

A classical result [Ole63] is that if the flux f is uniformly convex, say $f'' \geq c > 0$, then for every $t > 0$ the entropy solution $u(t)$ of (1) with $u_0 \in L^\infty(\mathbb{R})$ has locally bounded variation and it holds the one-sided Lipschitz estimate

$$D_x u(t) \leq \frac{\mathcal{L}^1}{ct}. \tag{2}$$

This is the most striking example where the nonlinearity in a PDE yields a regularizing effect on the solution. On the contrary a linear flux $f(u) = \lambda u$ does not generate additional regularity, indeed the solution to (1) is given by $u(t,x) = u_0(x - \lambda t)$.

However some of the qualitative properties of the solution u which hold true in the convex case make sense and can be investigated also in more generality: two important examples are the rectifiability of the entropy dissipation measure and the BV regularity of $f' \circ u$, or the corresponding regularity of u.

In this thesis we focus exactly on these two questions.
It is natural to split this research in two parts as follows:

(1) study the qualitative structure of L^∞-entropy solutions to (1) with general smooth fluxes;
(2) quantify the regularity of the entropy solution u to (1) in terms of an appropriate measure of the nonlinearity of the flux f.

The unifying framework of these two directions is the *Lagrangian representation*, which is essentially an extension of the method of characteristics to a non smooth setting.

At the end of this thesis we also introduce a first version of Lagrangian representation for entropy solutions to scalar conservation laws in several space dimension, studying its compactness and regularity properties. As an application we describe the structure of continuous entropy solutions.

We now present in more details our contribution to the problems above.

Structure of L^∞-entropy solutions

We focus our attention on entropy solutions: by definition for every smooth convex entropy $\eta : \mathbb{R} \to \mathbb{R}$, it holds in distributions

$$\eta(u)_t + q(u)_x \leq 0, \tag{3}$$

where $q'(u) = f'(u)\eta'(u)$ is the entropy flux. In particular the l.h.s. of (3) is a negative locally bounded measure μ, with the additional property that $\mu(B) = 0$ for all Borel sets B such that $\mathcal{H}^1(B) = 0$: this last property is a consequence of being the divergence of an L^∞ vector field.

For BV solutions, Volpert's formula together with the definition of the entropy flux q gives that

$$\begin{aligned}
&\eta(u)_t + q(u)_x = \eta'(u)\big(D_t^{\mathrm{cont}} u + f'(u) D_x^{\mathrm{cont}} u\big) \\
&+ \sum_{i\in\mathbb{N}} \bigg\{ - \dot\gamma_i(t) \Big[\eta(u(t,x+)) - \eta(u(t,x-))\Big] + \Big[q(u(t,x+)) - q(u(t,x-))\Big]\bigg\} g_i(t) \mathcal{H}^1 \llcorner_{\mathrm{Graph}(\gamma_i)} \\
&= \sum_{i\in\mathbb{N}} \bigg\{ - \dot\gamma_i(t) \Big[\eta(u(t,x+)) - \eta(u(t,x-))\Big] + \Big[q(u(t,x+)) - q(u(t,x-))\Big]\bigg\} g_i(t) \mathcal{H}^1 \llcorner_{\mathrm{Graph}(\gamma_i)},
\end{aligned}$$

where

(1) $D^{\mathrm{cont}} u = (D_t^{\mathrm{cont}} u, D_x^{\mathrm{cont}} u)$ is the continuous part of the measure Du,
(2) $u(t,x\pm)$ is the right/left limit of $u(t)$ at the point x,
(3) the curves γ_i are such that

$$D^{\mathrm{jump}} u = \sum_i \big(u(t,x+) - u(t,x-)\big) \begin{pmatrix} 1 \\ -\dot\gamma_i(t) \end{pmatrix} g_i(t) \mathcal{H}_i \llcorner_{\mathrm{Graph}(\gamma_i)}, \quad g_i(t) = \frac{1}{\sqrt{1 + |\dot\gamma_i(t)|^2}}.$$

In short we will say that *the entropy dissipation is concentrated*, meaning that the measure μ is concentrated on a countably 1-rectifiable set J. A simple superposition argument implies that J can be chosen to be independent on η.

For general L^∞-entropy solutions, if the flux is uniformly convex, by the Oleinik estimate (2), the solution has locally bounded variation for all positive times, therefore the above computation applies.

More general fluxes are considered in [**DLR03**]: here it is assumed that $f' \circ u$ has locally bounded variation and that the flux f has finitely many inflection points with polynomial degeneracy. The authors prove that for every smooth entropy η, the entropy dissipation measure μ is concentrated on the singular set of $f' \circ u$. The $\mathrm{BV}_{\mathrm{loc}}$ regularity of $f' \circ u$ has been proved in the case of one and two inflection points with polynomial degeneracy in [**Che86**]. We will come back on this point in the second part of this introduction.

It is worth to mention that the concentration of the measure μ is interesting also in the non entropic setting: the understanding of the structure of weak solutions to (1) such that μ is a Radon measure is important in models arising from different areas of physics (see [**DLR03**] and the references therein for more details on these topics).

A short way to state the main result of this part is the following:

THEOREM 1. If u is a bounded entropy solution of the scalar conservation law (1), then the entropy dissipation is concentrated.

No assumption on the flux function f have been made, i.e. it can have Cantor-like sets where $f'' = 0$. However such a statement is a corollary of a detailed description of the structure of bounded entropy solutions.

The first important result is that to every entropy solutions it is possible to associate a family of Lipschitz curves $t \mapsto \gamma(t)$ covering all $\mathbb{R}^+ \times \mathbb{R}$ with associated a set of values w, and a time function $\mathtt{T} = \mathtt{T}(\gamma, w)$ such that w is an entropy admissible boundary value on $\mathrm{Graph}(\gamma \llcorner_{[0,\mathtt{T}(\gamma,w)]})$. The set $\mathcal{K} \subset \mathrm{Lip}(\mathbb{R}^+, \mathbb{R}) \times \mathbb{R}$ made of the couples (γ, w) together with the function $\mathtt{T}(\gamma, w)$ is called *complete family of boundaries*: the precise definition is Definition 2.3, which contains also additional monotonicity, connectedness and regularity properties of the set of boundaries. In other words, the following diagram is commutative:

$$\begin{tikzcd}[column sep=huge]
u_0 \arrow[rr, "\text{Solve PDE } u_t + f(u)_x = 0 \text{ for } t \le \mathtt{T}"] \arrow[dr, "\substack{\text{Solve the PDE}\\ u_t+f(u)_x=0 \\ \text{in } \{x \lessgtr \gamma(t)\} \\ \text{with boundary data } w}"'] & & u(t) \arrow[dl, "\substack{\text{Restrict } u(t) \text{ to}\\ \{x\lessgtr\gamma(t)\}}"] \\
& u^{\pm}(t) &
\end{tikzcd}$$

The existence of a complete family of boundaries follows from quite easy compactness arguments, due to the stability of admissible boundaries under the convergence of the boundary value and the boundary set [Sze89]. The only technicality here is to prove that for a dense sets of initial data we can actually construct such a family of boundaries conditions: this is done by hand for wavefront tracking solutions, and then passed to the limit. It is interesting that the requirement to be admissible on both sides forces the curve $\gamma(t)$ to be a characteristic in the BV setting, where a suitable pointwise definition of speed is available by the Rankine-Hugoniot condition. Moreover, up to a $\mathcal{H}^1$-negligible set of points (and assuming for simplicity that f has not flat parts), the admissible boundary values of γ at time t are equal to the segment with extremal $u(t, \gamma(t)\pm)$, as the standard entropy conditions requires. In the general case, the admissible boundary values contain the previous segment, but it may have as well parts which are in the flat part of f to which $u(t, \gamma(t)\pm)$ belongs (see Lemma 2.11).

There are some important properties of the set $\mathcal{K}$ and the function $\mathtt{T}$ which play a role in describing the structure of the solution $u(t)$. One almost obvious requirement is that

$$\text{Graph } u \subset K := \Big\{(t,x,w) : \exists(\gamma,w) \in \mathcal{K}, \gamma(t) = x, \mathtt{T}(\gamma,w) > t\Big\}, \tag{4}$$

i.e. there exists at least one admissible boundary for each point and the value $u(t,x)$ is one of these admissible boundary values (up to redefining u in a $\mathcal{L}^2$-negligible set).

Consider moreover a region Ω bounded by two admissible curves γ_1, γ_2, such that

$$\gamma_1(\bar t) = \gamma_2(\bar t) \qquad \text{and} \qquad \forall t \in (\bar t, T] \,\big(\gamma_1(t) < \gamma_2(t)\big). \tag{5}$$

Then the function u inside Ω solves a boundary problem where the only data are the two boundary values w_1, w_2 associated to γ_1, γ_2 respectively (we need to assume that $\mathtt{T}(\gamma_i, w_i) > T$, $i = 1, 2$, but this is not restrictive due to (4) above). It is a simple generalization of the construction of the Riemann solver to give explicitly the unique monotone solution in Ω (Lemma 1.32). In particular $u\llcorner_\Omega$ is a BV function.

A second important property is that the curves γ can be taken totally ordered:

$$\exists s \in \mathbb{R}^+\big(\gamma(s) < \gamma'(s)\big) \quad \Longrightarrow \quad \forall t \in \mathbb{R}^+\big(\gamma(t) \le \gamma'(t)\big).$$

In particular they generate a monotone flow $\mathtt{X}(t,y)$, with $y \in \mathbb{R}$ not necessarily the initial position due possible future rarefactions: denote the curve $t \mapsto \mathtt{X}(t,y)$ by γ_y. This monotonicity allows us to define a maximal and minimal characteristic $\gamma^\pm_{t,x}$ passing through a point given (t,x), and then to show that if $\gamma_{\bar y}$ is an admissible boundary and $\gamma_{\bar y}(t) \neq \gamma_y(t)$ for all $y > \bar y$, then it is a segment in $[t_1, t_2]$ (Lemma 3.1). A symmetric result holds in the case $y < \bar y$.

As a corollary, it is possible to split the half plane $\mathbb{R}^+ \times \mathbb{R}$ into 4 parts, depending on the behavior of the sequences $\gamma \to \gamma^\pm_{t,x}$:

(1) a set A_1 such that for all $(t,x) \in A_1$
$$\gamma^-_{t,x} < \gamma^+_{t,x};$$
(2) an open set B made of regions bounded by two admissible curves satisfying (5): inside each component $u \in BV$;

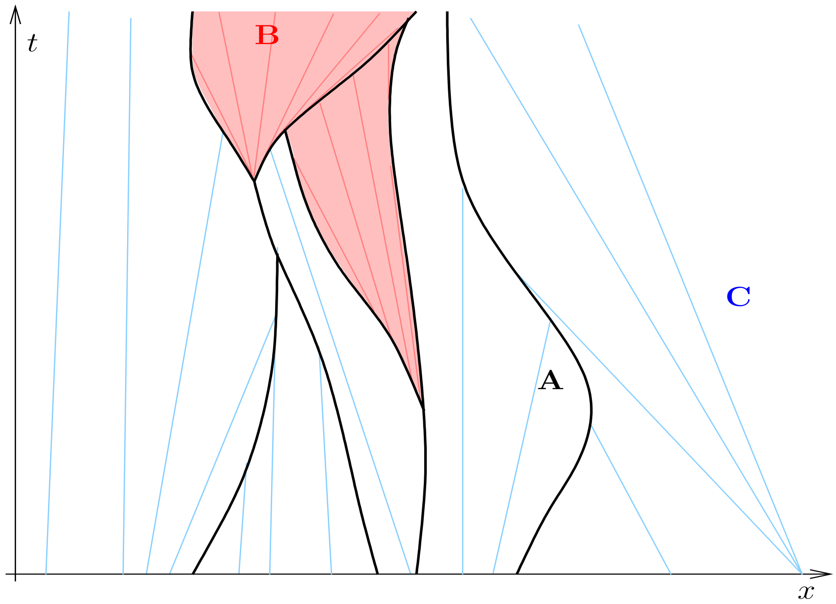


FIGURE 1. The structure of characteristics of an entropy solution u.

(3) a set of segments C, made of all admissible boundaries $\gamma_{\bar{y}}$ such that in $[0, \bar{t}]$

$$\forall t \in [0, \bar{t}], \forall y \neq \bar{y} \left(\gamma_{\bar{y}}(t) \neq \gamma_y(t)\right).$$

(4) a residual set A_2, where either the condition (3) above holds only on one side or it is the boundary of two BV regions.

Set $A = A_1 \cup A_2$.

The main result about this decomposition is the following (see Figure 1 and Section 3.1):

THEOREM 2. The exists a disjoint partition $\mathbb{R}^+ \times \mathbb{R} = A \cup B \cup C$ such that

(1) A is countably 1-rectifiable,
(2) B open and $u \llcorner_B$ is locally BV,
(3) C is made of disjoint segments starting from 0.

It is possible to compute the right and left limits for a given point up to the linearly degenerate components of the flux f: these are the connected components of the compact set $\{f'' = 0\}$. It follows that the characteristic speed has a BV structure: it is continuous outside A and it has L^1-right/left limits across every Lipschitz curve $\gamma(t)$ up to a $\mathcal{L}^1$-negligible set of t (Remark 3.8). In particular we conclude that the admissible boundaries are characteristics.

The jump set J is the set A together with the countably many segments in C which can dissipate, even if no characteristic is entering on both sides.

At this point one can prove Theorem 1. The only case requiring a careful analysis is the set of segments C. By partitioning the segments according to their length, and using the elementary fact that the slope of non intersecting segments of length 2ϵ is Lipschitz w.r.t. the distance of their middle points, the conjecture on the concentration of entropy dissipation is equivalent to require that the projection of the measure μ on the middle points has no continuous part. The Dirac deltas correspond to segment of C which belongs to J.

The basic idea is to use the two balances

$$u_t + f(u)_x = 0, \qquad \eta(u)_t + q(u)_x = \mu$$

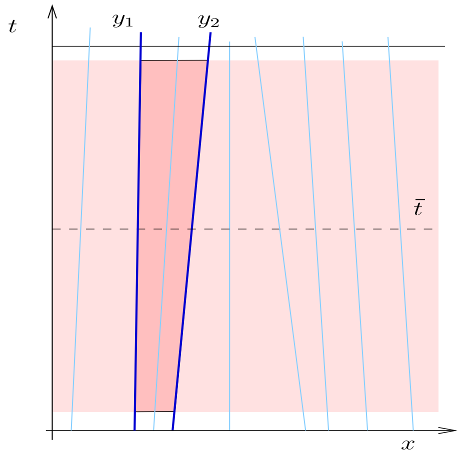


FIGURE 2. A model set of segments parameterized by their middle point and a cylinder.

on the cylinders made by segments in C (see Figure 2). Using the regularity of the slopes $\lambda(y)$ of the segments, which makes the bottom and top base equivalent, one concludes that the fluxes

$$Q(u) = q(u(y)) - \lambda(y)\eta(u(y)), \qquad F(u) = f(u(y)) - \lambda(y)u(y), \tag{6}$$

are BV and Lipschitz, respectively (Lemma 3.10). We note that $u(y)$ can be any value on the linearly degenerate component I containing u, because the quantities in the r.h.s. of (6) are constant in I.
From the balance of F one recovers also that $u(t, \gamma_y(t))$ is constant for $\mathcal{L}^1$-a.e. y, and then that the following holds (Lemma 3.14): denoting with $\mathtt{d}_y$ the limit of incremental ratio only using segments in C,

$$\mathtt{d}_y F(y) = -\mathtt{d}_y \lambda(y) u(y), \tag{7}$$

which is the correct version of the smooth chain rule $(f(u) - f'(u)u)_x = -(f'(u))_x u$.
A geometric lemma (Lemma 3.11), based on the assumption that if $\eta(u)/u \leq C$, then the curve

$$w \mapsto \begin{pmatrix} f(w) - f'(w)w \\ q(w) - f'(w)\eta(w) \end{pmatrix}$$

is rectifiable with tangent of bounded slope, implies that if $D_y F(y)$ has no Cantor part, then $D_y Q(y)$ has no Cantor part too. The chain rule (7) above gives also that

$$\mathtt{d}_y Q(y) = -\mathtt{d}_y \lambda(y) \eta(u(y)),$$

which shows that the dissipation μ has no absolutely continuous part too (Lemma 3.15).

To conclude the analysis, we have to study also the set of endpoints of segments which do not belongs to A: these are starting points of shocks. The analysis on cylinders allows to include some of these points: more precisely, the points which lie in the interior of a family of cylinders shrinking to the closed segment $\gamma_y([0, T_1(y)])$, where $T_1(y)$ is the last time before γ_y enters in A. Hence one can repeat the analysis above and conclude that no dissipation occurs in these points.
The remaining points lie on a countably 1-rectifiable set, because $T_1(y) < T_1(\bar{y}) + |y - \bar{y}|$ for either $y < \bar{y}$ or $y > \bar{y}$ close to $\bar{y}$: hence a standard criterion for rectifiability applies. One can thus use a blow up techniques, and show that the limiting solution has parallel characteristics (otherwise a shock appears) and then no dissipation is possible.

This concludes the proof of Theorem 1.

A second application of the existence of a full set of admissible boundaries is the fact that the time traces for measure valued entropy solutions are taken in a strong sense. Let

$$\nu = \int \nu_{t,x} dt dx$$

be a Young measure on $\mathbb{R}^+ \times \mathbb{R}$ such that

$$\mathrm{supp} \;\; \nu_{t,x} \subset [M, M],$$

and for all convex entropies η and corresponding entropy flux q it holds

$$\langle \nu_{t,x}, \eta \rangle_t + \langle \nu_{t,x}, q \rangle \leq 0,$$

where for all continuous functions $g : \mathbb{R} \mapsto \mathbb{R}$ we have used the notation

$$\langle \nu_{t,x}, g \rangle := \int g(v) \nu_{t,x}(dv).$$

Now let $\nu^+_{0,x}$ be the trace of $\nu_{t,x}$ as $t \searrow 0$. We prove the following result:

THEOREM 3. If $\mathfrak{d}$ is any bounded distance metrizing the weak topology on probability measures, it holds

$$\lim_{t \searrow 0} \int_{\mathbb{R}} \mathfrak{d}\big(\nu_{t,x}, \nu^+_{0,x}\big) dx = 0.$$

In particular if $\nu^+_{0,x} = \delta_{u(t,x)}$ is a Dirac solution, we recover the fact that $t \mapsto u(t)$ is continuous in L^1 also at $t = 0$, so that Theorem 3 is an extension to mv solutions of a result proved in [Pan09] in the case of f smooth.

The final result of this part is that a Lagrangian representation of the solution exists:

THEOREM 4. There exist a monotone flow $\mathtt{X}$ of characteristics and two functions $\mathtt{u}(y)$, $\mathtt{T}(y)$ such that,

$$u(t, \mathtt{X}(t, y)) = \mathtt{u}(y), \qquad t \leq \mathtt{T}(y),$$

for $\mathcal{L}^1$-a.e. x.

The difference with respect to a complete family of boundaries is that we remove the ambiguity of the linearly degenerate parts on segments.

Regularity estimates for L^∞-entropy solutions

In this second part we investigate the regularizing effect that the nonlinearity of the flux f has on the entropy solution u to (1). We already discussed the extremal cases of f uniformly convex, where the initial datum $u_0 \in L^\infty(\mathbb{R})$ is immediately regularized to a function of bounded variation and the case of f linear, where the initial datum is simply translated as time goes on. We will first deal with the case in which the flux function f has no flat parts, then we consider the more particular case of fluxes with isolated inflection points with polynomial degeneracy.

In order to fix the terminology, we say that f is *weakly genuinely nonlinear* if $\{w : f''(w) \neq 0\}$ is dense. Under this assumption on the flux, it is proved in [Tar79] that an equibounded family of entropy solutions to (1) is precompact in $L^1_{\mathrm{loc}}(\mathbb{R})$. Always relying on some nondegeneracy condition of the flux, regularity estimates in terms of fractional Sobolev spaces can be obtained also in several space dimensions, by means of the kinetic formulation and averaging lemmas (see [LPT94, DLM91]). The kinetic formulation is also one of the basic tools in [DLOW03], where the authors prove that solutions to scalar conservation laws in several space dimensions enjoy some fine properties of BV functions (see also [COW08]). In [CJ17] the regularity of the entropy solution u in the case of a strictly convex flux f is expressed in terms of BV^Φ spaces: the authors provide a convex function $\Phi : [0,+\infty) \to [0,+\infty)$ depending on the nonlinearity of f such that for every $t > 0$ and $[a,b] \subset \mathbb{R}$, the solution $u(t)$ satisfies

$$\Phi\text{-TV}_{[a,b]}u(t) := \sup_{n\in\mathbb{N},\, a<x_1<\ldots<x_n<b} \sum_{i=1}^{n-1} \Phi\left(|u(t,x_{i+1}) - u(t,x_i)|\right) < +\infty.$$

In order to state our first result in this direction, we need to introduce a convenient way to quantify the nonlinearity of the flux f: for any $h > 0$ let

$$\mathfrak{d}(h) := \min_{a\in[-\|u_0\|_\infty, \|u_0\|_\infty - h]} \mathrm{dist}(f\llcorner[a,a+h], \mathcal{A}(a,a+h)),$$

where $\mathcal{A}(a,a+h)$ denotes the set of affine functions defined on $[a,a+h]$ and the distance is computed with respect to the L^∞ norm. Moreover let Φ be the convex envelope of $\mathfrak{d}$ and set for every $\varepsilon > 0$

$$\Psi_\varepsilon(x) = \Phi\left(\frac{x}{2}\right)x^\varepsilon.$$

Then we prove the following result.

THEOREM 5. Let f be weakly genuinely nonlinear and u be the entropy solution of (1) with $u_0 \in L^\infty(\mathbb{R})$ with compact support. Let moreover $\varepsilon > 0$ and Ψ_ε be defined above. Then there exists a constant $C > 0$ depending on $\mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u_0))$, ε, $\|u_0\|_\infty$ and $\|f'\|_\infty$ such that for every $t > 0$, it holds

$$u(t) \in \mathrm{BV}^{\Psi_\varepsilon}(\mathbb{R}) \qquad \text{and} \qquad \Psi_\varepsilon\text{-TV}(u(t)) \le C\left(1 + \frac{1}{t}\right).$$

As already mentioned, the proof relies on the properties of the Lagrangian representation of the solution. The main novelty of this part is the following "length estimate": let $t > 0$ and $x_1 < x_2$ be such that $u(t,x_1) = u(t,x_2) = \bar{w}$ and consider the characteristics $\mathtt{X}(\cdot,y_1)$ and $\mathtt{X}(\cdot,y_2)$ with value $\bar{w}$ such that $\mathtt{X}(t,y_1) = x_1$ and $\mathtt{X}(t,y_2) = x_2$. Let

$$w_m := \inf_{(x_1,x_2)} u(t) \qquad \text{and} \qquad w_M := \sup_{(x_1,x_2)} u(t).$$

If we denote by $s = \max\{x_2 - x_1, \mathtt{X}(0,y_2) - \mathtt{X}(0,y_1)\}$ and

$$\mathfrak{d}(w_m,w_M) := \mathrm{dist}\left(f\llcorner[w_m,w_M], \mathcal{A}(w_m,w_M)\right),$$

then it holds

$$s \ge \frac{\mathfrak{d}(w_m,w_M)t}{\|u_0\|_\infty}. \tag{8}$$

Roughly speaking it means that an oscillation between two values at time t must occupy a space at time 0 or at time t of length bounded by below in terms of the nonlinearity of the flux f between the extremal values. In particular, by finite speed of propagation, if u_0 has compact support, then the estimate above provides an a priori bound on the number of oscillations of a given height. The proof of Theorem 5 is obtained by giving a uniform estimate on an approximating sequence of piecewise monotone solutions. After introducing an appropriate decomposition of piecewise monotone functions, we are in position to apply the previous estimate and this leads to the proof of Theorem 5. However we notice that the regularity obtained in Theorem 5 is not sharp for example in the case of Burgers' equation, i.e. if $f(u) = u^2$, and in general for any flux of the form $f(u) = u^k$, with $k \ge 2$. This case requires a more specific analysis.

Now we restrict again the class of flux functions that we are going to consider: we say that the flux f has *polynomial degeneracy* if $\{f''(w) = 0\}$ is finite and for each $w \in \{f''(w) = 0\}$ there exists $p \ge 2$ such that $f^{(p+1)}(w) \ne 0$. More precisely, for every $w \in \{f''(w) = 0\}$ let p_w be the minimal $p \ge 2$ such that $f^{(p+1)}(w) \ne 0$ and let $\bar{p} = \max_w p_w$. We say that $\bar{p}$ is the *degeneracy* of f.

As conjectured in [LPT94], it is proved in [Jab10] that if the flux f as above has degeneracy $p \in \mathbb{N}$, then for every $\varepsilon, t > 0$ the entropy solution $u(t) \in W^{s-\varepsilon,1}_{\mathrm{loc}}(\mathbb{R})$, with $s = \frac{1}{p}$. The result is proved actually in several space dimensions. However in this setting, it seems convenient to describe the regularity of u in terms of fractional BV spaces, i.e. BV^Φ spaces with $\Phi(u) = u^\alpha$ for some $\alpha \geq 1$. In [BGJ14], under the additional convexity assumption on the flux f, the authors prove that for every $t > 0$ the entropy solution $u(t) \in BV^s_{\mathrm{loc}}(\mathbb{R})$. In particular this implies that $u(t) \in W^{s-\varepsilon,p}_{\mathrm{loc}}(\mathbb{R})$ and that for every x, the function $u(t)$ admits both left and right limits. The strategy to prove this result is essentially to exploit the BV regularity of $f' \circ u(t)$ for $t > 0$ and then to deduce the corresponding regularity for the solution u itself.

Paying some attention in "inverting" f', we follow here the same strategy here to deal with the nonconvex case. Therefore we first show the following result:

THEOREM 6. Let f be a flux of polynomial degeneracy and let u be the entropy solution of (1) with $u_0 \in L^\infty(\mathbb{R})$ with compact support. Then there exists a constant $C > 0$ depending on $\mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u_0))$, $\|u_0\|_\infty$ and f such that for every $t > 0$, it holds

$$\mathrm{TV}(f' \circ u(T)) \leq C \left(1 + \frac{1}{T}\right). \tag{9}$$

As we have already mentioned, a proof of this result for fluxes with one or two inflection points is provided in [Che86]. We have to notice that, as it is shown here by an example at the end of Chapter 3, the assumption of polynomial degeneracy cannot be removed. Moreover, in the proof for the case of fluxes with one inflection point with polynomial degeneracy, the author makes implicitly some simplifying assumptions that do not hold in general. However the general argument is valid and we implement it here.

Before giving some comments about the proof of Theorem 9, we also mention that the case of fluxes with a single inflection point is also studied in [BC81] for homogeneous fluxes $f(u) = |u|^{\alpha-1}u$, by a scaling argument and in [Daf85] for fluxes with polynomial degeneracy at the inflection point, by an accurate description of the extremal backward characteristics. In both these works, the author gets the BV regularity for positive time of the following nonlinear function of the entropy solution:

$$F \circ u(t) := f \circ u(t) - u(t)(f' \circ u(t)).$$

This leads to a fractional regularity of the solution of one order less accurate then the sharp one: more precisely, if p is the degeneracy of the flux f, it is possible to deduce from the previous results that the entropy solution $u(t) \in BV^s(\mathbb{R})$ with $s = \frac{1}{p+1}$.

The argument of the proof of Theorem 6 is the following: as we did for Theorem 5, we give a uniform estimate on an approximating sequence of piecewise monotone entropy solutions. By means of the length estimate (8), we can uniformly bound from above the number of regions that we need to divide a bounded domain in $(\bar{t}, +\infty) \times \mathbb{R}$ for some $\bar{t} > 0$, in such a way that in each region the oscillation of the entropy solution u is smaller than a fixed quantity $\varepsilon > 0$. This reduces the analysis to the local behavior of the solution which takes values around an inflection point of polynomial degeneracy, or where the flux is strictly convex (or concave). In the last case the analysis is simpler and well-known, in the first case we rely on the structure proved in [Daf85] to apply the argument of [Che86]. We observe that also in these cases the interpretation of characteristics as admissible boundaries plays a role.

It remains open the problem of the optimal regularity of $f' \circ u$ with f smooth, out of the polynomial degeneracy assumption: the examples that we will present suggest that the right space could be $f' \circ u \in L^1(\mathbb{R}^+, \mathrm{BV}^\Phi(\mathbb{R}))$ with Φ such that in a neighborhood of 0 it holds $\Phi(-x \log x) = x$. One difficulty is that for a fixed time t, there is in general no uniform estimates of Φ-$\mathrm{TV}(f' \circ u(t))$.

After Theorem 6, we are in position to extend the result of [BGJ14] to the non-convex case: more precisely we prove the following result.

THEOREM 7. Let f be a flux of degeneracy p and let u be the entropy solution of (1) with $u_0 \in L^\infty(\mathbb{R})$ with compact support. Then there exists a constant $C > 0$ depending on $\mathcal{L}^1(\mathrm{conv}(\operatorname{supp} u_0))$, $\|u_0\|_\infty$ and f such that for every $t > 0$, it holds

$$u(t) \in \mathrm{BV}^{1/p}(\mathbb{R}) \qquad \text{and} \qquad \left(\mathrm{TV}^{1/p} u(t)\right)^p \leq C\left(1 + \frac{1}{t}\right).$$

If we additionally assume the flux to be strictly convex, then this result follows immediately by Theorem 6, by inverting f'. Also in the general case of fluxes with polynomial degeneracy, we can conclude in an elementary way for continuous solutions: in that case it suffices to invert f' locally on each convexity and concavity region. The analysis is a little more subtle because of the discontinuities of u. Given any $h > 0$ the number of jumps of height bigger than h in a bounded region is bounded uniformly by means of the length estimate (8). Moreover it is possible estimate the contribution to the fractional total variation of small entropy admissible jumps by the total variation of $f' \circ u(t)$ on these jumps and this will allow to conclude the proof.

REMARK. In order to slightly simplify the argument, the proofs of Theorems 5, 6 and 7 are provided for non negative solutions with bounded support. By finite speed of propagation, this is not a restrictive assumption.

A remarkable fact is that the BV regularity of $f' \circ u(t)$ can be improved to SBV regularity except an at most countable set $Q \subset (0, +\infty)$ of singular times. This regularity has been proved for the entropy solution u in [ADL04] in the case of a uniformly convex flux f, and extended to genuinely nonlinear hyperbolic systems in [BC12]. The proof in [ADL04] is based on the Lax-Oleinik formula for entropy solutions to (1) with uniformly convex flux f. This formula gives in particular the structure of characteristics in this setting: once you have it, the fundamental observation is that the slope of nonintersecting segments in a given time interval parametrized by the position of their middle points is a Lipschitz function. Recall that this observation was also relevant in the proof of the concentration of the entropy dissipation measure.

This part ends with the following theorem:

THEOREM 8. Let u be the entropy solution of (1) with f smooth and denote by

$$\mathcal{B} := \{t \in (0, +\infty) : f' \circ \bar{u}(t) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})\},$$
$$\mathcal{S} := \{t \in (0, +\infty) : f' \circ \bar{u}(t) \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R})\}.$$

Then $\mathcal{B} \setminus \mathcal{S}$ is at most countable.

Observe that no additional regularity on the flux is needed to prove this result. Indeed the argument is essentially the same as in [ADL04], relying on the structure of characteristics given by Theorem 2, instead of relying on the Lax-Oleinik formula. More in details, consider the partition $\mathbb{R}^+ \times \mathbb{R} = A \cup B \cup C$ as in Theorem 2. We will prove that B is the countable union of open sets where the velocity $f' \circ u$ is locally Lipschitz. It is in general not true that summing the total variation of $f' \circ u(t)$ for a given time $t > 0$, we get a finite quantity, i.e. in general $\mathcal{B} \neq \mathbb{R}^+$. However, when $f' \circ u(t) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$, the Cantor part of its derivative is concentrated on the section $A_t \cup C_t$ of $A \cup C$ at time t, and therefore on C_t, since A_t is at most countable.

Being C the union of segments starting from 0, we are now in the same position as in [ADL04], and we can similarly prove that if $\bar{t} \in \mathcal{B} \setminus \mathcal{S}$, a positive measure of segments that reach time $\bar{t}$ cannot be prolonged for $t > \bar{t}$. In particular, this can happen for a set of times at most countable.

This result depends only on the structure of the entropy solution u claimed in the previous section, we think however that it is natural to state it here taking into account Theorem 6, which provides a sufficient condition to have $\mathcal{B} = \mathbb{R}^+$ and it allows to prove the SBV regularity of $f' \circ u$ with respect to the space-time variable (t, x).

Lagrangian representation for multidimensional scalar conservation laws

In this part we present a suitable notion of Lagrangian representation for the non-negative entropy solutions to the multidimensional scalar equation,

$$\partial_t u + \operatorname{div}_x \mathbf{f}(u) = 0, \qquad \mathbf{f} : \mathbb{R} \to \mathbb{R}^d \text{ smooth.} \tag{10}$$

The key step is always to find an a priori compactness estimate and an approximating scheme exploiting this compactness: in this situation, the transport collapse method introduced by Brenier [Bre84].

This approximation method is based on the interpretation of the evolution of the solution as the action of two operators:

Transport map: a translation of each level set of u by the transport map

$$\text{hyp } u(t) := \big\{(x, h) : h \leq u(t, x)\big\} \mapsto \mathsf{Tr}(s, \text{hyp } u(t)) := \big\{(x, h) : h \leq u(t, x - f(h)s)\big\};$$

Collapse operator: the monotone mapping of each x section of a generic set $E \subset \mathbb{R}^d \times [0, +\infty)$ into an interval with the same measure,

$$(E, x, h) \mapsto \mathsf{C}(E, x, h) := \big(x, \mathcal{H}^1((\{x\} \times [0, h]) \cap E)\big).$$

The image is clearly an hypograph of a function.

The transport collapse method is then the standard operator splitting approximation applied to the two operators Tr, C: the solution $u(t)$ to (10) is the limit of approximate solutions u_n defined for $t \in 2^{-n}\mathbb{R}$ by

$$\operatorname{Graph} u_n([2^n t]2^{-n}) = \big(\mathsf{C}(\mathsf{Tr}(2^{-n}, \cdot), \|u\|_\infty)\big)^{[2^n t]} \text{hyp } u_0, \tag{11}$$

where $[\cdot]$ is the integer part of a real number. The composition $\mathsf{C}(\mathsf{Tr}(2^{-n}, \cdot), \|u\|_\infty)$ means that given a set, one first translates the level set according to the characteristic speed for a time 2^{-n}, and then find the total length on the vertical line at each pont $x \in \mathbb{R}^d$. Observe indeed that the projection operator C assign the new position of each point in a set $E \subset \mathbb{R}^{d+1}$, and does not just yield a function. A more detailed description is given in Section 5.2.3.

The natural compactness appears when interpreting the transport collapse method as a map acting on the whole hypograph of a function, i.e. assigning to every initial point $(x, h) \in \text{hyp } u_0$ a trajectory $(\gamma^1(t), \gamma^2(t)) \in \mathbb{R}^{d+1}$. Indeed, by inspection of (11), the curve $t \mapsto \gamma^1(t)$ is uniformly Lipschitz, with Lipschitz constant bounded by $\|\mathbf{f}'\|_\infty$, while the second trajectory $t \mapsto \gamma^2(t)$ is decreasing in time.
The set of trajectories described above are clearly compact in $L^1_{\text{loc}}([0, +\infty), \mathbb{R}^{d+1})$, so that one can apply standard compactness results to prove that there exists a bounded measure ω such that

(1) it is concentrated on the solutions to the "characteristic ODE"

$$\dot\gamma^1 = \mathbf{f}'(\gamma^2), \quad \dot\gamma^2 \leq 0,$$

(2) its push-forward $p_\sharp(\mathcal{L}^1 \times \omega)$ is the measure $\mathcal{L}^{d+2} \llcorner \text{hyp } u$, where

$$p(t, \gamma^1, \gamma^2) = (t, \gamma^1(t), \gamma^2(t)).$$

We can think the measure ω as a continuous version of the transport collapse operator splitting method, and following the nomenclature used in the one dimensional case, we call the measure ω a *Lagrangian representation* of the entropy solution $u(t)$.

As a first application of this construction, we consider the case of continuous solutions (see [**Daf06**] for the case of bounded distributional solutions in the one space dimension).

THEOREM 9. Let u be a continuous bounded entropy solution in $[0,T)\times\mathbb{R}^d$ to (10). Then for every $(t,x)\in[0,T)\times\mathbb{R}^d$, it holds

$$u(t,x)=u_0(x-\mathbf{f}'(u(t,x))t).$$

Moreover for every $\eta:\mathbb{R}\to\mathbb{R}$, $\mathbf{q}:\mathbb{R}\to\mathbb{R}^d$ Lipschitz such that $\mathbf{q}'=\eta'\mathbf{f}'$ a.e. with respect to $\mathcal{L}^1$, it holds

$$\eta(u)_t+\operatorname{div}_x\mathbf{q}(u)=0$$

in the sense of distributions.

This is a corollary of the fact that the Lagrangian representation in this case is unique because it satisfies $\dot{\gamma}^2=0$. In particular its graph is a bundle of characteristic curves as in the one dimensional case.

Structure of the thesis

In this section we indicate how the material presented in the introduction is organized in the main part of this thesis. The work is divided in five chapters.

In Chapter 1, we collect some preliminary results: this chapter is itself divided into two sections, the first one regarding general mathematical results and the second one about conservation laws.

In Section 1.1, we first recall the notion of convergence of sets in the sense of Kuratowski, which will be used in Chapter 2 in order to obtain compactness for the complete families of boundaries. Then we introduce a decomposition in *undulations* of piecewise monotone functions which is original, at least to the author knowledge, and will be used to implement the length estimate (8) in Chapter 4. Next we recall the notion of $\mathrm{BV}^\Phi(\mathbb{R})$ space, which generalizes the notion of function of bounded variation and it will be used to describe the regularity of the entropy solution u in Chapter 4. Finally we collect some technical and elementary results about smooth functions.

In Section 1.2, we introduce the concept of entropy solution and we mention the fundamental well-posedness result by Kruzkov. Then we recall the wave-front tracking algorithm because it can be interpreted as a preliminary version of the Lagrangian representation and it is in particular, the starting point to prove the existence of a Lagrangian representation in Chapter 2. It seems natural to study the problem in the setting of measure valued entropy solutions $\mathbb{R}^+\times\mathbb{R}\ni(t,x)\mapsto\nu_{t,x}$, with $\nu_{t,x}$ a probability in $\mathbb{R}$: this avoids additional computations when computing traces, due to the weak compactness of Young measure (Theorem 1.22). After recalling the definition of measure valued solution, we show indeed the existence of traces along every Lipschitz curve γ (Proposition 1.25), and we define the right and left admissible boundaries, Definition 1.28. The key argument for proving the existence of a complete family of boundaries is the stability of admissible boundaries w.r.t. the convergence of the parameters: the boundary curve, the boundary value, the solution $\nu_{t,x}$ and the flux function f (Proposition 1.29). The last preliminary is the analysis of the Riemann problem with two boundaries: more precisely, it is the unique entropy solution in the region delimited by two Lipschitz curves starting from the same point at $t=0$. The main result is a complete description of the solution, with a construction similar to the standard Riemann problem, Proposition 1.34: the main properties are the uniqueness in the family of measure valued solutions and the strict monotonicity of the solution and of the characteristic speed in an inner region.

In Chapter 2 we introduce the notion of *Lagrangian representation*, which is the main tool of this work.

In Section 2.1 we overview the formulations already present in the literature: the first version, called *wave representation*, has been introduced in [**BM14**] for the wave front tracking approximate solutions. After introducing this notion, we sketch its construction. Then we discuss various settings in which appropriate versions of this notion can be obtained for the exact entropy solutions by passing to the limit. In particular, we discuss the case of entropy solutions with bounded variation, we deduce some more properties in the particular case of piecewise monotone entropy solutions and finally we consider bounded entropy solution to (1) with continuous initial datum u_0.

This discussion also illustrates the difficulty of passing to the limit these formulations for general L^∞ entropy solutions and motivates the introduction of the concept of *complete family of boundaries*. This is the content of Section 2.2: the construction is essentially the same as in the previous formulations, the main novelty here is the interpretation of the values $\mathtt{u}(y)$ along the characteristic $\mathtt{X}(t, y)$ as an admissible boundary value. This requires to repeat the convergence analysis, starting from the front-tracking solutions, Lemma 2.7, and showing the stability of the set of boundaries when the flux and the solution converge weakly, Proposition 2.6. The precise definition of complete families of boundaries is given in Definition 2.3: it may seems quite strange that the only relation with the PDE is given by the values of the speed of the Lipschitz curves in the continuity points of the solution, but the requirement of the total ordering of the boundary curves is a strong requirement too.

Chapter 3 is the core of the analysis of the structure of L^∞ entropy solutions to (1).
Having shown the existence of a complete family of boundaries, we next consider its regularity in Section 3.1. The result is that there are boundary traces in the strong sense, up to the flat parts of f. We can split the results into two parts: existence of left/right traces in 1-d for all t fixed, and existence of left/right traces in 2-d for L^1-a.e. $t \in \mathbb{R}^+$ on a given curve. A common property is that we can take traces in the strong sense, once we quotient the real line w.r.t. the linearly degenerate components of f: in particular the traces are in C^0 if f is weakly genuinely nonlinear.
As we noticed in the introduction, the key observation is the decomposition of $\mathbb{R}^+ \times \mathbb{R}$ into the 3 sets A, B, C, page 34, and define from these sets the set of jumps J. Lemma 3.1 shows that C is the union of disjoint segments starting from 0 and Lemma 3.2 gives that J is rectifiable and $u \llcorner B \in \mathrm{BV}_{\mathrm{loc}}(B)$: this completes the proof of Theorem 2.
Outside this set, the blow up converges uniformly to a linearly degenerate component. At this point the regularity results are completely similar to the BV case, by just replacing the C^0-convergence with the uniform convergence to a linearly degenerate component: the existence of strong traces for every fixed time t (actually above the minimal/maximal characteristics, Lemma 3.4), the existence of strong traces outside a small cone around γ (Proposition 3.5). We note here that this is the best we can expect, due to the example 5.1.1 in [**BY15**] where a Cantor like shock is shown.

In Section 3.2 we prove Theorem 1. After selecting a family of segments $\gamma_y([0, T])$ in C which exist for a uniform time $T > 0$, and using the parameterization given by the intersection of $\gamma_y \cap \{t = T/2\}$, first we prove that the fluxes

$$Q(y) := q(u(y)) - f'(u(y))\eta(u(y)), \qquad F(y) = f(u(y)) - f'(u(y))u(y),$$

are well defined, being independent on the choice of $u(y)$ in the linearly degenerate component $I(y)$: here $f'(u(y)) = \lambda(y) = \dot{\gamma}_y$ by the properties of the complete family of boundaries. Moreover they are BV and Lipschitz continuous w.r.t. y, respectively (Lemma 3.10).
A geometric lemma (Lemma 3.11) implies that for entropies such that $\eta \leq \mathcal{O}(1)u$, the

only discontinuities of $Q(y)$ are jumps, and from the choice of jump set J this possibility is ruled out. One thus deduce that for solutions there is no dissipation in the inner part of the segments composing C, and for measure valued solutions the disintegration is a.c., Corollary 3.12. In particular one obtains the chain rule formulas for F and Q, Lemma 3.14 and formula (3.13). It is also possible to represent the dissipation measure for a measure valued solution along each segment in C as the derivative of the BV function $t \mapsto \int \eta(w) d\nu_{t,\gamma_y(t)}(w)$ (Lemma 3.15).
The analysis of the endpoints of the segments in C is split into two parts: either the endpoints are contained in the interior of a shrinking family of cylinders with sides in C, or they belong to a rectifiable set. In both cases one first prove that the disintegration has still an a.c. image measure (Lemma 3.17), and then in the case of entropy solution one deduce that no dissipation occurs (Theorem 3.18). When u is a Dirac solution, as a corollary we get Theorem 1 (Remark 3.19).

In Section 3.3 we prove that, in a suitable sense, the initial datum is taken strongly also for measure valued solutions. The key point is that the blow up around a constant state of a measure valued entropy solution is a constant Young measure, Lemma 3.20. At this point the argument is quite standard: find a suitable covering (Lemma 3.21), show that the limit occurs in average sense (Lemma 3.22), strengthen the result to have pointwise time continuity (Proposition 3.24). This concludes the proof of Theorem 3.
The last section shows that it is possible to construct a Lagrangian representation of a measure valued entropy solution with a complete family of boundaries, hence removing the ambiguity of the value w on the linearly degenerate components, Proposition 3.27. In particular, Theorem 4 follows. Moreover, once the structure is better understood, we refine the Lagrangian representation for piecewise monotone solutions introduced in Chapter 2.

The chapter ends with Section 3.5, where three examples are proposed: the first one proves that there is a set of positive measure which is not a starting point of segments, implying that the Lagrangian representation does not allow the reconstruction of the initial data by just tracing back the value function $u(t, x), t > 0$. The second example shows that the characteristic speed is not BV, even if the results contained in this chapter show that it still enjoys a BV structure similar to the 1-dimensional case i.e. C^0-continuity of the left and right traces. The last example is a refinement of the second one, where the same conclusion is obtained with a flux function with only an inflection point; we decided to include both the second and the third examples since the former is easier and it already contains all the main ideas and the latter is relevant in the following chapter.

Chapter 4 covers the second part of the project outlined at the beginning of the introduction: quantify the nonlinearity of the flux f and deduce the regularity of the entropy solution u to (1).

In Section 4.1, we prove the length estimate (8) for piecewise monotone entropy solutions. This estimate is only based on the existence of a Lagrangian representation for piecewise monotone entropy solutions. A key role in the proof is played by the properties of the sign function $\mathtt{S}$ proven in Proposition 3.27.

In Section 4.2, we prove Theorem 5. The regularity obtained in that statement depends only on the length estimate of the previous section. The proof is obtained providing a uniform regularity estimate for an approximating sequence of piecewise monotone entropy solutions. In order to apply the length estimate (8), we need to consider intervals where the solution takes the same value at their endpoints. This is where the decomposition in undulations of (nonnegative) piecewise monotone functions introduced in Chapter 1 is useful. The main argument is contained in Lemma 4.5, where a weak ℓ^1 estimate is proven for the terms defining the Φ-variation of the entropy solution u at a positive time t. Then it is standard to deduce Theorem 5.

Section 4.3 is devoted to the proof of Theorem 6; first we recall the structure of characteristics in the case of a convex flux (Lemma 4.8), then we consider the case of a flux with one inflection point: Lemma 4.13 summarizes the results obtained in [**Daf85**] about the structure of extremal backward characteristics for solutions with bounded variation. Once the structure of characteristics is established, we estimate the total variation of $f' \circ u(t)$ for piecewise monotone solutions to initial boundary value problems with constant boundary data. Proposition 4.14 deals with the case of a convex flux and and Proposition 4.15 with the case of a flux with an inflection point of polynomial degeneracy. In both proofs it is useful to recall the interpretation of characteristics as admissible boundaries and a fundamental step in Proposition 4.15 is the argument of [**Che86**]. The general case can be reduced to the cases studied in Proposition 4.14 and Proposition 4.15, by means of the length estimate 8 (Lemma 4.16) and this leads to the proof of Theorem 6.

In Section 4.4, we deduce Theorem 7 from the previous result: the argument consider separately the big and the small jumps of the entropy solution $u(t)$. The contribution of the big jumps is controlled by the length estimate and small jumps are considered in Lemma 4.20.

Theorem 8 is proved in Section 4.5 combining the structure obtained in Chapter 3 with the argument of [**ADL04**].

Finally in Section 4.6 some examples are provided: the first example is related to the possibility of repeating the analysis in [**DLR03**], without relying on Theorem 6. In order to be more precise we fix the notation in the kinetic representation:

$$\partial_t \chi_{\{u>w\}} + f'(w)\partial_x \chi_{\{u>w\}} = \partial_w \mu,$$

where μ is a negative measure. In [**DLR03**] it is proved that, under the assumptions of Theorem 6, the distribution $\partial_{ww}\mu$ can be represented as a finite measure. We exhibit an example with a general flux f such that $\partial_w \mu$ is not a finite measure.
The second example answers to a question raised in [**CJJ**]: we exhibit a function $u \in L^\infty((0,1))$ such that

$$\Phi\text{-TV}^+_{(0,1)} u := \sup_{n\in\mathbb{N},\, 0<x_1<\ldots<x_n<1} \sum_{i=1}^{n-1} \Phi\left((u(t,x_{i+1}) - u(t,x_i))^+ \right) < +\infty$$

and u does not belong to $\mathrm{BV}^\Phi((0,1))$.
Finally we provide here, for the sake of completeness, an example that shows that Theorem 7 is sharp. This result is already known, see e.g. [**CJ14**] for a similar construction and [**DLW03**], where it is shown in particular the sharpness of Theorem 7 in the setting of fractional Sobolev spaces.

In Chapter 5 we present a first result in the multidimensional case. Since the main part of the thesis deals with conservation laws in one space dimension, we decided to set the notation and discuss some preliminary results for the multidimensional case in the first section of this chapter. In particular in Section 5.1 we prove a suitable version of Ascoli-Arzelà theorem (Lemma 5.1) and a property of sets of finite perimeter (Lemma 5.2).

In Section 5.2 the notion of Lagrangian representation is presented in this setting and in Lemma 5.6 the entropy dissipation is expressed in terms of it. The construction of a Lagrangian representation for entropy solutions is obtained passing to the limit appropriate representations for approximate solutions given by the transport collapse method introduced in [**Bre84**]. The main compactness estimate, proven in Lemma 5.12 for approximate solutions with initial datum $u_0 \in \mathrm{BV}(\mathbb{R}^d)$, and the stability of the notion of Lagrangian representation (Proposition 5.8) give a Lagrangian representation for a limit function $u \in L^1((0,T), \mathrm{BV}(\mathbb{R}^d))$. By Proposition 5.5, we have that u is the entropy solution and this provides an alternative proof of the convergence of the transport collapse method and the existence of a Lagrangian representation for BV solutions. The stability proven in Proposition 5.8 implies that this can be extended to L^∞ solutions (Theorem 5.14).

The case of continuous entropy solutions is studied in Section 5.3: we first prove in two steps (Lemma 5.15 and Proposition 5.16) that the Lagrangian representation is unique and concentrated on straight lines, then we deduce Theorem 9.

CHAPTER 1

Preliminary results

Abstract. In this chapter we collect some preliminary and technical results that will be used in the main body of this thesis. More in details, Section 1.1 deals with several independent topics: first we recall the Kuratowski convergence of sets in a metric space, then we introduce a decomposition of piecewise monotone functions in "undulations". Next we recall the notion of $\mathrm{BV}^{\Phi}(\mathbb{R})$ space and we give estimates of the generalized variation of piecewise monotone functions in terms of their undulations. Finally we mention some elementary properties on smooth functions for future references.
In Section 1.2 we review some result about scalar conservation laws: the general theory is only mentioned, with some emphasis on the more relevant point for the following chapters. After recalling the fundamental theorem by Kruzkov, we introduce the wave-front tracking algorithm and the notion of measure valued entropy solution. Then we consider the problem in bounded domains: the related notion of admissible boundary is presented and some relevant estimates are recalled. An extension of the Riemann problem in bounded domains is studied in details and finally the notion of blow-up is introduced.

1.1. Mathematical preliminaries

1.1.1. Convergence of sets. We recall the notion of Kuratowski convergence. Let (X, d) be a metric space and $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X.

Definition 1.1. We define the *upper limit* and the *lower* limit of the sequence K_n respectively by the formulas

$$\limsup_{n\to+\infty} K_n = \Big\{x \in X : \liminf_{n\to+\infty} d(x, K_n) = 0\Big\},$$
$$\liminf_{n\to+\infty} K_n = \Big\{x \in X : \limsup_{n\to+\infty} d(x, K_n) = 0\Big\}.$$

We say that K_n converges to $K \subset X$ in the sense of Kuratowski if

$$K = \limsup_{n\to+\infty} K_n = \liminf_{n\to+\infty} K_n.$$

Equivalently $\limsup_{n\to+\infty} K_n$ is the set of cluster points of the of sequences $x_n \in K_n$ and $\liminf_{n\to+\infty} K_n$ is the set of limits of sequences $x_n \in K_n$.

A very general compactness result holds: see [Bee02].

Theorem 1.2 (Zarankiewicz). *Suppose that X is a separable metric space. Then for every sequence K_n of subsets of X there exists a convergent subsequence in the sense of Kuratowski.*

We remark that without any compactness assumption the Kuratowski limit may be empty.

1.1.2. Piecewise monotone functions.

Definition 1.3. A function $u : \mathbb{R} \to \mathbb{R}$ is said to be *piecewise monotone* if there exist $y_1 < \ldots < y_k$ in $\mathbb{R}$ such that for every $i = 1, \ldots, k-1$ the function u is monotone in the interval (y_i, y_{i+1}) and in the intervals $(-\infty, y_1)$ and $(y_k, +\infty)$.

We denote by X the set of piecewise monotone functions u such that the following assumptions are satisfied:

(1) u is bounded;
(2) u has compact support;
(3) $u \geq 0$;
(4) for every $x \in \mathbb{R}$,
$$u(x) = \limsup_{y \to x} u(y);$$
in particular u is upper semicontinuous.

We denote by $\mathrm{sc}^- u$ the lower semicontinuous envelope of u. It is well-known that the left and right limits of a piecewise monotone function exist at every point and in particular it has at most countably many discontinuity points. Under the boundedness assumption the limits are finite and we denote them by

$$u(x+) := \lim_{y \to x^+} u(y), \qquad u(x-) := \lim_{y \to x^-} u(y).$$

In the following proposition, we introduce a decomposition of the functions in X in terms of more elementary piecewise monotone functions.

Proposition 1.4. *Let $u \in X$. Then there exist $\tilde{N} = \tilde{N}(u) \in \mathbb{N}$ and $\{u_i\}_{i=1}^{\tilde{N}} \subset X$ non identically zero such that*

(1) it holds
$$u = \sum_{i=1}^{\tilde{N}} u_i; \tag{1.1}$$

(2) for every $i = 1, \dots \tilde{N}$ there exists $\bar{x}_i$ such that u_i is increasing in $(-\infty, \bar{x}_i]$ and decreasing in $[\bar{x}_i, +\infty)$;

(3) for every $i, j = 1, \dots, \tilde{N}$ with $i > j$, one of the following holds:
$$\operatorname{supp} u_i \subset \operatorname{supp} u_j \qquad \text{or} \qquad \operatorname{Int}(\operatorname{supp} u_i) \cap \operatorname{Int}(\operatorname{supp} u_j) = \emptyset.$$
If the first condition holds, then u_j is constant on the interior of the support of u_i and $u(\bar{x}_i) \leq u(\bar{x}_j)$.

Proof. First we introduce an operator $\mathcal{G} : X \to X$. Given $u \in X$, if $\max u = 0$ we set $\mathcal{G}(u) = u$. If instead $\max u > 0$ let

$$\bar{x} = \min\{x : u(x) = \max u\}.$$

The existence of $\bar{x}$ is guaranteed by definition of X. Let $v_l : (-\infty, \bar{x}] \to \mathbb{R}$ be the increasing envelope of $u \llcorner (-\infty, \bar{x}]$:

$$v_l = \sup\{v' : (-\infty, \bar{x}] \to \mathbb{R} \text{ such that } v' \text{ is increasing and } v' \leq u \llcorner (-\infty, \bar{x}]\}.$$

Similarly let $v_r : (\bar{x}, +\infty) \to \mathbb{R}$ be the decreasing envelope of $u \llcorner (\bar{x}, +\infty)$:

$$v_r = \sup\{v' : (\bar{x}, +\infty) \to \mathbb{R} \text{ such that } v' \text{ is decreasing and } v' \leq u \llcorner (\bar{x}, +\infty)\}.$$

Then let

$$\mathcal{G}(u) = \begin{cases} v_l & \text{in } (-\infty, \bar{x}], \\ v_r & \text{in } (\bar{x}, +\infty). \end{cases}$$

It is straightforward to check that $\mathcal{G}(u) \in X$.

Moreover $u - \mathcal{G}(u) \in X$ and this allows to iterate this procedure: we set $u_1 = \mathcal{G}(u)$ and by induction for $n > 1$

$$u_n = \mathcal{G}\left(u - \sum_{i=1}^{n-1} u_i\right).$$

We show now that there are only finitely many $n \in \mathbb{Z}^+$ such that u_n is not identically zero.

If $u = 0$ we set $k(u) = 0$ and for every $u \in X$ non identically zero, we set $k(u) = \bar{k}$ where $\bar{k}$ is the minimum value of $k \in \mathbb{Z}^+$ such that there exists $x_1 < \ldots < x_k$ for which u is monotone on $(-\infty, x_1), (x_k, +\infty)$ and (x_i, x_{i+1}) for every $i = 1, \ldots, k-1$.

It is easy to check that if $\bar{x} \in [x_i, x_{i+1})$ for some $i \in \{2, \ldots, k-1\}$, then $u - \mathcal{G}(u)$ is monotone on (x_{i-1}, x_{i+1}) and similarly if $\bar{x} \in [x_1, x_2)$, then u is monotone in $(-\infty, x_2)$ and if $\bar{x} \in [x_k, +\infty)$, then u is monotone in $(x_{k-1}, +\infty)$. Moreover, since $\mathcal{G}(u)$ is constant on each connected component of $\{x : u(x) \neq \mathcal{G}(u) = x\}$, the function $u - \mathcal{G}(u)$ is monotone on $(-\infty, x_1), (x_k, +\infty)$ and (x_i, x_{i+1}) for every $i = 1, \ldots, k-1$. Therefore

$$k(u - \mathcal{G}(u)) \leq k(u) - 1$$

and this proves that $u_n = 0$ for every $n > k(u)$.

Now we check that conditions (1), (2) and (3) in the statement are satisfied. Let $\tilde{N}(u) \in \mathbb{N}$ be such that $u_{\tilde{N}(u)} \neq 0$ and $u_{\tilde{N}(u)+1} = 0$. Then, since $\mathcal{G}(u) = 0 \Rightarrow u = 0$, by

$$0 = u_{\tilde{N}(u)+1} = \mathcal{G}\left(u - \sum_{i=1}^{\tilde{N}(u)} u_i\right),$$

it follows that (1.1) holds and this proves Condition (1). Condition (2) is clearly satisfied by construction. Consider $i, j \in \{1, \ldots, \tilde{N}(u)\}$ with $j < i$ such that there exists $x \in \mathrm{Int}(\operatorname{supp} u_i) \cap \mathrm{Int}(\operatorname{supp} u_j)$. Then if we denote by I the connected component of

$$\left\{x' : \sum_{l=1}^{j} u_l(x') < u(x')\right\}$$

containing x, it holds

$$\operatorname{supp} u_i \subset \bar{I} \subset \operatorname{supp} u_j$$

and u_j is constant on I.

It remains only to check that $u(\bar{x}_i) \leq u(\bar{x}_j)$: since for every $l = 1, \ldots, j-1$, u_l is constant on $\operatorname{supp} u_j$ and $\bar{x}_i \in \operatorname{supp} u_j$, it holds

$$\begin{aligned} u(\bar{x}_j) &= \left(\sum_{l=1}^{j-1} u_l(\bar{x}_j)\right) + \max\left(u - \sum_{l=1}^{j-1} u_l\right) \\ &= \left(\sum_{l=1}^{j-1} u_l(\bar{x}_i)\right) + \max\left(u - \sum_{l=1}^{j-1} u_l\right) \\ &\geq u(\bar{x}_i). \end{aligned}$$

This concludes the proof of Condition (3) and therefore the proof of the proposition. □

Definition 1.5. Let $u \in \mathtt{X}$ and $\{u_i\}_{i=1}^{\tilde{N}}$ be as in Proposition 1.4. We say that u_i is an *undulation* of u and that $h_i := \max u_i$ is its *height*. Moreover we say that u_i is a *descendant* of u_j if $\operatorname{supp} u_i \subset \operatorname{supp} u_j$.

1.1.3. BV^Φ **spaces.** In this section we recall the definition of BV^Φ spaces on the real line (see [MO59] for more details) and we see how the Φ-total variation of piecewise monotone functions can be estimated in terms of their undulations. Moreover we recall some basic properties of functions of bounded variations.

