SISSA · Scuola Internazionale Superiore di Studi Avanzati

Sector of Physics
PhD Programme in Astroparticle Physics

Black holes beyond general relativity: theoretical and phenomenological developments.

Supervisor:
Prof. Stefano Liberati

Submitted by:
Costantino Pacilio

Academic Year 2017/2018

Abstract

In four dimensions, general relativity is the only viable theory of gravity satisfying the requirements of diffeoinvariance and strong equivalence principle. Despite this aesthetic appeal, there are theoretical and experimental reasons to extend gravity beyond GR. The most promising tests and bounds are expected to come from strong gravity observations. The past few years have seen the rise of gravitational wave astronomy, which has paved the way for strong gravity observations. Future GW observations from the mergers of compact objects will be able to constrain much better possible deviations from GR. Therefore, an extensive study of compact objects in modified theories of gravitation goes in parallel with these experimental efforts.

In this PhD Thesis we concentrate on black holes. Black holes act as testbeds for modifications of gravity in several ways. While in GR they are extremely simple objects, in modified theories their properties can be more complex, and in particular they can have hair. The presence of hair changes the geometry felt by test fields and it modifies the generation of GW signals. Moreover, black holes are the systems in which the presence of singularities is predicted by classical gravity with the highest level of confidence: this is not only true in GR, but also in most of the modified gravity theories formulated in classical terms as effective field theories. Singularities are regarded as classical artifacts to be cured by quantum gravity effects. Therefore, considering mechanisms of singularity resolutions is a theoretical arena to study the form of these effects.

The Thesis presents theoretical contributions to all these aspects of black hole physics. The work is organized following three main topics: black holes with universal horizons, hairy black holes in Einstein-Maxwell-dilaton theory and regular black holes. These models originate from various motivations: black holes with universal horizons are found in modified gravity theories which break local Lorentz symmetry; Einstein-Maxwell-dilaton black holes originate in string theory and in lower dimensional compactifications, but they also serve as proxies for black holes in theories propagating additional degrees of freedom; regular black holes are motivated by the efforts to understand how quantum gravity solves the classical singularities.

In each of the above cases, we present results which appear to be relevant for the follow up research in their respective fields. We also emphasize that, besides the contextual significance of our results, we also developed thechniques for addressing the respective problems, which can be useful well beyond the specific cases considered in this Thesis.

Publication List

Here below you find a list of the publications on which the present PhD Thesis is based. In the main text, we will refer to them as P1, P2 and so on.

P1 Liberati, S; Pacilio, C.
Smarr formula for Lovelock black holes: A Lagrangian approach.
Physical Review D, (2016), 93.8: 084044.
arXiv:1511.05446

P2 Pacilio, C; Liberati, S.
Improved derivation of the Smarr formula for Lorentz-breaking gravity.
Physical Review D, (2017), 95.12: 124010.
arXiv:1701.04992

P3 Pacilio, C; Liberati, S.
First law of black holes with a universal horizon.
Physical Review D, (2017), 96.10: 104060.
arXiv:1709.05802

P4 Carballo-Rubio, R; Di Filippo, F; Liberati, S; Pacilio, C; Visser, M.
On the viability of regular black holes.
JHEP, (2018): 23.
arXiv:1805.02675

P5 Pacilio, C.
Scalar charge of black holes in Einstein-Maxwell-dilaton theory.
Submitted, (2018).
arXiv:1806.10238

P6 Brito, R; Pacilio, C.
Quasinormal modes of weakly charged Einstein-Maxwell-dilaton black holes.
Submitted, (2018).
arXiv:1807.09081

During my PhD I also published or submitted the following papers, whose contents are not part of this Thesis.

- De Lorenzo, T; Pacilio, C; Rovelli, C; Speziale, S.
  On the effective metric of a Planck star.
  General Relativity and Gravitation, (2015), 47(4), 41.
  arXiv:1412.6015

- Alesci, E; Pacilio, C; Pranzetti, D.
  Radial gauge fixing of first order gravity.
  Accepted for publication on PRD, (2018)
  arXiv:1802.06251

List of Abbreviations

ADM Arnowitt-Deser-Misner

AdS anti-deSitter

Æ Einstein-Aether

BH Black hole

BTZ Bañados-Teitelboim-Zanelli

CMB Cosmic microwave background

D^{2}GB Decoupled dynamical Gauss-Bonnet

DF Dudley-Finley

DOF Degree(s) of freedom

DTR Dray-t'Hooft-Redmount

EDGB Einstein-dilaton-Gauss-Bonnet

EM Einstein-Maxwell / Electromagnetic (which of the two is clear from the context)

EMD Einstein-Maxwell-dilaton

EOM Equation(s) of motion

GR General relativity

GSL Generalized second law

GW Gravitational wave

IR Infrared

KN Kerr-Newman

KRT Kastor-Ray-Traschen

LHS Left hand side

LV Lorentz-violation / Lorentz-violating

NS Neutron star

RHS Right hand side

QCD Quantum chromodynamics

QFT Quantum chromodynamics

QNM Quasinormal mode

UH Universal horizon

UV Ultraviolet

VEV Vacuum expectation value

Basic Conventions

In this Thesis we adapt all our conventions to the "mostly plus" metric signature $(- + ++)$. Moreover, unless otherwise specified, we work in units of $c = G = \hbar = 1$. Spacetime indices are labeled by lowercase latin letters a, b, c etc. The complete symmetrization and antisymmetrization of a rank-2 tensor t_{ab} are defined as

$$t_{(ab)} = \frac{t_{ab} + t_{ba}}{2} \quad \text{and} \quad t_{[ab]} = \frac{t_{ab} - t_{ba}}{2} \, .$$

Contents

1 Preface

In four dimensions, general relativity (GR) is the only viable theory of gravity satisfying the requirements of diffeoinvariance and strong equivalence principle [1]. Despite this aesthetic appeal, there are theoretical and experimental reasons to extend gravity beyond GR. On the theoretical side, GR faces two main problems: (i) it generically predicts the occurrence of singularities, both in cosmology and in black holes [2, 3, 4]; (ii) when quantized with standard metods, it is not renormalizable [5]. On the experimental side, the discovery of the dark sectors of the Universe makes clear that GR plus ordinary matter, the latter described by the standard model of particle physics, cannot be the end of fundamental physics. Merging this wealth of evidence all together seems to clearly point towards the need for new physics in both the ultraviolet (UV) and the infrared (IR). UV and IR extensions are not necessarily independent: one can devise mechanisms through which UV modifications percolate in the IR, where they are described by an effective field theory. For example, kronometric theory can be viewed as an effective IR description of Hořava gravity [6]; similarly, Einstein–dilaton–Gauss-Bonnet theory can be derived from an IR compactification of some string theory models [7].

“New physics” does not necessarily mean to modify the laws of gravity, in the form that they assume in GR. It can also mean the introduction of new fields interacting weakly enough with the ordinary matter and which are felt mainly through gravity: indeed this is the most common strategy when dealing with the dark matter problem. Therefore it is natural to ask how well and in which regimes GR is tested, to see whether and where there is still room for modifications. The most stringent tests up to date are the post–Newtonian constraints from Solar System and binary pulsars observations. However these tests are restricted to intermediate curvature scales [8]. While at the cosmological scales, i.e. at weak curvatures, we have already seen that one faces the dark matter and dark energy problems, much less is known at strong curvatures.

The past few years have seen the rise of gravitational wave (GW) astronomy [9, 10, 11, 12, 13, 14], which has paved the way for strong-gravity observations. Indeed, GW detectors receive waves emitted from the inspiral and merger of binary compact objects, such as black holes and neutron stars, the systems in which gravity manifests itself in its most extreme configurations. If gravity is modified, we expect to see corresponding corrections in the GW waveforms, the most notable of which involve [15]: a change in the energy balance of the system; a dephasing of the GWs; a fine splitting of the oscillation frequency of black holes and neutron stars; a deviation of the propagation speed of GWs from the speed of light. The event GW170817 and its associated electromagnetic counterpart GRB170817A have confirmed that GWs propagate at the speed of light on cosmological scales [14, 16] (at least at the very low energies characterizing the observed GW), thus already ruling out a wide range of modified theories [17, 18, 19]. Future GW observations will be able to constrain much better other possible deviations from GR, both in the inspiral-merger and in the post-merger regions of the waveform [20, 21]. Therefore, an extensive study of compact objects in modified theories of gravitation goes in parallel with these experimental efforts.

In this PhD Thesis we concentrate on black holes. Black holes act as testbeds for modifications of gravity in several ways. While in GR they are extremely simple objects, in modified theories their properties can be more complex, and in particular they can have hair. The presence of hair changes the geometry felt by test fields and it modifies the generation of GW signals, thus leading to the possibility of performing a "black hole spectroscopy" to detect deviations from standard GR.

Moreover, black holes are the systems in which the presence of singularities is predicted by classical gravity with the highest level of confidence: this is not only true in GR, but also in most of the modified gravity theories formulated in classical terms as effective field theories. Singularities are regarded as classical artifacts to be cured by quantum gravity effects. Therefore, considering mechanisms of singularity resolutions is a theoretical arena to study the form of these effects.

The Thesis presents theoretical contributions to all these aspects of black hole physics. The work is organized following three main topics:

- We present an original method to derive a Smarr formula for black holes in general diffeoinvariant gravity theories. The method is applied to black holes in Lovelock theory and Lorentz-violating gravity theory. In the latter case, we also explore the formulation of mechanical and theormodynamical laws for Lorentz-violating black holes. These studies will be the subject of Chapter 3;

- We study black holes in Einstein-Maxwell-dilaton theory, which, besides being motivated by string theory and by compactifications of higher dimensional theories, it also constitutes a proxy for theories in which scalar, vector and tensor degrees of freedom can propagate all together. Black holes in this theory are electrically charged and have scalar hair. We derive a formula for the scalar monopole charge and we investigate the spectrum of the quasinormal modes of oscillations. Both these studies are in connection with GW observations: while the scalar monopole influences the generation of the signal in the inspiral phase, the quasinormal modes characterize it in the ringdown phase. This material is presented in Chapter 4.

- In Chapter 5 we study a special class of black hole metrics, so called nonsingular or regular black holes. While not being explicit solutions of specific gravity theories, they are artificial modifications of the ususal black hole solutions, in which the singularity is smoothened and gives rise to an effective nonsingular classical spacetime. Therefore they are proposed as models of unknown quantum effects. We will study the consistency of this proposal under Hawking evaporation. We find that the presence of a mass inflation instability can make nonsingular models severely inconsistent, due to the fact that the Hawking evaporation time is infinite.

Before presenting our original material, in Chapter 2 we give a review of classical black holes both in GR and in modified theories. Finally, Chapter 6 contains concluding remarks and an overview of future research prospects.

2

Black holes in general relativity and beyond

In GR, black holes (BHs) are extremely simple objects, actually the simplest objects one can think of. This comes thanks to a remarkable series of results, collectively known under the name of "no-hair theorem" [22]. The no-hair theorem states that the most general isolated, stationary, asymptotically flat BH solution of Einstein–Maxwell theory is represented by the Kerr–Newman family. Kerr–Newman BHs are uniquely characterized by three conserved charges, namely the mass, the spin and the electric charge, while all the other multipole moments vanish identically. Moreover, astrophysical BHs are reasonably expected to have a negligible charge-to-mass ratio [23, 24], so the relevant solutions actually simplifies to Kerr BHs. The no-hair theorem has led to the "Kerr-hypothesis", that the endpoint of any gravitational collapse will be a Kerr BH.

What is the status of these arguments in modified gravity? The no-hair theorem has been extended to several modifications of GR: the most notable extension is to scalar-tensor theories of the Bergmann–Wagoner type; Horndeski theories with a shift symmetric scalar field are also known to satisfy the theorem, but so far only for static and slowly rotating BHs; the Kerr metric is also a solution of vacuum $f(R)$ theories, although not necessarily the only one. On the other hand, however, BHs do deviate from the Kerr solution in Lorentz-violating gravity, in massive gravity, in Einstein–dilaton–Gauss-Bonnet theory and in dilaton–Chern–Simons theory, just to mention several examples. Therefore, there exist classes of modified gravity theories that predict the appearence of hairy BHs. It must be stressed that such hair do not correspond necessarily to new conserved charges. One can also have nontrivial configuriations of the fields, in which the multipole moments are expressed as functions of the mass and the spin; in this case, we talk about "secondary hair". For a comprehensive overview of hairless and hairy BHs in modified gravity see [25, 26, 27, 28, 29].

Black hole hair emerge not only in modified gravity, but also in the phenomenology of some proposed extensions of the standard model of particle physics. One relevant example is the phenomenon of superradiant instability: massive ultralight bosons can accrete quasi-stable clouds around spinning BHs via superradiance, later decaying emitting GWs at characteristic frequencies [30, 31]. Alternatively, one can think to exotic forms of matter emerging as low energy descriptions of more fundamental physics: this is the case with Einstein–Maxwell–dilaton theory, in which a scalar field is dilatonically coupled to a Maxwell field [32, 33].

Even in theories where the stationary BH solutions are the same as in GR, the dynamics far from equilibrium will be sensitive to deviations from GR. For example, they would have an impact on the GW waveforms, both at the level of emission and propagation mechanisms [20, 21].

From all these considerations, it is clear that that BHs constitute important testbeds for modified gravity theories. This Chapter wants to offer an introduction to BHs, both in GR and in modified gravity theories. This is of course not an exhaustive review of the subject, but we choose to focus on those aspects which are more preparatory to the original contributions presented in the following Chapters.

The Chapter is organized as follows. Sec.2.1 introduces the main properties of the Kerr-Newman spacetime. Sec.2.2 reviews the laws of BH mechanics and thermodynamics in GR, and comments on their status in modified theories. Sec.2.3 describes the basic principles of BH perturbation theory, illustrates the main properties of the quasinormal modes of oscillations of BHs and explains how they can be used for theory testing. Finally, Sec.2.4 offers an overview of several modified gravity theories, along with their BH solutions.

2.1 The Kerr-Newman solution

In four spacetime dimensions, the Kerr-Newman (KN) spacetime is the only stationary asymptotically flat BH solution of Einstein-Maxwell theory

$$\mathcal{S}_{EM} = \int d^4 \frac{\sqrt{-g}}{16\pi} \left[R - F_{ab}F^{ab} \right] \tag{2.1}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the Maxwell strenght of the electromagnetic vector potential A_a. The KN solution is given by the line element [34, 35]

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) dt^2 - 2a\left(\frac{r^2 + a^2 - \Delta}{\Sigma}\right)\sin^2\theta dt d\varphi + \left[\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right]\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \quad (2.2)$$

and by the vector potential

$$A_a dx^a = \frac{Qr}{\Sigma}\left(dt - a\sin^2\theta d\psi\right) , \quad (2.3)$$

where

$$\Sigma = r^2 + a^2\cos^2\theta \, , \quad (2.4a)$$

$$\Delta = r^2 + a^2 + Q^2 - 2Mr \, . \quad (2.4b)$$

The solution describes an object with mass M, angular momentum $J = aM$ and electric charge Q. These three parameters specify the solution completely. When $M \neq 0$, the spacetime possess a true singularity at $\Sigma = 0$. If $a^2 + Q^2 > M^2$ the solution does not have any horizon and the singularity is naked. Therefore we assume $a^2 + Q^2 \leq M^2$, for which the solution describes a spinning BH with outer and and inner horizons of radius $R_\pm$, given by

$$\Delta|_{R_\pm} = 0 \implies R_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \, . \quad (2.5)$$

If $a^2 + Q^2 = M^2$ the inner and outer horizons coincide and the BH is said to be extremal. The outer event horizon is foliated by two-surfaces of spherical topology, with area

$$A_H = 4\pi\left(R_+^2 + a^2\right) \, . \quad (2.6)$$

The spacetime (2.2)–(2.3) is stationary and axysimmetric, with $t^a = (\partial/\partial t)^a$ being the Killing vector field associated to time translations and $\psi^a = (\partial/\partial\varphi)^a$ the one corresponding to rotations about the rotational axis $(\theta = 0, \pi)$. The event horizon is a null Killing hypersurface, generated by the Killing field

$$\chi^a = t^a + \Omega_H \psi^a \quad (2.7)$$

where Ω_H is the angular velocity of the event horizon

$$\Omega_H \equiv -\left.\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right|_{R_+, \theta=\pi/2} = \frac{a}{a^2 + R_+^2} \, . \quad (2.8)$$

When $a \neq 0$, the Killing field t^a is timelike at spatial infinity but spacelike at the event horizon, the transition occurring at

$$g_{tt}|_{\tilde{R}_\pm} = 0 \implies \tilde{R}_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta - Q^2}\,. \tag{2.9}$$

Notice that $\tilde{R}_- \leq R_- \leq R_+ \leq \tilde{R}_+$, with the first and last inequalities being saturated at the poles $\theta = 0, \pi$.

The portion of spacetime bounded by $\tilde{R}_+$ and R_+ is the so called ergoregion. In the ergoregion, while still being outside the BH, modes with negative Killing energy can be produced: this is at the core of the Penrose process and of its wave analog, superradiance [34, 36]. It can be shown that the process also extracts angular momentum, until the initial spinning BH has reduced to a static one. Interestingly, one can define an irreducible mass $M_{\rm irr}$, as the fraction of the initial BH mass that cannot be extracted via the Penrose process. It turns out that $M_{\rm irr} = \sqrt{A_H/16\pi}$ [37, 34]. This observation will play a role when discussing the laws of BH mechanics in Sec.2.2.

The causal structure of the KN BH is elucidated by its maximally extended conformal Penrose diagram [35, 3]. The diagram presents a different global structure, depending on whether one considers static ($a = 0$) and/or electrically neutral ($Q = 0$) BHs, and whether one considers nonextremal or extremal configurations. We concentrate on the domain of outer communication (see e.g. [3] for a complete description of the Penrose diagram for each of the mentioned cases). In the nonextremal case, the diagram has the general structure sketched in Fig.2.1.

Region I is the BH exterior: it is bounded at asymptotic infinity by *past and future null infinity*, $\mathscr{I}^-$ and $\mathscr{I}^-$, *past and future timelike infinity*, i^+ and i^-, and *spacelike infinity* i^0. Light cones propagate as straight lines at 90 degrees. Any causal curve originating from a point in region II cannot reach the future infinities $\mathscr{I}^+$ and i^+: therefore II is a black hole region and $\mathcal{H}^+$ is the *future event horizon*. Viceversa, any causal curve originating in region III either reaches the future infinities or falls in the black hole region: therefore III is a white hole region and $\mathcal{H}^-$ is the *past event horizon*. In the nonextremal configuration, the past and future event horizons intersect at the special surface $\mathcal{B}$, called the *bifurcation surface*: $\mathcal{B}$ has the notable property that the Killing field χ^a generating the horizon vanishes there. If the BH is extremal, a topology change occurs [38] and the bifurcation surface $\mathcal{B}$ is replaced by a special point at infinity [3].

An important property of the event horizon is that the surface gravity κ, defined by [34]

$$\chi^a \nabla_a \chi^b \big|_{\mathcal{H}_+} = \kappa \chi^b \tag{2.10}$$

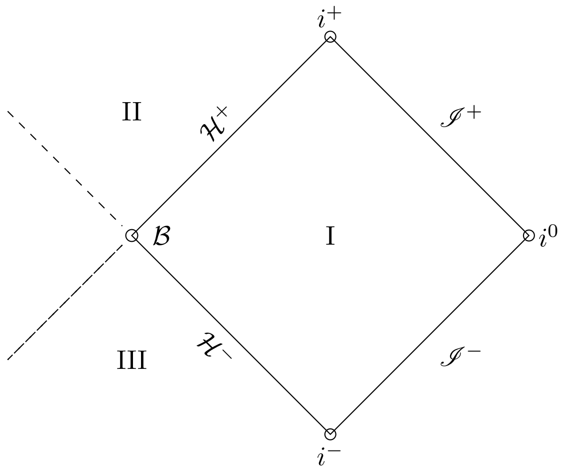

Figure 2.1: The outer domain of communication of a black hole in GR (Region I). Region II and Region III are, respectively, the black hole ad the white hole regions. If the black hole is not extremal, $\mathcal{B}$ is the bifurcation surface, otherwise it is a special point at infinity.

is constant on it. For the Kerr-Newmann black hole

$$\kappa = \frac{\sqrt{M^2 - Q^2 - a^2}}{R_+^2 + a^2} \,. \tag{2.11}$$

The surface gravity of a stationary BH measures the inaffinity of the Killing field χ^a at the event horizon; it is also quantifies of how much a congruence of light rays lingering $\mathcal{H}^+$ peels off [39]. Notice that, in the extremal limit, $\kappa = 0$.

As we shall see, κ plays a key role in the interpretation of the so called FIRST LAW OF BLACK HOLE MECHANICS. The first law is a variational identity relating linear changes of the mass, electric charge and angular momentum to the area growth of the event horizon [40]. It can be easily obtained by varying A_H linearly with respect to M, a and Q and using $J = Ma$, the result being

$$\boxed{\delta M = \frac{\kappa}{8\pi}\delta A_H + \Omega_H \delta J + V_E \delta Q} \tag{2.12}$$

where V_E is the electric potential at the event horizon

$$V_E = \chi^a A_a|_{R_+} = \frac{Q R_+}{R_+^2 + a^2} \,. \tag{2.13}$$

The first law can be intepreted mathematically, as relating two stationary KN solutions with infinitesimally different global charges, or physically, as governing the new equilibrium state of the BH after swallowing an infinitesimal amount of matter [41]. The physical process interpretation is the most enlightening: not only it shows that BHs are evolving objects, but it also hints that their evolution obeys simple mechanical laws. In the next section we will see that this is indeed the case and the dynamics of BHs can be codified in the so called four laws of black hole mechanics.

2.2 The laws of black hole mechanics and thermodynamics

2.2.1 Black hole mechanics in GR

Many properties of the KN BH can be derived without resorting to the explicit form of the solution. Most notably, one can prove that the mechanics obeys four laws, strongly resembling the ones of thermodynamics [42].

The fact that the event horizon of an asymptotically flat, stationary BH must be a Killing hypersuface was proved, with two different sets of assumptions, by Hawking [3] and Carter [43]. Hawking's proof makes use of the Einstein equations and assumes that matter satisfies the dominant energy condition[1]. Under these hypotheses, the spacetime is shown to be necessarily static or stationary-axisymmetric. Moreover there must exist a particular linear combination $\chi^a = t^a + \Omega_H \psi^a$ of the stationarity Killing field t^a and the axisymmetry Killing field ψ^a which is null on the horizon. Therefore the horizon is a null Killing hypersurface. Carter reaches analogous conclusions, but without using the Einstein equations nor any energy condition. Instead, he *assumes* that the spacetime is static or stationary-axisymmetric with a $t - \psi$ reflection symmetry; the last requirement means that the $t - \psi$ plane is orthogonal to a family of two dimensional surfaces. Carter's proof makes much stronger assumptions on the spacetime symmetries, yet it is remarkable that the result does not depend on specific equations of motion.

In both Hawking's and Carter's sets of assumptions, one can also prove that the surface gravity of the Killing horizon is constant [34, 43, 44]. The constancy of the surface gravity is known as the ZEROTH LAW OF BLACK HOLE MECHANICS. There is a connection between the zeroth law and the existence of a bifurcation surface. Racz and Wald [45, 44] showed that, if the surface gravity is constant and nonvanishing, the spacetime can be globally

[1] The dominant energy condition states that, for any timelike future-directed vector field ξ^a, the $-T^a{}_b \xi^b$ must be a future directed timelike or null vector, where T_{ab} is the matter stress energy tensor [34].

extended to one in which the horizon is part of a bifurcate Killing horizon; viceversa, it is easy to prove that the surface gravity of a bifurcate Killing horizon is constant. Taken together, these findings support the expectation that the event horizon is either a bifurcate Killing horizon with constant nonvanishing surface gravity or an extremal Killing horizon with vanishing surface gravity. To our knowledge, the only counterexamples so far are BH solutions in Lorentz-violating gravity theories. These BHs require a separate treatment and we postpone their discussion to Sec.2.4.

The modern derivation of the FIRST LAW OF BLACK HOLE MECHANICS is due to Iyer and Wald [46, 47] and, remarkably, it extends well beyond GR. They showed that, in any diffeoinvariant gravity theory, a bifurcate Killing horizon obeys a variational law, which is a generalization of the first law (2.12). The generalized first law connects the variations of the mass, angular momentum and other conserved charges to the variation of a specific functional at the horizon, so called Wald's entropy. In the GR limit Wald's entropy reduces to the area of the event horizon, in agreement with (2.12). We shall review Wald's derivation of the first law in Sec.2.2.2.

Hawking proved that, if the cosmic censorship conjecture[2] holds and matter satisfies the null energy condition[3] at the event horizon, the area of the latter is never decreasing [3, 34]. This is the famous Area Theorem, also known as the SECOND LAW OF BLACK HOLE MECHANICS. The Area Theorem makes explicit use of the Einstein equations and, in general, it does not extend to modified theories [48]. This theorem led Bekenstein [49] to conjecture that the area of a BH in GR has the physical meaning of an entropy. If a BH does not have an entropy, Bekenstein reasoned, then it would be possible to decrease the ordinary entropy of the outside Universe by carefully dropping entropic matter past the event horizon. This awful violation of the second law of thermodynamics could be avoided if the lack of matter entropy was compensated by a corresponding increase of the BH entropy. Now, the monotonic increase of the area is reminiscent of the monotonic increase of the entropy in ordinary thermodynamics. Moreover, for an isolated BH, an increase in the area corresponds to an increase in the irreducible mass, i.e. the amount of energy that cannot be converted into work from outside; this is reminiscent of the fact that an increase in entropy corresponds to a degradation of energy into heat. Therefore Bekenstein argued that the area of the

[2]The Cosmic Censorship Conjecture, originally due to Penrose, speculates that the collapse of any "reasonable" form of matter cannot produce a naked singularity and all the singularities are protected by an event horizon. See [34] for a discussion and for possible mathematical formulations.

[3]The null energy condition states that $T_{ab}\xi^a\xi^b \geq 0$ for every future directed null vector ξ^a, where T_{ab} is the matter stress energy tensor.

event horizon can represent the BH entropy and that the Area Theorem is the BH analogue of the second law of thermodynamics.

This temptative analogy between BH mechanics and thermodynamics is further supported by the formal resemblance between the first law (2.12) and the first law of ordinary thermodynamics, $\delta E = T\delta S - P\delta V$, which expresses the conservation of energy. The mass M obviously corresponds to the total energy E, A_H is proportional to the entropy S, while $\Omega_H \delta J + V_E \delta Q$ is a work term analogous to $-P\delta V$. The analogy fails only in that, classically, κ does not correspond to any temperature T; even more dramatically, the temperature outside a stationary black hole is classically expected to be absolute zero. This issue was later fixed by Hawking, who showed that, taking quantum mechanics into account, the surface gravity κ actually has the meaning of a physical temperature. We shall review these developments, along with their implications, in the Sec.2.2.3.

We close this discussion with a last, but more debated, analogy between BH mechanics and thermodynamics. The third law of ordinary thermodynamics, in its strong formulation, states that the entropy of a system at zero absolute temperature tends to a universal constant, which can be put to zero without loss of generality. In another formulation (weak formulation) it states the the absolute temperature of a system cannot be reduced to zero within a finite number of operation on its thermodynamical parameters. Guided by the surface gravity/temperature analogy, the zero temperature state corresponds to an extremal BH configuration ($\kappa = 0$), and one is thus tempted to formulate a THIRD LAW OF BLACK HOLE MECHANICS [42]. Israel [50] proved a weak version of the third law: extremal BHs are unattainable in a finite advanced time, if the matter stress energy tensor is bounded and respects the weak energy condition[4] in the neighborhood of the event horizon. This conclusion is reinforced by explicit calculations [51, 52]. Apparently, BHs cannot obey the strong version of the third law, because the area (2.6) (and thus the Bekenstein entropy) of an extremal is not a universal constant at extremality, but it depends on the parameters of the solution. However, it has been argued that extremal BHs should not be conceived as continuous limits of nonextremal ones. For example, as we saw above, there is an irreducible topological difference between the two configurations. Therefore, making a conceptual distinction between a black hole *tending to extremality* and an *exactly extremal* one, it might be possible to formulate a strong version of the third law in terms of the last notion. For a discussion along this line see [53].

[4]The weak energy condition states that $T_{ab}\xi^a\xi^b \geq 0$ for every future directed timelike vector ξ^a, where T_{ab} is the matter stress energy tensor.

2.2.2 Wald's derivation of the first law

In this subsection we review the derivation of the first law by Iyer and Wald [46, 47]. They consider a bifurcate Killing horizon, which serve as an inner boundary $\mathcal{H}^+$ of a globally hyperbolic, asymptotically flat portion of the spacetime. From the standard arguments recalled above in Sec.2.2, $\mathcal{H}^+$ has a constant surface gravity κ. The goal is to show that, in generic $D > 2$ spacetime dimensions, any gravitational theory specified by a diffeoinvariant action admits a first law of the form

$$\delta\mathcal{E} = \frac{\kappa}{2\pi}\delta S + \Omega_H \delta J \tag{2.14}$$

where Ω_H and J are the angular velocity and momentum of $\mathcal{H}^+$ respectively, S is a functional locally constructed out of the dynamical fields at $\mathcal{H}^+$ and $\mathcal{E}$ is a generalized notion of energy (which includes also contributions from long-range dynamical fields, such as gauge fields).

It proves more fitting to work in the language of differential forms. Let then $\mathbb{L}$ be the Lagrangian D-form, $\mathbb{L} = \mathcal{L}\,\epsilon$, where ϵ is the spacetime volume D-form

$$\epsilon = \frac{\epsilon_{a_1\ldots a_D}}{D!}\, dx^{a_1} \wedge \cdots \wedge dx^{a_D} \tag{2.15}$$

and $\epsilon_{a_o\ldots a_{D-1}}$ is the totally antysimmetric densitized Levi-Civita symbol with

$$\epsilon_{0\ldots D-1} = \sqrt{-g}\,.$$

The Lagrangian is a local functional of the dynamical fields collectively denoted as ϕ^α, where α is a collective index for the fields degrees of freedom (the fields ϕ^α include also the metric tensor g_{ab}). Under a first order variation $\delta\phi^\alpha$, $\mathbb{L}$ transforms as

$$\delta\mathbb{L} = \mathbb{E}_\alpha \delta\phi^\alpha + d\Theta(\phi, \delta\phi)\,. \tag{2.16}$$

The $(D-1)$-form $\Theta(\phi, \delta\phi)$, the "symplectic potential", is locally constructed out of ϕ^α and $\delta\phi^\alpha$, and it is linear in $\delta\phi^\alpha$ (but see [47] for discussions about possible ambiguities in the definition). From (2.16) we read the EOM $\mathbb{E}_\alpha \hat{=} 0$ $\forall\,\alpha$.[5]

Since the theory is diffeoinvariant, under a diffeomorphism $\pounds_\xi\phi^\alpha$ along an arbitrary fixed vector field ξ^a the Langrangian transforms as

$$\pounds_\xi\mathbb{L} = d\,(i_\xi\mathbb{L}) = \mathbb{E}_\alpha \pounds_\xi\phi^\alpha + d\Theta(\phi, \pounds_\xi\phi) \hat{=} d\Theta(\phi, \pounds_\xi\phi)\,. \tag{2.17}$$

[5]Hereafter, when we find it useful, we use the hatted symbol $\hat{=}$ to mean identites holding *on-shell*, i.e. when the equations of motion are statisfied.

In the first step we used Cartan's formula $\pounds_\xi = i_\xi d + d i_\xi$, where i_ξ denote the inner product with ξ, and the fact that $\mathbb{L}$ is a form of maximum degree D (which implies $d\mathbb{L} = 0$). From (2.17) we see that the $(D-1)$-form

$$\mathbb{J}[\xi] = \Theta(\phi, \pounds_\xi \phi) - i_\xi \mathbb{L} \tag{2.18}$$

is closed on-shell. $\mathbb{J}[\xi]$ is the Noether current associated with the invariance of the action under diffeomorphisms along ξ. Following [54], one can argue that $\mathbb{J}[\xi]$ is also exact, $\mathbb{J}[\xi] = d\mathbb{Q}[\xi]$, where the Noether charge $(D-2)$-form $\mathbb{Q}[\xi]$ will be crucial in the following. Varying (2.18) w.r.t. $\delta\phi^\alpha$ and using (2.16) we obtain

$$\delta d\mathbb{Q}[\xi] - d[i_\xi \Theta(\phi, \pounds_\xi \phi)] \hat{=} \omega(\phi, \delta\phi, \pounds_\xi \phi) \tag{2.19}$$

where $\omega(\phi, \delta\phi, \pounds_\xi \phi) = \delta\Theta(\phi, \pounds_\xi \phi) - \pounds_\xi \Theta(\phi, \delta\phi)$ is the symplectic current [55]. Integrating the symplectic current over a Cauchy surface Σ we obtain the symplectic form

$$\Omega(\phi, \delta\phi, \pounds_\xi \phi) = \int_\Sigma \omega(\phi, \delta\phi, \pounds_\xi \phi)\,. \tag{2.20}$$

By definition, Hamilton's equations of motion for the dynamics generated by the vector field ξ^a are

$$\delta H[\xi] = \Omega(\phi, \delta\phi, \pounds_\xi \phi)\,. \tag{2.21}$$

Now, we assume that $\delta\phi^\alpha$ solves the linearized EOM; then we can write $\delta d\mathbb{Q}[\xi] = d\delta\mathbb{Q}[\xi]$ and thus

$$\delta H[\xi] = \int_{\partial\Sigma} \left[\delta\mathbb{Q}[\xi] - i_\xi \Theta(\phi, \delta\phi)\right]\,. \tag{2.22}$$

The Hamiltonian $H[\xi]$ is well defined if there exists a form $\mathbb{B}(\phi)$ such that $i_\xi \Theta(\phi, \delta\phi) \equiv \delta\left[i_\xi \mathbb{B}(\phi)\right]$, so that

$$H[\xi] = \int_{\partial\Sigma} \left[\mathbb{Q}[\xi] - i_\xi \mathbb{B}\right]\,. \tag{2.23}$$

In an asymptotically flat spacetime, it is natural to identify the canonical energy $\mathcal{E}$ as the Hamiltonian associated with time translations at infinity

$$\mathcal{E} = \int_{S_\infty} \left[\mathbb{Q}[\xi] - i_t \mathbb{B}\right] \tag{2.24}$$

where t^a is the Killing field relative to time translations and the integration is over a two-sphere S_∞ at asymptotic spatial infinity. Similarly, the canonical angular momentum J is identified as the Hamiltonian associated with rotations (notice a conventional minus sign in the definition)

$$J = -\int_{S_\infty} \mathbb{Q}[\psi] \tag{2.25}$$

where ψ^a is the rotational Killing field and the second therm in the integrand of (2.23) does not contribute because ψ^a is tangent to the integration surface[6].

