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Inventory management system with On/Off control and phase-type distribution of purchases quantity

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Abstract. The purpose of this paper is to study mathematical model of an inventory management system with On/Off control. We consider the case in which input flow of product is continuous with fixed rate. Demand occurs according to a Poisson process with constant intensity and quantity of purchase have phase-type Distribution. We find an explicit form for a stationary probability density function of inventory level.

Keywords: Inventory management, On/Off control, mathematical modelling, phase-type distribution.

1. Problem statement

Inventory control models under various conditions have been studied intensively in the last century, for example, Single-period and Newsvendor problem are widely-known [1–5]. This models used to analyse systems with perishable products in airlines, fashion industries and other fields. In this paper we propose stochastic mathematical model of inventory management with on/off control.

Consider an product which is demanded and the product flow be continuous with fixed rate $\nu = 1$ (Fig. 1).

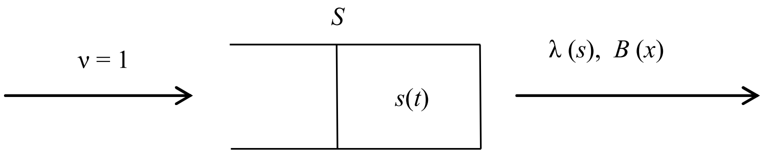

Figure 1. Inventory management system

Let $s(t)$ be inventory level at time t. The demand for the occurs according to a Poisson process with piecewise constant intensity $\lambda(s)$

$$\lambda(s) = \begin{cases} \lambda_1, s < S, \\ \lambda_2, s \geq S, \end{cases}$$

where S is the threshold inventory level of $s(t)$.

The values of purchases are independent and identically distributed random variables having Phase-type distribution

$$B(x) = 1 - \beta e^{\mathbf{G}x}\mathbf{E},$$

where $\beta_k > 0$, $\mathbf{G}$ is subgenerator matrix Markov chain that determines the Phase-type distribution and

$$\beta\mathbf{E} = 1.$$

In this paper, it is assumed that the process $s(t)$ can take the values $s(t) < 0$. In this situation, the customer waits for the required amount of product.

Condition for the existence of a stationary distribution has form

$$\lambda_1 b < 1 < \lambda_2 b,$$

where b is the first moment of the probability distribution (1).

Clearly, if condition if $\lambda_1 < 1/b < \lambda_2$ and $s(t) < S$ is satisfied then the stock level increases in the mean. Otherwise condition $s(t) \geq S$ means that the stock level decreases in the mean.

based on the problem statement, we conclude that $s(t)$ is a Markovian process with continuous time t and continuous state space $-\infty < s < \infty$.

We denote by

$$P(s,t) = \frac{\partial P\left\{s(t) < s\right\}}{\partial s}$$

the stationary probability density function of stock level.

We can derive the equation

$$P(s+\Delta t) = P(s)(1-\lambda(s)\Delta t) + \Delta t \int\limits_0^\infty \lambda(s+x)P(s+x)dB(x) + o(\Delta t).$$

We obtain Kolmogorov equation for the distribution $P(s)$

$$P\prime(s) + \lambda(s)P(s) = \int\limits_0^\infty \lambda(s+x)P(s+x)dB(x),$$

where the boundary conditions have following form

$$P(-\infty) = P(\infty) = 0.$$

Let us find a solution $P(s)$ of the Kolmogorov equation (1) in an explicit form, which is satisfied the boundary conditions (1).

Let us denote

$$P(s) = \begin{cases} P_1(s), s < S, \\ P_2(s), s > S. \end{cases}$$

Therefore, we can rewrite the equation (1) as two equations

$$P_2{}'(s) + \lambda_2 P_2(s) = \lambda_2 \int\limits_0^\infty P_2(s+x)dB(x), s > S,$$

and

$$P_1{}'(s) + \lambda_1 P_1(s) =$$
$$\lambda_1 \int\limits_0^{S-s} P_1(s+x)dB(x) + \lambda_2 \int\limits_{S-s}^\infty P_2(s+x)dB(x), s < S.$$

Now we find solutions of equations (1) and (1), that satisfy the boundary conditions

$$P_1(-\infty) = 0, P_2(\infty) = 0.$$

2. The solution $P_2(s)$ of equation (1)

Solution $P_2(s), s > S$ of equation (1) has to be sought in the form

$$P_2(s) = Ce^{-\gamma(s-S)}, s > S.$$

using substitution (2)) into (1)), we derive the equation

$$\lambda_2 - \gamma = \lambda_2 \int\limits_0^\infty e^{-\gamma x} dB(x).$$

Obviously, that equation (2) has extraneous zero root $\gamma = 0$, because we have the boundary condition (1) $P_2(\infty) = 0$.