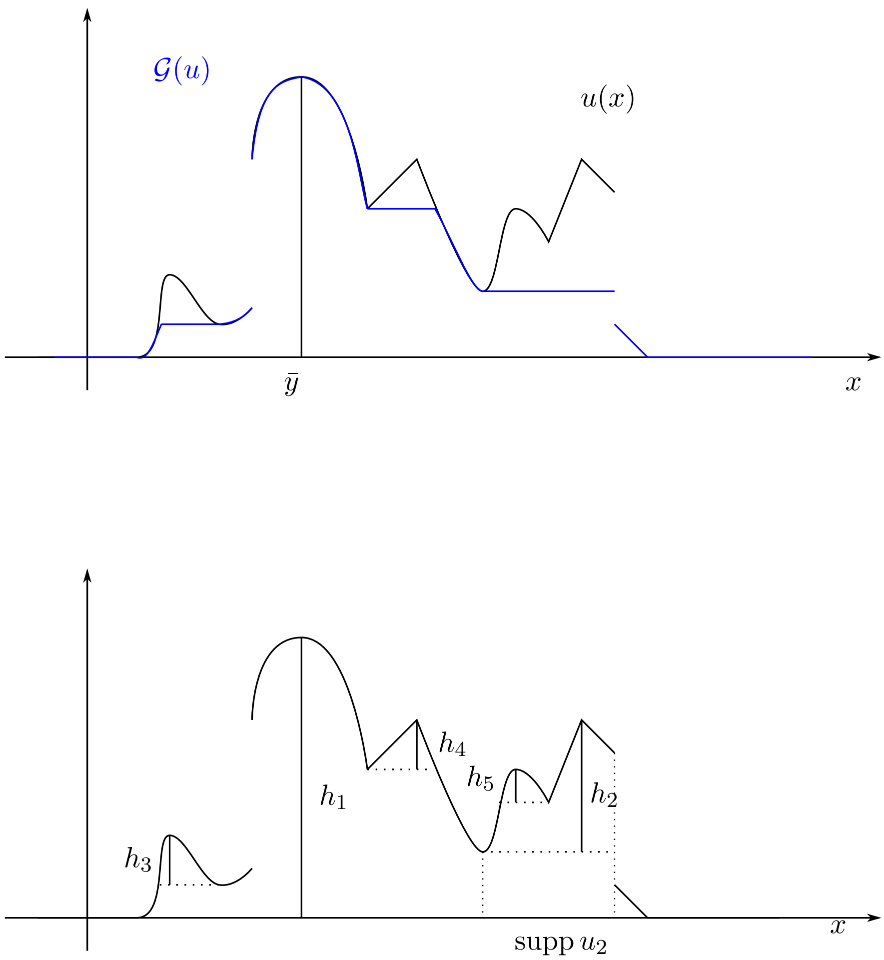


FIGURE 1.1. A representation of the decomposition: the figure above represents the operator $\mathcal{G}$ and in the figure below the heights of the undulations are represented.

DEFINITION 1.6. Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a convex function with $\Phi(0) = 0$ and $\Phi > 0$ in $(0, +\infty)$. Let $I \subset \mathbb{R}$ be a nonempty interval and for $k \in \mathbb{N}$ denote by

$$\mathcal{P}_k(I) = \left\{(x_1, x_2, \dots, x_k) \in I^k : x_1 < x_2 < \dots < x_k\right\} \qquad \text{and} \qquad \mathcal{P}(I) = \bigcup_{k \in \mathbb{N}} \mathcal{P}_k(I).$$

The Φ-total variation of u on I is

$$\Phi\text{-TV}_I(u) = \sup_{\mathcal{P}(I)} \sum_{i=1}^{k-1} \Phi(|u(x_{i+1}) - u(x_i)|).$$

If the supremum is finite we say that $u \in \mathrm{BV}^{\Phi}(I)$.

If Φ is the identity the Φ-total variation coincides with the classical total variation. It will be of particular interest also the case $\Phi(z) = z^p$ with $p > 1$. In this case if $\Phi\text{-TV}u(I) < \infty$ we write that $u \in \mathrm{BV}^{\frac{1}{p}}(I)$.

Let us recall an elementary lemma about convex functions due to Karamata.

PROPOSITION 1.7. *Let $\phi : [0, +\infty) \to \mathbb{R}$ be increasing and convex and let $a_k, b_k \in [0, +\infty)$ for $k = 1, \dots, n$. Assume that for every $k = 1, \dots, n-1$*

$$a_{k+1} \leq a_k, \qquad b_{k+1} \leq b_k$$

and for every $k = 1, \dots, n$

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i.$$

Then

$$\sum_{k=1}^{n} \phi(a_k) \geq \sum_{k=1}^{n} \phi(b_k).$$

Proof. For $i = 1, \dots k$ denote by

$$\Delta\phi_i = \begin{cases} \dfrac{\phi(a_i) - \phi(b_i)}{a_i - b_i} & \text{if } a_i \neq b_i, \\ \max\{\partial^- \phi(a_i+)\} & \text{if } a_i = b_i, \end{cases}$$

where $\partial^- \phi$ denotes the subdifferential of ϕ. Therefore

$$\Delta\phi_i(a_i - b_i) = \phi(a_i) - \phi(b_i).$$

Since ϕ is convex and increasing, for every $i \in 1, \dots, k-1$

$$0 \leq \Delta\phi_{i+1} \leq \Delta\phi_i.$$

We prove by (finite) induction that for every $k = 1, \dots, n$

$$\sum_{i=1}^{k} \phi(a_i) - \phi(b_i) \geq \Delta\phi_k \sum_{i=1}^{k} (a_i - b_i).$$

For $k = 1$ it holds by hypothesis, and if the claim holds for k, then

$$\sum_{i=1}^{k+1} \phi(a_i) - \phi(b_i) \geq \Delta\phi_k \sum_{i=1}^{k} (a_i - b_i) + \Delta\phi_{k+1}(a_{k+1} - b_{k+1}) \geq \Delta\phi_{k+1} \sum_{i=1}^{k+1} (a_i - b_i).$$

If $k = n$, we have

$$\sum_{i=1}^{n} \phi(a_i) - \phi(b_i) \geq \Delta\phi_n \sum_{i=1}^{n} (a_i - b_i) \geq 0,$$

which is the claim. □

Now we prove that it is possible to control the Φ-total variation of a function $u \in X$ in terms of its undulations. To simplify the exposition we assume the following additional properties about u:

(1) u is continuous;
(2) $\operatorname{supp} u = [a, b]$ for some $a, b \in \mathbb{R}$ and local minima and maxima of u assume different values.

The proof in general follows by a simple approximation argument.

Lemma 1.8. *Let $u \in X$ and let $(h_i)_{i=1}^{\tilde{N}(u)}$ be the heights of its undulations. Then*

$$\mathrm{TV}(u) = 2 \sum_{i=1}^{\tilde{N}(u)} h_i \qquad \textit{and} \qquad \Phi\text{-}\mathrm{TV}(u) \leq 2 \sum_{i=1}^{\tilde{N}(u)} \Phi(h_i).$$

Proof. Given two functions $v_1, v_2 : \mathbb{R} \to \mathbb{R}$ of bounded variation and $v = v_1 + v_2$ it holds

$$\mathrm{TV}(v) \leq \mathrm{TV}(v_1) + \mathrm{TV}(v_2).$$

If we also require that v_1 is constant on the support of v_2 then equality holds.

By Property (3) in Proposition 1.4 and the additional continuity assumption on u, if u_i is a descendant of u_j, then u_j is constant on $\operatorname{supp} u_i$ and obviously the same holds if the supports of u_i and u_j have disjoint interiors. In particular for every $k =$

$1,\dots \tilde N(u)-1$ the function $\sum_{i=1}^k u_i$ is constant on the support of u_{k+1}, therefore we can prove by induction that

$$\mathrm{TV}(u)=\mathrm{TV}\left(\sum_{i=1}^{\tilde N(u)} u_i\right)=\sum_{i=1}^{\tilde N(u)}\mathrm{TV}(u_i)=2\sum_{i=1}^{\tilde N(u)} h_i.$$

Now we consider the case of the Φ-total variation. Let $\varepsilon>0$ and $(x_1,\dots,x_k)\in\mathcal P$ be such that

$$\Phi\text{-}\mathrm{TV}(u)-\varepsilon<\sum_{i=1}^{k-1}\Phi(|u(x_{i+1})-u(x_i)|).$$

Denote by $(w_j)_{j=1,\dots,k-1}$ the non increasing rearrangement of $(|u(x_{j+1})-u(x_j)|)_{j=1,\dots,k-1}$ and by $(\tilde z_j)_{j=1,\dots,\tilde N(u)}$ the non increasing rearrangement of $(h_j)_{j=1,\dots,\tilde N(u)}$. Then let $(z_j)_{j\in\mathbb N}$ be the sequence defined by

$$z_{2j-1}=z_{2j}=\begin{cases}\tilde z_j & \text{if } 1\le j\le \tilde N(u),\\ 0 & \text{if } j>\tilde N(u),\end{cases}$$

and consider it restricted to $j=1,\dots,k-1$. The conclusion follows by Proposition 1.7 with $a_j=z_j$ and $b_j=w_j$: we only have to check that for every $\bar j=1,\dots,k-1$ it holds

$$\sum_{j=1}^{\bar j} z_j\ge\sum_{j=1}^{\bar j} w_j. \tag{1.2}$$

Consider $(x_1,\dots,x_{2k})\in\mathcal P_{2k}$ a maximum point in $\mathcal P_{2k}$ of the quantity

$$\sum_{i=1}^k u(x_{2i})-u(x_{2i-1}).$$

Then, if we denote by $\bar x_j$ the maximum point of the undulation u_j for $j=1,\dots,\tilde N(u)$, it clearly holds that for every $i=1,\dots,k$ there exists $j(i)$ such that $x_{2i}=\bar x_{j(i)}$. Moreover, by the maximality of the partition, it is fairly easy to prove that if $\bar x_j=\bar x_{j(i)}$ for some i and u_j is a descendant of another undulation $u_{j'}$, then there exists i' such that $j'=j(i')$. Set

$$\tilde u=\sum_{i=1}^k u_{j(i)}.$$

Since $u_j\ge 0$, it holds $\tilde u\le u$. Moreover, if $\bar x_j$ is a maximum point of u_j and $\bar j$ is such that u_j is not a descendant of $u_{\bar j}$, then $u_{\bar j}(\bar x_j)=0$. Therefore it holds

$$u(\bar x_{j(i)})=\sum_{l=1}^{\tilde N(u)} u_l(\bar x_{j(i)})=\sum_{i=1}^k u_i(\bar x_{j(i)})=\tilde u(\bar x_{j(i)})\qquad\text{for every } i=1,\dots,k.$$

It follows that

$$\sum_{i=1}^k u(x_{2i})-u(x_{2i-1})\le\sum_{i=1}^k\tilde u(x_{2i})-\tilde u(x_{2i-1})\le\frac12\mathrm{TV}(\tilde u)=\sum_{j\in I}h_j,$$

which is exactly (1.2) and this concludes the proof. □

Remark 1.9. Looking at the proof, we have that the positive and the negative parts

$$\mathrm{TV}^\Phi_+(u):=\sup_{\mathcal P(I)}\sum_{i=1}^{k-1}\Phi((u(x_{i+1})-u(x_i))^+)\quad\text{and}\quad\mathrm{TV}^\Phi_-(u):=\sup_{\mathcal P(I)}\sum_{i=1}^{k-1}\Phi((u(x_{i+1})-u(x_i))^-)$$

are separately bounded by $\sum_{i=1}^{N(u)} \Phi(h_i)$. The converse is not true, even up to a constant. The Φ-total variation of a piecewise monotone function depends not only the height of its undulations, but also how they are placed. In general the positive and the negative Φ-variations are not comparable with $\sum \Phi(h_i)$, and it may be that they are not comparable with each other. In Chapter 4 we provide an example where the increasing Φ-variation is finite and the decreasing Φ-variation is not. The question has been raised in [CJJ], where it has been observed that a counterexample like this precludes the possibility to obtain BV^Φ regularity by an Oleinik type estimate in the case of convex fluxes.

In the following lemma we collect some easy properties of functions of bounded variation that will be useful later.

LEMMA 1.10. *Let $g : \mathbb{R} \to \mathbb{R}$ be a piecewise monotone left-continuous function with bounded variation and suppose it does not have positive jumps, i.e. for every $x \in \mathbb{R}$*

$$g(x+) \leq g(x-).$$

Then, for every $a < b$ and $a = x_1 < \ldots < x_n = b$ it holds

$$\mathrm{TV}^+_{(a,b)} g = \sum_{i=1}^{n-1} \mathrm{TV}^+_{(x_i,x_{i+1})} g \qquad \text{and} \qquad \mathrm{TV}^-(g) - 2\|v\|_\infty \leq \mathrm{TV}^+(g) \leq \mathrm{TV}^-(g) + 2\|g\|_\infty.$$

Finally we state an easy lemma for future reference.

LEMMA 1.11. *Let $a, b \in \mathbb{R}$ with $a < b$ and let $g : (a,b) \to \mathbb{R}$ be increasing and bounded. Denote by*

$$h := \max_{x \in (a,b)} g(x+) - g(x-).$$

Then for every $\varepsilon > h$ there exists $\delta > 0$ such that

$$|x_2 - x_1| < \delta \qquad \Longrightarrow \qquad |g(x_2) - g(x_1)| < \varepsilon.$$

1.1.4. Weakly genuinely nonlinear fluxes and fluxes with polynomial degeneracy. The regularity of the entropy solution to (1) depends on the nonlinearity of the flux f; we introduce here some terminology.

DEFINITION 1.12. We say that $f : \mathbb{R} \to \mathbb{R}$ is *weakly genuinely nonlinear* if the set $\{w : f''(w) \neq 0\} \subset \mathbb{R}$ is dense.

We will also consider the case of a flux $f \in C^\infty(\mathbb{R})$ such that the set $\{w : f''(w) = 0\}$ is finite; let $w_1 < \ldots < w_S$ denote its elements.

DEFINITION 1.13. We say that f has *degeneracy* $p \in \mathbb{N}$, at the point w_s if $p \geq 2$ and

$$f^{(j)}(w_s) = 0 \quad \text{for} \quad j = 2, \ldots, p \qquad \text{and} \qquad f^{(p+1)}(w_s) \neq 0.$$

If there exists such a $p \in \mathbb{N}$, we say that f has *polynomial degeneracy at w_s*. If the set $\{w : f''(w) = 0\}$ is finite and f has polynomial degeneracy at each of its points, we say that f has *polynomial degeneracy*. Finally we say that f has *degeneracy p* if f has polynomial degeneracy and p is the maximum of the degeneracies of f at the points of $\{w : f''(w) = 0\}$.

In Section 4.3 it will be important to understand the behavior of f around its inflection points. The following lemma and its corollary will be useful to describe the small oscillations of the solution around an inflection point of the flux.

Lemma 1.14. *Let $f:\mathbb{R}\to\mathbb{R}$ be smooth and let $\bar w$ be such that for every $w\in\mathbb{R}\setminus\{\bar w\}$,*

$$f''(w)(w-\bar w)<0. \tag{1.3}$$

Then there exists $\delta>0$ such that for every $w\in(\bar w-r,\bar w+r)\setminus\{\bar w\}$, there exists a unique conjugate point *$w^*\in\mathbb{R}\setminus\{w\}$ such that*

$$f'(w^*)=\frac{f(w)-f(w^*)}{w-w^*}. \tag{1.4}$$

Assume moreover that $\bar w$ is of polynomial degeneracy, then there exist $\delta,\varepsilon>0$ such that for every $w,w'\in(\bar w-\delta,\bar w+\delta)\setminus\{\bar w\}$ with $w\neq w'$ it holds

$$\frac{w^*-\bar w}{w-\bar w}\in(-1+\varepsilon,0)\qquad \textit{and}\qquad \frac{f'(w^*)-f'(w'^*)}{f'(w)-f'(w')}\in(0,1-\varepsilon). \tag{1.5}$$

See Figure 1.2.

Proof. Suppose $w<\bar w$, being the opposite case analogous, and let $g_w:\mathbb{R}\to\mathbb{R}$ be defined by

$$g_w(t)=f(t)+f'(t)(w-t)-f(w).$$

Observe that (1.4) is equivalent to

$$w^*\neq w\qquad\text{and}\qquad g_w(w^*)=0. \tag{1.6}$$

Since $g_w'(t)=f''(t)(w-t)$, by (1.3), it holds

$$g_w'<0\quad\text{in }(w,\bar w)\qquad\text{and}\qquad g_w'>0\quad\text{in }(-\infty,w)\cup(\bar w,+\infty).$$

Moreover $g_w(w)=0$; by strict monotonicity this proves that there exists at most one w^* as in (1.6) and it exists if and only if $\lim_{t\to+\infty}g_w(t)>0$. Notice that

$$\begin{aligned}\lim_{t\to+\infty}g_w(t)&=g_w(\bar w)+\int_{\bar w}^{+\infty}g_w'(t)dt\\&\geq g_w(\bar w)+\int_{\bar w}^{+\infty}|f''(t)(t-\bar w)dt.\end{aligned} \tag{1.7}$$

Let $A:=\int_{\bar w}^{+\infty}|f''(t)(t-\bar w)dt>0$. Since the function $w\mapsto g_w(\bar w)$ is continuous and $g_{\bar w}(\bar w)=0$, there exists $\delta>0$ such that $w\in(\bar w-\delta,\bar w)\Rightarrow g_w(\bar w)>-A$ and therefore, by (1.7), for which w^* exists.

Now let us consider the case of f with polynomial degeneracy $p\in\mathbb{N}$ at $\bar w$. Since the statement is elementary, we only sketch the computations. Notice that since f'' changes sign at $\bar w$, p is even. It is sufficient to prove that

$$\lim_{w\to\bar w}\frac{w^*-\bar w}{w-\bar w}=\bar\rho, \tag{1.8}$$

with $\bar\rho\in(-1,0)$ and

$$\lim_{w_1,w_2\to\bar w}\frac{f'(w_2^*)-f'(w_1^*)}{f'(w_2)-f'(w_1)}=\bar\rho^p.$$

By assumption we have

$$f(w)\simeq f(\bar w)+f'(\bar w)(w-\bar w)+\alpha(w-\bar w)^{p+1},$$

with $\alpha\neq 0$.

By (1.4), it holds

$$\alpha(p+1)(w^*-\bar w)^p(w^*-w)\simeq\alpha\left[(w^*-\bar w)^{p+1}-(w-\bar w)^{p+1}\right].$$

Dividing by $(w-\bar w)^{p+1}$ and setting $\rho:=\frac{w^*-\bar w}{w-\bar w}$, we get

$$(p+1)\rho^p(\rho-1)\simeq\rho^{p+1}-1.$$

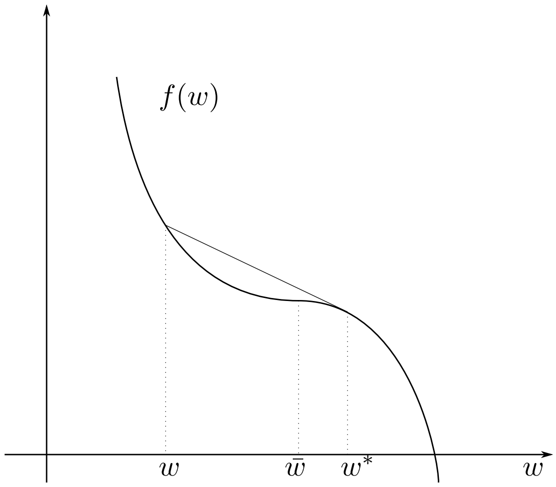


FIGURE 1.2. In this picture are represented the graph of a flux f with an inflection point in $\bar{w}$ and a point w with its conjugate w^*.

Setting $G(\rho) = p\rho^{p+1} - (p+1)\rho^p + 1$, the above formula is equivalent to $G(\rho) \simeq 0$. It is easy to show that the polynomial G has a double root in $\rho = 1$ and one root $\bar{\rho} \in (-1, 0)$. Since $\frac{w^* - \bar{w}}{w - \bar{w}} < 0$, the only possibility is that (1.8) holds. Moreover

$$\frac{f'(w_2^*) - f'(w_1^*)}{f'(w_2) - f'(w_1)} \simeq \frac{\alpha(p+1)(\bar{\rho}w_2)^p - \alpha(p+1)(\bar{\rho}w_1)^p}{\alpha(p+1)w_2^p - \alpha(p+1)w_1^p} = \bar{\rho}^p$$

and this concludes the proof. □

Applying the previous lemma around each inflection point of the flux we can easily obtain the following corollary for general fluxes with polynomial degeneracy.

COROLLARY 1.15. *Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and of polynomial degeneracy and let $w_1 < \ldots < w_S$ be the points where f'' vanishes. Then there exists $\delta > 0$ such that*

(1)
$$\delta < \frac{1}{2} \min_{s=1,\ldots,S-1} \big(w_s - w_{s-1}\big);$$

(2) for every $s = 1, \ldots, S$ and $w \in (w_s - \delta, w_s + \delta)$ there exists a unique $w^* \in (w_s - \delta, w_s + \delta)$ such that
$$f'(w^*) = \frac{f(w) - f(w^*)}{w - w^*};$$

(3) there exists $\varepsilon > 0$ such that for every $s = 1, \ldots, S$ and every $w, w' \in (w_s - \delta, w_s + \delta) \setminus \{w_s\}$ with $w \neq w'$ it holds
$$\frac{w^* - w_s}{w - w_s} \in (-1 + \varepsilon, 0) \qquad \textit{and} \qquad \frac{f'(w^*) - f'(w'^*)}{f'(w) - f'(w')} \in (0, 1 - \varepsilon).$$

1.2. Preliminaries about conservation laws

1.2.1. Entropy solutions and wave-front tracking algorithm. We consider the Cauchy problem for a scalar conservation law in one-space dimension:

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } (0, +\infty) \times \mathbb{R}, \\ u(0, \cdot) = u_0(\cdot). \end{cases} \tag{1.9}$$

The flux function f is given and supposed to be locally Lipschitz and $u : \mathbb{R}_t^+ \times \mathbb{R}_x \to \mathbb{R}$ is the unknown spatial density of the conserved quantity. Actually we consider smooth fluxes with a unique exception in the wave-front tracking algorithm, where the flux is piecewise affine.

It is well-known that the problem is well-posed in the classical setting only locally in time, therefore we consider solutions in the sense of distributions: let $u_0 \in L^\infty(\mathbb{R})$; we say that $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is a *weak* solution to (1.9), if for every $\varphi \in C_c^\infty([0,+\infty) \times \mathbb{R})$, it holds

$$\iint_{\mathbb{R}^+ \times \mathbb{R}} \big(u\varphi_t + f(u)\varphi_x\big) dx dt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)\, dx = 0.$$

Weak solutions exist for every initial datum $u_0 \in L^\infty(\mathbb{R})$ but uniqueness fails in this setting, therefore we impose additional constraints to select a unique weak solution to (1.9).

Definition 1.16. We say that (η, q) is an *entropy-entropy flux pair* if $\eta : \mathbb{R} \to \mathbb{R}$ is convex and $q : \mathbb{R} \to \mathbb{R}$ satisfies $q'(u) = \eta'(u) f'(u)$ for $\mathcal{L}^1$-a.e. $u \in \mathbb{R}$. In particular we will use the following notation: for every $k \in \mathbb{R}$ let

$$\eta_k^+(u) := (u-k)^+, \qquad \eta_k^-(u) := (u-k)^- \tag{1.10}$$

and the relative fluxes

$$q_k^+(u) := \chi_{[k,+\infty)}(u)\big(f(u) - f(k)\big), \qquad q_k^-(u) := \chi_{(-\infty,k]}(u)\big(f(k) - f(u)\big), \tag{1.11}$$

where χ_E denotes the characteristic function of the set E:

$$\chi_E(u) := \begin{cases} 1 & \text{if } u \in E, \\ 0 & \text{if } u \notin E. \end{cases}$$

We are now in position to define the notion of entropy solution.

Definition 1.17. A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an *entropy solution* to (1.9), if for every entropy-entropy flux pair (η, q) and every nonnegative $\varphi \in C_c^\infty([0,+\infty) \times \mathbb{R})$, it holds

$$\int_{\mathbb{R}^+ \times \mathbb{R}} (\eta(u)\varphi_t + q(u)\varphi_x) dx dt + \int_{\mathbb{R}} \eta(u_0(x))\phi(0,x) dx \geq 0.$$

The celebrated work of Kruzkov [Kru70] establishes well-posedness in the class of bounded entropy solutions, also in several space dimension: we summarize the result in the one space variable case in the following theorem.

Theorem 1.18. *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous. Then there exists a continuous semigroup $S : [0,\infty) \times L^1 \to L^1$ with the following properties.*

(i) $S_0(\bar{u}) = \bar{u}, \quad S_s(S_t\bar{u}) = S_{s+t}\bar{u}$.
(ii) $\|S_t\bar{u} - S_t\bar{v}\|_1 \leq \|\bar{u} - \bar{v}\|_1$.
(iii) For each $u_0 \in L^1 \cap L^\infty$, the trajectory $t \mapsto S_t u_0$ yields the unique, bounded, entropy solution of the corresponding Cauchy problem (1.9).
(iv) If $u_0(x) \leq v_0(x)$ for all $x \in \mathbb{R}$, then $S_t(u_0)(x) \leq S_t(v_0)(x)$ for every $x \in \mathbb{R}$, $t \geq 0$.

Remark 1.19. Actually the hyperbolic nature of the equation allows to localize Property (ii): if $u \in [-M, M]$ is a bounded entropy solution and $L := \max_{[-M,M]} |f'|$, then for every $a < b \in \mathbb{R}$ and $t > 0$, it holds

$$\int_a^b |S_t\bar{u} - S_t\bar{v}| dx \leq \int_{a-Lt}^{b+Lt} |\bar{u} - \bar{v}| dx.$$

The argument to prove the theorem above is the following: thanks to the abundance of entropies it is possible to prove the L^1-contraction property (ii). With such a priori estimate, one can obtain entropy solutions by constructing approximate solutions in several ways: we refer to [Daf16] for a discussion on several possible methods. Here we focus on the wave-front tracking algorithm introduced in [Daf72], and we follow

[Bre00]. We finally mention that Point (iv) and several other properties of the solution u can be obtained by passing to the limit the corresponding properties for the approximate solutions.

1.2.1.1. *Riemann problem.* The building block for the construction of approximate solutions by the wave-front tracking algorithm is the so called Riemann problem: it is by definition the Cauchy problem (1.9), with initial data of the form

$$u_0(x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0, \end{cases}$$

with $u^-, u^+ \in \mathbb{R}$.

Assume $u^- < u^+$ and let

$$\operatorname*{conv}_{[u^-,u^+]}(f)(u) := \sup \Big\{ g(u) | g \le f \text{ on } [u^-, u^+],\, g \text{ is convex} \Big\}.$$

be the *convex envelope* of f on the interval $[u^-, u^+]$. Let

$$\lambda(u) = \frac{d}{du} \operatorname*{conv}_{[u^-,u^+]} f(u).$$

The function λ is non-decreasing from $[u^-, u^+]$ into $[\lambda(u^-), \lambda(u^+)]$. So we can consider its pseudoinverse $\mathtt{u} : [\lambda(u^-), \lambda(u^+)] \to [u^-, u^+]$. The function $\mathtt{u} = \mathtt{u}(\lambda)$ is an increasing function, in particular it has at most countably many discontinuity points.

Consider now the function u defined by

$$u(t,x) = \begin{cases} u^- & \text{if } x/t < \lambda(u^-), \\ \mathtt{u}(\lambda) & \text{if } x/t = \lambda \text{ for some } \lambda \in [\lambda(u^-), \lambda(u^+)] \\ & \quad \text{which is not a discontinuity point of } \mathtt{u} = \mathtt{u}(\lambda), \\ u^+ & \text{if } x/t > \lambda(u^+). \end{cases} \tag{1.12}$$

It is possible to check that the function u is the entropy solution of the Riemann problem. We also mention that the case $u^+ < u^-$ can be treated in a similar way, replacing the convex envelope with the concave envelope.

1.2.1.2. *Wave-front tracking algorithm.* We recall now how to construct piecewise constant approximations by the method of wave-front tracking. Fixed $\nu \in \mathbb{N}$, let f^ν be the piecewise affine interpolation of f defined by

$$f^\nu(u) = \frac{u - 2^{-\nu} j}{2^{-\nu}} f(2^{-\nu}(j+1)) + \frac{2^{-\nu}(j+1) - u}{2^{-\nu}} f(2^{-\nu} j)$$

for $u \in [2^{-\nu} j, 2^{-\nu}(j+1)]$, with j integer.

Then we consider the approximate problem

$$\begin{cases} u_t + f^\nu(u)_x = 0, \\ u(0,\cdot) = u_0, \end{cases} \tag{1.13}$$

where $u_0 : \mathbb{R} \to 2^{-\nu}\mathbb{Z}$ has bounded variation and is compactly supported.

We prove that (1.13) has a piecewise constant entropy solution u. By (1.12) this is true for the Riemann problem. Indeed, if f_* is the convex (concave) envelope of f^ν, f_* is piecewise affine. Hence $\lambda = \frac{d}{du} f_*$ and $\mathtt{u} = \lambda^{-1}$ are piecewise constant.

Now we consider the general case. Denote by $x_1 < \ldots < x_N$ the jump points of u_0. Until the first time t_1 when two discontinuity lines meet, solution is obtained piecing together solutions of the Riemann problems at x_i. At time t_1 we have to solve the new Riemann problems generated by the solution with initial data taking values in $2^{-\nu}\mathbb{Z}$ again. This solution can be prolonged until a time t_2 when two or more discontinuity lines collide. We claim that this procedure defines a solution for all positive time since there are only finitely many collisions. We distinguish two cases.

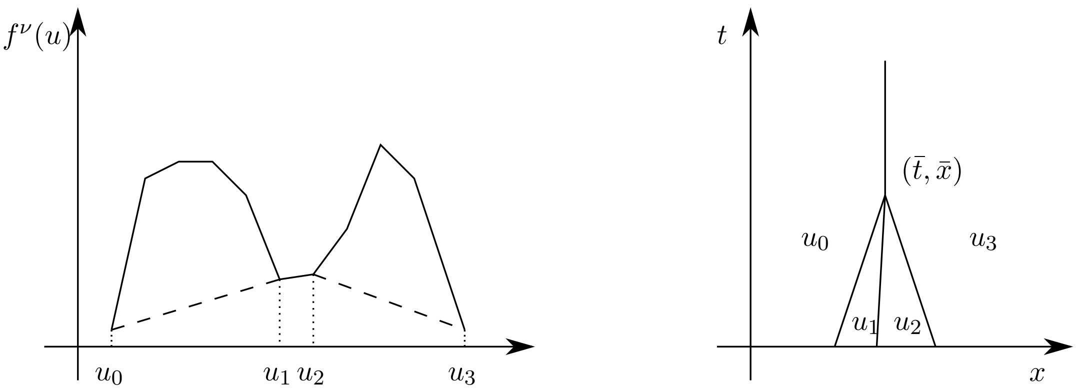


Figure 1.3. An interaction: there is only one discontinuity exiting from $(\bar t, \bar x)$, no matter how many are entering.

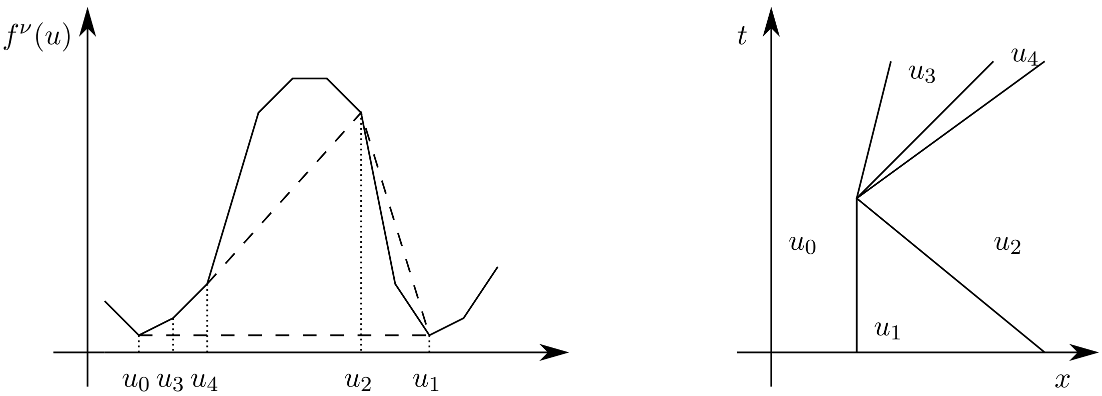


Figure 1.4. A cancellation: the number of discontinuity may increase, but the total variation decreases of at least $2 \cdot 2^{-\nu}$.

Case 1. All jumps colliding at a time t have the same sign: in this case we will say that an *interaction* occurs. In this case it is easy to see that solution of corresponding Riemann problem has only a discontinuity line, in particular total number of discontinuity lines decreases of at least 1. See Figure 1.3.

Case 2. At least two of the jumps colliding have opposite sign: in this case we will say that a *cancellation* occurs. In this case total variation decreases of a multiple of $2 \cdot 2^{-\nu}$. See Figure 1.4.

Observe that the total variation never increases, so Case 2 can occur only finitely many times. The number of discontinuity lines can increase only when Case 2 happens, therefore Case 1 can occur finitely many times too.

1.2.2. Measure valued solutions on bounded domains. The notion of entropy solution in bounded domains will play a central role through all the main part of this thesis. Since the natural setting in Chapter 3 is the one of measure valued (briefly mv) entropy solutions, we present the initial boundary value problem in this setting. For more details we refer to [**MNRR96**].

Denote by $\mathcal{P}(\mathbb{R})$ the set of probability measures on $\mathbb{R}$.

Definition 1.20. A *Young measure* on $\mathbb{R}^+ \times \mathbb{R}$ is a measurable map $\nu : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, in the sense that for all continuous functions g on $\mathbb{R}$

$$\langle \nu, g \rangle : (t,x) \mapsto \int g d\nu_{t,x}$$

is $\mathcal{L}^2$-measurable. A measurable function $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ induces the Young measure $\nu_{t,x} = \delta_{u(t,x)}$.

Definition 1.21. Let ν^n, ν be Young measures. We say that $\nu^n \to \nu$ *in the sense of Young measures* if for every $g \in C_c(\mathbb{R})$, the sequence $\langle \nu^n, g \rangle$ converges to $\langle \nu, g \rangle$ with respect to the weak* topology in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$.

This notion is motivated by the following compactness result: see [Bal89].

Theorem 1.22 (Young). *Let ν^n be a sequence of uniformly bounded Young measures, i.e. there exists $M > 0$ such that for every $n \in \mathbb{N}$ and for every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ it holds $\operatorname{supp} \nu^n_{t,x} \subset [-M, M]$. Then there exists a subsequence ν^{n_k} and a Young measure ν such that ν^{n_k} converges to ν in the sense of Young measures.*

The notion of mv entropy solution to

$$u_t + f(u)_x = 0 \tag{1.14}$$

has been introduced in [DiP85].

Definition 1.23. A bounded Young measure ν is a *mv entropy solution* of (1.14) if for all entropy-entropy flux pairs (η, q) it holds

$$\mu := \partial_t \langle \nu, \eta \rangle + \partial_x \langle \nu, q \rangle \leq 0 \tag{1.15}$$

in the sense of distributions on $(0, +\infty) \times \mathbb{R}$. We will say that ν is a *Dirac entropy solution* of (1.14) if $\nu_{t,x} = \delta_{u(t,x)}$ where u is an entropy solution of (1.14).

Remark 1.24. In Definition 1.23 we require that ν is bounded so that (1.15) makes sense for every entropy-entropy flux pair (η, q).

We observe that (1.15) implies that the vector field $(\langle \nu, \eta \rangle, \langle \nu, q \rangle)$ is a divergence measure field, in particular it has normal traces in the sense of Anzellotti [Anz83]. The argument in the next proposition is essentially taken from [Sze89] (see also [CF99]).

Proposition 1.25. *Let ν be a mv entropy solution of (1.14) in $\mathbb{R}^+ \times \mathbb{R}$ and let $\gamma : [0, +\infty) \to \mathbb{R}$ be a Lipschitz curve. Denote by Ω^- the set*

$$\Omega^- := \big\{ (t,x) \in \mathbb{R}^+ \times \mathbb{R} : x < \gamma(t) \big\}.$$

Then there exists a Young measure $\nu^- : \mathbb{R}^+ \to \mathcal{P}(\mathbb{R})$ such that for every Lipschitz φ with compact support and every entropy-entropy flux pair (η, q) it holds

$$\int_{\Omega^-} \varphi d\mu + \int_{\Omega^-} \varphi_t \langle \nu, \eta \rangle + \varphi_x \langle \nu, q \rangle dx dt = \int_0^{+\infty} \left(-\dot\gamma(t) \langle \nu_t^-, \eta \rangle + \langle \nu_t^-, q \rangle \right) \varphi(t, \gamma(t)) dt.$$

The same result (with a minus sign on the r.h.s.) holds on $\Omega^+ := \{(t,x) \in \mathbb{R}^+ \times \mathbb{R} : x > \gamma(t)\}$ and defines the right traces ν^+.

Proof. Given $\phi \in C_c^\infty(\mathbb{R}^+)$ consider the function

$$A_\phi(\varepsilon) := \int_{\mathbb{R}^+} \langle \nu_{t, \gamma(t) - \varepsilon}, -\dot\gamma(t) \eta + q \rangle \phi(t) dt.$$

It is defined for $\mathcal{L}^1$-a.e. $\varepsilon > 0$ by Fubini theorem. We show that it has bounded variation: in particular it coincides $\mathcal{L}^1$-a.e. with his right-continuous representative $\tilde A_\phi$. Test the equation (1.15) with

$$\varphi(t,x) = \phi(t) \psi(\gamma(t) - x)$$

for some $\phi, \psi \in C_c^\infty(\mathbb{R}^+)$.

$$\begin{aligned}\int_{\Omega^-} \varphi d\mu = & -\int_{\Omega^-} \Big[\langle \nu_{t,x}, \eta\rangle \big(\phi'(t)\psi(\gamma(t)-x) + \dot\gamma(t)\phi(t)\psi'(\gamma(t)-x)\big) \\ & - \langle \nu_{t,x}, q\rangle \phi(t)\psi'(\gamma(t)-x)\Big] dxdt \\ = & -\int_{(\mathbb{R}^+)^2} \langle \nu_{t,\gamma(t)-\varepsilon}, -\dot\gamma(t)\eta + q\rangle \phi(t)\psi'(\varepsilon) dt d\varepsilon \\ & + \int_{(\mathbb{R}^+)^2} \langle \nu_{t,\gamma(t)-\varepsilon}, \eta\rangle \phi'(t)\psi(\varepsilon) dt d\varepsilon \\ = & -\int_{\mathbb{R}^+} A_\phi(\varepsilon)\psi'(\varepsilon) d\varepsilon + \int_{\mathbb{R}^+}\left(\int_{\mathbb{R}^+} \langle \nu_{t,\gamma(t)-\varepsilon}, \eta\rangle \phi'(t) dt\right)\psi(\varepsilon) d\varepsilon.\end{aligned}$$

In particular this implies that A_ϕ has bounded variation and

$$|DA_\phi| \le C\mathrm{TV}(\phi)\mathcal{L}^1 + \|\phi\|_\infty p_\sharp |\mu| \llcorner \operatorname{supp}\varphi,$$

where $p : (t,x) \mapsto \gamma(t) - x$.

Let $D \subset C_c^1(\mathbb{R}^+)$ be a countable set dense in $C_c^0(\mathbb{R}^+)$. Repeating the argument in D we obtain that there exists a negligible set $E \subset \mathbb{R}^+$ such that for every $\phi \in D$ and for every $\varepsilon \in \mathbb{R}^+ \setminus E$,

$$\int_{\mathbb{R}^+} \langle \nu_{t,\gamma(t)-\varepsilon}, -\dot\gamma(t)\eta + q\rangle \phi(t) dt = \tilde A_\phi(\varepsilon).$$

Consider a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset \mathbb{R}^+ \setminus E$. By Theorem 1.22 there exist a subsequence ε_{k_l} and a Young measure $\nu^- : \mathbb{R}^+ \to \mathcal{P}(\mathbb{R})$ such that for every entropy-entropy flux pair (η, q)

$$\langle \nu_{t,\gamma(t)-\varepsilon_{k_l}}, -\dot\gamma(t)\eta + q\rangle \rightharpoonup \langle \nu_t^-, -\dot\gamma(t)\eta + q\rangle \quad w^* - L^\infty.$$

In particular for every $\phi \in D$, and using the boundedness of ν by density for every $\phi \in C_c^0(\mathbb{R}^+)$,

$$\lim_{\varepsilon\to 0} \tilde A_\phi(\varepsilon) = \int_{\mathbb{R}^+} \langle \nu_t^-, -\dot\gamma(t)\eta + q\rangle \phi(t) dt.$$

To prove the integration by parts formula consider a test function of this form:

$$(t,x) \mapsto \varphi(t,x)\psi_\varepsilon(\gamma(t)-x),$$

where φ is Lipschitz with compact support in $\mathbb{R}^+ \times \mathbb{R}$ and

$$\psi_\varepsilon(s) = \begin{cases} \frac{s}{\varepsilon} & \text{if } s \in (0,\varepsilon), \\ 1 & \text{if } s \ge \varepsilon. \end{cases}$$

Letting $\varepsilon \to 0$ in the divergence formula in the weak form and using $\varphi(t, \gamma(t)-\varepsilon) \to \varphi(t, \gamma(t))$, we get the claim. $\square$

Remark 1.26. The fact that $\langle \nu_t^-, -\dot\gamma(t)\eta + q\rangle$ is uniquely determined $\mathcal{L}^1$-a.e. for all entropy-entropy flux pairs (η, q) implies that ν^- is uniquely determined up to regions where $f' = \dot\gamma$: more precisely let

$$O = \{u : f'(u) \ne \dot\gamma(t)\}$$

and ν^1, ν^2 two measures such that for all entropy-entropy flux pairs (η, q)

$$\langle \nu^1, -\dot\gamma(t)\eta + q\rangle = \langle \nu^2, -\dot\gamma(t)\eta + q\rangle.$$

Then $\nu^1 = \nu^2$ on the σ-algebra generated by $\{(u, +\infty) : u \in O\}$.

It suffices to prove that $\nu^1(u^1,u^2) = \nu^2(u^1,u^2)$ for $u^1,u^2 \in O$. To see this we test with an entropy η_n such that

$$\eta_n'(u) = \frac{n}{f'(u)-\dot\gamma(t)}\Big(\chi_{(u^1,u^1+\frac{1}{n})}(u) - \chi_{(u^2,u^2+\frac{1}{n})}(u)\Big)$$

For n sufficiently large this defines an entropy and we can choose the entropy flux q_n such that letting $n\to+\infty$,

$$\begin{aligned}\eta_n &\to \frac{1}{f'(w_1)-\dot\gamma(t)}\chi_{(w_1,+\infty)} - \frac{1}{f'(w_2)-\dot\gamma(t)}\chi_{(w_2,+\infty)},\\ q_n &\to \frac{f'(w_1)}{f'(w_1)-\dot\gamma(t)}\chi_{(w_1,+\infty)} - \frac{f'(w_2)}{f'(w_2)-\dot\gamma(t)}\chi_{(w_2,+\infty)},\end{aligned}$$

therefore

$$-\dot\gamma(t)\eta_n + q_n \to \chi_{(w^1,w^2)}.$$

Remark 1.27. The same argument works for space-like curves. In particular a mv entropy solution ν has a representative such that for every $\bar t>0$ both the following limits exist in the sense of Young measures:

$$\nu^-_{t,x} = \lim_{t\to\bar t^-}\nu_{t,x}, \qquad \nu^+_{t,x} = \lim_{t\to\bar t^+}\nu_{t,x}$$

and they are equal for all t except at most countably many. We will denote in particular by $\nu^+_{0,x} = \lim_{t\to 0^+}\nu_{t,x}$ the trace at $t=0$. Notice that if we set $\nu_{0,x}=\nu^+_{0,x}$ the entropy condition (1.15) holds in the sense of distributions on $[0,+\infty)\times\mathbb{R}$ i.e. for every non negative $\varphi\in C^\infty_c([0,+\infty)\times\mathbb{R})$

$$\int_{\mathbb{R}^+\times\mathbb{R}}\big(\varphi_t\langle\nu_{t,x},\eta\rangle + \varphi_x\langle\nu_{t,x},q\rangle\big)dxdt + \int_{\mathbb{R}}\varphi(0,x)\langle\nu_{0,x},\eta\rangle dx \ge 0.$$

The notion of trace allows us to define in which sense a boundary condition is satisfied: see [BlRN79] where this notion has been introduced in the setting of bounded variation and [Ott96] for a treatment of the L^∞ case.

Definition 1.28. A couple (γ,w) with $\gamma:[0,+\infty)\to\mathbb{R}$ Lipschitz and $w\in\mathbb{R}$ is said to be an *admissible right boundary* if for $\mathcal{L}^1$ a.e. $t>0$,

$$\begin{aligned}-\dot\gamma\langle\eta^+_k,\nu^-\rangle + \langle q^+_k,\nu^-\rangle \ge 0 \quad \forall k\ge w,\\ -\dot\gamma\langle\eta^-_k,\nu^-\rangle + \langle q^-_k,\nu^-\rangle \ge 0 \quad \forall k\le w,\end{aligned} \tag{1.16}$$

where ν^- is a left trace of ν on γ. Similarly we say that (γ,w) is a *admissible left boundary* if for $\mathcal{L}^1$ a.e. $t>0$,

$$\begin{aligned}-\dot\gamma\langle\eta^+_k,\nu^+\rangle + \langle q^+_k,\nu^+\rangle \le 0 \quad \forall k\ge w,\\ -\dot\gamma\langle\eta^-_k,\nu^+\rangle + \langle q^-_k,\nu^+\rangle \le 0 \quad \forall k\le w.\end{aligned} \tag{1.17}$$

We simply say that (γ,w) is an *admissible boundary* if it is admissible left and right boundary.

We will also refer to (η,q) as *boundary entropy-entropy flux pair* with value w if $\eta=\eta^+_k$ for some $k\ge w$ or $\eta=\eta^-_k$ for some $k\le w$ and q is the corresponding flux defined by (1.11).

The following stability result will be useful to prove the existence of admissible boundaries for L^∞ entropy solutions.

Proposition 1.29 (Stability). *Let ν^n be mv entropy solutions of (1.14) with flux f^n and (γ^n,w^n) admissible boundaries for ν^n. Suppose that*

- *f^n are equi-Lipschitz and $f^n\to f$ uniformly;*
- *$\nu^n\to\nu$ in the sense of Young measures;*

- *$w^n \to w$;*
- *γ^n are equi-Lipschitz and $\gamma^n \to \gamma$ uniformly.*

Then (γ, w) is an admissible boundary for ν.

Proof. We show that the property of being an admissible right boundary is stable. Let $k < w$, $\eta = \eta_k^-$, $q^n = q_k^{-,n}$ the relative flux and

$$\mu^n = \langle \nu^n, \eta \rangle_t + \langle \nu^n, q^n \rangle_x$$

the dissipation. For n sufficiently large we have $w^n > k$ therefore, by hypothesis, for every nonnegative test function $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$,

$$\int_{(\Omega^n)^-} \varphi d\mu^n + \int_{(\Omega^n)^-} \left(\varphi_t \langle \nu_{t,x}^n, \eta \rangle + \varphi_x \langle \nu_{t,x}^n, q^n \rangle \right) dxdt \geq 0. \tag{1.18}$$

We want to pass to the limit the inequality above: since $\nu^n \to \nu$ in the sense of Young measures and $q^n \to q$ uniformly, by Young theorem

$$\langle \nu^n, \eta \rangle \rightharpoonup \langle \nu, \eta \rangle \quad w^* - L^\infty \qquad \text{and} \qquad \langle \nu^n, q^n \rangle \rightharpoonup \langle \nu, q \rangle \quad w^* - L^\infty.$$

Moreover $\chi_{(\Omega^n)^-} \to \chi_{\Omega^-}$ strongly in L^1, therefore

$$\int_{(\Omega^n)^-} \left(\varphi_t \langle \nu_{t,x}^n, \eta \rangle + \varphi_x \langle \nu_{t,x}^n, q^n \rangle \right) dxdt \to \int_{\Omega^-} \varphi_t \langle \nu_{t,x}, \eta \rangle + \varphi_x \langle \nu_{t,x}, q \rangle dxdt. \tag{1.19}$$

Let $\psi_\varepsilon \in C_c^\infty(\Omega^-)$ taking values in $[0,1]$ such that $\psi_\varepsilon(t,x) = 1$ for every (t,x) such that $\mathrm{dist}((t,x), (\Omega^-)^c) \geq \varepsilon$. Then, since μ^n are nonpositive,

$$\limsup_{n \to +\infty} \int_{(\Omega^n)^-} \varphi d\mu^n \leq \lim_{n \to +\infty} \int_{\Omega^-} \varphi \psi_\varepsilon d\mu^n = \int_{\Omega^-} \varphi \psi_\varepsilon d\mu.$$

Letting $\varepsilon \to 0$ we get

$$\limsup_{n \to +\infty} \int_{(\Omega^n)^-} \varphi d\mu^n \leq \int_{\Omega^-} \varphi d\mu. \tag{1.20}$$

By (1.19) and (1.20) we get that (1.18) holds in the limit, and this is equivalent to (1.16) by Proposition 1.25.

For η_k^+ the analysis is completely analogue. □

DiPerna [DiP85] showed that the doubling variable technique by Kruzkov [Kru70] applies also in the context of mv solutions: given ν_1, ν_2 mv entropy solutions of (1.14) in an open set, it holds

$$\partial_t \langle \nu_1 \times \nu_2, |w - w'| \rangle + \partial_x \langle \nu_1 \times \nu_2, \mathrm{sign}(w - w')(f(w) - f(w')) \rangle \leq 0$$

in the sense of distributions. This in particular implies the uniqueness of entropy solutions in the class of mv entropy solutions for the initial value problem with $u_0 \in L^\infty$.

Szepessy [Sze89] extended the result to bounded domains.

Proposition 1.30. *Let $T > 0$ and consider a domain*

$$\Omega = \Big\{ (t,x) \in (0,T) \times \mathbb{R} : \gamma_1(t) < x < \gamma_2(t) \Big\} \tag{1.21}$$

where $\gamma_1, \gamma_2 : [0,T] \to \mathbb{R}$ are Lipschitz and $\gamma_1 \leq \gamma_2$. Let ν and ν' be two solutions of (1.14) which satisfy the same boundary conditions w_1 on γ_1 and w_2 on γ_2: i.e. (γ_1, w_1) is an admissible left boundary for both ν and ν' and (γ_2, w_2) is an admissible right boundary for both ν and ν' as in Definition 1.28. Then

$$F(t) = \int_{\gamma_1(t)}^{\gamma_2(t)} \langle \nu_{t,x} \times \nu'_{t,x}, |u - v| \rangle dx$$

has non positive derivative in the sense of distributions in $(0,T)$.

In [**BMS02**] Proposition 1.30 is proven in the case that the curves γ_1 and γ_2 are constant: being the procedure the same in the case of moving boundaries we do not provide a proof.

It will also be useful the following estimate for solutions of bounded variation: in [**Ama97**] the estimate is proved for wave-front tracking approximate solutions in the context of systems. Since the analysis can be repeated in this setting, we omit the proof.

PROPOSITION 1.31. *Let Ω be as in (1.21) and let $u \in \mathrm{BV}(\Omega)$ be a bounded entropy solution of (1.14) with intial datum $u_0 \in \mathrm{BV}(\gamma_1(0), \gamma_2(0))$ and boundary data $u_l \in \mathrm{BV}(0,T)$ on $\mathrm{Graph}(\gamma_1)$ and $u_r \in \mathrm{BV}(0,T)$ on $\mathrm{Graph}(\gamma_2)$. Then*

$$\mathrm{TV}\,(u(T)) \leq \mathrm{TV}(u_0)+\mathrm{TV}(u_l)+\mathrm{TV}(u_r)+|u_l(0+)-u_0(\gamma_l(0)+)|+|u_r(0+)-u_0(\gamma_r(0)-)|,$$

where the total variations are computed on the domains of the corresponding functions.

Finally, we consider the particular case where γ_1, γ_2 are L-Lipschitz, $\gamma_1(0) = \gamma_2(0)$, $\gamma_1 < \gamma_2$ in $(0,T]$ and the boundary data are the constant a on γ_1 and the constant b on γ_2. We assume $a \leq b$, being the opposite case analogue. The solution can be expressed quite explicitly with a construction which generalizes the one for the classical Riemann problem.

LEMMA 1.32. *Let Ω be defined by (1.21). For every $(\bar{t}, \bar{x}) \in \Omega$ consider the length minimization problem*

$$\min_{\gamma \in \mathcal{A}_{\bar{t},\bar{x}}} \int_0^{\bar{t}} \sqrt{1+\gamma'(t)^2}dt, \quad \text{where} \quad \mathcal{A}_{\bar{t},\bar{x}} = \Big\{\gamma \in \mathrm{Lip}([0,\bar{t}]) : \gamma_1 \leq \gamma \leq \gamma_2, \gamma(\bar{t}) = \bar{x}\Big\}. \tag{1.22}$$

For every $(\bar{t}, \bar{x}) \in \Omega$ the minimizing curve $\gamma^{\bar{t},\bar{x}}$ in (1.22) exists and is unique.

Moreover the following properties hold:

(1) for every $(\bar{t}, \bar{x}) \in \Omega$, the function $\gamma^{\bar{t},\bar{x}}$ is L-Lipschitz;
(2) for every $\bar{t} \in (0,T)$ and for every $t \in (0,\bar{t})$, the map $\bar{x} \mapsto \gamma^{\bar{t},\bar{x}}(t)$ is non decreasing;
(3) for every $(\bar{t}, \bar{x}) \in \Omega$ the function $\dot{\gamma}^{\bar{t},\bar{x}}$ is constant on each connected component of

$$\big\{t \in (0,\bar{t}) : \gamma_1(t) < \gamma^x(t) < \gamma_2(t)\big\};$$

(4) the map $v : (\bar{t}, \bar{x}) \mapsto \dot{\gamma}^{\bar{t},\bar{x}}(\bar{t}-)$ is locally Lipschitz and has bounded variation in Ω. Moreover, for every $\bar{t} \in (0,T)$, the function $\bar{x} \mapsto v(\bar{t}, \bar{x})$ is strictly increasing;
(5) for every $\bar{t} \in (0,T)$ of differentiability for γ_1 and γ_2,

$$\lim_{\bar{x}\to\gamma_1(\bar{t})^+} v(\bar{t},\bar{x}) \leq \dot{\gamma}_1(\bar{t}) \qquad \text{and} \qquad \lim_{\bar{x}\to\gamma_2(\bar{t})^-} v(\bar{t},\bar{x}) \geq \dot{\gamma}_2(\bar{t}).$$

PROOF. We just sketch the proof.

Existence, uniqueness and Lipschitz regularity are standard.

Point (2) follows from uniqueness and Point (3) is trivial. Observe that uniqueness implies that for every $(\bar{t}, \bar{x}) \in \Omega$ and $t \in (0,\bar{t})$

$$\gamma^{\bar{t},\bar{x}} \llcorner [0,t] = \gamma^{t,\gamma^{\bar{t},\bar{x}}(t)}.$$

In particular each level sets of v is the union of segments with slope v and endpoints in $\partial\Omega$. Therefore v is locally Lipschitz. The strict monotonicity with respect to $\bar{x}$ is a consequence of minimality and then it follows that v has bounded variation. Point (5) is a consequence of minimality too. $\square$

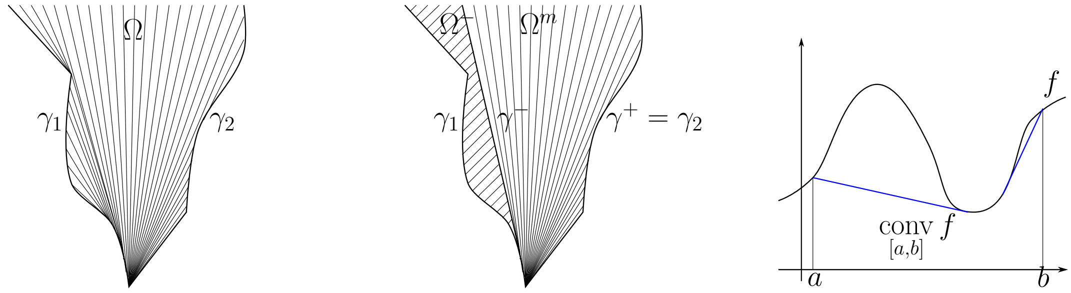


FIGURE 1.5. Riemann problem with boundaries: in the first figure there are the minimal curves, in the second the characteristics for the flux f as in the third picture.

Denote by $\operatorname{conv}_{[a,b]} f : [a,b] \to \mathbb{R}$ the convex envelope of f in $[a,b]$ and let $[\lambda^-,\lambda^+]$ the image of its derivative. The function $(\operatorname{conv}_{[a,b]} f)'$ is non decreasing, we denote its pseudo-inverse by $g : [\lambda^-,\lambda^+] \to [a,b]$. For every $(\bar t,\bar x) \in \Omega$, define

$$u(\bar t, x) = \begin{cases} a & \text{if } v(\bar t,\bar x) \le \lambda^-, \\ g(v(\bar t,\bar x)) & \text{if } v(\bar t,\bar x) \in (\lambda^-,\lambda^+), \\ b & \text{if } v(\bar t,\bar x) \ge \lambda^+. \end{cases} \tag{1.23}$$

Observe that g is strictly increasing (because $(\operatorname{conv}_{[a,b]} f)'$ is continuous) and $v(\bar t)$ is strictly increasing by the previous lemma, therefore $g \circ v(\bar t)$ is defined up to a countable set for $\mathcal{L}^1$-a.e. $\bar t$.

REMARK 1.33. By the strict monotonicity of v and the proof of Lemma 1.32, each level set of v is the union of at most countably many segments with velocity v and endpoints in $\partial\Omega$. Therefore, by the strict monotonicity of g, the same holds for the level sets $\{u = c\}$ with $c \in (a,b)$.

In the next proposition we show that u is the unique solution of the boundary problem and we list some of its properties that will be useful in the following sections.

PROPOSITION 1.34. *The function u defined by (1.23) is the unique solution of the boundary value problem in Ω in the class of mv entropy solutions.*

Moreover, there exist two L−Lipschitz curves γ^-,γ^+ such that

(1) for every $t \in [0,T]$, $\gamma_1(t) \le \gamma^-(t) \le \gamma^+(t) \le \gamma_2(t)$;

(2) $u(t,x) = a$ for every (t,x) in

$$\Omega^- := \big\{(t,x) \in \Omega : \gamma_1(t) < x < \gamma^-(t)\big\}$$

and $u(t,x) = b$ for every (t,x) in

$$\Omega^+ := \big\{(t,x) \in \Omega : \gamma^+(t) < x < \gamma_2(t)\big\};$$

(3) if $\gamma_1(t) < \gamma^-(t) < \gamma_2(t)$ then $\dot\gamma^-(t) = \lambda^-$ and similarly if $\gamma_1(t) < \gamma^+(t) < \gamma_2(t)$ then $\dot\gamma^+(t) = \lambda^+$;

(4) u and $f' \circ u$ are strictly increasing in

$$\Omega^m := \big\{(t,x) \in \Omega : \gamma^-(t) < x < \gamma^+(t)\big\}.$$

Moreover $f' \circ u$ is locally Lipschitz in Ω^m;

(5) for $\mathcal{L}^1$-a.e. $t \in (0,T)$ such that $\gamma^-(t) = \gamma_1(t)$, it holds $\dot\gamma^-(t) \ge \lambda^-$; similarly for $\mathcal{L}^1$-a.e. $t \in (0,T)$ such that $\gamma^+(t) = \gamma_2(t)$, it holds $\dot\gamma^+(t) \le \lambda^+$.