At this point we can make contact with BH mechanics by recalling that, for a stationary BH, t^a and ψ^a are global Killing vectors and that $\chi^a = t^a + \Omega_H \psi^a$ is null at the horizon $\mathcal{H}^+$. If we assume that all the fields respect the same symmetries of the metric, i.e. $\pounds_t \phi^\alpha \hat{=} \pounds_\psi \phi^\alpha \hat{=} 0$, then from (2.22) $\delta H[\chi]$ vanishes identically. Therefore, integrating (2.22) over a Cauchy surface extending from the bifurcation surface $\mathcal{B}$ to spatial infinity, we obtain

$$\delta\mathcal{E} - \Omega_H \delta J = \int_{\mathcal{B}} [\delta\mathbb{Q}[\chi] - i_\chi \Theta(\phi, \delta\phi)] = \int_{\mathcal{B}} \delta\mathbb{Q}[\chi] \tag{2.26}$$

where in the second step we used $\chi^a|_{\mathcal{B}} = 0$. It can be shown [47] that the RHS of (2.26) is equal to $\kappa\delta S_W/2\pi$, where

$$S_W = -2\pi \int_{\mathcal{B}} E_R^{abcd} \hat{n}_{ab} \hat{n}_{cd}\, \bar{\epsilon}\,, \tag{2.27}$$

$\hat{n}_{ab}$ is the binormal to $\mathcal{B}$, $\bar{\epsilon}$ is the surface element of Σ and $E_R^{abcd} = \delta\mathcal{L}/\delta R_{abcd}$ is the functional derivative of $\mathcal{L}$ w.r.t. R_{abcd}. Moreover S_W also coincides with

$$S_W = \frac{2\pi}{\kappa} \int_{\mathcal{B}} \mathbb{Q}[\chi]\,. \tag{2.28}$$

We refer to [46, 47] for the technical details. Finally, putting all the pieces togheter, we obtain

$$\delta\mathcal{E} = \frac{\kappa}{2\pi} \delta S_W + \Omega_H \delta J\,. \tag{2.29}$$

Some remarks are in order. First, formula (2.27) is obtained under the assumption that all the fields are regular at the bifurcation surface: this allows to drop many contributions using $\chi^a|_{\mathcal{B}} = 0$. However, when gauge fields are considered, one might be forced to single out a specific gauge choice in order to satisfy this condition. Second, the first law (2.29) is more general than (2.12) in that it applies also to nonstationary perturbations.

The quantity (2.27) is the so called Wald entropy: it is locally constructed out of the fields and their variations at $\mathcal{B}$ and reduces to $A_H/4$

[6] We are assuming that there is only one asymptotic rotational symmetry. However, when $D > 4$, a stationary spacetime admits several rotational symmetries in orthogonal directions, with associated Killing fields ψ^a_μ. The additional index μ labels the different rotational directions of symmetry. Correspondingly, there will be several angular momenta $J_\mu = -\int_\infty \mathbb{Q}[\psi_\mu]$. Therefore, for generic $D > 4$, the Killing field generating the horizon will be $\chi^a = t^a + \Omega_H^\mu \psi^a_\mu$, with Ω_H^μ the angular velocity of the horizon in the μ direction. In the main text, for simplicity, we take $\mu = 1$, the generalization to $\mu > 1$ being straightforward.

for the Einstein-Hilbert Lagrangian $\mathcal{L} = R/16\pi$. Indeed

$$S_{GR} = -\frac{1}{8}\int_{\mathcal{B}} \frac{\delta R}{\delta R_{abcd}} \hat{n}_{ab}\hat{n}_{cd}\bar{\epsilon} = -\frac{1}{8}\int_{\mathcal{B}} \hat{n}^{ab}\hat{n}_{ab}\bar{\epsilon} = \frac{A_H}{4} \tag{2.30}$$

where in the last step we used $\hat{n}^{ab}\hat{n}_{ab} = -2$.

Example: first law for a Kerr-Newman black hole

In the remaining of this section, we rederive the first law (2.12) for a KN BH using Wald's method. The Lagrangian 4-form is

$$\mathbb{L} = \frac{1}{16\pi}\left(R - F^{ab}F_{ab}\right)\epsilon = \frac{1}{16\pi}\left(R\epsilon - F \wedge \star F\right) \tag{2.31}$$

where $A = A_a dx^a$ and $F = dA = F_{ab}dx^a \wedge dx^b/2$. The generic linear variation of $\mathbb{L}$ is

$$\delta\mathbb{L} = \frac{1}{16\pi}\left[\mathbb{E}_{ab}\delta g^{ab} + \mathbb{E}^a\delta A_a\right] + d\Theta\,. \tag{2.32}$$

Here the EOM forms are

$$16\pi\mathbb{E}_{ab} = R_{ab} - 2F_a{}^c F_{cb} - \frac{1}{2}\mathcal{L}g_{ab}\,, \tag{2.33a}$$

$$4\pi\mathbb{E}^a = \nabla_b F^{ab}\,, \tag{2.33b}$$

while the symplectic potential Θ spits into a GR part and an electromagnetic part, $\Theta = \Theta_{GR} + \Theta_{EM}$, with[7]

$$\Theta_{GR} = \frac{1}{16\pi}\left(g_{bc}\nabla^a\delta g^{bc} - \nabla_b\delta g^{ab}\right)\epsilon_a\,, \tag{2.34a}$$

$$\Theta_{EM} = -\frac{1}{4\pi}F^{ab}\delta A_b\,\epsilon_a = -\frac{1}{4\pi}\delta A \wedge \star F\,. \tag{2.34b}$$

Using (2.18),(2.33) and (2.34) we obtain $\mathbb{Q}[\xi] = \mathbb{Q}_{GR}[\xi] + \mathbb{Q}_{EM}[\xi]$, with

$$\mathbb{Q}_{GR}[\xi] = -\frac{1}{16\pi}\nabla^a\xi^b\epsilon_{ab}\,, \tag{2.35a}$$

$$\mathbb{Q}_{EM}[\xi] = -\frac{1}{8\pi}\left(\xi^c A_c\right)F^{ab}\epsilon_{ab} = -\frac{1}{4\pi}\left(i_\xi A\right)\star F\,. \tag{2.35b}$$

Correspondingly, the expression for $\delta\mathcal{E}$ splits into $\delta\mathcal{E} = \delta\mathcal{E}_{GR} + \delta\mathcal{E}_{EM}$, with

$$\delta\mathcal{E}_{GR} = -\frac{1}{16\pi}\int_{S_\infty}\left[\delta\left(\nabla^a t^b\right) + \left(g_{cd}\nabla^a\delta g^{cd} - \nabla_c\delta g^{ac}\right)t^b\right]\hat{n}_{ab}\bar{\epsilon}\,, \tag{2.36a}$$

$$\delta\mathcal{E}_{EM} = -\frac{1}{4\pi}\int_{S_\infty}\left[\left(i_t A\right)\delta\left(\star F\right) - \delta A \wedge i_t\left(\star F\right)\right]. \tag{2.36b}$$

[7] We use the notation $\epsilon_a = \left[\epsilon_{ab_1\ldots b_{D-1}}dx^{b_1} \wedge \cdots \wedge dx^{b_{D-1}}\right]/(D-1)!$, $\epsilon_{ab} = \left[\epsilon_{abc_1\ldots b_{D-2}}dx^{c_1} \wedge \cdots \wedge dx^{c_{D-2}}\right]/(D-2)!$ and so on.

In order to proceed, we must recall some generic properties of the Maxwell field at the event horizon $\mathcal{H}^+$. We parallel the arguments of [43, 56, 57]. Assume that, in the background solution, the vector potential A_a respects the same symmetries of the underlying metric, $\pounds_t A_a \hat{=} \pounds_\psi A_a \hat{=} 0$. Since the generator $\chi^a = t^a + \Omega_H \psi^a$ is expansionless, shearless and twistless, the Raychauduri equation implies $R_{ab}\chi^a\chi^b = 0$ at $\mathcal{H}^+$. Using Einstein's equations (2.33), this gives $F_a{}^c F_{bc}\chi^a\chi^b\big|_{\mathcal{H}^+} = 0$, which is equivalent to the statement that the vector field $F^a{}_b\chi^b$ is null on the horizon. However, from the antisymmetry of the Maxwell tensor, $F_{ab}\chi^a\chi^b = 0$ everywhere. Therefore the pullback of $F^a{}_b\chi^b$ on the horizon vanishes: $i_\chi F|_{\mathcal{H}^+} = 0$. The same conclusion can be reached also about $\star F$, after noticing that the stress energy tensor of the Maxwell field can be equivalently rewritten as

$$T^{EM}_{ab} = 2F_a{}^c F_{bc} - \frac{1}{2}F^2 g_{ab} \equiv 2(\star F)_a{}^c(\star F)_{bc} - \frac{1}{2}(\star F)^2 g_{ab} \tag{2.37}$$

where $F^2 = F_{ab}F^{ab}$, $\star F_{ab} = F^{cd}\epsilon_{cdab}/2$ and $(\star F)^2 = (\star F)_{ab}(\star F)^{ab}$. Therefore, using $\pounds_\chi A = 0$ and Cartan's equation $\pounds_\chi = i_\chi d + d i_\chi$, we obtain

$$0 = \pounds_\chi A = i_\chi F + d\,(i_\chi A) \overset{\mathcal{H}^+}{=} d\,(i_\chi A)\ . \tag{2.38}$$

We have thus obtained the familiar result that electromagnetic potential $V_E = i_\chi A|_{\mathcal{H}^+}$ is constant on the the event horizon.

The above reasoning makes explicit use of the Einstein equations. Alternatively [58], if we *assume* the existence of a bifurcation surface, then $i_\chi F = 0 = i_\chi \star F$ on $\mathcal{B}$. It then follows from (2.38) that V_E is constant on $\mathcal{B}$. Moreover, observing that $\pounds_\chi A = 0 \implies \pounds_\chi(i_\chi A) = 0$, it follows that V_E is constant everywhere on the horizon.

From the constancy of V_E, we see that A_a cannot be regular at $\mathcal{B}$, unless we use a gauge in which the electromagnetic potential vanishes at the horizon. In such a gauge, the asymptotically flat falloff conditions for A_a read

$$\lim_{r\to\infty} A_a = \mathcal{C}_a + \frac{\mathcal{A}_a}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\ , \tag{2.39}$$

where $\mathcal{C}_a$ is a constant covecor such that $\chi^a\mathcal{C}_a = V_E$.

Using asymptotically flat falloff conditions for the metric, [47] showed that $\delta\mathcal{E}_{GR}$ is equal to the variation δM of the ADM mass [59]. Similarly, we see from (4.30) that the second term in (2.36b) does not contribute, while the first term gives

$$\delta\mathcal{E}_{EM} = \frac{1}{4\pi}V_E\int_{S_\infty}\delta(\star F) = -V_E\delta Q \tag{2.40}$$

where $Q = -1/4\pi \int_{S_\infty} \star F$ is the electric charge. Therefore the variation of the canonical energy is

$$\delta\mathcal{E} = \delta M - V_E \delta Q \,. \tag{2.41}$$

See also [60] for a similar derivation of (2.41) using the canonical Hamiltonian formalism. Combining (2.29) and (2.41) we eventually obtain the KN first law (2.12)

$$\delta M = \frac{\kappa}{8\pi}\delta A_H + \Omega_H \delta J + V_E \delta Q \,. \tag{2.42}$$

2.2.3 Hawking radiation

The gap between BH mechanics and its thermodynamical intepretation was closed by Hawking in [61, 62], with the discovery of quantum particle creation in a BH spacetime. Hawking employed a semiclassical approach: he studied the evolution of a quantum test field in the background of a *fixed* classical geometry. He found that, when the geometry describes the collapse of an initial matter state into a BH, an asymptotic observer at late time will see her vacuum as populated by a thermal radiation with temperature

$$T_H = \left(\frac{\hbar}{k_B c}\right) \frac{\kappa}{2\pi} \tag{2.43}$$

which is commonly referred as the *Hawking temperature*. We have restored physical units to emphasize the simultaneous interplay of relativity, quantum mechanics and thermodynamics in obtaining this result. The fact that Hawking radiation is exactly thermal [63, 64, 65], and so completely uncorrelated, implies that one cannot really extract any additional information from it: this fact can be regarded as yet another manifestation of the no-hair theorem.

The result does not rely on the Einstein equations nor on any other assumption regarding the gravitational equations of motion (with the obvious exception that a BH can form). From the point of view of the laws of gravity, Hawking radiation is a purely kinematical result, so it holds in large classes of modified gravity theories.

From the Hawking temperatire (2.43) and the first law (2.12), one identifies the BH entropy as the so called Bekenstein entropy

$$S_H = \left(\frac{k_B c^3}{\hbar}\right) \frac{A_H}{4G} \tag{2.44}$$

where again physical units have been restored. This is a huge entropy: for example, the Bekenstein entropy of a solar mass BH is $S_H/k_B \sim 10^{77}$, while the entropy of the Sun is $S_\odot/k_B \sim 10^{58}$. Such a difference is compatible with the educated guess, motivated by the no-hair theorem, that when a star collapses into a black hole most if its initial information is radiated away or made inaccessible from outside, except for the mass, the electric charge and the angular momentum.

Hawking radiation implies that a BH, while classically a stable object, is quantum mechanically unstable: the emission of quanta of radiation causes it to lose energy and shrink, eventually evaporating radiatively. Since Hawking temperature scales roughly as the inverse of the BH mass, for a solar mass BH $T_H \sim 10^{-8}$ K. This validates a posteriori the initial assumption that the backreaction of the test field on background geometry is negligible. Indeed, for an astrophysical BH, it is conceivable that sensible deviations from the semiclassical behaviour will occur only in the last stages of the evaporation, when the BH has shrunk down to Planckian size [66, 67].

It must be stressed that T_H is much smaller than the temperature of the CMB, therefore astrophysical BHs absorb much more radiation than they emit and they are currently not evaporating. Nevertheless, as we shall see in a moment, it is theoretically insightful to consider the ideal case of an isolated BH in its evaporation phase. We can obtain an order of magnitude estimate of the evaporation time by observing that, since the emission spectrum is thermal, the mass loss rate as seen from infinity is governed by Stefan's law

$$\frac{dM}{dt} = -\beta\sigma_{SB}T_H^4\left(4\pi R_+^2\right) \tag{2.45}$$

where $\sigma_{SB} = \pi^2 k_B^4/60\hbar^3c^2$ is the Stefan-Boltzman constant and β is a coefficient of order unity which accounts for effective the absorbtion cross section of the BH [34]. In the Schwarzschild case (2.45) implies $dM/dt \propto -M^{-2}$. Extrapolating up to the evaporation time τ (when $M = 0$) we find $\tau \sim M^3$. For a stellar mass BH, the evaporation time is much geater than the current age of the Universe. Therefore we do not expect to see any sign of BH evaporation in current astrophysical contexts.

This evaporation picture in GR is clearly in violation of the Area Theorem, because the area of the BH diminishes during the process. However, observe that this decrease is compensated by the emission of highly entropic radiation. Conversely, when ordinary matter is swallowed by a BH, the disappearence of the matter entropy is compensated by the increase in the area. This suggests the following extended formulation of the second law:

Generalized second law (GSL)**.** *The sum of the BH entropy $S_H = A_H/4$ and of the matter entropy S_m is never decreasing in the domain of outer* *communication*

$$\Delta\left(S_m + \frac{A_H}{4}\right) \geq 0\,. \tag{2.46}$$

Unruh and Wald [68] pointed out the important fact that the GSL cannot hold at the pure classical level, but Hawking radiation must be taken necessarily into account. Indeed, neglecting Hawking radiation, it is easy to conceive thought experiments in which the GSL is violated, as was already observed e.g. in [42, p.169]. The GSL was proved in [69], but restricted to linear quasistationary perturbations around a stationary BH. The restriction of quasistationarity of the perturbations was later removed by Wall, under some technical assuptions on the quantum matter fields [70].

There is a widespread belief that the occurrence of Hawking radiation poses a serious problem to the ordinary formulation of the physical laws. The problem, commonly known as information loss paradox, is that after the evaporation one is left with empty space filled with a maximally mixed quantum state (thermal radiation), *even if the initially collapsing matter was prepared in a pure state* [65, 71]. This is a violation of unitarity: stated otherwise, the initial information contaneid in the spacetime before the collapse is irremediably lost.

Several speculative solutions have been proposed, which essentially fall into two main categories: (i) the information is contained in a stable or quasistable remnant after the evaporation, or (ii) the information progressively leaks out in later stages of the evaporation, in the form of correlations between the early time and the late time quanta of the radiation. The first hypothesis seems disfavoured by the impossibility of encoding a huge amount of entropy into a Planckian remnant.

Assuming that the second hypothesis is correct, Page [72] argued that information starts to leak out when the initial black hole entropy has decreased by half. At this moment, the so called *Page time*, the BH is still macroscopic: therefore the first signs of deviation from the semiclassical behaviour manifest well before the BH has reached the Planckian regime. However, it was recently argued by [73] that, if this scenario is correct, an infalling observer experiences a “firewall” in the vicinity of the event horizon, thus violating the equivalence principle.

Alternatively, Hawking et al. [74] proposed that the missing information could be encoded into correlations with soft gauge bosons, corresponding to hidden degrees of freedom of the degenerate vacuum state. Finally, one can also accept with Unruh and Wald that information is really lost [75].

2.2.4 Black hole mechanics beyond GR

We want to close this section by commenting about the status of BH mechanics and thermodynamics in BHs beyond GR. We have already seen that the zeroth law is valid for any stationary BH, irrespective of the form of the gravitational action, provided the existence of a bifurcation surface. In the presence of a bifurcation surface, one can also prove a generalized first law, as discussed in Sec.2.2.2. Moreover, Hawking radiation is a kinematical effect and holds for any BHs even in modified gravity.

Therefore, one is tempted to identify Wald's entropy (2.27) as the BH entropy in the generalized second law. However, as observed in [76], S_W suffers from definition ambiguities that, while vanishing for stationary BHs and at $\mathcal{B}$, generically do not vanish for linear perturbations at a cross section of the horizon different from $\mathcal{B}$. Therefore the identification is not straightforward and, in general, it fails [76]. The ambiguites were fixed in [76, 77], showing that S_W must be corrected by additional terms depending on the extrinsic curvature of the horizon. Interestingly, the resulting entropy [77, Eq.(14)] matches the one proposed by Dong [78] from entanglement entropy calculations.

2.3 Black hole perturbation theory

In order for KN BHs to be physically viable, one must verify that they are stable against perturbations. Certainly, the most ambitious goal is to study the problem at the nonlinear level, which requires the development of appropriate numerical techniques. A first investigation, supporting nonlinear stability, was presented only recently in [79].

A more modest yet insightful approach is the study of linear mode stability: one considers small linear perturbations of arbitrary spin fields on top of a stationary BH background, to see if they decay sufficiently fast in time. BH perturbation theory dates back to the seminal work by Regge and Wheeler [80] and Zerilli [81, 82]. An extended account of the subject can be found in Chandrasekhar's milestone book [83]. For recent reviews see [84, 85, 86, 87].

This section is organized as follows. In 2.3.1 we introduce BH perturbation theory in the simple case of the Schwarzschild BH. In 2.3.2 we descibe the concept and the properties of the quasinormal mode of oscillations. In 2.3.3 we briefly review the status of BH perturbation theory for Kerr and KN black holes. Finally, in 2.3.3 we illustrate the connection between quasinormal modes and unstable photon orbits.

2.3.1 Perturbations of the Schwarzschild black hole

In order to introduce BH perturbation theory, we study the simplest case of linear perturbations of bosonic fields with spin s in the fixed background of the Schwarzschild spacetime, mainly following the reviews [86, 28]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{2.47}$$

with $f(r) = 1 - 2M/r$. We focus on scalar ($s = 0$), electromagnetic ($s = 1$) and gravitational ($s = 2$) perturbations. Since the background spacetime is spherically symmetric, we will expand the perturbations in spherical harmonics. When the multipole number l of the harmonic expansion is smaller than the spin, the problem must be treated separately from the general case. We first focus on the general case and comment later about $l < s$.

Consider a minimally coupled massless scalar field Φ. Without loss of generality, we can assume that the unperturbed scalar field vanishes and use the letter Φ for its perturbations as well. The equation of motion is just

$$\nabla_a\nabla^a\Phi = 0\,. \tag{2.48}$$

It is convenient to factorize the angular dependence of the field by expanding in scalar spherical harmonics $Y_l^m(\theta,\varphi)$, whith multipole number l and azimuthal number m. We study the perturbations in the Fourier time domain with (generically complex) frequency ω:

$$\Phi = \sum_{l,m}\int d\omega e^{-i\omega_{lm}t}\frac{Z_{l,m}(r)}{r}Y_l^m(\theta,\varphi)\,. \tag{2.49}$$

Inserting (2.49) into (2.48), the perturbed equation can be cast in the Schroedinger-like form

$$\left(\frac{d^2}{dr_\star^2} + \omega^2\right)Z(r) = V(r)Z(r) \tag{2.50}$$

where the potential $V(r)$ is

$$V(r) = f(r)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) \tag{2.51}$$

and $r_\star$ is the Regge radial coordinate defined by $dr_\star/dr = f(r)^{-1}$. Notice that, for $2M < r < +\infty$, $-\infty < r_\star < +\infty$. We have also omitted the additional subscripts (l,m) from Z and ω to lighten the notation.

Next, we consider the case of the electromagnetic potential A_a and perturb it as $A_a = A_a^{(0)} + \delta A_a$, where $A_a^{(0)}$ is the background part and δA_a is the perturbation. We can expand the angular dependence in vector spherical harmonics: the perturbations naturally separate into *axial perturbation* δA_a^A, which acquire a multiplicative factor $(-1)^{l+1}$ under a parity transformations, and *polar perturbations* δA_a^P, which acquire a factor $(-1)^l$. In the Fourier time domain they read

$$\delta A_a^A = \sum_{l,m} \int d\omega e^{-i\omega t} \left(0, 0, -\frac{u_4(r)\partial_\varphi Y_l^m}{\sin\theta}, u_4(r)\sin\theta\partial_\varphi Y_l^m\right), \tag{2.52a}$$

$$\delta A_a^P = \sum_{l,m} \int d\omega e^{-i\omega t} \left(\frac{u_1(r)Y_l^m}{r}, \frac{u_2(r)Y_l^m}{rf(r)}, 0, 0\right). \tag{2.52b}$$

We exploited the $U(1)$ gauge invariance to gauge out the angular components of δA_a^P. It is convenient to express the perturbations in terms of the Maxwell tensor F_{ab}. The axial and polar parts of the perturbed Maxwell tensor read

$$\delta F_{ab}^A = \sum_{l,m} \int d\omega e^{-i\omega t} \begin{bmatrix} 0 & 0 & -i\omega u_4(r)\partial_\varphi Y_l^m/\sin\theta & i\omega u_4(r)\sin\theta\partial_\theta Y_l^m \\ * & 0 & u_4'(r)\partial_\varphi Y_l^m/\sin\theta & -u_4'(r)\sin\theta\partial_\theta Y_l^m \\ * & * & 0 & l(l+1)u_4(r)\sin\theta Y_l^m \\ * & * & * & 0 \end{bmatrix}, \tag{2.53a}$$

$$\delta F_{ab}^P = \sum_{l,m} \int d\omega e^{-i\omega t} \begin{bmatrix} 0 & f_{01}(r)Y_l^m & f_{02}(r)\partial_\theta Y_l^m & f_{02}(r)\partial_\varphi Y_l^m \\ * & 0 & f_{12}(r)\partial_\theta Y_l^m & f_{12}(r)\partial_\varphi Y_l^m \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \tag{2.53b}$$

where the asterisks denote total antisymmetrization and we have defined the auxiliary variables

$$f_{01}(r) = \frac{ir\,\omega u_2(r) + f(r)\left[ru_1'(r) - u_1(r)\right]}{r^2 f(r)}, \tag{2.54a}$$

$$f_{02}(r) = \frac{u_1(r)}{r}, \tag{2.54b}$$

$$f_{12}(r) = \frac{u_2(r)}{rf(r)}. \tag{2.54c}$$

They are not independent from each other, but obey the Bianchi identity

$$f_{01}(r) = i\omega f_{12}(r) + f_{02}'(r). \tag{2.55}$$

Using the Bianchi identity, the perturbed Maxwell equations

$$\nabla_a \delta F^{ab} = 0 \tag{2.56}$$

can be separated into to two Schroedinger-like equations for axial and polar variables respectively

$$\left(\frac{d^2}{dr_\star^2}+\omega^2\right)Z_i(r)=\frac{l(l+1)}{r^2}Z_i(r) \qquad i=A,P \tag{2.57}$$

where the original perturbations are related to the variables $Z_{A,P}(r)$ through the relations

$$u_4(r)=\frac{iZ_A(r)}{\omega}\,, \qquad f_{01}(r)=-\frac{l(l+1)Z_P(r)}{r^2}\,, \tag{2.58a}$$

$$f_{02}(r)=-f(r)Z'_P(r)\,, \qquad f_{12}(r)=\frac{i\omega Z_P(r)}{f(r)}\,. \tag{2.58b}$$

Finally, let us turn to the metric perturbations. We perturb the metric as $g_{ab}=g^0_{ab}+h_{ab}$, where g^0_{ab} is the Schwarzschild metric (2.47). The perturbations h_{ab} can be expanded in tensor spherical harmonics. Out of the ten independent components of h_{ab}, four can be gauged out with an infinitesimal diffeomorphism, which is a symmetry of the Einstein–Hilbert Lagrangian. As before, the perturbations naturally split into axial h^A_{ab} and polar h^P_{ab}. We work in the Regge-Wheeler gauge [80], where

$$h^A_{ab}=\sum_{l,m}\int d\omega e^{-i\omega t}\begin{bmatrix}0 & 0 & -h_0(r)\partial_\varphi Y^m_l/\sin\theta & h_0(r)\sin\theta\partial_\theta Y^m_l\\ * & 0 & -h_1(r)\partial_\varphi Y^m_l/\sin\theta & h_1(r)\sin\theta\partial_\theta Y^m_l\\ * & * & 0 & 0\\ * & * & * & 0\end{bmatrix}, \tag{2.59a}$$

$$h^P_{ab}=\sum_{l,m}\int d\omega e^{-i\omega t}Y^m_l\begin{bmatrix}f(r)H_0(r) & H_1(r) & 0 & 0\\ * & H_2(r)/f(r) & 0 & 0\\ * & * & r^2K(r) & 0\\ * & * & * & r^2\sin^2\theta K(r)\end{bmatrix} \tag{2.59b}$$

and the asterisks now denote total symmetrization. As in the electromagnetic case, the perturbed Einstein equations can be separated in two Schroedinger-like equations for the axial and the polar variables

$$\left(\frac{d^2}{dr_\star^2}+\omega^2\right)Z_i(r)=V_i(r)Z_i(r) \qquad i=A,P \tag{2.60}$$

where the potentials $V_{A,P}(r)$ are given by

$$V_A(r)=f(r)\left(\frac{l(l+1)}{r^2}-\frac{6M}{r^3}\right) \tag{2.61a}$$

$$V_P(r)=f(r)\left[\frac{2\left(9M^3+9\lambda M^2r+3\lambda^2Mr^2+\lambda^2(\lambda+1)r^3\right)}{r^4(3M+\lambda r)^2}\right] \tag{2.61b}$$

and $\lambda = l(l+1)/2 - 1$. The axial potential (2.61a) was first derived by Regge and Wheeler in [80] and the corresponding axial equation is commonly refered to as the REGGE-WHEELER EQUATION. The derivation of the polar potential (2.61b) is much less trivial: it was obtained by Zerilli in [81] and the polar equation is refered as the ZERILLI EQUATION.

The scalar, electromagnetic and gravitational axial potentials can be collectively written as

$$V_A(r) = f(r)\left(\frac{l(l+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right) . \tag{2.62}$$

For $l \geq s$ the expression inside the big brackets is manifestly positive. This allows to show that a Schwarzschild BH cannot support stationary extra hair outside the event horizon. Indeed, rewrite the axial equation as

$$\frac{d}{dr}\left(\left(1-\frac{2M}{r}\right)Z_A'(r)\right) - \left(\frac{l(l+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right)Z_A(r) = 0 \tag{2.63}$$

where we put $\omega = 0$ to ensure stationarity. If we multiply by the complex conjugate wave function $Z_A^\dagger(r)$ and integrate from $r = 2M$ to $r \to \infty$ we obtain

$$\int_{2M}^{\infty}\left[\left(1-\frac{2M}{r}\right)\|Z_A'(r)\|^2 + \left(\frac{l(l+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right)\|Z_A(r)\|^2\right] = 0 \tag{2.64}$$

where we integrated the first term by parts and took into account that boundary terms vanish for well behaved perturbations. Then the impossibility of stationary hair directly follows from the fact that each member of the integrand is separately positive definite in the domain of integration. The same conclusion can be reached for the Zerilli equation.

As we said above, the case $l < s$ must be treated separately. In particular, for $l = 0$ electromagnetic perturbations, one obtains that the only nontrivial solution has $\omega = 0$ and represents the addition of an infinitesimal electric charge. Similarly, the only non trivial $l = 0$ and $l = 1$ gravitational perturbations represent the addition of an infinitesimal mass ($l = 0$), an infinitesimal spin ($l = 1$ axial) and an infinitesimal linear momentum ($l = 1$ polar) [82]. This is of course a manifestation of the no-hair theorem.

2.3.2 Quasinormal modes

Let us now restore ω and return to te general dynamical perturbations for $l > s$. One expects that, when initially perturbed, a BH will emit waves with characteristic oscillation frequency $\omega_R = \text{Re}(\omega)$ and damping $\omega_I = \text{Im}(\omega)$, i.e.

$\omega = \omega_R + i\,\omega_I$. The complex frequencies ω are the quasinormal modes (QNMs) of oscillation of the BH. They are obtained by solving (2.50) with outgoing boundary conditions at infinity and ingoing boundary conditions at the event horizon. These boundary conditions capture the physical interpretation of the horizon as a one-way membrane. Given that $V(r_\star = \pm\infty) = 0$, we have

$$Z(r) \sim \begin{cases} e^{-i\omega r_\star} \sim (r-2M)^{-2M\,i\omega} & \text{for } r_\star \to -\infty\,, \\ e^{i\omega r_\star} \sim e^{-i\omega r} r^{2M\,i\omega} & \text{for } r_\star \to +\infty\,, \end{cases} \tag{2.65}$$

where in the above asymptotic behaviours we are neglecting constant terms. The BH is stable outside the event horizon if $\omega_I > 0$ for every ω. As shown by [83, §35], for a Schroedinger-like equation of the form (2.50), subjected to the above boundary conditions, unstable modes are absent if $V(r) > 0$ in the domain $-\infty < r_\star < +\infty$. Therefore, all the above perturbed equations do not give rise to linear instabilities.

We restricted to single field perturbations of a Schwarzschild BH. If we allow for a nonvanishing electric charge Q, one cannot perturb the gravitational and electromagnetic fields separately, but must consider their interaction in a consistent treatment. As shown in [88, 83], axial and polar perturbations decouple and one obtains two pairs of Schroedinger-like equations. The gravitational and electromagnetic degrees of freedom are inherently coupled and therefore cannot be excited independently. Nevertheless, one can still distinguish in a sense between "gravitational modes" and "electromagnetic modes" as those which, in the limit $Q \to 0$, reduce respectively to the modes given by the Regge-Wheeler/Zerilli equations (2.60)-(2.61) and the electromagnetic equations (2.57).

A remarkable property of the Schwarzschild and Reissner-Nordstrom perturbations is that their quasinormal modes are isospectral, meaning that the axial and polar spectrum are the same. This property is rooted in a special relation between the Regge-Wheeler and the Zerilli potentials [83], which is a manifestation of a supersymmetry of the problem [89, 90, 91].

The QNM spectrum can be computed with several numerical and analytical approximate techniques [84, 87]. The most efficient one to date is Leaver's continued fraction method [92, 93, 94]. For symmetry reasons, the spectrum of perturbations of a spherically symmetric background must be degenerate with the azimuthal number m, as it is clear from the fact the the potentials above depend only on the multipole number l. For each pair l there is an infinite and discrete set of tones ω_n, where the overtone number $n = 0, 1, 2,$ etc. By convention, the tones are ordered from the least damped tone or *fundamental tone* ω_0 to the most damped one $\omega_{n\to\infty}$. The imaginary part ω_I decreases fastly with n, therefore the spectrum is effectively domininated by the fundamental tones.

2.3.3 Perturbations of a spinning black hole

The extension of the linear mode analysis to spinning BHs is more laborious. Teukolsky [95] showed that the angular and radial dependence of gravitational perturbations of Kerr are separable [83] using spinorial techniques. Although a mathematical proof has not yet been found that Kerr is linear mode stable, the numerical analysis in [96] supports this conclusion. The KN case is even more cumbersome, since nobody has been able to separate gravito-electromagnetic perturbations in this spacetime. This is an outstanding problem in BH perturbation theory [83]. The study of the QNMs of KN was addressed in [97, 98] under the simplifying restriction of small spin, and in [99] under the restriction of small electric charge. Finally, an analysis without such restrictions was performed in [100]. All these studies show convincing evidence that KN is linear mode stable. Moreover, [79] provided further indications in favour of stability at the nonlinear level. Therefore, although formally the problem is not solved, all the existing numerical studies confirm that the KN solution is stable.