It is easy to see that unique positive root $\gamma > 0$ of equation (2) exists for any distribution function $B(x)$ under the condition (1) $\lambda_2 b > 1$, consequently the solution of the equation (1) is a function (1) defined with multiplicative constant C accuracy, which value will be find later.

3. The solution $P_1(s)$ of equation (1)

Taking into account (2), we can rewrite equation (1) in the form

$$P_1{}'(s) + \lambda_1 P_1(s) =$$
$$\lambda_1 \int\limits_0^{S-s} P_1(s+x)dB(x) + \lambda_2 C e^{-\gamma(s-S)} \int\limits_{S-s}^{\infty} e^{-\gamma x} dB(x).$$

Using (1), we can find the integral on the right side of the equation (3)

$$\int\limits_{S-s}^{\infty} e^{-\gamma x} dB(x) = -\int\limits_{S-s}^{\infty} e^{-\gamma x} \beta e^{\mathbf{G}x} \mathbf{G}\mathbf{E} dx = -\int\limits_{S-s}^{\infty} \beta e^{(\mathbf{G}-\gamma\mathbf{I})x} \mathbf{G}\mathbf{E} dx =$$

$$= \beta e^{(\mathbf{G}-\gamma\mathbf{I})(S-s)} (\mathbf{G}-\gamma\mathbf{I})^{-1} \mathbf{G}\mathbf{E},$$

then (3) can be written as follows

$$P_1{}'(s) + \lambda_1 P_1(s) =$$

$$\lambda_1 \int\limits_0^{S-s} P_1(s+x)dB(x) + \lambda_2 C \beta e^{(\mathbf{G}-\gamma\mathbf{I})(S-s)} (\mathbf{G}-\gamma\mathbf{I})^{-1} \mathbf{G}\mathbf{E}.$$

Substituting the expression (1) for distribution functions $B(x)$ in this equation, we obtain the following equation

$$P_1{}'(s) + \lambda_1 P_1(s) =$$
$$\beta \left(\lambda_1 \int\limits_0^{S-s} P_1(s+x) e^{\mathbf{G}x} dx - \lambda_2 C e^{(\mathbf{G}-\gamma\mathbf{I})(S-s)} (\mathbf{G}-\gamma\mathbf{I})^{-1} \right) \mathbf{G}\mathbf{E}.$$

Theorem 1 *If the equation*

$$z + \lambda_1 = \lambda_1 \beta (\mathbf{G} + z\mathbf{I})^{-1} \mathbf{G}\mathbf{E}$$

has n simple roots with positive real parts, then solution $P_1(s)$ of equation (3) has form

$$P_1(s) = C \sum_{\nu=1}^{n} x_\nu e^{z_\nu (s-S)}, s < S,$$

where $z = z_\nu$, $\nu = \overline{1,n}$ is a nonzero roots of equation (1), $x_\nu, \nu = \overline{1,n}$ are solutions to a system of equations

$$\left(\lambda_1 \sum_{\nu=1}^{n} x_\nu (\mathbf{G} + z_\nu \mathbf{I})^{-1} - \lambda_2 (\mathbf{G}-\gamma\mathbf{I})^{-1} \right) \mathbf{G}\mathbf{E} = 0,$$

normalizing constant C is determined by the equation

$$C = \left(\frac{1}{\gamma} + \sum_{\nu=1}^{n} \frac{x_\nu}{z_\nu} \right)^{-1}.$$

Proof. Solution $P_1(s)$ of the equation (3) will be find in the form (1).