PROOF. Uniqueness is a corollary of Proposition 1.30, therefore we need to verify that u is an entropy solution in Ω, (γ_1,a) is an admissible left boundary for u and (γ_2,b) is an admissible right boundary for u. In the interior the analysis is the same as for the classical Riemann problem. Let us verify that (γ_1, a) is an admissible left boundary for u, namely conditions (1.17). Observe that the second condition is trivial, being $u \in [a,b]$. Denote by $u^+(t) \in \mathbb{R}$ the trace of u in $(t, \gamma_1(t))$ for $t \in (0,T)$ and let $\bar t$ be a differentiability point of γ_1. By Point (5) in Lemma 1.32 and the definition of u, it follows that one of the following holds:

$$u^+(\bar t) = a \qquad \text{or} \qquad f'(u^+(\bar t)) \le \dot\gamma_1(\bar t).$$

In the first case it is clear that (1.16) is satisfied, otherwise observe that u and in particular u^+ takes values only in the set $\{u : f(u) = \operatorname{conv}_{[a,b]} f(u)\} \subset \{u : f'(u) = (\operatorname{conv}_{[a,b]} f)'(u)\}$. Therefore for every $k \in [a, u^+(\bar t)]$, it holds

$$\dot\gamma_1(\bar t) \ge f'(u^+(\bar t)) = \Big(\operatorname*{conv}_{[a,b]} f\Big)'(u^+(\bar t)) \ge \Big(\operatorname*{conv}_{[a,b]} f\Big)'(k).$$

In particular (1.16) is satisfied.

In order to prove the second part of the statement, consider

$$\begin{aligned}\gamma^-(t) &= \inf\big\{\{x \in (\gamma_1(t), \gamma_2(t)) : u(t,x) > a\}, \gamma_2(t)\big\},\\ \gamma^+(t) &= \sup\big\{\{x \in (\gamma_1(t), \gamma_2(t)) : u(t,x) < b\}, \gamma_1(t)\big\}.\end{aligned}$$

These curves are Lipschitz because they are straight lines in the open set $\{\gamma_1 < \gamma^\pm < \gamma_2\}$ with bounded slope by (1.23).
Requirements (1) and (2) are satisfied by definition of γ^- and γ^+. Points (3), (4) and (5) follow from the respective points in Lemma 1.32. □

A widely used technique is to consider blow-ups; this is motivated by the fact that (1.14) is invariant under the rescaling $(t,x) \mapsto (\lambda t, \lambda x)$.

Definition 1.35. Let ν be a mv entropy solution on $\mathbb{R}^+ \times \mathbb{R}$. Given $(\bar t, \bar x) \in [0,+\infty) \times \mathbb{R}$ and $\varepsilon > 0$ consider

$$\nu^\varepsilon_{t,x} = \nu_{\bar t + \varepsilon t, \bar x + \varepsilon x}$$

defined for $\bar t + \varepsilon t \ge 0$. For all entropies η the dissipation μ^ε of ν^ε is given by

$$\mu^\varepsilon(B) = \frac{1}{\varepsilon}\mu((\bar t, \bar x) + \varepsilon B),$$

for a Borel set $B \subset \mathbb{R}^2$ such that $(\bar t, \bar x) + \varepsilon B \subset \mathbb{R}^+ \times \mathbb{R}$, where μ denotes the dissipation of ν. Every limit in the sense of Young measures of ν^ε as $\varepsilon \to 0$ is called *blow-up* of ν at $(\bar t, \bar x)$.

It is standard to check that every blow-up of a mv entropy solution is a mv entropy solution. An advantage of considering blow-ups is that often they have some simpler structure; a case of particular interest is the following: let $\bar u : \mathbb{R}^2 \to \mathbb{R}$ be such that

$$\bar u(t,x) = \begin{cases} u^- & \text{if } x < \lambda t, \\ u^+ & \text{if } x > \lambda t, \end{cases} \tag{1.24}$$

for some constants $u^-, u^+, \lambda \in \mathbb{R}$.

If $\bar u$ is a distributional solution of (1.14), then the following relation, called *Rankine-Hugoniot condition*, holds:

$$\lambda(u^+ - u^-) = f(u^+) - f(u^-). \tag{1.25}$$

Moreover, if μ is the dissipation measure relative to the entropy-entropy flux pair (η, q), it holds

$$\mu = \big(-\lambda(\eta(u^+) - \eta(u^-)) + q(u^+) - q(u^-)\big)\mathcal{H}^1 \llcorner \{x = \lambda t\}, \tag{1.26}$$

so that in particular, the entropy admissibility condition reduces to

$$-\lambda(\eta(u^+) - \eta(u^-)) + q(u^+) - q(u^-) \le 0, \tag{1.27}$$

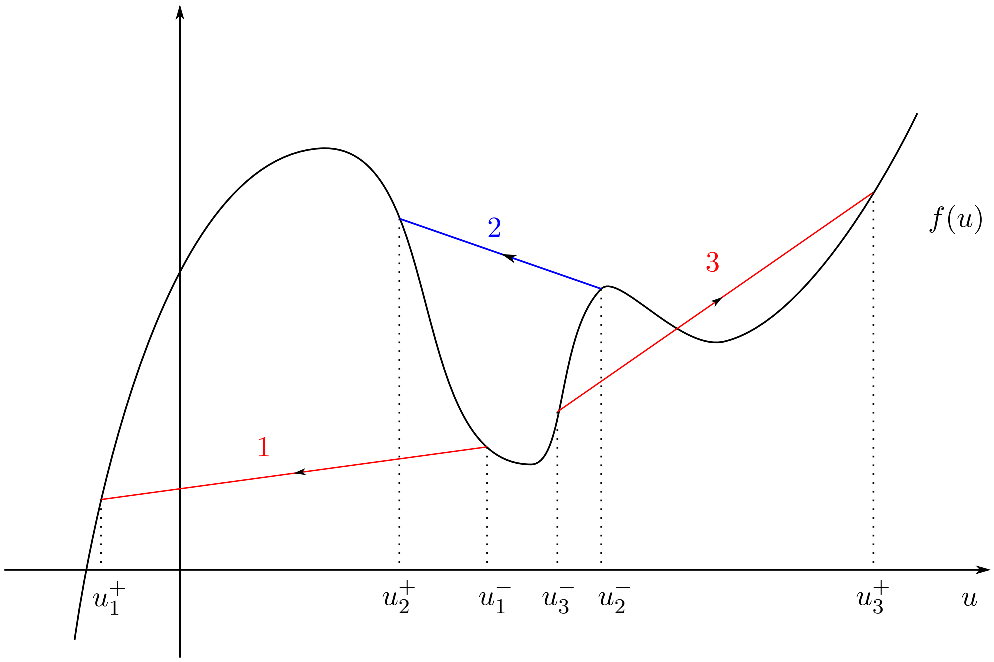


FIGURE 1.6. The shocks 1 between u_1^- and u_1^+ and the shock 3 between u_3^- and u_3^+ (in red) do not satisfy the chord admissibility condition; on the contrary shock 2 between u_2^- and u_2^+ (in blue) is admissible.

for every entropy-entropy flux pair (η, q).

Actually, if u is a distributional solution of (1.14), it is sufficient to verify that (1.27) holds with $\eta = \eta_k^+$ for every $k \in \mathbb{R}$, where η_k^+ is defined in (1.10). This condition is also called *chord admissibility condition* since, together with (1.25), it says that the chord between the two values u^- and u^+ on the graph of f lies below the graph of f if $u^- < u^+$ and above in the opposite case, see Figure 1.6.

We assume now some additional structure on the entropy solution u to (1.9): suppose that there exists a set $J \subset \mathbb{R}^+ \times \mathbb{R}$ such that

(1) there exists countably many Lipschitz curves $\gamma_n : \mathbb{R}_t^+ \to \mathbb{R}$ for $n \in \mathbb{N}$ such that
$$J \subset \bigcup_{n=1}^{\infty} \mathrm{Graph}(\gamma_n);$$
(2) u is continuous on $(\mathbb{R}^+ \times \mathbb{R}) \setminus J$;
(3) for $\mathcal{H}^1$-a.e. $(t,x) \in J$, there exists $\lambda, u^-, u^+ \in \mathbb{R}$ such that every blow-up of u at (t,x) is given by (1.24).

The previous analysis, and in particular (1.26) implies the following result.

PROPOSITION 1.36. *Under the assumptions above there exists $N \subset \mathbb{R}^+$ such that $\mathcal{L}^1(N) = 0$ and for every $t \in T \setminus N$ at each jump of the entropy solution $u(t)$ the chord condition is satisfied: more precisely if $u(t, \bar{x}-) = u^-$ and $u(t, \bar{x}+) = u^+$ then*
$$u^- < u^+ \qquad \Longrightarrow \qquad f(k) \geq f(u^-) + \frac{f(u^+) - f(u^-)}{u^+ - u^-}(k - u^-) \quad \forall k \in (u^-, u^+);$$
$$u^- > u^+ \qquad \Longrightarrow \qquad f(k) \leq f(u^+) + \frac{f(u^+) - f(u^-)}{u^+ - u^-}(k - u^+) \quad \forall k \in (u^+, u^-).$$

REMARK 1.37. We will prove that the regularity required in the Proposition above holds in particular for every L^∞-entropy solutions if the flux f is weakly genuinely nonlinear. Moreover in [BM16] it is proved that it holds for f smooth and u_0 continuous.

CHAPTER 2

Lagrangian representation and complete family of boundaries

Abstract. In this chapter we introduce the concepts of Lagrangian representation and of complete family of boundaries. In Section 2.1 we browse through various formulations of Lagrangian representations recently introduced in [BM14, BY15, BM16]; in particular we point out the main steps to pass from one to the other, starting from the by now well-established wave-front tracking scheme. In Section 2.2 we introduce the notion of complete family of boundaries [BM17], we show its existence for bounded entropy solutions by means of wave-front tracking approximations and we prove some of its properties.

The general underlying idea behind the notion of Lagrangian representation is the method of characteristics: by a formal computation we can reduce the equation (1.14) in quasilinear form:

$$u_t + f'(u)u_x = 0. \tag{2.1}$$

Let $\gamma : [0, T] \to \mathbb{R}$ be such that

$$\dot\gamma(t) = f'(u(t, \gamma(t))).$$

Then

$$\frac{d}{dt}u(t, \gamma(t)) = u_t(t, \gamma(t)) + f'(u(t, \gamma(t)))u_x(t, \gamma(t)) = 0,$$

i.e. u is constant along the characteristic curves of $f' \circ u$. So we introduce a flow $\mathtt{X} : \mathbb{R}^+_t \times \mathbb{R}_y \to \mathbb{R}$, where $\mathtt{X}(t, y)$ denotes the position of the characteristic starting from y at time t. We say that $(\mathtt{X}, u_0)$ represents the solution in the sense that

$$u(t, x) = u_0(\mathtt{X}(t)^{-1}(x)). \tag{2.2}$$

In general, the computations above are not allowed for entropy solutions; the goal is to prove the existence of a flow $\mathtt{X}$ such that (2.2) (or an equivalent version) holds. The strategy we adopt here, to prove the existence of such a flow, is to give an explicit construction for a sequence of approximating solutions and pass it to the limit. In order to proceed in that way, it is convenient to change formulation.

Differentiating formally (2.1) with respect to x and setting $v = u_x$ we get

$$v_t + \big(f'(u)v\big)_x = 0,$$

i.e. v solves the continuity equation driven by the vector field $f'(u)$. Therefore we can write the solution v as

$$v(t)\mathcal{L}^1 = \mathtt{X}(t)_\sharp(v_0\mathcal{L}^1), \qquad \text{where } v_0 = (u_0)_x,$$

and $\mathtt{X}(t)_\sharp(v_0\mathcal{L}^1)$ denotes the push-forward of the measure $v_0\mathcal{L}^1$ by the map $\mathtt{X}$.

Representing in this way the derivative of u, instead of the solution itself, has the advantage of being closer to wave-front tracking algorithm.

2.1. Previous notions of Lagrangian representation

In this section we briefly discuss some formulations of Lagrangian representation that have been recently introduced. The goal is to review the steps starting from the by now well-known wave-front tracking algorithm to the versions of Lagrangian representation that will be useful in the following chapters, see also the proceedings [BM] and [BBM17b]. Since we consider here results contained in previous works, the argument are only sketched.

2.1.1. Wave representation for wave-front tracking approximations. The first step towards the construction of a Lagrangian representation from the wave-front tracking algorithm is the introduction of a *wave representation*, see [BM14]: the main novelty with respect to the wave-front tracking is that you consider waves for their whole time of existence, instead of restarting the procedure at each interaction time.

Denote by u^ν the wave-front tracking approximate solution of (1.9).

DEFINITION 2.1. A *wave representation* of u^ν is a triple of functions

$$\mathtt{X}^\nu : \mathbb{R}^+ \times (0, \mathrm{TV}\{u_0^\nu\}] \supset E^\nu \to \mathbb{R}, \qquad \text{the } \textit{position of the wave } s,$$
$$\mathtt{u}^\nu : (0, \mathrm{TV}\{u_0^\nu\}] \to \mathbb{R}, \qquad \text{the } \textit{value of the wave } s,$$
$$\mathtt{a}^\nu : \mathbb{R}^+ \times (0, \mathrm{TV}\{u_0^\nu\}] \to \{-1, 0, 1\},$$
$$\text{the } \textit{signed existence interval of the wave } s,$$

satisfying the following conditions:

(1) the function $\mathtt{a}^\nu$ is of the form
$$\mathtt{a}^\nu(t, s) = \mathtt{S}^\nu(s)\chi_{[0,\mathtt{T}^\nu(s))}(t) \tag{2.3}$$
for some functions
$$\mathtt{S}^\nu : (0, \mathrm{TV}\{u_0^\nu\}] \to \{-1, 1\}, \qquad \text{the } \textit{sign of the wave } s,$$
$$\mathtt{T}^\nu : (0, \mathrm{TV}\{u_0^\nu\}] \to \mathbb{R}^+ \cup \{+\infty\}, \qquad \text{the } \textit{time of existence of the wave } s;$$

(2) the set E^ν is given by
$$E^\nu = \{(t, s) : t < \mathtt{T}^\nu(s)\};$$

(3) $s \mapsto \mathtt{X}^\nu(t, s)$ is increasing for all t, $t \mapsto \mathtt{X}^\nu(t, s)$ is Lipschitz for all s, and
   (a) $D_x u^\nu(t) = \mathtt{X}^\nu(t, \cdot)_\sharp\big(\mathtt{a}^\nu(t, \cdot)\mathcal{L}^1\llcorner(0, \mathrm{TV}\{u_0^\nu\}]\big)$, i.e. for all $t \geq 0$, $\varphi \in C^1(\mathbb{R})$
$$-\int_{\mathbb{R}} u^\nu(t, x) D_x\varphi(x)\, dx = \int_0^{\mathrm{TV}\{u_0^\nu\}} \varphi(\mathtt{X}^\nu(t, s))\mathtt{a}^\nu(t, s)\, ds, \tag{2.4}$$
   (b) $|D_x u^\nu(t)| = \mathtt{X}^\nu(t, \cdot)_\sharp\big(|\mathtt{a}^\nu(t, \cdot)|\mathcal{L}^1\llcorner(0, \mathrm{TV}\{u_0^\nu\}]\big)$;

(4) the value $\mathtt{u}^\nu$ satisfies for all $t < \mathtt{T}^\nu(s)$
$$\begin{aligned}\mathtt{u}^\nu(s) &= D_x u^\nu(t)(-\infty, \mathtt{X}^\nu(t, s)) + \int_{\{s' < s : \mathtt{X}^\nu(t, s') = \mathtt{X}^\nu(t, s)\}} \mathtt{a}^\nu(t, s')\, ds' \\ &= u^\nu(t, \mathtt{X}^\nu(t, s)-) + \int_{\{s' < s : \mathtt{X}^\nu(t, s') = \mathtt{X}^\nu(t, s)\}} \mathtt{a}^\nu(t, s')\, ds'.\end{aligned}$$
In particular $s \mapsto \mathtt{u}^\nu(s)$ is a 1-Lipschitz function satisfying
$$\frac{d}{ds}\mathtt{u}^\nu(s) = \mathtt{S}^\nu(s).$$

Now we sketch the construction of the wave-representation for a wave-front tracking approximate solution u^ν: we parametrize with $s \in (0, \mathrm{TV}u_0^\nu]$, the points $(x, w) : w \in$

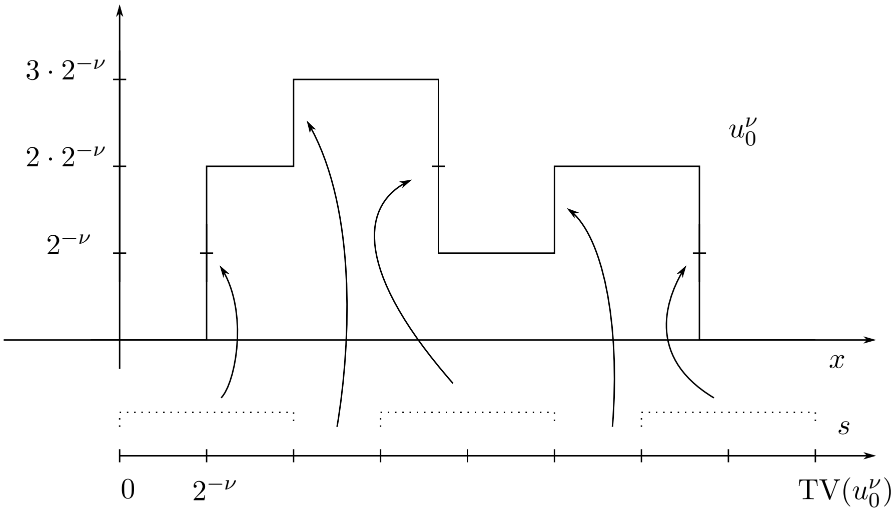


FIGURE 2.1. The parametrization of the waves in the wave-front tracking approximate solution u^ν.

$(\mathrm{sc}^- u_0^\nu(x), \mathrm{sc}^+ u_0^\nu(x))$, then we set $\mathtt{X}^\nu(0,s) = x$ and $\mathtt{u}^\nu(s) = w$, (see Figure 2.1). The sign function

$$\mathtt{S}^\nu := \frac{d}{ds}\mathtt{u}^\nu(s)$$

is equal to 1 if $u_0^\nu(\mathtt{X}(t,s)-) < u_0^\nu(\mathtt{X}(t,s)+)$ and equal to -1 if $u_0^\nu(\mathtt{X}(t,s)+) < u_0^\nu(\mathtt{X}(t,s)-)$. The position $\mathtt{X}(t,s)$ of the wave s at time $t>0$ is the position of the unique discontinuity that starts from $\mathtt{X}^\nu(0,s)$ and such that $\mathtt{u}^\nu(s) \in (u^\nu(t,\mathtt{X}(t,s)-), u^\nu(t,\mathtt{X}(t,s)+)]$ if $\mathtt{S}^\nu(s) = 1$ and $\mathtt{u}^\nu(s) \in (u^\nu(t,\mathtt{X}(t,s)+), u^\nu(t,\mathtt{X}(t,s)-)]$ if $\mathtt{S}^\nu(s) = -1$. If at a certain time t, a cancellation occurs and there are no shocks starting from $\mathtt{X}(t,s)$ such that one of the two condition above holds, then we set $\mathtt{T}^\nu(s) = t$ and, defining $\mathtt{a}^\nu$ by (2.3), this completes the construction. It is fairly easy to check that conditions (1),(3),(4) of the definition of wave representation are satisfied by $(\mathtt{X}^\nu, \mathtt{S}^\nu, \mathtt{a}^\nu)$ constructed above.

REMARK 2.2. In order to avoid ambiguities, we want to consider only binary collisions in wave-front tracking approximate solutions. This can be done by slightly modifying the speed of the shocks.

2.1.2. Wave representation to Lagrangian representation. We observe that it is possible to extend $\mathtt{X}^\nu$ of Definition 2.1 to $\mathbb{R}^+ \times (0, \mathrm{TV}(u_0^\nu)]$ maintaining the Lipschitz and monotonicity regularity with respect to t and s respectively and in such a way that if two waves s_1 and s_2 are canceled at the same collision, let us say at time $\bar t$, then for every $t > \bar t$, it holds $\mathtt{X}(t,s_1) = \mathtt{X}(t,s_2)$. In particular, since the amount of positive waves and negative waves that are canceled at the same point compensate each other, we can replace $\mathtt{a}^\nu(t,\cdot)$ with $\mathtt{S}^\nu(\cdot)$ in (2.4):

$$D_x u^\nu(t) = \mathtt{X}^\nu(t,\cdot)_\sharp\big(\mathtt{S}^\nu(\cdot)\mathcal{L}^1 \llcorner (0, \mathrm{TV}\{u_0^\nu\}]\big). \tag{2.5}$$

2.1.2.1. *BV setting.* The formulation in (2.5) is suitable to pass to the limit for solutions with bounded variation. Indeed for any $u_0 \in \mathrm{BV}(\mathbb{R})$ with compact support, it is possible to choose $\mathtt{S}^\nu \to \mathtt{S}$ in $L^1(0, \mathrm{TV}(u_0))$ and the sequence $(\mathtt{X}^\nu)_\nu$ is compact in L^1 by its regularity assumptions. Therefore, since $\mathtt{S}^\nu$ and $\mathtt{X}^\nu$ are uniformly bounded in L^∞, for every $\varphi \in C_c^\infty(\mathbb{R})$ and every $t>0$, it holds

$$\int_0^{\mathrm{TV}(u_0^\nu)} \mathtt{S}^\nu(s)\varphi(\mathtt{X}^\nu(t,s))ds \longrightarrow \int_0^{\mathrm{TV}(u_0)} \mathtt{S}(s)\varphi(\mathtt{X}(t,s))ds,$$

where $\mathtt{S}$ and $\mathtt{X}$ denote the limits of $(\mathtt{S}^\nu)_\nu$ and $(\mathtt{X}^\nu)_\nu$ respectively. Since by Kruzhkov stability estimate $u^n(t) \to u(t)$ in $L^1(\mathbb{R})$, we have that (2.5) passes to the limit, i.e.

$$D_x u(t) = \mathtt{X}(t,\cdot)_\sharp\big(\mathtt{S}(\cdot)\mathcal{L}^1\llcorner(0,\mathrm{TV}\{u_0\}]\big).$$

Imposing that u solves the equation 1.14, it is possible to deduce the characteristic equation: for every $s \in (0,\mathrm{TV}(u_0)]$, it holds

$$\partial_t \mathtt{X}(t,s) = \lambda(t,\mathtt{X}(t,s)),$$

for $\mathcal{L}^1$-a.e. $t>0$, where

$$\lambda(t,x) = \begin{cases} f'(u(t,x)) & \text{if } u(t) \text{ is continuous at } x, \\ \dfrac{f(u(t,x+)) - f(u(t,x-))}{u(t,x+)-u(t,x-)} & \text{if } u(t) \text{ has a jump at } x. \end{cases}$$

For a different approach which also takes care of the time existence function we refer to [BY15].

2.1.2.2. *Piecewise monotone solutions.* Coming back to wave-front tracking approximations, we consider now the case of piecewise monotone solutions.

In order to obtain a formulation of the form (2.2), we want to cover the whole domain $\mathbb{R}^+ \times \mathbb{R}$ with characteristics. For each continuity point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ of u^ν, we consider the segment passing through (t,x) with constant speed $f'(u^\nu(t,x))$. They are extended in the past and in the future until they do not meet a discontinuity; then they follow the closest existing wave that travels on that discontinuity. In such a way we can parametrize the characteristic with another flow (which we still denote by $\mathtt{X}^\nu : \mathbb{R}^+_t \times \mathbb{R}_y \to \mathbb{R}$) such that $\mathtt{X}^\nu$ is Lipschitz with respect to t and increasing and *continuous* with respect to y. We can also extend the time of existence and the sign functions in such a way that for every $t>0$,

$$\mathtt{X}^\nu(t,\{y : \mathtt{T}^\nu(y) \geq t\}) = \mathbb{R}, \tag{2.6}$$

and for every t such that at time t no cancellations occur,

$$\begin{aligned} \mathtt{S}^\nu(y) = 1 \quad &\Longrightarrow \quad \Big(\mathtt{u}^\nu(y) \leq u(t,\mathtt{X}^\nu(t,y)+) \quad \textit{or} \quad \mathtt{u}^\nu(y) \geq u(t,\mathtt{X}^\nu(t,y)-)\Big), \\ \mathtt{S}^\nu(y) = -1 \quad &\Longrightarrow \quad \Big(\mathtt{u}^\nu(y) \leq u(t,\mathtt{X}^\nu(t,y)-) \quad \textit{or} \quad \mathtt{u}^\nu(y) \geq u(t,\mathtt{X}^\nu(t,y)+)\Big). \end{aligned} \tag{2.7}$$

After the change of parametrization the representation formula (2.5) takes the following form:

$$D_x u^\nu(t) = \mathtt{X}^\nu(t,\cdot)_\sharp\big(D_y \mathtt{u}^\nu(\cdot)\mathcal{L}^1\llcorner(0,\mathrm{TV}\{u^\nu_0\}]\big). \tag{2.8}$$

We also observe that the number of connected components of $\{y : \mathtt{T}^\nu(y) \geq t\}$ is bounded by the number N of points $x_1 < \ldots < x_N$ such that u^ν_0 is monotone in $(-\infty,x_1),(x_N,+\infty)$ and (x_n,x_{n+1}) for $n=1,\ldots,N-1$. Therefore if u_0 is piecewise monotone with compact support it is possible to choose u^ν_0 and triples $(\mathtt{X}^\nu,\mathtt{T}^\nu,\mathtt{S}^\nu)$ as above such that for every $\mathtt{T}^\nu \to \mathtt{T}$ and $\mathtt{S}^\nu \to \mathtt{S}$ pointwise (except at most finitely many points), and $\mathtt{X}^\nu \to \mathtt{X}$ uniformly such that $(\mathtt{X},\mathtt{T},\mathtt{S})$ satisfies (2.6), (2.7), and (2.8). Moreover, by (2.6) and (2.8), it follows that for every $t>0$ there exists $Q(t)$ at most countable and $\tilde{u}(t) = u(t)$ $\mathcal{L}^1$-a.e. such that for every $x \in \mathbb{R}\setminus Q(t)$,

$$\mathtt{X}(t)^{-1}(x) = \{y\} \qquad \text{and} \qquad \tilde{u}(t,x) = \mathtt{u}(\mathtt{X}(t)^{-1}(x)).$$

2.1.2.3. *Solutions with continuous initial datum.* We recall that the final goal is to provide a Lagrangian representation for general L^∞-entropy solutions; here we sketch how to deal with the case of entropy solutions of (1.9) with continuous initial datum u_0. The failure of this argument for general $u_0 \in L^\infty$ motivates the analysis in the following section.

Suppose at the moment that $u_0 \in \mathrm{BV}(\mathbb{R})$ and let u be the entropy solution of (1.9). By (2.8), we have that for every $\varphi \in C_c^\infty(\mathbb{R})$, it holds

$$-\int_{\mathbb{R}} u(t,x) D_x\varphi(x)dx = \int_{\mathbb{R}} D_y \mathtt{u}(y)\varphi(\mathtt{X}(t,y))dy;$$

integrating by parts in the r.h.s. we get

$$\int_{\mathbb{R}} u(t,x) D_x\varphi(x)dx = \int_{\mathbb{R}} \mathtt{u}(y) dD_y(\varphi \circ \mathtt{X}(t))(y). \tag{2.9}$$

In particular we do not need any derivative of $\mathtt{u}$, but we need that $\mathtt{u}$ is defined almost everywhere with respect to the measure $D_y(\varphi \circ \mathtt{X}(t))$. More precisely if the representation formula (2.9) holds with $(\mathtt{X}^n, \mathtt{u}^n)$ and we want to pass it to the limit, relying only on the natural weak-star compactness of $(D_y(\varphi \circ \mathtt{X}^n(t)))_{n\in\mathbb{N}}$ as measures, we need the uniform convergence of $\mathtt{u}^n$.

The passage to the limit is indeed possible if $u_0 \in C^0(\mathbb{R})$ and also if u_0 has countably many jump points, since, as in the BV case, the solution can be represented with $\mathtt{u} \in C^0(\mathbb{R})$. See [**Mar14**] for more details.

In general we can expect only the L^1 convergence of $\mathtt{u}^n \to \mathtt{u}$, so we would need a uniform integrability estimate on the measures $D_y(\varphi \circ \mathtt{X}^n(t))$, but it seems difficult to prove.

2.2. Complete family of boundaries

In this section we introduce the notion of complete family of boundaries (see [**BM17**]): comparing to the Lagrangian representation, this notion is more stable and it allows to deal with the case of general bounded entropy solutions. In Section 3.4 we will see that it is actually possible to recover the original Lagrangian representation from the structure of the solution.

In order to include wavefront tracking approximate solutions, in the following definition we will consider fluxes f which are C^1 outside finitely many points and such that the left and right limits of f' exist everywhere. Here and in what follows we use the notation: $\exists\, a(B)$ to say $\exists\, a$ such that B holds.

Definition 2.3. Let ν be a mv entropy solution of (1.9). A *complete family of boundaries* is a couple $(\mathcal{K}, T)$, where

(a) $\mathcal{K}$ is a closed subset of $\mathrm{Lip}([0,+\infty),\mathbb{R}) \times \mathbb{R}$ with respect to the product topology of the topology of locally uniform convergence on $\mathrm{Lip}([0,+\infty),\mathbb{R})$ and the euclidean topology on $\mathbb{R}$,

(b) $T : \mathcal{K} \to \mathbb{R}$ is an upper semicontinuous function,

and the following properties hold:

(1) *monotonicity*: $\forall (\gamma_1, w_1), (\gamma_2, w_2) \in \mathcal{K}$,

$$\exists \bar t : \gamma_1(\bar t) < \gamma_2(\bar t) \qquad \Longrightarrow \qquad \forall t \geq 0,\ \gamma_1(t) \leq \gamma_2(t).$$

In particular there exists a total order on $\mathcal{K}_\gamma = \{\gamma : \exists w, (\gamma, w) \in \mathcal{K}\}$:

$$\gamma_1 \leq \gamma_2 \qquad \Longleftrightarrow \qquad \forall t \geq 0,\ \gamma_1(t) \leq \gamma_2(t);$$

(2) *entropy admissibility*: every $(\gamma, w) \in \mathcal{K}$ is an admissible boundary for the mv solution ν for $t \leq T(w, \gamma)$.

Moreover, setting

$$K = \Big\{(t,x,w) : \exists (\gamma, w) \in \mathcal{K} \text{ such that } \gamma(t) = x, T(\gamma, w) \geq t\Big\}$$

and its sections

$$K(t,x) = \{w : (t,x,w) \in K\},$$

it holds

(3) *completeness*:
$$\operatorname{supp}\left(\mathcal{L}^2 \otimes \nu_{t,x}\right) \subset K;$$
(4) *connectedness*: for all $t \geq 0$ and $\gamma_1 \leq \gamma_2$, the set
$$\Big\{(\gamma(t), w) : (\gamma, w) \in \mathcal{K}, \gamma_1 \leq \gamma \leq \gamma_2, t \leq T(w, \gamma)\Big\}$$
is connected;

(5) *consistency with the PDE*: for every $r > 0$
$$V_{\bar{t},\bar{x}}(r) := \left\{\dot{\gamma}(t) : (t, \gamma(t)) \in B_{\bar{t},\bar{x}}(r)\right\} \subset \bigcup_{w \in U_{\bar{t},\bar{x}}(r)} \left(D^+ f(w) \cup D^- f(w)\right), \tag{2.10}$$
where $D^{\pm} f(w)$ denotes the super/subdifferential of f at w and
$$U_{\bar{t},\bar{x}}(r) = \Big\{w : \exists t, \gamma \, \Big((\gamma, w) \in \mathcal{K}, (t, \gamma(t)) \in \bar{B}_{\bar{t},\bar{x}}(r) \text{ and } t \leq T(w, \gamma)\Big)\Big\}.$$

For a C^1 flux f, Condition (5) reduces to
$$\left\{\dot{\gamma}(t) : (t, \gamma(t)) \in B_{\bar{t},\bar{x}}(r)\right\} \subset \left\{f'(w) : w \in U_{\bar{t},\bar{x}}(r)\right\}.$$

Remark 2.4. The completeness property and the fact that K is closed imply that for every $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ the section $K(t, x) \neq \emptyset$.

Remark 2.5. The monotonicity and the covering properties imply that the curves in $\mathcal{K}_\gamma$ can be parametrized by $\mathbb{R}$: more precisely there exists a monotone invertible map $p : \mathbb{R} \to \mathcal{K}_\gamma$. We will denote $p(y)$ by γ_y.

Following the general strategy outlined in the previous section, we prove first the existence of a complete family of boundaries for a sequence of approximate solutions (again by wave-front tracking), then we show that this structure passes to the limit. The following proposition contains the stability result that we need in order to do this last step.

Proposition 2.6. *Let ν^n be mv entropy solutions of (1.9) and let $(\mathcal{K}^n, T^n)$ be a complete family of boundaries for ν^n. Assume that*

(1) $f^n \to f$ uniformly and $\mathrm{Graph}(D^+ f^n) \cup \mathrm{Graph}(D^- f^n) \to \mathrm{Graph}(f')$ *in the sense of Kuratowski,*

(2) $\nu^n \to \nu$ in the sense of Young measures,

(3) $\mathcal{K}^n \to \mathcal{K}$ in the sense of Kuratowski,

and set
$$T(\gamma, w) = \inf_{U \in \mathcal{U}(\gamma,w)} \limsup_{n \to +\infty} \sup_{(\gamma',w') \in U} T^n(\gamma', w') = -\Gamma\text{-}\liminf_{n \to +\infty} (-T^n(\gamma, w)).$$
Then $(\mathcal{K}, T)$ is a complete family of boundaries for ν.

Proof. We have to verify conditions (1) to (5) in Definition 2.3: Condition (2) follows from Proposition 1.29 and conditions (1), (3) and (4) follow from the very definition of Kuratowski convergence, convergence in the sense of Young measures and the definition of T. About Condition (5), each $\gamma \in \mathcal{K}_\gamma$ is the uniform limit of $\gamma^n \in \mathcal{K}^n_\gamma$, in particolar $\dot{\gamma}^n \to \dot{\gamma}$ weakly. Therefore, for every $r > 0$,
$$V_{\bar{t},\bar{x}}(r) \subset K\text{-}\limsup_{n\to\infty} \operatorname{conv} V^n_{\bar{t},\bar{x}}(r). \tag{2.11}$$
By the Kuratowski convergence of the complete families of boundaries,
$$K\text{-}\limsup_{n\to\infty} U^n_{\bar{t},\bar{x}}(r) \subset U_{\bar{t},\bar{x}}(r),$$

hence, by Assumption (1),

$$\bigcup_{w\in U_{\bar t,\bar x}(r)}\{f'(w)\}\supset K\text{-}\limsup_{n\to\infty}\bigcup_{w\in U^n_{\bar t,\bar x}(r)}\left(D^+f^n(w)\cup D^-f^n(w)\right). \tag{2.12}$$

Since $\bigcup_{w\in U^n_{\bar t,\bar x}(r)}(D^+f^n(w)\cup D^-f^n(w))$ is convex by the connectedness property, the claim follows from (2.11) and (2.12). ☐

In the following lemma we construct a complete family of boundaries for wave-front tracking approximate solutions: as before, we assume that only binary interactions among shocks occur.

LEMMA 2.7. *Let $u:\mathbb{R}^+\times\mathbb{R}\to 2^{-k}\mathbb{Z}$ be a wave-front tracking approximate solution of (1.9). Then there exists a complete family of boundaries for u such that*

(1) every $\gamma\in\mathcal{K}_\gamma$ is piecewise affine and for all except finitely many positive times it holds

$$\dot\gamma(t)=\lambda(t,\gamma(t)),$$

where

$$\lambda(t,x)=\begin{cases} f'(u(t,x)) & \text{if } u \text{ is continuous at } (t,x),\\ \dfrac{f(u(t,x+))-f(u(t,x-))}{u(t,x+)-u(t,x-)} & \text{if } u \text{ has a jump at } (t,x); \end{cases} \tag{2.13}$$

(2) for all $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ except the cancellation points

$$K(t,x)=\mathrm{conv}(u^-,u^+),$$

where u^- and u^+ denote the left limit and right limit respectively. At every cancellation point

$$K(t,x)=\mathrm{conv}(u^-,u^+)\cup I,$$

where I is the set of values of u that is canceled in (t,x).

PROOF. We just prove the existence because the properties (1) and (2) follow easily from the construction. *Step 1.* We first construct the candidate admissible boundaries on the set J of discontinuity points of u. Consider a shock starting at $t=0$ from $\bar x$ with left and right limits u^- and u^+ respectively. For every $w\in\mathrm{conv}(u^-,u^+)\setminus 2^{-k}\mathbb{Z}$ consider the unique Lipschitz continuous curve $\gamma_w: I_w\to\mathbb{R}$ such that for all $t\in I_w$, $w\in\mathrm{conv}(u(t,\gamma_w(t)-),u(t,\gamma_w(t)+))$, where $I_w=[0,\bar t]$ if the value w is canceled at time $\bar t$ and $I_w=[0,+\infty)$ if the value w is not canceled. Denote the set of pairs (γ_w,w) by $\tilde{\mathcal{K}}^1$ and set $\tilde T(\gamma_w,w)=\sup I_w$. The following monotonicity property holds: let $w_1<w_2$ and $t\in I_{w_1}\cap I_{w_2}$ such that $\gamma_{w_1}(t)=\gamma_{w_2}(t)=x$. Then $u(t,x-)<u(t,x+)$ implies $\gamma_{w_1}\le\gamma_{w_2}$ in $I_{w_1}\cap I_{w_2}$ and similarly $u(t,x-)>u(t,x+)$ implies $\gamma_{w_1}\ge\gamma_{w_2}$ in $I_{w_1}\cap I_{w_2}$. The proof is by direct inspection of binary interactions of shocks.

Step 2. Next we construct segments in $\mathbb{R}^+\times\mathbb{R}\setminus J$. For every $(\bar t,\bar x)\in\mathbb{R}^+\times\mathbb{R}\setminus J$ consider the straight line $\gamma^{\bar t,\bar x}:\mathbb{R}^+\to\mathbb{R}$ where $\gamma^{\bar t,\bar x}(\bar t)=\bar x$, ${\gamma^{\bar t,\bar x}}'(t)=f'(u(\bar t,\bar x))$. In order to have monotonicity we consider $\gamma^{\bar t,\bar x}$ restricted to the connected component $(t_1(\bar t,\bar x),t_2(\bar t,\bar x))$ of $\{t\in\mathbb{R}^+:(t,\gamma^{\bar t,\bar x}(t)\in\mathbb{R}^+\times\mathbb{R}\setminus J\}$ which contains $\bar t$. Denote the set of pairs $(\gamma^{\bar t,\bar x},u(\bar t,\bar x))$ by $\tilde{\mathcal{K}}^2$ and set $\tilde T(\gamma^{\bar t,\bar x},u(\bar t,\bar x))=t_2(\bar t,\bar x)$.

In order to construct a complete family of boundaries we begin to extend the curves in $\tilde{\mathcal{K}}^1$ and $\tilde{\mathcal{K}}^2$ to the whole $\mathbb{R}^+$.

Step 3. First, for every $(\bar t,\bar x)\in\mathbb{R}^+\times\mathbb{R}\setminus J$ we prolong $\gamma^{\bar t,\bar x}$ to $[0,t_2(\bar t,\bar x))$. Denote by γ^- and γ^+ the left and the right boundary of the connected component of $\mathbb{R}^+\times\mathbb{R}\setminus J$ which contains $(\bar t,\bar x)$. If $t_1(\bar t,\bar x)>0$ at least one of the following holds:

$$\gamma^-(t_1(\bar t,\bar x))=\gamma^{\bar t,\bar x}(t_1(\bar t,\bar x)+)\qquad\text{or}\qquad\gamma^+(t_1(\bar t,\bar x))=\gamma^{\bar t,\bar x}(t_1(\bar t,\bar x)+). \tag{2.14}$$

If the first condition holds we set $\gamma^{\bar{t},\bar{x}} = \bar{\gamma}$ in $[0, t_1(\bar{t},\bar{x})]$ where $\bar{\gamma} : [0, t_1(\bar{t},\bar{x})] \to \mathbb{R}$ is the unique curve such that there exists $(\gamma_w, w) \in \tilde{K}^1$ for which

$$\gamma_w = \bar{\gamma} \text{ in } [0, t_1(\bar{t},\bar{x})], \quad |w - u(\bar{t},\bar{x})| < 2^{-k},$$
$$\widetilde{T}(\gamma_w, w) > t_1(\bar{t},\bar{x}), \quad \gamma_w = \gamma^- \text{ in } (t_1(\bar{t},\bar{x}), t_1(\bar{t},\bar{x}) + \varepsilon)$$

for some $\varepsilon > 0$. If the first condition in (2.14) does not hold, the analogue extension can be done for $\bar{\gamma} = \gamma^+$ in a right neighborhood of $t_1(\bar{t},\bar{x})$. This extension maintains the monotonicity. Denote by $\tilde{\mathcal{K}}^3$ this extension of $\tilde{K}^2$.

For the extension in the future the only constraint is the monotonicity: denote by $\tilde{\mathcal{K}} = \tilde{\mathcal{K}}^1 \cup \tilde{\mathcal{K}}^3$ and let $(\gamma, w) \in \tilde{\mathcal{K}}$ and $t > \widetilde{T}(\gamma, w)$. Then we consider the following extension:

$$\gamma(t) = \sup_{(\gamma',w')\in\tilde{\mathcal{K}}} \left\{\gamma'(t) : \widetilde{T}(\gamma', w') \geq t \text{ and } \exists t' < T(\gamma, w) \text{ such that } \gamma'(t') < \gamma(t')\right\}.$$

It is fairly easy to check that with this extension the monotonicity is preserved.

Finally let $\mathcal{K}$ be the closure of the family constructed above with respect to the product of local uniform convergence topology and the standard topology on $\mathbb{R}$ and let T the minimal upper semicontinuous extension of $\widetilde{T}$.

To conclude, we have to verify the properties in Definition 2.3. Only the entropy admissibility is not straightforward but it is a consequence of the fact that for every $(\gamma, w) \in \mathcal{K}$ and $t < T(\gamma, w)$ it holds $w \in \mathrm{conv}(u(t, \gamma(t)-), u(t, \gamma(t)+))$. $\square$

The construction of a complete family of boundaries for approximations by wave-front tracking and the stability proven in Proposition 2.6 imply the following result.

THEOREM 2.8. *For every entropy solution of* (1.9) *with initial data* $u_0 \in L^\infty$*, there exists a complete family of boundaries.*

Since the linearly degenerate components of the flux play a significant role in what follows we introduce the following notation.

DEFINITION 2.9. We denote by $\mathcal{L}_f$ the set of maximal closed intervals on which f' is constant. For every $w \in \mathbb{R}$ we denote by I_w the unique element of $\mathcal{L}_f$ which contains w. Moreover if $I \in \mathcal{L}_f$, we write $f'(I)$ to indicate $f'(w)$ for some $w \in I$. Finally, when $\mathcal{L}_f$ is considered as a topological space it is endowed with the quotient topology obtained from the euclidean topology on $\mathbb{R}$ by the relation that identifies elements belonging to the same $I \in \mathcal{L}_f$.

By the completeness property of the complete family of boundaries we have that $\mathrm{supp}\,\left(\mathcal{L}^2 \otimes \nu_{t,x}\right) \subset K$; the next lemma is a first result about the opposite inclusion. We will see that it holds up to linearly degenerate components of the flux.

LEMMA 2.10. *Let* ν *be a mv entropy solution on* $\mathbb{R}^2$ *such that there exists* $I = [a, b] \in \mathcal{L}_f$ *for which* $\mathcal{L}^2$*-a.e.* $(t, x) \in \mathbb{R}^2$, $\mathrm{supp}\, \nu_{t,x} \subset I$ *and let* (γ, w) *an admissible boundary for* ν*. Then* $w \in I$*.*

PROOF. Assume by contradiction that there exists an admissible boundary (γ, w) with $w \notin I$ and let $\sigma = f'(I)$. First we prove that $\dot{\gamma} = \sigma$. Without loss of generality take $w < a$. By the admissibility condition (1.16) for every $w \leq k \leq a$,

$$\begin{aligned} 0 &\leq \langle \nu^-, f(\lambda) - f(k) - \dot{\gamma}(\lambda - k)\rangle \\ &= \langle \nu^-, f(\lambda) - f(k) - \sigma(\lambda - k) + (\sigma - \dot{\gamma})(\lambda - k)\rangle \\ &= f(a) - f(k) - \sigma(a - k) + (\sigma - \dot{\gamma})(\langle \nu^-, \lambda\rangle - k) \end{aligned}$$

because $f(w) - \sigma w$ is constant on I. Since $f(a) - f(k) - f'(a)(a - k) = o(|a - k|)$ as $k \to a$, $\sigma = f'(a)$ and $\langle \nu^-, \lambda\rangle \geq a$ we get $\sigma \geq \dot{\gamma}$. The same argument on the admissibility condition from the right of γ implies that $\sigma \leq \dot{\gamma}$ therefore $\sigma = \dot{\gamma}$. In particular the condition above reduces to $0 \leq f(a) - f(k) - \sigma(a-k)$ for all $w \leq k \leq a$ and the one on the right to $0 \geq f(a) - f(k) - \sigma(a-k)$ for all $w \leq k \leq a$. This means that w and a belongs to the same linearly degenerate component of the flux and this contradicts the maximality of $[a, b]$. □

In the next lemma we state some additional properties of the complete family of boundaries when the solution has bounded total variation. These results are based on a blow-up argument.

Lemma 2.11. *Let ν be a mv solution of (1.14) with a complete family of boundaries $(\mathcal{K}, T)$ and let $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$ be such that for every $(t, x) \in \Omega$,*

$$\nu_{t,x} = \delta_{u(t,x)},$$

where $u \in \mathrm{BV}(\Omega)$. Then for every $\gamma \in \mathcal{K}_\gamma$ and for $\mathcal{L}^1$-a.e. $t > 0$ such that $(t, \gamma(t)) \in \Omega$, it holds

$$\dot{\gamma}(t) = \lambda(t, \gamma(t)) \tag{2.15}$$

where λ is defined in (2.13). Moreover for $\mathcal{H}^1$-a.e. $(t, x) \in \Omega$,

$$\operatorname{conv}(u^-, u^+) \subset K(t, x) \subset \operatorname{conv}\left(I_{u^-}, I_{u^+}\right), \tag{2.16}$$

where u^- and u^+ denote the left and right limits if (t, x) is a jump point of u and $u^- = u^+ = a$ if u has a Lebesgue point with value a in (t, x).

The intervals I_{u^-} and I_{u^+} are defined in Definition 2.9.

Proof. For $\mathcal{H}^1$-a.e. $(\bar{t}, \bar{x}) \in \Omega$ there are two possibilities for the L^1-blow-up of u in $(\bar{t}, \bar{x})$:

(1) the limit is contained in I for some $I \in \mathcal{L}_f$;
(2) the limit is a jump with u^- and u^+ which do not belong to the same $I \in \mathcal{L}_f$.

In the first case (2.15) follows from Lemma 2.10 and (2.10). Moreover the first inclusion in (2.16) follows from the connectedness property in Definition 2.3 and the second inclusion follows from Lemma 2.10, being the blow-up a mv entropy solution.

In the second case let

$$\bar{u}(t, x) = \begin{cases} u^- & x < \lambda t, \\ u^+ & x > \lambda t, \end{cases}$$

be the L^1-blow-up. By Lemma 2.10, the speed of admissible boundaries in $\{(t, x) : x < \lambda t\}$ is $f'(u^-)$ and similarly the speed of admissible boundaries in $\{(t, x) : x > \lambda t\}$ is $f'(u^+)$. Moreover, since ν is a mv entropy solution, then

$$f'(u^-) \geq \lambda \geq f'(u^+)$$

and if γ is differentiable at $\bar{t}$, its blow-up is a straight line. So the unique velocity that γ can have, without violating the monotonicity property, is $\lambda(\bar{t}, \gamma(\bar{t}))$.

As in the previous case, the first inclusion in (2.16) follows from connectedness. About the second inclusion, let us consider an admissible boundary (γ, w) for $\bar{u}$. If $\gamma(t) \neq \lambda t$ the result follows from Lemma 2.10. Finally consider the case $\gamma(t) = \lambda t$ and suppose without loss of generality that $w < u^- < u^+$ and let $k \in (w, u^-)$. By the admissibility conditions (1.16) and (1.17),

$$f(u^+) - \lambda(u^+ - k) \leq f(k) \leq f(u^-) - \lambda(u^- - k) = f(u^+) - \lambda(u^+ - k),$$

therefore $w \in I_{u^-}$ and $\lambda = f'(u^-)$. □

Now we consider the particular case of the Riemann problem with two boundaries. With the same notation as in Proposition 1.34, the previous result implies that the complete family of boundaries is uniquely determined in Ω^m.

COROLLARY 2.12. *Let ν be a mv entropy solution of (1.14) with a complete family of boundaries $(\mathcal{K}, T)$ and let $(\gamma_1, a), (\gamma_2, b) \in \mathcal{K}$ such that for some $0 \leq t_1 < t_2$:*

(1) $\gamma_1(t_1) = \gamma_2(t_1)$;
(2) $\gamma_1(t) < \gamma_2(t)$ for every $t \in (t_1, t_2)$;
(3) $T(\gamma_1, a) > t_2, T(\gamma_2, b) > t_2$.

Let $(\gamma, w) \in \mathcal{K}$ be such that there exists $\bar{t} > 0$: $(\bar{t}, \gamma(\bar{t})) \in \Omega^m$ and $T(\gamma, w) \geq \bar{t}$.

Then $w \in (a, b)$ and γ coincides in $[t_1, \bar{t}]$ with the unique Lipschitz curve $\tilde{\gamma} : [t_1, \bar{t}] \to \mathbb{R}$ such that

(a) for all $t \in [t_1, \bar{t}]$, $\gamma_1(t) \leq \tilde{\gamma}(t) \leq \gamma_2(t)$;
(b) it holds

$$\mathrm{Graph}(\tilde{\gamma}) \cap \Omega^m = \left\{ (t, x) \in \Omega : v(t, x) = \left(\operatorname*{conv}_{[a,b]} f \right)' (w) \right\},$$

where v is defined in Lemma 1.32.

Moreover for every $(t, x) \in \Omega^m$, it holds

$$K(t, x) = \mathrm{conv}(u^-, u^+),$$

where u^-, u^+ denote the left and right limits of u at time t in x.

PROOF. Let $(\gamma, w) \in \mathcal{K}$ be as in the statement. By Point (4) of Proposition 1.34, it follows that for every $(t, x) \in \Omega^m$ it holds

$$\mathrm{conv}(u^-, u^+) = \mathrm{conv}(I_{u^-}, I_{u^+}),$$

otherwise $f' \circ u$ would not be strictly increasing, therefore the last part of the statement is a consequence of (2.16) and it implies that $w \in (a, b)$.

Since v is strictly increasing with respect to x in Ω^m, the curve $\tilde{\gamma}$ that satisfies (a) and (b) in the statement is unique. Indeed suppose by contradiction that $\gamma(\tilde{t}) \neq \tilde{\gamma}(\tilde{t})$ for some $\tilde{t} \in (t_1, \bar{t})$, then u solves the boundary Riemann problem with boundary data equal to w on γ and $\tilde{\gamma}$. In particular $u(t, x) \equiv w$ for every (t, x) in the open region delimited by γ and $\tilde{\gamma}$ and this contradicts the strict monotonity of u in Ω^m. □

REMARK 2.13. A complete family of boundaries for a Riemann problem with two boundaries is not uniquely determined in Ω^- and Ω^+ if I_a and I_b are non trivial. However we will see that $\Omega^- = \Omega^+ = \emptyset$ in the setting of the corollary above.

CHAPTER 3

Structure of L^∞-entropy solutions to scalar conservation laws in one space dimension

ABSTRACT. In this chapter we present the results about the structure of the entropy solution of (1) obtained in [BM17]. In Section 3.1 we study the structure of characteristics: they are segments outside a countably 1-rectifiable set and the left and right traces of the solution exist in a C^0-sense up to the degeneracy due to the intervals where $f''=0$. In Section 3.2 we prove that the entropy dissipation of an entropy solution u is a measure concentrated on countably many Lipschitz curves. In Section 3.3 we show that the initial data is taken in a suitably strong sense. In Section 3.4 we refine the notion of Lagrangian representation introduced in the previous chapter taking advantage of the structure proved here, and finally in Section 3.5, we give some examples which show that these results are sharp.

3.1. Structure of $\mathcal{K}$

In this section we see that a complete family of boundaries for a mv entropy solution enjoys additional properties than the ones required in the definition. More precisely we prove that $\mathbb{R}^+\times\mathbb{R}$ is covered by characteristics which are straight lines outside a 1-rectifiable set of jumps, similarly to the case of solutions with bounded variations.

First we introduce some notation: given $\gamma\in\mathcal{K}_\gamma$, a differentiability point $\bar t$ of γ and $r,\delta>0$, let

$$B^{\delta+}_{\bar t,\gamma}(r):=\Big\{(t,x)\in B_{\bar t,\gamma(\bar t)}(r): x>\gamma(\bar t)+\dot\gamma(\bar t)(t-\bar t)+\delta|t-\bar t|\Big\},$$

$$B^{\delta-}_{\bar t,\gamma}(r):=\Big\{(t,x)\in B_{\bar t,\gamma(\bar t)}(r): x<\gamma(\bar t)+\dot\gamma(\bar t)(t-\bar t)-\delta|t-\bar t|\Big\}.$$

Accordingly we define

$$U_{\bar t,\bar x}(r):=\Big\{w\in\mathbb{R}:\exists t\in\mathbb{R}^+,(\gamma,w)\in\mathcal{K}\text{ such that }T(\gamma,w)>t,(t,\gamma(t))\in B_{\bar t,\bar x}(r)\Big\},$$

$$U^{\delta\pm}_{\bar t,\bar\gamma}(r):=\Big\{w\in\mathbb{R}:\exists t\in\mathbb{R}^+,(\gamma,w)\in\mathcal{K}\text{ such that }T(\gamma,w)>t,(t,\gamma(t))\in B^{\delta\pm}_{\bar t,\bar\gamma}(r)\Big\}.$$

For every $(t,x)\in\mathbb{R}^+\times\mathbb{R}$, let us introduce the maximal characteristic $\gamma^+_{t,x}$ as the maximum in $\mathcal{K}_\gamma$ of the curves γ such that $\gamma(t)=x$. The maximum exists being $\mathcal{K}$ closed. Similarly let $\gamma^-_{t,x}$ be the minimal characteristic, and denote by

$$U^+_{t,x}(r):=\Big\{w\in\mathbb{R}:\exists t\in\mathbb{R}^+,(\gamma,w)\in\mathcal{K}\Big(T(\gamma,w)>t,(t,\gamma(t))\in B_{t,x}(r)\text{ and }\gamma>\gamma^+_{t,x}\Big)\Big\},$$

$$U^-_{t,x}(r):=\Big\{w\in\mathbb{R}:\exists t\in\mathbb{R}^+,(\gamma,w)\in\mathcal{K}\Big(T(\gamma,w)>t,(t,\gamma(t))\in B_{t,x}(r)\text{ and }\gamma<\gamma^-_{t,x}\Big)\Big\}.$$

See Figure 3.1. Notice that this notion of maximal and minimal characteristics is different from the one introduced in [Daf89].

LEMMA 3.1. *Consider $\bar\gamma\in\mathcal{K}_\gamma$ and $[t_1,t_2]\subset\mathbb{R}^+$. Suppose that $\forall\gamma\in\mathcal{K}_\gamma$,*

$$\exists\,\bar t\in[t_1,t_2]:\gamma(\bar t)>\bar\gamma(\bar t)\quad\Longrightarrow\quad\forall t\in[t_1,t_2]:\gamma(t)>\bar\gamma(t).$$

Then there exists an interval $I\in\mathscr{L}_f$ such that for every sequence $(\gamma^n,u^n)\in\mathcal{K}$ satisfying

(1) $\gamma^n\llcorner_{[t_1,t_2]}>\bar\gamma\llcorner_{[t_1,t_2]}$,
(2) $\gamma^n\to\bar\gamma$ uniformly in $[t_1,t_2]$,

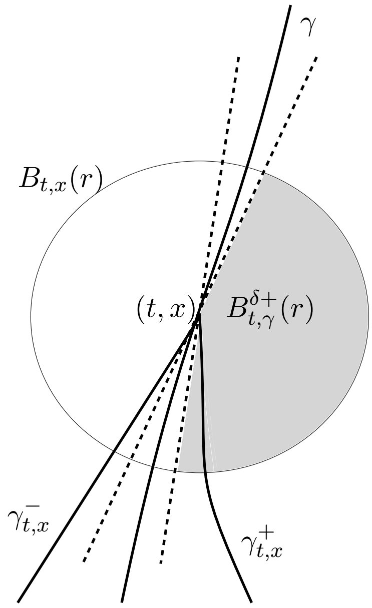


FIGURE 3.1. Maximal and minimal curves and cone.

(3) $\liminf_{n\to+\infty} T(\gamma^n, w^n) = \tilde t > t_1$,

it holds

$$\lim_{n\to+\infty} \mathrm{dist}(w^n, I) = 0. \tag{3.1}$$

In particular $\bar\gamma$ *is a segment in* $[t_1, t_2]$ *with velocity* $f'(I)$.

PROOF. **Claim 1**. Let (γ^n, w^n) be a sequence as in the statement and let $\bar w$ be a cluster point of the sequence w^n. Let η be an entropy of this type:

$$\eta^+_k(u) = (u-k)^+ \text{ with } k > \bar w \quad \text{or} \quad \eta^-_k(u) = (u-k)^- \text{ with } k < \bar w$$

and denote the relative flux by q.

Then the flux from the right side of η across $\bar\gamma$ in $[t_1, \tilde t]$ is zero: i.e. for almost every $t \in [t_1, \tilde t]$,

$$-\dot{\bar\gamma}\langle \nu^+_t, \eta\rangle + \langle \nu^+_t, q\rangle = 0.$$

Proof of Claim 1. Roughly speaking the proof is the following: consider the amount of entropy between $\bar\gamma$ and γ^n at time t_1. The flux across both boundaries is non negative, in particular the flux across $\bar\gamma$ from the right is less than the total amount of entropy at time t_1 between $\bar\gamma$ and γ^n. Since γ^n is arbitrarily close to $\bar\gamma$ the flux must be 0.

Consider η as in the statement of the claim and compute the balance in the region delimited by $\bar\gamma$ and γ^n for $t \in [t_1, \tilde t]$: using that $\eta_t + q_x \le 0$,

$$\int_{\gamma(\tilde t)}^{\gamma^n(\tilde t)} \langle \nu_{\tilde t,x}, \eta\rangle dx - \int_{\gamma(t_1)}^{\gamma^n(t_1)} \langle \nu_{t_1,x}, \eta\rangle dx + \int_{t_1}^{\tilde t} (\langle \nu^-_{n,t}, -\dot\gamma^n(t)\eta + q\rangle - \langle \nu^+_t, -\dot{\bar\gamma}(t)\eta + q\rangle) dt \le 0, \tag{3.2}$$

where ν^+ denotes the right trace of ν on $\bar\gamma$ and ν^-_n denotes the left trace on γ^n. Since $w^n \to \bar w$, η is an admissible boundary entropy also for (γ^n, w^n) for n sufficiently large, so that the flux across γ^n is non-negative: for $\mathcal{L}^1$-a.e. $t \in (t_1, \tilde t)$,

$$\langle \nu^-_{n,t}, -\dot\gamma^n(t)\eta + q\rangle \ge 0.$$

Moreover for $\mathcal{L}^1$-a.e. $t \in (t_1, \tilde t)$,

$$\langle \nu^+_t, -\dot{\bar\gamma}(t)\eta + q\rangle \le 0$$

because $\bar{w}$ is an admissible boundary on $\bar{\gamma}$. To prove the other inequality take the limit as $n \to \infty$ in (3.2): since

$$\lim_{n\to\infty} \int_{\bar{\gamma}(\tilde{t})}^{\gamma^n(\tilde{t})} \langle \nu_{\tilde{t},x}, \eta \rangle dx = \lim_{n\to\infty} \int_{\bar{\gamma}(t_1)}^{\gamma^n(t_1)} \langle \nu_{t_1,x}, \eta \rangle dx = 0,$$

it holds

$$0 \le \lim_{n\to\infty} \int_{t_1}^{\tilde{t}} \langle \nu_{n,t}^-, -\dot{\gamma}^n(t)\eta + q \rangle dt \le \int_{t_1}^{\tilde{t}} \langle \nu_t^+, -\dot{\bar{\gamma}}(t)\eta + q \rangle dt$$

and this concludes the proof of Claim 1.