Regarding the properties of the QNMs, the nonvanishing angular momentum induces an additional dependence on the azimuthal number m and a corresponding Zeeman-like splitting of the spectrum. It is not clear if isospectrality persists also in the rotating case, and there is no a priori reason to believe it. The only indication that isospectrality may indeed hold comes from the slow-rotation numerical analysis of [97, 98].

Let us mention that there is actually a way to induce instabilities in KN BHs. Recall that rotating BHs exhibit the phenomenon of superradiance, by means of which one can extract energy and angular momentum from the ergoregion. It turns out that massive bosonic fields, with a Compton wavelenght of the same order of the horizon radius, can trap superradiant modes in a potential barrier and grow as a condensate cloud around the BH: this is a superradiant instability [36, 30, 31]. Superradiant instabilities can grow sufficiently to have astrophysical implications: in the case of a real bosonic field, the instability slowly decays though GWs, leading to potentially observable signals from advanced GW detectors [101, 102]; if the field is complex, the GW decay is suppressed and the field can effectively give rise to an hairy BH [103, 104, 105]. For astrophysical BHs, the bosonic mass range of interest is the ultralight range $10^{-20} \div 10^{-10}$ eV. As a consequence, superradiant instability has been proposed as a possible signature of some dark matter candidates, such as dark photons or QCD axions; alternatively, the absence of superradiance can constraint the viability of these models

[102, 106].

2.3.4 The light ring correspondence

Goebel [107] proposed an appealing physical interpretation of QNMs in the eikonal limit $l \gg 1$: they can be considered as the oscillation and damping frequencies of unstable photon orbits around the BH. This connection was further explored by Mashoon and Ferrari [108, 109], who considered the leakage of perturbed light rays from unstable orbits in the KN spacetime, showing that the correspondence is exact in the slow rotating limit. The result was later extended to generic asymptotically flat BHs beyond general relativity in [110].

The light ring correspondence has been extensively explored in the Kerr spacetime [111, 112, 113, 114]. Although it is formally valid only in the limit $l \gg 1$, it provides surprisingly accurate predictions also at small $l \gtrsim 2$. See for example [24, Appendix A], in which it is shown that the exact $l = m = 2$ modes for Kerr and KN BHs differ from their light ring estimates only by 4% or less. We illustrate the procedure in the case of slowly rotating and weakly charged KN BHs. For simplicity we consider equatorial orbits ($\theta = \pi/2$), which correspond to $l = |m|$ QNMs.

The first corrections of KN with respect to Schwarzschild occur at first order in the spin and at second order in the electric charge. In this approximation the metric is specified by the line element

$$ds^2 = -f(r)dt^2 - 2a\,\Omega(r)\sin^2\theta dt d\varphi + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (2.66)$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}\,, \qquad \Omega(r) = \frac{2M}{r} - \frac{Q^2}{r^2}\,. \quad (2.67)$$

At order $\mathcal{O}(a)$, the equations of a general null geodesic in the equatorial plane are

$$\dot{t} = \frac{1}{f(r)} \mp \frac{a\,\Omega(r)L}{r^2 f(r)}\,, \quad (2.68a)$$

$$\dot{\varphi} = \pm\frac{L}{r^2} + \frac{a\,\Omega(r)}{r^2 f(r)}\,, \quad (2.68b)$$

$$\dot{r}^2 = V_{\rm geo}(r) = 1 - \frac{f(r)L^2}{r^2} \mp \frac{2a\,\Omega(r)L}{r^2} \quad (2.68c)$$

where L is the angular Killing angular momentum and, without loss of generality, we set the Killing energy to unity. The plus/minus sign refers to corotating/counterrotating orbits. The radius r_c of the unstable circular orbit and the corresponding Killing angular momentum L are determined by the equations $V_{\rm geo}(r_c) = 0 = V'_{\rm geo}(r_c)$. As shown in [110], unstable null geodesic decay with a pricipal Lyapunov exponent Γ_c given by

$$\gamma_c = \sqrt{\frac{V''_{\rm geo}(r_c)}{2\dot{t}^2}}\,. \tag{2.68d}$$

Then, in the light ring correspondence, the QNMs are given by [110]

$$\omega_n = \pm l\,\omega_c - i\left(n + \frac{1}{2}\right)\gamma_c \tag{2.69}$$

where $\omega_c = \dot{\varphi}/\dot{t}$ is the angular frequency of the orbit.

When specifying to the spacetime (2.66)-(2.67), it is convenient to introduce the adimensional quantities $v = Q/M$ and $\tilde{a} = a/M$, which represent the charge-to-mass and spin-to-mass ratio respectively, in such a way to isolate a universal mass scale M. Then an explicit calculation shows that the radius of the unstable equatorial orbit is

$$M^{-1}r_c = 3\left(1 - \frac{2v^2}{9}\right) \mp \frac{2\tilde{a}}{\sqrt{3}}\left(1 + \frac{v^2}{6}\right) \tag{2.70}$$

while the oscillation and damping frequencies are

$$M\omega_c = \frac{1}{3\sqrt{3}}\left(1 + \frac{v^2}{6}\right) \pm \frac{2\tilde{a}}{27}\left(1 + \frac{v^2}{2}\right)\,, \tag{2.71a}$$

$$M\gamma_c = \frac{1}{3\sqrt{3}}\left(1 + \frac{v^2}{18}\right) \pm \frac{\tilde{a}\,v^2}{243}\,. \tag{2.71b}$$

It is understood that all the expressions hold at order $O(\tilde{a})$ and $O(v^2)$.

2.3.5 Black hole spectroscopy

The existence of QNMs led Detweiler [115] to propose BH spectroscopy as a method for testing GR and the no-hair theorem from GW detections. It is based on the observation that QNMs describe the spectrum of linear perturbations away from a stationary BH *in a universal way*. The typical situation of interest is the formation of a BH in the merger of two binary compact objects: the late part of the waveform (ringdown), is well described as a superposition of QNM oscillations [116]

$$h(t) \sim \mathrm{Re}\left[\sum_{lmn} A_{lmn} e^{-i(\omega_{lmn}t + \phi_{lmn})}\right] \tag{2.72}$$

where the sum is over the multipole and azimuthal numbers l, m and the overtone number n. While the amplitude A_{lmn} and the phase ϕ_{lmn} depend on the particular initial conditions of the system, the (complex) oscillation frequencies ω_{lmn} are universally given by the QNM spectrum, as described in the previous section. In GR the spectrum is dominated by the fundamental $l = 2$ modes, while the subdominant radiation is well approximately accounted by cutting the above sum to $l = 4$ [116, Table I].

Now, if the remnant BH is described by GR, the no-hair theorem holds and the spectrum depends only on the three conserved charges: the mass M, the spin a and the electric charge Q. Considering that astrophysical BHs are expected to have a negligible charge-to-mass ratio, the independent parameters reduce to two. This is a huge constraint on the form of the spectrum and any deviation would signal a departure from the standard GR description. BH spectroscopy then consists in testing GR by detecting and fitting (at least two complex) QNMs [117, 118]. In the words of Detweiler [115]

> After the advent of gravitational wave astronomy, the observation of these resonant frequencies might finally provide direct evidence of BHs with the same certainty as, say, the 21 cm line identifies interstellar hydrogen.

Notice that this is not just and not necessarily a test of the no-hair theorem: indeed, several modified theories have the same stationary solutions as in GR, but the presence of additional degrees of freedom alters the perturbed equations and as a consequence also the QNM spectrum [119].

BH spectroscopy is expected to play a key role in second and third generation GW detectors [21]. These future observational prospects call for a parallel theoretical effort to compute QNMs for black holes beyond GR. Unfortunately, this is a very challenging task. To date, QNMs have been computed only in dynamical Chern-Simons gravity [120, 121, 122, 123] and in Einstein-dilaton-Gauss-Bonnet gravity [124, 125, 126]. A general formalism for perturbations in Horndeski theory was developed in [127, 128] and in [119, 129]. However, in all these cases, the computation is restricted to perturbations of spherically symmetric or slowly rotating BHs.

A theory-agnostic approach to treat fully rotating BHs was proposed in [130]: the authors consider generic small deviations from the Kerr metric and derive approximate QNM frequencies through the light ring correspondence. The main limitation is that the light ring approximation captures only the QNMs of the gravitational family, while it is blind to additional modes induced by additional degrees of freedom. All these considerations show that the determination of QNMs for alternative theories is a fully active research area. In Chapter 4 we will contribute by studying QNMs for BHs in Einstein-Maxwell-dilaton theory.

2.4 A survey of black holes in modified theories

In this section we describe several theories of modified gravity along with their BH solutions. The aim is to present the various types of modifications that can occur at the level of the no hair theorem and of the causal structure. This is not at all intended as a complete coverage of the subject and the reader is referred to e.g. [25, 26, 29, 27] for an extensive overview.

2.4.1 Scalar-tensor theory

Horndeski theory is the most general scalar-tensor theory with second order field equations in four spacetime dimensions [131, 132]. The generic from of the action is

$$\mathcal{S} = \int d^4x \frac{\sqrt{-g}}{16\pi} \sum_{i=2}^{5} L_i \tag{2.73}$$

where

$$L_2 = G_2(\Phi, X)\,, \tag{2.74a}$$
$$L_3 = -G_3(\Phi, X)\Box\Phi\,, \tag{2.74b}$$
$$L_4 = G_4(\Phi, X)R + G_{4,X}(\Phi, X)\left[(\Box\Phi)^2 - \Phi^{ab}\Phi_{ab}\right]\,, \tag{2.74c}$$
$$L_5 = G_5(\Phi, X)G_{ab}\Phi^{ab} - \frac{1}{6}G_{5,X}(\Phi, X)\left[(\Box\Phi)^3 - 3\Phi^{ab}\Phi_{ab} + 2\Phi^{ab}\Phi_{bc}\Phi_a{}^c\right]\,. \tag{2.74d}$$

R is the Ricci scalar, G_{ab} is the Einstein tensor, Φ is a real scalar field, $X = -\nabla^a\Phi\nabla_a\Phi/2$, $\Phi_{ab} = \nabla_a\nabla_b\Phi$, $\Box\Phi = \nabla^a\nabla_a\Phi$, the G_i's are arbitrary functions of Φ and X and $G_{i,X}$ is the derivative of G_i w.r.t. X. All the indices are raised and lowered with the metric tensor g_{ab} and its inverse. Given the arbitrariness of the G_i's, it is not surprising that no general characterization of Horndeski BHs exist. One must rely on specific motivated subclasses of the theory.

A well studied subset of Horndeski is the generalized Brans-Dicke theory, or Bergmann-Wagoner scalar-tensor theory [133, 134]

$$\mathcal{S}_{BD} = \int d^4 \frac{\sqrt{-g}}{16\pi}\left[\Phi R - \frac{\omega(\Phi)}{\Phi}\nabla^a\Phi\nabla_a\Phi - V(\Phi)\right]\,. \tag{2.75}$$

This is the most general subclass with a Lagrangian at most quadratic in the scalar derivatives. In Brans-Dicke, asymptotically flat BHs obey a no-hair theorem [25, 135, 136]: BH solutions are the same as in GR, endowed with a trivial scalar field, provided suitable stability conditions on the potential $V(\Phi)$ are imposed.

A similar no-hair theorem was also proved [25, 137] for static asymptotically flat black holes in SHIFT-SYMMETRIC HORNEDSKI THEORY, i.e. the subclass of Horndeski which is invariant under a constant shift of the scalar $\Phi \to \Phi + \text{const}$. The theorem makes several assumptions, the most notable of which is that the Noether current associated to shift-symmetry does not contain Φ-independent terms. It was shown in [138] that, relaxing this assumption, a counterexample can be found in DECOUPLED DYNAMICAL GAUSS-BONNET GRAVITY (D^2GB)

$$\mathcal{S}_{D^2GB} = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[R - \frac{1}{2} \nabla^a \Phi \nabla_a \Phi + \alpha \Phi \mathcal{G} \right] \tag{2.76}$$

where $\mathcal{G} = R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}$ is the Gauss-Bonnet invariant and α is a coupling constant. In D^2GB theory the scalar field must have a nontrivial profile outside the horizon: explicit hairy solution were constructed in [139] in the decoupling limit, and they were shown to form dynamically in [140, 141].

BHs with scalar hair are particularly relevant for theory testing [20]: indeed, monopole scalar charges are crucial to understanding the emission of scalar radiation in a binary system and allow to put potentially strong constraints on the parameters of the modified theory [142, 15]. In D^2GB case, an elegant proof of hairiness was given in [143] for BHs, together with the key result that the monopole scalar charge Q_S is given by

$$Q_S = \frac{1}{2} \alpha \kappa \mathrm{Euler}(\mathcal{B}) \tag{2.77}$$

where κ is the surface gravity and $\mathrm{Euler}(\mathcal{B})$ is the Euler characteristic of the bifurcation surface $\mathcal{B}$ ($\mathrm{Euler}(\mathcal{B}) = 2$ for spherical topology). The result is valid even beyond the decoupling limit, at all orders in α. Moreover, it does not rely on the details of the gravitational EOM, therefore it is valid even if the kinetic term of the gravitational field differs from the usual Einstein-Hilbert one. We refer to [143] to the details of the proof.

On the ohter hand, horizonless compact objects, such as NSs, in D^2GB cannot have a monopole scalar charge (hereafter simply scalar charge). This is easily seen by integrating the scalar EOM

$$\Box \Phi = -\alpha \mathcal{G} \tag{2.78}$$

over a compact Cauchy hypersurface. While the integral of the LHS gives the scalar charge, the integral of the RHS vanishes identically for flat asymptotics because $\mathcal{G}$ is a topological invariant in four dimensions [142, 144].

It is instructive to consider a simplified version of the theorem in [143], in which the Gauss-Bonnet invariant in the action is replaced by the topological invariant of an abelian gauge connection. This variant is covered in [143] as well. Consider the Lagrangian form

$$\mathbb{L} = \mathbb{L}_g + \mathbb{L}_\Phi \tag{2.79}$$

where $\mathbb{L}_g$ is the gravitational Lagrangian, whose detailed structure is irrelevant, and

$$16\pi\mathbb{L}_\Phi = \frac{1}{2} \star d\Phi \wedge d\Phi + \frac{\alpha}{2}\Phi F \wedge F \tag{2.80}$$

is the dilaton Lagrangian, with $F = dA$ the field strenght of the vector potential A. Using $dF = 0$, we easily see that $F \wedge F = d\,(A \wedge F)$ *off-shell*, i.e. $F \wedge F$ is a topological invariant. Thus the scalar EOM is

$$d\,(\star d\Phi) = -\frac{\alpha}{2} F \wedge F = -\frac{\alpha}{2} d\,(A \wedge F)\ . \tag{2.81}$$

Now, consider a stationary BH solution such that the scalar and the vector fields respect the same symmetries of the background: in particular, they are Lie dragged along the Killing vector field χ^a generating the event horizon, $\pounds_\chi \Phi = \pounds_\chi A = 0$. Contracting both sides of (2.81) with χ^a and using Cartan's identity $\pounds_\chi = di_\chi + i_\chi d$, we get

$$d\,[i_\chi(\star d\Phi) + \alpha(i_\chi A)F] = 0. \tag{2.82}$$

Finally, integrating (2.82) from $\mathcal{B}$ to infinity and assuming asymptotically flat boundary conditions we get [143]

$$Q_S = \alpha V_E Q_M \tag{2.83}$$

where

$$Q_S = \frac{1}{4\pi} \int_{S_\infty} i_t\,(\star d\Phi) \tag{2.84}$$

is the scalar charge, $V_E = i_\chi A$ is the electric potential, which is constant on the bifurcation surface [58], and Q_M is the magnetic charge

$$Q_M = \frac{1}{4\pi} \int_{\mathcal{B}} F\,. \tag{2.85}$$

Observe that the derivation does not make use of the gravitational and electromagnetic EOM, the only essential ingredient being the linearity of coupling between the scalar and the gauge topological invariant. Along the same lines one shows that NSs have no scalar charge: indeed, if the BH is replaced by a NS, the integration of (2.82) is over a compact hypersurface with no internal boundaries and so $Q_S = 0$. In Chapter 4 we will use similar arguments in the context of Einstein-Maxwell-dilaton theory.

By the way, notice that a much stronger result holds for NSs in shift-symmetric Horndeski theory, of which D^2GB is a particular case: in this subclass, NSs have no hair at all [145].

D^2GB can also be viewed as a linear realization of EINSTEIN-DILATON-GAUSS-BONNET THEORY (EDGB) [146]

$$\mathcal{S}_{EDGB} = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[R - \frac{1}{2}\nabla^a\Phi\nabla_a\Phi + \frac{\alpha}{4} f\left(\Phi\right)\mathcal{G} \right] , \quad f\left(\Phi\right) = e^{\gamma\Phi} \tag{2.86}$$

which emerges in the framework of low-energy effective string theories. It is clear, from the fact that $\mathcal{G}$ does not vanish on GR BH solutions, that BHs in EDGB cannot support a trivial scalar profile and therefore they cannot obey a no-hair theorem. As it was shown in [147, 148], this is not necessarily the case if the exponential coupling is replaced by a different $f(\Phi)$: for example, if $f(\Phi) \sim \Phi^2$, it is clear that Kerr BHs with $\Phi = 0$ are solutions of the EOM, although not the only ones: Refs. [147, 148] discovered that, for certain ranges of the BH charges, the GR solution is unstable and hairy BHs becomes dynamically favoured. This is the phenomenon of *spontaneous scalarization*.

Finally, let us mention that EDGB can also be viewed as a particular example of QUADRATIC GRAVITY, which is not a subclass of Horndeski. In quadratic gravity which the scalar field is coupled to all the independent curvature invariants

$$\mathcal{S}_{QG} = \int d^4x \frac{\sqrt{-g}}{16\pi} \Big[R - \frac{1}{2}\nabla^a\Phi\nabla_a\Phi + f_1(\Phi)R^2 + f_2(\Phi)R^{ab}R_{ab} \\ f_3(\Phi)R^{abcd}R_{abcd} + f_4(\Phi)R^\star R \Big] \tag{2.87}$$

where $R^\star R = R^{abcd}{R_{ab}}^{ef}\epsilon_{cdef}$ is the Pontryagin invariant. In general, both BHs and NSs are hairy in quadratic gravity, although the hair cannot be expressed in a simple form even for BHs [26, 144].

2.4.2 Einstein-Maxwell-dilaton theory

EINSTEIN-MAXWELL-DILATON THEORY (EMD) is not a modified gravity theory *per se*: the kinetic gravity term is the ordinary Einstein-Hilbert one and no further curvature terms are present in the Lagrangian. However, it constitutes a proxy for theories in which all the three bosonic DOF (scalar, vector and tensor) can propagate. The action of EMD is

$$\mathcal{S}_{EMD} = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[R - 2\nabla^a\Phi\nabla_a\Phi - e^{-2\eta\Phi}F^{ab}F_{ab}\right] \tag{2.88}$$

where Φ is a real scalar field (the dilaton) and $F_{ab} = \partial_a A_b - \partial_b A_a$ is the field strenght of the vector potential A_a. The action depends on the coupling constant η. Some specific values have been widely considered in the literature: for $\eta = 0$, it trivially reduces to the Einstein-Maxwell action (2.1); for $\eta = 1$ it emerges as a low energy effective action in string modeles, while $\eta = \sqrt{3}$ corresponds to the four-dimensional compactification of five-dimensional Kaluza-Klein theory [32, 33, 149].

Static spherically symmetric asymptotically flat BH solutions in EMD have been studied in [32, 150, 33] for all values of η. If the electric charge does not vanish, they are hairy, otherwise they reduce to the Schwarzschild solution with trivial vector and scalar fields. It must be stressed that the potential A_a must not necessarily coincide with the photon field of the standard model. Therefore standard arguments for the smallness of the electric charge do not necessarily apply [24]. The extension to rotating BHs is not straightforward: fully rotating solutions have been found so far only for the Kaluza-Klein value $\eta = \sqrt{3}$ [33, 149]. On the other hand, slowly rotating solutions have been classified for all the values of η [33]. Remarkably, a uniqueness theorem was proved in EMD [151] for $0 \leq \eta^2 \leq 3$.

EMD BH hair are of secondary type: no new conserved charge is associated with the dilaton field. In Chapter 4.3 we will show that the monopole scalar charge Q_S is related to the electric potential at the event horizon by the simple relation

$$Q_S = -\eta\, V_E\, Q_E \tag{2.89}$$

where V_E is the electric potential at the horizon and Q_E is the electric charge. The proof proceeds along the same lines of the proof of (2.83), although in EMD it is not possible to conclude that NSs have no scalar charge.

2.4.3 Black holes in Lorentz-violating gravity

Lorentz symmetry is regarded as a fundamental symmetry of nature. Departures from this symmetry are highly constrained in the matter sector of particle physics [152, 153]. There is still the possibility that Lorentz symmetry is broken only in the gravity sector, withs mechanisms suppressing its percolation in the matter sector [154]. Another motivation for considering Lorentz violation (LV) in gravity is that it can alleviate the nonrenormalizability of GR: as originally proposed by Hořava, the introduction of a preferred foliation can make the theory power counting renormalizable, while still consistent with the ordinary IR gravitational physics [155, 156, 157]. The two main frameworks to treat LV in gravity at low energies are Einstein-Aether theory and khronometric theory.

Einstein-Aether (Æ) theory [158] introduces a preferred timelike direction u^a at every point of spacetime, thus breaking local Lorentz invariance (LLI) at the level of the boosts. The vector u^a is the "aether vector" and it induces a preferred frame up to spatial rotations. The action for Æ theory is[8]

$$\mathcal{S}_{Æ} = \int d^4x \frac{\sqrt{-g}}{16\pi G} \left[R + \mathcal{L}_u + \lambda \left(u^a u_a + 1 \right) \right] \tag{2.90}$$

where

$$\mathcal{L}_u = -K^{ab}{}_{cd} \nabla_a u^c \nabla_b u^d \,, \tag{2.91a}$$
$$K^{ab}{}_{cd} = c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c - c_4 u^a u^b g_{cd} \,. \tag{2.91b}$$

This is the most general action formed with g_{ab} and u^a and their derivatives, which gives second-derivative EOM and preserves general covariance. The physical units of u^a can be chosen in such a way that the c_i's are pure numbers. The parameter λ is a Lagrange multiplier enforcing the unit-timelike condition $u^a u_a = -1$. A direct coupling of matter with the aether vector u^a is expected to be highly suppressed, because it would lead to unobserved violations of Lorentz symmetry in particle physics [152, 153].

Khronometric theory [156] emerges as a low energy limit of Hořava gravity. In the IR regime, it has the same action of Æ theory, with the additional restriction that the aether vector is assumed to be hypersurface orthogonal [6]

$$u_a = -N \nabla_a T \,, \quad N = \left(-\nabla^a T \nabla_a T \right)^{-1/2} \tag{2.92}$$

where T is a preferred time, the "khronon", and N is the lapse of the preferred foliation. We don't restate the action because it identical to (2.90); the only difference is that the unit-timelike constraint is already enforced by the definition (2.92), so there is no need of the Lagrange multiplier in the action. In khronometric theory u^a defines a preferred folitation, which is a stronger requirement than defining just a preferred frame as in Æ theory. Notice that u^a is invariant under a redefinition $T \to f(T)$ of the preferred time: in other words, the preferred time is not uniquely specified, but the preferred foliation is.

In both theories, Lorentz invariance is broken *dynamically*. Indeed, the Æ and the khronometric action are generally covariant, but the aether field

[8]For LV theories we explicitate Newton's constant G in the gravitational action.

acquires a nonvanishing VEV in the solutions of the EOM. The Langrangian $\mathcal{L}_u$ can also be rewritten as

$$-\mathcal{L}_u = \frac{c_\theta}{3}\theta^2 + c_\sigma \sigma^{ab}\sigma_{ab} + c_\omega \omega^{ab}\omega_{ab} + c_\alpha a^a a_a \tag{2.93}$$

where $a_a = u^b \nabla_b u_a$ is the aether acceleration, while $\theta = \nabla_a u^a$, $\sigma_{ab} = \underleftarrow{\nabla_{(a} u_{b)}}$ and $\omega_{ab} = \underleftarrow{\nabla_{[a} u_{b]}}$ are respectively the expansion, the shear and the twist of the aether field. An underleft arrow denotes projection on the local hypersurface orthogonal to u^a. Obviously for khronomoetric theory $\omega_{ab} = 0$ because the field is globally hypersurface orthogonal. Moreover, for khronometric theory $a_a = N^{-1}\underleftarrow{\nabla_a N} = \underleftarrow{\nabla_a \ln N}$. The new c_i's are related to the original ones by the relations

$$c_\sigma = c_{13}\,, \quad c_\omega = c_1 - c_3\,, \quad c_\theta = 3c_2 + c_{13}\,, \quad c_\alpha = c_{14} \tag{2.94}$$

where $c_{ij} = c_i + c_j$. Although in principle the parameters c_i's are free, they have been severely constrained with theoretical and experimental observations, such as absence of instabilities, Čerenkov radiation, ppN constraints, binary pulsars, big bang nucleosynthesis and GWs. In particular, from the recent tight constaint on the speed of GW170817, one deduces the very stringest restriction $|c_\sigma| < 10^{-15}$. An updated discussion of such contraints after GW170817 can be found in [159, 160].

For physical applications, the preferred frame can be identified with the rest frame of the CMB. Æ theory propagates two tensor, two vector and one scalar DOF with the following speeds in the aether frame

$$s_T^2 = \frac{1}{1-c_\sigma}\,, \quad s_V^2 = \frac{c_\sigma^2 + (1-c_\sigma)(c_\sigma + c_\omega)}{2c_\alpha(1-c_\sigma)}\,, \quad s_S^2 = \frac{(c_\sigma + c_2)(2-c_\alpha)}{c_\alpha(1-c_\sigma)(2+c_\theta)} \tag{2.95}$$

which in general are all different from the speed of light $s = 1$. Khronometric theory propagates only the tensor and the scalar DOF, with the same speeds s_T and s_S as above, while there is no propagation of vector DOF. Notice that the absence of Čerenkov radiation requires $s_{T,V,S} \geq 1$, i.e. the propagating modes cannot be subluminal.

The existence of modes with different propagating speeds makes already clear that the notion of BH in LV gravity is not as simple as in GR. Indeed, to each mode with speed s_i will correspond a different event horizon, defined as the metric horizons of the modified metrics $g_{ab}^{(i)} = g_{ab} + (1 - s_i^2)u_a u_b$, such that there will be a nested triplet of horizons. However, if Lorentz symmetry is broken, there is no reason why the dispersion relation of the fields must be quadratic in the momentum, $\omega^2 \sim k^2$. One could as well consider modified dispersion relations of the form

$$\omega^2 \sim k^2 + \alpha k^4 \,. \tag{2.96}$$

In the IR limit the quadratic term in (2.96) dominates, but in the UV the modes can propagate at arbitrarily high speed. The presence of modified dispersion relations of the aboke kind appears natural the full Hořava theory beyond the IR limit, while formally it is not necessary in Æ theory. Such dispersion relations do note admit a maximum limiting speed, and the very concept of BH seems to lose meaning.

At least in the presence of a preferred foliation, this naive expectation proves to be wrong. Indeed, it has been shown by numerical and analytical calculations [161, 162, 163, 164] that BHs with a preferred foliation possess a special surface, the so called universal horizon (UH), acting as an event horizon for modes of arbitrary speeds. UHs were first discovered in [161, 162] in the static case; in particular [162] showed that UHs are always present if the parameter c_i's are of astrophysical relevance. Since then, UHs have been found in a variety of configurations: with non-flat asymptotics [164], whith slow rotation [165], in lower dimensions [166, 167], at extremality [168]. A general characterization of UHs in asymptotically flat spacetimes with a preferred foliation was presented in [169].

The mechanism behind the existence of a UH is the following. The aether vector u^a defines causality: causal modes propagate forward w.r.t. the preferred time slices, the forward direction being defined precisely by the direction of u^a. Therefore, if there is a time slice which is disconnected from asymptotic infinity and where u^a points "inward", that slice acts as a future event horizon. This is a universal horizon. The situation is depicted in Fig.2.2 for the case of spherical symmetry.

Brown continuous lines represent slices of constant preferred time T: they are plotted in advanced null coordinates (v, r)[9]. The vertical continuous line is the UH: since the aether u^a points inward, modes inside the UH are disconnected from asymptotic infinity. It is clear from the figure that the vector $t^a = (\partial/\partial v)^a$ is timelike at infinity but spacelike at the UH, thus indicating that a Killing horizon must exist in between (vertical dashed line), where t^a is null. This figure is well representative of the analytic static solutions presented in [163].

It is not clear that UHs and Killing horizons always cohexist. It seems reasonable to demand that a viable LV BH must possess a Killing horizon, in

[9]The advanced null coordinates (v, r) are coordinates adapted to ingoing null rays: r is the areal radius of the spherical cross sections, while v is a null time such that ingoing null rays move at $v = \text{const}$.

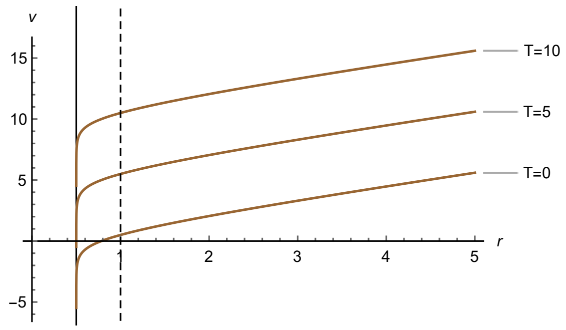

Figure 2.2: The preferred slices T = const. (brown continuous lines) of a spacetimes containing a universal horizons. The diagram is plotted in advanced null coordinates (v, r). The continuous and dashed vertical lines represent respectively the universal horizon and the Killing horizon. We do not plot the slices for $r < r_{\rm UH}$ to keep the diagram clean.

order to reconnect with standard physics when ordinary matter is considered. To avoid confusion, let us keep calling BH a conventional Killing horizon, and use the word UH for the LV event horizon. Numerics in [162] indicate that static LV BHs always cloak an UH in situations of astrophysical relevance, both in khronometric ans Æ theory. Since Æ theory does not assume the existence of a preferred foliation *per se*, it is not trivial that UHs are admitted also there. A strong result states that spherically symmetric BHs with a regular UHs in khronometric theory are also solutions of Æ theory [164]; the viceversa is also true, due to the more general fact that hypersurface orthogonal solutions in Æ theory are also solutions of khronometric theory [6]. Therefore the two theories share the same static spherically symmetric UHs. However, this correspondence is broken in the spinning case: it was shown in [170] that four-dimensional asymptotically flat slowly rotating BHs in Æ theory generically do not admit a UH, except for very special choices of the couplings c_i's; on the contrary, UHs are present in khronometric theory even after the inclusion of rotation [165].

As observed in [171], stationary Killing horizons in khronometric theory and Æ theory cannot have a regular bifurcation surface. This is because, at the bifurcation surface, only tangent vectors are invariant under the flow of Killing field χ^a generating the horizon. However, u^a is timelike everywhere and therefore cannot be tangent to $\mathcal{B}$. For the same reason, u^a cannot be parallel to χ at $\mathcal{B}$, and it cannot vanish because it is normalized to unity. Therefore, if a bifurcation surface exists, u^a must blow up there, contradicting the initial regularity assumption. This is the reason why Wald's derivation of the first law cannot be carried on in the case of LV BHs.

2.4.4 Black holes in higher and lower dimensions

The Einstein-Hilbert gravitational action, with the possible addition of a cosmological constant term, gives rise to the EOM

$$G_{ab} = 8\pi G\, T_{ab} \tag{2.97}$$

where $G_{ab} = R_{ab} - Rg_{ab}/2 + \Lambda g_{ab}$ is the Einstein tensor. The two key properties of the Einstein tensor are that: (i) it is divergence-free, $\nabla_a G^a{}_b = 0$, in such a way to ensure the weak equivalence principle (WEP) $\nabla_a T^a_b = 0$; (ii) it is locally constructed out of g_{ab} and its first and second derivates, in such a way to give rise to second-order EOM. It is meaningful to ask if G_{ab} is the only symmetric, divergence-free tensor purely constructed out of the metric and its first and second derivatives. In four spacetime dimensions, the answer is in the affirmative. This is a particular case of a theorem due to Lovelock [172, 173], who wrote the most general gravitational action giving rise to an Einstein tensor with the above properties, in arbitrary D spacetime dimensions. The result is the Lovelock action

$$\mathcal{S}_L = \int d^D x \frac{\sqrt{-g}}{16\pi} \sum_{k=0}^{[(D-1)/2]} c_k \mathcal{L}^{(k)}\,, \tag{2.98a}$$

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \dots a_k b_k}_{c_1 d_1 \dots c_k d_k} R_{a_1 b_1}{}^{c_1 d_1} \dots R_{a_k b_k}{}^{c_k d_k} \tag{2.98b}$$

where the c_k's are coupling constants and $\delta^{a_1 b_1 \dots a_k b_k}_{c_1 d_1 \dots c_k d_k}$ is the generalized Kronecker delta, defined by

$$\delta^{a_1 b_1 \dots a_k b_k}_{c_1 d_1 \dots c_k d_k} = \det \begin{bmatrix} \delta^{a_1}_{c_1} & \dots & \delta^{b_k}_{c_1} \\ \vdots & \vdots & \vdots \\ \delta^{a_1}_{d_k} & \dots & \delta^{b_k}_{d_k} \end{bmatrix} = 2^k \delta^{[a_1}_{c_1} \delta^{b_1}_{d_1} \dots \delta^{a_k}_{c_k} \delta^{b_k]}_{d_k}\,. \tag{2.99}$$

The sum is restricted to the integer part of $(D-1)/2$ because the generalized delta vanishes for $k > [D/2]$; moreover, in even dimensions, $\mathcal{L}^{(D/2)}$ is a topological density and therefore, although formally admitted, it does not conrtibute to the EOM and it can be omitted. See [174] for a review of Lovelock theory and its black hole solutions.