Substituting (1) into (3) we obtain the equation

$$\sum_{\nu=1}^{n} x_\nu e^{z_\nu(s-S)} \left\{ z_\nu + \lambda_1 - \lambda_1 \beta (\mathbf{G} + z_\nu \mathbf{I})^{-1} \mathbf{GE} \right\} =$$

$$= \beta e^{\mathbf{G}(S-s)} \left(\lambda_1 \sum_{\nu=1}^{n} x_\nu (\mathbf{G} + z_\nu \mathbf{I})^{-1} - \lambda_2 (\mathbf{G} - \gamma \mathbf{I})^{-1} \right) \mathbf{GE}.$$

By equating the coefficients to zero in the linear combination of exponents $e^{z_\nu(s-S)}$ in this expression, we get

$$z_\nu + \lambda_1 = \lambda_1 \beta (\mathbf{G} + z_\nu \mathbf{I})^{-1} \mathbf{GE}, \ \nu = \overline{1, n}.$$

Obviously that this expression and (1) have the same form. Consequently, $z_\nu, \nu = \overline{1, n}$ are the roots of the equation (1).

Analogically, we obtain

$$\left(\lambda_1 \sum_{\nu=1}^{n} x_\nu (\mathbf{G} + z_\nu \mathbf{I})^{-1} - \lambda_2 (\mathbf{G} - \gamma \mathbf{I})^{-1} \right) \mathbf{GE} = 0.$$

Using the normalization condition we derive the constant C

$$1 = \int_{-\infty}^{\infty} P(s) ds = \int_{-\infty}^{S} P_1(s) ds + \int_{S}^{\infty} P_2(s) ds =$$

$$= C \sum_{\nu=1}^{n} x_\nu \int_{-\infty}^{S} e^{z_\nu(s-S)} ds + C \int_{S}^{\infty} e^{-\gamma(s-S)} ds =$$

$$= C \sum_{\nu=1}^{n} x_\nu \int_{-\infty}^{0} e^{z_\nu x} dx + C \int_{0}^{\infty} e^{-\gamma x} dx = C \left\{ \sum_{\nu=1}^{n} \frac{x_\nu}{z_\nu} + \frac{1}{\gamma} \right\}.$$

Finally, we get

$$C = \left(\sum_{\nu=1}^{n} \frac{x_\nu}{z_\nu} + \frac{1}{\gamma} \right)^{-1}.$$

It is easy to see that this expression coincides with (1).

The theorem is proved. ■

Probability density function $P(s)$ of stock-level process has form

$$P(s) = \left(\sum_{\nu=1}^{n} \frac{x_\nu}{z_\nu} + \frac{1}{\gamma} \right)^{-1} \cdot \begin{cases} \sum\limits_{\nu=1}^{n} x_\nu e^{z_\nu (s-S)}, s < S, \\ e^{-\gamma(s-S)}, s > S, \end{cases}$$

where z_ν is a nonzero roots of equation (1), γ is unique positive root of equation (2), x_ν are solutions equations (1).

4. Numerical results

Let us consider Phase-type distribution of random demand with 3 phases.

For the following values of the parameters $\lambda_1 = 0.8$ and $\lambda_2 = 1.2$, $S = 10$ We found the roots of equations (3) and (1). Thus, the equation (2) has a unique positive solution $\gamma = 0.198$, the equation (1) has three real roots $z_1 = 0.198, z_2 = 6.327, z_3 = 8.345$. Let us find probability density function of inventory level for the given parameters.

The parameters $x_\nu, \nu = \overline{1, n}$ and normalizing constant of distribution (3), have the form

$$x_1 = 1; x_2 = -0.004; x_3 = 0.003, C = 0.99,$$

resulting distribution is shown in Fig. 2.

The explicit expression (3) for the solution $P(s)$ of the equation (1) completely solves the problem of the study of mathematical inventory control model with following restrictions: on/off control and Phase-type distribution of values of product purchases.

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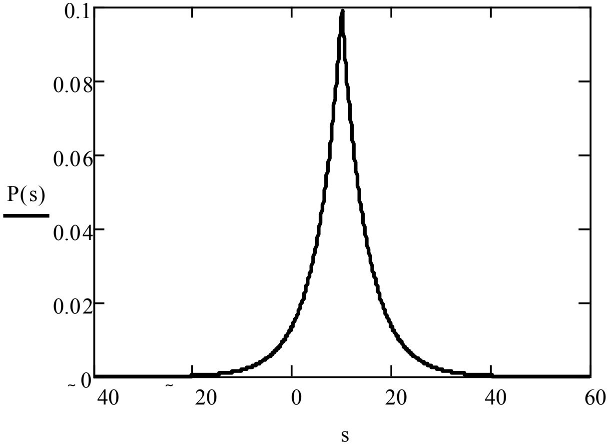

Figure 2. probability density function $P(s)$

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