Claim 2. Let $\bar{w}$ be a cluster point of the sequence w^n. Then the Young measures

$$\nu_{t,x}^1 = \begin{cases} \nu_{t,x} & \text{if } x < \bar{\gamma}(t), \\ \delta_{\bar{w}} & \text{if } x > \bar{\gamma}(t), \end{cases} \qquad \text{and} \qquad \nu_{t,x}^2 = \begin{cases} \delta_{\bar{w}} & \text{if } x < \bar{\gamma}(t), \\ \nu_{t,x} & \text{if } x > \bar{\gamma}(t), \end{cases}$$

are mv entropy solutions of (1.14) in $(t_1, \tilde{t}) \times \mathbb{R}$.

Proof of Claim 2. We need to verify that ν^1 and ν^2 are mv entropy solutions on $\bar{\gamma}$. More precisely we have to verify that for all convex entropy-entropy flux (η, q), for $\mathcal{L}^1$-a.e. $t \in (t_1, \tilde{t})$

$$Q_{\eta,q}^{1-}(t) := \langle \nu_t^{1-}, -\dot{\bar{\gamma}}(t)\eta + q \rangle \ge \langle \nu_t^{1+}, -\dot{\bar{\gamma}}(t)\eta + q \rangle =: Q_{\eta,q}^{1+}(t) \tag{3.3}$$

and similarly for ν^2.

By the previous step we know that for every boundary entropy-entropy flux pair (η, q) with value $\bar{w}$, $Q_{\eta,q}^{1+} = Q_{\eta,q}^{+} = 0$, where $Q_{\eta,q}^{+}(t) = \langle \nu_t^+, -\dot{\bar{\gamma}}(t)\eta + q \rangle$ is the flux for the real solution. We claim that this implies that $Q_{\eta,q}^{+} = Q_{\eta,q}^{1+}$ for every entropy-entropy flux pair (η, q). This is sufficient to conclude since (3.3) holds for ν.

Observe that

$$Q_{\eta_1 \pm \eta_2, q_1 \pm q_2}^{+} = Q_{\eta_1,q_1}^{+} \pm Q_{\eta_2,q_2}^{+}$$

and that the family of finite sums with sign of boundary entropies is dense in the family of Lipschitz entropies with $\eta(\bar{w}) = 0$. Using the fact that if η^n and q^n converges uniformly to η and q respectively then $Q_{\eta^n,q^n}^{+} \to Q_{\eta,q}^{+}$ almost everywhere, by density $Q_{\eta,q}^{+} = Q_{\eta,q}^{1+} = 0$ for every entropy-entropy flux pair with $\eta(\bar{w}) = q(\bar{w}) = 0$ and the claim for ν^1 easily follows.

Consider the entropy $\mathbb{I} - \bar{w}$ and the flux $f - f(\bar{w})$. By the previous step it follows that $Q^+ = 0$, therefore $Q^{2+} = 0$ by conservation. By definition $Q^{2-} = 0$ therefore ν^2 is a distributional solution. Moreover it does not dissipate any of the boundary entropies on γ and it is fairly easy to prove that a solution that does not dissipate any boundary entropy does not dissipate any entropy. In particular ν^2 is a mv entropy solution.

Proof of Lemma 3.1. In order to prove (3.1) suppose there exists a sequence as in the statement such that w^n has two cluster points $a \neq b$ and $\liminf_{n\to+\infty} T(\gamma^n, w^n) = \tilde{t} > t_1$. We need to prove that a and b belong to the same linearly degenerate component of the flux. Let

$$\bar{t} = \min\left\{\liminf_{n\to+\infty} T(\gamma^n, w^n), t_2\right\}.$$

Applying Claim 2 twice we get that

$$u^1(t,x) = \begin{cases} a & \text{if } x < \bar{\gamma}(t), \\ b & \text{if } x > \bar{\gamma}(t), \end{cases} \qquad \text{and} \qquad u^2(t,x) = \begin{cases} b & \text{if } x < \bar{\gamma}(t), \\ a & \text{if } x > \bar{\gamma}(t), \end{cases}$$

are both entropy solutions of (1.14) in $[t_1, \bar{t}] \times \mathbb{R}$. This implies that a and b belong to the same linearly degenerate component of f and that $\bar{\gamma}$ is a segment with velocity $f'(a)$.

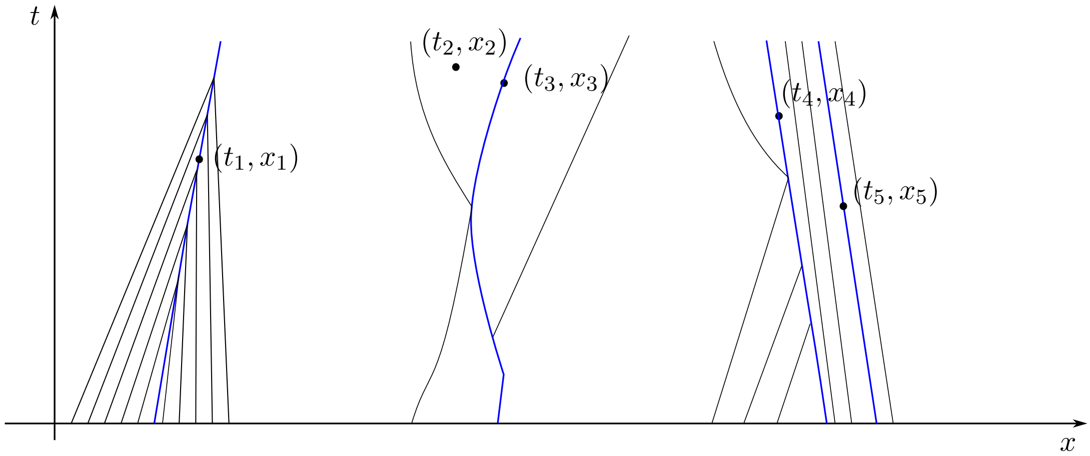


FIGURE 3.2. Example of points of the partition: $(t_1, x_1) \in A_1$, $(t_2, x_2) \in B$, $(t_3, x_3) \in A_2''$, $(t_4, x_4) \in A_2'$ and $(t_5, x_5) \in C$.

By the completeness property in Definition 2.3, there exists a sequence as in the statement with the additional assumption that

$$\liminf_{n\to+\infty} T(\gamma^n, w^n) \geq t_2.$$

Then repeating the argument above in $[t_1, \tilde{t}]$ we get that every limit of u^n belongs to I_a, in particular $\bar{\gamma}$ is a segment with constant velocity $f'(a)$ for $t \in [t_1, t_2]$. □

We introduce the following partition of the half-plane.

(1) The set A_1 is given by points belonging to at least two curves in $\mathcal{K}_\gamma$:

$$A_1 = \Big\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} : \ \exists \gamma \neq \gamma' \in \mathcal{K}_\gamma \big(\gamma(t) = \gamma'(t) = x\big)\Big\}.$$

For every $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R} \setminus A_1$, let $\bar{\gamma} = \gamma_{\bar{y}}$ be the unique curve in $\mathcal{K}_\gamma$ such that $\bar{\gamma}(\bar{t}) = \bar{x}$.

(2) The open set B is given by

$$B = \Big\{(\bar{t}, \bar{x}) : \exists \tilde{t} < \bar{t}, \ y_1 < \bar{y} < y_2 \,\big(\gamma_{y_1}(\tilde{t}) = \gamma_{y_2}(\tilde{t}) \text{ and } \gamma_{y_1}(\bar{t}) < \bar{x} < \gamma_{y_2}(\bar{t})\big)\Big\}.$$

(3) The set C is given by the points $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}$ such that

$$\forall t < \bar{t}, \ \forall y > \bar{y} \ \ (\gamma_y(t) > \gamma_{\bar{y}}(t)) \quad \text{and} \quad \forall t < \bar{t}, \ \forall y < \bar{y} \ \ (\gamma_y(t) < \gamma_{\bar{y}}(t)) \,. \tag{3.4}$$

By Lemma 3.1 the set C is obtained as the union of segments starting from 0.

(4) Let A_2 be the complement: $A_2 = \mathbb{R}^+ \times \mathbb{R} \setminus (A_1 \cup B \cup C)$.

Setting $A = A_1 \cup A_2$ we have $\mathbb{R}^+ \times \mathbb{R} = A \cup B \cup C$.

We will need a further distinction: let $A_2 = A_2' \cup A_2''$, where A_2' is the set of points (t, x) for which only one condition in (3.4) holds and A_2'' is the set of points (t, x) such that

(1) there exists a unique curve $\gamma \in \mathcal{K}_\gamma$ such that $\gamma(t) = x$;
(2) there exist $t^- < t$ and γ^- such that $\gamma^-(t^-) = \gamma(t^-)$ and $\gamma^-(t) < \gamma(t)$;
(3) there exist $t^+ < t$ and γ^+ such that $\gamma^+(t^+) = \gamma(t^+)$ and $\gamma^+(t) > \gamma(t)$;
(4) there are no $\tilde{\gamma}^-, \tilde{\gamma}^+ \in \mathcal{K}_\gamma$ and $\tilde{t} < t$ such that $\tilde{\gamma}^-(\tilde{t}) = \tilde{\gamma}^+(\tilde{t})$ and $\tilde{\gamma}^-(t) < \gamma(t) < \tilde{\gamma}^+(t)$.

See Figure 3.2 for an illustration of the above decomposition.

The candidate jump set J is the set of points $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ such that one of the following possibilities happens:

(1) $(t, x) \in A$;

(2) $(t,x) \in B$ and the solution is not continuous in (t,x);
(3) $(t,x) \in C$ such that the left and right limits obtained by Lemma 3.1 belong to different linearly degenerate components $I^-, I^+ \in \mathcal{L}_f$ of the flux f.

LEMMA 3.2. *There exists a countable subset $N \subset \mathcal{K}_\gamma$ such that*

$$J \subset \bigcup_{\gamma \in N} \mathrm{Graph}(\gamma).$$

Moreover $u \llcorner B \in \mathrm{BV}_{\mathrm{loc}}(B)$.

PROOF. The set A_1 is covered by countably many curves thanks to monotonicity:

$$A_1 \subset \bigcup_{y \in \mathbb{Q}} \mathrm{Graph}(\gamma_y).$$

Let $(\bar{t}, \bar{x}) \in A_2'$ such that only the first condition in (3.4) holds: then there exist a left neighborhood $U_{\bar{y}}$ of $\bar{y}$ in $\mathbb{R}$ and a neighborhood $U_{\bar{t}}$ of $\bar{t}$ such that for every $(t,y) \in U_{\bar{t}} \times U_{\bar{y}}$, the first condition in (3.4) is not satisfied. Since $\mathbb{R}^+ \times \mathbb{R}$ is separable, it has at most countably many disjoint open subsets and this proves the claim for A_2'. For every $(\bar{t}, \bar{x}) \in A_2''$, there exists a neighborhood in $\mathbb{R}^+ \times \mathbb{R}$ such that the points on γ are the only points not belonging to B and this concludes the proof of the statement for A_2, again by separability.

By definition B is open and for every $(t,x) \in B$ there exists a neighborhood $B'_{t,x} \subset B$ such that $u \llcorner B'_{t,x}$ is the the restriction of the solution of a Riemann problem with two boundaries. In particular $u \in \mathrm{BV}_{\mathrm{loc}}(B)$. The structure of the complete family of boundaries for solution of Riemann problems with two boundaries is described in Corollary 2.12 and it implies the result for $J \cap B$.

Write

$$C = \bigcup_n C_n,$$

where C_n is the subset of C such that (3.4) holds with $\bar{t} \geq 2^{-n}$: clearly

$$\overline{C}_n \subset C_n \cup A_2' \cup A_1.$$

Since every point in $\overline{C}_n$ has both limits left and right I^- and I^+ respectively from Lemma 3.1, they can be different at most on countably many segments of $\overline{C}_n$ because for every $m \in \mathbb{N}$ the points such that $\mathrm{dist}(I^-, I^+) > 1/m$ is discrete. $\square$

The following proposition states a sort of continuity outside J.

PROPOSITION 3.3. *For every $(t,x) \in \mathbb{R}^+ \times \mathbb{R} \setminus J$ there exists $I \in \mathscr{L}_f$ such that*

$$\forall \varepsilon > 0 \ \exists r > 0 \ \big(U_{t,x}(r) \subset I + (-\varepsilon, \varepsilon)\big).$$

PROOF. If $(t,x) \in B \setminus J$ the claim follows from Lemma 2.11. If $(t,x) \in C \setminus J$, the claim follows from the definition of J. $\square$

Similarly, the next lemma corresponds to an extension of left/right continuity at a fixed time $\bar{t}$.

LEMMA 3.4. *For every $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}$ there exist $I^+, I^- \in \mathscr{L}_f$ such that*

$$\forall \varepsilon > 0 \, \exists r > 0 \ \big(U^{\pm}_{\bar{t},\bar{x}}(r) \subset I^{\pm} + (-\varepsilon, \varepsilon)\big).$$

PROOF. One of the following cases occurs:

(1) for all $\gamma > \gamma^+_{\bar{t},\bar{x}}$ and $t < \bar{t}$, it holds $\gamma(t) > \gamma^+_{\bar{t},\bar{x}}(t)$;
(2) there exists $\gamma > \gamma^+_{\bar{t},\bar{x}}$ and $t < \bar{t}$ such that $\gamma(t) = \gamma^+_{\bar{t},\bar{x}}(t)$.

Case (1): the claim immediately follows from Lemma 3.1.
Case (2): in this case $\gamma^+_{\bar t,\bar x}$ is the left boundary of a Riemann problem with two boundaries. Consider a monotone decreasing sequence $\gamma_n \to \gamma^+_{\bar t,\bar x}$ and values w_n such that

$$\liminf_{n\to\infty} T(\gamma_n, w_n) \geq \bar t.$$

By Corollary 2.12, the sequence w_n is monotone and this implies the claim. □

Finally a result similar to the L^1 blow-up of BV functions.

Proposition 3.5. *For every $\gamma \in \mathcal{K}_\gamma$, for $\mathcal{L}^1$-a.e. $t>0$, there exist $I^+, I^- \in \mathscr{L}_f$ such that*

$$\forall\delta>0\ \forall\varepsilon>0\ \exists r>0\ \big(U^{\delta\pm}_{t,\gamma}(r) \subset I^\pm + (-\varepsilon,\varepsilon)\big).$$

Recall that $U^{\delta\pm}_{t,\gamma}(r)$ is defined if $\dot\gamma(t)$ exists.

Proof. Fixed $\bar\gamma \in \mathcal{K}_\gamma$, we distinguish two cases as in the proof of the previous lemma: let $\mathbb{R}^+ = T_m \cup T_s$, where

(a) $\bar t \in T_m$ if there exists $\gamma > \gamma^+_{\bar t,\bar\gamma(\bar t)}$ and $t<\bar t$ such that $\gamma(t) = \gamma^+_{\bar t,\bar\gamma(\bar t)}(t)$;
(b) $\bar t \in T_s$ if for all $\gamma > \gamma^+_{\bar t,\bar\gamma(\bar t)}$ and $t<\bar t$, it holds $\gamma(t) > \gamma^+_{\bar t,\bar\gamma(\bar t)}(t)$.

Case (a). Since for every $\bar t \in T_m$ there exists $\lim_{x\to\bar\gamma(\bar t)^+} u(\bar t, x)$ as in case (2) above, by a standard application of Egorov theorem, for $\mathcal{L}^1$-a.e. $\bar t \in T_m$ there exists the trace u^+ in the following sense:

$$\lim_{r\to 0}\frac{1}{r^2}\int_{B^+_{(\bar t,\bar\gamma(\bar t))}(r)}\int_{\mathbb{R}} |w-u^+(\bar t)|\,d\nu_{t,x}(w)\,dx\,dt = 0,$$

where $B^+_{(\bar t,\bar\gamma(\bar t))}(r) = B_{(\bar t,\bar\gamma(\bar t))}(r) \cap \{(t,x) : x > \bar\gamma(t)\}$. In particular the blow-up at $(\bar t,\bar\gamma(\bar t))$ is constant $u^+(\bar t)$ on the right side of the straight line $\{x = \dot{\bar\gamma}(\bar t)t\}$.
Assume by contradiction that the statement of the proposition is false in $\bar t \in T_m$ as above. Then there exist $\delta,\varepsilon>0$ and a subsequence of rescaled solutions with an admissible boundary in

$$\left\{(t,x) : |t| \leq \frac{1}{\delta}, x = \dot{\bar\gamma}(\bar t)t + 1\right\}$$

with value in $\mathbb{R}\setminus(I_{u^+(\bar t)} + (-\varepsilon,\varepsilon))$. Therefore the blow-up has an admissible boundary in $\{x > \dot{\bar\gamma}(\bar t)t\}$ with value not belonging to the linearly degenerate component $I_{u^+(\bar t)} \in \mathcal{L}_f$. This contradicts Lemma 2.10.

Case (b). By Lemma 3.1, for every $\bar t \in T_s$, the maximal characteristic $\gamma^+_{\bar t,\bar\gamma(\bar t)}$ is a segment in $[0,\bar t]$ belonging to $C \cup A_2'$ and there exists $I^+(\bar t) \in \mathcal{L}_f$ such that the admissible boundary values from the right of $\gamma^+_{\bar t,\bar\gamma(\bar t)}$ converge to $I^+(\bar t)$. We fix an arbitrary $\varepsilon>0$ and we prove the statement for $\bar t \in T_s(\varepsilon) := T_s \cap (2\varepsilon, +\infty)$.
For every $\bar t \in T_s(\varepsilon)$, denote by $y(\bar t) := \gamma^+_{\bar t,\bar\gamma(\bar t)}(\varepsilon)$ and let $\gamma_{y(\bar t)} = \gamma^+_{\bar t,\bar\gamma(\bar t)}$. In the proof of Lemma 3.2, we observed that there exists an at most countable set $N = \{y^n\}_{n\in\mathbb{N}} \subset y(T_s(\varepsilon))$ such that for every $y \in y(T_s(\varepsilon)) \setminus N$, $\gamma_y \in C$ and $I^- = I^+$. It is easy to prove that the function $y(t)$ is monotone, in particular it is continuous except an at most countable subset of $T_s(\varepsilon)$. Therefore we can write

$$T_s(\varepsilon) = E \cup \bigcup_{n=0}^{+\infty} T^n_s(\varepsilon),$$

where

(1) $\mathcal{L}^1(E) = 0$;

(2) $t \in T_s^0(\varepsilon)$ if and only if $t \in T_s(\varepsilon)$, t is a differentiability point of γ, t is a continuity point of y and $y(t) \notin N$;
(3) for every $n > 0$, $t \in T_s^n(\varepsilon)$ if and only if $t \in T_s(\varepsilon)$, t is a differentiability point of γ, t is a continuity point of y and $y(t) = y^n$.

We prove the statement for points of Lebesgue density one of T_s^n for every $n \geq 0$. If $n > 0$ it immediately follows from Lemma 3.4, being the Lebesgue points of T_s^n times where $\bar\gamma$ is tangent to γ_{y^n}.
It remains to consider the case $n = 0$. Let R be the region

$$R = \bigcup_{\bar t \in T_s(\varepsilon)} \Big\{(t,x) \in [0,\bar t] \times \mathbb{R} : x > \gamma^+_{\bar t, \bar\gamma(\bar t)}(t)\Big\}.$$

By definition of $T_s^0(\varepsilon)$ it follows that for every sequence $R \ni (t_n, x_n) \to (\bar t, \bar\gamma(\bar t))$ with $\bar t \in T_s^0(\varepsilon)$ and every $\gamma_n \in \mathcal{K}_\gamma$ such that $\gamma_n(t_n) = x_n$ it holds $\gamma_n \to \gamma^+_{\bar t, \bar\gamma(\bar t)}$. In particular, since the limits of admissible boundaries are admissible boundaries, it suffices to verify that for every $\bar t \in T_s^0(\varepsilon)$ of density one

$$\forall \delta > 0 \, \forall \varepsilon > 0 \, \exists r > 0 \, \big(B^{\delta +}_{\bar t, \bar\gamma}(r) \subset R\big).$$

By finite speed of propagation, it follows from the fact that $\bar t$ has density one in $T_s^0(\varepsilon)$. $\square$

The next result in this section describes the structure of the solution ν that follows from the corresponding structure of the complete family of boundaries.

Corollary 3.6. *Let ν be a mv entropy solution of (1.14) with a complete family of boundaries. Then there exists a representative of ν such that*

(1) $\langle \nu, f' \rangle$ is continuous in $\mathbb{R}^+ \times \mathbb{R} \setminus J$;
(2) for $\mathcal{H}^1$-a.e. $(t,x) \in J$, there exists $\lambda^-, \lambda^+ \in \mathbb{R}$ and $\gamma \in \mathcal{K}_\gamma$ such that $\gamma(t) = x$ and for every $\delta > 0$

$$\lim_{r \to 0} \|\langle \nu, f' \rangle - \lambda^-\|_{L^\infty(B^{\delta -}_{t,\gamma})} = 0, \qquad \lim_{r \to 0} \|\langle \nu, f' \rangle - \lambda^+\|_{L^\infty(B^{\delta +}_{t,\gamma})} = 0;$$

(3) for every $(\bar t, \bar x) \in \mathbb{R}^+ \times \mathbb{R}$ there exist left and right limits

$$\lambda^- = \lim_{x \to \bar x^-} \langle \nu_{\bar t, x}, f' \rangle, \qquad \lambda^+ = \lim_{x \to \bar x^+} \langle \nu_{\bar t, x}, f' \rangle.$$

If f is weakly genuinely nonlinear then $\nu = \delta_u$ is a Dirac solution and the same regularity can be deduced for u.

The proof is just the observation that f' is constant on $I \in \mathcal{L}_f$ plus the fact that weak genuine nonlinearity implies that each $I \in \mathcal{L}_f$ is a singleton.

Remark 3.7. Let ν be a mv solution for which there exists a complete family of boundaries. Then for almost every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$,

$$\text{supp } \nu_{t,x} \subset I$$

for some $I \in \mathscr{L}_f$.

Suppose additionally that f is weakly genuinely nonlinear and $u^n \to \nu$ as Young measures where u^n are entropy solutions of (1.14). Then Remark 3.7 implies that $\nu_{t,x} = \delta_{u(t,x)}$ for an L^∞ entropy solution u of (1.14) and $u^n \to u$ strongly in $L^1(\mathbb{R}^+ \times \mathbb{R})$.

Remark 3.8. Consider a curve $\gamma \in \mathcal{K}_\gamma$. In Chapter 1 a notion of left and right trace has been defined for ν on γ . The results in this section allow to compute the speed of γ and the dissipation μ on γ for every entropy η.

By Proposition 3.5 it follows that for $\mathcal{L}^1$-almost every $t > 0$ there exist $I^+(t)$ and $I^-(t)$ in $\mathcal{L}_f$ such that $\operatorname{supp} \nu_t^\pm \subset I^\pm(t)$. If $I^+(t) \neq I^-(t)$ then the Rankine-Hugoniot condition implies that

$$\dot\gamma(t) = \frac{\langle \nu_t^+, f\rangle - \langle \nu_t^-, f\rangle}{\langle \nu_t^+, \mathbb{I}\rangle - \langle \nu_t^-, \mathbb{I}\rangle}.$$

Observe that the denominator is non zero since $I^-(t)$ and $I^+(t)$ are disjoint. Moreover for every entropy-entropy flux pair (η, q) the dissipation along γ is given by

$$\mu\llcorner \mathrm{Graph}(\gamma) = \Big(\langle \nu^+, q\rangle - \langle \nu^-, q\rangle - \dot\gamma(t)\big(\langle \nu^+, \eta\rangle - \langle \nu^-, \eta\rangle\big)\Big)\frac{\mathcal{H}^1\llcorner \mathrm{Graph}(\gamma)}{\sqrt{1+\dot\gamma(t)^2}}. \tag{3.5}$$

If $I^+(t) = I^-(t)$ then, by Point (5) in Definition 2.3, it holds $\dot\gamma(t) = f'(I^+(t))$ moreover since $q - \dot\gamma\eta$ is constant on $I^+(t)$, by (3.5), it follows $\mu(\mathrm{Graph}(\gamma)) = 0$.

Remark 3.9. If the boundaries of the Riemann problem with two boundaries belong to a complete family of boundaries, we can refine Proposition 1.34: in particular using the same notation we can prove that $\Omega^m = \Omega$. Roughly speaking this means that no constant region can appear.

By properties (3) and (5) in Proposition 1.34 it suffices to prove that $\mathcal{L}^1$-a.e. in $(0,T)$, it holds $\dot\gamma_1 \geq \lambda^-$ and $\dot\gamma_2 \leq \lambda^+$.
Denote by $T^- \subset (0,T)$ the set of points where $\dot\gamma_1(t) < \lambda^-$. In particular for every $t \in T^-$, $\dot\gamma_1(t) < f'(a)$. By Points (2) and (5), for $\mathcal{L}^1-$a.e. $t \in T^-$, the right trace $u^+(t) = a$. Therefore for every entropy-entropy flux pair (η, q) the dissipation on γ_1 for $t \in T^-$ is

$$\big(-\dot\gamma_1(t)(\eta(a) - \langle \nu_t^-, \eta\rangle) + q(a) - \langle \nu_t^-, q\rangle\big)\,\frac{1}{\sqrt{1+\dot\gamma_1(t)^2}}\mathcal{H}^1\llcorner \mathrm{Graph}(\gamma_1\llcorner T^-).$$

We will obtain $\mathcal{L}^1(T^-) = 0$ by checking that $-\dot\gamma_1(t)(\eta(a) - \langle \nu_t^-, \eta\rangle) + q(a) - \langle \nu_t^-, q\rangle \leq 0$ for every entropy-entropy flux pair (η, q).
Indeed we already know by Corollary 3.6 that for $\mathcal{L}^1$-a.e. $t \in T^-$, there exists $I_t \in \mathcal{L}_f$ such that $\operatorname{supp} \nu_t^- \subset I_t$. First we observe that $a \in I_t$ for $\mathcal{L}^1$-a.e. $t \in T^-$. Indeed if $a < w$ for every $w \in I_t$, check the entropy inequality for η_k^- with $k \in (a, \inf I_t)$:

$$\begin{aligned} 0 \geq\ & -\dot\gamma_1(t)(\eta_k^-(a) - \langle \nu_t^-, \eta_k^-\rangle) + q_k^-(a) - \langle \nu_t^-, q_k^-\rangle \\ =\ & -\dot\gamma_1(t)(k-a) + f(k) - f(a) \\ =\ & (f'(a) - \dot\gamma_1(t))(k-a) + o(|k-a|). \end{aligned}$$

Since $\dot\gamma_1(t) < f'(a)$, the inequality above cannot be satisfied for k in a right neighborhood of a. Similarly the case $a > w$ for $w \in I_t$ is excluded. Then the conclusion easily follows: by Property (5) in Definition 2.3, we have that $\dot\gamma^1(t) = f'(a)$ for $\mathcal{L}^1$-a.e. $t \in T^-$, therefore $\mathcal{L}^1(T^-) = 0$.

3.2. Concentration

In this section we study the structure of the dissipation measure $\mu = \langle \nu, \eta\rangle_t + \langle \nu, q\rangle_x$, where (η, q) is an entropy-entropy flux pair and ν is a mv entropy solution with a complete family of boundaries.

Consider the decomposition of $\mathbb{R}^+ \times \mathbb{R}$ introduced in Section 3.1: the dissipation on J can be computed by means of the traces given in Proposition 1.25: see Remark 3.8. Moreover $\mu(B \setminus J) = 0$ by Volpert chain rule for functions of bounded variation. Here we analyze $\mu\llcorner(C \setminus J)$.

Let $\varepsilon > 0$, $T > 2\varepsilon$ and consider the set $S_T = \{x \in \mathbb{R} : (T,x) \in C \setminus J\}$. By Lemma 3.1, for all $x \in S_T$ the unique curve $\gamma \in \mathcal{K}_\gamma$ such that $\gamma(T) = x$ has constant velocity $f'(I)$ in $[0,T]$ where I is the unique element of $\mathcal{L}_f$ such that $K(T,x) \subset I$. Denote this set of curves by $\mathcal{K}_\gamma(T)$ and parametrize it by the position $y = \gamma_y(\varepsilon)$ of the curves at time ε. Moreover set

$$Y_T := \Big\{ y \in \mathbb{R} : \gamma_y \in \mathcal{K}_\gamma(T) \Big\}$$

and let $I(y)$ be the corresponding element of $\mathcal{L}_f$.

LEMMA 3.10. *There exists $U \in L^\infty(\mathbb{R})$ such that for every $y \in Y_T$, $U(y) \in K(\varepsilon, y)$ and*

(1) for every entropy-entropy flux pair (η, q) the function

$$Q(y) = q(U(y)) - f'(U(y))\eta(U(y))$$

has locally bounded variation;

(2) in the particular case with $(\eta, q) = (\boldsymbol{I}, f)$ the function

$$F(y) = f(U(y)) - f'(U(y))U(y)$$

has no Cantor part.

PROOF. Since the segments do not cross in $(0, T)$, by monotonicity, for every $y_1, y_2 \in Y_T$,

$$\big| f'(I(y_2)) - f'(I(y_1)) \big| \le \frac{1}{\varepsilon} |y_2 - y_1|.$$

Then consider the domain

$$\mathcal{C}_{y_1,y_2}(T) = \Big\{ (t, x) : t \in (0, T), \gamma_{y_1}(t) < x < \gamma_{y_2}(t) \Big\}.$$

Proposition 1.25 allows the application of the divergence theorem on $\mathcal{C}_{y_1,y_2}(T)$ so we get

$$\int_{\gamma_{y_1}(T)}^{\gamma_{y_2}(T)} \langle \nu^-_{T,x}, \eta \rangle dx - \int_{\gamma_{y_1}(0)}^{\gamma_{y_2}(0)} \langle \nu^+_{0,x}, \eta \rangle dx + T(Q(y_2) - Q(y_1)) = \mu(\mathcal{C}_{y_1,y_2}(T)), \tag{3.6}$$

where $Q(y) = q(I(y)) - f'(I(y))\eta(I(y))$ is well-defined, being constant on each $I \in \mathcal{L}_f$.

Since $\{\gamma_y\}_{y \in Y_T}$ are segments in $[0, T]$ which do not cross in $(0, T)$ thanks to the monotonicity property, for every $y_1 < y_2$ in Y_T

$$0 \le \gamma_{y_2}(T) - \gamma_{y_1}(T) < \frac{T}{\varepsilon} \qquad \text{and} \qquad \gamma_{y_2}(0) - \gamma_{y_1}(0) < \frac{T}{T - \varepsilon} < \frac{T}{\varepsilon}. \tag{3.7}$$

Therefore from (3.6), it follows that there exists a constant C depending on $f, \eta, \|\nu\|_\infty$ such that

$$\big| Q(y_2) - Q(y_1) \big| \le \frac{C}{\varepsilon}(y_2 - y_1) + \frac{1}{T} |\mu|(\mathcal{C}_{y_1,y_2}(T)).$$

It follows that for every $L > 0$

$$\begin{aligned} \sup_{y_1 < \ldots < y_n \in Y_T \cap [-L, L]} \sum_{i=1}^{n-1} |Q(y_{i+1}) - Q(y_i)| \le{}& \frac{2CL}{\varepsilon} + \frac{1}{T} |\mu| \Big((0, T) \times (-L - CT, L + CT) \Big) \\ <{}& +\infty \end{aligned}$$

and that $F \llcorner Y_T$ is Lipschitz for every section U of $K(\varepsilon, \cdot)$. Then it is easy to show that U can be extended maintaining the required properties e.g. taking $U(y) \in I(y)$ where

$$I(y) = \lim_{y' \searrow \inf\{Y_T \cap [y, +\infty)\}} I(y').$$

The limit exists by Lemma 3.4. $\square$

The following geometric lemma has a quite standard proof. We give it for completeness.

Lemma 3.11. *Let $\alpha : [-M,M] \to \mathbb{R}^2$ be a smooth curve and assume that there exists a constant $C > 0$ such that*

$$|\dot\alpha^2| \le C|\dot\alpha^1|. \tag{3.8}$$

Let $I \subset \mathbb{R}$ be an interval and $\gamma = (\gamma^1,\gamma^2) = \alpha \circ \varphi$ for some Borel $\varphi : I \to [-M,M]$; suppose that γ has bounded variation and $\gamma^1 \in \mathrm{SBV}(I)$. Then $\gamma^2 \in \mathrm{SBV}(I)$ and there exists $c : I \to \mathbb{R}$ such that for $\mathcal{L}^1$-a.e. $y \in I$

$$D_y\gamma(y) = c(y) D_w\alpha(\varphi(y)).$$

Proof. Let $\tilde\gamma = (\tilde\gamma^1,\tilde\gamma^2) : [0,\mathrm{TV}(\gamma)] \to \mathbb{R}^2$ be the unique curve 1-Lipschitz curve such that there exists a monotone increasing function $\psi : I \to [0,\mathrm{TV}(\gamma)]$ satisfying $\tilde\gamma \circ \psi = \gamma$. For $\mathcal{L}^1$-a.e. $s \in \psi(I)$ there exists a unique $y \in I$ such that

$$\psi(y) = s, \quad |D_w\alpha(\varphi(y))| \neq 0 \quad \text{and} \quad D_s\tilde\gamma(s) = \frac{D_w\alpha(\varphi(y))}{|D_w\alpha(\varphi(y))|}, \tag{3.9}$$

because ψ is monotone, $\tilde\gamma(\psi(I)) \subset \mathrm{Graph}\,\alpha$ and $\mathcal{H}^1(\alpha(\{D_w\alpha = 0\})) = 0$. Since $|\dot\alpha^2| \le C|\dot\alpha^1|$, this implies that

$$|D_s\tilde\gamma^2| \llcorner \psi(I) \le C|D_s\tilde\gamma^1| \llcorner \psi(I).$$

Therefore $|D^c_y\gamma^2| \le C|D_y\gamma^1|$, in particular $\gamma^2 \in \mathrm{SBV}(I)$. Moreover it follows from (3.9) that there exists $c : I \to \mathbb{R}$ such that for $\mathcal{L}^1$-a.e. $y \in I$

$$D_y\gamma(y) = c(y) D_w\alpha(\varphi(y)). \qquad \square$$

In our context, let (η,q) be an entropy-entropy flux pair, $\alpha : [-M,M] \to \mathbb{R}^2$ defined by

$$\alpha(w) = \begin{pmatrix} f(w) - f'(w)w \\ q(w) - f'(w)\eta(w) \end{pmatrix},$$

and $\varphi = U$ introduced in Lemma 3.10. The hypothesis on γ^1 are satisfied by Lemma 3.10, moreover

$$\dot\alpha(w) = \begin{pmatrix} -f''(w)w \\ -f''(w)\eta(w) \end{pmatrix},$$

therefore (3.8) is satisfied for every (η,q) with $\eta(0) = 0$. This is not a restrictive condition since in general it is sufficient to consider $\eta - \eta(0)$.

Denote by

$$\widetilde{C}(T) = \bigcup_{y \in Y_T} \mathrm{Graph}(\gamma_y \llcorner (0,T))$$

and let $P : \widetilde{C}(T) \to \mathbb{R}$ be the map which assigns to each $(t,x) \in \widetilde{C}(T)$ the parameter $y \in Y_T$ such that $\gamma_y(t) = x$.

The following corollary is a first result toward the concentration of entropy dissipation for mv entropy solutions with a complete family of boundaries. The analysis of the endpoints of segments in $\widetilde{C}(T)$ will be done in Lemma 3.17.

Corollary 3.12. *For every entropy-entropy flux pair (η,q) with dissipation measure μ the Cantor part of $P_\sharp(\mu\llcorner_{\widetilde{C}(T)})$ vanishes.*

Proof. By (3.6) and (3.7) it follows that there exists a constant C such that

$$|P_\sharp(\mu\llcorner_{\widetilde{C}(T)})| \le T|D_yQ| + C\frac{T}{\varepsilon}\mathcal{L}^1$$

and Lemma 3.11 implies that Q belongs to SBV_{loc}, therefore $P_\sharp(\mu\llcorner_{\widetilde{C}(T)})$ has no Cantor part. $\square$

Denote by

$$L_T = \Big\{ y \in Y_T : \exists w \in I \text{ for some nontrivial } I \in \mathcal{L}_f \text{ such that } (\gamma_y, w) \in \mathcal{K} \Big\}.$$

Observe that, being the isolated points of a subset of $\mathbb{R}$ at most countably many, we can find a set $\tilde{L}_T$ such that $L_T \setminus \tilde{L}_T$ is at most countable and for every $y \in \tilde{L}_T$ there exist a sequence $y_n \to y$, an interval $I \in \mathcal{L}_f$ and $u_n \in I$ such that $(\gamma_{y_n}, u_n) \in \mathcal{K}$.
In particular for $\mathcal{L}^1$-a.e. $y \in L_T$

$$\mathtt{d}_y \dot{\gamma}_y = 0, \tag{3.10}$$

where $\mathtt{d}_y$ denotes the limit of incremental ratios with values in Y_T.

In the following lemma we prove that the average $\langle \nu, \mathbf{I} \rangle$ is a constant $\bar{u}(y)$ on γ_y in $(0, T)$ for every $y \in Y_T$.

Lemma 3.13. *For $\mathcal{L}^1$-a.e. $y \in Y_T$ there exists $\bar{u}(y) \in K(\varepsilon, y)$ such that for $\mathcal{L}^1$-a.e. $t \in (0, T)$,*

$$\langle \nu_{t,\gamma_y(t)}, \boldsymbol{I} \rangle = \bar{u}(y).$$

Proof. By the previous analysis, we already know that for $\mathcal{L}^1$-a.e. $y \in Y_T$ there exists $I(y)$ such that for $\mathcal{L}^1$-a.e. $t \in (0, T)$,

$$\langle \nu_{t,\gamma_y(t)}, \mathbf{I} \rangle \in I(y).$$

In particular the claim is trivial if $y \in Y_T \setminus L_T$, where $I(y)$ is a singleton. Now consider $y \in \tilde{L}_T$ such that $\partial_y \dot{\gamma}_y = 0$ and $0 < t_1 < t_2 < T$ such that $\nu_1 := \nu_{t_1,\gamma_y(t_1)}$ and $\nu_2 := \nu_{t_2,\gamma_y(t_2)}$ are Lebesgue points of ν_{t_1} and ν_{t_2} respectively. Consider a sequence $y_n \to y$ as above: the conservation in $\mathcal{C}_{y_n,y}(t_1, t_2)$ gives

$$\begin{aligned} 0 &= \frac{1}{y - y_n} \left(\int_{\gamma_{y_n}(t_2)}^{\gamma_y(t_2)} \langle \nu_{t_2,x}, \mathbf{I} \rangle dx - \int_{\gamma_{y_n}(t_1)}^{\gamma_y(t_1)} \langle \nu_{t_1,x}, \mathbf{I} \rangle dx + (t_2 - t_1)(F(y) - F(y_n)) \right) \\ &= \frac{1}{y - y_n} \left(\int_{\gamma_{y_n}(t_2)}^{\gamma_y(t_2)} \langle \nu_{t_2,x}, \mathbf{I} \rangle dx - \int_{\gamma_{y_n}(t_1)}^{\gamma_y(t_1)} \langle \nu_{t_1,x}, \mathbf{I} \rangle dx \right) \end{aligned} \tag{3.11}$$

because F is constant on $I(y)$. Since $\gamma_y(t) = y + (t - \varepsilon)\dot{\gamma}_y$ and $\partial_y \dot{\gamma}_y = 0$ by (3.10),

$$\lim_{n\to\infty} \frac{\gamma_y(t_2) - \gamma_{y_n}(t_2)}{y - y_n} = \lim_{n\to\infty} \frac{\gamma_y(t_1) - \gamma_{y_n}(t_1)}{y - y_n} = 1,$$

therefore taking the limit as $n \to \infty$ in (3.11), we get $\langle \nu_1, \mathbf{I} \rangle = \langle \nu_2, \mathbf{I} \rangle$. $\square$

At this point we can obtain the chain rule corresponding to $(f(u) - f'(u)u)_y = -(f'(u))_y u$.

Lemma 3.14. *For $\mathcal{L}^1$-a.e. $y \in Y_T$ it holds*

$$\mathtt{d}_y F(y) = -\bar{u}(y) \mathtt{d}_y \dot{\gamma}_y.$$

Proof. If $y \in L_T$ the claim follows from (3.10). If $y \in Y_T \setminus L_T$ consider again the conservation (3.11). In this case only the first equality holds but, by Lemma 3.13, we can compute

$$\lim_{n\to\infty} \frac{1}{y - y_n} \left(\int_{\gamma_{y_n}(t_2)}^{\gamma_y(t_2)} \langle \nu_{t_2,x}, \mathbf{I} \rangle dx - \int_{\gamma_{y_n}(t_1)}^{\gamma_y(t_1)} \langle \nu_{t_1,x}, \mathbf{I} \rangle dx \right) = \bar{u}(y)(t_2 - t_1) \mathtt{d}_y \dot{\gamma}_y$$

and this completes the proof. $\square$

We introduce the set D_T of points $y \in Y_T$ for which $\nu_{t,\gamma_y(t)} = \delta_{\bar{u}(y)}$ for $\mathcal{L}^1$-a.e. $t \in (0, T)$. In particular we have that $Y_T \setminus L_T \subset D_T$.

LEMMA 3.15. *For every $\varphi \in C_c(\mathbb{R}^+ \times \mathbb{R})$,*

$$\begin{aligned}\int_{\widetilde{C}(T)} \varphi d\mu &= \int_{Y_T} \int_0^T \varphi(t,\gamma_y(t))\, d\big(\partial_t \langle \nu_{t,\gamma_y(t)}, \eta\rangle\big) dy \\ &= \int_{Y_T \setminus D_T} \int_0^T \varphi(t,\gamma_y(t))\, d\big(\partial_t \langle \nu_{t,\gamma_y(t)}, \eta\rangle\big) dy.\end{aligned} \tag{3.12}$$

In particular if ν is a Dirac entropy solution then $\mu \llcorner \widetilde{C}(T) = 0$.

PROOF. It suffices to prove that for $\mathcal{L}^2$-a.e. $0 < t_1 < t_2 < T$,

$$P_\sharp \mu \llcorner \big(\widetilde{C}(T) \cap ((t_1,t_2] \times \mathbb{R})\big) = \big\langle \nu_{t_2,\gamma_y(t_2)} - \nu_{t_1,\gamma_y(t_1)}, \eta\big\rangle\, \mathcal{L}^1(dy).$$

By Corollary 3.12 and the definition of $\widetilde{C}(T)$, we have that

$$P_\sharp \mu \llcorner \big(\widetilde{C}(T) \cap ((t_1,t_2) \times \mathbb{R})\big) \ll \mathcal{L}^1,$$

so we have to check that for $\mathcal{L}^1$-a.e. $y \in Y_T$, the Radon-Nykodim derivative is $\big\langle \nu_{t_2,\gamma_y(t_2)} - \nu_{t_1,\gamma_y(t_1)}, \eta\big\rangle$.

As before we distinguish the cases $y \in L_T$ and $y \in Y_T \setminus L_T$. Consider $y \in \tilde{L}_T$ which is a Lebesgue point for $\nu_{t_1,\gamma_y(t_1)}$ and $\nu_{t_2,\gamma_y(t_2)}$. Then, similarly to Lemma 3.13, for every entropy-entropy flux pair (η, q),

$$\begin{aligned}&\frac{1}{y-y_n} \mu\big(\mathcal{C}_{y_1,y_2}(T) \cap ((t_1,t_2)\times\mathbb{R})\big) = \\ &\qquad = \frac{1}{y-y_n}\left(\int_{\gamma_{y_n}(t_2)}^{\gamma_y(t_2)} \langle \nu_{t_2,x}, \eta\rangle dx - \int_{\gamma_{y_n}(t_1)}^{\gamma_y(t_1)} \langle \nu_{t_1,x}, \eta\rangle dx + (t_2-t_1)(Q(y)-Q(y_n))\right) \\ &\qquad = \frac{1}{y-y_n}\left(\int_{\gamma_{y_n}(t_2)}^{\gamma_y(t_2)} \langle \nu_{t_2,x}, \eta\rangle dx - \int_{\gamma_{y_n}(t_1)}^{\gamma_y(t_1)} \langle \nu_{t_1,x}, \eta\rangle dx\right),\end{aligned}$$

because Q is constant on $I(y)$ and taking the limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{y-y_n} \mu\big(\mathcal{C}_{y_1,y_2}(T) \cap ((t_1,t_2)\times\mathbb{R})\big) = \big\langle \nu_{t_2,\gamma_y(t_2)} - \nu_{t_1,\gamma_y(t_1)}, \eta\big\rangle\,.$$

Now we consider the case $y \in Y_T \setminus L_T$: by Lemma 3.11 and 3.14 for $\mathcal{L}^1$-a.e. $y \in Y_T$,

$$\mathtt{d}_y Q(y) = -\eta(\bar{u}(y))\mathtt{d}_y \dot{\gamma}_y. \tag{3.13}$$

Since $L_T \setminus \tilde{L}_T$ is at most countable, it is sufficient to consider $y \in Y_T \setminus L_T$ of $\mathcal{L}^1$ density one so that $\nu_{t_1,\gamma_y(t_1)} = \delta_{u(y)} = \nu_{t_2,\gamma_y(t_2)}$ are Lebesgue points of ν and assume that (3.13) holds.

For every sequence $y_n \to y$ in Y_T and for every $t \in (0,T)$ it holds

$$\lim_{n\to\infty} \frac{\gamma_y(t) - \gamma_{y_n}(t)}{y-y_n} = 1 + (t-\varepsilon)\mathtt{d}_y \dot{\gamma}_y,$$

therefore the balance in $\mathcal{C}_{y_1,y_2}(T) \cap ((t_1,t_2)\times\mathbb{R})$ gives

$$\lim_{n\to\infty} \frac{1}{y-y_n} \mu\big(\mathcal{C}_{y_1,y_2}(T) \cap ((t_1,t_2)\times\mathbb{R})\big) = (t_2-t_1)\eta(\bar{u}(y))\mathtt{d}_y\dot{\gamma}_y + (t_2-t_1)\mathtt{d}_y Q(y) = 0. \quad \square$$

REMARK 3.16. Consider the function $P_0(y) = \gamma_y(0)$ defined on Y_T. Observe that P_0 is monotone and for $\mathcal{L}^1$-a.e. $y \in L_T$, it holds $P_0'(y) = 1$. In particular

$$P_{0\sharp}\mathcal{L}^1 \llcorner L_T = \mathcal{L}^1 \llcorner P_0(L_T).$$

Therefore we can write the formula (3.12) in the following form:

$$\int_{\widetilde{C}(T)} \varphi d\mu = \int_{P_0(L_T)} \int_{(0,T)} \varphi(t,\gamma^x(t)) d\left(\partial_t \langle \nu_{t,\gamma^x(t)}, \eta\rangle\right) dx,$$

where γ^x denote the curve in $\mathcal{K}$ for which $\gamma^x(0) = x$: we already observed that it is well-defined on a set whose complement is at most countable.

In the last part of this section we study the endpoints of the segments in C. For every $\gamma \in \mathcal{K}_\gamma$ let

$$T_1(\gamma) = \inf \Big\{ t : \exists \gamma' \in \mathcal{K}_\gamma \text{ such that } \gamma' \neq \gamma, \gamma'(t) = \gamma(t) \Big\}.$$

Therefore, denoting by $\gamma^x \in \mathcal{K}_\gamma$ the curve starting from x as in Remark 3.16, write

$$S = C \setminus J = \bigcup_{x \in X_1} \big\{ (t, \gamma^x(t)) : t \in (0, T_1(\gamma^x)] \big\} \cup \bigcup_{x \in X_2} \big\{ (t, \gamma^x(t)) : t \in (0, T_1(\gamma^x)) \big\},$$

where $x \in X_2$ if the infimum in the definition of $T_1(\gamma^x)$ is an actual minimum, and $x \in X_1$ when it is not a minimum. Let $X = X_1 \cup X_2$, and denote by

$$E := \big\{ (T_1(\gamma^x), \gamma^x(T_1(\gamma^x))) : x \in X_1 \big\}$$

the set of endpoints of segments in $C \setminus J$ and let $\tilde{S} = S \setminus E$.

Iterating the argument above on a countable dense set of t in $\mathbb{R}^+$, we obtain that

$$\int_{\tilde{S}} \varphi d\mu = \int_X \int_{(0, T_1(\gamma^x))} \varphi(t, \gamma^x(t)) \, d(\partial_t \langle \nu_{t, \gamma^x(t)}, \eta \rangle) dx.$$

It remains to analyze the dissipation on E. In [ADL04] it is provided an example for which $\mathcal{L}^1(X_1) > 0$.

As in the previous argument, fix $\varepsilon > 0$ and consider $E_\varepsilon = \{(t, x) \in E : t \geq 2\varepsilon\}$. Denote by $P : E_\varepsilon \to \mathbb{R}$ the map that at each $(t, x) \in E_\varepsilon$ assigns the unique $y = \gamma_y(\varepsilon) \in \mathbb{R}$ such that $\gamma_y(t) = x$ and by Y_ε the image $P(E_\varepsilon)$. Moreover we denote by $D_\varepsilon \subset Y_\varepsilon$ the set of points y of density 1 for $\mathcal{L}^1 \llcorner Y_\varepsilon$ such that $\nu_{\varepsilon, y} = \delta_{\bar{u}(y)}$ is a Lebesgue point of ν_ε. We also introduce the function

$$\Phi : E_\varepsilon \to \mathrm{Graph}(T_1 \llcorner X_1) =: G, \qquad \Phi(t, x) = (t, P(t, x)).$$

We observe that Φ is invertible and, since the segments γ_y for $y \in Y_\varepsilon$ do not cross, it is fairly easy to check that Φ^{-1} is L-Lipschitz.

We say that $(t, y) \in G$ can be approximated from the right if there exists a sequence $(t_n, y_n) \in G$ converging to (t, y) such that $y_n > y$ and $t_n > t + y_n - y$. Similarly we say that (t, y) can be approximated from the left if there exists a sequence $(t_n, y_n) \in G$ converging to (t, x) such that $y_n < y$ and $t_n > t + y - y_n$. We denote by F the set of points of G which can be approximated from both sides. A standard argument proves that $R := G \setminus F$ is countably 1-rectifiable and being Φ^{-1} Lipschitz, $\Phi^{-1}(R)$ is also countably 1-rectifiable. See for example [AFP00, Chapter 2].

Lemma 3.17. *The image measure $m := P_\sharp(\mu \llcorner E_\varepsilon)$ is absolutely continuous with respect to $\mathcal{L}^1$. Moreover*

$$m \llcorner D_\varepsilon = 0.$$

Proof. We consider separately $\mu \llcorner \Phi^{-1}(F)$ and $\mu \llcorner \Phi^{-1}(R)$: in the first case we take advantage of the fact that these points can be approximated from both sides to repeat the argument of the previous section including end-points, in the second case, being $\Phi^{-1}(R)$ rectifiable, we can use a blow-up technique.

Non-rectifiable part F. Denote by π_y the projection with respect to the y variable and assume by contradiction that there exists $A \subset \pi_y(F)$ such that $m(A) > 0$ and $\mathcal{L}^1(A) = 0$. Without loss of generality we can take A compact and $T_1 \llcorner A$ continuous.

We first prove that

$$\forall \bar{y} \in A \, \forall \varepsilon > 0 \, \exists \, y^- < \bar{y} < y^+ : |y^+ - y^-| < 2\varepsilon$$

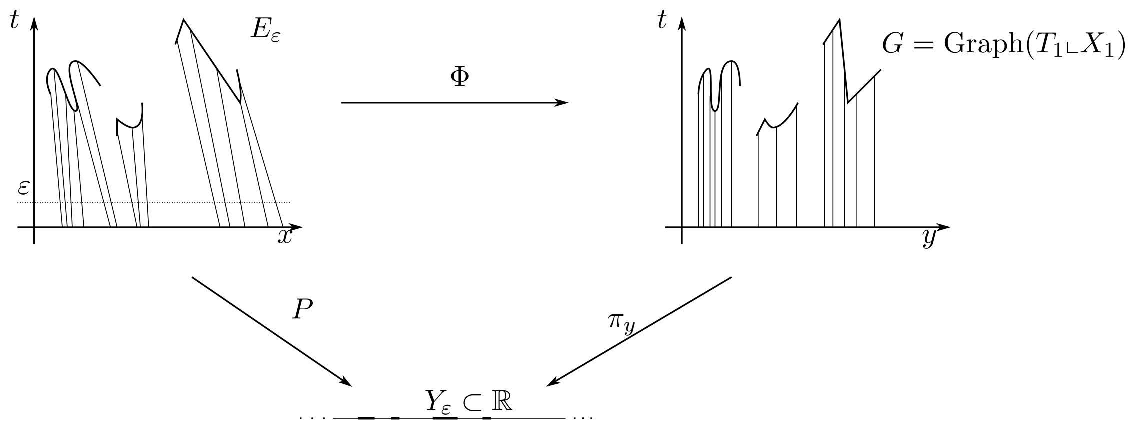


FIGURE 3.3. The set of endpoints in the two coordinate systems.

and

$$\mathcal{C}_{y^-,y^+}(T_1(\bar{y})+\varepsilon) \supset \mathrm{Graph}(T_1 \llcorner (A \cap (y^-,y^+)). \tag{3.14}$$

Define the function

$$\varpi(l) = \begin{cases} \max\left\{T_1(y) : y \in [\bar{y}+l,\bar{y}] \cap A\right\} & l<0, \\ \max\left\{T_1(y) : y \in [\bar{y},\bar{y}+l] \cap A\right\} & l \geq 0. \end{cases}$$

The function ϖ is upper semicontinuous, so that

$$\bar{y} \in (\tilde{y}^-,\tilde{y}^+) = \varpi^{-1}([0,T_1(\bar{y})+\varepsilon)).$$

Define

$$y^+ \begin{cases} = \tilde{y}^+ & \tilde{y}^+ \leq \varepsilon, \\ \in (\bar{y},\bar{y}+\varepsilon] \cap T_1^{-1}([T_1(\bar{y})+\varepsilon,+\infty)) & \text{otherwise.} \end{cases}$$

The last set is nonempty by the assumption on F. For y^- the definition is analogue and this gives (3.14).

Being a fine cover, for every $\delta>0$ there exists y_i^-, y_i^+ for $i=1,\dots,n$ such that

(1) $\displaystyle\sum_{i=1}^n y_i^+ - y_i^- < \delta$,

(2) $\displaystyle\sum_{i=1}^n |D_yQ|(y_i^-,y_i^+) < \delta$ by Corollary 3.12,

(3) $\displaystyle\bigcup_{i=1}^n \mathcal{C}_{y_i^-,y_i^+}(T^i) \supset \mathrm{Graph}(T_1 \llcorner A)$, where $T^i = \min(T_1(\gamma_{y_i^-}),T_1(\gamma_{y_i^+}))$.

Computing the balance in each cylinder we get

$$\begin{aligned} |\mu|(\mathcal{C}_{y_i^-,y_i^+}(T^i)) \leq{}& \int_{\gamma_{y_i^-}(0)}^{\gamma_{y_i^+}(0)} \langle \nu^+_{0,x},\eta\rangle dx - \int_{\gamma_{y_i^-}(T^i)}^{\gamma_{y_i^+}(T^i)} \langle \nu^-_{T^i,x},\eta\rangle dx + T^i|Q(y_i^+)-Q(y_i^-)| \\ \leq{}& C(y_i^+-y_i^-) + |D_yQ|(y_i^+-y_i^-). \end{aligned}$$

Summing in i we get $m(A)<(C+1)\delta$ and, by arbitrariness of $\delta>0$, this proves that $m \llcorner \pi_y(F) \ll \mathcal{L}^1$.

Moreover the same covering argument allows to repeat computations in Lemma 3.15 and this yields that the Radon-Nikodym derivative of m with respect to $\mathcal{L}^1$ vanishes in D_ε.

Rectifiable part R. The dissipation measure on $\Phi^{-1}(R)$ has the form $\mu \llcorner \Phi^{-1}(R) = g\mathcal{H}^1 \llcorner \Phi^{-1}(R)$ for some $g \in L^\infty(\Phi^{-1}(R), \mathcal{H}^1)$, being the divergence of an L^∞ vector field. We consider a blow-up ν^∞ of ν at the points $z \in \Phi^{-1}(R)$ such that z is a Lebesgue point of g and the blow-up of $\Phi^{-1}(R)$ at z is a straight line R^∞. Since $z \notin J$, there exists $I(z) \in \mathcal{L}_f$ such that $\operatorname{supp} \nu^\infty \subset I(z)$.

We consider two cases:

(1) the tangent to $\Phi^{-1}(R)$ at z has the same direction of $(1, f'(I(z)))$;
(2) the tangent to $\Phi^{-1}(R)$ at z is (α^1, α^2), not parallel to $(1, f'(I(z)))$.

By Remark 3.8 it follows immediately that in the first case the dissipation μ^∞ of ν^∞ on $\mathbb{R}^\infty$ is zero. In particular $g(z) = 0$ for $\mathcal{H}^1$-a.e. point in $\Phi^{-1}(R)$ such that the tangent has direction $(1, f'(I(z)))$. Denote the image of this set through Φ by $R_\|$. Then it follows that

$$m \llcorner \pi_y(R_\|) = 0.$$

In the second case an easy computation shows that

$$m = m \llcorner \pi_y(R \setminus R_\|) = \frac{g(P^{-1}(y))}{\left|\alpha^2(P^{-1}(y)) - \alpha^1(P^{-1}(y)) f'(I_{P^{-1}(y)})\right|} \mathcal{L}^1(dy),$$

in particular it is absolutely continuous.

To prove that $m \llcorner D_\varepsilon = 0$ we show that $g(P^{-1}(y)) = 0$ for $\mathcal{L}^1$-almost every $y \in D_\varepsilon$. Consider a blow-up ν^∞ of ν at a point $z \in \Phi^{-1}(R \setminus R_\|)$ as above with the additional requirement that $P(z) \in D_\varepsilon$. By the dissipation formula (3.12) and the definition of D_ε, it follows that ν^∞ is a mv entropy solution on the plane with $\nu^\infty = \delta_{\bar{u}}$ for some constant $\bar{u}$ on the half-plane $\alpha^2(z)t - \alpha^1(z)x < 0$, where the sign of α has been chosen so that

$$\alpha^2(z) > f'(I(z))\alpha^1(z). \tag{3.15}$$

The dissipation on $R^\infty = \{t, f'(I(z))t\}_{t \in \mathbb{R}}$ is given by

$$\alpha^2(z)(\langle \nu^+, \eta \rangle - \eta(\bar{u})) - \alpha^1(z)(\langle \nu^+, q \rangle - q(\bar{u})), \tag{3.16}$$

where ν^+ is the trace on $\mathbb{R}^\infty$ of ν^∞ from the half-plane $\alpha^1(z)t + \alpha^2(z)x > 0$. Since $\operatorname{supp} \nu^+ \subset I(z)$, imposing that the dissipation (3.16) is nonpositive for every entropy-entropy flux pair (η, q), by the condition (3.18) it follows that $\nu^+ = \delta_{\bar{u}}$. In particular the dissipation on R^∞ is 0 and this concludes the proof. $\square$

For every nontrivial $I \in \mathcal{L}_f$ and $t > 0$ let $L(t, I)$ be the set of points $x \in \mathbb{R}$ for which (t, x) is a Lebesgue point for ν, $\operatorname{supp} \nu_{t,x} \subset I$ and $\nu_{t,x}$ is not a Dirac delta. By the previous analysis it follows that for every nontrivial $I \in \mathcal{L}_f$, there exists

$$L(0, I) := \lim_{t \to 0} L(t, I) \quad \text{in } L^1.$$

Denote by $L(0)$ the union of $L(0, I)$ for $I \in \mathcal{L}_f$ nontrivial and let $D(0) = \mathbb{R} \setminus L(0)$.