One can easily verify that the first few terms are $\mathcal{L}^{(0)} = 1$, $\mathcal{L}^{(1)} = R$ and $\mathcal{L}^{(2)} = R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd} = \mathcal{G}$. Therefore, in $D = 4$, we recover that the most general Lagrangian is $\mathcal{L} = c_0 + c_1 R$, while $\mathcal{G}$ is the Gauss-Bonnet topological density.

Lovelock theory is also connected to two four dimensional modified theory that we already mentioned, namely Einstein-dilaton-Gauss-Bonnet and

Einstein-Maxwell-dilaton. Indeed, it was shown in [7] that they can be obtained in a four-dimensional reduction of the five dimensional Lovelock theory.

Static BH solutions in D-dimensional Lovelock theory were studied in [175, 176] in spherical symmetry and were generalized to planar and hyperbolic symmetry in [177]. For simplicity, let us restrict to the spherically symmetric case. It was shown in [175] that the general form of the metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2 \tag{2.100}$$

where $f(r) = 1 - r^2 g(r)$ and $g(r)$ solves the polinomial equation

$$\sum_{k=0}^{[(D-1)/2]} \hat{c}_k g^k = \frac{\mu}{r^{D-1}}\,. \tag{2.101}$$

Here μ is a constant related to the asymptotic mass of the solution, while the hatted constants $\hat{c}_i$'s are defined through

$$\hat{c}_0 = \frac{c_0}{c_1}\frac{1}{(D-1)(D-2)}\,, \quad \hat{c}_1 = 1\,, \tag{2.102a}$$

$$\hat{c}_k = \frac{c_k}{c_1}\prod_{n=3}^{2k}(D-n) \quad \text{for } k > 1\,. \tag{2.102b}$$

In $D = 4$, the solution reduces to the standard (anti-)deSitter Schwarzschild metric

$$f(r) = 1 - \frac{\Lambda}{3}r^2 - \frac{\mu}{r} \tag{2.103}$$

where we adopt the standard convention $\hat{c}_0 = -2\Lambda$. If we consider a less trivial case, say $D = 5$, the solution reads

$$f(r) = 1 + \frac{r^2}{2\hat{c}_2}\left(1 \pm \sqrt{1 - 4\hat{c}_2\left(\hat{c}_0 - \frac{\mu}{r^4}\right)}\right) \tag{2.104}$$

and we see that there are two branches, the "$-$ branch' and the "$+$ branch". The $+$ branch is not perturbatively connected to the Einstein-Hilbert theory, i.e. it is not perturbative in the paramter $\hat{c}_2$. When $\hat{c}_0 = 0$, the two branches scale at infinity like

$$\lim_{r\to\infty} f(r) = \begin{cases} 1 - \frac{\mu}{r^2} & \text{"$-$ branch"} \\ 1 + \frac{r^2}{\hat{c}_2} + \frac{\mu}{r^2} & \text{"$+$ branch"} \end{cases} \tag{2.105}$$

and we see that, in the + branch, one can generate an effective cosmological constant even if the bare one c_0 vanishes. It must be stressed that an event horizon does not necessarily exist for all the values of μ and/or $\hat{c}_2$ [176].

In the general case, maximally symmetric backgrounds are determined by solutions of (2.101) in the form $g = g_0 =$ const. for a vanishing sourde μ. If $c_0 = 0$, it is clear that the flat vacuum $g_0 = 0$ is always admitted. If $c_0 \neq 0$, however, it is not even clear that maximally symmetric vacua exist. Assuming that this is the case, then [176] showed that at infinity the solution goes like

$$\lim_{r\to\infty} f(r) = 1 - g_0 r^2 - \frac{\tilde{\mu}}{r^{D-3}} \tag{2.106}$$

where $\tilde{\mu}$ is just a rescaling of μ.

As a last example, let us consider what appens in lower dimensions, for $D = 3$. If we solve (2.101) for $c_0 = 0$, we see that $f = 1 - \mu$ and no BH exists. On the other hand, if we allow for a nonvanishing bare cosmological constant, we obtain

$$f(r) = -M - \Lambda r^2 \tag{2.107}$$

whee we rescaled $\mu = M + 1$ and we see that a BH exists iff $\Lambda < 0$, i.e. only for anti-deSitter asymptotics. This is the famous Bañados-Teitelboim-Zanelli (BTZ) solution [178, 179], which generalizes to the spinning case as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(\Omega(r)dt + d\varphi\right)^2 , \tag{2.108a}$$

$$f(r) = -M - \Lambda r^2 + \frac{J^2}{4r^2}, \quad \Omega(r) = -\frac{J}{2r^2} . \tag{2.108b}$$

3

Smarr formula and black hole mechanics for modified theories

3.1 Introduction

In this Chapter we describe a systematic procedure to derive the Smarr formula for BH solutions in diffeoinvariant gravity theories. The procedure is based on Wald's formalism for the derivation of the first law, introduced in Ch.2.2.2 (to which we refer for the notation). The content of the Chapter is based on the papers "Smarr formula for Lovelock black holes: A Lagrangian approach" (P1), "Improved derivation of the Smarr formula for Lorentz-breaking gravity" (P2) and "First law of black holes with a universal horizon" (P3).

The main idea is simple. Consider a stationary nonextremal BH solution and assume that the matter fields respect the same symmetries of the background metric, i.e. $\pounds_t\phi = \pounds_\psi\phi = 0$, where t^a ans ψ^a are the Killing vectors associated to time translations and axi-rotations respectively. From the fact that the symplectic potential $\Theta\left(\phi, \pounds_\xi\phi\right)$ is linear and homogeneous in $\pounds_\xi\phi$, it follows that $\Theta(\phi, \pounds_\chi\phi)$ vanishes on shell, with $\chi^a = t^a + \Omega_H\psi^a$ being the vector field generating the event horizon[1]. Eq.(2.18), $\mathbb{J}[\xi] = \Theta(\phi, \pounds_\xi\phi) - i_\xi\mathbb{L}$, then implies

$$\mathbb{J}[\chi] + \chi\cdot\mathbb{L} = d\mathbb{Q}[\chi] + \chi\cdot\mathbb{L} = \Theta\left(\phi, \pounds_\chi\phi\right) = 0 \tag{3.1}$$

on shell. Integrating (3.1) over a spatial hypersurface Σ extending from the

[1] As in Ch.2.2.2, we are assuming that there exists only one rotational axis of symmetry. See footnote 3 of Ch.2.2.2.

bifurcation surface $\mathcal{B}$ to asymptotic infinity, we obtain

$$\int_{S_\infty} \mathbb{Q}[\chi] = \int_{\mathcal{B}} \mathbb{Q}[\chi] - \int_\Sigma \chi \cdot \mathbb{L} \tag{3.2}$$

or equivalently

$$\int_{S_\infty} \mathbb{Q}[t] = \frac{\kappa}{2\pi} S_W + \Omega_H J - \int_\Sigma \chi \cdot \mathbb{L}\,. \tag{3.3}$$

Eq.(3.3) follows from the definition (2.25) of the angular momentum J and from the expression (2.28) of the Wald entropy S_W.

That (3.3) reproduces the Smarr formula in four-dimensional vacuum GR follows straighforwardly from the fact that the vacuum Einstein equations imply $R = 0$, and so $\mathbb{L} = 0$ on shell. Moreover, Iyer and Wald [47] showed that, for an asymptotically flat spacetime, $\int_{S_\infty} \mathbb{Q}[t] = M/2$, where M is the ADM mass. Therefore (3.3) for vacuum GR reduces to the standard Smarr formula

$$M = \frac{\kappa A_H}{4\pi} + 2\Omega_H J\,. \tag{3.4}$$

If we perform the most minimal modification, namely the addition of a cosmological constant, we see that things are not so straightfoward anymore. Indeed Einstein equations

$$R_{ab} - \frac{R}{2} g_{ab} + \Lambda g_{ab} = 0 \tag{3.5}$$

imply $R = 4\Lambda$. Then the on shell Lagrangian is $\mathcal{L} = (R - 2\Lambda)/16\pi = \Lambda/8\pi$. If $\Lambda < 0$, one has AdS asymptotics and the outer domain of communication extends up to infinity. In this case, one can easily convince herself that the integral of $\chi \cdot \mathbb{L}$ over Σ diverges.

Let us restrict for simplicity to the static case, where $\chi^a = t^a$ and $J = 0$. The line element is given by (2.100), with the $f(r)$ (2.103). Therefore, choosing Σ as a $t =$ constant hypersurface, the integral of $t\cdot\mathbb{L}$ over Σ diverges as the volume of an euclidean sphere with infinite radius. Of course, since κ and S_W are finite, the divergence is compensated by an equally divergent contribution from the LHS of (3.3). It is nevertheless desiderable to have an equivalent of Eq.(3.3) in which each single term is finite definite.

To this aim, one can simply subtract from (3.3) the same identity evaluated on the vacuum solution, i.e. on the globally AdS spacetime, thus obtaining

$$\int_{S_\infty - AdS} \mathbb{Q}[t] = \frac{\kappa}{2\pi} S_W + \int_{\Sigma - AdS} t \cdot \mathbb{L}\,. \tag{3.6}$$

In the case of a static BH, similarly to the asymptotically flat case, the LHS of (3.6) is equal to $M/2$, while the integral on the RHS is equal to $-\Lambda V_H/8\pi$, where $V_H = 4\pi r_H^3/3$ is the "Euclidean volume" of the black hole. Therefore the Smarr formula of a static BH in GR in the presence of a cosmological constant is

$$M = \frac{\kappa A_H}{4\pi} + \frac{\Lambda V_H}{4\pi}\,. \tag{3.7}$$

This is a standard result. Eq.(3.7) was rederived by Kastor, Ray and Traschen (KRT) [180] by means of a "Killing potential", namely an antisymmetric tensor χ^{ab} such that $\chi^a = \nabla_b \chi^{ab}$, whose (at least local) existence is guaranteed by the fact that $\nabla_a \xi^a = 0$ for any Killing vector ξ^a (to see this, just take the trace of the Killing equation). KRT [181, 182] applied the same technique to the more general case of asymptotically AdS BHs in Lovelock gravity. The derivation of KRT is based on the Hamiltonian framework for Lovelock theory: in particular, it is based on a Hamiltonian definition of mass *à la* Regge-Teitelboim [183] and a Hamiltonian derivation of a first law of mechanics [184]. In this derivation, the use of Lovelock-adapted Killing potentials [185] plays a fundamental role in reducing the Smarr formula to a purely boundary integral, with contributions at infinity and at the event horizon.

In Sec.3.2 we give an alternative derivation of the Smarr formula for static Lovelock BHs, based on Wald's Lagrangian formalism. In particular, we will show that, for static BHs, the volume integral of $t \cdot \mathbb{L}$ over Σ reduces to a surface integral over $\partial\Sigma = S_\infty \cup \mathcal{B}$, thus making contact between Eq.(3.6) and the Smarr formula of [181, 182]. We also show how the expression of the ADM mass can be obtained in a Lagrangian framework, using a background subtraction technique similar to the one employed in the standard Hamiltonian procedure. These results were obtained in P1. Besides their inherent interest, they also serve as a testbed for developing the necessary formalism.

The formalism will be then applied in Sec.3.3 to the case of Lorentz-violating BHs in Einstein-aether theory and (infrared) Hořava gravity. The motivation comes from the fact that a mechanical and/or thermodynamical interpretation of LV BHs is still controversial. The concomitant existence of a Smarr formula and an associated first law is is widely considered a basic requirement for such an interpretation. In Sec.3.3.1 we provide a derivation of the Smarr formula for LV BHs which is applicable to any BH solution, thus extending previous results which were restricted to the static case, while in Sec.3.3.2 we specify to some exact LV BH solutions, in order to see if they admit a first law with a thermodynamical interpretation. These Sections are based on P2 and P3.

3.2 Smarr Formula for Lovelock black holes

Consider the Lovelock action (cf. Ch.2.4.4)

$$\mathcal{S}_L = \int d^D x \frac{\sqrt{-g}}{16\pi} \sum_{k=0}^{m} c_k \mathcal{L}^{(k)}\,, \tag{3.8a}$$

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k} R_{a_1 b_1}{}^{c_1 d_1} \ldots R_{a_k b_k}{}^{c_k d_k} \tag{3.8b}$$

where $m \leq [(D-1)/2]$ is the highest integer such that $c_m \neq 0$. Varying $\mathcal{S}_L$ w.r.t. the metric we obtain the EOM

$$\begin{aligned} G^r{}_s &= \sum_{k=0}^{m} \frac{c_k}{2^{k+1}} \delta^{r a_1 b_1 \ldots a_k b_k}_{s c_1 d_1 \ldots c_k d_k} R_{a_1 b_1}{}^{c_1 d_1} \ldots R_{a_k b_k}{}^{c_k d_k} \\ &= \sum_{k=1}^{m} \left(\frac{k c_k}{2^k} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 s \ldots c_k d_k} R_{a_1 b_1}{}^{c_1 r} \ldots R_{a_k b_k}{}^{c_k d_k} \right) - \frac{1}{2} \delta^r_s \mathcal{L} = 0 \end{aligned} \tag{3.9}$$

as well as the symplectic potential

$$\Theta\left(g_{ab}, \delta g_{ab}\right) = \sum_{k=1}^{m} \frac{k c_k}{2^{k-1}} \delta^{a_1 b_1 \ldots a_k c_k}_{c_1 d_1 \ldots c_k d_k} \nabla^{d_1} \delta g^{c_1}{}_{b_1} \ldots R_{a_k b_k}{}^{c_k d_k} \epsilon_{a_1}\,. \tag{3.10}$$

From the first line of (3.9) it is immediate to see that the generalized Einstein tensor $G^r{}_s$ is divergence free, $\nabla_r G^r{}_s = 0$.

From the definitions of the Noether current (2.18) $\mathbb{J}[\xi] = \Theta(\phi, \pounds_\xi \phi) - i_\xi \mathbb{L}$ and of the Noether charge $\mathbb{J}[\xi] = d\mathbb{Q}[\xi]$, we derive the following expression of the Noether charge in Lovelock theory

$$\mathbb{Q}[\xi] = \sum_{k=1}^{m} \frac{k c_k}{2^{k-1}} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k} \nabla_{[a_1} \xi^{d_1]} \ldots R_{a_k b_k}{}^{c_k d_k} \epsilon_{b_1}{}^{c_1}\,. \tag{3.11}$$

Using (2.27) one can easily show that the Wald entropy of Lovelock BHs is given by

$$S_W = \sum_{k=1}^{m} \frac{k c_k}{4} \int_{\mathcal{B}} \tilde{\mathcal{L}}^{(k-1)} \bar{\epsilon} \tag{3.12}$$

where a tilde indicates that $\tilde{\mathcal{L}}^{(k-1)}$ is evaluated w.r.t. to the $(D-2)$-dimensional intrinsic metric of the bifurcation surface $\mathcal{B}$, and $\bar{\epsilon}$ is the surface element of $\mathcal{B}$. The expression (3.12) for the Lovelock BH entropy was originally found in [184] via an Hamiltonian derivation of the first law.

Given the symplectic potential, the Noether current and the Wald entropy, we have all the ingredients to proceed to the derivation of the Smarr formula for static BHs from Eq.(3.6). Recall from Ch.2.4.4 that static Lovelock BHs are described by a line element of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2 \,. \tag{3.13}$$

This allows to express $\int_\Sigma t\cdot\mathbb{L}$ in a simple form as an integral over $\partial\Sigma$. Indeed, it can be shown that, for a metric of the form (3.13),

$$\mathcal{L}^{(k)} = \frac{\gamma_k}{r^{D-2}} \frac{d^2}{dr^2}\left[(1-f)^k\, r^{D-2k}\right] \,, \quad \gamma_k = \frac{(D-2)!}{(D-2k)!} \,. \tag{3.14}$$

The proof is a bit involved and we present it in Appendix A. With the help of (3.14), we have

$$\begin{aligned}\int_{\Sigma-AdS} t\cdot\mathbb{L} &= \int_{\Sigma-AdS} \mathcal{L}\, t^a \epsilon_a = \int_{\Sigma-AdS} \mathcal{L}\, r^{D-2} dr d\Omega_{D-2} = \\ &= \sum_{k=0}^{m} \frac{c_k}{16\pi} \int_{\Sigma-AdS} \mathcal{L}^{(k)} r^{D-2} dr d\Omega_{D-2} = \\ &= \sum_{k=0}^{m} \frac{c_k \gamma_k}{16\pi} \int_{\partial\Sigma-AdS} \frac{d}{dr}\left[(1-f)^k r^{D-2k}\right] d\Omega_{D-2} = \\ &= \sum_{k=0}^{m} \frac{c_k \gamma_k}{16\pi} \frac{d}{dr}\left[(1-f)^k r^{D-2k}\right] \Omega_{D-2}\end{aligned} \tag{3.15}$$

where

$$\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)} \tag{3.16}$$

is the total solid angle spanned by an euclidean $(D-2)$-sphere. The sum in the expression (3.15) can be reduced by one: tracing (3.9), we obtain

$$\mathcal{L} = \sum_{\substack{k=0\\k\neq\bar{m}}}^{m} \frac{c_k}{16\pi} \left(\frac{2k-2\bar{m}}{D-2\bar{m}}\right) \mathcal{L}^{(k)} \tag{3.17}$$

for any $0 \leq \bar{m} \leq m$. It then follows that

$$\begin{aligned}\mathcal{W} &:= \int_{\Sigma-AdS} t\cdot\mathbb{L} = \\ &= \sum_{\substack{k=0\\k\neq\bar{m}}}^{m} \frac{c_k\gamma_k}{16\pi} \left(\frac{2k-2\bar{m}}{D-2\bar{m}}\right) \frac{d}{dr}\left[(1-f)^k r^{D-2k}\right]\Big|_{\partial\Sigma-AdS} \Omega_{D-2} \,.\end{aligned} \tag{3.18}$$

Summing up, for static Lovelock BHs, we have been able to reduce our candidate Smarr formula (3.6) to the form

$$\int_{S_\infty - AdS} \mathbb{Q}[t] = \frac{\kappa}{2\pi} S_W - \mathcal{W}\,. \tag{3.19}$$

where $\mathcal{W}$ is the surface term (3.18). What remains to do is to connect the integral on the LHS to the mass of the BH.

In general, we don't know the explicit solution for $f(r)$, which is determined implicitly by the polinomial equation (2.101). However, from the discussion of Ch.2.4.4, we can impose asymptotic boundary conditions of the form

$$\lim_{r\to\infty} f(r) = 1 + \frac{r^2}{l^2} - \frac{\mu}{r^{D-3}} + \dots \tag{3.20}$$

Here l^{-2} plays the role of an effective cosmological constant (modulo numerical factors), but it does not necessarily coincide with the bare cosmological contant $\Lambda = -c_0/2c_1$. Rather, l is determined by which branch of the solution (2.101) we are considering. (Notice that the constant μ is not the same as in (2.101), but they coincide up to a rescaling, cf. (2.106); we use the same letter only for the sake on notational simplicity.)

Using the asymptotic scaling (3.20), [182] derived an expression for the total mass M in terms of l, μ and the coupling constants c_k. The derivation of [182] uses the Regge-Teitelboim canonical Hamiltonian formalism. We shall now rederive M using the covariant Hamiltonian formalism of Wald. As we saw in Ch.2.2.2, one can identify the variation of the Hamiltonian energy as

$$\delta\mathcal{E} = \int_{S_\infty} [\delta\mathbb{Q}[t] - i_t\Theta(\phi, \delta\phi)] \tag{3.21}$$

where we specified (2.22) to $\xi^a = t^a$ and we used (2.24). The quantity $\mathcal{E}$ defined in (2.24) is not what we would call the "mass" of the BH, because in general $\mathcal{E}$ contains also contributions from the non-flat background. But obvioulsy, for a fixed asymptotic background, $\delta\mathcal{E}$ *is* the variation δM of the mass. Therefore, if we consider variations of the solution in the form

$$\lim_{r\to\infty} \delta f(r) = -\frac{\delta\mu}{r^{D-3}} \qquad \delta\mu = \text{constant}, \tag{3.22}$$

from (3.21) we have

$$\delta\mathcal{E} \equiv \delta M = \#\delta\mu \tag{3.23}$$

where # denotes a proportionality constant which depends only on l and on the c_k's. Then we can simply remove the delta's from (3.23) and the mass will be $M = \#\mu$. Let us now see this at work.

Using Eqs.(3.11),(A.4),(A.5) and (A.9) from Appendix A, it easy to show that

$$\int_{S_\infty} \mathbb{Q}[t] = \lim_{r\to\infty} \sum_{k=1}^{m} \left[\frac{k c_k \gamma_k (1-f)^{k-1} f'}{16\pi\, r^{2k-D}} \right] \Omega_{D-2} \tag{3.24}$$

from which it follows that

$$\begin{aligned} \int_{S_\infty} \delta\mathbb{Q}[t] &= \lim_{r\to\infty} \sum_{k=1}^{m} \frac{k c_k \gamma_k}{16\pi\, r^{2k-D}} \frac{d}{dr} \left[(1-f)^{k-1} \delta f \right] \Omega_{D-2} = \\ &= \sum_{k=0}^{m} (-1)^{k+1} \frac{k c_k \gamma_k}{16\pi\, l^{2k-2}} (D-2k-1) \delta\mu\, \Omega_{D-2} = (\sigma-\gamma) \frac{\delta\mu\, \Omega_{D-2}}{16\pi} \end{aligned} \tag{3.25}$$

where in the second step we used the asymptotic scalings (3.20) and (3.22), and in the last step we defined

$$\sigma = \sum_{k=1}^{m} (-1)^{k+1} \frac{k c_k}{l^{2k-2}} \frac{(D-2)!}{(D-2k-1)!}\,, \tag{3.26a}$$

$$\gamma = \sum_{k=1}^{m} (-1)^{k+1} \frac{k c_k}{l^{2k-2}} \frac{(D-2)!}{(D-2k)!}\,. \tag{3.26b}$$

Similarly, from (3.10)

$$\begin{aligned} -\int_{S_\infty} t\cdot\Theta &= \lim_{r\to\infty} \sum_{k=1}^{m} \frac{k c_k \gamma_k (1-f)^{k-1} f}{16\pi\, r^{2k-D}} 2\nabla_{[a}\delta g_{r]}{}^{a} \Omega_{D-2} = \\ &\lim_{r\to\infty} \sum_{k=1}^{m} \frac{k c_k \gamma_k (1-f)^{k-1}}{16\pi\, r^{2k-D}} \left(-\frac{d\delta f}{dr} - \frac{(D-2)\delta f}{r} \right) \Omega_{D-2} = \\ &= \frac{\gamma\, \delta\mu\, \Omega_{D-2}}{16\pi}\,. \end{aligned} \tag{3.27}$$

Therefore, putting (3.24) and (3.27) together, we find that the mass is given by

$$M = \frac{\sigma\, \mu\, \Omega_{D-2}}{16\pi} \tag{3.28}$$

which agrees with the result of [182]. The relation between M and the LHS of (3.19) can be already read from (3.24), replacing $\delta f \to f$ and subtracting the same expression evaluated at $\mu = 0$. The result is

$$\int_{S_\infty - AdS} \mathbb{Q}[t] = \left(1 - \frac{\gamma}{\sigma}\right) M \tag{3.29}$$

and therefore the Smarr formula becomes

$$\left(1-\frac{\gamma}{\sigma}\right)M=\frac{\kappa}{2\pi}S_W-\mathcal{W}\,. \tag{3.30}$$

The above Smarr formula can be put in the more standard form

$$(D-3)M=(D-2)\frac{\kappa}{2\pi}S_W-\hat{\mathcal{W}} \tag{3.31}$$

where $\hat{\mathcal{W}}$ is

$$\hat{\mathcal{W}}=\sum_{k=0}^{m}2(k-1)c_k\Psi^{(k)} \tag{3.32}$$

and $\Psi^{(k)}$ are suitably defined "potentials". In this form, the Smarr formula is in direct connection with the generalized first law of [181, 182]. Indeed, by dimensional analysis, the k-th coupling c_k has dimension [lenght]$^{2(k-1)}$. Moreover, M and S_W have dimensions [lenght]$^{D-3}$ and [lenght]$^{(D-2)}$. Therefore, a scaling argument suggests a generalized first law of the form

$$\delta M=\frac{\kappa}{2\pi}\delta S_W-\sum_{k=0}^{m}\Psi^{(k)}\delta c_k\,. \tag{3.33}$$

Actually, [181, 182] derived the Smarr formula (3.31) starting from the first law (3.33) and reversing our argument. For an explicit derivation of $\hat{\mathcal{W}}$ from (3.30) see the Appendix A of P1. Observe that, in the asymptotically flat limit $l\to\infty$, $\gamma\to c_1$ and $\sigma\to c_1(D-2)$, and so the Smarr fomula acquires naturally the form (3.31) with $\hat{\mathcal{W}}=(D-2)\mathcal{W}$.

3.3 Black holes with Lorentz violation

The aim of this Section is to explore whether BHs with Lorentz violation admit the formulation of suitable laws of mechanics, and under which conditions a thermodynamical interpretation is possible. In order to set things up and fix the notation, we first introduce the most simple example of BHs in LV gravity, namely static and asymptotically flat BHs. Recall from Ch.2.4.3 that the action for Einstein-Aether theory is

$$\mathcal{S}_{Æ}=\int d^4x\frac{\sqrt{-g}}{16\pi G}\left[R+\mathcal{L}_u+\lambda\left(u^a u_a+1\right)\right] \tag{3.34}$$

where

$$\mathcal{L}_u=-K^{ab}{}_{cd}\nabla_a u^c\nabla_b u^d\,, \tag{3.35a}$$

$$K^{ab}{}_{cd}=c_1g^{ab}g_{cd}+c_2\delta^a_c\delta^b_d+c_3\delta^a_d\delta^b_c-c_4u^au^bg_{cd}\,. \tag{3.35b}$$

The variation of (3.34) w.r.t. to g^{ab}, u^a and λ is (up to boundary terms)

$$\delta\mathcal{S}_{Æ} = \int d^4x \frac{\sqrt{-g}}{16\pi G}\left[\delta g^{ab}E_{ab} + 2\delta u^a\left(Æ_a + \lambda u_a\right) + \delta\lambda(u^2+1)\right] , \tag{3.36}$$

from which we read the EOM

$$u_a u_b g^{ab} = -1\,, \tag{3.37a}$$
$$Æ_a + \lambda u_a = 0\,, \tag{3.37b}$$
$$E_{ab} = G_{ab} - T^u_{ab} = 0\,. \tag{3.37c}$$

The first EOM just expresses that the aether vecor u^a is unit-timelike. When explicitating the second and third EOM, it is convenient to define the following quantities

$$Y^a_{\ b} = K^{ac}{}_{bd}\nabla_c u^d\,, \tag{3.38a}$$
$$X^m{}_{ab} = u^m Y_{(ab)} + u_{(a}Y^m{}_{b)} - u_{(b}Y^m{}_{a)}\ . \tag{3.38b}$$

The covector $Æ_a$ in (3.37b) reads

$$Æ_a = \frac{1}{2}\frac{\delta\mathcal{L}_u}{\delta u^a} = c_4\, a^m\nabla_a u_m + \nabla_m Y^m{}_a \tag{3.39}$$

where $a^m = u^a\nabla_a u^m$ is the aether acceleration. In (3.37c), $G_{ab} = R_{ab} - Rg_{ab}/2$ is the usual Einstein tensor, while the aether stress energy tensor is

$$T^u_{ab} = c_1\left(\nabla_a u_m\nabla_b u^m - \nabla_m u_a\nabla^m u_b\right) + c_4 a_a a_b + \nabla_m X^m{}_{ab} + \\ + \lambda u_a u_b + \frac{1}{2}\mathcal{L}_u\, g_{ab}. \tag{3.40}$$

Eqs.(3.37b)-(3.37c) can be simplified by solving (3.37a) for λ. This yields $\lambda = (u\cdot Æ)$, and the aether and gravitational EOM become

$$\underleftarrow{Æ}_a = \left(\delta^b_a + u_a u^b\right)Æ_b = 0\,, \tag{3.41a}$$

$$\begin{aligned} G_{ab} &= c_1(\nabla_a u_m\nabla_b u^m - \nabla_m u_a\nabla^m u_b) + c_4 a_a a_b + \\ &+ \nabla_m X^m{}_{ab} + (u\cdot Æ)u_a u_b + \frac{1}{2}\mathcal{L}_u\, g_{ab}\,. \end{aligned} \tag{3.41b}$$

BH solutions to the EOM (3.41) are not known analitically for generic values of the couplings c_i's. A generic ansatz for a static solutions is

$$ds^2 = -e(r)dt^2 + \frac{f^2(r)}{e(r)}dr^2 + r^2 d\Omega^2\,, \tag{3.42a}$$
$$u_a = \left((u\cdot t), -\frac{(s\cdot t)f(r)}{e(r)}, 0, 0\right)\,, \tag{3.42b}$$
$$s^a = \left(-\frac{(s\cdot t)}{e(r)}, -\frac{(u\cdot t)}{f(r)}, 0, 0\right)\,. \tag{3.42c}$$

For later convenience, we have introduced s^a as the unit-spacelike normal vector to u^a. The four functions $e(r)$, $f(r)$, $(u\cdot t)$ and $(s\cdot t)$ in the above ansatz are not all independent from each other. In particular, the unit-timelike constraint on u^a, or equivalently the unit-spacelike constraint on s^a, enforce the relation

$$e(r) = (u\cdot t)^2 - (s\cdot t)^2\,. \tag{3.43}$$

The Killing field associated to time translations is

$$t^a = \left(\frac{\partial}{\partial t}\right)^a = -(u\cdot t)u^a + (s\cdot t)s^a\,. \tag{3.44}$$

Assuming asymptotic flatness and expanding in inverse powers of r at infinity, numerical analyses [162] show that the solution depends on two parameters r_0 and $r_{\rm ae}$. At order $\mathcal{O}(r^{-2})$ the expansion reads [162, 163]

$$e(r) = 1 - \frac{r_0}{r} + \mathcal{O}(r^{-3})\,, \tag{3.45a}$$

$$f(r) = 1 + \frac{c_{14} r_0^2}{16 r^2} + \mathcal{O}(r^{-3})\,, \tag{3.45b}$$

$$(u\cdot t) = -1 + \frac{r_0}{2r} + \frac{r_0^2}{8r^2} + \mathcal{O}(r^{-3})\,, \tag{3.45c}$$

$$(s\cdot t) = \frac{r_{\rm ae}^2}{r^2} + \mathcal{O}(r^{-3})\,. \tag{3.45d}$$

However, a close numerical inspection shows that the solution diverges at the s_S-horizon[2], unless $r_{\rm ae}$ is fine tuned as a function of r_0. Therefore, once regularity is imposed, the solution depends only upon one free parameter r_0. This parameter is connected to the Hamiltonian mass at infinity by [186, 187]

$$M = \frac{r_0}{2G}\left(1 - \frac{c_{14}}{2}\right)\,. \tag{3.46}$$

The expansion (3.45) assumes that $c_{123} \neq 0$ and $c_{14} \neq 0$. Remarkably, when $c_{123} = 0$ or $c_{14} = 0$, exact BH solutions can be found in a close analytic form [163].

In both the numerical and the analytical solutions, a special hypersurface exists at $r = r_{\rm UH} =$ constant, to which the aether vector u^a is orthogonal. As explained in Ch.2.4.3, this is the universal horizon: it represents a causal barrier for modes propagating at arbitrarily high speeds in the aether frame (i.e. in the frame in which the time direction is aligned with u^a). From

[2] Recall from Ch.2.4.3 that the s_S-horizon is defined as the outer event horizon for modes of speed s_S, namely as the Killing horizon of the metric $g^{(S)}_{ab} = g_{ab} + (1 - s_S^2)u_a u_b$.

(3.42) we see that this is possible if and only if $(u \cdot t) = 0$ at $r = r_{\rm UH}$. From (3.44) this means that the Killing filed t^a becomes spacelike at the UH and lies tangent to it.

Recall from Ch.2.4.3 that Æ-theory and khronometric theory share the same static BH solutions with a regular UH. Therefore all the above considerations extend also to khronometric theory. In particular, staticity implies that the solution (3.42) possess a preferred foliation also in Æ-theory, a property that does not hold in general in this theory (for example, it is not true in general when rotation is switched on [170]).

It was shown in [169, Theorem 4] that $(u \cdot t) = 0$ is a general condition at the UH, even beyond staticity. More specifically it was shown that, for stationary asymptotically flat spacetimes with a preferred foliation and with time translational Killing field t^a, $(u \cdot t) = 0$ and $(a \cdot t) \neq 0$ form a set of necessary and sufficient conditions for an hypersurface to be a UH (here $(a \cdot t)$ denotes the contraction of the aether acceleration with t^a). Moreover, [169] also showed that $(a \cdot t)$ must be constant on the UH.

The fact that static LV BHs depend only on the single lenght scale r_0, or equivalently on the UH radius $r_{\rm UH}$, makes the existence of a first law and of a Smarr formula associated to the UH a somewhat trivial fact [163]. Indeed, consider two solutions differing by an infinitesimal variation δr_0 of r_0. Then it follows from (3.46) that

$$\delta M = \frac{1}{2G}\left(1 - \frac{c_{14}}{2}\right)\frac{\partial r_0}{\partial r_{\rm UH}}\delta r_{\rm UH} \tag{3.47}$$

which can be rearranged as

$$\delta M = \frac{q_{\rm UH}}{8\pi G}\delta A_{\rm UH} \tag{3.48}$$

where $A_{\rm UH} = 4\pi r_{\rm UH}^2$ is the area of the UH and $q_{\rm UH}$ is a parameter with the dimensions of an inverse lenght. Given that M depends only on $r_{\rm UH}$ and that there are no dimensionful parameters in the Lagrangian apart from G, we can apply a scaling argument an derive the Smarr formula [163]

$$M = \frac{q_{\rm UH} A_{\rm UH}}{4\pi G}\,. \tag{3.49}$$

Of course, since (3.48) and (3.49) are derived under strong symmetry assumptions, they can be taken as an hint but not as a strong indication that the UH admits a first law and a corresponding Smarr formula. Moreover, the first law (3.48) describes variations between infinitesimally close solutions, therefore it is a first law in a more restricted sense than a physical process version, or than a phase-space version *à la* Wald. We therefore take (3.48)-(3.49) as a motivation for exploring the Smarr formula and the first law for LV BHs in more general and less symmetric configurations.

3.3.1 Derivation of the Smarr formula

In order to derive the Smarr formula for LV gravity, we start from the general equation

$$d\mathbb{Q}[\xi] + \xi \cdot \mathbb{L} = \Theta\left(\phi, \pounds_\xi \phi\right) \tag{3.50}$$

valid for a generic vector field ξ^a in any diffeoinvariant gravity theory, and therefore also in Æ-theory and in khronometric theory. As explained in the introduction, our basic idea to derive the Smarr formula is to identify ξ^a with a properly chosen Killing field of the stationary solution. If we can further show that $\xi \cdot \mathbb{L}$ reduces to a total divergence, the Smarr formula follows from the integration of the LHS over a spacelike hypersurface with boundaries at infinity and at the horizon.

However, we immediately recognize that, while this procedure applies safely to Æ-theory, a difficulty arises in khronometric theory. Indeed, in Æ-theory the dynamical fields are g_{ab} and u^a and therefore it is true that $\Theta_{Æ}\left(\phi, \pounds_\xi \phi\right) = 0$ for any Killing field ξ^a. On the ohter hand, in khronometric theory the dynamical variables are g_{ab} and T, which is the preferred time up to redefinitions $T \to \tilde{T}(T)$. In the presence of a preferred foliation, it has no physical meaning to impose symmetries on T, which is just a label for the leaves; rather, the symmetries must be imposed on the normal vector u^a defining the foliation. Therefore in general it is not true that $\pounds_\xi T = 0$, and in particular it cannot be true that $\pounds_t T = 0$. This implies that the RHS of (3.50) does not vanish in khronometric theory, which therefore must be treated separately from Æ-theory.

Smarr formula for Æ-theory

From the boundary terms neglected in (3.36), we can read the symplectic potential for Æ-theory [187]

$$\Theta_{Æ}\left(\phi, \delta\phi\right) = \frac{1}{16\pi G}\left[g_{ab}\nabla^m \delta g^{ab} - \nabla_a \delta g^{ma} + X^m{}_{ab}\delta g^{ab} - 2Y^m{}_a\,\delta u^a\right]\epsilon_m \tag{3.51}$$

from which it follows, using

$$d\mathbb{Q}[\xi] = \Theta\left(\phi, \pounds_\xi \phi\right) - i_\xi \mathbb{L}\,, \tag{3.52}$$

that

$$\mathbb{Q}_{Æ}[\xi] = \frac{-1}{16\pi G}\left[\nabla^a \xi^b + u^a Y^b_{\ c}\,\xi^c + u^a Y_c{}^b \xi^c + Y^{ab}(u\cdot\xi)\right]\epsilon_{ab}. \tag{3.53}$$

In order to show that $\xi \cdot \mathbb{L}$ is exact on-shell, we make the important observation that the Lagrangian itself is exact on-shell. In order to show this, it is sufficient to trace the gravitational EOM (3.41b), thus obtaining

$$\left(\frac{2-D}{2}\right) R = c_4 a^2 + \nabla_m \left(g^{ab} X^m{}_{ab}\right) - (u \cdot \text{Æ}) + \frac{D}{2}\mathcal{L}_u =$$
$$= \nabla_m \left(u^m Y^a{}_a\right) - u^a \nabla_m Y^m{}_a + \frac{D}{2}\mathcal{L}_u =$$
$$= \nabla_m \left(u^m Y^a{}_a - u^a Y^m{}_a\right) + \left(\frac{D-2}{2}\right)\mathcal{L}_u \quad (3.54)$$

where we work in generic $D > 2$ spacetime dimensions. Therefore the on-shell Lagrangian is equal to

$$\mathbb{L}_{\text{Æ}} = \left(\frac{R + \mathcal{L}_u + \lambda(u^a u_a + 1)}{16\pi G}\right)\epsilon =$$
$$= \frac{1}{16\pi G}\left(\frac{2}{D-2}\right)\nabla_m \left(u^a Y^m{}_a - u^m Y^a{}_a\right)\epsilon =$$
$$= d\left[\frac{\left(u^a Y^m{}_a - u^m Y^a{}_a\right)\epsilon_m}{8\pi G(D-2)}\right] := d\mathbb{A}_{\text{Æ}} \quad (3.55)$$

where in the last step we defined the $(D-1)$-form

$$\mathbb{A}_{\text{Æ}} = \frac{\left(u^a Y^m{}_a - u^m Y^a{}_a\right)\epsilon_m}{8\pi G(D-2)}\,. \quad (3.56)$$

Now, suppose that ξ^a is a Killing field such that $\pounds_\xi g_{ab} = \pounds_\xi u^a = 0$. From the observation that $\mathbb{A}$ is constructed only with g_{ab}, u^a and their covariant derivatives, it follows that on-shell

$$\xi \cdot \mathbb{L}_{\text{Æ}} = \xi \cdot d\mathbb{A}_{\text{Æ}} = \pounds_\xi \mathbb{A}_{\text{Æ}} - d\left(\xi \cdot \mathbb{A}_{\text{Æ}}\right) = -d\left(\xi \cdot \mathbb{A}_{\text{Æ}}\right)\,. \quad (3.57)$$

Then Eq.(3.50) becomes

$$d\left(\mathbb{Q}_{\text{Æ}}[\xi] - \xi \cdot \mathbb{A}_{\text{Æ}}\right) = 0 \quad (3.58)$$

and, integrating over an hypersurface Σ, we obtain

$$\int_{\partial\Sigma}\left(\mathbb{Q}_{\text{Æ}}[\xi] - \xi \cdot \mathbb{A}_{\text{Æ}}\right) = 0\,. \quad (3.59)$$

Is there a deeper reason for the seemingly miracolous property that the Æ Lagrangian is a total derivative on shell? We shall now show that the answer is in the affirmative and is rooted in the scale invariance of the action (3.34). Consider the following redefinitions of the fields

$$g_{ab} \to \Omega\, g_{ab}, \tag{3.60a}$$
$$u^a \to \Omega^{-1/2}\, u^a, \tag{3.60b}$$
$$\lambda \to \Omega^{-1}\, \lambda, \tag{3.60c}$$

where Ω is a constant. Since the Christoffel symbol and the Riemann tensor are unaffected by this transformation, and since $\sqrt{-g} \to \Omega^{D/2}\sqrt{-g}$, the Æ Lagrangian D–form transforms as

$$\mathbb{L}_{Æ} \to \Omega^{(D-2)/2}\, \mathbb{L}_{Æ}\,, \tag{3.61}$$

i.e. it experiences a constant rescaling itself: the theory is thus scale invariant (this was already noted in [188], where the parameter A there in Eq.(3) corresponds to our Ω). Now consider the corresponding infinitesimal dilatation around the identity, $\Omega \approx 1 + \omega$,

$$\left.\begin{aligned} \delta_\omega g_{ab} &= \omega\, g_{ab} \\ \delta_\omega u^a &= -\frac{\omega}{2} u^a \\ \delta_\omega \lambda &= -\omega\lambda \end{aligned}\right\} \implies \delta_\omega \mathbb{L}_{Æ} = \omega \left(\frac{D-2}{2}\right) \mathbb{L}_{Æ}\,. \tag{3.62}$$

Under the infinitesimal transformation (3.62), the symplectic potential (3.51) becomes

$$\Theta_{Æ}\left(\phi, \delta_\omega \phi\right) = -\omega \frac{\left(g^{ab} X^m{}_{ab} - u^a Y^m{}_a\right)\epsilon_m}{16\pi G} = \omega \frac{\left(u^a Y^m{}_a - u^m Y^a{}_a\right)\epsilon_m}{16\pi G}\,. \tag{3.63}$$

Therefore, from the fact that the variation of any Lagrangian on-shell is equal to $\delta\mathbb{L} = d\Theta(\phi, \delta\phi)$, Eq.(3.55) immediately follows.

By the way, this derivation shows that any scale invariant Lagrangian is a total derivative on-shell: a Lagrangian is scale invariant if, under a transformation $\phi^\alpha \to \Omega^{p_\alpha}\phi^\alpha$ of the dynamical fields, where the p_α's are real numbers and α labels the different fields, it transforms as $\mathbb{L} \to \Omega^p \mathbb{L}$ for some real number p. Therefore, considering the corresponding infinitesimal transfromation $\Omega \approx 1+\omega$, it follows that $\mathbb{L} = p^{-1} d\Theta\left(\phi^\alpha, p^\alpha \phi_\alpha\right)$ on-shell (here the field index α is obviously not summed over).

Before passing to khronometric theory, let us briefly discuss the inclusion of a bare cosmological constant. In this case the Lagrangian acquires an additional contribution

$$\mathbb{L} = \mathbb{L}_{Æ} + \mathbb{L}_\Lambda\,, \quad \mathbb{L}_\Lambda = -\frac{\Lambda\epsilon}{8G}\,. \tag{3.64}$$

Then $\mathbb{L}$ is not anymore scale invariant, but under (3.63) transforms as

$$\delta_\omega \mathbb{L} = \omega \left(\frac{D-2}{2}\right) \mathbb{L}_{Æ} - \omega \frac{D}{2} \mathbb{L}_\Lambda = \omega \left(\frac{D-2}{2}\right) \mathbb{L} + \omega \mathbb{L}_\Lambda \,. \tag{3.65}$$

Since the symplectic potential $\Theta_{Æ}$ is not modified, it follows that the Lagrangian on shell is

$$\mathbb{L} = d\mathbb{A}_{Æ} - \left(\frac{2}{D-2}\right) \mathbb{L}_\Lambda \tag{3.66}$$

and Eq.(3.59) becomes

$$0 = \int_{\partial\Sigma} \left(\mathbb{Q}_{Æ}[\xi] - \xi \cdot \mathbb{A}_{Æ}\right) + \frac{\Lambda}{8G(D-2)} \int_\Sigma \xi \cdot \epsilon \,. \tag{3.67}$$

As explained in the introduction, the last integral can be turned into a surface integral over $\partial\Sigma$ by means of a Killing potential, so the inclusion of a cosmological constant does not spoil our reduction of the Smarr formula to an indentity between surface integrals.

Smarr formula for khronometric theory

In khronometric theory, we must vary the action

$$\mathcal{S}_T = \int_d x^D \frac{\sqrt{-g}}{16\pi G} \left[R + \mathcal{L}_u\right] \tag{3.68}$$

w.r.t. g_{ab} and T. Here $\mathcal{L}_u$ is the same as in (3.35), but the aether vector u_a is viewed as constructed out of the preferred time T as in (2.92), $u_a = -\left(-g^{ab}\nabla_a T \nabla_b T\right)^{-1/2} \nabla_a T$. The resulting EOM are

$$\nabla_a \left(N \underleftarrow{Æ}^a\right) = 0 \,, \tag{3.69a}$$

$$\begin{aligned} G_{ab} = {} & c_1 \left(\nabla_a u_m \nabla_b u^m - \nabla_m u_a \nabla^m u_b\right) + c_4 a_a a_b + \\ & + \nabla_m X^m{}_{ab} - (u \cdot Æ)\, u_a u_b - 2Æ_{(a} u_{b)} + \frac{1}{2}\mathcal{L}_u g_{ab} \,. \end{aligned} \tag{3.69b}$$

The symplectic potential is

$$\begin{aligned} \Theta_T\left(\phi, \delta\phi\right) = {} & \frac{1}{16\pi G} \Big[g_{ab} \nabla^m \delta g^{ab} - \nabla_a \delta g^{ma} + X^m{}_{ab} \delta g^{ab} - 2Y^m{}_a\, u_b \delta g^{ab} \\ & - Y^m{}_c\, u^c u_a u_b \delta g^{ab} + 2N Y^m{}_a\, \underleftarrow{\nabla^a \delta T} - 2N \underleftarrow{Æ}^m \delta T\Big]\, \epsilon_m \end{aligned} \tag{3.70}$$

where $N = \left(-g^{ab}\nabla_a T \nabla_b T\right)^{-1/2}$ is the lapse of the preferred foliation. Moreover, the Noether current $\mathbb{Q}_T[\xi]$ coincides with $\mathbb{Q}_{Æ}[\xi]$ as derived from Æ-theory.

That the Lagrangian form is exact on-shell can be seen again by tracing the gravitational EOM (3.69b),

$$\mathbb{L}_T = \left(\frac{R+\mathcal{L}_u}{16\pi G}\right)\epsilon = \frac{1}{16\pi G}\left(\frac{2}{D-2}\right)\nabla_m\left(u^a Y^m{}_a - u^m Y^a{}_a\right)\epsilon =$$
$$= d\left[\frac{\left(u^a Y^m{}_a - u^m Y^a{}_a\right)\epsilon_m}{8\pi G(D-2)}\right] \equiv d\mathbb{A}_{Æ} \quad (3.71)$$

exactly as in Æ-theory. The same result can be obtained by noticing that the action (3.68) is invariant under the scale transformation

$$\left.\begin{aligned} g_{ab} &\to \Omega g_{ab} \\ T &\to \Omega^{1/2} T \end{aligned}\right\} \implies \mathbb{L}_T \to \Omega^{(D-2)/2}\mathbb{L}_T\,. \quad (3.72)$$

Considering the infinitesimal transformation $\Omega \approx 1+\omega$, from the on-shell equality $\omega(D-2)\mathbb{L}_T/2 = \Theta_T(\phi, \delta_\omega\phi)$ we get

$$\mathbb{L}_T = d\left[\frac{\left(u^a Y^m{}_a - u^m Y^a{}_a - N \underleftarrow{Æ^m} T\right)\epsilon_m}{8\pi G(D-2)}\right]. \quad (3.73)$$

This coincides with (3.71) after noticing that the last term vanishes due to the aether EOM:

$$d\left(N\underleftarrow{Æ^m} T\epsilon_m\right) = \nabla_m\left(N\underleftarrow{Æ^m} T\right)\epsilon =$$
$$= \left[\nabla_m\left(N\underleftarrow{Æ^m}\right)T + N\underleftarrow{Æ^m}\nabla_m T\right]\epsilon = 0 \quad (3.74)$$

where the last equality follows from (3.69a) and from $\underleftarrow{Æ^m} u_m = 0$. Therefore the reduction of the LHS of (3.50) to a surface integral proceeds as in Æ-theory,

$$\int_\Sigma \left(d\mathbb{Q}_T[\xi] + \xi\cdot\mathbb{L}_T\right) = \int_{\partial\Sigma}\left(\mathbb{Q}_{Æ}[\xi] - \xi\cdot\mathbb{A}_{Æ}\right). \quad (3.75)$$

The problem is that the RHS of (3.50) does not vanish anymore. From (3.70), assuming $\pounds_\xi g_{ab} = \pounds_\xi u_a = 0$, $\Theta_T(\phi, \pounds_\xi\phi)$ it is equal to

$$8\pi G\,\Theta_T\left(\phi, \pounds_\xi\phi\right) = \left[-Y^{ma}\underleftarrow{\nabla_a(u\cdot\xi)} + (u\cdot\xi)Y^{ma}\underleftarrow{\nabla_a \ln N} + (u\cdot\xi)\underleftarrow{Æ^m}\right]\epsilon_m =$$
$$= \left[Y^{ma}\left((u\cdot\xi)a_a - \underleftarrow{\nabla_a(u\cdot\xi)}\right) + (u\cdot\xi)\underleftarrow{Æ^m}\right]\epsilon_m =$$
$$= \left[-Y^{ma}\underleftarrow{\pounds_\xi u_a} + (u\cdot\xi)\underleftarrow{Æ^m}\right]\epsilon_m = (u\cdot\xi)\underleftarrow{Æ^m}\epsilon_m \quad (3.76)$$

where we used $\pounds_\xi T = \xi^a \nabla_a T = -(u\cdot\xi)/N$, $a_a = \underleftarrow{\nabla_a \ln N}$ and the relation $\underleftarrow{\pounds_\xi u_a} = \underleftarrow{\nabla_a (u\cdot\xi)} - (u\cdot\xi) a_a$.

We obtain that, on-shell, the integral of (3.50) in khronometric theory becomes

$$\int_{\partial\Sigma} (\mathbb{Q}_{Æ}[\xi] - \xi\cdot\mathbb{A}_{Æ}) = \frac{1}{8\pi G}\int_\Sigma (u\cdot\xi)\underleftarrow{Æ^m}\epsilon_m\,. \tag{3.77}$$

The RHS of (3.77) vanishes when $\epsilon_m \propto u_m$. Therefore a Smarr formula, as a relation between terms at infinity and terms at the inner boundary, is generally defined only on the preferred slices Σ_T at $T=$ const. This is a manifestation of the fact that, while we are formally dealing with a diffeoinvariant formulation of IR Hořava, the theory "knows" about the existence of a preferred foliation and certain properties are well defined or accessible only in that foliation.

Smarr formula for spherically symmetric black holes

We want to show that, when restricted to spherically symmetric configurations, our Smarr formula reproduces the one originally found in [189, 163]. Using (3.53) and (3.56), the form $\mathbb{Q}_{Æ}[\xi] - \xi\cdot\mathbb{A}_{Æ}$ reads

$$\begin{aligned}\mathbb{Q}_{Æ}[\xi] - \xi\cdot\mathbb{A}_{Æ} = -\frac{1}{16\pi G}\Big[&\nabla^a\xi^b + 2c_{13}u^a\xi_c\nabla^{(b}u^{c)} + c_{13}(u\cdot\xi)u^a b^b \\ &-2c_{14}(u\cdot\xi)u^a a^b + (u\cdot\xi)c_\omega\omega^{ab} - \frac{2}{D-2}\left(c_{123}(\nabla\cdot u)u^a\xi^b - c_{13}\xi^a a^b\right)\Big]\epsilon_{ab}\end{aligned} \tag{3.78}$$

where $\omega_{ab} = \underleftarrow{\nabla_{[a}u_{b]}}$ is the twist of u^a, and $c_\omega = c_1 - c_3$.

For a spherically symmetric configuration, the expression simplifies. First, the twist ω^{ab} vanishes because spherical symmetry implies hypersurface orthogonality of u^a. Second, let s_a be the spherically symmetric unit-spacelike normal to the spherical surfaces. Then, using $\epsilon_{ab} = \hat{n}_{ab}\bar{\epsilon}$, where $\hat{n}_{ab} = -2u_{[a}s_{b]}$ is the binormal to the spatial spherical sections and $\bar{\epsilon}$ is their surface element, we get

$$\begin{aligned}\mathbb{Q}_{Æ}[\xi] - \xi\cdot\mathbb{A}_{Æ} = -\frac{1}{16\pi G}\Big[&\nabla^a\xi^b\hat{n}_{ab} + c_{13}(s\cdot\xi)\left(s^a s^b\nabla_a u_b\right) \\ &-\frac{2\,c_{123}}{D-2}(s\cdot\xi)\,(\nabla\cdot u) - \frac{2(D-3)c_{14}}{D-2}(u\cdot\xi)(a\cdot s)\Big]\,\bar{\epsilon}\,.\end{aligned} \tag{3.79}$$

Moreover, since ξ^a is a Killing vector and $\pounds_\xi u_a = 0$, we can rewrite the first term as

$$\frac{\nabla^a \xi^b \hat{n}_{ab}}{2} = \nabla^a \xi^b s_a u_b = s^a \nabla_a (u \cdot \xi) + s^a \xi^b \nabla_a u_b = \\ = (u \cdot \xi)(a \cdot s) - (s \cdot \xi)\left(s^a s^b \nabla_a u_b\right) \quad (3.80)$$

where in the last line we used $\underleftarrow{\pounds_\xi u_a} = \underleftarrow{\nabla_a (u \cdot \xi)} - (u \cdot \xi) a_a$. Therefore

$$\mathbb{Q}_{Æ}[\xi] - \xi \cdot \mathbb{A}_{Æ} = \frac{1}{8\pi G}\left[-\left(1 - \frac{(D-3)c_{14}}{(D-2)}\right)(u \cdot \xi)(a \cdot s) \\ + (1 - c_{13})(s \cdot \xi)\left(s^a s^b \nabla_a u_b\right) + \frac{c_{123}}{D-2}(s \cdot \xi)(\nabla \cdot u)\right] \bar{\epsilon}\,. \quad (3.81)$$

Specializing to four dimensions we get

$$\mathbb{Q}_{Æ}[\xi] - \xi \cdot \mathbb{A}_{Æ} = \frac{1}{8\pi G}\left[-\left(1 - \frac{c_{14}}{2}\right)(u \cdot \xi)(a \cdot s) \\ + (1 - c_{13})(s \cdot \xi)\left(s^a s^b \nabla_a u_b\right) + \frac{c_{123}}{2}(s \cdot \xi)(\nabla \cdot u)\right] \bar{\epsilon}\,. \quad (3.82)$$

Eq.(3.82) makes direct contact with Eqs.(33)-(34) in [163]. Indeed, defining

$$q = -\left(1 - \frac{c_{14}}{2}\right)(u \cdot \xi)(a \cdot s) + (1 - c_{13})(s \cdot \xi)\left(s^a s^b \nabla_a u_b\right) + \frac{c_{123}}{2}(s \cdot \xi)(\nabla \cdot u) \quad (3.83)$$

and using $\hat{n}^{ab}\hat{n}_{ab} = -2$, the identity $d\left(\mathbb{Q}_{Æ}[\xi] - \xi \cdot \mathbb{A}_{Æ}\right) = 0$ can be rewritten as

$$0 = -\frac{1}{16\pi G} d\left(q\, \hat{n}^{ab} \epsilon_{ab}\right) = -\frac{1}{8\pi G} \nabla_a \left(q\, \hat{n}^{ab}\right) \epsilon_b \quad (3.84)$$

from which Eqs.(33)-(34) of [163] follow.

Finally, let us evaluate the Smarr formula on a leaf of the preferred foliation, for the four-dimensional static BHs (3.42), for which $\xi^a = t^a$. Integrating (3.82) on a two-sphere at infinity we get precisely $M/2$, where M is the same as in (3.46) and we used the falloffs (3.45), $(a \cdot s) = r_0/2r^2 + \mathcal{O}(r^{-3})$ and the fact that $\nabla_a u_b = \mathcal{O}(r^{-2})$. On the other hand, integrating (3.82) over a two-sphere at the UH, using $(u \cdot t)_{\mathrm{UH}} = 0$, we obtain $q_{\mathrm{UH}} A_{\mathrm{UH}}/8\pi G$ where q_{UH} is the value of q at the UH

$$q_{\mathrm{UH}} = (s \cdot \xi)_{\mathrm{UH}} \left[(1 - c_{13})\left(s^a s^b \nabla_a u_b\right) + \frac{c_{123}}{2}(\nabla \cdot u)\right]_{\mathrm{UH}}. \quad (3.85)$$

Therefore, equating the two integrals, we reproduce Eq.(3.49), where q_{UH} is given by (3.85).

3.3.2 A thermodynamical interpretation?

A mechanical and/or thermodynamical interpretation of BHs in Lorentz violating gravity is problematic. First of all, we have already discussed in Ch.2.4.3 that LV BHs cannot extend to a regular bifurcation surface, a fact which forbids Wald's derivation of the first law in these BHs [187]. Second, Æ-theory and IR Hořava gravity should be considered as effective theories, in which higher order Lorentz-violating operators are neglected in the infrared; in particular, this is exactly the way khronometric theory is obtained from the full Hořava action [6]. These operators become relevant at high energies, thus inducing modified superluminal dispersion relations in the ultraviolet regime. Therefore a Killing horizon is not anymore a true causal horizon and its thermodynamical significance in terms of entropy and associated Hawking temperature becomes questionable.

The discovery of the UH opens the way for a possible solution of these issues. Indeed, since the UH is now a true causal boundary, one may conjecture that a notion of entropy as a measure of the missing information should rather be associated with it. Two questions then arise: (i) Is there a notion of "Hawking temperature" associated with UHs? (ii) Is there a first law of mechanics associated with UHs, possibly with a thermodynamical interpretation?

The first question was addressed in [190, 189, 191, 192]. Refs.[190, 189, 191] studied the problem from the point of view of particle creation near the UH and the subsequent generation of a thermal radiation via quantum tunneling of the antiparticles beyond the horizon. The general outcome is that the tunneling gives rise to a thermal radiation with temperature

$$T_{\rm UH} = \left(\frac{N-1}{N}\right)\frac{\kappa_{\rm UH}}{\pi}\,. \tag{3.86}$$

Here N represents the leading behaviour of the modified dispersion relation of the test field in the ultraviolet,

$$\omega \sim k^N \quad \text{for } k \to \infty,\ N > 1 \tag{3.87}$$

where the freqeuncy ω and the momentum k in (3.87) are understood to be defined in the aether frame. The quantity $\kappa_{\rm UH}$ is defined as

$$\kappa_{\rm UH} = \frac{1}{2}u^a\nabla_a(u\cdot t) \equiv \frac{(a\cdot t)}{2} \tag{3.88}$$

where in the last step we used the Killing identity $\nabla_{(a}t_{b)} = 0$. Notice that the factor $N-1$ at the numerator of $T_{\rm UH}$ is crucial to exclude that relativistic modes, for which $\omega \sim k$, are emitted from the UH. Indeed, it was shown in [191] that they continue to be emitted from the Killing horizon with the usual Hawking temperature $T_{\rm KH} = \kappa_{\rm KH}/2\pi$.

As we anticipated, [169] showed that $(a \cdot t)$, and thus $\kappa_{\rm UH}$, must be constant on the UH. Moreover, [193, 194] showed that $\kappa_{\rm UH}$ can be intepreted as a surface gravity of the UH, in the sense that it quantifies how much a congruence of infinite-speed modes peels off at the UH. Therefore it seems that not only a temptative identification of the UH temperature exists, but it is also connected to a notion of surface gravity and obeys a zeroth law.

It must be noted that $T_{\rm UH}$ is a local notion which does not necessarily correspond to the temperature perceived by an observer at infinity. It is not even clear that the spectrum remains thermal after a reprocessing of the rays occurs at the Killing horizon before reaching infinity [193, 190]. Actually, by means of a calculation based on a collapsing null shell, [192] found that the details of the spectrum for an observer at infinity are independent of the UH and the temperature is given by $T_{\rm KH}$, up to corrections of order $\kappa_{\rm KH}/\Lambda_{LV}$, where Λ_{LV} is the LV scale entering the modified dispertion relation.

From the above results, it is not really clear what is the role played by the UH in a possible formulation of BH thermodynamics in LV theories. A general derivation of a first law would surely be of great help, because one could then try to identify the temperature and entropy contributions from it. Such a derivation still lacking, one can adopt the more modest approach of scrutinizing the existing exact solutions, in order to see if an empirical pattern emerges pointing towards the existence of a first law. This is precisely the approach that we carry on here.

Some methodological calrifications are necessary. In order for a thermodynamical interpretation to be viable, we expect to find a first law of the form

$$dM = T_{\rm UH}\, dS_{\rm UH} + (\text{work terms}). \tag{3.89}$$

We interpret (3.89) as a differential equation to be solved for $S_{\rm UH}$: this means that we do not assume *a priori* that $S_{\rm UH}$ is proportional to the area of the UH. Clearly there is some vagueness here, because we could always ascribe anyawkward term in dM to a not better specified form of work in (3.89). For this reason we restrict ourselves to the simplest and most natural choices of work terms. In particular, since the static solutions we consider have a one parameter dependence, we assume in analogy with GR that no work term is involved. Similarly, when considering rotating solutions, we allow only for the work term due to the change in angular momentum. Moreover, we do not consider variations of the cosmological constant.

Another matter of concern is the fact that, from (3.86), $T_{\rm UH}$ is N-dependent.

This in turn would induce an N-dependence also on $S_{\rm UH}$, which implies an awful species dependence of the UH entropy. However, in a UV completion of Hoăva-Lifshitz gravity, N becomes a universal constant dictated by the Lifshitz symmetry of the theory in the UV [155]. Therefore the N-dependence is not problematic and we can simply work at fixed $N > 1$. Here, for definiteness, we take $N \to \infty$, the case of a finite N differing just by factors of $(N-1)/N$.

The exact solutions we will work with are: (i) four-dimensional static asymptotically flat UHs with $c_{14} = 0$ or $c_{123} = 0$; (ii) four-dimensional static asymptotically AdS UHs with $c_{14} = 0$; (iii) three-dimensional rotating asymptotically AdS UHs with $c_{14} = 0$. Case (i) has been already treated in [163] and we just review it; as anticipated, it admits a first law of the form (3.89) with a straightforward interpretation. Case (ii) was treated in [164], but we claim that the treatment suffered from a imprecise definition of mass; anyway, even after adjusting the mass, we find that a first law of the form (3.89) does not seem to be admitted. Finally, case (iii) is the only one for which exact rotating UH solutions have been found in Æ-theory and khronometric theory [166]: we find that, while a meaningful first law of the form (3.89) is admitted in the static limit, things get messy as soon as rotation is switched on.

Four-dimensional static asymptotically flat UHs

Refs.[189, 163] showed that, for $c_{14} = 0$ or $c_{123} = 0$, static asymptotically flat BH solutions with a regular UH can be found in four dimensions. The solutions are summarized in Table (3.90). The parameter $f(r)$ in the line element (3.42a) turns out to be equal to 1. The other metric and aether functions are presented in terms of $r_{\rm UH}$ rather that r_0, because this facilitates the computation of the first law.

$$
\begin{array}{c|c|c}
 & c_{14}=0 & c_{123}=0 \\
\hline
e(r) & 1-\frac{4r_{\rm UH}}{3r}-\frac{c_{13}}{3(1-c_{13})}\frac{r_{\rm UH}^4}{r^4} & 1-\frac{2r_{\rm UH}}{r}-\frac{(c_{14}-2c_{13})}{2(1-c_{13})}\frac{r_{\rm UH}^2}{r^2} \\
\hline
(u\cdot t) & -\left(1-\frac{r_{\rm UH}}{r}\right)\sqrt{1+\frac{2r_{\rm UH}}{3r}+\frac{r_{\rm UH}^2}{3r^2}} & -1+\frac{r_{\rm UH}}{r} \\
\hline
(s\cdot t) & \frac{r_{\rm UH}^2}{\sqrt{3(1-c_{13})}r^2} & \frac{r_{\rm UH}}{r}\sqrt{\frac{2-c_{14}}{2(1-c_{13})}}
\end{array}
\tag{3.90}
$$

In terms of $r_{\rm UH}$, the mass M and the temperature $T_{\rm UH}$ (for $N \to \infty$) read

$$
M = \begin{cases} \frac{2r_{\rm UH}}{3G} & \text{if } c_{14}=0 \\ \left(1-\frac{c_{14}}{2}\right)\frac{r_{\rm UH}}{G} & \text{if } c_{123}=0 \end{cases}, \tag{3.91}
$$

and

$$T_{\rm UH} = \frac{(a \cdot t)}{2\pi} = \begin{cases} \frac{1}{2\pi r_{\rm UH}} \sqrt{\frac{2}{3(1-c_{13})}} & \text{if } c_{14} = 0 \\ \frac{1}{2\pi r_{\rm UH}} \sqrt{\frac{2-c_{14}}{2(1-c_{13})}} & \text{if } c_{123} = 0 \end{cases} . \tag{3.92}$$

Therefore, integrating (3.89) for $S_{\rm UH}$ with no work term, we obtain

$$S_{\rm UH} = \alpha \frac{A_{\rm UH}}{4G}, \quad \alpha = \begin{cases} \sqrt{\frac{2(1-c_{13})}{3}} & \text{if } c_{14} = 0 \\ \sqrt{(1-c_{13})\left(1-\frac{c_{14}}{2}\right)} & \text{if } c_{123} = 0 \end{cases} . \tag{3.93}$$

This is the expected result. In order to see how it is affected in less simple or less symmetric spacetimes, let us consider the inclusion of a negative cosmological constant and the effects of rotation.

Four-dimensional static asymptotically AdS UHs

It was shown in [164] that static asymptotically AdS UHs in Æ-theory and khronometric theory are possible only if $c_{14} = 0$. The solution can be expressed in the following closed analytic form

$$(u \cdot t) = -\frac{r}{l}\left(1 - \frac{r_{\rm UH}}{r}\right)\sqrt{1 + \frac{2r_{\rm UH}}{r} + \frac{(3r_{\rm UH}^2 + l^2)(r_{\rm UH}^2 + 2r_{\rm UH}r + 3r^2)}{3r^4}}\,, \tag{3.94a}$$

$$(s \cdot t) = \frac{r}{\lambda} + \frac{r_{\rm UH}^2}{r^2\sqrt{3(1-c_{13})}}\sqrt{\frac{3r_{\rm UH}^2 + l^2}{l^2}}\,, \tag{3.94b}$$

$$e(r) = 1 - \frac{\bar{\Lambda} r^2}{3} - \frac{r_0}{r} - \frac{c_{13}}{3(1-c_{13})}\left(\frac{3r_{\rm UH}^2 + l^2}{l^2}\right)\frac{r_{\rm UH}^4}{r^4}\,, \tag{3.94c}$$

$$r_0 = \frac{2r_{\rm UH}(3r_{\rm UH}^2 + 2l^2)}{3l^2} + \frac{2r_{\rm UH}^2}{\lambda\sqrt{3(1-c_{13})}}\sqrt{\frac{3r_{\rm UH}^2 + l^2}{l^2}}\,. \tag{3.94d}$$

and again $f(r) = 1$. The solution depends on the UH radius $r_{\rm UH}$ and ont he three further parameters $\bar{\Lambda}$, l and λ. These are not independent from each other, but are related to the bare comsological constant Λ by the relation

$$\frac{\bar{\Lambda}}{3} = \frac{\Lambda}{3} - \frac{c_{13} + 3c_2}{2\lambda^2} = \frac{1}{\lambda^2} - \frac{1}{l^2}\,. \tag{3.95}$$

The parameter λ act as a “misalignement” coefficient, in the sense that u^a is not aligned with t^a at spatial infinity unless $\lambda = 0$. Although the latter is the most natural choice, we leave λ unspecified because our results are independent of its specific value.