In the following statement we summarize the results on concentration of entropy dissipation obtained in this section.

THEOREM 3.18. *Let ν be a mv entropy solution with a complete family of boundaries. Then for every entropy-entropy flux pair (η, q) the dissipation measure $\mu = \langle \nu, \eta \rangle_t + \langle \nu, q \rangle_x$ can be decomposed as $\mu = \mu_{\mathrm{diff}} + \mu_{\mathrm{jump}}$ where*

(1) μ_{jump} is concentrated on J,
(2) the image $P_{0\sharp}\mu_{\mathrm{diff}} \ll \mathcal{L}^1$ and $P_{0\sharp}\mu_{\mathrm{diff}}(D(0)) = 0$.

REMARK 3.19. If ν is a Dirac entropy solution then $D(0) = \mathbb{R}$ therefore Theorem 1 immediately follows from this result.

3.3. Initial data

We show that a mv entropy solution endowed with a compete family of boundaries assumes the initial datum in a strong sense. The fact that the solution has a complete family of boundaries is used in the following lemma.

LEMMA 3.20. *Let $\bar\nu$ be a constant Young measure on $\mathbb{R}$ and let ν be a mv entropy solution with a complete family of boundaries on $\mathbb{R}^+\times\mathbb{R}$ such that for all entropy-entropy flux pairs (η,q) and $\varphi\in C_c^\infty([0,+\infty)\times\mathbb{R})$*

$$\int_{\mathbb{R}^+\times\mathbb{R}}(\langle\nu,\eta\rangle\varphi_t+\langle\nu,q\rangle\varphi_x)\,dxdt+\int_{\mathbb{R}}\langle\bar\nu,\eta\rangle\varphi(0,x)dx=0.$$

Then $\operatorname{supp}\bar\nu\subset I$ *for some* $I\in\mathcal{L}_f$ *and* $\nu_{t,x}=\bar\nu$ *for* $\mathcal{L}^2$*-a.e.* $(t,x)\in\mathbb{R}^+\times\mathbb{R}$.

PROOF. Since there is no dissipation, for every $(\gamma,w)\in\mathcal{K}$, for every $t\in(0,T(\gamma,w))$ we have $u\in I^-=I^+$, where $I^\pm$ are given by Proposition 3.5. In particular γ has constant speed $f'(u)$ in $(0,T(\gamma,w))$ and $\langle\nu,f'\rangle$ is continuous in $\mathbb{R}^+\times\mathbb{R}$.

We claim that every $\gamma\in\mathcal{K}_\gamma$ has constant speed in $(0,+\infty)$. Fix a positive time T and for each $x\in\mathbb{R}$ let $(\gamma_x,w_x)\in\mathcal{K}$ be such that $\gamma_x(T)=x$ and $T(\gamma_x,w_x)\geq T$. The velocity $\dot\gamma_x$ is continuous in x thanks to (2.10), therefore for every $t\in(0,T)$ we have

$$\bigcup_{x\in\mathbb{R}}\gamma_x(t)=\mathbb{R}.$$

By arbitrariness of T we have the claim.

By the dissipation formula (3.12) we know that ν is constant on each straight line and therefore the initial condition implies that $\nu_{t,x}=\bar\nu$ for $\mathcal{L}^2$-a.e. $(t,x)\in\mathbb{R}^+\times\mathbb{R}$. In particular the curves $\gamma\in K_\gamma$ are parallel. □

LEMMA 3.21. *Let ν be a mv entropy solution with a complete family of boundaries. For $\mathcal{L}^1$-a.e. $x\in\mathbb{R}$ the blow-up of ν about $(0,x)$ is the constant Young measure $\bar\nu$, where $\nu^+_{0,x}=\bar\nu$ is the Lebesgue value of the trace ν^+_0.*

PROOF. We first observe that for $\mathcal{L}^1$-a.e. $x\in\mathbb{R}$ the blow-up at $(0,x)$ is a mv entropy solution with a complete family of boundaries which does not dissipate any entropy: in fact with a standard application of Vitali covering theorem it is possible to prove that for $\mathcal{L}^1$-a.e. $x\in\mathbb{R}$

$$\lim_{\varepsilon\to0}\frac{\mu(B_{0,x}(\varepsilon)\cap\{t>0\})}{\varepsilon}=0 \tag{3.17}$$

and this implies the claim.

Consider a point $x\in\mathbb{R}$ such that (5.31) holds and x is a Lebesgue point of the trace ν^+_0 with value $\bar\nu$. With the notation introduced in Definition 1.35, for every entropy-entropy flux pair (η,q) and every test function $\varphi\in C_c^\infty([0,+\infty)\times\mathbb{R})$,

$$\int_{\mathbb{R}^+\times\mathbb{R}}(\langle\nu^\varepsilon,\eta\rangle\varphi_t+\langle\nu^\varepsilon,q\rangle\varphi_x)\,dxdt+\int_{\mathbb{R}}\langle\nu_0^{\varepsilon+},\eta\rangle\varphi(0,x)dx=\int_{\mathbb{R}^+\times\mathbb{R}}\varphi d\mu^\varepsilon.$$

Passing to the limit as $\varepsilon\to0$ we get

$$\int_{\mathbb{R}^+\times\mathbb{R}}(\langle\nu^0,\eta\rangle\varphi_t+\langle\nu^0,q\rangle\varphi_x)\,dxdt+\int_{\mathbb{R}}\langle\bar\nu,\eta\rangle\varphi(0,x)dx=0.$$

By Lemma 3.20 this concludes the proof. □

Let $\mathfrak{d}$ be a bounded distance on $\mathcal{P}([-M,M])$ which induces the weak topology, for example the Wasserstein distance W_2.

LEMMA 3.22. *The initial datum is assumed in the following sense: for every $L > 0$*

$$\lim_{T\to 0} \frac{1}{T} \int_0^T \int_{-L}^{L} \mathfrak{d}(\nu_{t,x}, \nu_{0,x}^+) dx dt = 0,$$

where $\nu_{0,x}^+$ is the trace at $t = 0$ of ν.

PROOF. By Lemma 3.21 and Egorov theorem, for every $\varepsilon > 0$ there exists $A_\varepsilon \subset [-L, L]$ and $\bar{r} > 0$ such that $\mathcal{L}^1([-L, L] \setminus A_\varepsilon) < \varepsilon$ and for all $x \in A_\varepsilon$, $r \in (0, \bar{r})$,

$$\frac{1}{2r^2} \int_0^r \int_{x-r}^{x+r} \mathfrak{d}(\nu_{t,x'}, \nu_{0,x}^+) dx' dt < \varepsilon \qquad \text{and} \qquad \frac{1}{2r} \int_{x-r}^{x+r} \mathfrak{d}(\nu_{0,x'}^+, \nu_{0,x}^+) dx' < \varepsilon.$$

It is easy to see that for every $r \in (0, \bar{r})$ it is possible to choose $x_1 < \ldots < x_N$ in A_ε such that every $x \in [-L, L]$ belongs to at most two of the intervals $(x_i - r, x_i + r)$ and

$$\mathcal{L}^1\left([-L, L] \setminus \bigcup_{i=1}^{N} (x_i - r, x_i + r)\right) < \varepsilon.$$

Therefore

$$\begin{aligned}
\frac{1}{r} \int_0^r \int_{-L}^{L} \mathfrak{d}(\nu_{t,x}, \nu_{0,x}^+) dx dt &= \frac{1}{r} \int_0^r \left(\int_{A_\varepsilon} \mathfrak{d}(\nu_{t,x}, \nu_{0,x}^+) dx + \int_{[-L,L]\setminus A_\varepsilon} \mathfrak{d}(\nu_{t,x}, \nu_{0,x}^+) dx \right) dt \\
&\le \frac{1}{r} \sum_{i=1}^{N} \int_0^r \int_{x_i - r}^{x_i + r} \left[\mathfrak{d}(\nu_{t,x}, \nu_{0,x_i}^+) + \mathfrak{d}(\nu_{0,x_i}^+, \nu_{0,x}^+) \right] dx dt \\
&\quad + \bar{M} \mathcal{L}^1([-L, L] \setminus A_\varepsilon) \\
&\le 8L\varepsilon + \bar{M}\varepsilon,
\end{aligned}$$

where $\bar{M}$ is the supremum of d in $\mathcal{P}([-M, M])^2$, and this concludes the proof. □

REMARK 3.23. The lemma above implies that there exists a sequence $t_n \to 0$ such that for every $L > 0$

$$\lim_{n\to\infty} \int_{-L}^{L} \mathfrak{d}(\nu_{t_n,x}, \nu_{0,x}^+) dx = 0.$$

In particular we can deduce from Proposition 3.3 that for $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$ there exists $I \in \mathcal{L}_f$ such that

$$\operatorname{supp} \nu_{0,x}^+ \subset I$$

and, if ν is a Dirac entropy solution, then

$$\nu_{0,x}^+ = \delta_{u_0(x)}$$

for some $u_0 \in L^\infty$ and ν represents the unique entropy solution of (1.9) with initial datum u_0.

Now we are ready to prove Theorem 3.

PROPOSITION 3.24. *Let ν a mv solution with a complete family of boundaries. Then for every $L > 0$*

$$\lim_{t\to 0} \int_{-L}^{L} \mathfrak{d}(\nu_{t,x}, \nu_{0,x}^+)\, dx = 0,$$

where $\nu_{t,x} = \nu_{t,x}^+$ is continuous from the right.

PROOF. We use the same notation introduced at the end of the previous section. We prove separately the convergence in $\tilde{L} = L(0) \cap [-L, L]$ and $\tilde{D} = D(0) \cap [-L, L]$. For every nontrivial $I \in \mathcal{L}_f$, for $\mathcal{L}^1$-a.e. $x \in L(0, I)$ there exists the limit

$$\nu_{0,x}^+ = \lim_{t\to 0} \nu_{t,x+f'(I)t},$$

because by entropy dissipation the function $t \mapsto \nu_{t,x+f'(I)t}$ is BV_t for $\mathcal{L}^1$-a.e. $x \in L(0,I)$ when tested with $\mathcal{C}^2$ functions. Since translations are continuous in L^1 it follows that

$$\lim_{t\to 0} \int_{\tilde{L}} \mathfrak{d}(\nu_{t,x}, \nu^+_{0,x}) dx = 0.$$

Hence it remains to prove the convergence on $D(0)$. It is sufficient to prove the claim with $\mathfrak{d}$ equal to the Wasserstein distance. For every $x \in D(0)$, let $\nu^+_{0,x} = \delta_{u_0(x)}$ and consider a sequence $t_n \to 0$. We already know that $\nu_{t_n} \to \nu^+_0$ in the sense of Young measures by Proposition 1.25: in particular this implies

$$\begin{aligned}\lim_{n\to\infty} \int_{\tilde{D}} \langle (w-u_0(x))^2, \nu_{t_n,x}\rangle dx &= \lim_{n\to\infty} \bigg(\int_{\tilde{D}} \langle w^2, \nu_{t_n,x}\rangle dx + \int_{\tilde{D}} u_0^2(x) dx \\ &\qquad - 2\int_{\tilde{D}} u_0(x)\langle w, \nu_{t_n,x}\rangle dx \bigg) \\ &= 0\end{aligned}$$

and this concludes the proof. □

We conclude with the case of Dirac solutions.

Corollary 3.25. *Suppose that u is an entropy solution of (1.14) in the open set $\mathbb{R}^+ \times \mathbb{R}$ and suppose that the initial datum is attained weakly* in L^∞: for every sequence $t_n \to 0^+$*

$$u(t_n) \rightharpoonup u_0 \quad w^* - L^\infty.$$

Then the initial datum is attained in a strong sense: for every sequence $t_n \to 0^+$

$$u(t_n) \to u_0 \quad s - L^1_{\mathrm{loc}}.$$

Proof. By Remark 3.23 it is sufficient to show that u has a complete family of boundaries. In order to prove it, we construct a sequence of entropy solutions that converges to the Dirac solution δ_u in the sense of Young measures.

Assume for simplicity u compactly supported. Observe that since $\|u(t)\|_{L^2(\mathbb{R})}$ is non increasing with respect to t and $u : \mathbb{R}^+ \to L^1(\mathbb{R})$ is weakly continuous, u can have at most countably many discontinuity points with respect the strong topology $s - L^1$. Consider the entropy solutions u_n with initial datum $u(t_n)$ for a sequence $t_n \to 0$ of strong continuity points of u. Then by Kruzkov theorem, $u_n(t) = u(t+t_n)$ and in particular this implies that $u : (0,+\infty) \to L^1(\mathbb{R})$ is strongly continuous and it has a complete family of boundaries. Therefore by Proposition 3.24, $\delta_{u_n} \to \delta_u$ in the sense of Young measures and this concludes the proof. □

3.4. Lagrangian representation revisited

In this section we introduce a notion of Lagrangian representation for L^∞-entropy solutions, completing the discussion in Section 2.1, and we provide some additional properties in the case of piecewise monotone solutions.

3.4.1. Lagrangian representation. Here we deduce the existence of a suitable notion of Lagrangian representation for a mv entropy solution with a complete family of boundaries, and in particular we obtain Theorem 4.

Proposition 3.26. *Let ν be a mv entropy solution with a complete family of boundaries. Then there exists a couple of functions $(\mathtt{X}, \mathtt{u})$ such that*

(1) $\mathtt{X} : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, $t \mapsto \mathtt{X}(t,y)$ is Lipschitz for every y and $y \mapsto \mathtt{X}(t,y)$ is non-decreasing for every t;

(2) $\mathtt{u} \in L^\infty(\mathbb{R})$;

(3) *there exists a representative of* ν *such that for every* $(t,x) \in \mathbb{R}^+ \times \mathbb{R} \setminus J$

$$\langle \nu_{t,x}, \mathbb{I} \rangle = \bar{u}, \quad \text{where } \bar{u} = \mathtt{u}(\mathtt{X}(t)^{-1}(x));$$

(4) *the flow* $\mathtt{X}$ *satisfies the characteristic equation: for every* $y \in \mathbb{R}$ *for* $\mathcal{L}^1$*-a.e.* $t > 0$ *it holds*

$$D_t\mathtt{X}(t,y) = \begin{cases} f'(I_t^+) & \text{if } I_t^+ = I_t^-, \\ \dfrac{\langle \nu_t^+, f\rangle - \langle \nu_t^-, f\rangle}{\langle \nu_t^+, \mathbb{I}\rangle - \langle \nu_t^-, \mathbb{I}\rangle} & \text{if } I_t^+ \neq I_t^-, \end{cases}$$

where $I_t^\pm \in \mathcal{L}_f$ *contains the support of the trace* $\nu^\pm$ *at the point* $(t, \mathtt{X}(t,y))$ *from the left and the right of* $\mathtt{X}(\cdot, y)$ *(see Remark 3.8).*

Proof. Since $\mathcal{K}_\gamma$ is a closed monotone family of Lipschitz curves that covers the whole $\mathbb{R}^+ \times \mathbb{R}$, there exists a function $\mathtt{X}$ as in the statement such that for every $\gamma \in \mathcal{K}_\gamma$ there exists a unique $y \in \mathbb{R}$ for which $\gamma_y(t) = \mathtt{X}(t,y)$ in $(0,+\infty)$. So we only need to check that $\langle \nu_{t,\gamma_y(t)}, \mathbb{I}\rangle$ is constant for t such that $(t,\gamma_y(t)) \in \mathbb{R}^+ \times \mathbb{R} \setminus J$. Denote this set of times by T_y.

Since $(\gamma_y, \langle \nu_{t,\gamma_y(t)}, \mathbb{I}\rangle)$ is an admissible boundary in $(0,t)$ for every $t \in T_y$, we have that there exists $I_y \in \mathcal{L}_f$ such that $\langle \nu_{t,\gamma_y(t)}, \mathbb{I}\rangle \in I_y$ for every $t \in T_y$. This in particular implies the claim for all y such that $I_y = \{u\}$ for some $u \in \mathbb{R}$, therefore it suffices to consider the set where ν takes values in a linearly degenerate component of the flux. In this case Lemma 3.13 implies that the claim is true in $(0, T_1(\gamma_y))$ and from Remark 3.9 it follows that $(t, \gamma_y(t)) \in J$ for every $t > T_1(\gamma_y)$. □

3.4.2. Lagrangian representation for piecewise monotone solutions. In this Section we refine the presentation given in Section 2.1, after the analysis done in this chapter. The main improvement is that if the initial datum is continuous we can parametrize the Lagrangian representation with $\mathtt{X}(0) = \mathbb{I}$, and therefore $\mathtt{u} = u_0$ and this follows from Remark 3.9.

Proposition 3.27. *Let* $u_0 \in X$ *(defined in Section 1.1.2) be continuous and let* u *be the entropy solution of (1). Then there exists a Lagrangian representation* $(\mathtt{X}, \mathtt{u})$ *of* u *as above and such that*

$$\mathtt{X}(0) = \mathbb{I}, \qquad \text{and} \qquad \mathtt{u} = u_0.$$

Moreover there exists a set $Q' \subset (0,+\infty)$ *at most countable and a function* $\mathtt{T} : \mathbb{R} \to [0,+\infty)$ *(which we call* existence time function*) such that*

(1) *for every* $t \in [0,+\infty)$,

$$\{(x,w) : w \in [\mathrm{sc}^- u(t,x), u(t,x)]\} \subset \{(\mathtt{X}(t,y), u_0(y)) : \mathtt{T}(y) \geq t\}.$$

(2) *for every* $t \in [0,+\infty) \setminus Q'$ *and for every* $(x,w) \in \mathbb{R}^2$ *such that* $u(t)$ *is continuous at* x *and* $u(t,x) = w$*, or* $w \in (\mathrm{sc}^- u(t,x), u(t,x))$*, there exists a unique* $y(t,x,w) \in \mathbb{R}$ *such that*

$$\mathtt{T}(y(t,x,w)) \geq t, \quad \mathtt{X}(t, y(t,x,w)) = x, \quad \text{and} \quad u_0(y(t,x,w)) = w.$$

Moreover, if $u(t,x-) < u(t,x+)$ *the function* $w \mapsto y(t,x,w)$ *is increasing in* $(u(t,x-), u(t,x+))$ *and if* $u(t,x+) < u(t,x-)$ *the function* $w \mapsto y(t,x,w)$ *is decreasing in* $(u(t,x+), u(t,x-))$*;*

(3) *for every* $y \in \mathbb{R}$ *the pair* $(\mathtt{X}(\cdot,y), u_0(y))$ *is an admissible boundary of* u *in* $[0, \mathtt{T}(y)]$.

Moreover there exists a piecewise constant sign function $\mathtt{S} : \mathbb{R} \to \{-1,0,1\}$ *such that*

(4) for every $t \in [0, \mathtt{T}(y)] \setminus Q'$,

$$\begin{aligned} \mathtt{S}(y) = 1 \quad &\Longrightarrow \quad \Big(u_0(y) \le u(t, \mathtt{X}(t,y)+) \quad \text{or} \quad u_0(y) \ge u(t, \mathtt{X}(t,y)-)\Big), \\ \mathtt{S}(y) = -1 \quad &\Longrightarrow \quad \Big(u_0(y) \le u(t, \mathtt{X}(t,y)-) \quad \text{or} \quad u_0(y) \ge u(t, \mathtt{X}(t,y)+)\Big); \end{aligned} \tag{3.18}$$

(5) if $y_1, y_2 \in \{y : \mathtt{T}(y) \ge T\}$ with $y_1 < y_2$ and there exists $t \in [0,T)$ such that $\mathtt{X}(t,y_1) = \mathtt{X}(t,y_2)$, then u is strictly monotone in $(\mathtt{X}(T,y_1), \mathtt{X}(T,y_2))$ and $f' \circ u(T)$ is strictly increasing in $(\mathtt{X}(T,y_1), \mathtt{X}(T,y_2))$.

Proof. We only sketch this proof since all the properties we need habe already been presented. From the analysis in Section 2.1 and the argument in Proposition 3.27, the only things that we need to prove is that we can choose the parametrization such that $\mathtt{X}(0) = \mathbb{I}$ (and therefore $\mathtt{u} = u_0$) and the uniqueness property in Point (2). Both follow from Remark 3.9: more in details first we consider that parametrization with the appropriate change of variables, then we have to redefine $\mathtt{T}$ so that Point (2) holds and the other properties are preserved. In this proof we write $\mathtt{T}$ to refer to the existence function given in Section 2.1 and to $\tilde{\mathtt{T}}$ for the one we introduce now and such that satisfies the properties above. The set Q' is the set of times t such that a nontrivial interval of waves are canceled at time t, therefore it is at most countable. By Remark 3.9, there exist no $y_1 < y_2$ and $0 < t_1 < t_2$, such that $u_0(y_1) = u_0(y_2)$, $\mathtt{X}(t_1, y_1) = \mathtt{X}(t_1, y_2)$, $\mathtt{X}(t_2, y_1) < \mathtt{X}(t_2, y_2)$ and $\mathtt{T}(y_1), \mathtt{T}(y_2) \ge t_2$, therefore at every cancellation point (t,x), it suffices to choose a unique y for each value $w \in (\mathrm{sc}^- u(t,x), u(t,x))$ for which $\tilde{\mathtt{T}}(y) > t$ such that the monotonicity property in Point (2) is satisfied. □

3.5. Examples

We present three examples in this section: the first is about the possibility of reconstructing the initial datum tracing back the value of the solution on characteristics, and the other two examples show that in general $f' \circ u \notin \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$.

3.5.1. Example 1. Consider an entropy solution u of (1.9). Observe that at time 0 there is a set $J_0 \subset \mathbb{R}$ at most countable such that every point of J_0 is the starting point of two different curves. For every $x \in \mathbb{R} \setminus J_0$ denote by $y_x = \mathtt{X}(0)^{-1}(x)$. In Section 3.1 we saw that every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ either belongs to a segment starting from 0 or the rectifiable set J or a domain of a Riemann problem with two boundaries. Nevertheless it is in general not true that for $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$ there exists $t_x > 0$ such that $\partial_t \mathtt{X}(t, y_x)$ is constant in $(0, t_x)$. In particular it is not true that for $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$ the value $u_0(x)$ is transported along a characteristic for a positive time.

The example is the entropy solution u of Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

where the initial datum is the characteristic function of a Cantor set C of positive measure, and for every $t > 0$ the level set $\{u(t) = 1\}$ has Lebesgue measure 0.

It is well-known that the function

$$U(t,x) = \int_{-\infty}^x u(t,z)dz$$

is the viscosity solution of the Hamilton-Jacobi equation

$$U_t + \left(\frac{U_x}{2}\right)^2 = 0.$$

In particular U can be obtained by Lax formula:

$$U(t,x) = \min_{y\in\mathbb{R}} \left\{ U(0,y) + \frac{|x-y|^2}{2t} \right\}. \tag{3.19}$$

We provide an example of C such that for $\mathcal{L}^1$-a.e. $y \in C$ there are no $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ such that y is the minimizer in (3.19).

Claim. Let y be a point of density one for C. If there exists $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ such that y is a minimizer in (3.19), then for every $\theta > 0$,

$$\frac{1}{\theta^2}\int_y^{y+\theta} \chi_{C^c}(z)dz \leq \frac{1}{2t}. \tag{3.20}$$

Proof of the claim. Let $\theta \in \mathbb{R}$, by minimality

$$U(0,y) + \frac{|x-y|^2}{2t} \leq U(0,y+\theta) + \frac{|x-y-\theta|^2}{2t},$$

therefore

$$U(0,y) - U(0,y+\theta) \leq \frac{|x-y-\theta|^2}{2t} - \frac{|x-y|^2}{2t} = -\frac{(x-y)\theta}{t} + \frac{\theta^2}{2t}. \tag{3.21}$$

Since (4.48) holds for every θ positive and negative and $U(0)$ has derivative equal to 1 at y we get

$$\frac{x-y}{t} = 1, \qquad x = y + t.$$

We get (3.20) from (4.48) observing that

$$\int_y^{y+\theta} \chi_{C^c}(z)dz = U(0,y) - U(0,y+\theta) + \theta.$$

The last step is the construction of a Cantor set C of positive measure such that for every $y \in C$

$$\limsup_{\theta\to 0} \frac{1}{\theta^2}\int_y^{y+\theta} \chi_{C^c}(z)dz = +\infty.$$

On the interval $C_0 = [0,2]$ consider the standard Cantor construction where C_n is obtained from C_{n-1} removing the middle interval of size 3^{-n} in each connected component of C_{n-1}. Then

$$C = \bigcap_{n\in\mathbb{N}} C_n$$

has Lebesgue measure equal to 1. Fix $\bar{y} \in C$, for every $n \in N$ let y_n be the minimal $y > \bar{y}$ such that y is the left endpoint of a connected component of C_n. Since the length of every connected component of C_n is bounded by 2^{-n+1}, by direct checking

$$\frac{1}{(y_n-y)^2}\int_y^{y_n} \chi_{C^c}(z)dz \geq \frac{3^{-n}}{(2^{-n+1}+3^{-n})^2} \to +\infty.$$

3.5.2. Example 2. Here we present an example of an L^∞ entropy solution u of (1.9) such that $f' \circ u$ has no bounded variation locally in $\mathbb{R}^+ \times \mathbb{R}$.

The building block. Consider a function $g \in C^\infty([-1,1])$ such that

(1) $g(-1) = 0, \quad g(0) = \frac{1}{2}, \quad g(1) = 1;$
(2) g is convex in $[-1,0]$ and concave in $[0,1]$;
(3) $g'(0) = 1$;
(4) all derivatives vanishes at the points -1 and 1;
(5) $g - \frac{1}{2}$ is odd.

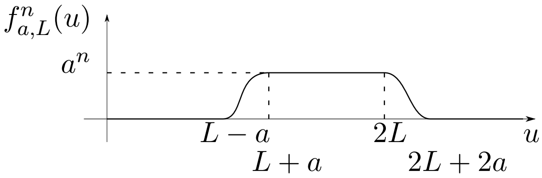


FIGURE 3.4. Flux $f^n_{a,L}$.

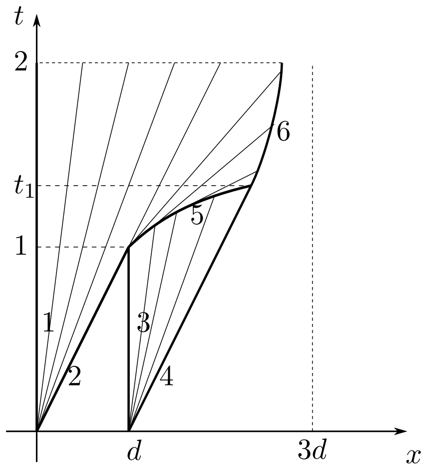


FIGURE 3.5. Solution to $u_0 = 2L\chi_{[0,d]}$ with flux $f^n_{a,L}$.

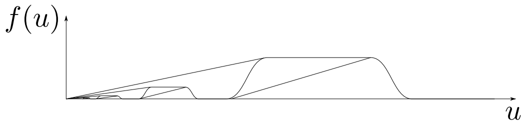


FIGURE 3.6. Flux f.

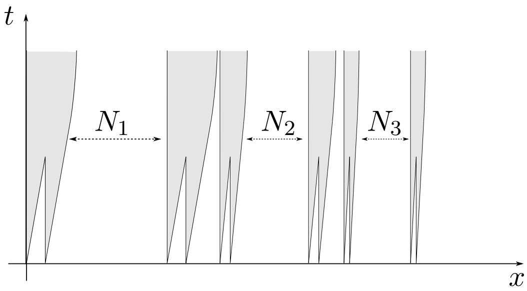


FIGURE 3.7. Solution with $\mathrm{TV}(f' \circ u) = +\infty$.

Let $a, L > 0$ and $n \in \mathbb{N}$ be parameters such that $3a \leq L$ and consider the smooth flow $f^n_{a,L}$ as in Figure 3.4:

$$f^n_{a,L}(u) = \begin{cases} 0 & \text{if } u \leq L-a, \\ a^n g\left(\frac{u-L}{a}\right) & \text{if } L-a < u \leq L+a, \\ a^n & \text{if } L+a < u \leq 2L, \\ a^n g\left(\frac{L+a-u}{a}\right) & \text{if } 2L < u \leq 2L+2a, \\ 0 & \text{if } u > 2L+2a. \end{cases}$$

The initial datum is

$$u_0 = 2L\chi_{[0,d]},$$

where $d > 0$ will be fixed below. For t small the solution is obtained solving separately the two Riemann problems (see Figure 3.5): the problem in 0 has a first shock $[0, L-a]$ of velocity 0, then a rarefaction from $L-a$ to $L-b$ for some $b \in (0,a)$ and a second shock $[L-b, 2L]$. Let

$$d = \frac{f(2L) - f(L-b)}{L+b}$$

be equal to the velocity of the second shock in 0. Since $b \in (0,a)$ we have

$$d \in \left(\frac{a^n}{L+a}, \frac{a^n}{L}\right).$$

The solution of the second Riemann problem has the same structure. It follows that there is a cancellation in $(1, d)$ from which it starts the shock number 5 of Figure 3.5. Let t_1 be the time for which the shock 5 collides with the shock 4. Since the shock 4 has constant velocity equal to d and the shock 5 has velocity $v(t) \in (\frac{a^{n-1}}{2}, a^{n-1})$, we have

$$t_1 < 1 + \frac{2a}{L - 2a}.$$

For every $t \in (t_1, 2)$ the maximal velocity $v_{\max}(t)$ at time t is the velocity of characteristics which enters in shock 6 at time t: in particular $v_{\max}(t)(t-1) \geq d \geq \frac{a^n}{L+a}$. Moreover observe that the solution $u(t)$ has support contained in $[0, 3d] \subset [0, \frac{3a^n}{L}]$ for every $t \in [0, 2]$. The estimate on the total variation is

$$\int_1^2 \int_0^{\frac{3a^n}{L}} |D_x f'(u(t,x))| dx dt \geq \int_{1+\frac{2a}{L-2a}}^2 \frac{a^n}{(L+a)(t-1)} dt = \frac{a^n}{L+a} \log\left(\frac{L-2a}{2a}\right).$$

The point is that for $a \ll L$ in an interval of length of the order $\frac{a^n}{L}$ the total variation is of the order of $\frac{a^n}{L} \log\left(\frac{L}{a}\right)$.

The general case. Consider the flux (Figure 3.6)

$$f = \sum_{n=1}^{\infty} f^n_{a_n, L_n}.$$

Observe that if $4L_{n+1} \leq L_n$ the supports of $f^n_{a_n,L_n}$ are disjoint and $f \in C_c^\infty(\mathbb{R})$. The initial datum is obtained by placing side by side N_1 initial data of the form $2L_1\chi_{[0,d_1]}$, N_2 initial data of the form $2L_2\chi_{[0,d_2]}$ and so on, see Figure 3.7.

The condition

$$a^{n+1}_{n+1} < \frac{L_n}{a^n_n} L_{n-1} \tag{3.22}$$

guarantees that for every n the solution with initial datum $2L_n\chi_{[0,d_n]}$ with flux f is the same as the solution with the same initial datum and flux $f^n_{a_n,L_n}$. In order to have infinite total variation in an interval of finite length, it suffices to provide three sequences $(a_n)_{n\in\mathbb{N}}$, $(L_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ such that (3.22) holds, $3a_n \leq L_n$, $4L_{n+1} \leq L_n$,

$$\sum_{n=1}^{\infty} N_n \frac{a_n^n}{L_n} < +\infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} N_n \frac{a_n^n}{L_n + a_n} \log\left(\frac{L_n - 2a_n}{2a_n}\right) = +\infty.$$

For example consider $a_n = 4^{-n}$, $L_n = 3 \cdot 4^{-n}$ and N_n equal to the integer part of $\frac{L_n}{n^2 a_n^n}$.

3.5.3. Example 3. Here we provide an example similar to the one before: the difference in this case is that the flux f has only one inflection point.

3.5.3.1. *Building block.* For every $n \in \mathbb{N}$ let $g_n : [-1, 1] \to \mathbb{R}$ be odd and such that

$$g_n(x) = \begin{cases} 0 & \text{if } x \in (-1, -a_{n-1}), \\ \varepsilon_n & \text{if } x \in (-a_{n-1}, -2a_n), \\ b_n & \text{if } x \in (-2a_n, -a_n), \\ 0 & \text{if } x \in (-a_n, 0), \end{cases}$$

with

$$a_1 < \frac{1}{2}, \quad a_n < \frac{a_{n-1}}{2}, \quad \sum_n \varepsilon_n < 1, \quad \sum_n b_n < 1$$

and let $f : [-1, 1] \to \mathbb{R}$ the unique continuous function for which for $\mathcal{L}^1$-almost every $x \in [-1, 1]$

$$f''(x) = \sum_{n=1}^{\infty} g_n(x), \tag{3.23}$$

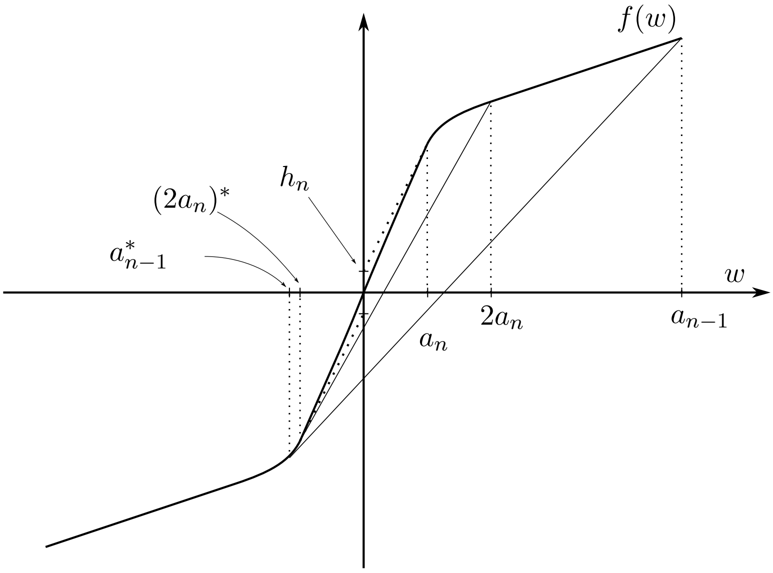


FIGURE 3.8. The flux f in the interval $[-a_{n-1}, a_{n-1}]$.

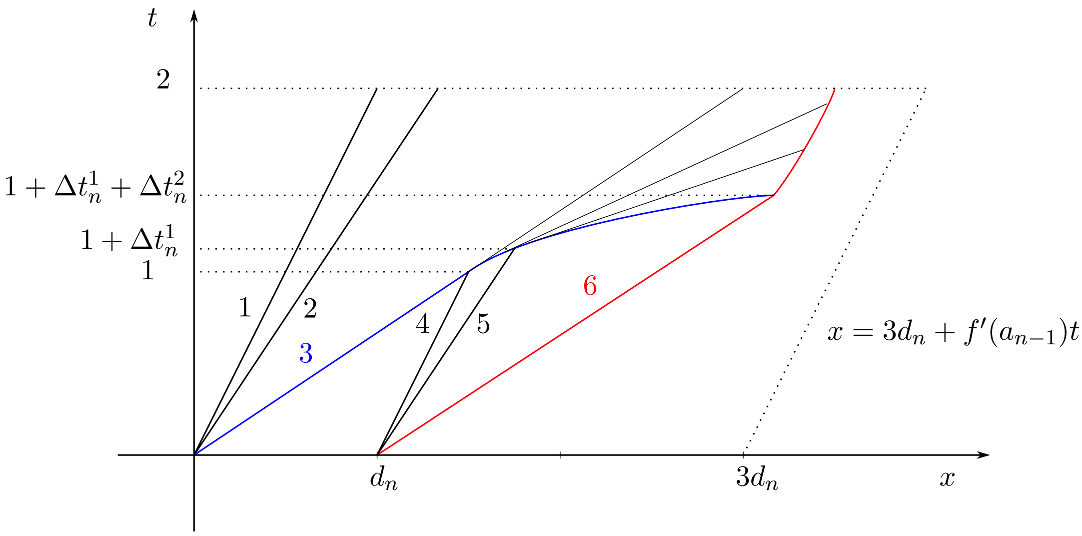


FIGURE 3.9. The solution u for $t \in (0, 2)$.

with $f(0) = 0$ and $f'(-1) = 0$. We consider the solution u^n with initial datum

$$u^n_0(x) = \begin{cases} -a_n & \text{if } x < 0, \\ a_n & \text{if } x \in (0, d_n), \\ -a_n & \text{if } x > d_n, \end{cases}$$

where $d_n > 0$ will be chosen.

The parameters ε_n, a_n, b_n will be chosen in particular in such a way that

$$a_n < (-2a_n)^* < -|a_{n-1}|^* < 2a_n, \tag{3.24}$$

where $(-2a_n)^*$ denotes the conjugate point of $-2a_n$ defined in Lemma 1.14. We assume it at the moment and we describe the entropy solution (see Figure 3.8 and 3.9): for small $t > 0$ the solution is obtained solving the two Riemann problems at $x = 0$ and $x = d_n$. Being f odd, it suffices to discuss the Riemann problem at $x = 0$. The solution has a strict rarefaction between the curves 1 and 2 with values $-a_{n-1}$ and $-2a_n$ respectively, then another rarefaction between the values $-2a_n$ and a^*_{n-1} and finally a left-contact discontinuity 3 that travels with speed $f'(a^*_{n-1})$. We set

$$d_n = f'(a^*_{n-1}) - f'(a_{n-1}) \tag{3.25}$$

so that the left-contact discontinuity starting from $x = 0$ interact with the rarefaction starting from $x = d_n$ at time $t = 1$. Then the left-contact discontinuity cancels the rarefaction and increases its speed. In particular it interacts with the characteristic with value a_{2n} at time $t + \Delta t^1_n$ with

$$\Delta t^1_n \leq \frac{f'(2a_n) - f'(a_{n-1})}{f'(a^*_{n-1}) - f'(2a_n)}, \tag{3.26}$$

indeed $f'(2a_n) - f'(a_{n-1})$ is the distance of the two curves at time $t = 1$ and $f'(a^*_{n-1}) - f'(2a_n)$ is smaller than the difference of their speeds. After time $1+\Delta t^1_n$ the left contact discontinuity moves with speed bigger than $f'(2a^*_n)$. Moreover by convexity of the curve 3, the distance between the curves 3 and 6 at time $1 + \Delta t^1_n$ is less than d_n. Therefore, recalling (3.25), curve 3 interacts with curve 6 at time $1 + \Delta t^1_n + \Delta t^2_n$ with

$$\Delta t^2_n \leq \frac{f'(a^*_{n-1}) - f'(a_{n-1})}{f'((2a_n)^*) - f'(a^*_{n-1})}. \tag{3.27}$$

Finally observe that the speed of curve 6 decreases after the collision with curve 3, in particular $u(t, x) = -a_{n-1}$ for every

$$(t, x) \in \{(t, x) \in (0, 2) \times \mathbb{R} : x < f'(a_{n-1})t \text{ or } f'(a_{n-1})t + 3d_n < x)\}. \tag{3.28}$$

Now we estimate $\mathrm{TV} f' \circ u^n(t)$ for $t \in (1 + \Delta t^1_n + \Delta t^2_n, 2)$: given t as before, consider the characteristic $\mathtt{X}(\cdot, y_t)$ entering in curve 6 from the left at time t. By monotonicity of the flow, the distance at time 1 between the characteristic and curve 6 is at least d_n. Moreover, since the speed of curve 6 for every $t \in (0, 2)$ is bigger than $f'(a_{n-1})$ and since the characteristic is convex, the speed $v_{\max}(t)$ of the characteristic at time t is such that

$$v_{\max}(t) - f'(a_{n-1}) \geq \frac{d_n}{t-1}.$$

Therefore, if we denote by

$$A_n := \{(t, x) \in (1 + \Delta^1_n + \Delta^2_n, 2) \times \mathbb{R} : f'(a_{n-1})t < x < f'(a_{n-1})t + 3d_n\},$$

it holds

$$|D_x(f' \circ u^n)|(A_n) \geq \int_{(1+\Delta^1_n+\Delta^2_n)}^{2} \big(v_{\max}(t) - f'(a_{n-1})\big)\, dt \geq d_n \log\left(\frac{1}{\Delta t^1_n + \Delta t^2_n}\right).$$

This additional logarithm allows to conclude the example after choosing in an appropriate way the parameters a_n, ε_n, b_n.

3.5.3.2. *General example.* In order to build the general counterexample, we consider an initial datum of the following form:

$$u_0 = -\chi_{(-3\|f'\|_\infty, 0)} + \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \left(a_{n-1}\chi_{[x^n_i, x^n_i + d_n]} - a_{n-1}\chi_{(x^n_i + d_n, x^n_i + 3d_n)}\right),$$

where x^n_i is defined inductively by

$$\begin{cases} x^1_1 = 0, \\ x^n_{i+1} = x^n_i + 3d_n & \text{for } n \geq 1,\ i = 1, \dots, N_n - 1, \\ x^{n+1}_1 = x^n_{N_n} + 3d_n & \text{for } n \geq 1. \end{cases}$$

For every $n \geq 1$ and $i = 1, \dots, N_n - 1$, denote by

$$S^n_i := \{(t, x) \in (0, 2) \times \mathbb{R} : x^n_i + f'(a_{n-1})t < x < x^n_i + f'(a_{n-1})t + 3d_n\}.$$

By (3.28), for every $(t,x) \in S_i^n$,

$$u(t,x) = u^n(t, x - x_i^n),$$

where u^n is the solution described in the previous step. Therefore

$$|D_x(f' \circ u)|((1,2) \times \mathbb{R}) \geq \sum_{n=1}^{\infty} N_n d_n \log \left(\frac{1}{\Delta t_n^1 + \Delta t_n^2} \right).$$

In order to have u_0 with bounded support, we need

$$\sum_{n=1}^{\infty} N_n d_n < +\infty \tag{3.29}$$

and finally, choosing $\varepsilon_n, b_n \leq a_n^n$ we have that $f^{(p)}(0) = 0$ for every $p \geq 2$. Therefore we conclude by proving that there exists $\varepsilon_n, a_n, b_n > 0$ such that

$$\varepsilon_n, b_n \leq a_n^n, \qquad a_n < |2a_n|^* < |a_{n-1}|^* < 2a_n,$$

$$\sum_{n=1}^{\infty} N_n d_n < +\infty, \qquad \sum_{n=1}^{\infty} N_n d_n \log \left(\frac{1}{\Delta t_n^1 + \Delta t_n^2} \right) = +\infty,$$

where we recall that d_n is defined by (3.25). In particular we need to estimate from above Δt_n^1 and Δt_n^2. By (3.26), (3.27) and (3.23),

$$\Delta t_n^1 \leq \frac{\varepsilon_n(a_{n-1} - 2a_n)}{b_n(2a_n - |a_{n-1}|^*)} \qquad \text{and} \qquad \Delta t_n^2 \leq \frac{b_n(2a_n - |a_{n-1}|^*) + \varepsilon_n(a_{n-1} - 2a_n)}{b_n(|a_{n-1}|^* - |(2a_n)^*|)}. \tag{3.30}$$

We estimate now $|(2a_n)^*|$ and $|a_{n-1}|^*$. Imposing $|(2a_n)^*| = (1+\alpha_n)a_n$ for some $\alpha_n \in (0,1)$ we get by definition of $|(2a_n)^*|$,

$$f'((1+\alpha)a_n))((3+\alpha)a_n) = f((1+\alpha)a_n) + f(2a_n). \tag{3.31}$$

Let $h_n := f(a_n) - f'(a_n)a_n$; by elementary computations

$$h_n = \sum_{i=n+1}^{\infty} \Delta_i, \qquad \text{where } \Delta_i := \frac{\varepsilon_i}{2}(a_{i-1}^2 - 4a_i^2) + \frac{3}{2} a_i^2 b_i.$$

Using this notation, (3.31) is equivalent to

$$(\alpha_n^2 + 6\alpha_n - 1)a_n^2 b_n + 2h_n = 0.$$

If we denote by α the positive root of $\alpha^2 + 6\alpha - 1 = 0$, i.e. $\alpha = \sqrt{10} - 3$, we have that

$$\alpha_n = \alpha + r^1 \left(\frac{h_n}{a_n^2 b_n} \right), \tag{3.32}$$

and $r^1(s) \to 0$ as $s \to 0$. Similarly we impose $|a_{n-1}|^* = (2 - \beta_n)a_n$ and we get

$$\left[\frac{\beta_n^2}{2} - (2 + R_n)\beta_n + 1 \right] a_n^2 b_n = 2h_n - \varepsilon_n \frac{(a_{n-1} - 2a_n)^2}{2},$$

where $R_n = \frac{a_{n-1}}{a_n}$. Therefore

$$\beta_n = \frac{a_n}{a_{n-1}} + r_n^3 \left(\frac{a_n}{a_{n-1}} \right) + r^2 \left(\frac{\varepsilon_n + h_n}{a_n^2 b_n} \right), \tag{3.33}$$

with $r^2(s) \to 0$ as $s \to 0$ and $r_n^3(s) = O(s^2)$ as $s \to 0$.

Let us take now $b_n = a_n^n$ for every $n \geq 1$. Therefore if

$$a_n < \frac{a_{n-1}}{3} \quad \text{and} \quad \varepsilon_{n+1} < \varepsilon_n,$$

then $\Delta_n < \frac{\Delta_{n-1}}{2}$, so that $h_n < 2\Delta_{n+1}$. Therefore (3.32) reduces to

$$\alpha_n = \alpha + \tilde{r}^1_n\left(\frac{\varepsilon_{n+1}}{a_n^{n+2}}\right) + \tilde{r}^2_n\left(\frac{a_{n+1}^{n+3}}{a_n^{n+2}}\right) \tag{3.34}$$

and (3.33) reduces to

$$\beta_n = \frac{a_n}{a_{n-1}} + r^3\left(\frac{a_n}{a_{n-1}}\right) + \tilde{r}^4_n\left(\frac{\varepsilon_n}{a_n^{n+2}}\right) + \tilde{r}^5_n\left(\frac{a_{n+1}^{n+3}}{a_n^{n+2}}\right). \tag{3.35}$$

Fix $\varepsilon' \in (0, \alpha/2)$. We choose the parameters a_n and ε_n. By definition $a_0 = 1$; let $a_1 \in (0, 1/3)$ such that $|\tilde{r}^3(a_1)| < \varepsilon' a_1/3$. The existence is granted by the fact that $r^3(s) = O(s^2)$ as $s \to 0$. Moreover let $\varepsilon_1 \in (0, 1/2)$ be such that

$$\tilde{r}^1_1\left(\frac{\varepsilon_1}{a_1^3}\right) < \frac{\varepsilon'}{2} \qquad \text{and} \qquad \tilde{r}^4_1\left(\frac{\varepsilon_1}{a_1^3}\right) < \frac{\varepsilon' a_1}{3}.$$

Inductively, since for every $n \geq 1$ the remainders $\tilde{r}^1_n, \tilde{r}^2_n, \tilde{r}^4_n, \tilde{r}^5_n$ are infinitesimal at 0 and $\tilde{r}^3_n(s) = O(s^2)$ as $s \to 0$, it is possible to choose a_n and ε_n (for every n first choose a_n then ε_n) such that for every $n \geq 1$,

(1)
$$\sum_{n=1}^{\infty} \varepsilon_n < 1, \qquad \sum_{n=1}^{\infty} a_n^n < 1, \qquad \varepsilon_n < a_n^n,$$

(2)
$$\left|\tilde{r}^1_n\left(\frac{\varepsilon_{n+1}}{a_n^{n+2}}\right)\right| < \frac{\varepsilon'}{2}, \qquad \left|\tilde{r}^2_n\left(\frac{a_{n+1}^{n+3}}{a_n^{n+2}}\right)\right| < \frac{\varepsilon'}{2};$$

(3)
$$\left|\tilde{r}^3_n\left(\frac{a_n}{a_{n-1}}\right)\right| < \varepsilon'\frac{a_n}{3a_{n-1}}, \qquad \left|\tilde{r}^4_n\left(\frac{\varepsilon_n}{a_n^{n+2}}\right)\right| < \varepsilon'\frac{a_n}{3a_{n-1}}, \qquad \left|\tilde{r}^5_n\left(\frac{a_{n+1}^{n+3}}{a_n^{n+2}}\right)\right| < \varepsilon'\frac{a_n}{3a_{n-1}};$$

(4)
$$\log\left(\frac{a_n}{a_{n+1}}\right) > n; \tag{3.36}$$

(5)
$$\log\left(\frac{a_n^{n+2}}{\varepsilon_n}\right) > n; \tag{3.37}$$

Conditions (1), (2) and (3) implies in particular that all the assumptions we made in the previous parts (in particular (3.24)) are satisfied. Condition (4) and (5) will be useful in a moment.

With this choice, by (3.30) we have that

$$\Delta t^1_n \leq \frac{\varepsilon_n}{a_n^{n+1}\beta_n} \leq \frac{\varepsilon_n}{a_n^{n+2}(1-\varepsilon')},$$

and

$$\Delta t^2_n \leq \frac{\beta_n a_n^{n+1} + \varepsilon_n}{a_n^{n+1}(1 - \alpha_n - \beta_n)} \leq \frac{a_{n-1}}{c a_n} + \frac{\varepsilon_n}{c a_n^{n+1}},$$

where $c > 0$ is a constant such that $1 - \alpha_n - \beta_n > c$. Such a constant exists by (3.34), (3.35) and the choice of the parameters. Therefore by (3.36) and (3.37), there exists $\tilde{c} > 0$ such that

$$\log\left(\frac{1}{\Delta t^1_n + \Delta t^2_n}\right) \geq \tilde{c}n. \tag{3.38}$$

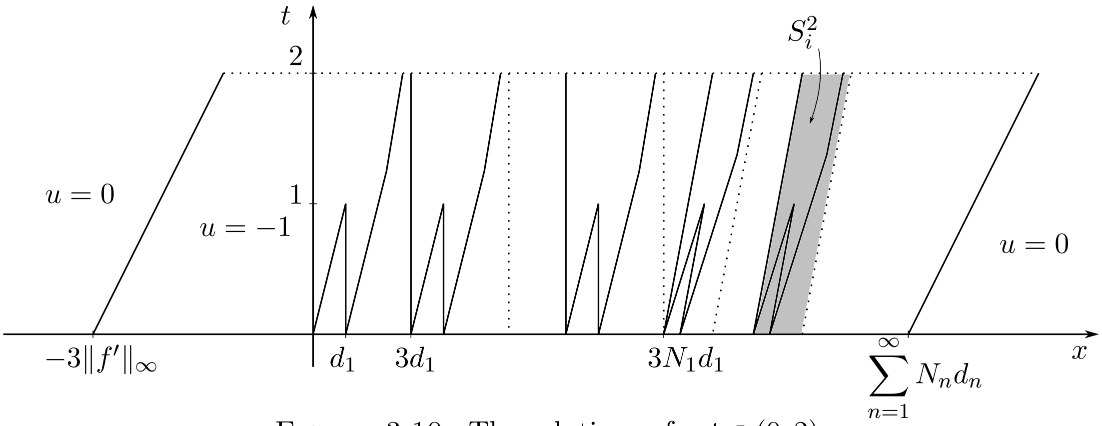


FIGURE 3.10. The solution u for $t \in (0,2)$.

Choosing finally

$$N_n = \left\lfloor \frac{1}{n^2 d_n} \right\rfloor$$

we have that

$$\sum_{n=1}^{\infty} N_n d_n \approx \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$$

and by (3.38),

$$\sum_{n=1}^{\infty} N_n d_n \log\left(\frac{1}{\Delta t_n^1 + \Delta t_n^2}\right) \gtrsim \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

This concludes the analysis of this example.

CHAPTER 4

Regularity estimates for scalar conservation laws in one space dimension

ABSTRACT. In this chapter we present the regularity estimates of the entropy solution u to (1) obtained in [Mar17]. In Section 4.1 we bound the oscillation of u in the region delimited by two characteristics with the same value, in terms of the nonlinearity of the flux f and the distance between the two characteristics. In Section 4.2 we deduce an a priori estimate of Φ-TV$(u(t))$, where Φ depends on the nonlinearity of f. In Section 4.3 we prove that, under polynomial degeneracy of the flux f, the velocity $f' \circ u(t) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ and in Section 4.4 we deduce the optimal regularity of u in terms of fractional BV spaces. In Section 4.5, we prove that under polynomial degeneracy assumption, the $\mathrm{BV}_{\mathrm{loc}}$ regularity of $f' \circ u$ can be improved to $\mathrm{SBV}_{\mathrm{loc}}$ regularity and finally in Section 4.6 we provide some examples regarding the previous estimates and some related problems.

4.1. Length estimate

In this section we only assume that the flux f is smooth. We quantify the non-linearity of f between two values $w_1 \leq w_2$ by considering twice the C^0 distance of $f \llcorner [w_1, w_2]$ from the set of affine functions on $[w_1, w_2]$:

$$\mathfrak{d}(w_1, w_2) := \min_{\lambda \in \mathbb{R}} \max_{\{w, w'\} \in [w_1, w_2]} \big(f(w) - f(w') - \lambda(w - w')\big). \tag{4.1}$$

In the statement and in the proof of the following theorem, we will refer to the objects introduced in Proposition 3.27: $\mathtt{X}$ is the flow of the Lagrangian representation of the entropy solution u to (1) with $u_0 \in X$ and continuous, $\mathtt{T}$ denotes the existence time function and $\mathtt{S}$ denotes the sign function. We recall that the set X has been introduced in Section 1.1.2.

THEOREM 4.1. *Let $T > 0$ and $u, \mathtt{X}, \mathtt{T}$ be as above. Let $y_l < y_r$ such that*

$$u_0(y_l) = u_0(y_r) = \bar{w}, \quad \mathtt{T}(y_l) \geq T, \quad \mathtt{T}(y_r) \geq T,$$

and let

$$s := \max\{y_r - y_l, \mathtt{X}(T, y_r) - \mathtt{X}(T, y_l)\}.$$

Then

$$\mathfrak{d}(w_m, w_M) \leq \frac{2s\|u_0\|_\infty}{T},$$

where

$$[w_m, w_M] = \big\{w : \exists y \in [y_l, y_r]\big(u_0(y) = w, \mathtt{T}(y) \geq T\big)\big\}.$$

REMARK 4.2. The set $[w_m, w_M]$ contains the closure of the convex hull of the image $u(t, (\mathtt{X}(T, y_l), \mathtt{X}(T, y_r)))$ and the inclusion may be strict.

PROOF. Observe that by Proposition 1.36, we can assume that $T > 0$ is such that each discontinuity of $u(T)$ satisfies the chord admissibility condition. The general case follows considering the same y_l and y_r for a sequence of time $T_n \to T^-$ for which the chord admissibility condition is satisfied.

Fix $\varepsilon > 0$ and let

$$\lambda = \frac{\mathtt{X}(T, y_l) - \mathtt{X}(0, y_l)}{T}.$$

By Proposition 3.27 it immediately follows that for every $t > 0$ the solution $u(t)$ is piecewise monotone. In particular we can choose $w_1, w_2 \in [w_m, w_M]$ different from $\bar{w}$ such that

$$\mathfrak{d}(w_m, w_M) - \varepsilon \leq f(w_2) - f(w_1) - \lambda(w_2 - w_1)$$

and such that w_1, w_2 are not local maximum or minimum values of $u(T)$. We consider the case $w_1 \leq w_2$, being the opposite case analogous. Since $u_0(y_l) = u_0(y_r)$ there exists $y_1 \in [y_l, y_r]$ such that $u_0(y_1) = w_1$, $\mathtt{S}(y_1) = 1$ and $\mathtt{T}(y_1) \geq T$ and similarly $y_2 \in [y_l, y_r]$ such that $u_0(y_2) = w_2$, $\mathtt{S}(y_2) = -1$ and $\mathtt{T}(y_2) \geq T$.

The proof in the two cases $y_1 < y_2$ and $y_2 < y_1$ differs only in some sign, therefore we only consider the case $y_1 < y_2$. Let $w \in \mathbb{R}$ be such that w is not a value of local minimum or local maximum for u_0. Let $t \in [0, T]$ and compute

$$m(t, w) := \mathcal{L}^1(\{x : u(t, x) > w\} \cap [\mathtt{X}(t, y_1), \mathtt{X}(t, y_2)]).$$

By Proposition 3.27, there exist $\bar{y}_1 < \ldots < \bar{y}_{2k}$ such that

$$\{x : \mathrm{sc}^- u(t, x) > w\} = \bigcup_{i=1}^{k} (\mathtt{X}(t, \bar{y}_{2i-1}), \mathtt{X}(t, \bar{y}_{2i})).$$

Let us consider

$$I^+ := [y_1, y_2] \cap \mathtt{S}^{-1}(1), \quad I^- := [y_1, y_2] \cap \mathtt{S}^{-1}(-1).$$

and let

$$\lambda^{\pm}(t, w) = \sum_{y \in I^{\pm} \cap u_0^{-1}(w)} \partial_t \mathtt{X}(t, y).$$

For every w the function $m(t, w)$ is Lipschitz with respect to t because the characteristics are Lipschitz and for $\mathcal{L}^1$-a.e. $t \in (0, T)$

$$\partial_t m(t, w) = \lambda^-(t, w) - \lambda^+(t, w) - \partial_t \mathtt{X}(t, y_1)\chi_{[0, w_1)}(w) + \partial_t \mathtt{X}(t, y_2)\chi_{[0, w_2)}(w),$$

where χ_E denotes the characteristic function of the set E.

Fix $w \in [0, \|u_0\|_\infty]$ which is not an extremal value; integrating with respect to t we get

$$\begin{aligned} m(T, w) - m(0, w) &= \int_0^T \partial_t m(t, w) dt \\ &= \int_0^T (\lambda^-(t, w) - \lambda^+(t, w)) dt - \Delta\mathtt{X}_1 \chi_{[0, w_1)}(w) + \Delta\mathtt{X}_2 \chi_{[0, w_2)}(w), \end{aligned}$$

where for $i = 1, 2$

$$\Delta\mathtt{X}_i = \mathtt{X}(T, y_i) - \mathtt{X}(0, y_i).$$

Integrating with respect to $w \in [0, \|u_0\|_\infty]$, we get

$$\begin{aligned} \int_0^{\|u_0\|_\infty} (m(T, w) - m(0, w)) dw = &\int_0^{\|u_0\|_\infty} \int_0^T (\lambda^-(t, w) - \lambda^+(t, w)) dt dw \\ &+ \Delta\mathtt{X}_2 w_2 - \Delta\mathtt{X}_1 w_1. \end{aligned} \tag{4.2}$$

Now consider a fixed time $t \in [0, T]$. We claim that

$$-\int_0^{\|u_0\|_\infty} (\lambda^-(t, w) - \lambda^+(t, w)) dw \geq f(w_2) - f(w_1). \tag{4.3}$$

This follows by the fact that $\mathtt{S}(y_1) = 1$ and $\mathtt{S}(y_2) = -1$. See Figure 4.1 and Figure 4.2 to get a graphic intuition of the proof. The solution u at time t is piecewise monotone so denote by $x_1 < \ldots < x_k$ the local minimum and maximum points of $u(t)$ in the interval $(\mathtt{X}(t, y_1), \mathtt{X}(t, y_2))$. For every $i = 1, \ldots, k$ set $a_i = u(t, x_i)$ and let $a_0 = u(t, \mathtt{X}(t, y_1)+)$,

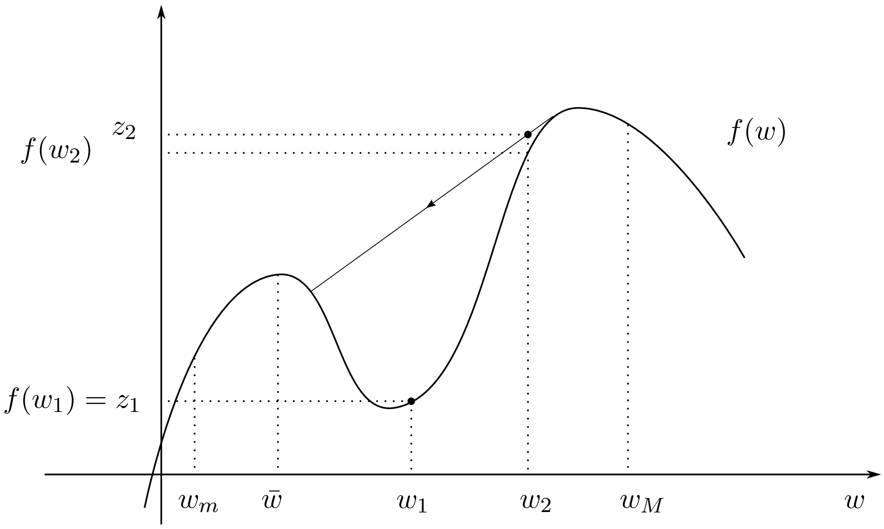


FIGURE 4.1. The flow f and the secant denoting the shock at the point $\mathtt{X}(t, y_2)$. The difference $z_2 - z_1$ is equal to the l.h.s. in (4.3); since $\mathtt{S}(y_2) = -1$ the secant passes above the graph of f, and similarly if there is a shock in $\mathtt{X}(t, y_1)$ it passes below. Therefore $f(w_2) - f(w_1) \leq z_2 - z_1$ and this is (4.3).

$a_{k+1} = u(t, \mathtt{X}(t, y_2)-)$. Since $\mathtt{S}(y_1) = 1$, by (3.18), it holds $a_1 \geq w_1$ and similarly $a_{k+1} \leq w_2$. Therefore

$$
\begin{aligned}
&-\int_0^{\|u_0\|_\infty} (\lambda^-(t,w) - \lambda^+(t,w))dw \\
&\qquad\qquad = (a_0 - w_1)\partial_t \mathtt{X}(t, y_1) + \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} f'(w)dw - (a_{k+1} - w_2)\partial_t \mathtt{X}(t, y_2) \\
&\qquad\qquad = (a_0 - w_1)\partial_t \mathtt{X}(t, y_1) - f(a_0) + f(a_{k+1}) - (a_{k+1} - w_2)\partial_t \mathtt{X}(t, y_2).
\end{aligned}
$$

By the chord admissibility condition condition and the characteristic equation,

$$
(a_0 - w_1)\partial_t \mathtt{X}(t, y_1) - f(a_0) \geq -f(w_1) \quad \text{and} \quad f(a_{k+1}) - (a_{k+1} - w_2)\partial_t \mathtt{X}(t, y_2) \geq f(w_2),
$$

therefore we get (4.3).

Integrating this relation with respect to t we get

$$
-\int_0^T \int_0^{\|u_0\|_\infty} (\lambda^-(t,w) - \lambda^+(t,w))dwdt \geq T(f(w_2) - f(w_1)). \tag{4.4}
$$

Comparing (4.2) and (4.4):

$$
\begin{aligned}
T(\mathfrak{d}(w_m, w_M) - \varepsilon) &\leq T(f(w_2) - f(w_1)) - T\lambda(w_2 - w_1) \\
&\leq -\int_0^T \int_0^{\|u_0\|_\infty} (\lambda^-(t,w) - \lambda^+(t,w))dwdt - T\lambda(w_2 - w_1) \\
&= -\int_0^{\|u_0\|_\infty} (m(T,w) - m(0,w))dw + (\Delta\mathtt{X}_2 - \Delta\mathtt{X}_1)w_1 \\
&\quad + (\Delta\mathtt{X}_2 - \lambda T)(w_2 - w_1) \\
&\leq s\|u_0\|_\infty + sw_1 + s(w_2 - w_1) \\
&\leq 2s\|u_0\|_\infty.
\end{aligned}
$$

Letting $\varepsilon \to 0$ we conclude the proof. $\square$

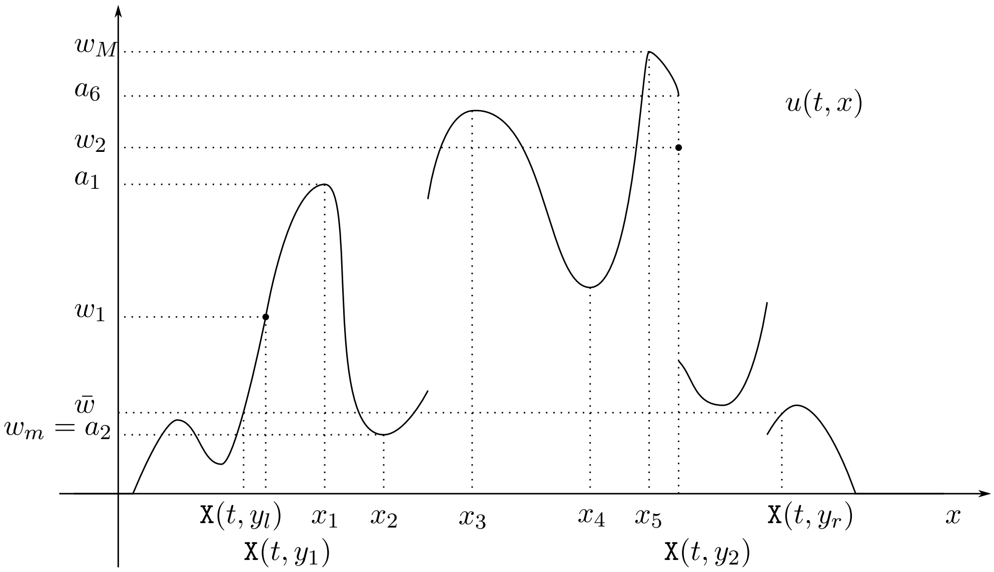


FIGURE 4.2. The graph of the solution $u(t)$ corresponding to the argument in Figure 4.1 with the notation used in the proof of Theorem 4.1.

4.2. BV^Φ regularity of the solution

In this section we obtain the regularity of the entropy solution in terms of BV^Φ spaces by means of Theorem 4.1. The definition of Φ depends on the nonlinearity of f. In particular we assume in this section that f is weakly genuinely nonlinear (see Definition 1.12).

We also define $\mathfrak{d} : \mathbb{R}^+ \to [0, +\infty)$ by

$$\mathfrak{d}(h) = \inf_{a \in \mathbb{R}^+} \mathfrak{d}(a, a+h), \tag{4.5}$$

where $\mathfrak{d}(a, a+h)$ is defined in (4.1). This quantity quantifies the nonlinearity of the flux f. Since we consider bounded nonnegative solutions, the inf in (4.5) can be computed only on $[0, \|u\|_\infty - h]$; in that case it is a minimum and $\mathfrak{d}(h) > 0$ for every $h > 0$ if and only if $f \llcorner [0, \|u_0\|_\infty]$ is weakly genuinely nonlinear.

Given a sequence $(h_n)_{n\in\mathbb{N}}$ with $h_n \geq 0$ for every $n \in \mathbb{N}$ and $h > 0$, let

$$N(h) := \#\{n : h_n \geq h\}.$$

LEMMA 4.3. *Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a convex function with $\Phi(0) = 0$ and $\Phi > 0$ in $(0, +\infty)$ such that for every $h > 0$*

$$N(h) \leq \frac{1}{\Phi(h)}. \tag{4.6}$$

Then, if we denote by $\bar{h} := \max_n h_n$, for every $\varepsilon > 0$,

$$\sum_i \Psi_\varepsilon(h_i) \leq \bar{h}^\varepsilon \frac{2^\varepsilon}{2^\varepsilon - 1}, \qquad \textit{where} \qquad \Psi_\varepsilon(x) = \Phi\left(\frac{x}{2}\right) x^\varepsilon. \tag{4.7}$$

REMARK 4.4. In the case with $\varepsilon = 0$ you only get a weak ℓ^1 estimate of the sequence $\Psi_\varepsilon(h_i)$. Observe also that, by (4.6), $\bar{h} \leq \Phi^{-1}(1)$.

PROOF. Since ψ_ε is increasing, for every $n \in \mathbb{N}$,

$$N\left(\frac{\bar{h}}{2^{n+1}}\right) \psi_\varepsilon\left(\frac{\bar{h}}{2^n}\right) \geq \sum_{i \in I_n} \psi_\varepsilon(h_i),$$

where I_n denotes the set of indexes i for which $h_i \in (2^{-n-1}\bar{h}, 2^{-n}\bar{h})$. Finally

$$\sum_i \psi_\varepsilon(h_i) \leq \sum_n N(2^{-n-1}\bar{h})\psi_\varepsilon(2^{-n}\bar{h}) \leq \sum_n \frac{\psi_\varepsilon(2^{-n}\bar{h})}{\phi(2^{-n-1}\bar{h})} = \bar{h}^\varepsilon \sum_n \left(\frac{1}{2^\varepsilon}\right)^n = \bar{h}^\varepsilon \frac{2^\varepsilon}{2^\varepsilon - 1}.$$

This concludes the proof of the lemma. □

We want to apply the lemma above to the height of the undulations of the entropy solution u. The existence of such a function Φ is proved in the following lemma as a corollary of Theorem 4.1 in the case of weakly genuinely nonlinear fluxes.

Lemma 4.5. *Let u be the entropy solution of (1) with $u_0 \in X$ and let $t > 0$. Then the number $N(u(t), h)$ of undulations of $u(t)$ of height strictly bigger than $h > 0$ is bounded by*

$$N(u(t), h) \leq \frac{4\|u_0\|_\infty(\mathcal{L}^1(\mathrm{conv}(\operatorname{supp} u_0)) + \|f'\|_\infty t)}{t\mathfrak{d}(h)}.$$

Proof. The idea of the proof is the following: the measure of the support of an undulation of height bigger than h is bounded from below by Theorem 4.1. The inequality we want to prove states that the number of such undulations is bounded by the measure of the support of u divided by the space occupied by each of them. Actually the supports of the undulations are not disjoint in general and the proof consists in finding pairwise disjoint subsets of them with the appropriate measure.

Denote for brevity by $N = N(u(t), h)$ and up to rearrangements we can assume that for $i = 1, \dots, N$ the undulations u_i are the ones with height strictly bigger than h, let moreover

$$\bar{x}_i = \min \arg\max u_i = \min\{x : u_i(x) = \max u_i\}$$

and let

$$a_i = \sup\{x < \bar{x}_i : u(x) \leq u(t, \bar{x}_i) - h\}, \quad b_i = \inf\{x > \bar{x}_i : u(x) \leq u(t, \bar{x}_i) - h\}.$$

It may happen $a_i = \bar{x}_i$ or $b_i = \bar{x}_i$, but it holds $a_i < b_i$. Moreover, since $h_i \geq h$, it holds

$$(a_i, b_i) \subset \operatorname{supp} u_i. \tag{4.8}$$

We claim that the intervals $([a_i, b_i])_{i=1}^N$ are pairwise disjoint. Consider two undulations $u_i \neq u_j$ with $i, j = 1, \dots N$. If $\operatorname{supp} u_i \cap \operatorname{supp} u_j$ has empty interior, then by (4.8),

$$(a_i, b_i) \cap (a_j, b_j) = \emptyset.$$

Suppose instead that u_j is a descendant of u_i and assume without loss of generality that $\bar{x}_j < \bar{x}_i$. Then by point (3) in Proposition 1.4, $u(t, \bar{x}_i) \geq u(t, \bar{x}_j)$, therefore

$$\mathrm{sc}^- u(t, b_j) \leq u(t, \bar{x}_j) - h \leq u(t, \bar{x}_i) - h$$

and since $b_j \leq \bar{x}_i$, by definition of a_i, it holds $b_j \leq a_i$. This proves that the intervals $((a_i, b_i))_{i=1}^N$ are pairwise disjoint. Finally we check that there exist no $i \neq j$ such that $a_i = b_j$. In fact notice that by definition of a_i the function $u(t)$ cannot have a decreasing jump at a_i and similarly it cannot have an increasing jump at b_j. In particular if $a_i = b_j$, it must be a point of continuity of $u(t)$ and therefore $u(t, a_i) = u(t, \bar{x}_i) - h = u(t, \bar{x}_j) - h$. In particular by definition of a_i and b_j it holds $u(t, x) \geq u(t, \bar{x}_i) - h$ for every $x \in (\bar{x}_j, \bar{x}_i)$ and this is in contradiction with the fact that both the undulations u_i and u_j have height strictly bigger than h.

By Proposition 3.27, for every $i = 1, \dots, N$ there exists $y_i^- \in \mathbb{R}$ such that

$$\mathtt{X}(t, y_i^-) = a_i, \quad u_0(y_i^-) = u^+(t, \bar{x}_i) - h \quad \text{and} \quad \mathtt{T}(y_i^-) \geq t.$$

Similarly there exists $y_i^+ \in \mathbb{R}$ such that

$$\mathtt{X}(t, y_i^+) = b_i, \quad u_0(y_i^+) = u^+(t, \bar{x}_i) - h \quad \text{and} \quad \mathtt{T}(y_i^+) \geq t.$$

For every $i =, \dots, N$ apply Theorem 4.1 with $y_l = y_i^-$ and $y_r = y_i^+$. Letting

$$s_i := \max\{y_i^+ - y_i^-, \mathtt{X}(t, y_i^+) - \mathtt{X}(t, y_i^-)\},$$

it holds

$$s_i \geq \frac{t\mathfrak{d}(h)}{2\|u_0\|_\infty}. \tag{4.9}$$

Since the intervals $([a_i, b_i])_{i=1,\dots,N}$ are pairwise disjoint, the same holds for the intervals $((y_i^-, y_i^+))_{i=1,\dots,N}$ and for $((\mathtt{X}(t, y_i^-), \mathtt{X}(t, y_i^+)))_{i=1,\dots,N}$ by monotonicity of the flow $\mathtt{X}$.

Moreover, by finite speed of propagation,

$$\mathcal{L}^1(\operatorname{conv}(\operatorname{supp} u(t))) \leq \mathcal{L}^1(\operatorname{conv}(\operatorname{supp} u_0)) + 2\|f'\|_\infty t, \tag{4.10}$$

therefore, from (4.9), (4.10) and the fact that we have disjoint intervals, it follows that

$$N(u(t), h)\frac{t\mathfrak{d}(h)}{2\|u_0\|_\infty} \leq \sum_{i \in I_h} s_i \leq 2(\mathcal{L}^1(\operatorname{conv}(\operatorname{supp} u_0)) + \|f'\|_\infty t),$$

and this concludes the proof. □

The main result of this section is the following:

THEOREM 4.6. *Let $u_0 \in L^\infty(\mathbb{R})$ be nonnegative and with compact support, and let u be the entropy solution of (1). Let Φ be the convex envelope of $\mathfrak{d}$, i.e. denote by*

$$\mathcal{G} = \{\varphi : [0, +\infty) \to [0, +\infty] \ \textit{convex} \ : \varphi(0) = 0 \ \textit{and} \ \varphi(h) \leq \mathfrak{d}(h)\}$$

and let $\Phi = \sup_{\varphi \in \mathcal{G}} \varphi$. Then there exists a constant $C > 0$ depending only on $\|u_0\|_\infty$, $\mathcal{L}^1(\operatorname{conv}(\operatorname{supp} u_0))$, and $\|f'\|_\infty$ such that for every $t > 0$ and every $\varepsilon > 0$ it holds

$$u(t) \in \mathrm{BV}^{\Psi_\varepsilon} \qquad \textit{and} \qquad \Psi_\varepsilon\text{-TV}\, u(t) \leq C\left(1 + \frac{1}{t}\right)\frac{2^\varepsilon}{2^\varepsilon - 1},$$

where Ψ_ε is defined in (4.7).

PROOF. Let $u_0 \in L^\infty(\mathbb{R})$ be nonnegative and with compact support. Since X is dense in the space of nonnegative L^∞ functions with compact support with respect to the L^1-topology, there exists a sequence $(u_0^n)_{n\in\mathbb{N}}$ in X such that $u_0^n \to u_0$ strongly in $L^1(\mathbb{R})$. Moreover we can also assume that for every $n \in \mathbb{N}$

(1) u_0^n is continuous;
(2) it holds
$$\operatorname{conv}(\operatorname{supp} u_0^n) \subset \operatorname{conv}(\operatorname{supp} u_0);$$
(3) $\|u_0^n\|_\infty \leq \|u_0\|_\infty$.

By Lemma 4.5 and the choice of the approximation, there exists a constant $C' > 0$ such that for every $n \in \mathbb{N}$ and for every $h > 0$

$$N(u^n(t), h) \leq C'\left(1 + \frac{1}{t}\right)\frac{1}{\Phi(h)}, \tag{4.11}$$

therefore

$$\begin{aligned}\Psi_\varepsilon\text{-TV}\, u^n(t) &\leq 2\sum_{i=1}^{N(u(t))} \Psi^\varepsilon(h_i) \\ &\leq C'\|u_0\|_\infty\left(1 + \frac{1}{t}\right)\frac{2^\varepsilon}{2^\varepsilon - 1},\end{aligned}$$

where the first inequality holds by Lemma 1.8 and the second one holds by (4.11) and Lemma 4.3. Finally, setting $C = C'\|u_0\|_\infty$, the result follows by lower semicontinuity of the Ψ_ε-total variation with respect to L^1 convergence. □

REMARK 4.7. We give some comment on the previous result:

(1) the regularity of u depends crucially on the nonlinearity of f. Such dependence is encoded here in the condition $\Phi(h) \leq \mathfrak{d}(h)$.
(2) the upper bound for $\mathrm{TV}^{\Psi_\varepsilon}$ blows up as $t \to 0$, as we expect for L^∞ entropy solutions;
(3) in the case of f of polynomial degeneracy $p \in \mathbb{N}$ (see Definition 1.13), it is not hard to prove that there exists $c > 0$ such that for every $h > 0$

$$\mathfrak{d}(h) \geq ch^{p+1}.$$

Therefore, by Theorem 4.6, we get that for every $t > 0$,

$$u(t) \in \mathrm{BV}^{\frac{1}{p+1+\varepsilon}}(\mathbb{R}).$$

Relying on the BV regularity of $f' \circ u(t)$ (Section 4.3), we will prove in Section 4.4 that in this case the regularity of $u(t)$ can be improved to

$$u(t) \in \mathrm{BV}^{\frac{1}{p}}(\mathbb{R}).$$

However in Section 4.6, we prove that in general, even if f is weakly genuinely nonlinear, $f' \circ u(t) \notin \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$.

4.3. BV regularity of $f' \circ u$

In this section we prove that if the flux f has finitely many inflection points of polynomial degeneracy (see Definition 1.13), then for every $T > 0$ the velocity $f' \circ u(T)$ has bounded variation. In particular in this section we always assume that f has degeneracy $p \in \mathbb{N}$.

We are going to prove a uniform estimate of $\mathrm{TV}(f' \circ u(T))$ for the entropy solutions u of (1) with $u_0 \in X$ (defined in Section 1.1) and with $\|u_0\|_\infty$ and $\mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u_0))$ uniformly bounded.

The strategy is the following: we will consider separately small and big undulations of the solution u. The number of big undulations is bounded a priori by Theorem 4.1. The contribution of small undulations is more delicate: if u takes values in an interval where f is convex, the structure of characteristics is well-known and it implies a one-sided Lipschitz estimate on $f' \circ u$. If instead u oscillates around an inflection point of f we adapt the argument of [Che86].

We start by recalling the structure of characteristics in the convex case. We omit the proof of the following lemma that can be found in [Daf16]: it can be proved either by means of Lax-Oleinik formula or with the method of generalized characteristics by Dafermos.

Let $0 \leq \bar{t} < T$ and let $\gamma_l, \gamma_r : [\bar{t}, T] \to \mathbb{R}$ be Lipschitz curves with $\gamma_l \leq \gamma_r$ and consider the domain

$$\Omega := \{(t, x) \in (\bar{t}, T) \times \mathbb{R} : \gamma_l(t) < x < \gamma_r(t)\}. \tag{4.12}$$

LEMMA 4.8. *Let u be a piecewise monotone solution of (1.14). Suppose that $u \llcorner \Omega$ takes values in $[u^-, u^+]$ and that $f \llcorner [u^-, u^+]$ is strictly convex. Then for every $x \in (\gamma_l(T), \gamma_r(T))$ there exists $\bar{y}$ and $t_0 \in [\bar{t}, T)$ such that for every $t \in [t_0, T]$*

$$\mathtt{X}(t, \bar{y}) = x - (T - t) f'(u_0(\bar{y})) \qquad \text{and} \qquad \mathtt{X}(t_0, \bar{y}) \in \partial\Omega.$$

The characteristic structure of solutions with bounded variation when the flux has only one inflection point is studied in [Daf85]. We introduce some terminology and recall his result, then we translate his result in our language.

DEFINITION 4.9. A *generalized characteristic* of (1.14) associated with the admissible BV solution u is a Lipschitz trajectory $\chi : [a,b] \to \mathbb{R}$, $0 \le a < b < \infty$ such that for $\mathcal{L}^1$-a.e. $t \in [a,b]$

$$\dot{\chi}(t) \in [f'(u(t,\chi(t)+)), f'(u(t,\chi(t)-))].$$

DEFINITION 4.10. A generalized characteristic $\chi : [a,b] \to \mathbb{R}$ of (1.14), associated with the admissible BV solution u, is called a *left contact* or a *right contact* if

$$\dot{\chi}(t) = f'(u(t,\chi(t)-)) \qquad \text{or} \qquad \dot{\chi}(t) = f'(u(t,\chi(t)+))$$

for $\mathcal{L}^1$-a.e. $t \in [a,b]$ respectively.

The existence of generalized characteristics is granted by Filippov theory. In general uniqueness fails, in the following two theorems, whose proof can be found in [Daf85], it is described the structure of maximal and minimal backward characteristics respectively (see also [Daf16], Section 11.12).

As the author did in [Daf85], we assume in Theorems 4.11 and 4.12 that:

(1) it holds $f(0) = f'(0) = f''(0) = 0$;
(2) it holds $uf''(u) < 0$ for every $u \neq 0$;
(3) the function f'' is nonincreasing in a neighborhood of 0.

THEOREM 4.11. *Let ξ denote the maximal backward characteristic through any point $(\bar{t},\bar{x}) \in (0,+\infty) \times \mathbb{R}$. When $u(\bar{t},\bar{x}-) \neq 0$ or $u(\bar{t},\bar{x}+) \neq 0$, then there is a finite mesh $0 = a_0 < a_1 < \ldots < a_{N+1} = \bar{t}$ such that ξ is a convex polygonal line with vertices at the point $(a_n, \xi(a_n))$, $n = 0,\ldots,N+1$. Furthermore,*

$$\begin{aligned}
u(t,\xi(t)-) = u(t,\xi(t)+) &= u(a_{n+1},\xi(a_{n+1})+), && a_n < t < a_{n+1},\ n = 0,\ldots,N,\\
u(a_n,\xi(a_n)-) &= u(a_{n+1},\xi(a_{n+1})+), && n = 1,\ldots,N,\\
u_0(\xi(0)) &\ge u(a_1,\xi(a_1)+), && \textit{if } u(a_1,\xi(a_1)+) > 0,\\
u_0(\xi(0)) &\le u(a_1,\xi(a_1)+), && \textit{if } u(a_1,\xi(a_1)+) < 0,\\
\dot{\xi}(t) &= f'(a_{n+1},u(\xi(a_{n+1})+)), && a_n < t < a_{n+1},\ n = 0,\ldots,N,\\
f'(u(a_n,\xi(a_n)-)) &= \frac{f(u(a_n,\xi(a_n)+)) - f(u(a_n,\xi(a_n)-))}{u(a_n,\xi(a_n)+) - u(a_n,\xi(a_n)-)}, && n = 1,\ldots,N.
\end{aligned} \tag{4.13}$$

When $u(\bar{t},\bar{x}-) = u(\bar{t},\bar{x}+) = 0$, then there is $a \in [0,\bar{t}]$ such that $\xi(t) = \bar{x}$, $t \in [a,\bar{t}]$, and $u(t,\xi(t)-) = u(t,\xi(t)+) = 0$, $t \in (a,\bar{t}]$ (also at $t = a$ if $a > 0$). Moreover, if $a > 0$, there is an increasing sequence $0 = a_0 < a_1 < \ldots$ with $a_n \to a$ and $\xi(a_n) \downarrow \bar{x}$ as $n \to \infty$, such that (4.13) all hold for $n = 1,2,\ldots$. In particular,

$$|u(t,\xi(t))| \downarrow 0, \quad f'(u(t,\xi(t))) \uparrow 0, \quad \textit{as } t \uparrow a.$$

THEOREM 4.12. *Let ζ denote the minimal backward characteristic through any point $(\bar{t},\bar{x}) \in (0,+\infty) \times \mathbb{R}$. Then $u(t,\zeta(t)-)$ is a continuous function on $(0,\bar{t}]$, which is nondecreasing when $u(\bar{t},\bar{x}-) < 0$, nonincreasing when $u(\bar{t},\bar{x}-) > 0$ and constant equal to 0 when $u(\bar{t},\bar{x}-) = 0$. For $t \in (0,\bar{t})$,*

$$\dot{\zeta}(t) = f'(t,\zeta(t)-)$$

so, in particular, ζ is a convex C^1 curve. Furthermore, the interval $(0,\bar{t})$ is decomposed into the union of two disjoint subset O and C with the following properties: O is the (at most) countable union of pairwise disjoint open intervals, $O = \bigcup_n (\alpha_n,\beta_n)$, such that

$$u(t,\zeta(t)-) = u(t,\zeta(t)+) = u(\alpha_n,\zeta(\alpha_n)) = u(\beta_n,\zeta(\beta_n))$$

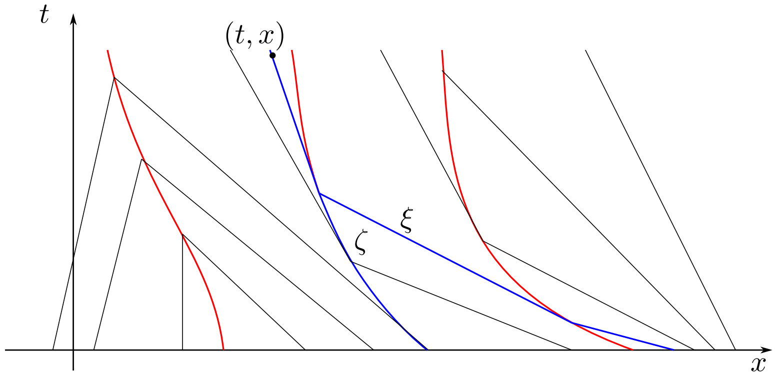


FIGURE 4.3. The structure of the characteristic curves: the minimal and maximal backward characteristic ζ and ξ are blue and a shock and two left contact discontinuities are red.

for all $t \in (\alpha_n, \beta_n)$ so the restriction of ζ on (α_n, β_n) is a straight line with slope $f'(u(\alpha_n, \zeta(\alpha_n)-))$. For any point $t \in C$, $u(t, \zeta(t)-) \neq u(t, \zeta(t)+)$ and

$$f'(u(t, \zeta(t)-)) = \frac{f(u(t, \zeta(t)+)) - f(u(t, \zeta(t)-))}{u(t, \zeta(t)+) - u(t, \zeta(t)-)}.$$

Now we restrict our attention to the case of piecewise monotone initial data and we formulate Theorem 4.11 in terms of the Lagrangian representation.

LEMMA 4.13. *Let u be the entropy solution of (1) with $u_0 \in X$, and let Ω be as in (4.12). Suppose that $u \llcorner \Omega$ takes values in $[u^-, u^+]$ with $\bar{w} \in (u^-, u^+)$,*

$$f''(w)(w - \bar{w}) < 0 \quad \text{in} \quad [u^-, u^+] \setminus \{\bar{w}\}$$

and that f'' is nonincreasing in a neighborhood of $\bar{w}$. Then for every $\bar{x} \in (\gamma_l(T), \gamma_r(T))$ the maximal backward generalized characteristic $\xi_{\bar{x}}$ from $(T, \bar{x})$ enjoys the following properties: there exists $N = N(\bar{x}) \in \mathbb{N}$, $\bar{t} \le t_0 < t_1 < \ldots < t_N = T$ and $y_1 > \ldots > y_N$ such that

(1) for every $n = 1, \ldots, N$, for every $t \in [t_{n-1}, t_n]$

$$\xi_{\bar{x}}(t) = \mathtt{X}(t, y_n) = \mathtt{X}(t_{n-1}) + (t - t_{n-1}) f'(u_0(y_n)),$$

in particular $\xi_{\bar{x}}$ is piecewise affine;

(2) for every $t \in (t_0, T]$, $\xi_{\bar{x}}(t) \in (\gamma_l(t), \gamma_r(t))$ and $(t_0, \xi_{\bar{x}}(t_0)) \in \partial\Omega$;

(3) for every $n = 1, \ldots, N$

$$u_0(y_n) = u(t, \mathtt{X}(t, y_n)-) \quad \text{for every } t \in (t_{n-1}, t_n],$$
$$u_0(y_n) = u(t, \mathtt{X}(t, y_n)+) \quad \text{for every } t \in [t_{n-1}, t_n) \setminus \{t_0\};$$

(4) for every $n = 2, \ldots, N$

$$u_0(y_n) = u_0(y_{n-1})^*, \tag{4.14}$$

where $u_0(y_{n-1})^*$ is defined by Lemma 1.14;

(5) for every $\bar{x}_1, \bar{x}_2 \in (\gamma_l(T), \gamma_r(T))$ with $\bar{x}_1 < \bar{x}_2$ it holds

$$\xi_{\bar{x}_1}(t) < \xi_{\bar{x}_2}(t) \qquad \forall t \in (t_0(\bar{x}_1) \vee t_0(\bar{x}_2), T].$$

Moreover if $u(T, \bar{x}-) = u(T, \bar{x}+) = \bar{w}$, then the conditions above hold with $N = 1$: in particular $\xi_{\bar{x}} \llcorner (t_0, T)$ has constant velocity.

PROOF. We first observe that for every $n = 1, \ldots, N$ there exists y_n such that $\xi_{\bar{x}} \llcorner (t_{n-1}, t_n) = \mathtt{X}(\cdot, y_n) \llcorner (t_{n-1}, t_n)$. Let $t \in (t_{n-1}, t_n)$ and y_n be such that $\mathtt{X}(t, y_n) = \xi_{\bar{x}}(t)$ and $\mathtt{T}(y_n) \ge t$. Then $u_0(y_n) = u(t, \xi_{\bar{x}}(t))$ and, since $\mathtt{X}(\cdot, y_n)$ satisfies the characteristic equation and u is continuous, it holds $\xi_{\bar{x}}\llcorner(t_{n-1},t_n) = \mathtt{X}(\cdot,y_n)\llcorner(t_{n-1},t_n)$. By monotonicity of the flow $\mathtt{X}$, the maximality of $\xi_{\bar{x}}$ and the fact that for every $n = 2,\dots,N$,

$$u_0(y_n) = u(t_{n-1},\mathtt{X}(t_n,y_n)+) \neq u(t_{n-1},\mathtt{X}(t_n,y_n)-) = u(y_{n-1})$$

we have $y_n > y_{n-1}$. Observe that, by (4.14), the value $u_0(y_{n-1})$ is uniquely determined by $u_0(y_n)$ and in particular

$$(u_0(y_{n-1}) - \bar{w})(u_0(y_n) - \bar{w}) < 0. \tag{4.15}$$

Since the initial datum is piecewise monotone and $n \mapsto y_n$ is strictly decreasing, (4.15) implies that N is bounded by the number of monotone regions of the initial datum. In particular if $u(T,\bar{x}-) = u(T,\bar{x}+) = 0$, the existence of a sequence as in Theorem 4.11 is excluded. It remains to prove the monotonicity in (5): Let $t \in (t_0(\bar{x}_1) \vee t_0(\bar{x}_2), T]$ the maximal time such that $\xi_{\bar{x}_1}(t) = \xi_{\bar{x}_2}(t)$. By monotonicity of the flow and since the maximal characteristics have piecewise constant speed, the point $(t,\xi_{\bar{x}_1}(t))$ must belong to a left-discontinuity curve. Since the left-discontinuity curve has C^1 regularity it holds $\partial_t\xi_{\bar{x}_1}(t+) = \partial_t\xi_{\bar{x}_2}(t)$ and this implies that $t = T$. But this is in contradiction with the hypothesis $\bar{x}_1 < \bar{x}_2$ and this concludes the proof. □

In the following two propositions we deduce by the structure of the characteristics, an estimate the total variation of $f' \circ u(T)$ in the two cases of a convex flux or of a flux with an inflection point of polynomial degeneracy.

PROPOSITION 4.14. *Let u be the entropy solution of (1) with $u_0 \in X$; let $\bar{t}, T, \gamma_l, \gamma_r$ and Ω be defined as in (4.12). Assume that there exists $a, b \in [0, \|u_0\|_\infty]$ such that $u\llcorner\Omega$ solves the initial-boundary value problem*

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \Omega, \\ u(t,\gamma_l(t)) = a & \text{for } t \in (\bar{t},T), \\ u(t,\gamma_r(t)) = b & \text{for } t \in (\bar{t},T). \end{cases}$$

Denote by

$$I := \operatorname{conv}\big(\{a,b\} \cup u(\bar{t},(\gamma_l(\bar{t}),\gamma_r(\bar{t})))\big)$$

and assume moreover that $f\llcorner I$ is strictly convex. Then

$$\mathrm{TV}_{(\gamma_l(T),\gamma_r(T))} f' \circ u(T) \leq 6\|f''\|_\infty\|u_0\|_\infty + 2\frac{\gamma_r(T) - \gamma_l(T)}{T - \bar{t}}.$$

PROOF. Let $\mathtt{X}$ be a Lagrangian representation of u and consider the following decomposition:

$$(\gamma_l(T),\gamma_r(T)) = A_l \cup A_m \cup A_r,$$

where

$$\begin{aligned} A_l &:= \left\{x \in (\gamma_l(T),\gamma_r(T)) : \exists t_0 \in [\bar{t},T), \exists \bar{y} \in \mathbb{R}\big(\mathtt{X}(t_0,\bar{y}) = \gamma_l(t_0), \mathtt{X}(T,\bar{y}) = x\big)\right\}, \\ A_r &:= \left\{x \in (\gamma_l(T),\gamma_r(T)) : \exists t_0 \in [\bar{t},T), \exists \bar{y} \in \mathbb{R}\big(\mathtt{X}(t_0,\bar{y}) = \gamma_r(t_0), \mathtt{X}(T,\bar{y}) = x\big)\right\}, \\ A_m &:= (\gamma_l(T),\gamma_r(T)) \setminus (A_l \cup A_r). \end{aligned}$$

By monotonicity and continuity of the flow $\mathtt{X}$ with respect to y, there exist $x_l, x_r \in [\gamma_l(T),\gamma_r(T)]$ such that

$$A_l = (\gamma_l(T), x_l] \qquad \text{and} \qquad A_r = [x_r\gamma_r(T)).$$

Observe that it may be $x_r \leq x_l$; in that case $A_m = \emptyset$.

Assume A_l is nonempty and let $\bar{y} \in \mathbb{R}$ and $t_0 \in [\bar{t},T)$ be such that

$$\mathtt{X}(T,\bar{y}) = x_l \qquad \text{and} \qquad \mathtt{X}(t_0,\bar{y}) = \gamma_l(t_0).$$

Moreover let

$$y^- := \max\{y : \mathtt{X}(T,y) = \gamma_l(T)\} \qquad \text{and} \qquad \bar{w}^- := \lim_{x \to \gamma_l(T)^+} u(T,x)$$

and similarly

$$y^+ := \min\{y : \mathtt{X}(T,y) = x_l\} \qquad \text{and} \qquad \bar{w}^+ := \lim_{x\to x_l^-} u(T,x).$$

Denote by

$$\Omega_l := \{(t,x) \in (\bar{t},T)\times\mathbb{R} : \mathtt{X}(t,y^-) < x < \mathtt{X}(t,y^+)\}.$$

By definition of y^-, y^+ and the monotonicity of $\mathtt{X}$ with respect to y there exists $t_0 \in [\bar{t},T)$ such that $\mathtt{X}(t_0,y^-) = \mathtt{X}(t_0,y^+) = \mathtt{X}(t_0,y_j)$. Since the limit of admissible boundaries is an admissible boundary in the sense of Proposition 1.29, $(\mathtt{X}(\cdot,y^-),\bar{w}^-)$ and $(\mathtt{X}(\cdot,y^+),\bar{w}^+)$ are admissible boundaries of u for $t\in[0,T)$. Therefore the restriction $u(T)\llcorner(\gamma_l(T),x_l)$ is the entropy solution at time T of the boundary value problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \Omega_l,\\ u(t,\mathtt{X}(t,y^-)) = \bar{w}^- & \text{for } t\in(t_0,T),\\ u(t,\mathtt{X}(t,y^+)) = \bar{w}^+ & \text{for } t\in(t_0,T). \end{cases}$$

By Proposition 1.31, this implies that $u(T)\llcorner(\gamma_l(T),x_l)$ is monotone, therefore

$$\mathrm{TV}_{(\gamma_l(T),x_l)}(f'\circ u(T)) \le \|f''\|_\infty\|u_0\|_\infty. \tag{4.16}$$

Similarly we can prove that

$$\mathrm{TV}_{(x_r,\gamma_r(T))}(f'\circ u(T)) \le \|f''\|_\infty\|u_0\|_\infty, \tag{4.17}$$

therefore it remains to estimate the total variation on A_m. Assume that $A_m\neq\emptyset$ i.e. $x_l<x_r$. This case is well-known, we take advantage of the fact that by Lemma 4.8, the characteristics starting from $x\in A_m$ are segments in $[\bar{t},T]$, and we deduce a one-sided Lipschitz estimate. Denote by

$$\bar{y}^- := \max\{y : \mathtt{X}(T,y) = x_l\} \qquad \text{and} \qquad \bar{y}^+ := \max\{y : \mathtt{X}(T,y) = x_r\}.$$

By Lemma 4.8, for every $x\in A_m$ there exists $y(x)\in(\bar{y}^-,\bar{y}^+)$ such that for every $t\in[\bar{t},T]$, it holds

$$\mathtt{X}(t,y(x)) = x - f'(u(T,x))(T-t).$$

By monotonicity of the flow, for every $x_l<x_1<x_2<x_r$, it holds

$$x_1 - f'(u(T,x_1))(T-\bar{t}) = \mathtt{X}(\bar{t},y(x_1)) \le \mathtt{X}(\bar{t},y(x_2)) = x_2 - f'(u(T,x_2))(T-\bar{t}),$$

which gives the one-sided Lipschitz estimate

$$f'(u(T,x_2)) \le f'(u(T,x_1)) + \frac{x_2-x_1}{T-\bar{t}}.$$

This implies that the positive total variation

$$\mathrm{TV}^+_{(x_l,x_r)}(f'\circ u(T)) \le \frac{x_r-x_l}{T-\bar{t}}.$$

Hence, by Lemma 1.10, the whole total variation can be estimate by

$$\mathrm{TV}_{(x_l,x_r)}(f'\circ u(T)) \le 2\mathrm{TV}^+_{(x_l,x_r)}(f'\circ u(T)) + 2\|f''\|_\infty\|u_0\|_\infty \le 2\frac{x_r-x_l}{T-\bar{t}} + 2\|f''\|_\infty\|u_0\|_\infty. \tag{4.18}$$

Adding (4.16), (4.17), (4.18) and taking into account the possible jumps of $f'\circ u(T)$ at the points x_l and x_r we get

$$\mathrm{TV}_{(\gamma_l(T),\gamma_r(T))}(f'\circ u(T)) \le 6\|f''\|_\infty\|u_0\|_\infty + 2\frac{\gamma_r(T)-\gamma_l(T)}{T-\bar{t}}$$

that is the claimed estimate. □

The case of a flux with an inflection point is more elaborate and it is based on the structure of maximal characteristics.

Proposition 4.15. *Let u be the entropy solution of (1) with $u_0 \in X$; let $\bar{t}, T, \gamma_l, \gamma_r$ and Ω be defined as in (4.12). Assume that there exists $a, b \in [0, \|u_0\|_\infty]$ such that $u \llcorner \Omega$ solves the initial-boundary value problem*

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \Omega, \\ u(t, \gamma_l(t)) = a & \text{for } t \in (0, T), \\ u(t, \gamma_r(t)) = b & \text{for } t \in (0, T). \end{cases}$$

Denote by

$$I := \operatorname{conv}\big(\{a, b\} \cup u(\bar{t}, (\gamma_l(\bar{t}), \gamma_r(\bar{t})))\big).$$

Assume moreover that there exists a unique inflection point $\bar{w} \in I$ of f and that $\bar{w}$ has degeneracy $p \in \mathbb{N}$. Let $\delta', \varepsilon' > 0$ be given by Lemma 1.14 and assume finally that $I \subset (\bar{w} - \delta', \bar{w} + \delta')$.

Then there exists a constant $C > 0$ depending on ε', $\|f'\|_\infty$, $\|f''\|_\infty$, $\|u_0\|_\infty$, $\mathcal{L}^1(\operatorname{conv}(\operatorname{supp} u_0))$ such that

$$\mathrm{TV}_{(\gamma_l(T), \gamma_r(T))}(f' \circ u(T)) \le C \left(1 + \frac{1}{T - \bar{t}}\right). \tag{4.19}$$

Proof. The structure of the proof of this proposition is similar to the one of Proposition 4.14. By Proposition 1.36, we can assume that the chord admissibility condition is satisfied at time T. Here we reach the final estimate studying the behavior of maximal backward characteristics. Let $(\mathtt{X}, \mathtt{T})$ be a Lagrangian representation of u and for every $x \in (\gamma_l(T), \gamma_r(T))$ let ξ_x be the maximal backward characteristic from (T, x). Consider moreover the corresponding $y_1(x), \dots y_{N(x)}$ and $t_0(x), \dots, t_{N(x)}$ given by Lemma 4.13. Consider the decomposition

$$(\gamma_l(T), \gamma_r(T)) = A_l \cup A_m \cup A_r,$$

where

$$\begin{aligned} A_l &:= \{\bar{x} \in (\gamma_l(T), \gamma_r(T)) : \xi_{\bar{x}}(t_0(\bar{x})) = \gamma_l(t_0(\bar{x}))\}, \\ A_r &:= \{\bar{x} \in (\gamma_l(T), \gamma_r(T)) : \xi_{\bar{x}}(t_0(\bar{x})) = \gamma_r(t_0(\bar{x}))\}, \\ A_m &:= (\gamma_l(T), \gamma_r(T)) \setminus (A_l \cup A_r). \end{aligned}$$

Let $\bar{x} \in A_l$ and set

$$\Omega_l := \{(t, x) \in (t_0(\bar{x}), T] \times \mathbb{R} : \gamma_l(t) < x < \xi_{\bar{x}}(t)\}.$$

Then $u(T) \llcorner (\gamma_l(T), \bar{x})$ is the entropy solution at time T of the boundary value problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \Omega_l, \\ u(t, \gamma_l(t)) = a & \text{for } t \in (t_0, T), \\ u(t, \xi_{\bar{x}}(t)) = u^+(t) & \text{for } t \in (t_0, T), \end{cases}$$

where $u^+(t) = u(t, \xi_{\bar{x}}(t))$, and this definition makes sense since by Lemma 4.13, for every $t \notin \{t_i\}_{i=1}^n$ the solution u is continuous at $(t, \xi_{\bar{x}}(t))$. By Proposition 1.31, it holds

$$\mathrm{TV}_{(\gamma_l(T), \bar{x})}(u(T)) \le \mathrm{TV}_{(t_0, T)}(u^+) + |u^+(t_0+) - a| \tag{4.20}$$

and by Lemma 4.13, we have that

$$\mathrm{TV}_{(t_0, T)}(u^+) = \sum_{n=2}^{N(\bar{x})} |u_0(y_n) - u_0(y_{n-1})| \le 2 \sum_{n=1}^{N(\bar{x})} |u_0(y_n) - \bar{w}|.$$

Since for every $n = 2, \dots, N(\bar{x})$, it holds $u_0(y_n(\bar{x})) = u_0(y_{n-1}(\bar{x}))^*$, by Lemma 1.14 and (4.14), it holds

$$|u_0(y_n(\bar{x})) - \bar{w}| \le |u_0(y_1(\bar{x})) - \bar{w}|(1 - \varepsilon')^{h-1}.$$

So we finally have that

$$\mathrm{TV}_{(t_0,T)}(u^+) \leq 2|u_0(y_1(\bar{x})) - \bar{w}| \sum_{n=1}^{N(\bar{x})} (1-\varepsilon')^{n-1} \leq \frac{2\|u_0\|_\infty}{\varepsilon'}. \tag{4.21}$$

By (4.20) and (4.21), we get

$$\mathrm{TV}_{(\gamma_l(T),\bar{x})}(u(T)) \leq \|u_0\|_\infty \left(1 + \frac{2}{\varepsilon'}\right),$$

and since the estimate is independent of $\bar{x} \in \mathrm{Int}A_l$, it holds

$$\mathrm{TV}_{\mathrm{Int}A_l}(u(T)) \leq \|u_0\|_\infty \left(1 + \frac{2}{\varepsilon'}\right).$$

Therefore it immediately follows

$$\mathrm{TV}_{\mathrm{Int}A_l}(f' \circ u(T)) \leq \|f''\|_\infty \|u_0\|_\infty \left(1 + \frac{2}{\varepsilon'}\right). \tag{4.22}$$

and the same argument proves that

$$\mathrm{TV}_{\mathrm{Int}A_r}(f' \circ u(T)) \leq \|f''\|_\infty \|u_0\|_\infty \left(1 + \frac{2}{\varepsilon'}\right). \tag{4.23}$$

It remains to prove the estimate in A_m. Again we take advantage of the fact that the generalized characteristics $\xi_{\bar{x}}$ for $\bar{x} \in \mathrm{Int}A_m$ are defined on the whole time interval $[\bar{t}, T]$. The estimate is obtained by partitioning $(x_l, x_r) = \mathrm{Int}A_m$ in regions where the maximal characteristics of each region cross the same set of minimal backward characteristic, bounding $\mathrm{TV}^+(f' \circ u)$ on each of these regions and adding them.

Let

$$y^- := \max\{y : \mathtt{X}(T,y) = \inf A_m\} \qquad \text{and} \qquad y^+ := \min\{y : \mathtt{X}(T,y) = \sup A_m\}.$$

Since u_0 is piecewise monotone, there exist $L \in \mathbb{N}$ and $y^- = \bar{y}_0 < \ldots < \bar{y}_L = y^+$ such that

(1) for every $l = 1, \ldots, L$,

$$\bar{y}_l \in \{y : \mathtt{T}(y) \geq \bar{t}\} \qquad \text{and} \qquad u_0(\bar{y}_l) = \bar{w}.$$

(2) the function u_0 alternates the sign on $((y_l, y_{l+1}))_{l=1}^{L-1}$, i.e. the function

$$\sum_{l=1}^{L-1} (-1)^l u_0 \llcorner \{y \in (\bar{y}_l, \bar{y}_{l+1}) : \mathtt{T}(y) \geq \bar{t}\}$$

has constant sign; without loss of generality we assume that it is nonnegative.

(3) for every $l = 1, \ldots, L-2$, there exist $y' \in \{y \in (\bar{y}_l, \bar{y}_{l+1}) : \mathtt{T}(y') \geq \bar{t}\}$ and $y'' \in \{y \in (\bar{y}_{l+1}, \bar{y}_{l+2}) : \mathtt{T}(y) \geq \bar{t}\}$ such that $u_0(y')u_0(y'') < 0$.

For every $x \in (x_l, x_r)$ and for every $n = 1, \ldots, N(x)$, let $l(x,n)$ be the unique value in $\{1, \ldots, L\}$ such that

$$y_n(x) \in [\bar{y}_{l(x.n)}, \bar{y}_{l(x_n)+1}) \qquad \text{and let} \qquad \mathbf{l}(x) := \{l(x,n) : n = 1, \ldots, N(x)\}.$$

For every $\mathbf{l} \in \mathcal{P}(\{1, \ldots, L-1\})$, let

$$A(\mathbf{l}) := \{x \in \mathrm{Int}A_m : \mathbf{l}(x) = \mathbf{l}\}.$$

Clearly it holds

$$\bigcup_{\mathbf{l} \in \mathcal{P}(1,\ldots,L)} A(\mathbf{l}) = (x_l, x_r),$$

now we check that for every $\mathbf{l} \in \mathcal{P}(\{1, \ldots, L\})$ the set $A(\mathbf{l})$ is an interval.

In order to do this let us introduce a partial ordering on $\mathcal{P}(\{1, \ldots, L-1\})$: we say that $\mathbf{l}_1 \preceq \mathbf{l}_2$ if

(1) $\min \mathbf{l}_1 \leq \min \mathbf{l}_2$;
(2) $\max \mathbf{l}_1 \leq \max \mathbf{l}_2$;
(3) for every $j \in [\min \mathbf{l}_2, \max \mathbf{l}_1]$

$$j \in \mathbf{l}_2 \quad \Rightarrow \quad j \in \mathbf{l}_1.$$

It is standard to check that $\preceq$ is a partial ordering, so in order to prove that $A(\mathbf{l})$ are intervals, it suffices to prove that for every $x_1, x_2 \in (x_l, x_r)$ it holds

$$x_1 < x_2 \qquad \Longrightarrow \qquad \mathbf{l}(x_1) \preceq \mathbf{l}(x_2).$$

The conditions (1) and (2) of the definition of $\preceq$ immediately follow from the monotonicity of $x \mapsto \xi_x$ (Point (5) of Lemma 4.13). Finally by Proposition 3.27, it follows that if $\mathtt{X}(t', \bar{y}_{l_1}) = \mathtt{X}(t', \bar{y}_{l_2})$ for some $t' \in (\bar{t}, T)$, then for every $t \in [t', T]$ it holds $\mathtt{X}(t, \bar{y}_{l_1}) = \mathtt{X}(t, \bar{y}_{l_2})$ and, by Point (5) in Lemma 4.13, this implies that if $j \in [\min \mathbf{l}_2, \max \mathbf{l}_1]$ is such that $j \notin \mathbf{l}_1$ then $j \notin \mathbf{l}_2$. This proves Condition (3) in the definition of $\preceq$.

Claim 1. There exist $V \in \mathbb{N}$ and $x_l = \bar{x}_1 < \ldots < \bar{x}_V = x_r$ such that for every $v = 1, \ldots, V-1$ there exists $\mathbf{l}(v) \in \mathcal{P}(\{1, \ldots, L-1\})$ such that

$$(\bar{x}_v, \bar{x}_{v+1}) \subset A(\mathbf{l}(v)). \tag{4.24}$$

Moreover if $v = 1, \ldots, V-1$ is such that $\#\mathbf{l}(v) \geq 2$, then

(1) for every $x_1, x_2 \in (\bar{x}_v, \bar{x}_{v+1})$ it holds

$$\sum_{n=1}^{\#\mathbf{l}(v)} |t_n(x_2) - t_n(x_1)| < \frac{T - \bar{t}}{2}. \tag{4.25}$$

(2) the velocity $f' \circ u(T)$ is strictly increasing in $(\bar{x}_v, \bar{x}_{v+1})$;

Proof of Claim 1. Let

$$\{\tilde{x}_1, \ldots, \tilde{x}_Q\} := \{\inf A(\mathbf{l}) : \mathbf{l} \in \mathcal{P}(\{1, \ldots, L-1\}) \text{ and } A(\mathbf{l}) \neq \emptyset\} \cup \{x_l, x_r\}$$

with $\tilde{x}_1 < \ldots, \tilde{x}_Q$. Since $(A(\mathbf{l}))_{\mathbf{l} \in \mathcal{P}(\{1,\ldots,L-1\})}$ is a family of pairwise disjoint intervals, the sequence $(\tilde{x}_q)_{q=1}^Q$ satisfies (4.24).
Therefore, in order to prove (1), it is sufficient to prove that for every $q = 1, \ldots, Q-1$ there exists $V' \in \mathbb{N}$ and $\tilde{x}_q = x'_1 < \ldots < x'_{V'} = \tilde{x}_{q+1}$ such that condition (2) holds for every $v' = 1 \ldots V' - 1$. Let $q \in \{1, \ldots, Q-1\}$ and let $\mathbf{l} \in \mathcal{P}(\{1, \ldots, L-1\})$ be such that

$$(\tilde{x}_q, \tilde{x}_{q+1}) \subset A(\mathbf{l}).$$

Since for every $x \in \mathrm{Int}A(\mathbf{l})$ and for every $n = 1, \ldots, \#\mathbf{l}(v)$ the function $x \mapsto t_n(x)$ is increasing, the set

$$F := \left\{ x : \exists n = 1, \ldots, \#\mathbf{l}(v) : \left(t_l(x+) - t_l(x-) \geq \frac{T - \bar{t}}{2L} \right) \right\}$$

is finite. For every $n = 1, \ldots, \#\mathbf{l}(v)$ let $\tilde{\delta}(n)$ be given by Lemma 1.11 with

$$g = t_n : \mathrm{Int}A(\mathbf{l}) \to \mathbb{R} \qquad \text{and} \qquad \tilde{\varepsilon} = \frac{T - \bar{t}}{2L},$$

and fix

$$\tilde{\delta} = \min_{n=1,\ldots,\#\mathbf{l}(v)} \delta(n).$$

Then, by Lemma 1.11, any sequence $(x'_{v'})_{v'=1}^{V'}$ with $x'_1 = \tilde{x}_q$, $x' = \tilde{x}_{q+1}$ and such that for every $v' = 1, \ldots, V'-1$

$$0 < x'_{v'+1} - x'_{v'} < \tilde{\delta}$$

satisfies condition (1) of Claim 1. The Point (2) follows by the Point (5) in Proposition 3.27 and this concludes the proof of Claim 1.

Claim 2. Let $(\bar{x}_v, \bar{x}_{v+1})$ as in Claim 1. There exist $\bar{\delta} > 0$ and a constant C as in the statement of Proposition 4.15 such that the positive total variation

$$\mathrm{TV}^+_{(\bar{x}_v,\bar{x}_{v+1})}(f' \circ u(T)) \leq \frac{C}{(T-\bar{t})^2} A_v,$$

where A_v denotes the area of the region

$$\Omega_v := \{(t,x) \in (\bar{t}, T) : \xi_{\bar{x}_v}(t) < x < \xi_{\bar{x}_{v+1}}(t)\}.$$

Proof of Claim 2. If $\#\mathbf{l}(v) = 1$ we are in the same position as in Proposition 4.14: in particular $f' \circ u(T)$ is one-sided Lipschitz and

$$\mathrm{TV}^+_{(\bar{x}_v,\bar{x}_{v+1})}(f' \circ u(T)) \leq \frac{\bar{x}_{v+1} - \bar{x}_v}{T - \bar{t}} \leq \frac{A_v}{2(T-\bar{t})^2}. \tag{4.26}$$

Now we consider the case $\#\mathbf{l}(v) \geq 2$ so that, by Claim 1, for every $x_1 < x_2$ in $(\bar{x}_v, \bar{x}_{v+1})$ it holds

$$\mathrm{TV}_{(x_1,x_2)}(f' \circ u(T)) = \mathrm{TV}^+_{(x_1,x_2)}(f' \circ u(T)) = f'(u(T,x_2)) - f'(u(T,x_1)).$$

For every $n = 2, \dots, \#\mathbf{l}(v)$, consider the time $t'_n \in \mathbb{R}$ for which the straight-line extensions of the segments $\mathtt{X}(\cdot, y_n(x_1)) \llcorner [t_{n-1}(x_1), t_n(x_1)]$ and $\mathtt{X}(\cdot, y_n(x_2)) \llcorner [t_{n-1}(x_2), t_n(x_2)]$ intersect. Since they are tangent to the same convex curve at the time $t_{n-1}(x_1)$ and $t_{n-1}(x_2)$ respectively it holds

$$t'_n \in (t_{n-1}(x_1), t_{n-1}(x_2)). \tag{4.27}$$

See Figure 4.4. Moreover for every $n = 2, \dots, \#\mathbf{l}(v)$, set

$$\tau_n := (t_n(x_1) - t'_n)^+ \qquad \text{and let} \qquad \tau_1 := t_1(x_1) - \bar{t}.$$

Let $\Delta_{\#\mathbf{l}(v)}$ be the area of the triangle bounded by the following three lines:

$$\begin{aligned}
&\{(t,x) : x = x_1 - f'(u_0(y_{\#\mathbf{l}(v)}(x_1)))(T-t)\},\\
&\{(t,x) : x = x_2 - f'(u_0(y_{\#\mathbf{l}(v)}(x_2)))(T-t)\},\\
&\{(t,x) : t = T\}.
\end{aligned}$$

If $n = 2, \dots, \#\mathbf{l}(v) - 1$ is such that $\tau_n > 0$ let Δ_n be the area of the triangle bounded by the following three lines:

$$\begin{aligned}
&\{(t,x) : x = \mathtt{X}(t_n(x_1), y_n(x_1)) - f'(u_0(y_n(x_1)))(t_n(x_1) - t)\},\\
&\{(t,x) : x = \mathtt{X}(t_n(x_2), y_n(x_2)) - f'(u_0(y_n(x_2)))(t_n(x_2) - t)\},\\
&\{(t,x) : t = t_n(x_1)\}.
\end{aligned}$$

If $n = 2, \dots, \#\mathbf{l}(v) - 1$ is such that $\tau_n = 0$ let $\Delta_n = 0$ and finally let Δ_1 be the area of the trapezoid delimited by the lines

$$\begin{aligned}
&\{(t,x) : x = \mathtt{X}(t_1(x_1), y_1(x_1)) - f'(u_0(y_1(x_1)))(t_1(x_1) - t)\},\\
&\{(t,x) : x = \mathtt{X}(t_1(x_2), y_1(x_2)) - f'(u_0(y_1(x_2)))(t_1(x_2) - t)\},\\
&\{(t,x) : t = t_1(x_1)\},\\
&\{(t,x) : t = \bar{t}\}.
\end{aligned}$$

For every $n = 2, \dots, \#\mathbf{l}(v)$ the area of the triangle is given by

$$\Delta_n = \frac{\tau_n^2}{2}\big(f'(u_0(y_n(x_2))) - f'(u_0(y_n(x_1)))\big), \tag{4.28}$$

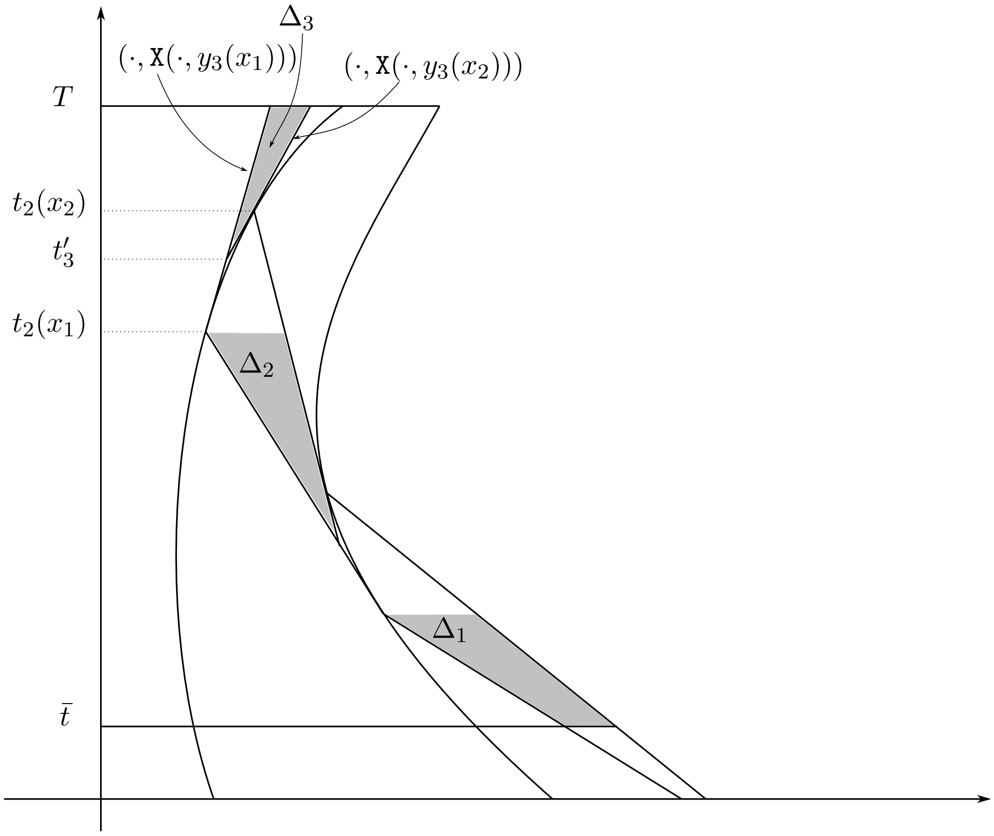


FIGURE 4.4. The notation of the construction to estimate $\mathrm{TV}^+(f' \circ u(T))$ in A_m for fluxes with an inflection point of polynomial degeneracy.

and for $n = 1$

$$\begin{aligned}\Delta_1 &= \frac{\tau_1^2}{2}\big(f'(u_0(y_1(x_2))) - f'(u_0(y_1(x_1)))\big) + \tau_1\big(\mathtt{X}(\bar{t}, y_1(x_2)) - \mathtt{X}(\bar{t}, y_1(x_1))\big) \\ &\geq \frac{\tau_1^2}{2}\big(f'(u_0(y_1(x_2))) - f'(u_0(y_1(x_1)))\big).\end{aligned}$$

We now prove that

$$\sum_{n=1}^{\#\mathbf{l}(v)} \tau_n \geq \frac{T - \bar{t}}{2}. \tag{4.29}$$

Recalling that $t_{\#\mathbf{l}(v)}(x_1) = T$, $t_0(x_2) = \bar{t}$ and (4.27), we have that

$$\begin{aligned}\sum_{n=1}^{\#\mathbf{l}(v)} \tau_n &\geq \sum_{n=1}^{\#\mathbf{l}(v)} \big(t_n(x_1) - t_n'\big) \\ &\geq \sum_{n=1}^{\#\mathbf{l}(v)} \big(t_n(x_1) - t_{n-1}(x_2)\big) \\ &= T - \bar{t} + \sum_{n=1}^{\#\mathbf{l}(v)-1} \big(t_n(x_1) - t_n(x_2)\big) \\ &\geq \frac{T - \bar{t}}{2},\end{aligned}$$

where the last inequality follows by (4.25).