We now derive an expression for the mass of the above spacetime. The formula (3.46) is valid only for flat asymptotics and, therefore, it cannot be used for the spacetime (3.94). We use the same method introduced in Sec.3.2 for Lovelock theory: we identify the variation of the mass δM with the variation of the covariant energy

$$\delta\mathcal{E} = \int_{S_\infty} \left[\delta\mathbb{Q}[t] - i_t\Theta\left(\phi, \delta\phi\right)\right] \tag{3.96}$$

with respect to r_{UH}. Using (3.51) and (3.53) it can be shown that, for two infinitesimally close static solutions of the form (3.42), $\delta\mathcal{E}$ is equal to

$$\begin{aligned}\delta\mathcal{E} = \frac{\Omega_{D-2}r^{D-2}}{8\pi G}\Bigg[&-\frac{(D-2)\delta e(r)}{2r} + \frac{(D-2)e(r)}{rf^2(r)}\frac{d\delta f(r)}{dr} + c_{14}(u\cdot t)\delta(a\cdot s) \\ &- c_2(s\cdot t)\delta\left(\nabla\cdot u\right) - c_{13}(s\cdot t)\delta\left(s^a s^b\nabla_a u_b\right)\Bigg]\end{aligned} \tag{3.97}$$

and moreover

$$(a\cdot s) = -\frac{(s\cdot t)(u\cdot t)'}{f(r)}\,, \tag{3.98a}$$

$$(\nabla\cdot u) = -\frac{\left[r^{D-2}(s\cdot t)\right]'}{r^{D-2}f(r)} \tag{3.98b}$$

$$(s^a s^a\nabla_a u_b) = -\frac{(u\cdot t)'}{f(r)} \tag{3.98c}$$

where the primes denote derivatives w.r.t. r. Therefore, using (3.97)-(3.98), the mass of the solution (3.94) is

$$M = \frac{r_{\mathrm{UH}}\left(3r_{\mathrm{UH}}^2 + 2l^2\right)}{3Gl^2} + \frac{\sqrt{1-c_{13}}\,r_{\mathrm{UH}}^2}{\sqrt{3}\lambda G}\sqrt{\frac{3r_{\mathrm{UH}}^2 + l^2}{l^2}}\,. \tag{3.99}$$

Notice that this mass is different from the one identified in Eq.(57) of [164]. The discrepancy comes from the fact that [164] identifies the mass with the (finite part) of $\int_{S_\infty}\left(\mathbb{Q}_{\AE}[t] - t\cdot\mathbb{A}_{\AE}\right)$, modulo a constant proportionality factor. However, while these two notions coincide for flat asymptotics, they are not anymore guaranteed to match in an AdS spacetime. Therefore, it looks more consistent to compute the mass using a proper Hamiltonian definition.

Since we do not vary the cosmological constant, the only variable parameter in the first law (3.89) is r_{UH}. For this reason, we assume that there is no work term in (3.89). The temperature T_{UH} is

$$T_{\mathrm{UH}} = \frac{1}{2\pi r_{\mathrm{UH}}\sqrt{3(1-c_{13})}}\left(\frac{r_{\mathrm{UH}}\sqrt{3(1-c_{13})}}{\lambda} + \sqrt{\frac{3r_{\mathrm{UH}}^2 + l^2}{l^2}}\right)\sqrt{\frac{9r_{\mathrm{UH}}^2 + 2l^2}{l^2}} \tag{3.100}$$

from which it follows that

$$\frac{dS_{\rm UH}}{dr_{\rm UH}} = \frac{1}{T_{\rm UH}}\frac{dM}{dr_{\rm UH}} = \frac{2\pi r_{\rm UH}\sqrt{1-c_{13}}}{G}\sqrt{\frac{9r_{\rm UH}^2+2l^2}{9r_{\rm UH}^2+3l^2}}\,. \tag{3.101}$$

This equation can be integrated to give

$$S_{\rm UH} = \frac{\pi\sqrt{1-c_{13}}}{18\,G}\left[2\sqrt{3\left(2l^4+15l^2r_{\rm UH}^2+27r_{\rm UH}^4\right)}+ \right. \\ \left. -l^2\ln\left(5l^2+18r_{\rm UH}^2+2\sqrt{3\left(2l^4+15l^2r_{\rm UH}^2+27r_{\rm UH}^4\right)}\right)\right]\,. \tag{3.102}$$

Expression (3.102) is certainly awkward. While it is true that, in general, we should not expect the entropy to be proportional to the area, it is also true that it is not easy to intepret (3.102) without some numerology. Therefore one should be very cautious and be open to the fact that (3.102) is signaling a problem in the thermodynamical interpretation based on the putative first law (3.89).

Three-dimensional rotating asymptotically AdS UHs

Fully rotating solutions in three-dimensional khronometric theory were found in [166] for $c_{14} = 0$ and AdS asymptotics. They are the equivalent of the BTZ solution (2.108) in three-dimensional GR. The line element and aether vector have the form

$$ds^2 = -e(r)dt^2 + \frac{dr^2}{e(r)} + r^2\left(d\varphi + \Omega(r)dt\right)^2\,, \tag{3.103a}$$

$$u_a dx^a = (u\cdot t)dt - \frac{(s\cdot t)}{e(r)}dr \tag{3.103b}$$

where

$$e(r) = -r_0 + \frac{\bar{\mathcal{J}}^2}{4r^2} - \bar{\Lambda}r^2\,, \tag{3.104a}$$

$$\Omega(r) = -\frac{\mathcal{J}}{2r^2}\,, \tag{3.104b}$$

$$(u\cdot t) = \frac{1}{l}\left(\frac{r^2-r_{\rm UH}^2}{r}\right)\,, \tag{3.104c}$$

$$(s\cdot t) = \frac{r}{\lambda} + \frac{1}{r}\sqrt{\frac{r_{\rm UH}^4}{l^2(1-c_{13})} - \frac{\mathcal{J}^2}{4}} \tag{3.104d}$$

and $r_{\rm UH}$, $\mathcal{J}$, l and λ are integration constants. The quantities r_0 and $\bar{\mathcal{J}}$ are given in terms of $r_{\rm UH}$, $\mathcal{J}$, l and λ by

$$r_0 = \frac{2r_{\rm UH}^2}{l^2} + \frac{2}{\lambda}\sqrt{\frac{r_{\rm UH}^4}{l^2(1-c_{13})} - \frac{\mathcal{J}^2}{4}}\,, \tag{3.105a}$$

$$\bar{\mathcal{J}}^2 = \mathcal{J}^2 - \frac{4c_{13}r_{\rm UH}^4}{l^2(1-c_{13})}\,. \tag{3.105b}$$

As in the four-dimensional case, the solution is controlled by a misalignment paramter λ and by an effective cosmological constant $\bar{\Lambda}$, which are related to l and to the bare cosmological constant Λ by

$$\bar{\Lambda} = \Lambda - \frac{2(c_2+c_{13})}{\lambda^2} = \frac{1}{\lambda^2} - \frac{1}{l^2}\,. \tag{3.106}$$

The solution is stationary and axisymmetric with Killing vectors $t^a = (1,0,0)$ and $\psi^a = (0,0,1)$. Introducing the unit-spacelike vector s_a orthogonal both to the aether u^a and to ψ^a,

$$s_a dx^a = (s\cdot t)dt - \frac{(u\cdot t)}{e(r)}dr\,, \tag{3.107}$$

the expression (3.44) for timelike Killing vector t^a can be generalized to the rotational case as

$$t^a = -(u\cdot t)u^a + (s\cdot t)s^s + \Omega(r)\psi^a\,. \tag{3.108}$$

The mass of the solution can be computed using the formula (3.97) for $D=3$, finding

$$M = \frac{1}{4G}\left(\frac{r_{\rm UH}^2}{l^2} + \frac{(1-c_{13})}{\lambda}\sqrt{\frac{r_{\rm UH}^4}{l^2(1-c_{13})} - \frac{\mathcal{J}^2}{4}}\right)\,. \tag{3.109}$$

From Eq.(2.25), the total angular momentum J is

$$J = \frac{(1-c_{13})\mathcal{J}}{8G}\,. \tag{3.110}$$

Finally, $T_{\rm UH}$ is equal to

$$T_{\rm UH} = \frac{1}{l\pi r_{\rm UH}}\left(\frac{r_{\rm UH}^2}{\lambda} + \sqrt{\frac{r_{\rm UH}^4}{l^2(1-c_{13})} - \frac{\mathcal{J}^2}{4}}\right)\,. \tag{3.111}$$

Notice that the solution is well defined only if

$$\frac{r_{\rm UH}^4}{l^2(1-c_{13})} - \frac{\mathcal{J}^2}{4} \geq 0\,. \tag{3.112}$$

If $\lambda \to \infty$, an interesting fact happens when the bound (3.112) is saturated: the quantity $(s \cdot t)$ vanishes identically, and the preferred time coincides with the coordinate time t. Moreover, the Killing horizon and the UH coincide: in this case the BH degenerates into an extremal configuration where the UH is null. Notice that this is not in contradiction with the result of [189], spelled above, that the UH must be a timelike leaf. Indeed, imposing $\lambda \to \infty$ in (3.111) and saturating the bound (3.112), gives $T_{\rm UH}$, which in turn imples $(a \cdot t)_{\rm UH} = 0$ via (3.86) and (3.88)[3]. From now on, we assume that the inequality (3.112) holds strictly, i.e. that the BH is nonextremal.

In order to study the viability of the first law (3.89), we consider separately the static case $J = 0$ from the rotating case $J \neq 0$. In the static case, it is immediate to verify that (3.89) is satisfied with no work term and with

$$S_{\rm UH} = \frac{\sqrt{1-c_{13}}P_{\rm UH}}{4G} \tag{3.113}$$

where $P_{\rm UH} = 2\pi r_{\rm UH}$ is the perimeter of the UH. The situation changes dramatically when $J \neq 0$. In this case, we posit a first law of the form

$$dM = T_{\rm UH} dS_{\rm UH} + \Omega_{\rm UH} dJ \tag{3.114}$$

which is the analogous to the one arising in GR. Here $\Omega_{\rm UH} = -\Omega(r_{\rm UH})$ is the frame dragging of locally nonrotating observers w.r.t. the aether frame, whose velocity is given by the aether vector u^a. From (3.109)-(3.111), the first law (3.114) implies the Clausius relations

$$\frac{\partial S_{\rm UH}}{\partial r_{\rm UH}} = \frac{1}{T_{\rm UH}}\frac{\partial M}{\partial r_{\rm UH}} = \frac{\pi r_{\rm UH}^2}{2G\,l}\left(\frac{r_{\rm UH}^4}{l^2(1-c_{13})} - \frac{\mathcal{J}^2}{4}\right)^{-1/2}, \tag{3.115a}$$

$$\frac{\partial S_{\rm UH}}{\partial \mathcal{J}^2} = \frac{1}{T_{\rm UH}}\frac{\partial M}{\partial \mathcal{J}^2} - \frac{(1-c_{13})\Omega_{\rm UH}}{16\pi G\, T_{\rm UH}\mathcal{J}} = -\frac{(1-c_{13})\pi l}{32G\, r_{\rm UH}}\left(\frac{r_{\rm UH}^4}{l^2(1-c_{13})} - \frac{\mathcal{J}^2}{4}\right)^{-1/2}. \tag{3.115b}$$

Eq.(3.115b) can be integrated to give

$$S_{\rm UH} = \frac{(1-c_{13})\pi l}{32G\, r_{\rm UH}}\sqrt{\frac{r_{\rm UH}^4}{l^2(1-c_{13})} - \frac{\mathcal{J}^2}{4}} + \rho(r_{\rm UH}) \tag{3.116}$$

[3] The existence of extremal UHs was first pointed out in [195] for $c_{14} = 0$ and in [168] in the general coupling case.

where $\rho(r_{\rm UH})$ is an arbitrary function of $r_{\rm UH}$ that must be fixed integrating (3.115a). The problem is that, if we differentiate (3.116) w.r.t. $r_{\rm UH}$, the result differs from the RHS of (3.115a) by terms depending explicitely on $\mathcal{J}$, which cannot be compensated by any choice of $\rho(r_{\rm UH})$, leading to a contradiction.

One could think to replace (3.114) with a more general first law of the form

$$dM = T_{\rm UH} dS_{\rm UH} + \alpha \Omega_{\rm UH} dJ \tag{3.117}$$

where α is a generic constant. However, working for simplicity in the limit $\lambda \to \infty$, one can show that (3.117) leads to the same $S_{\rm UH}$ as in (3.105), with the only addition of a factor of α on the RHS. Therefore, (3.115a) cannot be satisfied anyway.

Summing up, we conclude that, in the three-dimensional rotating solution under consideration, there is no first law that can be satisfied at the UH, under the restrictions stated below Eq.(3.89). Those restrictions are dictated by a simple generalization of GR: therefore, even if one does not want to abandon the hope that UHs respect a first law of the form (3.89), our analysis shows that more exotic solutions (e.g. in the form of work terms) are necessary.

Some remarks are necessary, in order to not overestimate prematurely our results. First of all, the problems that we encountered in the interpretation of the first law are restricted to the particular branch of the theory $c_{14} = 0$. It might be the case that higher order terms in the full Hořava theory would always end up introducing a nonzero c_{14} via radiative corrections: in this case, setting this parameter to zero in the infrared action would be inconsistent with the UV completed theory. Second, AdS is not a natural asymptotic for Hořava theory. Indeed we expect that (i) astrophysical BHs are modeled by flat asymptotics, and (ii) if we use Hořava theory as an holographic dual of a Lifshitz QFT, we should consider asymptotic Lifshitz symmetry rather than AdS (see e.g. [196, 197, 198]).

Therefore our results signal problems that *can* occur but, in order to see if they constitute *actual* drawbacks of the theory, one must investigate what happens when more physicl asymptotics are considered. For astrophysical BHs, this implies the study of fully rotating asymptotically flat four-dimensional solutions, which have not yet been obtained neither analytically nor numerically. Regarding the applications to holography, static asymptotically Lifshitz UHs in three dimensions were analyzed in [198]. It was shown that these UHs possess a first law of the form

$$dM \propto T_{\rm UH} dP_{\rm UH} \,, \tag{3.118}$$

in analogy with their static three-dimensional static AdS counterparts. Whether they are better behaved when rotation is switched on, is a matter for future research; nonetheless, from our previous considerations, we expect the case of Lifshitz asymptotics to be more promising.

4 Einstein–Maxwell–dilaton black holes

4.1 Introduction

In this Chapter we will study black holes in Einstein-Maxwell-dilaton theory (EMD). The material is taken from the papers "Scalar charge of black holes in Einstein-Maxwell-dilaton theory" (P5) and "Quasinormal modes of weakly charged Einstein-Maxwell-dilaton black holes" (P6).

EMD was introduced in Ch.2.4.2. Let us briefly describe the main motivations for considering this theory and its BH solutions. Recall from Ch.2.4.2 that EMD is described by the action

$$\mathcal{S}_{EMD} = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[R - 2\nabla^a\Phi\nabla_a\Phi - e^{-2\eta\Phi}F^{ab}F_{ab}\right] . \tag{4.1}$$

EMD can be viewed as an extension of the ordinary Einstein-Maxwell theory, recovered in the limit of vanishing scalar coupling $\eta = 0$. When $\eta = 1$ it emerges in a low energy limit of string theory [32, 33], while the case $\eta = \sqrt{3}$ corresponds to the compactification of the five-dimensional Kaluza-Klein theory [33, 149, 7]. More generally, it constitutes a proxy for theories where all the three bosonic degrees of freedom (scalar, vector and tensor) can be dynamical. Recently the physics of single and binary BHs in EMD has received new attention, mainly due to the perspective of observational tests offered by GW astronomy [199, 200, 201, 202]. Moreover, Ref.[203] considered EMD with the addition of an axion field as an example of a modified BH, in relation to the theory-testing prospects of the Event Horizon Telescope.

Despite this increasing attention, the characterization of BHs in EMD is understood only partially. The main limitation is that explicit solutions are essentially confined to static or slow rotating configurations [32, 33], with the only exception of the single case $\eta = \sqrt{3}$, for which fully rotating solutions were derived in [149]. Notably, [151] proved an uniqueness theorem for stationary EMD BHs under the condition $0 \leq \eta^2 \leq 3$. The general picture that emerges from these solutions is that EMD BHs are electrically charged and, most importantly, hairy: the scalar field (the dilaton) acquires a nontrivial configuration outside the event horizon. The hair is of secondary type, i.e. the multipole moments of the dilaton are not new conserved charges, but they depend depend on the mass, spin and electric charge of the BH. Finally, when the electric charge vanishes, the dilaton configuration trivializes and the BH ceases to be hairy.

It is then clear that the no-hair theorem can be evaded in EMD, if the elctric charge of the BH is nonzero. There are standard arguments in the literature predicting that BHs of astrophysical interest should have a very small, if not vanishing, electric charge-to-mass ratio [23]. However, one must not exclude the possibility that EMD describes the dynamics of a novel gauge vector field, different from the photon field of the standard model (see for example [24], in which "dark photons" and "dark electric charges" are studied as dark matter candidates). In the latter case, anyway, one can appeal to the guiding principle of small deviations from GR, and still argue that the EMD electric charge is a perturbative parameter, although not necessarily as small as in the photon case [24].

The natural question then arises how one can constrain the deviations of BHs in EMD from their standard GR counterparts. As anticipated, GW observations constitute a natural testing tool. This comes essentially from two physical mechanisms. On the one hand, it is known that the presence of scalar hair, and in particular the presence of a monopole scalar charge, triggers the activation of dipole radiation. In the inspiral phase of a merger event, dipole radiation occurs at negative post-Newtonian order and can superseed the ordinary quadrupolar radiation, thus allowing to put efficient constraints of the generation mechanism [204, 26, 142, 15, 20]. On the other hand, the presence of hair modifies the QNM spectrum in the ringdown part of the waveform, thus allowing to constrain deviations through BH spectroscopy, as explained in Ch.2.3.5.

Unfortunately, the application of these testing strategies to EMD is obstructed by a lack of results. The main limitation to the inspiral-phase tests is that they require the knowledge of the monopole scalar charge in terms of other measurable parameters of the BH. Since, in the generic case, only static and slowly rotating solutions are known, and since astrophysical BHs are generally fully rotating, an explicit knowledge of the monopole scalar charge is not at hand. Finding rotating solutions in modified theories is notoriously a difficult task. Therefore it is desiderable to develop a strategy to compute the monopole charge which does not rely on explicit solutions, but only on the structure of the theory. For example, we saw in Ch.2.4.1 that this is possible in theories where the dilaton is linearly coupled to a topological invariant (cf. Eq.2.83). In P5 we have shown that this is possible also in EMD, thus filling a gap in the sense just explained. This is our first constribution to EMD BH theory.

The second contribution concerns the computation of the QNMs. A detailed characterization of QNMs for EMD BHs was so far limited to the particular value $\eta = 1$, and even in that case only to static BHs [205]. Although this study already provides relevant informations, one would like to go beyond particular realizations of the theory and encompass the general case. This is precisely what we have done in P6: we computed numerically QNMs for weakly charged BHs in EMD, both in the static and in the slowly rotating cases, without restricting to a specific value of η. The restriction to slow rotation is obviously due to the lack of fully rotating solutions, but we shall see that our results already shed light on various peculiar features. Therefore they represent an important step towards a general understanding of QNMs in EMD.

The rest of this Chapter is organized as follows. In Sec.4.2 we review the explicit stationary BH solutions in EMD existing in the literature. We also discuss the definition and the limitations of the weak electric charge limit in these BHs. In Sec.4.3 we derive a general relation between the monopole scalar charge and the electric potential energy at the BH horizon, which was anticipated in Ch.2.4.2 (cf. Eq.2.89), and which constitutes the main result of P5. Finally, in Sec.4.4 we investigate the properties of QNMs in EMD BHs, as they were originally presented in P6.

4.2 Black hole solutions in EMD theory

In this section we review the existing stationary BH solutions in EMD. By variational principle, the action (4.1) gives the following EOM

$$S \equiv \nabla^a \nabla_a \Phi + \frac{\eta}{2} e^{-2\eta\Phi} F_{ab} F^{ab} = 0\,, \tag{4.2a}$$

$$J_a \equiv \nabla_b \left(e^{-2\eta\Phi} F^b{}_a \right) = 0\,, \tag{4.2b}$$

$$E_{ab} \equiv G_{ab} - T^{\Phi}_{ab} - T^{F}_{ab} = 0\,, \tag{4.2c}$$

where $G_{ab} = R_{ab} - g_{ab}R/2$ is the Einstein tensor. The scalar and EM stress energy tensors are respectively

$$T^{\Phi}_{ab} = 2\nabla_a\Phi\nabla_b\Phi - (\nabla\Phi)^2 g_{ab}\,, \tag{4.3a}$$

$$T^{F}_{ab} = e^{-2\eta\Phi}\left(2F_{ac}F_b{}^c - \frac{1}{2}F^2 g_{ab}\right)\,, \tag{4.3b}$$

where we used the shorthand notations $(\nabla\phi)^2 = g^{ab}\nabla_a\phi\nabla_b\phi$ and $F^2 = F_{ab}F^{ab}$.

Static black holes Static asymptotically flat BH solutions of the EOM (4.45) where derived in [32, 150], see also [33]. The line element is

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 G(r)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{4.4}$$

where

$$F(r) = \left(1 - \frac{R_+}{r}\right)\left(1 - \frac{R_-}{r}\right)^{(1-\eta^2)/(1+\eta^2)}\,, \tag{4.5a}$$

$$G(r) = \left(1 - \frac{R_-}{r}\right)^{2\eta^2/(1+\eta^2)}\,. \tag{4.5b}$$

The surface $r = R_+$ is the BH event horizon. The surface $r = R_-$ is a curvature singularity, except for the case $\eta = 0$ when it is a nonsigular inner horizon. The dilaton and the vector field are given by

$$e^{2\Phi} = e^{2\Phi_\infty}\left(1 - \frac{R_-}{r}\right)^{2\eta/(1+\eta^2)}\,, \tag{4.6a}$$

$$A_a dx^a = \frac{e^{2\eta\Phi_\infty}Q_E}{r}dt\,. \tag{4.6b}$$

where Q_E is the electric charge of the BH. We see that, when $\eta = 0$, the dilaton reduces to a constant Φ_∞ and the solution correctly reproduces the Reissner-Nordström BH. Notice that, since the original action (4.1) is invariant under the constant shift $\Phi \to \Phi + \Phi_\infty$ and the concomitant rescaling $A_a \to e^{\eta\Phi_\infty}A_a$, it follows that Φ_∞ can be eliminated through a redefinition of Q_E. Therefore, for simplicity, from now on we put $\Phi_\infty = 0$.

The quantities R_+ and R_- are related to the mass M and to the electric charge Q_E by

$$M = \frac{R_+}{2} + \left(\frac{1-\eta^2}{1+\eta^2}\right)\frac{R_-}{2}\,, \tag{4.7a}$$

$$Q_E = \left(\frac{R_+R_-}{1+\eta^2}\right)^{1/2}\,. \tag{4.7b}$$

Eqs.(4.7) can be inverted to give

$$R_+ = M\left[1+\sqrt{1-(1-\eta^2)v^2}\right], \tag{4.8a}$$

$$R_- = \left(\frac{1+\eta^2}{1-\eta^2}\right) M\left[1-\sqrt{1-(1-\eta^2)v^2}\right] \tag{4.8b}$$

where we introduced the electric charge-to-mass ratio $v = Q_E/M$. The value of v is not arbitrary: requiring that $R_+ > R_-$ (in order to avoid naked singularities) and that $R_\pm$ are real, we obtain the bound

$$v^2 \leq 1+\eta^2\,. \tag{4.9}$$

When the upper bound is satisfied, R_+ and R_+ coincide and the BH is extremal. Notice however that, when $\eta \neq 0$, extremal BHs in EMD can have very different properties from the extremal Reissner-Nordström solution. In particular, the surface gravity of the event horizon

$$\kappa_+ = \frac{1}{2R_+}\left(1-\frac{R_-}{R_+}\right)^{(1-\eta^2)/(1+\eta^2)} \tag{4.10}$$

vanishes for $\eta < 1$, is finite for $\eta = 1$ and diverges for $\eta > 1$. Moreover, for $\eta \neq 0$, the area of the event horizon of an extremal EMD BH vanishes. In this chapter we focus exclusively on nonextremal EMD BHs, but a discussion of the properties of extremal BHs can be found in [206].

Slowly rotating black holes The above static solution can be generalized to slow rotation [33]. The resulting line element has the form

$$ds^2 = ds^2_{\text{static}} - 2a\,\Omega(r)\sin^2\theta dtd\varphi + \mathcal{O}(a^2) \tag{4.11}$$

where ds^2_{static} is the same as in (4.4), a is a spin parameter and $\Omega(r)$ is given by

$$\Omega(r) = \frac{(1+\eta^2)^2}{(1-\eta^2)(1-3\eta^2)}\frac{r^2}{R_-^2}\left(1-\frac{R_-}{r}\right)^{2\eta^2/(1+\eta^2)} - \left(1-\frac{R_-}{r}\right)^{(1-\eta^2)/(1+\eta^2)} \times$$
$$\times\left[1-\frac{R_+}{r}+\frac{(1+\eta^2)^2}{(1-\eta^2)(1-3\eta^2)}\frac{r^2}{R_-^2}+\frac{(1+\eta^2)}{(1-\eta^2)}\frac{r}{R_-}\right]\,. \tag{4.12}$$

The vector potential becomes

$$A_a dx^a = \frac{Q_E}{r}\left(dt - a\,\sin^2\theta d\varphi\right) + \mathcal{O}(a^2) \tag{4.13}$$

while the dilaton field remains unaffected at $\mathcal{O}(a)$. As a consequence, the relations between $R_\pm$ and the mass and electric charge are still the same as in (4.8). The system has now also an angular momentum J, given by

$$J = \frac{a}{2}\left(R_+ + \frac{3-\eta^2}{3(1+\eta^2)}R_-\right) = Ma\left(1 + \frac{\eta^2}{3(1+\eta^2)}\frac{R_-}{M}\right). \tag{4.14}$$

Weak charge limit As we anticipated in the introduction, we will consider EMD BHs in the weak electric charge limit. Observe that the line element (4.4) reduces to Schwarzschild for $v \to 0$, while the first corrections appear at order $\mathcal{O}(v^2)$. Therefore we define the weak charge limit as a second order expansion in v at fixed M. In this limit, $F(r)$ and $G(r)$ become

$$F(r) = 1 - \frac{2M}{r} + (1-\eta^2)\frac{M^2v^2}{r^2} + \mathcal{O}(v^4)\,, \tag{4.15a}$$

$$G(r) = 1 - \frac{\eta^2 M v^2}{r} + \mathcal{O}(v^4)\,. \tag{4.15b}$$

The vector field is unchanged since it is already linear in v, while the dilaton becomes

$$\Phi = -\frac{\eta\, M v^2}{2r} + \mathcal{O}(v^4)\,. \tag{4.16}$$

The lenghty expression (4.12) for $\Omega(r)$ drastically simplifies to

$$\Omega(r) = \frac{2M}{r} + \frac{[\eta^2 r - 3M(1-\eta^2)]\, M v^2}{3r^2} + \mathcal{O}(v^4)\,, \tag{4.17}$$

while the angular momentum becomes

$$J = Ma\left(1 + \frac{\eta^2 v^2}{6}\right) + \mathcal{O}(v^4)\,. \tag{4.18}$$

For later convenience, let us introduce the "physical spin parameter" a_J, defined in such a way to satisfy the Kerr-like relation $J = Ma_J$. From (4.18)

$$a_J = a\left(1 + \frac{\eta^2 v^2}{6}\right) + \mathcal{O}(v^4)\,. \tag{4.19}$$

Moreover, we redefine $a_J = M\tilde{a}$, so that we work with an adimensional spin parameter $\tilde{a}$ and the only dimensional scale is the mass M.

At this point, it must be observed that the above expansions have been made too lightly. The reason is easily understood: from (4.5) and (4.6), we are actually expanding in powers of R_-/r. This expansion is not meaningful for all values of r and it certainly fails for $r \to 0$. Since we are interested in meaningful approximations outside the event horizon, it is sufficient that the expansion holds for $r \sim M$. From (4.8), this implies the condition $|(1 - \eta^2)v^2| \ll 1$. We thus find that, for a fixed v, a weak charge expansion must be restricted to a corresponding upper limit of η. Of course, the exact value of such an upper limit dipends on how much precision we want to achieve. We will return to this issue in Section 4.4, where we apply the $\mathcal{O}(v^2)$ expansion to simplify the perturbed EOM and compute the QNMs.

4.3 The monopole scalar charge

In this section we prove that the monopole scalar charge of an isolated asymptotically flat BH in EMD is related to the electric potential energy at the event horizon by the simple relation

$$\boxed{Q_S = -\eta\, V_E\, Q_E} \tag{4.20}$$

as anticipated in Chapter (2.4.2). The proof follows closely the derivation of Eq.(2.83). We start from the EMD Lagrangian written in the language of differential forms

$$\mathbb{L}_{EMD} = \mathbb{L}_g + \mathbb{L}_{\Phi A} \tag{4.21}$$

where $\mathbb{L}_g$ is the Einstein-Hilbert term, whose differential form we don't need explicitely in the following, and $\mathbb{L}_{\Phi A}$ is the Maxwell-dilaton part of the Lagrangian

$$8\pi G\,\mathbb{L}_{\Phi A} = \star d\Phi \wedge d\Phi - e^{-2\eta\Phi} F \wedge \star F\,. \tag{4.22}$$

By varying (4.22) w.r.t. to Φ and A, we derive the EOM

$$\Phi: \qquad d \star d\Phi + \eta e^{-2\eta\Phi} F \wedge \star F = 0\,, \tag{4.23a}$$

$$A: \qquad d\left[e^{-2\eta\Phi} \star F\right] = 0. \tag{4.23b}$$

Using (4.23b) into (4.23a), we obtain the divergence identity

$$d\left[\star d\Phi + \eta e^{-2\eta\Phi} A \wedge \star F\right] = 0. \tag{4.24}$$

At this point, we assume that a stationary BH exist, with timelike Killing field t^a and axisymmetry Killing field ψ^a, such that the horizon is a null hypersurface generated by the linear combination $\chi^a = t^a + \Omega_H \psi^a$. We also assume that the dilaton and the vector fields respect the same symmetries of the metric, $\pounds_t \Phi = \pounds_t A = 0$ and similarly for ψ^a. Then, contracting (4.24) with χ^a and using Cartan's identity $\pounds_\chi = i_\chi d + d i_\chi$, we get

$$\begin{aligned} 0 = d\left[i_\chi \star d\Phi + \eta e^{-2\eta\Phi}(i_\chi A) \star F - \eta e^{-2\eta\Phi} A \wedge i_\chi \star F\right] + \\ - \pounds_\chi \left[\star d\Phi + \eta e^{-2\eta\Phi} A \wedge \star F\right] = \\ = d\left[i_\chi \star d\Phi + \eta e^{-2\eta\Phi}(i_\chi A) \star F - \eta e^{-2\eta\Phi} A \wedge i_\chi \star F\right] \end{aligned} \quad (4.25)$$

where in the last step we used the fact that $\pounds_\chi g_{ab} = \pounds_\chi \Phi = \pounds_\chi A = 0$. Therefore we have reduced the dilaton and vector EOM to a divergence-free rank-2 differential form. The last step of the proof consists into integrating (4.25) over a spacelike hypersurface with boundaries at the bifurcation surface $\mathcal{B}$ and at spatial infinity. Applying the Gauss' theorem, the integration reduces to two boundary contributions. The boundary contribution at the bifurcation surface is

$$\begin{aligned} \mathcal{I}_{\mathcal{B}} = \int_{\mathcal{B}} \left[i_\chi \star d\Phi + \eta e^{-2\eta\Phi}(i_\chi A) \star F - \eta e^{-2\eta\Phi} A \wedge i_\chi \star F\right] = \\ = \int_{\mathcal{B}} \eta e^{-2\eta\Phi}(i_\chi A) \star F = 4\pi\eta\, V_E\, Q_E \end{aligned} \quad (4.26)$$

where $V_E = i_\chi A|_{\mathcal{B}}$ is the electric potential at $\mathcal{B}$, and Q_E is the electric charge

$$Q_E = -\frac{1}{4\pi}\int_{\mathcal{B}} e^{-2\eta\Phi} \star F\,. \quad (4.27)$$

In the first step of (4.26) we used the fact that $\chi^a|_{\mathcal{B}} = 0$ and that $i_\chi \star F|_{\mathcal{B}} = 0$, while in the second step we used the constancy of $i_\chi A$ on the event horizon (see Chapter 2.2.2 for the derivation of these properties).