Since for every $n = 2, \dots, \#\mathbf{l}(v)$ and $s = 1, 2$, $u_0(y_n(x_s)) = u_0(y_{n-1}(x_s))^*$, by (4.28) and iterating (1.5), we have that for every $n = 1, \dots, \#\mathbf{l}(v)$ it holds

$$\Delta_n \geq \frac{\tau_n^2}{2}\big(f'(u_0(y_{\#\mathbf{l}(v)}(x_2))) - f'(u_0(y_{\#\mathbf{l}(v)}(x_1)))\big)\left(\frac{1}{1-\varepsilon'}\right)^{\#\mathbf{l}(v)-n}. \tag{4.30}$$

Let us for brevity denote by

$$\lambda := \frac{1}{1-\varepsilon'} > 1.$$

Since the $(\Delta_n)_{n=1}^{\#\mathbf{l}(v)}$ are the area of pairwise disjoint regions contained in Ω_{x_1,x_2}, it holds

$$\sum_{n=1}^{\#\mathbf{l}(v)} \Delta_n \leq A_{x_1,x_2}.$$

Therefore adding for $n = 1, \dots, \#\mathbf{l}(v)$ the inequality (4.30) we obtain

$$f'(u_0(y_{\#\mathbf{l}(v)}(x_2))) - f'(u_0(y_{\#\mathbf{l}(v)}(x_1)) \leq A_{x_1,x_2}\left(\sum_{n=1}^{\#\mathbf{l}(v)} \frac{\tau_n^2}{2}\lambda^{\#\mathbf{l}(v)-n}\right)^{-1}.$$

Hence the proof of Claim 2 reduces to proving that there exists a constant $C > 0$ as in the statement of Proposition 4.15 such that

$$\left(\sum_{n=1}^{\#\mathbf{l}(v)} \frac{\tau_n^2}{2}\lambda^{\#\mathbf{l}(v)-n}\right)^{-1} \leq C,$$

or equivalently that there exists $c > 0$ such that

$$\sum_{n=1}^{\#\mathbf{l}(v)} \frac{\tau_n^2}{2}\lambda^{\#\mathbf{l}(v)-n} \geq c.$$

This follows by (4.29) and $\lambda > 1$. In fact let $a, b \in \mathbb{R}^{\#\mathbf{l}(v)}$ be the vectors of components

$$a_n = \tau_n \lambda^{\frac{\#\mathbf{l}(v)-n}{2}}, \qquad \text{and} \qquad b_n = \lambda^{-\frac{\#\mathbf{l}(v)-n}{2}}.$$

Then, by the Cauchy-Schwarz inequality,

$$\left(\sum_{n=1}^{\#\mathbf{l}(v)} \tau_n\right)^2 \leq \left(\sum_{n=1}^{\#\mathbf{l}(v)} \tau_n^2\lambda^{\#\mathbf{l}(v)-n}\right)\sum_{n=1}^{\#\mathbf{l}(v)} \lambda^{\#\mathbf{l}(v)-n},$$

so that by (4.29),

$$\sum_{n=1}^{\#\mathbf{l}(v)} \tau_n^2\lambda^{\#\mathbf{l}(v)-n} \geq \left(\frac{T-\bar{t}}{2}\right)^2\left(\sum_{n=1}^{\infty}\lambda^n\right)^{-1}$$

and this concludes the proof of Claim 2. Since the chord condition holds at time T, the function $f' \circ u(T)$ does not have jumps of positive sign, therefore applying Point(1) of Lemma 1.10 with $n = V$ and $x_i = \bar{x}_i$ for $i = 1, \dots, V$, we get

$$\mathrm{TV}^+_{A_m}(f' \circ u(T)) \leq \frac{C}{(T-\bar{t})^2}A_v. \tag{4.31}$$

Finally by (4.22), (4.23) and (4.31), it follows (4.19) and this concludes the proof of Proposition 4.15. □

The next lemma will be used to reduce the estimate of the total variation of $f' \circ u(T)$ to the estimate on the regions where the oscillation of the solution is small. The smallness parameter $\delta > 0$ will be chosen later.

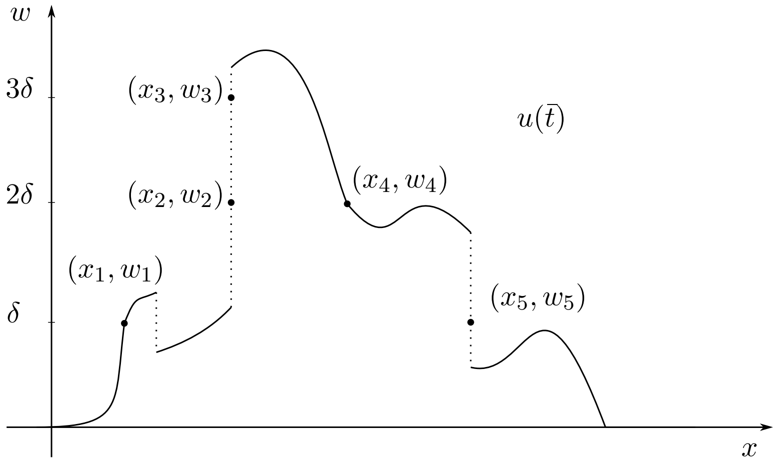


FIGURE 4.5. A representation of the construction of $(y_m)_m$: for $m = 1, \dots, 5$, $y_m = y(x_m, w_m)$.

LEMMA 4.16. *Let u be the entropy solution of (1) with $u_0 \in X$ and let $\bar{t}, \delta > 0$ with $\bar{t}$ generic. Then there exists $M \in \mathbb{N}$ depending only on $\delta, f, \|u_0\|_\infty, \mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u_0)), \bar{t}$ and there exist $y_1, \dots, y_{M(u_0)}$ with $M(u_0) \le M$ such that for every $m = 1, \dots M-1$ there exists $k = k(m) \in \mathbb{N}$ for which for every $t > \bar{t}$, it holds*

$$u(t, (\mathtt{X}(t, y_m), \mathtt{X}(t, y_{m+1}))) \subset [(k-2)\delta, (k+2)\delta] \tag{4.32}$$

and

$$u(t, (-\infty, \mathtt{X}(t, y_1))) \subset [0, 2\delta], \qquad u(t, (\mathtt{X}(t, y_m), +\infty)) \subset [0, 2\delta]. \tag{4.33}$$

PROOF. Let $\mathtt{X}$ be a Lagrangian representation of u and let $\mathtt{T}$ be a time existence function as in Proposition 3.27. Consider the map $y = y(t, x, w)$ defined in Proposition 3.27 and for every $w \in \mathbb{R}$ let

$$A_w := \{y \in \mathbb{R} : \mathtt{T}(y) \ge \bar{t} \text{ and } u_0(y) = w\}.$$

Let $y_1 := \min A_\delta$ and for $m \in \mathbb{N}$ with $l \ge 2$ we define recursively

$$y_m := \min\left((A_{u_0(y_{m-1})+\delta} \cup A_{u_0(y_{m-1})-\delta}) \cap [y_m, +\infty)\right)$$

if the set on the right hand side is nonempty, otherwise we set $y_m = +\infty$ (see Figure 4.5).

By definition it is obvious that the sequence $(y_m)_{m\in\mathbb{N}}$ is increasing. For every $u_0 \in X$ denote by $M(u_0)$ the number of indexes m such that y_m is finite; by construction we have the estimate

$$M(u_0)\delta \le \mathrm{TV}(u(\bar{t})) \le \mathrm{TV}(u_0).$$

Since $\|u(\bar{t})\|_\infty \le \|u_0\|_\infty$ the number $N(u(\bar{t}), \delta)$ of undulations of $u(\bar{t})$ of height bigger than δ is bounded by below by

$$N(u(\bar{t}), \delta) \ge \frac{M(u_0)\delta}{2\|u_0\|_\infty}. \tag{4.34}$$

Moreover, by Lemma 4.5,

$$N(u(\bar{t}), \delta) \le \frac{4\|u_0\|_\infty(\mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u_0)) + \|f'\|_\infty \bar{t})}{\bar{t}\mathfrak{d}(\delta)}. \tag{4.35}$$

By (4.34) and (4.35), we have that $M(u_0)$ is uniformly bounded by a constant M as in the statement. Now it remains to prove (4.32) and (4.33) and they follow by the definition of Lagrangian representation and the construction of $(y_m)_{m\in\mathbb{N}}$. $\square$

We now have all the ingredients to prove the main result of this section.

THEOREM 4.17. *Let f be a flux with polynomial degeneracy and let u be the entropy solution of (1) with $u_0 \in L^\infty(\mathbb{R})$ nonnegative and with compact support. Then there exists a constant $C > 0$ depending on f, $\|u_0\|_\infty$ and $\mathcal{L}^1(\mathrm{conv}(\operatorname{supp} u_0))$ such that for every $T > 0$*

$$\mathrm{TV}(f' \circ u(T)) \leq C\left(1 + \frac{1}{T}\right). \tag{4.36}$$

PROOF. Observe that it is enough to prove the statement for $u_0 \in X$: indeed for every u_0 as in the statement there exists a sequence $(u_0^n)_{n\in\mathbb{N}}$ contained in X such that $u_0^n \to u_0$ in $L^1(\mathbb{R})$ and $\|u_0^n\|_\infty, \mathcal{L}^1(\mathrm{conv}(\operatorname{supp} u_0^n))$ are uniformly bounded by $\|u_0\|_\infty$ and $\mathcal{L}^1(\mathrm{conv}(\operatorname{supp} u_0))$ respectively.

Then notice also that, since $t \mapsto u(t)$ is continuous with respect to the L^1 topology and the total variation is lower semicontinuous with respect to the same topology, it is sufficient to prove (4.36) for a dense set of $T > 0$. In particular we assume that the chord admissibility condition is satisfied at time T.

Since every inflection point w_s of f has polynomial degeneracy, if f'' changes sign at w_s, then there exists $\delta_s > 0$ such that f'' is monotone in $(w_s - \delta_s, w_s + \delta_s)$. Consider $\delta', \varepsilon' > 0$ given by Corollary 1.15 and apply Lemma 4.16 with

$$\delta < \left(\frac{\delta'}{2} \wedge \min_{s=1,\dots,S-1} \delta_s\right).$$

Taking into account the possible jumps of $f' \circ u(T)$ at the points $\mathtt{X}(T, y_m)$ for $m = 1, \dots, M(u_0)$ we have that

$$\mathrm{TV}(f' \circ u(T)) \leq \sum_{m=1}^{M(u_0)-1} \mathrm{TV}_{(\mathtt{X}(T,y_m)\mathtt{X}(T,y_{m+1}))}(f' \circ u(T)) + M\|f''\|_\infty \|u_0\|_\infty, \tag{4.37}$$

where M and $y_1, \dots, y_{M(u_0)}$ are given by Lemma 4.16. By the choice of δ, it holds in particular that for every $m = 1, \dots M(u_0) - 1$, there exists at most one inflection point w_s of f such that

$$w_s \in u(\bar{t}, (\mathtt{X}(\bar{t}, y_m), \mathtt{X}(\bar{t}, y_{m+1}))). \tag{4.38}$$

We can therefore distinguish the following cases:

(1) there exists no s such that (4.38) holds;
(2) there exists $s = 1, \dots, S$ such that (4.38) holds and f' does not change sign at w_s;
(3) there exists $s = 1, \dots, S$ such that (4.38) holds and f' changes sign at w_s.

Now we check that in Case (1) and in Case (2) we can apply Proposition 4.14 (or its obvious version in the concave case). Let

$$y^- := \max\{y : \mathtt{X}(T, y) = \mathtt{X}(T, y_m)\} \qquad \text{and} \qquad w^- := \lim_{x \to \mathtt{X}(T,y_m)^+} u(T, x).$$

By the stability of the notion of admissible boundary (Proposition 1.29), we have that $(\mathtt{X}(\cdot, y^-), w^-)$ is an admissible boundary of u. Similarly, if we let

$$y^+ := \min\{y : \mathtt{X}(T, y) = \mathtt{X}(T, y_{m+1})\} \qquad \text{and} \qquad w^+ := \lim_{x \to \mathtt{X}(T,y_{m+1})^-} u(T, x),$$

we have that $(\mathtt{X}(\cdot, y^+), w^+)$ is an admissible boundary of u. Therefore we can apply Proposition 4.14 and we get that

$$\mathrm{TV}_{(\mathtt{X}(T,y_m),\mathtt{X}(T,y_{m+1}))}(f' \circ u(T)) \leq 5\|f''\|_\infty \|u_0\|_\infty + 2\frac{\mathtt{X}(T, y_{m+1}) - \mathtt{X}(T, y_m)}{T - \bar{t}}. \tag{4.39}$$

The same argument shows that in Case (3) we can apply Proposition 4.15 and it implies that there exists a constant $C > 0$ depending on $f, \|u_0\|_\infty, \mathcal{L}^1(\mathrm{conv}(\operatorname{supp} u_0))$

such that

$$\mathrm{TV}_{(\mathtt{X}(T,y_m),\mathtt{X}(T,y_{m+1}))}(f'\circ u(T)) \le C\left(1+\frac{1}{T-\bar t}\right). \tag{4.40}$$

By finite speed of propagation

$$\begin{aligned}\sum_{m=1}^{M(u_0)-1}(\mathtt{X}(T,y_{m+1})-\mathtt{X}(T,y_m)) &= \mathtt{X}(T,y_{M(u_0)})-\mathtt{X}(T,y_1)\\ &\le \mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u(T)))\\ &\le \mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u_0)) + 2\|f'\|_\infty T.\end{aligned}$$

Therefore choosing $\bar t = T/2$ and combining (4.39), (4.40) and (4.37), we get that there exists a constant C depending on $f, \|u_0\|_\infty, \mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u_0))$ such that

$$\mathrm{TV} f'\circ u(T) \le C\left(1+\frac{1}{T}\right),$$

and this concludes the proof. □

We conclude this section with the following remark.

Remark 4.18. Proposition 4.15 requires the polynomial degeneracy assumption of f at the inflection point; on the contrary Proposition 4.14 does not. Moreover the structure of characteristics described in Lemma 4.8 holds for every strictly convex flux f. In particular Theorem 4.17 holds under the following assumptions on the flux: there exists $w_1 < \ldots < w_S$ such that $f\llcorner(w_s, w_{s+1})$ is strictly convex or strictly concave for every $s = 1,\dots,S-1$ and that f has polynomial degeneracy at w_s for every $s = 1,\dots,S$.

4.4. Fractional BV regularity of the solution

In this section we want to deduce a $\mathrm{BV}^{1/p}$ regularity result of the solution u from the BV regularity of $f'\circ u$ obtained in Section 4.3, where p is the degeneracy of f.

We briefly describe the argument. If the flux is strictly convex, then the polynomial degeneracy of f implies an Hölder type estimate for $(f')^{-1}$:

$$(b-a)^{p-1} \le C|f'(b)-f'(a)| \tag{4.41}$$

for some $C>0$ and this is sufficient to conclude. Of course (4.41) does not hold for general fluxes f of polynomial degeneracy, but it holds for every $a<b$ for which f has no inflection points in (a,b), (Lemma 4.19). This is sufficient to conclude the proof for continuous solutions.

It remains to consider jumps. As before we distinguish among big and small jumps: big jumps are treated as in Section 4.3 by means of Theorem 4.1 and small jumps around the inflection point between two different values w, w' with $f'(w)\simeq f'(w')$ are excluded by the entropy admissibility condition (Lemma 4.20 and Figure 4.6).

Lemma 4.19. *Let $g:[0,M]\to\mathbb{R}$ be a smooth function and $p\ge 2$ be an integer such that*

$$g'\neq 0 \text{ in } (0,M], \qquad g^{(j)}(0)=0 \text{ for } j=1,\dots,p-1 \qquad \text{and} \qquad g^{(p)}(0)\neq 0.$$

Then for every $l\ge p$ there exists a constant $C>0$ such that for every $0\le a\le b\le M$,

$$(b-a)^l \le C|g(b)-g(a)|.$$

Proof. The result easily follows for $l=p$ and hence for every $l\ge p$ by Taylor expansion in a right neighborhood $[0,\delta)$ of zero and by the fact that

$$\min_{[\delta,M]}|g'|>0.$$

□

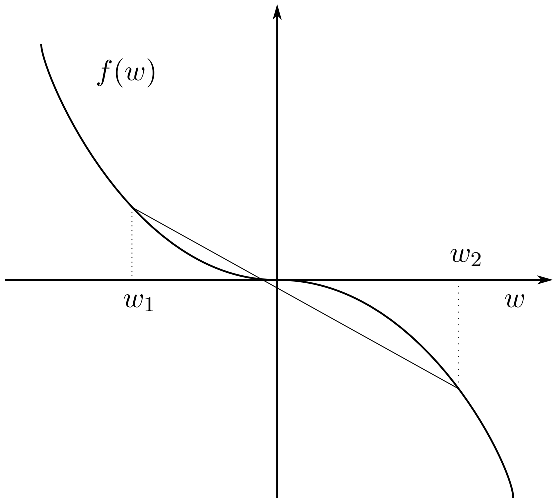


FIGURE 4.6. If $f'(w_1) \approx f'(w_2)$ with $w_1 < \bar{w} < w_2$ the shocks between w_1 and w_2 are not admissible.

LEMMA 4.20. *Let u be an entropy solution of (1) with $u_0 \in L^\infty$ and f of degeneracy $p \in \mathbb{N}$ and let $\bar{t} > 0$ be generic. Then there exist two constants $c, \delta' > 0$, depending on f and $\|u_0\|_\infty$, such that for every x_1, x_2 with*

$$w_s - \delta' < u(\bar{t}, x_1) < w_s < u(\bar{t}, x_2) < w_s + \delta'$$

for some $s = 1, \dots, S$, it holds

$$\mathrm{TV}_{\mathcal{I}(x_1,x_2)}(f' \circ u(\bar{t})) \geq c|u(\bar{t}, x_2) - u(\bar{t}, x_1)|^p,$$

where $\mathcal{I}(x_1, x_2) = (x_1, x_2) \cup (x_2, x_1)$ denotes the open interval with endpoints x_1 and x_2.

PROOF. We assume for simplicity that $w_s = 0$ and $f'(0) = 0$. Moreover it is not restrictive to assume that $x_1 < x_2$. Let δ, ε be given by Corollary 1.15. Then

$$|w| < \delta \qquad \Rightarrow \qquad \frac{|w^*|}{|w|} \in (0, 1 - \varepsilon).$$

Therefore there exists $\tilde{c} > 0$ such that if $|w| < \delta$, then

$$|f'(w) - f'(w^*)| \geq \tilde{c}|w|^p \qquad \text{and in particular} \qquad |f'(w)| \geq \tilde{c}|w|^p.$$

We distinguish three cases:

(1) there exists $\bar{x} \in (x_1, x_2)$ such that $u(\bar{t}, \bar{x}) = 0$;
(2) there exists $\bar{x} \in (x_1, x_2)$ such that $u(\bar{t}, \bar{x}) \notin (-2\delta, 2\delta)$.
(3) there exists $\bar{x} \in (x_1, x_2)$ such that

$$-2\delta < u(\bar{t}, \bar{x}+) < 0 < u(\bar{t}, \bar{x}-) < 2\delta;$$

Case (1): it holds

$$\begin{aligned}
\mathrm{TV}_{[x_1,x_2]}(f' \circ u(\bar{t})) &\geq |f'(u(\bar{t}, x_1)) - f'(u(\bar{t}, \bar{x}))| + |f'(u(\bar{t}, \bar{x})) - f'(u(\bar{t}, x_2))| \\
&= |f'(u(\bar{t}, x_1))| + |f'(u(\bar{t}, x_2)))| \\
&\geq \tilde{c}(|u(\bar{t}, x_1)|^p + |u(\bar{t}, x_2)|^p) \\
&\geq \tilde{c}2^{-p}|u(\bar{t}, x_1) - u(\bar{t}, x_2)|^p.
\end{aligned}$$

Case (2): similarly to the case above, it holds

$$\begin{aligned}\mathrm{TV}_{[x_1,x_2]}(f'\circ u(\bar t)) &\geq |f'(u(\bar t,x_1)) - f'(u(\bar t,\bar x))| \\ &\geq \tilde c|u(\bar t,x_1) - u(\bar t,\bar x)|^p \\ &\geq \tilde c\delta^p \\ &\geq \tilde c 2^{-p}|u(\bar t,x_1) - u(\bar t,x_2)|^p.\end{aligned}$$

Case (3): suppose additionally that $\max\{|u(\bar t,x_1)|,|u(\bar t,x_2)|\} \leq 2|u(\bar t,\bar x+)|$. Then

$$\begin{aligned}\mathrm{TV}_{[x_1,x_2]}(f'\circ u(\bar t)) &\geq |f'(u(\bar t,\bar x+)) - f'(u(\bar t,\bar x-))| \\ &\geq |f'(u(\bar t,\bar x+)) - f'(u(\bar t,\bar x+)^*)| \\ &\geq \tilde c|u(\bar t,\bar x+)|^p \\ &\geq \tilde c 2^{-p}\max\{|u(\bar t,x_1)|,|u(\bar t,x_2)|\}^p \\ &\geq \tilde c 4^{-p}|u(\bar t,x_1) - u(\bar t,x_2)|^p.\end{aligned}$$

If instead for definitness $|u(\bar t,x_1)| = \max\{|u(\bar t,x_1)|,|u(\bar t,x_2)|\} \geq 2|u(\bar t,\bar x+)|$, then

$$\begin{aligned}\mathrm{TV}_{[x_1,x_2]}(f'\circ u(\bar t)) &\geq |f'(u(\bar t,\bar x+)) - f'(u(\bar t,x_1))| \\ &\geq \tilde c|u(\bar t,\bar x+) - u(\bar t,x_1)|^p \\ &\geq \tilde c 2^{-p}|u(\bar t,x_1)|^p \\ &\geq \tilde c 4^{-p}|u(\bar t,x_1) - u(\bar t,x_2)|^p.\end{aligned}$$

Setting $c = \tilde c 4^{-p}$, the lemma is proved. □

The main result of this section is stated in the following theorem.

THEOREM 4.21. *Let u be the entropy solution of (1) with $u_0 \in L^\infty(\mathbb{R})$ nonnegative and with compact support and let p the degeneracy of the flux f. Then for every $t > 0$ the solution*

$$u(t) \in \mathrm{BV}^{1/p}(\mathbb{R}),$$

and there exists a constant $C > 0$ depending on $f, \|u_0\|_\infty$ and $\mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u_0))$ such that for every $t > 0$

$$\left(\mathrm{TV}^{1/p}u(t)\right)^p \leq C\left(1 + \frac{1}{t}\right).$$

PROOF. By lower semicontinuity of the TV^p, it suffices to prove the estimate for a dense set of $t > 0$; in particular we can assume that the chord admissibility condition is satisfied at time t. Let $\delta > 0$ be given so that the conclusion of Lemma 4.20 holds. By Theorem 4.1 there exists a constant $\bar N = \bar N(\mathcal{L}^1(\mathrm{conv}(\mathrm{supp}\, u_0)), \delta, \|u_0\|_\infty, f)$ such that for every $x_1 < \ldots < x_m$,

$$\#\{i : |u(t,x_{i+1}) - u(t,x_i)| \geq \delta\} \leq \bar N\left(1 + \frac{1}{t}\right).$$

So

$$\sum_{i=1}^{m-1} |u(t,x_{i+1}) - u(t,x_i)|^p \leq \bar N(2\|u_0\|_\infty)^p + \sum_{I_\delta} |u(t,x_{i+1}) - u(t,x_i)|^p,$$

where $I_\delta = \{i : |u(t,x_{i+1}) - u(t,x_i)| < \delta\}$. If $f'' \neq 0$ in $\mathcal{I}(u(t,x_{i+1}), u(t,x_i))$, by Lemma 4.19 with $g = f'$, we get

$$|u(t,x_{i+1}) - u(t,x_i)|^p \leq C|f'(u(t,x_{i+1})) - f'(u(t,x_i))|.$$

Similarly if $w_s \in \mathcal{I}(u(t,x_{i+1}), u(t,x_i))$ for some $s = 1, \ldots, S$, by Lemma 4.20,

$$\mathrm{TV}_{[x_1,x_2]}(f'\circ u(t)) \geq c|u(t,x_2) - u(t,x_1)|^p.$$

Finally

$$\sum_{i=1}^{m-1} |u(t, x_{i+1}) - u(t, x_i)|^p \leq \bar{N} \left(1 + \frac{1}{t}\right) (2\|u_0\|_\infty)^p + \tilde{C}\mathrm{TV}(f' \circ u(t)),$$

where $\tilde{C} = \max\{C, 1/c\}$. The conclusion follows by Theorem 4.17. □

4.5. SBV regularity of $f' \circ u$

In this section we prove that, under the only smoothness assumption on the flux f, the BV regularity of $f' \circ u$ can be improved to SBV regularity. The proof is based on the decomposition

$$\mathbb{R}^+ \times \mathbb{R} = A \cup B \cup C,$$

obtained in Section 3.1 and the argument in [ADL04].

For every $t > 0$ we denote by A_t, B_t, C_t the time sections

$$A_t := \{x : (t, x) \in A\}, \qquad B_t := \{x : (t, x) \in B\}, \qquad C_t := \{x : (t, x) \in C\}.$$

Proposition 4.22. *Let the flux f be smooth and let u be f the entropy solution u to (1) with $u_0 \in L^\infty$. Denote by*

$$\mathcal{B} := \{t \in (0, +\infty) : f' \circ u(t) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})\},$$
$$\mathcal{S} := \{t \in (0, +\infty) : f' \circ u(t) \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R})\}.$$

Then $\mathcal{B} \setminus \mathcal{S}$ is at most countable.

Proof. By finite speed of propagation, it is not restrictive to assume that $\operatorname{supp} u_0 \subset [a, b]$ for some $a, b \in \mathbb{R}$. Moreover we consider a representative $\bar{u}$ of u as in Corollary 3.6. For every $t > 0$, we set

$$F(t) := \mathcal{L}^1 \left(\{\mathtt{X}(0, y) \in [a, b] : \mathtt{X}(t, y) \in C_t\}\right).$$

Observe that F is decreasing. We are going to prove that if $t \in \mathcal{B} \setminus \mathcal{S}$, then $F(t+) < F(t-)$ and this easily implies the claim. Assume that $t \in \mathcal{B} \setminus \mathcal{S}$; denote by $v := f' \circ \bar{u}(t)$ and by μ the Cantor part of Dv. By the structure of the solution in B, and the fact that A_t is at most countable, we have that the measure μ is concentrated on C_t. Moreover, as already observed in the proof of Proposition 4.14, for every $x_1 < x_2$ in C_t it holds the one-sided Lipschitz estimate

$$v(x_2) - v(x_1) \leq \frac{x_2 - x_1}{t}.$$

Therefore μ is a negative measure. Fix $\varepsilon \in \left(0, \frac{1}{3}\right)$; since μ is negative and it is singular to $\mathcal{L}^1 + |Dv - \mu|$, by Besicovitch differentation theorem, there exists $E \subset \mathbb{R}$ such that

(1) μ is concentrated on E;
(2) $\mathcal{L}^1(E) = 0$;
(3) for every $x \in E$ there exists two sequences $z_i^1(x) \to x$ and $z_i^2(x) \to x$ with $z_i^1(x) < x$, $z_i^2(x) > x$ and such that for every $i \geq 1$,

$$\begin{aligned} v(z_i^1(x)) - v(x) &\geq (1 - \varepsilon)|Dv|([z_i^1(x), x]), \\ v(x) - v(z_i^2(x)) &\geq (1 - \varepsilon)|Dv|([x, z_i^2(x)]). \end{aligned} \tag{4.42}$$

Observe that since μ has no atoms, up to removing a countable set from E, we can assume that the sequences z_i^1 and z_i^2 are contained in C_t.

The next step is to give a lower bound on $\mathcal{L}^1(\mathtt{X}(0, y)) : \mathtt{X}(t, y) \in E\}$, see Figure 4.7a. Denote by

$$Y := \{y : \mathtt{X}(t, y) \in E\} \qquad \text{and} \qquad \nu := \mathtt{X}(t, \cdot)_\sharp \left(\mathcal{L}^1 \llcorner \mathtt{X}(0, \mathbb{R} \setminus Y)\right);$$

since μ is concentrated on E, it holds $\mu \perp \nu$. Therefore, by Besicovitch covering theorem, there exist $x_1, \ldots, x_N \in E$ and $a_n := z_i^1(x_n)$, $b_n := z_j^2(x_n)$ for some $i, j \geq 1$ such that $([a_n, b_n])_{n=1}^N$ is a pairwise disjoint family of intervals and

$$|\mu| \left(\bigcup_{n=1}^N [a_n, b_n] \right) \geq (1-\varepsilon)\|\mu\|, \qquad \nu \left(\bigcup_{n=1}^N [a_n, b_n] \right) \leq t\varepsilon\|\mu\|. \tag{4.43}$$

Since $a_n, b_n \in C_t$ for every $n = 1, \ldots, N$, it holds

$$y(t, b_n) - y(t, a_n) = b_n - a_n + t(v(a_n) - v(b_n)) > t(v(a_n) - v(b_n)).$$

Moreover, by (4.42), we have $t(v(a_n) - v(b_n)) > (1-\varepsilon)|\mu|([a_n, b_n])$. Set

$$U := \mathtt{X}(t)^{-1} \left(\bigcup_{n=1}^N [a_n, b_n] \right).$$

By (4.43), summing on $n = 1, \ldots, N$, we get

$$\mathcal{L}^1(U) \geq t(1-\varepsilon) \sum |\mu_t|([a_n, b_n]) \geq t(1-\varepsilon)^2 \|\mu\| \qquad \text{and} \qquad \mathcal{L}^1(U \setminus \mathtt{X}(0, Y)) < t\varepsilon\|\mu\|.$$

Therefore we have

$$\mathcal{L}^1(\mathtt{X}(0, Y)) \geq t(1 - 3\varepsilon)\|\mu\|. \tag{4.44}$$

Then we conclude by the following geometrical observation: let $\tilde{Y} \subset \mathbb{R}$ be such that

$$\mathcal{L}^1(\mathtt{X}(0, \tilde{Y})) > 0, \qquad \text{and} \qquad \mathcal{L}^1(\mathtt{X}(t, \tilde{Y})) = 0.$$

Let $\tau > t$ and consider the set $\tilde{Y}(\tau)$ of points $y \in \tilde{Y}$ such that $\mathtt{X}(\cdot, y)$ has constant speed in $[0, \tau]$; then $\mathcal{L}^1(\mathtt{X}(0, \tilde{Y}(\tau))) = 0$.

This follows from the monotonicity of the map $\mathtt{X}$, see Figure 4.7b. Indeed for any $y_1 < y_2$ in $\tilde{Y}(\tau)$, since $\mathtt{X}(0, y_1) < \mathtt{X}(0, y_2)$, we have

$$\begin{aligned}
(\mathtt{X}(0, y_2) - \mathtt{X}(0, y_1))(\tau - t) &= (\mathtt{X}(\tau, y_2) - \mathtt{X}(\tau, y_1))(\tau - t) - (\partial_t \mathtt{X}(t, y_2) - \partial_t \mathtt{X}(t, y_1))(\tau - t)\tau \\
&\leq (\mathtt{X}(\tau, y_2) - \mathtt{X}(\tau, y_1))\tau - (\partial_t \mathtt{X}(t, y_2) - \partial_t \mathtt{X}(t, y_1))(\tau - t)\tau \\
&= (\mathtt{X}(t, y_2) - \mathtt{X}(t, y_1))\tau,
\end{aligned}$$

i.e. the map

$$\mathtt{X}(t, y) \mapsto \mathtt{X}(0, y)$$

is $\tau/(\tau - t)$ Lipschitz on $\tilde{Y}(\tau)$. In particular, since $\mathcal{L}^1(\mathtt{X}(t, \tilde{Y}(\tau))) = 0$, then $\mathcal{L}^1(\mathtt{X}(0, \tilde{Y}(\tau))) = 0$.

Applying this observation to our case with $\tilde{Y} = Y$ and an arbitrary $\tau > t$, we get that

$$\mathcal{L}^1(\{y \in Y : \mathtt{X}(\tau, y) \in C_\tau\}) = 0.$$

Since $\tau > t$ is arbitrary, by (4.44), we have that

$$F(t) - F(t+) \geq t(1 - 3\varepsilon)\|\mu\| > 0,$$

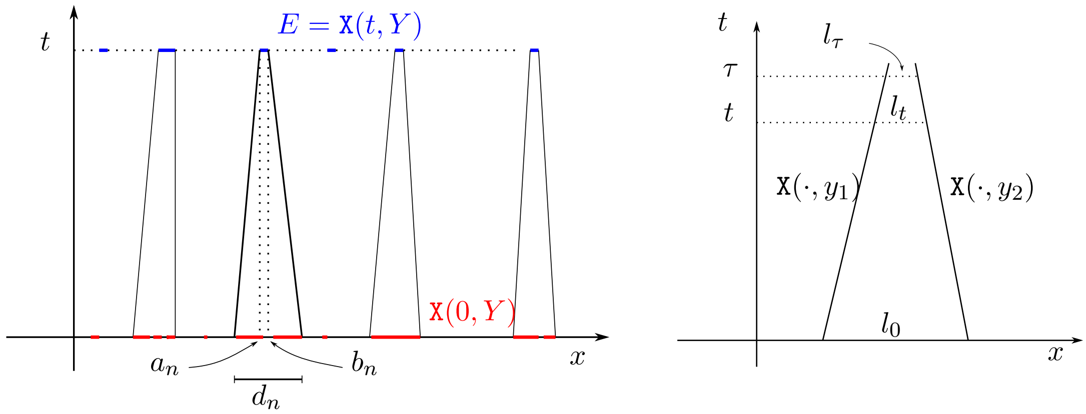
and this concludes the proof. $\square$

Corollary 4.23. *Let u be the entropy solution of (1) with $u_0 \in L^\infty$ and f of polynomial degeneracy (or more in general as in Remark 4.18). Then*

$$f' \circ u \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}).$$

Proof. By Theorem 4.17 and Proposition 4.22, it immediately follows that there exists an at most countable set $Q \subset \mathbb{R}^+$ such that for every $t \in \mathbb{R}^+ \setminus Q$,

$$f' \circ u(t) \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}).$$



(A) $d_n = b_n - a_n - tDv((a_n, b_n))$, therefore $d_n \approx t|\mu|((a_n, b_n))$ and $\mathcal{L}^1(\mathtt{X}(0, Y)) \approx |\mu|(E)$.

(B) Since $l_\tau \geq 0$, it holds $l_0 \leq \frac{\tau l_t}{\tau - t}$.

FIGURE 4.7. SBV regularity of $f' \circ u$

By slicing theory (see [AFP00]), the Cantor part of $(f' \circ u)_x$ vanishes. Moreover, denoting by μ_k^+ the dissipation measure of the entropy $\eta_k^+(w) = (w-k)^+$, we have that the velocity $f' \circ u$ satisfies the following equation:

$$f'(u)_t + \bar{q}(u)_x = \bar{\mu}, \tag{4.45}$$

where

$$\bar{q}(w) = \frac{f'(w)^2}{2} \qquad \text{and} \qquad \bar{\mu} = \int_{\mathbb{R}} \left(f^{(3)}(w)\mu_w^+ \right) dw.$$

By Volpert chain rule for functions of bounded variation $\bar{q} \circ u(t) \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R})$ for every $t \in \mathbb{R}^+ \setminus Q$; in particular the Cantor part of the measure $\bar{q}(u)_x$ vanishes. Moreover, by Theorem 3.18, $\bar{\mu}$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner J$, with J countably 1-rectifiable. In particular it has no Cantor part, therefore by (4.45), it follows that the measure $D_t(f' \circ u)$ has no Cantor part, and this concludes the proof. □

4.6. Examples

In this section we present three examples: the first is about regularity in the kinetic formulation, the second concerns BV^Φ spaces and the third shows the sharpness of Theorem 4.21.

4.6.1. $\partial_w \mu$ is not a measure. We start by briefly recalling the kinetic formulation of (1.14) (see [LPT94]): $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution of (1.14) if and only if

$$\partial_t \chi_{\{u>w\}} + f'(w)\partial_x \chi_{\{u>w\}} = \partial_w \mu,$$

where $\mu \in \mathcal{M}(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$ is the nonnegative Radon measure obtained as

$$\mu = \mathcal{L}^1 \otimes \mu_k^+$$

with μ_k^+ the dissipation of the entropy $\eta_k^+(w) = (w-k)^+$.

In [DLR03] it is proved that Theorem 4.17 implies that there exists a constant $C > 0$ such that

$$\|\partial_w^2 \mu\|_{\mathcal{M}} \leq C.$$

Then this has been used to get a refined averaging lemma and finally to deduce the rectifiability of the entropy dissipation measure.

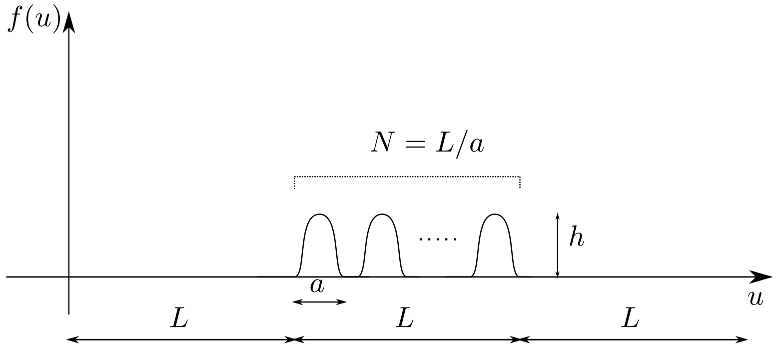


FIGURE 4.8. The flux f for the basic step of the construction.

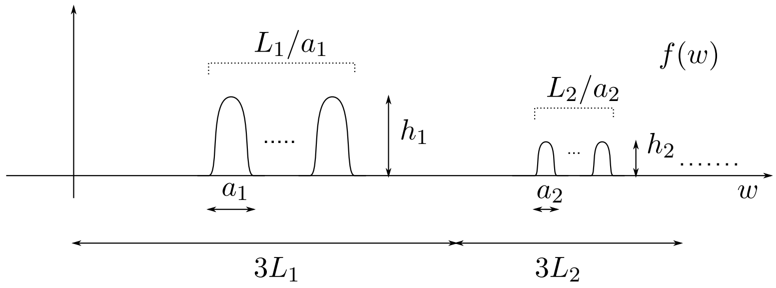


FIGURE 4.9. The flux f for the general example.

The following example shows that there exists a degenerate flux f such that even the first derivative $\partial_w \mu$ can not be represented as a Radon measure.

4.6.1.1. *Building block.* Consider a flux as in Figure 4.8.

Now we consider the initial datum

$$u_0 = 3L\chi_{[0,A]}.$$

The solution has a shock starting from 0 moving with velocity 0 between the values 0 and $3L$ that does not interact with anything for $t \in [0, AL/h]$. In particular we choose

$$A = \frac{h}{L} \qquad \text{so that} \qquad \operatorname{supp} u(t) \subset [0, 2A] \quad \text{and} \quad u(t, 0-) = 0, \quad u(t, 0+) = 3L$$

for every $t \in [0, 1]$. We compute $|\partial_w \mu|$ along the shock at $x = 0$: by standard computations,

$$\mu_k^+ \llcorner (\{0\} \times [0,1]) = (f(3L) - f(k))\mathcal{H}^1 \llcorner (\{0\} \times [0,1]) = -f(k)\mathcal{H}^1 \llcorner (\{0\} \times [0,1]),$$

therefore

$$|\partial_w \mu|(\{0\} \times [0,1]) = \int_L^{2L} |f'(w)| dw = 2\frac{hL}{a}.$$

4.6.1.2. *General example.* The flux is obtained repeating the construction above with smaller and smaller parameters and the initial datum is obtained placing side by side N_n multiples of characteristic functions at step n. See Figure 4.9 and Figure 4.10.

At step n we set $h_n = a_n^n$ so that the flux f is C^∞. In order to have an L^∞ initial datum we need

$$\sum_n L_n < \infty.$$

In order to have the initial datum with bounded support it suffices to have

$$\sum N_n \frac{a_n^n}{L_n} < \infty$$

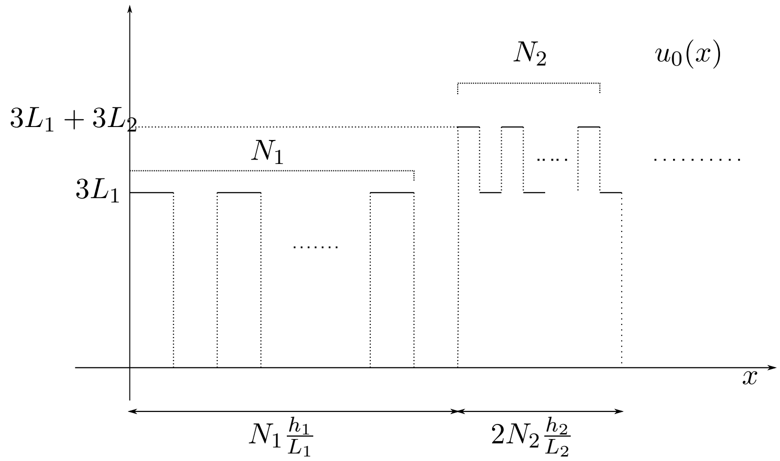


FIGURE 4.10. The initial datum u_0 for the general example.

and finally the distribution $\partial_w \mu$ is not a Radon measure if

$$\sum_n N_n a_n^{n-1} L_n = \infty.$$

A possible choice is

$$L_n = 2^{-n}, \qquad a_n = 8^{-n}, \qquad N_n = \frac{8^{(n^2)}}{4^n}.$$

4.6.2. Positive and negative fractional total variation. In this section we provide an example that proves the following Proposition.

PROPOSITION 4.24. *For every $p > 1$ there exists a function $u : [0,1] \to [0,1]$ such that*

$$\left(\mathrm{TV}_+^{1/p} u\right)^p := \sup_{\mathcal{P}([0,1])} \sum_{i=1}^{k-1} \left[(u(x_{i+1}) - u(x_i))^+\right]^p = 1$$

and

$$\left(\mathrm{TV}_-^{1/p} u\right)^p := \sup_{\mathcal{P}([0,1])} \sum_{i=1}^{k-1} \left[(u(x_{i+1}) - u(x_i))^-\right]^p = +\infty.$$

REMARK 4.25. As we already mentioned the conclusion of the proposition above cannot hold for $p = 1$. In fact the trivial relation holds:

$$\mathrm{TV}_+ u = \mathrm{TV}_- u + u(1) - u(0). \tag{4.46}$$

This proves that a bounded function with finite positive total variation is a function of bounded variation. In particular if a bounded function $u : \mathbb{R} \to \mathbb{R}$ is one-sided Lipschitz, then it has locally finite total variation. By the Oleinik estimate, this argument applies to entropy solutions to (1) with f uniformly convex. In the more general case of strictly convex fluxes with polynomial degeneracy, it holds a one-sided Hölder estimate. As observed in [**CJJ**], an analogous of (4.46) for $\mathrm{TV}^{1/p}$ would allow to apply the same argument to get fractional BV regularity. The example shows that this cannot be done, however as in [**CJJ**] for the convex case and here in Section 4.3, it is enough to rely on the BV regularity of $f' \circ u$.

Let $n \geq 0$ and C_n be the n-th step in the construction of the Cantor set:

$$C_n = \left\{ x : x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \text{ with } x_i \in \{0,2\}\, \forall i = 1, \dots, n \text{ and } x_i \in \{0,1,2\}\, \forall k > n \right\}.$$

Let $u_0 = \mathbb{I}_{\llcorner}[0,1]$ and fix $\alpha = 2^{\frac{p-1}{p}} - 1 < 1$. Define by induction u_n for $n \geq 1$:

$$u_n = \begin{cases} u_{n-1} & \text{in } [0,1] \setminus C_{n-1} \\ v_n & \text{in } C_{n-1} \end{cases}$$

where v_n is defined on each connected component $[a,b]$ of C_{n-1} as the piecewise affine interpolation between the points

$$(a, u_{n-1}(a)), \quad \left(a + \frac{b-a}{3}, u_{n-1}(a) + \frac{1+\alpha}{2}(u_{n-1}(b) - u_{n-1}(a))\right)$$
$$\left(a + \frac{2}{3}(b-a), u_{n-1}(a) + \frac{1-\alpha}{2}(u_{n-1}(b) - u_{n-1}(a))\right), \quad (b, u_{n-1}(b)).$$

See Figure 4.11 for the first two steps of the construction.

Observe that α has been chosen in such a way that

$$\mathrm{TV}_+^{1/p} u_1 = \mathrm{TV}_+^{1/p} u_0 = 1.$$

Let $n \geq 1$ and let $[a,b]$ be a connected component of C_n. A straightforward computation leads to

$$\begin{aligned} \|u_n - u_{n-1}\|_\infty &= u_n\left(\frac{2a}{3} + \frac{b}{3}\right) - u_{n-1}\left(\frac{2a}{3} + \frac{b}{3}\right) \\ &= \left(\frac{1+\alpha}{2} - \frac{1}{3}\right)(u_{n-1}(b) - u_{n-1}(a)) \\ &= \left(\frac{1+\alpha}{2} - \frac{1}{3}\right)\left(\frac{1+\alpha}{2}\right)^{n-1} \end{aligned}$$

Since $\alpha < 1$ this implies that the sequence u_n converges uniformly to a continuous function u.

We estimate from below the negative $1/p$-variation of u by considering $(\bar{x}_1, \ldots, \bar{x}_{2^{n+1}}) \in \mathcal{P}$ where $\{\bar{x}_1, \ldots, \bar{x}_{2^{n+1}}\} = \partial C_n$:

$$\sum_{i=1}^{2^{n+1}-1} [(u(x_{i+1}) - u(x_i))^-]^p = \sum_{i=1}^{2^{n+1}-1} [(u_n(x_{i+1}) - u_n(x_i))^-]^p = \sum_{j=1}^{n} 2^j \alpha^p \left(\frac{1+\alpha}{2}\right)^{jp} = n\alpha^p.$$

In particular the negative $1/p$-variation of u is not finite.

Now we prove that the positive s-variation is equal to 1. It is sufficient to prove

$$\mathrm{TV}_+^{1/p} u_n \leq 1$$

for every n by lower semicontinuity of $\mathrm{TV}_+^{1/p}$ with respect to pointwise convergence.

Since u_n is piecewise monotone and $\phi(u) = |u|^p$ is convex it is easy to show that there exits $(x_1, \ldots, x_{2k}) \in \mathcal{P}$ such that

$$\mathrm{TV}_+^{1/p} u_n = \sum_{i=1}^{2k-1} [(u_n(x_{i+1}) - u_n(x_i))^+]^p = \sum_{i=1}^{k} [u_n(x_{2i}) - u_n(x_{2i-1})]^p \tag{4.47}$$

and for every $i = 1, \ldots, 2k$

$$x_i = \bar{x}_{\sigma(i)},$$

where $\sigma : [1, 2k] \cap \mathbb{N} \to [1, 2^{n+1}] \cap \mathbb{N}$ is strictly increasing. By (4.47), it immediately follows that $\sigma(1) = 1$, $\sigma(2k) = 2^{n+1}$ and σ maps even indexes into even indexes and odd indexes into odd indexes.

We are going to prove that given $(x_1, \ldots, x_{2k}) \in \mathcal{P}$ such that (4.47) holds with $k > 1$ there exists $(y_1, \ldots, y_{2k-2}) \in \mathcal{P}$ which realizes the $\mathrm{TV}_+^{1/p} u_n$ too. $(y_1, \ldots, y_{2k-2})$ is obtained eliminating two consecutive points in $(x_1, \ldots, x_{2k})$.

We need also the following property that follows by the optimality of the partition: given $(x_1, \dots, x_{2k}) \in \mathcal{P}$ such that (4.47) holds, for every $j = 1, \dots, k-1$

$$\sigma(2j) = 2l \implies \sigma(2j+1) = 2l+1.$$

Let $\bar{j} \in [1, k-1] \cap \mathbb{N}$ such that

$$u_n(x_{2\bar{j}}) - u_n(x_{2\bar{j}+1}) = \min_{j \in [1, \dots k-1]} u_n(x_{2j}) - u_n(x_{2j+1}). \tag{4.48}$$

Claim. If $(x_1, \dots, x_{2k}) \in \mathcal{P}$ is optimal, then $(x_1, \dots, x_{2k}) \setminus (x_{2\bar{j}}, x_{2\bar{j}+1})$ is still optimal.

First we observe that iterating this argument $k-1$ times we get that (4.47) holds for $(0,1) \in \mathcal{P}$ so that

$$\mathrm{TV}_s^+ u_n \leq 1$$

and this reduces the proof of Proposition 4.24 to the proof of the claim.

The claim is a consequence of the convexity of $\phi(u) = |u|^p$, which is exploited in the following lemma.

LEMMA 4.26. *Let $w < z$ and*

$$u_1 \leq w, \quad u_2 \geq z, \quad v_1 = w + \frac{1+\alpha}{2}(z-w), \quad v_2 = w + \frac{1-\alpha}{2}(z-w).$$

Then

$$(v_1 - u_1)^p + (u_2 - v_2)^p \leq (u_2 - u_1)^p.$$

PROOF. By elementary computations,

$$\begin{aligned}
(v_1 - u_1)^p + (u_2 - v_2)^p &= \int_0^{v_1 - u_1} pt^{p-1} dt + \int_0^{u_2 - v_2} pt^{p-1} dt \\
&= \int_0^{v_1 - w} pt^{p-1} dt + \int_{v_1 - w}^{v_1 - u_1} pt^{p-1} dt \\
&\quad + \int_0^{z - v_2} pt^{p-1} dt + \int_{z - v_2}^{u_2 - v_2} pt^{p-1} dt.
\end{aligned}$$

Since $(v_1 - w)^p + (z - v_2)^p = (z - w)^p$,

$$\begin{aligned}
(v_1 - u_1)^p + (u_2 - v_2)^p &= (z - w)^p + \int_{v_1 - w}^{v_1 - u_1} pt^{p-1} dt + \int_{z - v_2}^{u_2 - v_2} pt^{p-1} dt \\
&\leq (z - w)^p + \int_{z - w}^{u_2 - u_1} pt^{p-1} dt \\
&= (u_2 - u_1)^p,
\end{aligned}$$

where the inequality holds since pt^{p-1} is increasing with respect to t. $\square$

We want to apply the previous lemma with

$$u_1 = u(x_{2\bar{j}-1}), \quad u_2 = u(x_{2\bar{j}+2}), \quad v_1 = u(x_{2\bar{j}}), \quad v_2 = u(x_{2\bar{j}+1})$$

and

$$w = u(x_{2\bar{j}} - (x_{2\bar{j}+1} - x_{2\bar{j}})), \quad z = u(x_{2\bar{j}+1} + (x_{2\bar{j}+1} - x_{2\bar{j}})).$$

The two equalities

$$v_1 = w + \frac{1+\alpha}{2}(z-w) \quad \text{and} \quad v_2 = w + \frac{1-\alpha}{2}(z-w)$$

hold by construction. Therefore it remains to check that $u_1 \leq w$ and $u_2 \geq z$. Since they are similar we prove only the first inequality: by the minimality in (4.48) it follows that $x_{2\bar{j}-1} \leq x_{2\bar{j}} - (x_{2\bar{j}+1} - x_{2\bar{j}})$ and therefore by optimality in (4.47) it follows that

$$u_1 = u(x_{2\bar{j}-1}) \leq u(x_{2\bar{j}} - (x_{2\bar{j}+1} - x_{2\bar{j}})) = w.$$

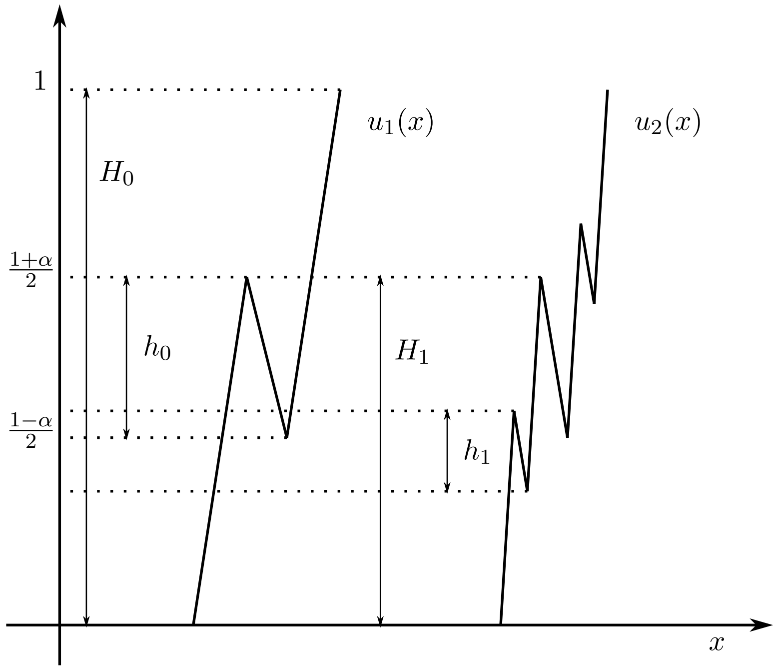


FIGURE 4.11. The first two steps of the construction of the function u in Proposition 4.24.

Hence we can apply the lemma and this implies the claim, therefore the proof of Proposition 4.24 is complete.

4.6.3. Theorem 4.21 is sharp. We show here for completeness, the already known sharpness of Theorem 4.21.

Let $p \in \mathbb{N}$ and consider the flux $f(u) = u^{p+1}$ of degeneracy p. We provide a bounded initial datum u_0 with compact support such that for every $q \in [1, p)$ the entropy solution u at time 1 does not belong to $\mathrm{BV}^{1/q}(\mathbb{R})$.

Consider first the entropy solution of (1) with $f(u) = u^{p+1}$ and $u_0 = a\chi_{[0,L]}$ for some $a, L > 0$. The solution for small $t > 0$ is given by a rarefaction starting from $x = 0$ and a shock starting from $x = L$. The maximal speed of the rarefaction is $f'(a) = (p+1)a^p$ and the the velocity λ of the shock is given by Rankine-Hugoniot:

$$\lambda = \frac{f(a) - f(0)}{a - 0} = a^p.$$

Therefore

$$t < \frac{L}{pa^p} \qquad \Longrightarrow \qquad \max u(t) = a \quad \text{and} \quad \operatorname{supp} u(t) \subset [0, L + ta^p]. \tag{4.49}$$

Consider now an initial datum of the form

$$u_0 = \sum_{n=1}^{\infty} a_n \chi_{[x_n, x_n + L_n]}.$$

Choose

$$L_n = (p+1)a_n^p > pa_n^p, \qquad x_1 = 0, \quad x_n = x_{n-1} + L_n + a_n^p.$$

Let $q \geq 1$, by the choice above and (4.49), it holds

$$\operatorname{supp} u_0 \subset \left[0, (p+2)\sum_{n=1}^{\infty} a_n^p\right], \qquad \left(\mathrm{TV}^{\frac{1}{q}} u(1)\right)^q = 2\sum_{n=1}^{\infty} a_n^q.$$

Therefore in order to conclude the example it suffices to consider a nonnegative sequence $(a_n)_{n\in\mathbb{N}} \in \ell^p \setminus \ell^q$ for every $q < p$. For example let

$$a_n = \left[\frac{1}{n[\log(1+n)]^2}\right]^{\frac{1}{p}}.$$

CHAPTER 5

A Lagrangian approach for scalar multidimensional conservation laws

Abstract. We introduce a notion of Lagrangian representation for scalar multidimensional conservation laws and we study its compactness and stability properties. In Section 5.1 the notation is fixed and some preliminary results are presented, in Section 5.2 we construct the Lagrangian representation and we prove some of its properties and finally in Section 5.3 an application to the case of continuous solution is presented. The work of this chapter is taken from [BBM17a].

5.1. Preliminaries and notations

In the following, if $f\colon X \to [0, +\infty)$ is a non-negative function defined on some set X, we will denote its *hypograph* by

$$\operatorname{hyp} f := \{(x, h) \in X \times [0, +\infty) : 0 \le h \le f(x)\}.$$

Conversely, if $U \subset X \times [0, +\infty)$ we will use the notation

$$\operatorname{hyp}^{-1}(U) = f \tag{5.1}$$

to indicate that the set U is the hypograph of the function f. The power set of X will be denoted by $\mathcal{P}(X)$.

If X is a measurable space, the space of finite measures over X will be written as $\mathscr{M}(X)$ and as usual the total variation is defined for every measurable $E \subset X$ as

$$|\mu|(E) := \sup\left\{ \sum_{i=1}^{k} |\mu(E_i)| : E_i \cap E_j = \emptyset \text{ for } i \neq j,\ \bigcup_{i=1}^{k} E_i = E \right\}.$$

The norm of a measure $\mu \in \mathscr{M}(X)$ will be written as $\|\mu\|_{\mathscr{M}} := |\mu|(X)$ and the space of non-negative measures over X will be denoted by $\mathscr{M}^+(X)$.

Often we will consider X to be the d-dimensional Euclidean space $\mathbb{R}^d$ or a suitable space of curves that will be denoted by Γ. In the former case, $\mathcal{L}^d$ will be the Lebesgue measure and $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure; in the latter, elements of the space and measures will be generically denoted by greek letters, namely we will use γ for a generic curve and ω for a measure on the space of curves. Recall also that there are natural "projection" operators defined on the space of curves, namely the *evaluation map* at time $t > 0$

$$\begin{aligned} e_t\colon \Gamma &\to \mathbb{R}^{d+1} \\ \gamma &\mapsto \gamma(t) \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} p\colon (0, +\infty) \times \Gamma &\to (0, +\infty) \times \mathbb{R}^{d+1} \\ (t, \gamma) &\mapsto \big(t, \gamma(t)\big). \end{aligned} \tag{5.3}$$

Usually, the curves we will consider are not necessarily continuous, but they enjoy BV regularity. Accordingly, we will use the symbols $\gamma(t\pm)$ for the right/left limits at t; for the derivative we will write

$$D_t\gamma = \tilde{D}_t\gamma + D_t^{\mathrm{jump}}\gamma \tag{5.4}$$

where $\tilde{D}_t\gamma$ is the *continuous* (or diffuse) part and $D_t^{\mathrm{jump}}\gamma$ is the *jump* part.

Finally, we will use the standard language of measure theory. In particular, a.e. (if not otherwise stated) refers to the Lebesgue measure. The Lebesgue spaces are denoted in the usual way L^p and the notation L^p_+ will be used for the space of non-negative functions with integrable p-power. The *essential interior* of a set $\Omega \subset \mathbb{R}^d$, $\operatorname{ess\,Int}(\Omega)$, is the set of points $x \in \mathbb{R}^d$ for which there exists a Lebesgue negligible set N such that $x \in \operatorname{Int}(\Omega \cup N)$, being Int the standard topological interior.

We now prove the technical lemmas that will be useful in the following.

LEMMA 5.1. *Let $I = [a,b] \subset \mathbb{R}$ be a closed interval in $\mathbb{R}$. Let $(D_n)_n$ be an increasing sequence of finite sets $D_1 \subset D_2 \subset \ldots \subset I$ such that their union*

$$D := \bigcup_n D_n$$

is dense in I. Let moreover $(f_n)_{n\in\mathbb{N}}$ be a sequence of maps $f_n\colon I \to X$ where (X,d) is a complete metric space. Assume that:

1. *$a \in D_1$;*
2. *there exists a compact set $K \subset X$ such that for every $n,m \in \mathbb{N}$ with $n \le m$ and for every $q \in D_n$, $f_m(q) \in K$;*
3. *there exists a constant $C > 0$ such that for every $n,m \in \mathbb{N}$ with $n \le m$, for every $q \in D_n$ and for every $x \in I$ with $q < x$, it holds*
$$d\big(f_m(q), f_m(x)\big) \le C(x-q).$$

Then there exist a subsequence $(n_k)_k$ and a C-Lipschitz function $f\colon I \to X$ such that

$$f_{n_k} \to f \qquad \textit{uniformly on } I \textit{ as } k \to +\infty.$$

PROOF. By Condition (2) and the standard diagonal argument there exists a subsequence f_{n_k}, that we will denote by f_k, which converges pointwise in D. Therefore, for every $q \in D$, the sequence $(f_k(q))_{k\in\mathbb{N}}$ is a Cauchy sequence in X. Since D_n is finite for every $n \in \mathbb{N}$, the convergence is uniform on each D_n. In particular for every $n \in \mathbb{N}$, there exists $N_n : [0,+\infty) \to \mathbb{N}$ such that for every $\varepsilon > 0$, for every $l,m \ge N_n(\varepsilon)$ and for every $q \in D_n$, it holds $d(f_l(q), f_m(q)) \le \varepsilon$.

Now we prove that actually the sequence $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to the sup-norm. Fix $\varepsilon > 0$. Then by Condition (1), the monotonicity of the sequence $(D_n)_{n\in\mathbb{N}}$ and the density of $D \subset I$ there exists $\bar{n}$ such that for every $x \in I$ there exists $q \in D_{\bar{n}}$ such that $0 < x - q < \varepsilon$. Then for every $l,m \ge \bar{n} \vee N_{\bar{n}}(\varepsilon)$, it holds

$$\begin{aligned} d(f_l(x), f_m(x)) &\le d(f_l(x), f_l(q)) + d(f_l(q), f_m(q)) + d(f_m(q), f_m(x)) \\ &\le C(x-q) + \varepsilon + C(x-q) \\ &\le (2C+1)\varepsilon. \end{aligned}$$

Therefore the sequence f_k converges uniformly to a function f. Now we check that f is C-Lipschitz. For every $x,y \in I$ with $x < y$ and for every $q \in D$ with $q < x$, it holds

$$\begin{aligned} d(f(x), f(y)) &\le d(f(x), f(q)) + d(f(q), f(y)) \\ &\le C(x-q+y-q). \end{aligned}$$

Letting $q \to x$ from below we get that f is C-Lipschitz and this concludes the proof. $\square$

We will also need the following standard result in the theory of sets of finite perimeter.

LEMMA 5.2. *Let $E \subset \mathbb{R}^d$ be a set of finite measure and of finite perimeter and let $v \in \mathbb{R}^d$ with $|v| = 1$. Then for every $\bar{t} \ge 0$ if $E_{\bar{t}v} := \{x + \bar{t}v : x \in E\}$ it holds*

$$\mathcal{L}^d(E \,\Delta\, E_{\bar{t}v}) \le 2\bar{t}\operatorname{Per}(E).$$

PROOF. By Anzellotti-Giaquinta Theorem [**AFP00**, Theorem 3.9] there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty \cap W^{1,1}(\mathbb{R}^d)$ such that $u_n \to \chi_E$ in $L^1(\mathbb{R}^d)$ and $Du^n \rightharpoonup D\chi_E$ in duality with continuous, bounded functions over $\mathbb{R}^d$ and $\|Du^n\| \to \|D\chi_E\|$. We want to compute

$$\mathcal{L}^d(E \,\Delta\, E_{tv}) = 2\int_{\mathbb{R}^d} (1-\chi_E(x))\chi_{E_{tv}}(x)\,dx.$$

Now we set

$$g_n(t) := \int_{E^c} u^n(x-tv)\,dx, \qquad g(t) := \int_{E^c} \chi_{E_{tv}}(x)\,dx.$$

For $\phi \in C_c^\infty((0,+\infty))$ we have

$$-\langle D_t g_n, \phi\rangle = \int_0^{+\infty}\int_{E^c} u^n(x-tv)\,\phi'(t)\,dx\,dt = \int_{E^c}\int_0^{+\infty} \nabla u^n(x-tv)\cdot v\,\phi(t)\,dt\,dx.$$

This shows that

$$D_t g_n = -\int_{E^c} \nabla u^n(x-tv)\cdot v\,dx.$$

In particular,

$$|D_t g_n| \le \int_{E^c} |\nabla u^n(x-tv)\cdot v|\,dx \le \|Du^n\|.$$

We thus have

$$g_n(\bar t) - g_n(0) \le \int_0^{\bar t} \|Du^n\| dt = \bar t \|Du^n\|.$$

By observing that $g_n \to g$ pointwise and using that $\|Du^n\| \to \|D\chi_E\| = \operatorname{Per} E$, we conclude the proof. $\square$

5.2. Lagrangian representation

We consider *scalar multidimensional conservation laws*, i.e. first order partial differential equations of the form

$$\partial_t u + \operatorname{div}_x(\mathbf{f}(u)) = 0 \qquad \text{in } (0,+\infty)\times\mathbb{R}^d, \tag{5.5}$$

where $u\colon (0,+\infty)\times\mathbb{R}^d \to \mathbb{R}$ is a scalar function and $\mathbf{f}\colon \mathbb{R}\to\mathbb{R}^d$ is a smooth map, called the *flux function*.

5.2.1. Definition and properties of the Lagrangian representation.

Since we only consider L^∞ solutions, up to a translation in the flux $\mathbf{f}$, we can assume $u \ge 0$. We denote by

$$\Gamma := \left\{ \gamma = (\gamma^1,\gamma^2)\colon (0,+\infty) \to \mathbb{R}^d \times [0,+\infty) : \gamma^1 \text{ is continuous and } \gamma^2 \text{ is decreasing} \right\}$$

equipped with the product of the uniform convergence on compact sets topology and of the L^1_{loc}-topology.

DEFINITION 5.3. A *Lagrangian representation* of a solution u to (5.5) is a measure $\omega \in \mathscr{M}^+(\Gamma)$ such that:

(1) it holds

$$p_\sharp(\mathcal{L}^1 \times \omega) = \mathcal{L}^{d+2} \llcorner \operatorname{hyp} u, \tag{5.6}$$

where we recall p is the projection map defined in (5.3);

(2) ω is concentrated on the set of curves $\gamma = (\gamma^1,\gamma^2) \in \Gamma$ such that

$$\begin{cases} \dot\gamma^1(t) = \mathbf{f}'(\gamma^2(t)) & \mathcal{L}^1\text{-a.e. } t \in [0,+\infty), \\ \dot\gamma^2 \le 0 \ \text{ in the sense of distributions.} \end{cases} \tag{5.7}$$

The following lemma shows that the condition expressed in (5.6) is equivalent to its pointwise version.

LEMMA 5.4. *Assume that $t \mapsto u(t)$ is strongly continuous in L^1. Then in Definition 5.3, Condition (1) can be replaced with the following:*

(1') *for every $t > 0$, it holds*

$$e_{t\sharp}\omega = \mathcal{L}^{d+1} \llcorner \operatorname{hyp} u(t), \tag{5.8}$$

where we recall e_t is the evaluation map defined in (5.2).

PROOF. Condition (1') clearly implies (1). On the other hand, by Fubini, Condition (1) gives that (5.8) for $\mathcal{L}^1$-a.e. t. By exploiting the L^1-continuity in time of u, we now show that (5.8) holds indeed for every $t \in [0, +\infty)$. To do this, we write $\gamma(t) = (\gamma^1(t), \gamma^2(t))$ and we fix $\bar{t}$; we take as test function the following

$$\varphi(t, x, h) = \phi(x, h)\psi_\delta(t),$$

where $\phi\colon \mathbb{R}^{d+1} \to \mathbb{R}$ is arbitrary, $\psi_\delta\colon [0, +\infty) \to \mathbb{R}$ is a non negative smooth function, with $\operatorname{supp}\psi_\delta \subset (\bar{t}, \bar{t} + \delta)$ and $\int_{\mathbb{R}^+} \psi_\delta = 1$. Taking the limit as $\delta \to 0^+$ of (5.6) tested against φ, we have

$$\int_{\mathbb{R}^{d+1}} \phi(x, h)\, d\mathcal{L}^{d+1} \llcorner \operatorname{hyp} u(\bar{t}) = \int_\Gamma \phi(\gamma(\bar{t}+))\, d\omega,$$

where $\gamma(\bar{t}+)$ denotes the right limit (which exists because γ^1 is continuous and γ^2 is decreasing). Similarly, on the left side, we get

$$\int_{\mathbb{R}^{d+1}} \phi(x, h) d\mathcal{L}^{d+1} \llcorner \operatorname{hyp} u(\bar{t}) = \int_\Gamma \phi(\gamma(\bar{t}-))\, d\omega$$

thus, in particular,

$$0 = \int_\Gamma \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}-)\big) - \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}+)\big)\, d\omega.$$

Let us fix a compact set $K \subset \mathbb{R}^d$ and choose $\phi \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\partial_h \phi \geq 1$ in $K \times (0, \|u\|_\infty)$ and $\partial_h \phi \geq 0$ in $\mathbb{R}^d \times (0, \|u\|_\infty)$: being γ^2 decreasing, we have

$$\begin{aligned}
0 &= \int_\Gamma \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}-)\big) - \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}+)\big)\, d\omega \\
&\geq \int_{\Gamma \setminus \Gamma_K} \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}-)\big) - \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}+)\big)\, d\omega + \int_{\Gamma_K} \big(\gamma^2(\bar{t}-) - \gamma^2(\bar{t}+)\big)\, d\omega \\
&\geq \int_{\Gamma_K} |\gamma^2(\bar{t}-) - \gamma^2(\bar{t}+)|\, d\omega,
\end{aligned}$$

where $\Gamma_K \subset \Gamma$ is the set of curves such that $\gamma^1(\bar{t}) \in K$. This shows that for every $t \in (0, +\infty)$, ω-a.e. γ is continuous in t: in particular, we have $(e_t)_\sharp \omega = \mathcal{L}^{d+1} \llcorner \operatorname{hyp} u(t)$ for every t. □

We now present the following proposition, which says that Conditions (1), (2) in Definition (5.3) imply that u is an entropy solution to (5.5).

PROPOSITION 5.5. *Let $\omega \in \mathscr{M}_+(\Gamma)$ be a non-negative measure on the space of curves and assume there exists a non-negative, bounded function $u\colon (0, +\infty) \times \mathbb{R}^d \to [0, +\infty)$ such that Conditions (1), (2) of Definition 5.3 hold. Then u is an entropy solution to (5.5).*

PROOF. Let $(\eta, \mathbf{q})$ be an entropy-entropy flux pair with η convex (w.l.o.g. $\eta(0) = 0, \mathbf{q}(0) = 0$). Using the elementary identities

$$u(t, x) = \int_0^{+\infty} \chi_{[0, u(t,x)]}(h)\, dh$$

and

$$\eta(u(t,x)) = \int_0^{+\infty} \chi_{[0,u(t,x)]}(h)\eta'(h)\,dh, \qquad \mathbf{q}(u(t,x)) = \int_0^{+\infty} \chi_{[0,u(t,x)]}(h)\mathbf{q}'(h)\,dh$$

and recalling that $\mathbf{q}' = \eta'\mathbf{f}'$, we can write, for any non-negative test function $\phi \in C_c^1([0,+\infty)\times\mathbb{R}^d)$,

$$\begin{aligned}
-\langle \eta(u)_t + &\operatorname{div}_x(\mathbf{q}(u)), \phi\rangle \\
&= \int_{\mathbb{R}^d}\int_0^{+\infty} \eta(u(t,x))\phi_t(t,x) + \mathbf{q}(u(t,x))\cdot\nabla_x\phi(t,x)\,dt\,dx \\
&= \int_{\mathbb{R}^d}\int_0^{+\infty}\left[\int_0^{+\infty} \chi_{[0,u(t,x)]}(h)\eta'(h)\phi_t(t,x) + \mathbf{q}'(h)\cdot\nabla_x\phi(t,x)\,dh\right]dt\,dx \\
&= \int_{\mathbb{R}^d}\int_0^{+\infty}\int_0^{+\infty} \chi_{[0,u(t,x)]}(h)\eta'(h)\big(\phi_t(t,x) + \mathbf{f}'(h)\cdot\nabla_x\phi(t,x)\big)\,dh\,dt\,dx \\
&= \int_{\mathbb{R}^{d+2}} \eta'(h)\big(\phi_t(t,x) + \mathbf{f}'(h)\cdot\nabla_x\phi(t,x)\big)\,d\big(\mathcal{L}^{d+2}\llcorner\text{hyp } u\big).
\end{aligned}$$

By Condition 1 we have $p_\sharp(\mathcal{L}^1\times\omega) = \mathcal{L}^{d+2}\llcorner\text{hyp } u$, so that

$$\begin{aligned}
-\langle \eta(u)_t + &\operatorname{div}_x(\mathbf{q}(u)), \phi\rangle \\
&= \int_{\mathbb{R}^{d+2}} \eta'(h)\big(\phi_t(t,x) + \mathbf{f}'(h)\cdot\nabla_x\phi(t,x)\big)\,d\big(\mathcal{L}^{d+2}\llcorner\text{hyp } u\big) \\
&= \int_\Gamma\int_0^{+\infty} \eta'(\gamma^2(t))\Big(\phi_t(t,\gamma^1(t)) + \mathbf{f}'(\gamma^2(t))\cdot\nabla_x\phi(t,\gamma^1(t)\Big)\,dt\,d\omega.
\end{aligned}$$

Moreover, let us define for a.e. $t\in(0,+\infty)$ and for ω-a.e. γ the function

$$g_\gamma(t) := \eta'\big(\gamma^2(t)\big). \tag{5.9}$$

Recall that η is convex and that for ω-a.e. γ the function γ^2 is decreasing by Condition (2); thus we have that g_γ is decreasing for ω-a.e. γ. Hence it holds $g'_\gamma \le 0$ in the sense of distributions. By Fubini Theorem, we finally have

$$\begin{aligned}
-\langle \eta(u)_t + &\operatorname{div}_x(\mathbf{q}(u)), \phi\rangle \\
&= \int_\Gamma\int_0^{+\infty} \eta'(\gamma^2(t))\Big(\phi_t\big(t,\gamma^1(t)\big) + \mathbf{f}'\big(\gamma^2(t)\big)\cdot\nabla_x\phi\big(t,\gamma^1(t)\big)\Big)\,dt\,d\omega \\
&= \int_\Gamma\int_0^{+\infty} \eta'(\gamma^2(t))\Big(\phi_t\big(t,\gamma^1(t)\big) + \dot\gamma^1(t)\cdot\nabla_x\phi\big(t,\gamma^1(t)\big)\Big)\,dt\,d\omega \\
&= \int_\Gamma\int_0^{+\infty} \eta'(\gamma^2(t))\frac{d}{dt}\phi(t,\gamma^1(t))\,dt\,d\omega \\
&= \int_\Gamma\int_0^{+\infty} g_\gamma(t)\phi'_\gamma(t)\,dt\,d\omega \ge 0
\end{aligned} \tag{5.10}$$

where the last inequality comes from the distributional definition of derivative for the function g_γ, being $\phi_\gamma(t) := \phi(t,\gamma^1(t))$ an admisible, non-negative test function. Thus we have established that, for any convex entropy η, it holds in the sense of distributions

$$\eta(u)_t + \operatorname{div}_x(\mathbf{q}(u)) \le 0. \tag{5.11}$$

In particular, by taking $\eta(s) = \pm s$ and repeating the computation above, we get

$$u_t + \operatorname{div}_x(\mathbf{f}(u)) = 0. \tag{5.12}$$

Having established the two conditions (5.11) and (5.12), we have that u is by definition an entropy solution to (5.5), hence the proof is complete. ☐

This proof shows also how the dissipation measure can be decomposed along the characteristic curves. Since this fact will be useful, we fix some notation and explicit this decomposition.

Let η be a convex entropy and set

$$\mu_\gamma^\eta = (\mathbb{I},\gamma)_\sharp\Big((\eta'\circ\gamma^2)\tilde{D}\gamma^2\Big) + \eta''(h)\mathcal{H}^1\llcorner\{(t,x,h) : \gamma^1(t) = x, h \in (\gamma^2(t+),\gamma^2(t-))\}.$$

Accordingly define

$$\nu^\eta := \int_\Gamma \mu_\gamma^\eta \, d\omega. \tag{5.13}$$

Lemma 5.6. *It holds*

$$(\pi_{t,x})_\sharp \nu^\eta = \mu^\eta,$$

where the map $\pi_{t,x} \colon \mathbb{R}^d \times [0,+\infty) \times [0,+\infty) \ni (t,x,h) \mapsto (t,x) \in \mathbb{R}^d \times [0,+\infty)$ *is the projection on the* t,x *variables.*

Proof. By definition we immediately get

$$(\pi_{t,x})_\sharp(\mu_\gamma) = (\mathbb{I},\gamma^1)_\sharp(D_t g_\gamma), \tag{5.14}$$

where g_γ is defined in (5.9). Including (5.14) in (5.10) we get

$$\begin{aligned}
\langle \eta(u)_t + \mathrm{div}_x(\mathbf{q}(u)),\phi\rangle = & -\int_\Gamma\int_0^{+\infty} g_\gamma(t)\phi_\gamma'(t)\,dt\,d\omega \\
= & \int_\Gamma\int_{[0,+\infty)\times\mathbb{R}^d} \phi\, d((\pi_{t,x})_\sharp\mu_\gamma)d\omega \\
= & \int_{[0,+\infty)\times\mathbb{R}^d} \phi\, d((\pi_{t,x})_\sharp\nu^\eta),
\end{aligned}$$

where in the last inequality we used the definition of ν (5.13) and the relation

$$\int_\Gamma (\pi_{t,x})_\sharp \mu_\gamma^\eta \, d\omega = (\pi_{t,x})_\sharp\left(\int_\Gamma \mu_\gamma^\eta\, d\omega\right). \qquad \square$$

We will use the notation $\bar{\nu}$ to indicate ν^η with $\eta(u) = u^2/2$.

Proposition 5.7. *The dissipation* $\bar{\nu}$ *in the essential interior of* hyp u *is zero.*

Proof. Let $\psi : \mathbb{R}^d \times [0,+\infty) \to [0,+\infty)$ such that for every $t \in (t_1,t_2)$, $\mathrm{supp}\,\psi \subset \mathrm{ess\,Int}(\mathrm{hyp}\ u(t), \mathbb{R}^d \times [0,+\infty))$, then

$$t \mapsto \int_{\mathbb{R}^{d+1}} \psi(x,h)\, d(e_t)_\sharp\omega$$

is constant. Take $(\bar{t},\bar{x},\bar{h})$ in the essential interior of hyp u. Take $\psi(x,h) = \psi_1(x)\psi_2(h)$, where

$$\psi_1(x) = \sigma(|x-\bar{x}|), \qquad \partial_h\psi_2 < 0 \text{ in } [0,\bar{h}) \qquad \text{and} \qquad \psi_2(h) = 0 \text{ for } h > \bar{h},$$

where σ is smooth and nonnegative and $\sigma > 0$ in $[0,r)$, where $r \ll 1$. For every $\phi \in C_c^1((t_1,t_2))$, it holds

$$\begin{aligned}
0 = & -\int_{t_1}^{t_2}\int_{\mathbb{R}^{d+1}} \phi'(t)\psi(x,h)d(e_t)_\sharp\omega dt \\
= & \int_\Gamma\int_{(t_1,t_2)} \phi(t)d(D_t(\psi\circ\gamma))d\omega \\
= & \int_\Gamma\int_{(t_1,t_2)} \phi(t)\nabla\psi(\gamma(t))d(\tilde{D}_t\gamma) + \int_\Gamma\sum_i \phi(t_i)\left(\psi(\gamma(t_i^+)) - \psi(\gamma(t_i^-))\right)d\omega,
\end{aligned}$$

by Volpert chain rule, where $\tilde{D}_t\gamma$ is the continuous part of the derivative defined in (5.4). For every $\phi \geq 0$, and using the assumptions on ψ

$$\int_\Gamma \int_{(t_1,t_2)} \phi(t)\nabla\psi(\gamma(t))d(\tilde{D}_t\gamma)d\omega = \int_\Gamma \int_{t_1}^{t_2} \phi(t)\nabla_x\psi(\gamma(t))\cdot \mathbf{f}'(\gamma^2(t))dtd\omega + \int_\Gamma \int_{(t_1,t_2)} \phi(t)\partial_h\psi(\gamma(t))d(\tilde{D}_t\gamma^2)d\omega,$$

by splitting horizontal and vertical components. We prove that the horizontal contribution is zero.

$$\begin{aligned}\int_\Gamma \int_{t_1}^{t_2} \phi(t)\nabla_x\psi(\gamma(t))\cdot \mathbf{f}'(\gamma^2(t))dtd\omega &= \int_{\mathbb{R}^{d+1}} \int_{t_1}^{t_2} \phi(t)\nabla_x\psi(x,h)\cdot \mathbf{f}'(h)dtd\mathcal{L}^{d+1}\llcorner(\text{hyp } u(t)) \\ &= \int_{t_1}^{t_2} \phi(t)\int_0^{+\infty} \mathbf{f}'(h)\cdot \int_{B_r(\bar{x})} \nabla_x\psi(x,h)d\mathcal{L}^d dhdt \\ &= 0.\end{aligned}$$

We conclude that

$$\begin{aligned}0 &= -\int_{t_1}^{t_2} \int \phi'(t)\psi(x,h)\, d(e_t)_\sharp\omega\, dt \\ &= \int_\Gamma \int_{t_1}^{t_2} \phi(t)\partial_w\psi(\gamma(t))\, d(\tilde{D}_t\gamma^2) + \int_\Gamma \sum_i \phi(t_i)\left(\psi(\gamma(t_i^+)) - \psi(\gamma(t_i^-))\right) d\omega \\ &= \int_{\mathbb{R}^{d+2}} \phi(t)\partial_h\psi\, d\bar{\nu}.\end{aligned}$$

By arbitrariness of ϕ, ψ or by using $\bar{\nu} \leq 0$ we get $\bar{\nu} = 0$ in the interior of the hypograph. □

5.2.2. Compactness and stability of Lagrangian representations. We now turn to analyze stability properties that, in particular, will be useful in the construction of Lagrangian representations. In the following proposition, we show how the compactness of approximate solutions translates into tightness of the corresponding Lagrangian measures and how conditions (1) and (2) pass to the limit.

Actually, we present the result in the more general framework in which the push forward of the measure $\mathcal{L}^1 \times \omega$ through the evaluation map p is merely the Lebesgue measure $\mathcal{L}^{d+2}$ restricted to a set U, and not necessarily an hypograph. This allows more freedom in the construction of approximate solutions (e.g. Brenier's Transport-Collapse scheme will fit in this setting).

PROPOSITION 5.8 (Compactness and stability). *Let $(\omega^n)_{n\in\mathbb{N}} \subset \mathscr{M}_+(\Gamma)$ be a sequence of bounded measures such that Condition (2) in Definition 5.3 holds. Assume that*

$$p_\sharp(\mathcal{L}^1 \times \omega^n) = \mathcal{L}^{d+2}\llcorner U^n$$

for some set $U^n \subset \mathbb{R}^{d+2}$ and assume that there exists $M > 0$ such that $U^n \subset (0,+\infty)\times \mathbb{R}^d \times [0,M]$ for every $n \in \mathbb{N}$. Assume furthermore that

$$\chi_{U^n} \to \chi_U \qquad \text{in } L^1(\mathbb{R}^{d+2}),$$

for some set $U \subset \mathbb{R}^{d+2}$. Then $(\omega^n)_{n\in\mathbb{N}}$ is tight, every limit point ω satisfies Condition (2) in Definition 5.3 and it holds

$$p_\sharp(\mathcal{L}^1 \times \omega) = \mathcal{L}^{d+2}\llcorner U.$$

PROOF. Since ω^n satisfies Condition (2) in Definition 5.3, we have that

$$\operatorname{supp}\omega^n \subset \operatorname{Lip}((0,+\infty),\mathbb{R}^d) \times \mathcal{D}$$

with local uniform bounds, hence $(\omega^n)_n$ is locally tight. Using a diagonal argument, we construct a measure ω which is the limit of ω^n. We now show that

$$p_\sharp(\mathcal{L}^1 \times \omega) = \mathcal{L}^{d+2} \llcorner U.$$

where p is the evaluation map defined in (5.3). Indeed, let $\varphi = \varphi(t,x,h)$ be a test function; we get

$$\begin{aligned}
\int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,x,h)\, dp_\sharp(\mathcal{L}^1 \times \omega)(t,x,h) &= \int_\Gamma \int_{\mathbb{R}^+} \varphi(t,\gamma(t))\, dt\, d\omega \\
&= \int_\Gamma \Phi(\gamma) d\omega(\gamma) \\
&= \lim_n \int_\Gamma \Phi(\gamma) d\omega^n(\gamma) \\
&= \lim_n \int_\Gamma \int_{\mathbb{R}^+} \varphi(t,\gamma(t))\, dt\, d\omega^n \\
&= \lim_n \int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,x,h)\, dp_\sharp(\mathcal{L}^1 \times \omega^n) \\
&= \lim_n \int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,x,h)\, d(\mathcal{L}^{d+2} \llcorner U^n) \\
&= \int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,x,h)\, d(\mathcal{L}^{d+2} \llcorner U),
\end{aligned}$$

where we have used in the second line the continuous function

$$\Phi(\gamma) := \int_0^{+\infty} \phi(t,\gamma(t))\, dt. \qquad \square$$

We conclude this paragraph by pointing out the following corollary, whose proof can be obtained particularizing Proposition 5.8 in the case where U^n are hypographs of entropy solutions.

COROLLARY 5.9. *Let $(u^n)_{n\in\mathbb{N}}$ be a sequence of uniformly bounded entropy solutions to (5.5) and assume it is given a sequence $(\omega^n)_{n\in\mathbb{N}}$ of corresponding Lagrangian representations. If $u^n \to u$ locally in L^1, then $(\omega^n)_{n\in\mathbb{N}}$ is tight and every limit point ω is a Lagrangian representation of u.*

5.2.3. Existence of Lagrangian representations for initial data in L^∞. The compactness properties stated in Corollary 5.9 and standard approximation results imply that, in order to prove the existence of Lagrangian representations for solutions with initial data in L^∞, it is enough to construct them for solutions with bounded variation. In order to do this, we exploit a numerical scheme which was proposed by Brenier in [Bre84] and is called "transport-collapse". We consider the initial value problem

$$\begin{cases} \partial_t u + \operatorname{div}_x(\mathbf{f}(u)) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = u_0(\cdot) \end{cases} \tag{5.15}$$

with $u_0 \in L^\infty(\mathbb{R}^d) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and we denote by u the entropy solution to (5.15). As before, we assume that $u \geq 0$.

We define the following *transport* map

$$\begin{aligned}
\mathsf{Tr}\colon [0,+\infty)\times\mathbb{R}^d\times[0,+\infty) &\to \mathbb{R}^d\times[0,+\infty) \\
(t,x,h) &\mapsto (x + t\mathbf{f}'(h), h),
\end{aligned}$$

which moves a point in $\mathbb{R}^d \times [0,+\infty)$ with the characteristic speed. Observe that, in general, if $v = v(x)$ is a function of x then, for $t > 0$, the image

$$\mathsf{Tr}(t, \text{hyp } v) := \bigcup_{(x,h)\in \text{hyp } v} \mathsf{Tr}(t,x,h) \subset \mathbb{R}^d \times [0,+\infty)$$

is not necessarily an hypograph.

Then we introduce the *collapse* operator: we first define the set

$$X := \Big\{(E,x,h) \in \mathcal{P}(\mathbb{R}^d \times [0,+\infty)) \times \mathbb{R}^d \times [0,+\infty) : (x,h) \in E\Big\},$$

where we recall $\mathcal{P}$ denotes the power set and then

$$\begin{aligned} \mathsf{C}\colon X &\mapsto \mathbb{R}^d \times [0,+\infty) \\ (E,x,h) &\mapsto \big(x, \mathcal{H}^1((\{x\}\times[0,h])\cap E)\big), \end{aligned}$$

where $\mathcal{H}^1$ is the (outer) 1-dimensional Hausdorff measure. The collapse operator moves points vertically in the negative direction. Moreover the image of a set is always an hypograph (possibly taking value $+\infty$) and $\mathsf{C}(E,\cdot,\cdot)$ is the identity if and only if E is an hypograph.

We now set

$$Y := \Big\{(v,x,h) \in L^\infty_+(\mathbb{R}^d) \times \mathbb{R}^d \times [0,+\infty) : (x,h) \in \text{hyp } v\Big\}.$$

We define the transport-collapse map at time $t > 0$ in the following way:

$$\begin{aligned} \mathsf{TC}_t\colon Y &\to \mathbb{R}^d \times [0,+\infty) \\ (v,x,h) &\mapsto \mathsf{C}(\mathsf{Tr}(t,\text{hyp } v), \mathsf{Tr}(t,x,h)) \end{aligned}$$

Remark 5.10. The contruction above is only a Lagrangian rephrase of the Transport-Collapse scheme proposed by Brenier in [Bre84]. There, the author defines the Transport-Collapse operator as the family of operators $\{\mathsf{T}(t)\}_{t>0}$ on $L^1(\mathbb{R}^d)$ whose restriction to the space of non-negative, integrable functions $L^1_+(\mathbb{R}^d)$ is

$$\begin{aligned} \mathsf{T}(t)\colon L^1_+(\mathbb{R}^d) &\to L^1_+(\mathbb{R}^d) \\ v &\mapsto (\mathsf{T}(t)v)(x) := \int_{\mathbb{R}} jv(x - t\mathbf{f}'(h), h)\, dh \end{aligned}$$

where

$$jv(x,h) := \chi_{\text{hyp } v}(x,h) = \begin{cases} 1 & \text{if } 0 < h < v(x), \\ 0 & \text{else.} \end{cases}$$

The link between the two formulations is the following:

$$\text{hyp } (\mathsf{T}(t)v) = \mathsf{TC}_t(v, \text{hyp } v).$$

On the other hand, the map TC_t chooses the image of *each* point in the hypograph and not only the image of the whole hypograph (see Figure 5.1) .

We are now in position to define an approximating sequence (TC^n_t) of the Kruzkov semigroup. We define first them inductively for $t \in 2^{-n}\mathbb{N}$:

$$\begin{cases} \mathsf{TC}^n_0(v,x,h) = (x,h), \\ \mathsf{TC}^n_{(k+1)\cdot 2^{-n}}(v,x,h) = \mathsf{TC}_{2^{-n}}\big(\text{hyp}^{\,-1}(\mathsf{TC}^n_{k\cdot 2^{-n}}(v,\text{hyp } v)), \mathsf{TC}^n_{k\cdot 2^{-n}}(v,x,h)\big), \end{cases}$$

where $\text{hyp}^{-1}(\cdot)$ is defined in (5.1).

For the intermediate times $t = s + k\cdot 2^{-n}$, with $s \in (0, 2^{-n})$, we set

$$\mathsf{TC}^n_t := \mathsf{Tr}(s) \circ \big(\mathsf{TC}^n_{k\cdot 2^{-n}}\big).$$

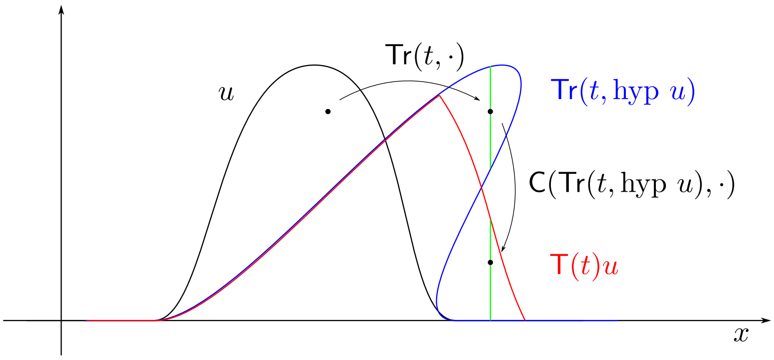


FIGURE 5.1. Picture of the transport collapse scheme.

Taking now $u_0 \in L^\infty(\mathbb{R}^d) \cap \mathrm{BV}(\mathbb{R}^d)$, we define accordingly for every $(x,h) \in \text{hyp } u_0$ and for every $t > 0$,

$$\gamma^n_{(x,h)}(t) := \mathsf{TC}^n_t(u_0, x, h),$$

and we set

$$\omega^n := \int_{\text{hyp } u_0} \delta_{\gamma^n_{(x,h)}} \, dx \, dh. \tag{5.16}$$

Since the Transport Collapse scheme is measure preserving, there exists $U^n \subset [0,+\infty) \times \mathbb{R}^d \times [0,+\infty)$ such that

$$(e_t)_\sharp \omega^n = \mathcal{L}^d \llcorner U^n(t), \tag{5.17}$$

where

$$U^n(t) := \big\{(x,h) \in \mathbb{R}^d \times [0,+\infty) : (t,x,h) \in U\big\}.$$

5.2.3.1. *Total variation along Transport-Collapse.* A crucial property in [Bre84] is that the total variation decreases along the Transport-Collapse scheme. This is indeed stated and proved in the following lemma and we present the proof for the sake of completeness.

LEMMA 5.11. *For every $t \geq 0$ and $u \in L^1_+(\mathbb{R}^d)$ it holds*

$$\mathrm{TV}(\mathsf{T}(t)u) \leq \mathrm{TV}(u).$$

PROOF. For every $t \geq 0$, for any test vector field $\Phi \in C^1_c(\mathbb{R}^d; \mathbb{R}^d)$, with $\|\Phi\|_\infty \leq 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} (\mathsf{T}(t)u)(x) \operatorname{div} \Phi(x) \, dx &= \int_{\mathbb{R}^d} \int_0^{+\infty} ju(x - t\mathbf{f}'(h), h) \operatorname{div} \Phi(x) \, dh \, dx \\
&= \int_{\mathbb{R}^d} \int_0^{+\infty} ju(x,h) \operatorname{div} \Psi_h(x) \, dh \, dx \\
&\leq \int_0^{+\infty} \mathrm{TV}(ju(\cdot,h)) \, dh,
\end{aligned}$$

where we have set $\Psi_h(x) = \Phi(x + t\mathbf{f}'(h))$ and the last inequality holds by definition of total variation (together with the trivial fact that $\|\Psi_h\|_\infty \leq 1$). Finally, by coarea formula, we have

$$\int_0^{+\infty} \mathrm{TV}(ju(\cdot,h)) dh = \mathrm{TV}(u).$$

Being Φ arbitrary, the proof is complete. $\square$

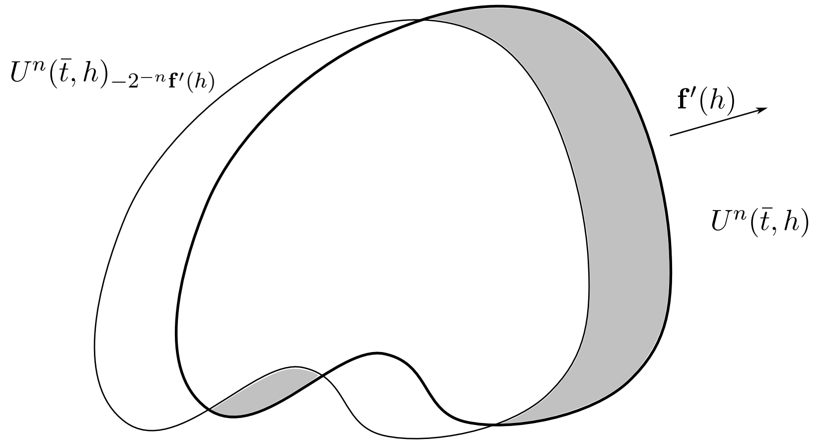


FIGURE 5.2. The set in grey is $U^n(\bar t, h) \cap \big(U^n(\bar t, h)_{-2^{-n}\mathbf{f}'(h)}\big)^c$.

5.2.3.2. *Passage to the limit of Transport-Collapse.* In this section we give an alternative proof of the fact that the iterated Transport-Collapse scheme converges to the Kruzkov semigroup, based on the Lagrangian representation. As a byproduct, we obtain the existence of Lagrangian representations for BV initial data and, as already noticed, this suffices for the general L^∞ case.

Let us also fix $D_n := \{\frac{k}{2^n} : k \in \mathbb{N}_{\geq 0}\}$ so that for every $\bar t \in D_n$ there exists $u^n(\bar t) \in L^\infty(\mathbb{R}^d)$ such that

$$U^n(\bar t) = \mathrm{hyp}\ u^n(\bar t).$$

The key point to prove the compactness of the family $(U^n)_{n\in\mathbb{N}}$ is contained in the following lemma.

LEMMA 5.12. *Let $\bar n \in \mathbb{N}$ and $\bar t \in D_{\bar n}$. Then for every $t > \bar t$ and for every $n \geq \bar n$, it holds*

$$\|(e_t)_\sharp\omega^n - (e_{\bar t})_\sharp\omega^n\|_{\mathscr{M}} = \mathcal{L}^{d+1}\big(U^n(t)\,\Delta\,U^n(\bar t)\big) \leq 2\|f'\|_\infty\big(t-\bar t\big)\mathrm{TV}(u_0). \tag{5.18}$$

PROOF. Let us now write $t - \bar t = k \cdot 2^{-n} + s$ for $s \in [0, 2^{-n})$. For $j = 0, \dots, k-1$ set

$$I_j := [t_{j,n}, t_{j+1,n}], \qquad \text{where} \qquad t_{j,n} := \bar t + j2^{-n}.$$

Observe that it holds

$$\mathcal{L}^{d+1}\big(U^n(t)\,\Delta\,U^n(\bar t)\big) = 2\omega^n\left(\{\gamma : \gamma(\bar t) \in U^n(\bar t), \gamma(t) \notin U^n(\bar t)\}\right).$$

Being $U(\bar t)$ the hypograph of $u^n(\bar t)$, for every $j = 0, \dots, k-1$ and $\gamma \in \operatorname{supp}\omega^n$,

$$\gamma(t_{j,n}-) \in U^n(\bar t) \qquad \Longrightarrow \qquad \gamma(t_{j,n}+) \in U^n(\bar t). \tag{5.19}$$

For any $j = 0, \dots, k-1$ we set

$$\mathcal{G}_{j,n} := \Big\{\gamma \in \operatorname{supp}\omega^n : \gamma(t_{j,n}+) \in U^n(\bar t), \gamma(t_{j+1,n}-) \notin U^n(\bar t)\Big\}.$$

Finally, if $s = 0$ we set $\mathcal{G}_k = \emptyset$ and if $s > 0$,

$$\mathcal{G}_{k,n} := \Big\{\gamma \in \operatorname{supp}\omega^n : \gamma(t_{k,n}+) \in U^n(\bar t), \gamma(t) \notin U^n(\bar t)\Big\}.$$

By (5.19), it holds

$$\big\{\gamma : \gamma(\bar t) \in U^n(\bar t), \gamma(t) \notin U^n(\bar t)\big\} \subset \bigcup_{j=0}^{k} \mathcal{G}_{j,n}.$$

Let us fix $j = 0, \dots, k-1$. By (5.17) and definition of ω^n,

$$\begin{aligned}\omega^n(\mathcal{G}_{j,n}) &= \mathcal{L}^{d+1}\left(\left\{(x,h) \in U^n(\bar{t}) \cap U^n(t_{j,n}) : (x + \mathbf{f}'(h)2^{-n}, h) \notin U^n(\bar{t})\right\}\right)\\ &= \int_0^{\|u_0\|_\infty} \mathcal{L}^d\left(\left\{x \in U^n(\bar{t},h) \cap U^n(t_{j,n},h) : x + \mathbf{f}'(h)2^{-n} \notin U^n(\bar{t},h)\right\}\right) dh,\end{aligned} \tag{5.20}$$

where we have set $U(t,h) := \{x : (t,x,h) \in U\}$ and used Fubini theorem. Now we observe that

$$\left\{x \in U^n(\bar{t},h) \cap U^n(t_{j,n},h) : x + \mathbf{f}'(h)2^{-n} \notin U^n(\bar{t},h)\right\} \subset U^n(\bar{t},h) \cap (U^n(\bar{t},h)_{-2^{-n}\mathbf{f}'(h)})^c,$$

where we recall that $E_{\mathbf{v}} := E + \mathbf{v}$ (see Figure 5.2). Since

$$\mathcal{L}^d\left(U^n(\bar{t},h) \cap (U^n(\bar{t},h)_{-2^{-n}\mathbf{f}'(h)})^c\right) = \frac{1}{2}\mathcal{L}^d\big(U^n(\bar{t},h)\Delta(U^n(\bar{t},h)_{-2^{-n}\mathbf{f}'(h)})\big),$$

by applying Lemma 5.2, we have

$$\mathcal{L}^d\big(U^n(\bar{t},h)\Delta(U^n(\bar{t},h)_{-2^{-n}\mathbf{f}'(h)})\big) \le 2\|f'\|_\infty 2^{-n}\,\mathrm{Per}(U^n(\bar{t},h)).$$

Taking into account (5.20), by coarea formula for functions of bounded variation

$$\begin{aligned}\omega^n(\mathcal{G}_{j,n}) &\le \int_0^{\|u_0\|_\infty} \|f'\|_\infty 2^{-n}\,\mathrm{Per}(U^n(\bar{t},h))\,dh\\ &= 2^{-n}\|f'\|_\infty \mathrm{TV}(u^n(\bar{t}))\\ &\le 2^{-n}\|f'\|_\infty \mathrm{TV}(u_0),\end{aligned}$$

where the last inequality follows by Lemma 5.11. Similarly we can prove that

$$\omega^n(\mathcal{G}_{k,n}) \le s\|f'\|_\infty \mathrm{TV}(u_0),$$

therefore summing over $j = 0, \dots, k$ we get

$$\begin{aligned}\mathcal{L}^{d+1}\big(U^n(t)\,\Delta\,U^n(\bar{t})\big) &\le 2\sum_{j=0}^{k} \omega^n(\mathcal{G}_{j,n})\\ &\le 2((2^{-n}k + s)\|f'\|_\infty \mathrm{TV}(u_0)\\ &= 2(t-\bar{t})\|f'\|_\infty \mathrm{TV}(u_0).\end{aligned}$$

□

We now combine the estimate (5.18) together with Lemma 5.1 to deduce the existence of a Lagrangian representation for BV solutions.

PROPOSITION 5.13. *The sequence $(\omega^n)_{n\in\mathbb{N}}$ constructed in (5.16) is tight and every limit point ω is a Lagrangian representation of the entropy solution to (5.15).*

PROOF. As in the proof of Proposition 5.8, the tightness of the family follows from Condition (2) in Definition 5.3 together with uniform bounds. Let ω be any limit point.

We now want to apply Lemma 5.1: set $I = [0,T]$ and let $D_n := \{\frac{k}{2^n} : k = 0, \dots, 2^n T\}$. Let then $X := L^1(\mathbb{R}^{d+1})$ and accordingly define

$$\begin{aligned}f_n\colon I &\to L^1(\mathbb{R}^{d+1})\\ t &\mapsto \chi_{\mathrm{supp}\,(e_t)_\sharp\omega^n}(\cdot)\end{aligned}$$

Condition (1) is trivially satisfied; let us verify Assumption (2). For any $n \in \mathbb{N}$, for every $t \in D_n$ and every $m > n$ we have $(e_t)_\sharp\omega^m$ is concentrated on the hypograph of some function $u^m(t)$. By Lemma 5.11 the functions $(u^m(t))_{m\ge n}$ have uniformly bounded total variation, hence they are compact in $L^1(\mathbb{R}^d)$ and therefore the hypographs are compact in $L^1(\mathbb{R}^{d+1})$. To verify Condition (3), it is enough to apply Lemma 5.12.

Thus we obtain a Lipschitz function $f\colon I \to L^1(\mathbb{R}^{d+1})$; since $f(t)$ is the characteristic function of an hypograph for every $t \in D$, by continuity, there exists $u \in \mathrm{Lip}([0,T];\mathrm{BV}(\mathbb{R}^d))$ such that

$$f(t) = \chi_{\mathrm{hyp}\ u(t)}$$

for every $t \in [0,T]$.

Thanks to Proposition 5.8 we obtain that

$$(e_t)_\sharp \omega = \mathcal{L}^{d+1} \llcorner \mathrm{hyp}\ u(t)$$

for every $t \geq 0$. Finally, a direct application of Proposition 5.5 shows that the function u is the entropy solution to (5.15) and concludes the proof. □

The compactness and stability properties of Lagrangian representations stated in Corollary 5.9, together with standard approximation results, yield immediately the following

THEOREM 5.14. *Let u be the entropy solution to the initial value problem (5.15) with $u_0 \in L^\infty(\mathbb{R}^d)$. Then there exists a Lagrangian representation of u.*

5.3. The case of continuous solutions

In this section we prove that if u is a continuous entropy solution of (5.5) then for every entropy-entropy flux pair $(\eta, \mathbf{q})$ with $\eta \in C^1(\mathbb{R})$, the dissipation measure μ vanishes, namely

$$\mu = \eta(u)_t + \mathrm{div}(\mathbf{q}(u)) = 0.$$

Consider the jump part of $\bar{\nu}$ defined by

$$\nu^j := \int_\Gamma \mu^j_\gamma d\omega, \qquad \text{where} \quad \mu^j_\gamma = \mathcal{H}^1 \llcorner \{(t,x,h) : \gamma^1(t) = x, h \in (\gamma^2(t+), \gamma^2(t-))\}.$$

As an intermediate step we prove that $\nu^j = 0$, which is equivalent, by definition, to the fact that ω is concentrated on continuous curves.

LEMMA 5.15. *Let $u\colon [0,T) \times \mathbb{R}^d \to \mathbb{R}$ be a continuous solution of (5.5) and let ω be a Lagrangian representation of u. Then ω is concentrated on* continuous *characteristic curves.*

PROOF. Since the solution u is continuous, for every $(t,x,h) \in [0,+\infty) \times \mathbb{R}^d \times (0,+\infty)$ such that $h < u(t,x)$, it holds $(t,x,h) \in \mathrm{Int}(\mathrm{hyp}\ u)$. Hence for every $\gamma \in \mathrm{supp}\,\omega$,

$$\mu^j_\gamma = \mu^j_\gamma \llcorner \mathrm{Int}(\mathrm{hyp}\ u).$$

Therefore

$$\nu^j = \nu^j \llcorner \mathrm{Int}(\mathrm{hyp}\ u) = 0,$$

by Proposition 5.7. This concludes the proof of this lemma. □

In the following proposition we show that for the continuous solutions the hypograph at time t is the translation of hyp u_0 along segments with characteristic speed.

PROPOSITION 5.16. *Let $u\colon [0,T) \times \mathbb{R}^d \to \mathbb{R}$ be a continuous entropy solution of (5.5). Then*

$$\bar{\omega} = \int_{\mathrm{hyp}\ u_0} \delta_{\bar{\gamma}_{x,h}}\, dx\, dh,$$

where

$$\bar{\gamma}_{x,h}(t) = (x + t\mathbf{f}'(h), h), \qquad t \in [0,T)$$

is a Lagrangian representation of u.

Proof. To begin we notice that there exists a set E with $\mathcal{L}^{d+2}(\mathrm{hyp}\, u \setminus E) = 0$ such that for every $z = (t, x, h) \in E$ there exists a curve $\gamma_z \colon [0, \bar{t}] \to \mathbb{R}^d \times [0, +\infty)$ with the following properties:

(1) $\gamma_z(t) = (x, w)$;
(2) γ_z is a continuous characteristic curve;
(3) $\gamma_z\big([0, \bar{t}]\big) \subset \mathrm{hyp}\, u$;
(4) γ_z^2 is constant on the connected components of $\gamma_z^{-1}(\mathrm{Int}(\mathrm{hyp}\, u))$.

In fact, (1) follows from the definition of Lagrangian representation and (2) follows from Lemma 5.15. From the definition of Lagrangian representation ω is concentrated on curves that lie in hyp u for $\mathcal{L}^1$-a.e. $t \in [0, T]$. By continuity of u, we thus get (3). Finally (4) follows by Proposition 5.7.

Let $\bar{t} > 0$ and for every $(x, h) \in \mathrm{hyp}\, u(\bar{t})$ we consider the function

$$\begin{aligned} \sigma_{(x,h)} \colon [0, \bar{t}] &\to \mathbb{R}^d \times [0, +\infty) \\ t &\mapsto \big(x - (\bar{t} - t)\mathbf{f}'(h), h\big). \end{aligned}$$

We first prove that for every $(x, h) \in \mathrm{hyp}\, u(\bar{t})$ the segments

$$\sigma_{(x,h)}\big([0, \bar{t}]\big) \subset \mathrm{hyp}\, u.$$

Fix $\varepsilon > 0$ and let us construct by iteration a curve contained in the hypograph which approximates the segment. By uniform continuity of u there exists $\delta \in (0, 1)$ such that

$$|(t, x) - (t', x')| \le \delta \quad \Rightarrow \quad |u(t, x) - u(t', x')| \le \varepsilon.$$

Let $\varepsilon' < \delta\varepsilon$ and fix $(t_1, x_1) \in [0, +\infty) \times \mathbb{R}$ and $\bar{h} > 0$ such that $(t_1, x_1, \bar{h}) \in \mathrm{hyp}\, u$. For $k \ge 1$ we define by recursion the points $\tilde{z}_k$, t_k and x_k in the following way:

$$\tilde{z}_k = (\tilde{t}_k, \tilde{x}_k, \tilde{h}_k) \in B_{\varepsilon'}((t_k, x_k, \bar{h} - \varepsilon)) \cap E, \tag{5.21}$$

with $\tilde{t}_k < t_k$ and

$$t_{k+1} := \inf\{t \in [0, \tilde{t}_k] : \gamma_{\tilde{z}_k}(t) < \bar{h} + \varepsilon\}, \qquad x_{k+1} := \gamma^1_{\tilde{z}_k}(t_{k+1}+).$$

The procedure ends when $t_{k+1} = 0$. The existence of points $\tilde{z}_k$ is ensured by the fact that E has full measure. We now prove that the procedure ends in finitely many steps. Since for every $k \ge 0$, $\gamma^2_{\tilde{z}_k}$ is constant on each connected component of $\gamma^{-1}_{\tilde{z}_k}(\mathrm{Int}(\mathrm{hyp}\, u))$ and $\gamma^2_{\tilde{z}_k}(\tilde{t}_k) < u(\tilde{t}_k, \tilde{x}_k) - \varepsilon$, by the uniform continuity of u

$$\tilde{t}_k - t_{k+1} \ge \frac{\delta}{\|f'\|_\infty} \wedge \tilde{t}_k,$$

therefore the number of steps N after which the procedure ends is bounded by

$$N \le 1 + \frac{\|f'\|_\infty \bar{t}}{\delta}. \tag{5.22}$$

We now prove the following claim, which states that $\gamma_{\tilde{z}_k}$ approximates $\sigma_{(\bar{x},\bar{h})}$ in $(t_{k+1}, \tilde{t}_k)$.

Claim. There exists $C > 0$ independent of ε such that for every $t \in [0, \bar{t}]$ there exists $k = 1, \dots, N$ and $s \in (t_{k+1}, \tilde{t}_k)$ for which

$$|(s, \gamma_{\tilde{z}_k}(s)) - (t, \sigma_{(\bar{x},\bar{h})}(t))| < C\varepsilon. \tag{5.23}$$

First we observe that for every $k = 1, \dots, N$ and for every $s \in (t_{k+1}, \tilde{t}_k)$ it holds

$$|\gamma^2_{\tilde{z}_k}(s) - \bar{h}| < 2\varepsilon. \tag{5.24}$$

The estimate for the first components follows by (5.24) and (5.7): for every $k = 1, \dots, N$,

$$\begin{aligned} |\gamma^1_{\tilde{z}_k}(t_{k+1}) - \sigma^1_{(\bar{x},\bar{h})}(t_{k+1})| &= \left| \gamma^1_{\tilde{z}_k}(\tilde{t}_k) - \sigma^1_{(\bar{x},\bar{h})}(\tilde{t}_k) - \int_{t_{k+1}}^{\tilde{t}_k} \left(\dot{\gamma}^1_{\tilde{z}_k}(t) - \mathbf{f}'(\bar{h})\right) dt \right| \\ &\leq |\gamma^1_{\tilde{z}_k}(\tilde{t}_k) - \sigma^1_{(\bar{x},\bar{h})}(\tilde{t}_k)| + 2\varepsilon(\tilde{t}_k - t_{k+1})\|\mathbf{f}''\|_\infty. \end{aligned} \tag{5.25}$$

Moreover, by (5.21),

$$\begin{aligned} |\gamma^1_{\tilde{z}_k}(\tilde{t}_k) - \sigma^1_{(\bar{x},\bar{h})}(\tilde{t}_k)| &\leq |\gamma^1_{\tilde{z}_k}(\tilde{t}_k) - \gamma^1_{\tilde{z}_k}(t_k)| + |\gamma^1_{\tilde{z}_k}(t_k) - \sigma^1_{(\bar{x},\bar{h})}(t_k)| + |\sigma^1_{(\bar{x},\bar{h})}(t_k) - \sigma^1_{(\bar{x},\bar{h})}(\tilde{t}_k)| \\ &\leq 2\|\mathbf{f}'\|_\infty \varepsilon' + |\gamma^1_{\tilde{z}_k}(t_k) - \sigma^1_{(\bar{x},\bar{h})}(t_k)|. \end{aligned} \tag{5.26}$$

By (5.25) and (5.26), it follows that for every $k = 1, \dots, N-1$ it holds

$$|\gamma^1_{\tilde{z}_k}(t_{k+1}) - \sigma^1_{(\bar{x},\bar{h})}(t_{k+1})| \leq |\gamma^1_{\tilde{z}_k}(t_k) - \sigma^1_{(\bar{x},\bar{h})}(t_k)| + 2\varepsilon(\tilde{t}_k - t_{k+1})\|\mathbf{f}''\|_\infty + 2\|\mathbf{f}'\|_\infty \varepsilon'. \tag{5.27}$$

For every $t \in [0, \bar{t}]$ let $\bar{k} = 1, \dots, N-1$ and $s \in (t_{\bar{k}+1}, \tilde{t}_{\bar{k}})$ be such that $|s - t| < \varepsilon'$. Then, iterating (5.27) for $k = \bar{k}, \dots, N-1$ and by (5.22), we have

$$\begin{aligned} |\gamma^1_{\tilde{z}_k}(s) - \sigma^1_{\bar{z}}(t)| &\leq |\gamma^1_{\tilde{z}_k}(s) - \sigma^1_{\bar{z}}(s)| + |\sigma^1_{\bar{z}}(s) - \sigma^1_{\bar{z}}(t)| \\ &\leq 2\varepsilon\|f''\|_\infty(\bar{t} - s) + 2(N - \bar{k})\varepsilon'\|f'\|_\infty + \|f'\|_\infty|t - s| \\ &\leq 2\varepsilon\|f''\|_\infty T + 2\varepsilon'\|\mathbf{f}'\|_\infty + 2\varepsilon\|f'\|^2_\infty \bar{t} + \|f'\|_\infty \varepsilon' \\ &\leq C\varepsilon, \end{aligned} \tag{5.28}$$

where $C = 2\|f''\|_\infty T + 2\|\mathbf{f}'\|_\infty + 2\|f'\|^2_\infty T + \|f'\|_\infty$. The estimates (5.24) and (5.28) prove (5.23).

Since hyp u is closed, letting $\varepsilon \to 0$ we obtain that for every $(\bar{x}, \bar{h}) \in \text{hyp } u(\bar{t})$, the segment

$$\sigma_{(\bar{x},\bar{h})}([0, \bar{t}]) \subset \text{hyp } u.$$

Let

$$\tilde{\omega} = \int_{\text{hyp } u(\bar{t})} \delta_{\sigma_{x,h}} dx dh.$$

Since the translations are area-preserving, for every $t \in [0, \bar{t}]$, there exists $U(t) \subset [0, +\infty) \times \mathbb{R}^d$ such that

$$(e_t)_\sharp \tilde{\omega} = \mathcal{L}^{d+1} \llcorner U(t)$$

and

$$\mathcal{L}^{d+1}(U(t)) = \int_{\mathbb{R}^d} u(\bar{t}, x) dx. \tag{5.29}$$

Since we proved that for every $t \in [0, \bar{t}]$ it holds $U(t) \subset \text{hyp } u(t)$, (5.29) implies that $U(t) = \text{hyp } u(t)$. This proves that $\tilde{\omega} = \bar{\omega}$ and it is a Lagrangian representation of u. $\square$

THEOREM 5.17. *Let u be a continuous bounded entropy solution in $[0, T) \times \mathbb{R}^d$ to (5.5). Then for every $(t, x) \in [0, T) \times \mathbb{R}^d$, it holds*

$$u(t, x) = u_0(x - \mathbf{f}'(u(t, x))t). \tag{5.30}$$

Moreover for every $\eta : \mathbb{R} \to \mathbb{R}$, $\mathbf{q} : \mathbb{R} \to \mathbb{R}^d$ Lipschitz such that $\mathbf{q}' = \eta'\mathbf{f}'$ a.e. with respect to $\mathcal{L}^1$, it holds

$$\eta(u)_t + \text{div}_x \, \mathbf{q}(u) = 0 \tag{5.31}$$

in the sense of distributions.

Proof. The validity of (5.30) is an immediate consequence of Proposition (5.16). Concerning the second claim, if η is a convex C^2 entropy, then (5.31) follows by Lemma 5.6 and Proposition 5.16, since $\mu_\gamma^\eta = 0$ for every $\gamma \in \operatorname{supp} \omega$. If η is C^2, then there exist η_1, η_2 of class C^2 and convex such that $\eta = \eta_1 - \eta_2$ and thus it is enough to apply the previous result to both η_1 and η_2. Finally, in order to prove that (5.31) holds for Lipschitz $(\eta, \mathbf{q})$, we consider a sequence $(\eta^n)_{n\in\mathbb{N}}$ such that $\eta^n \to \eta$ uniformly on $\mathbb{R}$ and $(\eta^n)' \to \eta'$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ with the associated $\mathbf{q}^n$ such that $\mathbf{q}^n(0) = \mathbf{q}(0)$. We have that $\mathbf{q}^n \to \mathbf{q}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and hence, for every test function $\phi \in C_c^\infty([0,T) \times \mathbb{R}^d)$,

$$\begin{aligned} -\langle \eta(u)_t + \operatorname{div}_x \mathbf{q}(u), \phi \rangle &= \int_0^T \int_{\mathbb{R}^d} \phi_t \eta(u) + \mathbf{q}(u) \cdot \nabla \phi \, dx \, dt \\ &= \lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}^d} \phi_t \eta^n(u) + \mathbf{q}^n(u) \cdot \nabla \phi \, dx \, dt = 0, \end{aligned}$$

and this completes the proof. $\square$

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