In order to compute the boundary contribution at infinity, we must impose asymptotic flatness boundary condition. The dilaton field falls off as

$$\lim_{r\to\infty} \Phi = \Phi_\infty + \frac{\Phi_1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \quad (4.28)$$

where Φ_∞ is a constant, while Φ_1 is related to the monopole scalar charge by

$$Q_S = \frac{1}{4\pi}\int_\infty \frac{\Phi_1}{r^2}\epsilon_2 \quad (4.29)$$

where ϵ_2 is the surface element at asymptotic infinity. We can fix the gauge of the vector potential such that it scales as

$$\lim_{r\to\infty} A = \frac{\mathcal{A}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\,. \quad (4.30)$$

Using the asymptotic expansions (4.28) and (4.30), the contribution to the integral of (4.25) at infinty is

$$\mathcal{I}_\infty = \int_{S_\infty} \left[i_\chi \star d\Phi + \eta e^{-2\eta\Phi} (i_\chi A) \star F - \eta e^{-2\eta\Phi} A \wedge i_\chi \star F \right] = \\ = \int_{S_\infty} i_\chi \star d\Phi = 4\pi Q_S \quad (4.31)$$

where in the first step we used the falloff (4.30) to drop out the terms involving the vector potential and the Maxwell field strenght, while in the last step we used the definition (4.29) of the monopole scalar charge. Finally, from $\mathcal{I}_\infty = \mathcal{I}_\mathcal{B}$, we obtain the identity (4.20). This completes the derivation. □

It can be easily seen that (4.20) is satisfied in the BH solutions presented in the previous section. Indeed from (4.6) (taking for simplicity $\Phi_\infty = 0$) we have

$$\Phi = \frac{\eta}{1+\eta^2} \ln\left(1 - \frac{R_-}{r}\right) . \quad (4.32)$$

Expanding at infinity and taking (4.7) into account, we get

$$\lim_{r\to\infty} \Phi = -\frac{\eta}{1+\eta^2}\frac{R_-}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) = -\frac{\eta\, Q_E^2}{R_+ r} + \mathcal{O}\left(\frac{1}{r^2}\right) . \quad (4.33)$$

It follows that $Q_S = \eta\, Q_E^2/R_+$, which coincides with (4.20) once observing that $V_E = Q_E/R_+$.

Horizonless compact objects A comment is in order. It may seem that the above proof could be adapted to prove that horizonless compact objects, such as neutron stars, cannot hold a monopole scalar hair in EMD. Indeed, the spacetime of an horizonless compact objects has no inner boundary. Therefore, if we integrate (4.25) over a compact Cauchy hypersurface with a single boundary at infinity, we obtain $Q_S = 0$. This argument would be exactly the same as the one in Ch.2.4.1 for the Lagrangian (2.80). However, a crucial difference arises. For the Lagrangian (2.80), the reduction of the dilaton EOM to an exact rank-2 differential form did not make use of the EOM for the vector potential, but merely follows from the topological nature of the electromagnetic part of the action. Instead, in the case of EMD, we made explicit use of (4.23b) in order to reduce the dilaton EOM to the form (4.24). Now, since we are implicitely assuming that charged BHs in EMD can form from a gravitational collapse, it means that the collapsing matter carries electric charge and therefore it must be coupled to the vector field A_a, even if the latter does not coincide with the photon field of the standard model. Such a coupling would generate a current term on the RHS of

(4.23b), and consequently also on the RHS of (4.24). Therefore, the integral of (4.25) would not vanish, but instead it would be equal to a volume integral involving electric currents. These additional terms are not present in vacuum, but one cannot exclude their presence for an horizonless compact object. Therefore, in conclusion, we cannot conclude that neutron stars in EMD do not support a monopole scalar charge.

Weak charge limit Let us now return to BHs. We want to prove a further important result: the scalar charge of weakly charged EMD BHs does not depend on the spin (by weak electric charge we mean a $\mathcal{O}(v^2)$ expansion, as explained in the previous section). To see why we first observe that, for a static BH,(4.33) implies

$$Q_S = -\eta \frac{Q_E^2}{2M} + \mathcal{O}(v^4)\,. \tag{4.34}$$

When the BH is spinning, we can write the scalar charge as

$$Q_S = -\eta \frac{Q_E^2}{2M} f(M, v, a, \eta) + \mathcal{O}(v^3) \tag{4.35}$$

where $f(M, v, a, \eta)$ is a function which accounts for corrections to the static charge due to the spin, i.e. $f(M, v, a = 0, \eta) = 1$. However, since we are interested in knowing Q_S only up to terms quadratic in v, and since in the static limit Q_S is already quadratic in v, we can neglect the dependence of f on v. This, in turn, implies that we can also neglect the dependence on η: indeed, from the EOM (4.45), all the terms containing η vanish when the electric charge vanishes. Therefore we conclude that, in the weak charge limit, the factor f reduces to f_{Kerr}, the one given by the Kerr-Newman solution in the limit $Q_E \to 0$.

The factor f_{Kerr} can be derived by contracting the Kerr-Newman vector potential (2.3) with the Killing field χ^a given by (2.7)-(2.8), thus obtaining the Kerr electromagnetic potential. Then, writing

$$V_{E,\text{Kerr}} = \frac{Q_E}{2M} f_{\text{KN}}(M, a, Q_E)\,, \tag{4.36}$$

where $Q_E/2M$ is the electromagnetic potential in the static limit at $\mathcal{O}(Q_E)$, and taking the limit of f_{KN} for $Q_E \to 0$, we obtain

$$f_{\text{Kerr}}(M, a) = \lim_{Q_E \to 0} f_{\text{KN}}(M, a, Q_E) = \left.\frac{2MR_+}{R_+^2 + a^2}\right|_{\text{Kerr}} = 1\,. \tag{4.37}$$

Therefore

$$Q_S = -\eta\, Q_E^2/2M + \mathcal{O}(v^3) \tag{4.38}$$

irrespective of the spin parameter a. □

This result is significant when testing the theory against GW observations. Indeed, GW events so far constrain the spin only loosely, thus preventing the extraction of efficient contraints from dipolar radiation emission [15]. The fact that in the weak charge limit Q_S is spin-independent suggests that less degenerate bounds can be obtained in EMD.

We conclude by noticing that (4.38) contrasts with the claim in [200] that the scalar charge varies with the spin. The scalar charge for spinning EMD BHs was estimated in [200] numerically, by evolving suitable initial data towards a stationary BH configuration. The initial conditions chosen are compatible with the weak charge approximation ($Q_E \sim 10^{-3}$), but the authors find that the monopole charge decreases with the spin. This discrepancy might be due to propagating numerical errors in the initial conditions (see the discussion in the first paragraph of [200, Sec.IV.A]).

4.4 Quasinormal modes of EMD black holes

4.4.1 Review of the previous studies

In this section we study the quasinormal modes of oscillations of BHs in EMD. The problem of finding QNMs for EMD BHs has been already addressed before in [206, 205, 207]. Ref.[206] derived the perturbed axial EOM and sketched the derivation of the polar ones. The computation of the QNMs was not the main concern, and the EOM were used to study the Hawking emission of EMD BHs in the extremal limit. Ref.[207] used the axial EOM derived in [206] to compute the gravitational QNMs and noticed that, unless the charge-to-mass ratio approaches unity, the results are almost independent of η. The polar case wass not treated, due to the complicated form of the polar EOM.

The first detailed study was [205]: here the QNMs are computed for the special case $\eta = 1$ and only for static BHs, but the axial and polar spectrum are both treated and compared to each other. Moreover, the analysis does not restrict to gravitational modes only, but it extends also to electromagnetic and dilaton ones. The most notable result of [205] is that the presence of the dilaton breaks the isospectrality between axial and polar spectrum. The level of isospectrality breaking (henceforth ISO-breaking) grows with v and it is almost negligible for small v, as it should be expected from the fact that the perturbations reduce to the ones on a Schwarzschild background in the limit $v \to 0$. Additionally, ISO-breaking is not very pronounced in the gravitational sector, while it is much more evident in the EM modes: this is also expected from the fact that, in the action 4.1, the dilaton couples directly only to the EM field.

The main limitation of [205] is that it focuses on the specific value $\eta = 1$. It would be certainly interesting to see how the properties of the spectrum vary with η. Some considerations in this direction are contained in [200], which numerically simulates the collision of static BHs in EMD for a wide range of η and extract the ringdown frequencies. However, [200] considers BHs with an electric charge as weak as $Q_E \sim 10^{-3}$: for such small charges, ISO-breaking and η-dependence of the spectrum lie within the numerical error and therefore the spectrum is not distinguishable in practice from the standard Reissner-Nordström one.

Plan of the Section The rest of this Section is organized as follows. Sec.4.4.2 contains a detailed overview of our results, to facilitate the rest of the reading. In Sec.4.4.3 we give an estimate of gravitational QNMs using the light ring correspondence. In Sec.4.4.4 we compute the QNMs numerically for static BHs. In Sec.4.4.5 we employ a modified version of the DF approximation scheme. In Sec.4.4.6 we extend the numerical computation of the QNMs of slowly rotating BHs. Finally, in Sec.4.4.7 we address the possible presence of dilaton instabilities.

4.4.2 Overview of the results

In P6 we have extended the analysis of [205] to generic values of η, for both the static and the slowly rotating BHs introduced in Sec.4.2. In order to deal with the complexity of the polar EOM, we resorted to the weak charge approximation, in the sense explained in Sec.4.2. On general grounds, we must expect that the presence of the dilaton induces ISO-breaking for all η except $\eta = 0$, for which the theory reduces to electrovacuum GR. Moreover, from the form of the dilaton coupling, the effect should grow not only with v, but also with η.

This is exactly what we find. In agreement with [205], we observe that ISO-breaking is almost negligible in the gravitational sector, while it becomes relevant in the EM one. However, differently from [205], we are also able to monitor how the spectrum varies with η. The gravitational modes are scarsely dependent on η, the differences being within the estimated numerical error. On the contrary, the EM part of the spectrum exhibits a marked difference between axial and polar modes: in particular, ISO-breaking grows much beyond the numerical error already for moderate values of η. This provides in principle a way to distinguish among doifferent realizations of the theory through BH spectroscopy.

It is interesting to compare the exact numerical results with suitable approximation schemes. After all, an exact study of QNMs in modified theories is not possible in general. Therefore it is important to test approximation schemes in those few cases in which the exact computation is viable. Moreover, our study restricts to static and slowly rotating BHs: the difficulties that we already encountered with such restrictions let us believe that an extension to fully rotating configurations would not be possible without suitable simplification techniques. We concentrated on two (complementary) approximations: the light ring correspondence and a modification of the so called Dudley-Finley scheme.

The ligh ring correspondence was introduced in Ch.2.3.4. There we saw that in GR the gravitational part of the spectrum is well approximated, with an accuracy of some percents, by the properties of unstable null geodesics around the BH. As argued in [130], the correspondence should continue to be valid in modified theories, provided that GWs propagate at the speed of light and that deviations of the background geometry from Kerr are parametrized by a perturbative small parameter (these conditions certainly hold for weakly charged EMD BHs). Then [130] proposes to use the light ring to estimate BH QNMs, in order to bypass the difficulty of computing them rigorously in arbitrarily modified theories. We verify that the light ring approximation gives a good description of the gravitational QNMs, in both the static and the slowly rotating cases.

The light ring correspondence is limited in that it describes only gravitational modes. However, as discussed above, EM modes contain important informations about the dynamics of the theory. Therefore it is desiderable to develop a complementary approximation scheme to deal with the non-gravitational QNMs. To this aim, we employ a modified version of the so called Dudley-Finley (DF) approximation. In its original version, the DF approximation consists in perturbing the dynamical fields independently from each other: this is particularly appropriate when the backreaction of matter fields on the vacuum geometry is small. Since the coupling between the EM field and the dilaton plays a key role in EMD, we devise a modified DF scheme: in our scheme, one perturbs the metric as an independent field, but the vector and the dilaton fields are perturbed together, keeping on their mutual coupling. Our DF scheme captures the essential physics behind ISO-breaking in the EM sector, also providing a simpler way for its estimation.

As a last task we investigate the stability of EMD BHs. In particular, we critically discuss the claim, formulated in [200], that EMD BHs are subjected to dilaton instabilities. According to [200], instabilities occur when the electric charge is bigger than a certain threshold, and it is enhanced for monopolar ($l = 0$) perturbations. In order to study this possibility, we derive the $l = 0$ pertubed EOM at all oders in v. We find that the effective potential is such that no instability occurs. We argue that the wrong claim of [200] is rooted in an inconsistent application of the weak electric charge approximation.

4.4.3 The light ring approximation

We need to generalize the light ring estimation of Chapter 2.3.4 to the class of metrics of the form (4.4) and (4.11) (the formal difference is the presence of the factor $G(r)$ in the line element). As in Chapter 2.3.4, we will work in the weak electric charge approximation $\mathcal{O}(v^2)$ and at slow rotation $\mathcal{O}(a)$. At order $\mathcal{O}(a)$, the equations of a general null geodesic in the equatorial plane become

$$\dot{t} = \frac{1}{F(r)} \mp \frac{a\,\Omega(r)L}{r^2F(r)G(r)}\,, \tag{4.39a}$$

$$\dot{\varphi} = \pm\frac{L}{r^2G(r)} + \frac{a\,\Omega(r)}{r^2F(r)G(r)}\,, \tag{4.39b}$$

$$\dot{r}^2 = V_{\rm geo}(r) = 1 - \frac{F(r)L^2}{r^2G(r)} \mp \frac{2a\,\Omega(r)L}{r^2G(r)}\,. \tag{4.39c}$$

Then, repeating the same steps as in Chapter 2.3.4, and expressing the results in terms of the physical (adimensional) spin parameter $\tilde{a}$, we find that the complex light ring frequencies are given by

$$\omega_n = \pm l\,\omega_c - i\left(n+\frac{1}{2}\right)\gamma_c \tag{4.40}$$

where

$$M\omega_c = \frac{1}{3\sqrt{3}}\left(1+\frac{v^2}{6}\right) \pm \frac{2\tilde{a}}{27}\left(1+\frac{v^2}{2}\right)\,, \tag{4.41a}$$

$$M\gamma_c = \frac{1}{3\sqrt{3}}\left(1+\frac{v^2}{18}\right) \pm \frac{\tilde{a}v^2}{243}\,. \tag{4.41b}$$

Remarkably, (4.41) is exactly the same as (2.71). In other words, the light ring frequencies for slowly rotating, weakly charged BHs in EMD are degenerate with η and reproduce the Kerr-Newman result. Therefore, if we take the light ring approximation as a good indicator of the behaviour of the gravitational QNMs, we deduce that they depend very weakly on η. We emphasize that this conclusion would not have been reached if, instead of a_J[1], one had used the “unphysical spin parameter” a. However, it is obvious

[1] Recall that $\tilde{a} = a_J M^{-1}$.

that the use of a_J must be preferred, because it is the one connected with the physical definition of the angular momentum.

As we shall see in 4.4.4 and 4.4.6, explicit numerical computations show that gravitational QNMs are well approximately isospectral within the numerical error and in agreement with (4.41), thus validating the use of the light ring correspondence.

4.4.4 Quasinormal modes I: static case

We now compute the EMD QNMs numerically using the perturbed EOM at order $\mathcal{O}(v^2)$ and Leaver's continued fraction method. For conceptual cleareness, we distinguish the static case, addressed here, from the slowly rotating case, which is treated in Sec.4.4.6.

The perturbed equations The derivation of the perturbed EOM is a matter of pedantic and technical manipulations. Here we just illustrate their main features (a detailed derivation is contained in a specific MATHEMATICA® notebook [208]). We find that the axial perturbed EOM can be casted in the form

$$\left[\frac{d^2}{dr_\star^2}+\omega^2\right]\vec{Z}^{\rm ax}(r)=\mathbb{V}^{\rm ax}(r)\vec{Z}^{\rm ax}(r) \tag{4.42}$$

where $\vec{Z}^{\rm ax}$ is a vector-like wavefunction with two components, while $\mathbb{V}^{\rm ax}(r)$ is a 2×2 potential matrix. The tortoise coordinate $r_\star$ is defined through $dr_\star/dr = F^{-1}(r)$ and takes values in the domain $r_\star \in]-\infty,+\infty[$ for $r \in]R_+,+\infty[$. The first and second entry of the matricial Eq.(4.42) reduce respectively to the Maxwell equation and to the Regge-Wheeler equation, in the limit where the background reduces to a Schwarzschild BH. This clearly shows that there are two families of QNMs: the modes of gravitational type and the modes of EM type, distinguished according to their limit as $v\to 0$.

Similarly, the polar perturbed EOM can be casted in the form

$$\left[\frac{d^2}{dr_\star^2}+\omega^2\right]\vec{Z}^{\rm pol}(r)=\mathbb{V}^{\rm pol}(r)\vec{Z}^{\rm pol}(r)+\mathbb{U}^{\rm pol}(r)\frac{d\vec{Z}^{\rm pol}(r)}{dr_\star} \tag{4.43}$$

where now, since the polar sector propagates also the dilaton degree of freedom, $\vec{Z}^{\rm pol}(r)$ is a vector wave function with three components and, correspondingly, $\mathbb{V}^{\rm pol}(r)$ is a 3×3 potential matrix. Moreover, the polar EOM present also a friction-like potential $\mathbb{U}^{\rm pol}(r)$. In the polar sector there are three families of QNMs: gravitational QNMs, EM QNMs and scalar QNMs; in particular, in the limit $v\to 0$, scalar QNMs reduce to the modes of the massless Klein-Gordon equation on a Schwarzschild BH.

The perturbed EOM (4.42) and (4.43) are valid in the general case $l \geq 2$. The particular cases $l = 0$ and $l = 1$ must be treated separately. For $l = 0$ there is no axial propagating mode and only one polar propagating mode, described by the EOM

$$\left[\frac{d^2}{dr_\star^2} + \omega^2\right] Z^0(r) = V^0(r) Z^0(r) \tag{4.44}$$

where the potential $V^0(r)$ reduces to the Klein-Gordon one for $v \to 0$. For $l = 1$ there are one axial and two polar propagating modes, and the respective EOM can be casted in the form

$$\left[\frac{d^2}{dr_\star^2} + \omega^2\right] Z^{\text{az},1}(r) = V^{\text{ax},1}(r) Z^{\text{ax},1}(r)\,, \tag{4.45a}$$

$$\left[\frac{d^2}{dr_\star^2} + \omega^2\right] \vec{Z}^{\text{pol},1}(r) = \mathbb{V}^{\text{pol},1}(r) \vec{Z}^{\text{pol},1}(r)\,. \tag{4.45b}$$

In the limit $v \to 0$, the $l = 1$ axial EOM reduces to the Maxwell equation on a Scwarzschild background, while the $l = 1$ polar EOM reduce to the Klein-Gordon and Maxwell equations respectively.

The explicit expressions of all the above equations and potentials at $\mathcal{O}(v^2)$ are provided in a specific MATHEMATICA® notebook [208]. A key property of all the potentials $\mathbb{V}(r)$, $\mathbb{U}(r)$ and $V(r)$ is that they vanish both at infinity and at the event horizon, up to subleading terms of order $\mathcal{O}(v^3)$. Therefore, imposing ingoing boundary conditions at the event horizon and outgoing boundary conditions at infinity, the generic eigenfunction $\vec{Z}(r)$ will scale as

$$\vec{Z}(r) \sim \begin{cases} e^{-i\omega r_\star} \sim (r - R_+)^{-2\mu\, i\omega} & \text{for } r \to R_+\,, \\ e^{i\omega r_\star} \sim e^{-i\omega r} r^{2M\, i\omega} & \text{for } r \to +\infty\,. \end{cases} \tag{4.46}$$

The constant μ can be obtained by integrating $dr_\star/dr = F^{-1}(r)$ close to the horizon, and it is equal to

$$\mu = M \left(\frac{\sqrt{1 + (\eta^2 - 1)\, v^2} - \eta^2}{1 - \eta^2}\right)^{\frac{\eta^2 - 1}{\eta^2 + 1}} \left(\frac{\sqrt{1 + (\eta^2 - 1)\, v^2} + 1}{2}\right)^{\frac{2}{\eta^2 + 1}}\,. \tag{4.47}$$

However, at order $\mathcal{O}(v^2)$, μ reduces to M and (4.46) greatly simplifies. The scalings (4.46) can be used as a starting point to make a convenient ansatz for the wave functions, in the form of a series expansion around the event horizon

$$\vec{Z}(r) = e^{-i\omega(r - R_+)}\, r^{2(M+\mu)\, i\omega} (r - R_+)^{-2\mu\, i\omega} \sum_{k=0}^{\infty} \vec{a}_k \left(1 - \frac{R_+}{r}\right)^k\,. \tag{4.48}$$

The ansatz (4.48) is the point of departure for the numerical implementation of Leaver's continued fraction method. We applied the method as described in [87]. It must be noted that, in the case of the axial EOM (4.42), we were able to find an homogeneous redefinition of the fields, such that the system diagonalizes and one is left with two decoupled EOM. Obviously, when casted in this form, the system is more tractable. However, we were not able to find a similarly convenient redefinition for the polar EOM (4.43). Therefore, in order to agevolate a more consistent comparison between the axial and the polar spectra, we decided to use the nondiagonalized form of the perturbed EOM also in the axial case.

In Sec.4.2 we saw that we cannot expand in v without implicitely restricting the values of η under consideration, according to $|(1-\eta^2)v^2| \ll 1$. We must now specify which degree of precision we want to achieve in our concrete computation. We adopt the convention to consider the expansion meaningful if $v \leq 0.6$ and $|(1-\eta^2)v^2| \leq 0.5$, which translates into

$$0 \leq \eta \leq \sqrt{1+0.5/v^2}\,. \tag{4.49}$$

For example, when $v = 0.5$, according to (4.49) the expansion is valid for $\eta \leq \sqrt{3}$.

We shall now describe the main features of the QNM spectrum, as resulting from our numerical analysis. According to their limit for $v \to 0$, we will divide the modes in five families: axial and polar gravitational modes, $\omega_{AG,l}$ ans $\omega_{PG,l}$; axial and polar electromagnetic modes, $\omega_{AE,l}$ and $\omega_{PE,l}$; polar scalar modes $\omega_{PS,l}$. We explicitated the multipole number l but omitted the overtone number n, because we will focus only on the fundamental tones $n = 0$.

Gravitational modes The gravitational modes exist for $l \geq 2$. In the limit $v \to 0$, the least-damped modes are the $l = 2$ and $l = 3$ fundamental tones, given respectively by $M\omega = 0.3737 - i\,0.0890$ and $M\omega = 0.5994 - i\,0.0927$ [26]. They are expected to be the dominant modes in a pure GR GW signal. In Fig. 4.1 we show the real and imaginary parts of the axial QNM frequencies for $l = 2$ and $l = 3$ as a function of the dilaton coupling η, for different values of v. One can see that the frequencies depend very weakly on η: the relative difference between the maximum and the minimum for the curves shown in Fig. 4.1 remains below 0.6%, for both real and imaginary parts. The corresponding polar modes are analogous qualitatively and quantitatively, as we show in Fig. 4.2 where we plot the relative percentage difference

$$\Delta\mathrm{Re}\left(\omega_{G,l}\right) = 100 \times \frac{\mathrm{Re}\left(\omega_{PG,l}\right) - \mathrm{Re}\left(\omega_{AG,l}\right)}{\left|\mathrm{Re}\left(\omega_{AG,l}\right)\right|} \tag{4.50}$$

and similarly for the imaginary part. We see that ISO-breaking grows only moderately with v, being almost negligible at small $v \leq 0.2$ and becoming of the order of the percent for $v = 0.6$. Therefore we observe a very weak ISO-breaking in the gravitational sector, in agreement with [205, 200].

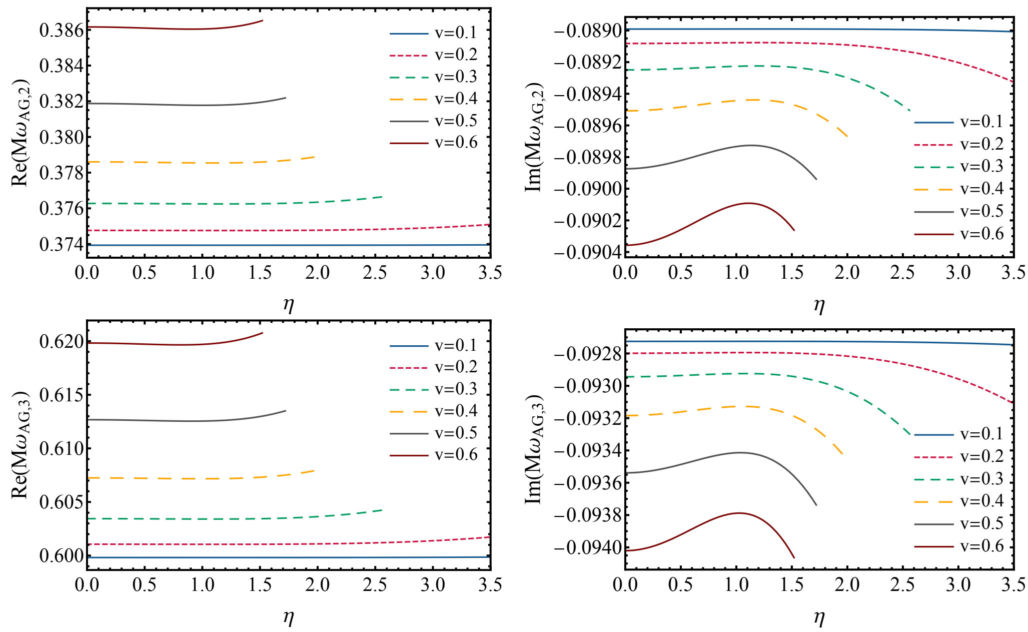

Figure 4.1: Real (left) and imaginary (right) parts of the axial gravitational QNMs, $M\omega_{AG,l}$, as a function of the dilaton coupling η, for $l = 2$ (top) and $l = 3$ (bottom) for different BH's charge-to-mass ratios v, computed using the $\mathcal{O}(v^2)$ equations.

It is important to estimate the percentage error made in working with the $\mathcal{O}(v^2)$ approximation. To this aim, we derived the exact form of the axial perturbed EOM at all orders in v and we compared the $l = 2$ axial gravitational modes with the ones from the $\mathcal{O}(v^2)$ EOM (the exact axial EOM are listed in a MATHEMATICA® notebook [208]). For concreteness we focus on the values $\eta = 0$, 1 and $\sqrt{3}$ and we restrict to $v \leq 0.8$. In Fig.4.3 we plot the relative percentage difference

$$\delta\,\mathrm{Re}(\omega) = 100 \times \left| \frac{\mathrm{Re}(\omega_{\mathrm{FULL}}) - \mathrm{Re}(\omega_{\mathrm{SMALL}})}{\mathrm{Re}(\omega_{\mathrm{FULL}})} \right| \tag{4.51}$$

and similarly for the imaginary part, where ω_{FULL} corresponds to the QNM frequency computed without any approximation and ω_{SMALL} to the QNM frequency computed at $\mathcal{O}(v^2)$. We see that for the real part the error remains below 0.2% for $v \leq 0.6$, while it becomes of the order of half the percent for

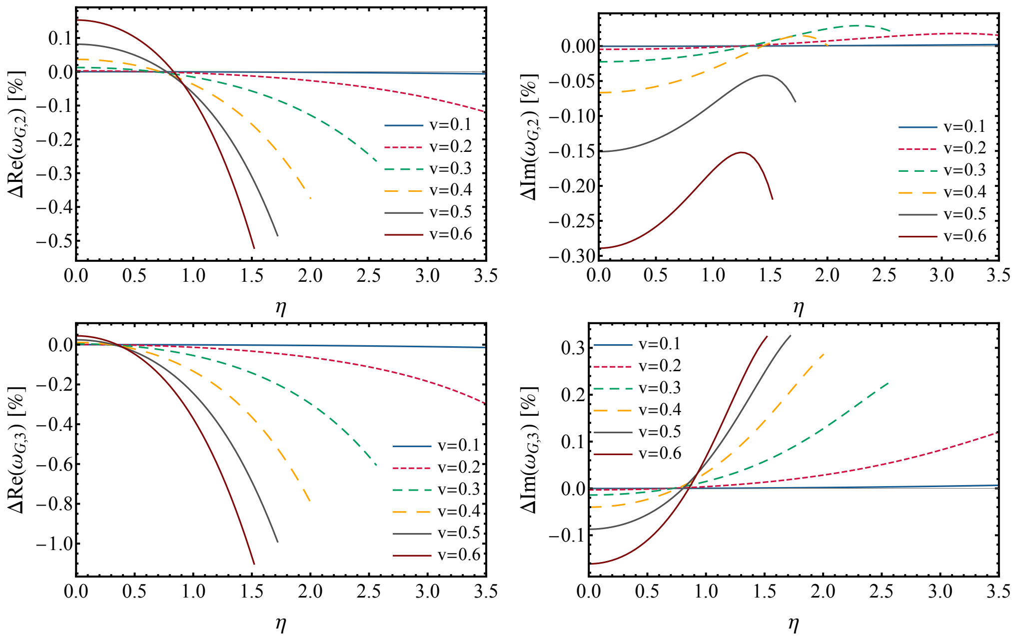

Figure 4.2: ISO-breaking [cf. Eq. (4.50)] between the polar and axial gravitational QNMs for $l = 2$ (top) and $l = 3$ (bottom).

the imaginary part. In particular, we note that this error is comparable to the amount of ISO-breaking that we find in the gravitational sector. It is reasonable to expect that similar errors will occur also in the EM and scalar sectors.

Electromagnetic modes The EM modes exist for $l \geq 1$. As shown in Refs. [209, 210, 211], these modes can become significant for the radiation emitted by the merger of charged BHs. In particular, Refs. [209, 210] numerically studied head-on BH collisions in Einstein-Maxwell theory ($\eta = 0$) for equal [209] and opposite [210] charge-to-mass ratio, while Ref. [211] simulated the inspiral of weakly-charged Reissner-Nordström BHs for different initial configurations. A generic prediction of these studies is that the process is always accompanied by the emission of both EM waves and GWs, with the ringdown part being described by a superposition of both EM and GW QNM frequencies. In addition, for the head-on collisions, it was shown that while for equal charges the EM wave emission is always subdominant w.r.t. GWs [209], for opposite charges, $l = 1$ EM waves become the dominant channel of radiative emission already for moderate values of $|v| \geq 0.37$ [210]. Therefore, depending on the initial binary parameters, EM wave emission and EM QNMs can constitute a non-negligible part of the radiation and

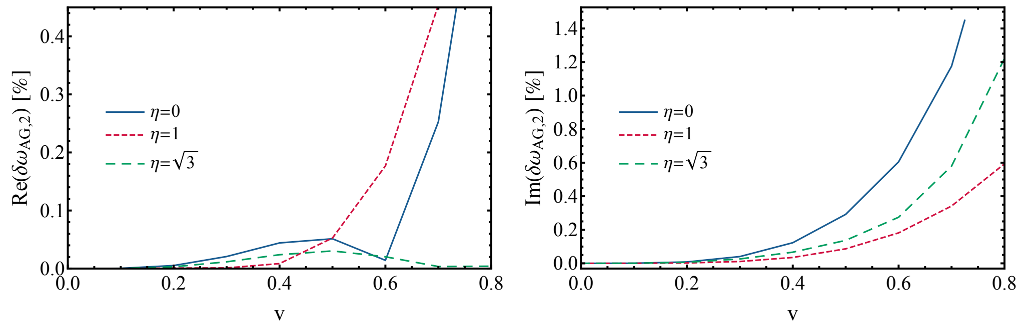

Figure 4.3: Real (left) and imaginary (right) part of $\delta\omega_{AG}$ for $l = 2$, as defined in Eq. (4.51), as a function of the BH's charge-to-mass-ratio, v, for different values of η.

their study can be relevant for the purposes of BH spectroscopy.

For concreteness, we focus on the axial and polar EM modes for $l = 1$ and $l = 2$. Our results, for $0.1 \leq v \leq 0.6$, are shown in Fig. 4.4 (for $l = 1$) and Fig. 4.5 (for $l = 2$). In the limit $v \to 0$, these QNMs coincide with the fundamental EM modes on a Schwarzschild background, $M\omega = 0.2483 - i\,0.0925$ and $M\omega = 0.4576 - i\,0.0950$ [26]. As can be easily seen already at a qualitative level, there is a marked difference between the axial and the polar modes for sufficiently high η. In particular, the polar QNMs have a much stronger dependence on η, which can be understood from the fact that the dilaton only couples directly to the polar EOM.

This difference is more easily seen in Fig. 4.6 where we show the percentage ISO-breaking, evaluated as in Eq. (4.50). The difference between polar and axial modes is very small for $\eta \sim 0$ but grows monotonically with η and v.[2] Isospectrality of the real part of the polar and axial frequencies is broken up to $\sim 15\%$ for $l = 1$ and $\sim 8\%$ for $l = 2$, while for the imaginary part the effect is smaller, but still more pronounced than in the gravitational sector. Therefore ISO-breaking in the EM sector provides a clear signature to distinguish EMD BHs in the (v, η) plane.

Scalar modes Unless $\eta = 0$, the dilaton perturbations couple dynamically to the other fields, therefore inducing the presence of scalar modes. From the action (4.1), one expects that the importance of the scalar radiation grows with η, being almost negligible when $\eta \ll 1$ [200]. This is already visible in the above analysis of EM QNMs, where we saw that larger values of η are

[2] We note that the difference should be exactly zero for $\eta = 0$ because of the known isospectrality of the Reissner-Nordström QNMs [83]. The very small departure from zero at large v can be ascribed to the small charge approximation that we employed.

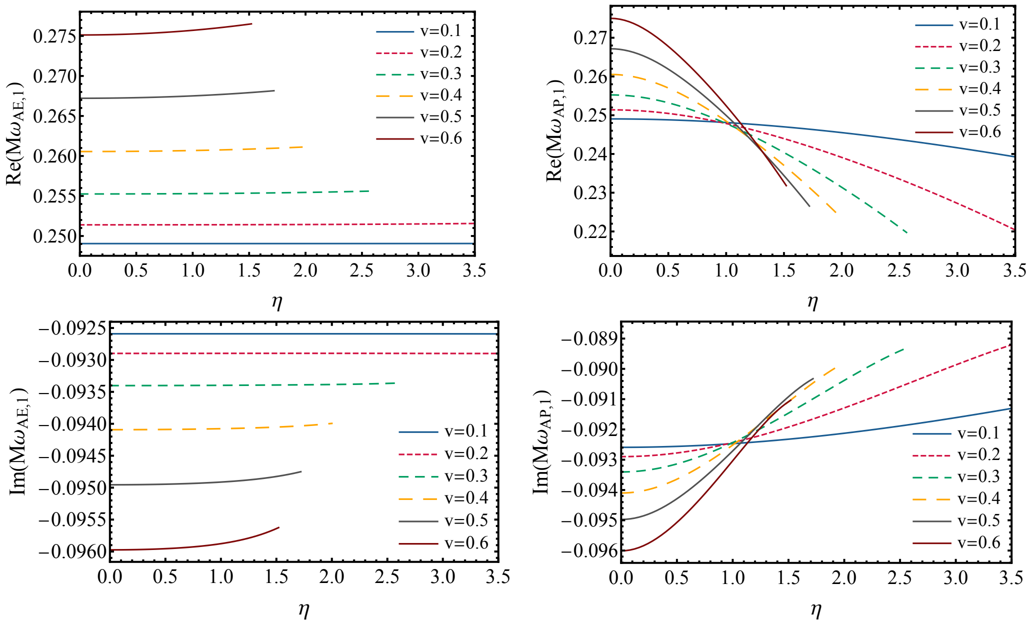

Figure 4.4: Real (top) and imaginary (bottom) parts of the EM axial QNMs (left), $M\omega_{AE,l}$, and polar QNMs (right), $M\omega_{AP,l}$, for $l = 1$, computed using the $\mathcal{O}(v^2)$ equations.

also accompanied by an increasing of EM ISO-breaking.

A possible consequence of the presence of the dilaton is the possibility that it could induce instabilities in this BH spacetime. In fact, it was argued in Ref. [200] that the presence of the dilaton could induce tachyonic-like instabilities for sufficiently large coupling constant η. We did not find any evidence for an instability when computing the scalar QNMs. In particular, in Fig. 4.7 we show the scalar QNM for $l = 0$, where it can be seen that the imaginary part is always negative, thus indicating that these modes always decay and are therefore stable (the same conclusion remains valid for $l = 1$ and $l = 2$). For reference, we note that the fundamental Klein-Gordon mode on a Schwarzschild background is given by $M\omega = 0.1105 - i\,0.1049$.

A more detailed analysis of the possible presence of dilaton instabilities, which goes beyond the weak charge approximation, is postponed to Sec.4.4.7.

4.4.5 The Dudley-Finley approximation

In Refs. [212, 213] an approximate approach to compute the perturbed equations was introduced by Dudley and Finley (DF), motivated by the difficulty of separating radial and angular perturbations in the Kerr-Newman space-

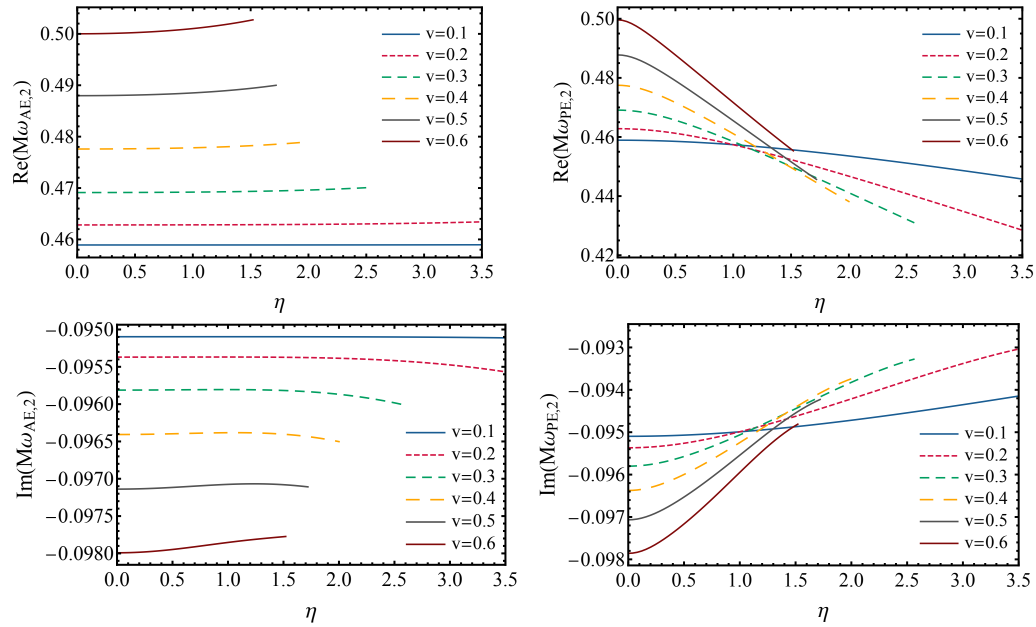

Figure 4.5: Same as Fig. 4.4 but for $l = 2$.

time. In the DF approximation the metric and the matter fields are perturbed separately. This method should be valid as long as the matter fields do not induce large deviations from vacuum GR, i.e. when the effects of matter are already weak at the background level. In the case of the Reissner-Nordström black hole this expectation was confirmed in [214], where the DF QNMs were found in good agreement with the exact ones for $v \lesssim 0.5$. It is reasonable to expect that a similar agreement remains valid in the more general case of EMD theory.

The original DF method consists in perturbing each field independently from the others. We have seen that, while the gravitational modes are only weakly sensitive to the presence of the dilaton, EM modes are quite sensitive to the coupling to the dilaton. It is then reasonable to employ a modified DF scheme in which (i) the gravitational field is varied independently and (ii) the vector and scalar fields are varied togheter but independently from the metric.[3]

Using the DF approximation, we derived the $l = 1$ EOM at $\mathcal{O}(v^2)$ for the system of coupled scalar and EM fields and computed their QNM spectrum[4].

[3]Notice that, in the DF approach, metric or matter perturbations are turned off from the very beginning when one derives the perturbed EOM. An alternative approach could be to turn off the degrees of freedom at the end, once the EOM have already be obtained. For a discussion and a comparison of these two approaches see [215].

[4]The EOM can be found in a supplemental MATHEMATICA® notebook [208].

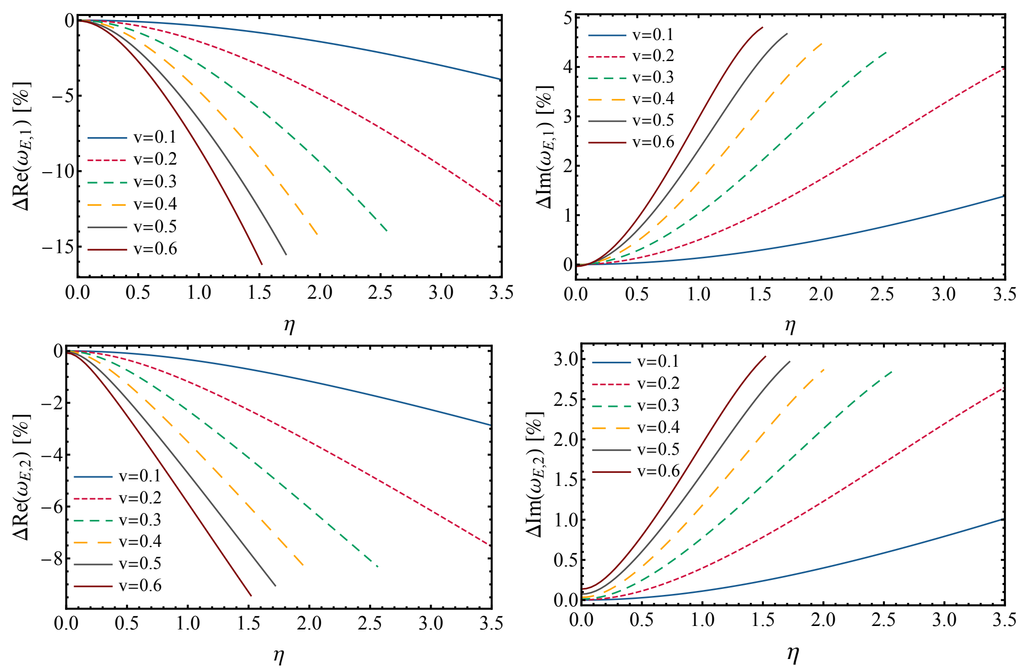

Figure 4.6: ISO-breaking [cf. Eq. (4.50)] of the EM QNMs for $l = 1$ (top) and $l = 2$ (bottom).

In Fig. 4.8 we plot the relative percentage difference for the real part of the $l = 1$ EM QNM frequencies between the DF approximation and the exact QNMs (similar results also hold for the imaginary part). The error due to DF approximation is almost negligible for small v and remains quite accurate even for $v \approx 0.6$, i.e. when we already expect the DF approximation to break down. Moreover, the difference is not very sensitive to the particular value of η. Similar results also hold for the gravitational and EM $l = 2$ QNMs.

In Fig. 4.9 we show the ISO-breaking for the real part of the QNM for $l = 1$, as estimated using the DF approximation. By comparison with Fig. 4.6, we can see that the DF prediction remains quite accurate even for $v \approx 0.6$. We thus conclude that the DF approximation captures the main qualitative and quantitative features of the QNM spectrum of EMD BHs, under the approximation of weak electric charge. In particular, it allows a computationally simpler study of ISO-breaking in the EM channel. In the next subsection we will therefore rely on the DF approximation to compute EM QNMs in the presence of slow rotation.

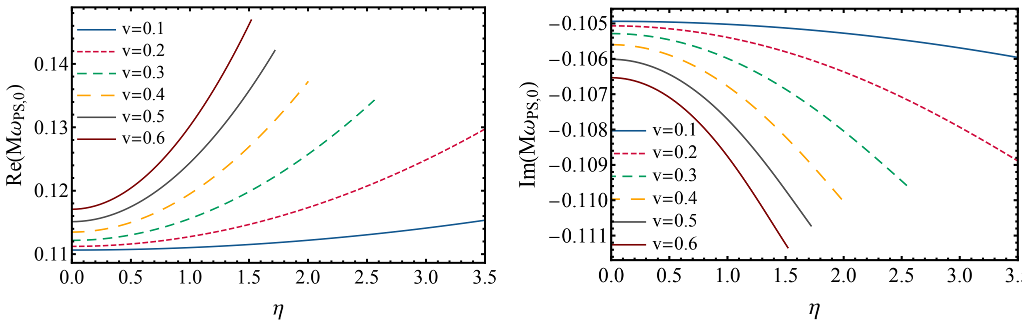

Figure 4.7: Real (left) and imaginary (right) part of the scalar QNM, $M\omega_{PS,l}$, for $l = 0$.

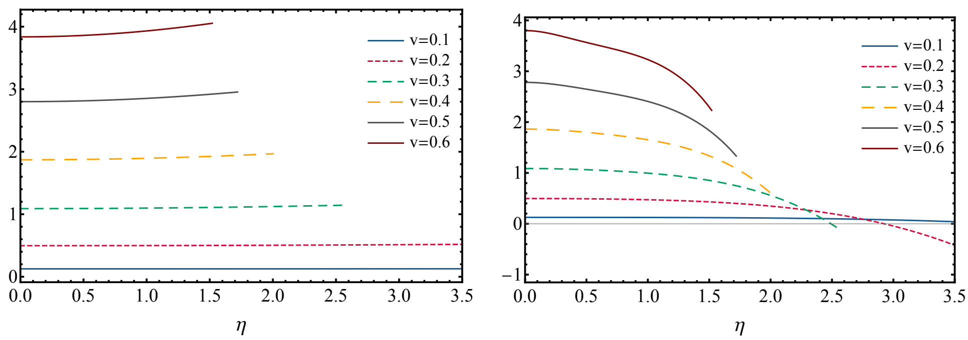

Figure 4.8: Percentage deviation of QNM frequencies for the real part of the axial (left) and polar (right) EM QNMs for $l = 1$, as derived from the $\mathcal{O}(v^2)$ true equations compared to the QNMs computed using the DF approximation. As expected, the DF approximation works better when $v \to 0$.

4.4.6 Quasinormal modes II: slowly rotating case

To derive the perturbed EOM for slowly rotating BHs we follow the procedure described in [87, 216, 214, 217]. In Ref. [216] it was shown that, at linear order in the spin, the radial and angular components of the perturbations are separable, axial and polar modes decouple and the couplings between different multipoles do not affect the QNM frequencies. The resulting equations, which can be found in the supplemental MATHEMATICA® notebook [208], are sufficiently similar to the static ones to be addressed with the same techniques, the only difference being that the asymptotic behaviour of the wave functions reads [218, 216, 219]

$$Z(r) \sim \begin{cases} e^{-i(\omega - m\Omega_H)r_\star} & \text{for } r \to R_+\,, \\ e^{i\omega r_\star} & \text{for } r \to \infty\,. \end{cases} \tag{4.52}$$

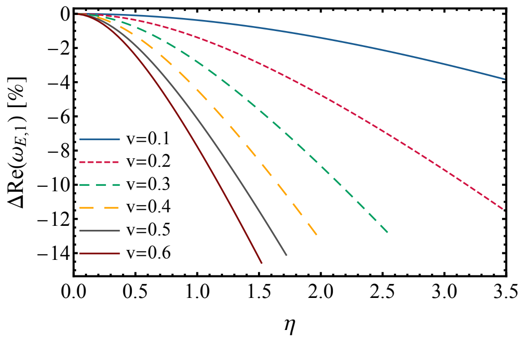

Figure 4.9: ISO-breaking for $l = 1$ EM modes as derived from the DF approximate equations (cf. top left panel of Fig. 4.6).

Here m is the azimutal number of the spherical harmonics and Ω_H is the angular velocity of the BH event horizon

$$\Omega_H = -\left.\frac{g_{t\phi}}{g_{\phi\phi}}\right|_{r=r_H} = \frac{a\,\Omega(R_+)}{R_+^2\, g(R_+)} = \tilde{a}\left(\frac{2+v^2}{8M}\right) + \mathcal{O}(v^3)\,. \tag{4.53}$$

At first order in the spin we can expand the QNM frequencies $\omega_{l,m}$ as [216]

$$\omega_{l,m} = \omega_l^{(0)} + \tilde{a} m\, \omega_l^{(1)} + \mathcal{O}(\tilde{a}^2)\,, \tag{4.54}$$

where $\omega_l^{(0)}$ is the frequency of the static BH, while $\omega_l^{(1)}$ is the first order correction to the QNM frequency due to the BH spin. The quantity $\omega_l^{(1)}$ depends only on the multipole number l, the dilaton coupling η, and on the BH mass and electric charge, while the dependence on $\tilde{a}$ and m factors out at first order. Therefore the computation of the slow-rotation QNMs reduces to the determination of $\omega_l^{(1)}$.

This approximation was used in Ref. [216] to compute the EM QNMs in a slowly rotating Kerr BH background, while Refs. [214, 219] used it to compute the QNMs of Kerr-Newmann BHs. In particular, they found that the $\mathcal{O}(a)$ approximation predicts QNMs frequencies that deviate from their exact values by less than 1% for $a \lesssim 0.3$ and 3% when $a \lesssim 0.5$. Within this error, they also showed that axial and polar sectors are still isospectral even when including spin.

Here we extend these computations for the slowly-rotating EMD BHs described by the metric (4.11), although limiting our analysis to the weak charge limit. When $v \lesssim 0.6$ and $\eta = 0$, the results of [214, 219] coincide with ours. For concreteness let us focus on the gravitational and the EM modes since the behavior for the scalar QNMs is completely analogous.

Gravitational modes. We start by computing the axial gravitational QNMs for $l = 2$ (similar results apply to $l > 2$). In Fig. 4.10 we show the real part of $\omega_2^{(1)}$. When $\eta = 0$, these results are in good agreement with the ones plotted in Fig.1 of Ref. [214] where the QNMs of Kerr-Newman where computed within the slowly-rotating approximation but without any approximation for the BH charge. As in the static case, the dependence on η is weak and the modes are very close to those of a Kerr-Newmann BH in Einstein-Maxwell.

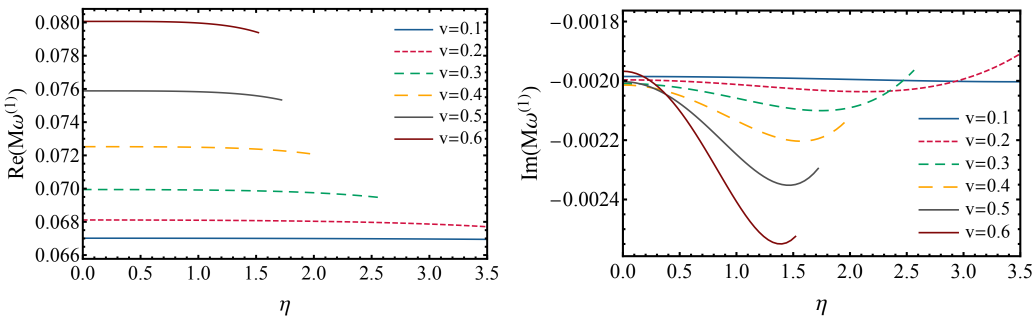

Figure 4.10: Real (left) and imaginary (right) part of the leading-order spin corrections, $M\omega_l^{(1)}$, for the $l = 2$ axial gravitational QNMs. Similar results can be obtained for $l > 2$.

It is instructive to compare the results of Fig. 4.10 with the light ring approximation (4.41). Eq. (4.41) is in agreement with the fact that the correction to the imaginary part of the QNM frequency depends very weakly on the spin in the small-charge approximation. Moreover, we also see that the leading-order correction due to $\tilde{a}$ for the real part of the QNM ranges from 0.074 for $v = 0.1$ to 0.087 for $v = 0.6$, yielding quite accurate results when compared with Fig. 4.10. Overall we find that Eq. (4.41) predicts the $l = 2$ gravitational QNM complex frequencies with relative errors always smaller than $\sim 5\%$ for the real part and $\sim 8\%$ for the imaginary part, within the parameter space we consider.

The polar equations are rather cumbersome to treat. However, guided by the intuition of the static case and the results in Refs. [214, 219], we expect that the difference with the axial modes will be small.

Electromagnetic modes. It is perhaps more interesting to investigate the difference between axial and polar modes in the EM spectrum, to see how our conclusions in Sec.4.4.4 are modified. To this aim, we simplify the problem using the DF scheme, as explained in Sec.4.4.5. We concentrate on the real part of the QNMs because it displays the larger effects. Fig. 4.11 shows the

EM ISO-breaking for $l = 1$, $\tilde{a} = 0.2$ and $m = \pm 1$ (when $m = 0$, Eq.(4.54) implies that the spectrum is unchanged). It is clear from a comparison with Fig. 4.6 that the spin does not substantially change the degree of ISO-breaking.

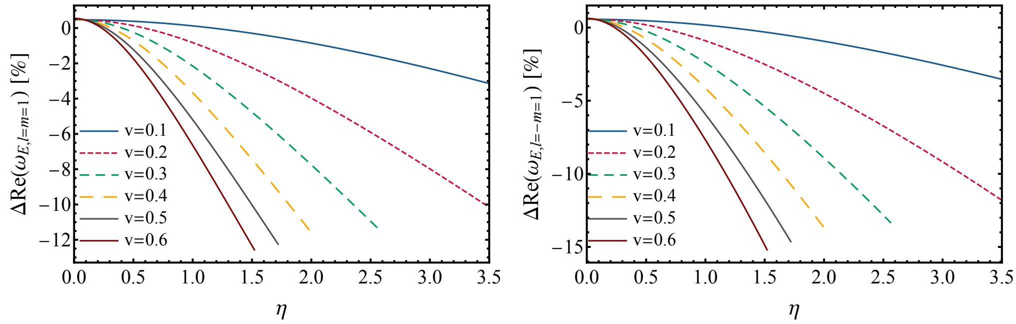

Figure 4.11: ISO-breaking of the real part of the EM QNMs for $l = 1$, $m = 1$ (left), $m = -1$ (right) and $\tilde{a} = 0.2$ [cf. top left panel of Fig. 4.6].

4.4.7 Dilatonic (in-)stability in Einstein-Maxwell-dilaton

In [200] an argument was presented for the existence of dilaton instabilities in EMD BHs. The argument goes as follows. If we perturb the scalar EOM (4.2a) w.r.t. to Φ, keeping the metric and the vector field constant, we obtain the perturb EOM

$$\Box\delta\Phi = \eta^2 F^2 \delta\Phi \,. \tag{4.55}$$

Ref.[200] assumes that the electric charge Q_E is small and evaluates the D'Alambertian on the LHS of (4.55) on the Schwarzschild background, thus obtaining

$$\left[\frac{d^2}{dr_\star^2} + \omega^2\right] Z(r) = V(r)Z(r) \,, \tag{4.56a}$$

$$V(r) = F(r)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \eta^2\frac{Q_E^2}{r^4}\right) \tag{4.56b}$$

where $Z(r) = r\delta\Phi(r)$ and $F(r) = 1 - 2M/r$. A sufficient condition for the Eq.(4.56a) to generate an instability is [220, 221]

$$\mathcal{I} = \int_{R_+}^{\infty} \mathcal{V}(r)dr < 0 \,, \tag{4.57}$$

where $\mathcal{V}(r) = V(r)/F(r)$. It is clear from (4.56b) that the integral $\mathcal{I}$ will be most negative for $l = 0$. Therefore, putting $l = 0$ and using $R_+ = 2M$, one can verify that the instability occurs for

$$\eta^2 v^2 > 3 \tag{4.58}$$

where $v = Q_E/M$ as usual. This is the prediction of [200].

The obvious criticism to this argument is that the conclusion is not self-consistent with the initial assumptions. Indeed, one assumed that a weak electric charge expansion was possible: but the final inequality (4.58) is not consistent with the condition $|(1-\eta^2)v^2| \ll 1$ for the reliability of the expansion (cf. Sec.4.2). Therefore, a more rigorous analysis is needed, and in particular we must consider perturbations at all orders in v.

In order to be as much rigorous as possible, we have derived the $l = 0$ perturbed EOM at all orders in v, without freezing the metric and the vector perturbation. In other words, our derivation is the most general without approximations. We find that the effective potential $V(r)$ has the rather lengthy expression

$$\begin{aligned} V^0(r) = &\left(1-\frac{R_+}{r}\right)\left(1-\frac{R_-}{r}\right)^{\frac{-4\eta^2}{\eta^2+1}} \frac{1}{\left(\eta^2+1\right)^2 r^5 \left[r(1+\eta^2)-R_-\right]^2}\times \\ &\times\Big\{\left(\eta^2+1\right)^3 r^4 \left[(1+\eta^2)R_+ - (\eta^2-1)R_-\right] \\ &+\left(\eta^2+1\right)^2 r^3 R_- \left[\left(2\eta^2-5\right)\left(\eta^2+1\right)R_+ - 3R_-\right] \\ &+\left(\eta^2+1\right) r^2 R_-^2 \left[\left(2\eta^4+\eta^2+3\right)R_- - \left(\eta^2+1\right)\left(2\eta^4+\eta^2-9\right)R_+\right] \\ &-rR_-^3\left[\left(2\eta^2+7\right)\left(\eta^2+1\right)R_+ + R_-\right] + \left(\eta^2+2\right)R_+R_-^4\Big\}\ , \end{aligned} \tag{4.59}$$

where $R_\pm$ are given by (4.8). Scanning the parameter space we were unable to find evidence for an unstable mode. In fact, by evaluating numerically the potential (4.59) for generic values of η and v we find that the potential is always positive definite outside the event horizon. This implies that (4.57) cannot hold. Moreover, the positivity of the effective potential is a proof that the modes do not suffer from instabilities [83].

The positivity check can be done analitically in the special limit $\eta \to \infty$, in which (4.59) reduces to

$$\lim_{\eta\to\infty} V^0(r) = \left(1-\frac{R_+}{r}\right)\left(1-\frac{R_-}{r}\right)^{-4} \frac{\left[r^2(R_+-R_-)+2R_+R_-(r-R_-)\right]}{r^5}\ . \tag{4.60}$$

It is clear that the RHS of (4.60) is positive everywhere for $r > R_+$. A possible issue is that, formally, $R_\pm$ diverge in the limit $\eta \to \infty$ (see Eq.4.8). However, this can be fixed by rescaling $v \to \sigma/\eta$. With this rescaling one has

$$R_\pm = M\left(\sqrt{1+\sigma^2} \pm 1\right)\ , \tag{4.61}$$

and (4.60) is positive for any value of σ. In contrast, Eq.(4.58) would have predicted an instability for $\sigma > \sqrt{3}$.

It is instructive to repeat the same computation by only perturbing the dilaton field while keeping the metric and the vector field fixed, similarly to what was done in Ref. [200]. In this case the potential $V^0(r)$, whose general expression is presented in a supplemental MATHEMATICA® notebook [208], in the limit $\eta \to \infty$ reduces to

$$\lim_{\eta\to\infty} V^0(r) \to \left(1 - \frac{R_+}{r}\right)\left(1 - \frac{R_-}{r}\right)^{-4} \frac{[r^2(R_+ - R_-) - 2R_+R_-(r - R_-)]}{r^5}\,. \tag{4.62}$$

It can be checked that this expression is not always positive for $r > R_+$. Indeed, using (4.61), one can easily see that the factor $r^2(R_+ - R_-) - 2R_+R_-(r - R_-)$ can be negative for $\sigma > 2\sqrt{2}$. Moreover, a numerical inspection reveals that (4.58) is satisfied for $\sigma \gtrsim 3.08$, thus giving rise to an instability. This shows that a consistent treatment of all the perturbations is essential in order to not produce wrong estimates.

In conslusion, we do not find any evidence of dilatonic instabilities in EMD BHs. We believe that this discrepancy with [200] might be due to the fact that, for the values of η for which Ref. [200] finds an instability, the small-charge approximation they employ is not valid, as we argued in Sec. 4.2.

5

Nonsingular Black Holes

5.1 Motivations

As we saw in Chapter 2, black holes in GR are singular. From the astrophysical point of view, the singularity is not worrysome because it is protected by the event horizon and it cannot influence the physics in the outer domain. However, from a theoretical point of view, the occurrence of a singularity signals a breakdown of the theory. Since singularities are common also to BHs in modified gravity theories, it is widely believed that they are classical artifacts and they must be solved by quantum gravitational effects. Since we do not have a complete theory of quantum gravity, we do not know how these modifications occur in concrete.

We can roughly estimate at which scales we expect the modifications. The working principle is that the classical theory breaks down when the curvature and density become Planckian. Consider for simplicity a Schwarzschild BH. The curvature invariant $\sqrt{|R^{abcd}R_{abcd}|}$ scales with the radial coordinate r as the energy density M/r^3. Therefore they become Planckian at a radius $\tilde{r}$ such that $M/\tilde{r}^3 \sim l_P^{-2}$, where l_P is the Planck length, i.e. at $\tilde{r} \sim \left(Ml_p^2\right)^{1/3}$.

One of the approaches to the problem assumes that, irrespective of the nature of the quantum gravitational theory, the modifications can be modeled via a regular effective classical metric g_{ab}, which results from the averaging of fluctuating quantum geometries and which obeys quantum modified Einstein equations. This is by no means guaranteed, and therefore it is just an assumption. Can we test it?

At the astrophysical level, a modified nonsingular BH will be indistinguishable from a pure GR one. This is because the mass of ordinary BHs is much bigger than the Planck mass, and therefore quantum corrections start to manifest well inside the event horizon[1]. This indicates that we must resort to theoretical considerations. We saw in Ch.2.2.3 that, theoretically, BHs tend to evaporate via Hawking radiation. As we stressed there, although the evaporation process is not expected to happen in realistic stellar mass or supermassive BHs, its abstract theoretical consideration can lead to important insights in BH physics: for example, it is precisely in this setting that the generalized second law is formulated and argued for.

In this Chapter we will study the internal consistency of the regular BH paradigm when Hawking radiation is taken into account. For simplicity, we will restrict to static BHs. Such studies are of course not new, and important conclusions have already been reached. It is a general feature of nonsingular black holes that they possess an outer horizon, identified with the event horizon, and an inner horizon, which in the stationary case plays the role of a Cauchy horizon. The distance between the horizons increases with the mass and, in particular, the inner horizon is at a Planckian distance from the BH regular center when the mass is much higher than the Planck mass. As the BH evaporates, the two horizons get closer and closer and, in principle, they can merge and subsequently disappear, leading to an horizonless massive remnant or to a Minkowski spacetime plus radiation. This is the standard qualitative regular BH evaporation picture, as it is usually presented, see e.g. [223, 224].

We will question this picture in two steps. First, we show that the inner horizon develops an instability of the mass-inflation type, as it happens for Reissner-Nordström and Kerr BHs. This is a dynamical instability, which develops within a characteristic time scale $\tau_{\rm in}$. Therefore it is crucial to compare $\tau_{\rm in}$ with the BH evaporation time scale $\tau_{\rm ev}$: if $\tau_{\rm ev} > \tau_{\rm in}$, then the instability has enough time to grow and the very original assumption of a regular effective geometry is invalidated. We actually demonstrate that, unless the metric components are chosen in special ways, Hawking evaporation of a nonsingular BH takes an infinite time to complete. This is of course a major drawback and it constitutes our main result.

We then discuss critically the assumptions behind such conclusion, to see wether they can be relaxed. The result essentially lays on the two assumptions that the evaporation is adiabatic and quasi-static, in a sense to be specified below. We argue that these conditions are generically satisfied through all the evaporation process. Therefore an effective semiclassical description is inconsistent and, at least in the last stages of the evaporation, the

[1]In Ch.2.2.3 we mentioned the firewall hypothesis, according to which Hawking evaporation and unitarity cannot be conciliated without invoking dramatic modifications even at the event horizon scale. This is also argued in models of black hole- white hole transitions [222]. Here we neglect these possibilities and stick to more conservative scenarios.

explicit nature of quantum gravity effects must be considered. This seems to affect the real utility of the regualr BH paradigm, because it fails exactly where it was supposed to provide an effetive description.

At the end, we discuss special non-generic conditions under which regular BHs may still be viable. Although such conditions seem rather artificial, it cannot be excluded that they belong to some universal class with features independent of the particular model. We do not attempt to explore such a classification, but we limit ourselves to state the relevant conditions. All the results of this Chapter are cointained in the paper "On the viability of regular black holes" (P4).

5.2 General aspects of regular black holes

We start by reviewing general properties of static regular BHs [225, 224]. Without loss of generality, the line element of a static spherically symmetric regular BH can be written as

$$ds^2 = -e^{-2\Phi(r)}F(r)dt^2 + \frac{dr^2}{F(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{5.1}$$

where $\Phi(r)$ and $F(r)$ are real functions. It is convenient to introduce the "local mass function" $m(r)$ defined by

$$F(r) = 1 - \frac{2m(r)}{r}\,. \tag{5.2}$$

Several considerations constrain the shape of $m(r)$ and, in turn, the number of horizons. Using the Einstein equations, the effective energy density associated with the geometry (5.1) is $m'(r)/4\pi r^2$, where a prime denotes derivative w.r.t. the radial coordinate. Finiteness at $r = 0$ requires that $m(r)$ vanishes at least as r^3 for $r \to 0$. On the other hand, imposing that the Schwarzschild solution is recovered at large distances from the center, implies $m(r) \to M$ for $r \to \infty$, where M is the usual ADM mass. Therefore $F(r) = 1$ for both $r \to 0$ and $r \to \infty$. By a simple counting of the roots, $F(r)$ must then have an even number of zeroes, so at least two horizons are present.

A more extended analysis of the generic properties of regular BHs can be found in [226], where it is shown that, if $\Phi(r) = 0$ and the DEC holds, then the number of horizons is exactly two. Here for simplicity we assume that there are exactly two horizons also in the generic case, but our conclusions do not depend crucially on this assumption.

A simple explicit model is Hayward's metric [223], given by $\Phi(r) = 0$ and

$$F(r) = 1 - \frac{2Mr^2}{r^3 + 2Ml^2} \tag{5.3}$$

where M is the ADM mass and l is a constant with the dimensions of a lenght. From the discussion of Sec.5.2 we may identify $\tilde{r} = 2Ml^2$ with the characteristic radius at which the metric starts to deviate significantly from the Schwarzschild one, and so we identify l with the Planck lenght. The position of the two horizons $r_\pm$ is determined by the algebraic equation

$$F(r_\pm) = 0 \implies r_\pm^3 - 2Mr_\pm^2 + 2Ml^2 = 0\,. \tag{5.4}$$

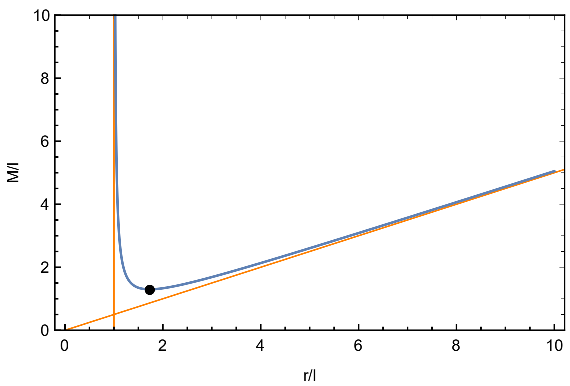

Figure 5.1: The blue line shows the positions of the horizons as a function of the mass M. For large masses, the horizons tend to $r_- \to l$ and $r_+ \to 2M$ (orange lines). The black spot is the locus of the extremal point $r_\star = \sqrt{3}l$ and $M_\star = 3\sqrt{3}l/4$. The plot is drawn in units of l.

The position of the horizons as a function of the mass is shown in Fig.5.1. We see that there are two horizons as long as $M > M_\star = 3\sqrt{3}l/4$, for which the two horizons degenerate into a single horizon at $r = r_\star = \sqrt{3}$. For $M < M_\star$ there is no horizon and the solution represents an horizonless compact object. The locus of the extremal point is univoquely determined by the two equations

$$F(r_\star) = 0\,, \tag{5.5a}$$
$$F'(r_\star) = 0\,. \tag{5.5b}$$

The surface gravities of the outer and inner horizons are given by

$$\kappa_\pm = \frac{1}{2}F'(r_\pm)\,, \tag{5.6}$$

from which we see that, at the extremal point where $F'(r_\star) = 0$ and the two horizons coincide, the surface gravity $\kappa_\star = 0$. Since the solution tends to Schwarzschild for $M \to \infty$, and since in this limit the Schwarschild surface gravity $\kappa = 1/4M$ tends to zero, then the surface gravity κ_+ of the outer horizon of Hayward's BH must have a global maximum. Fig.5.2 plots κ_+ as a function of M.

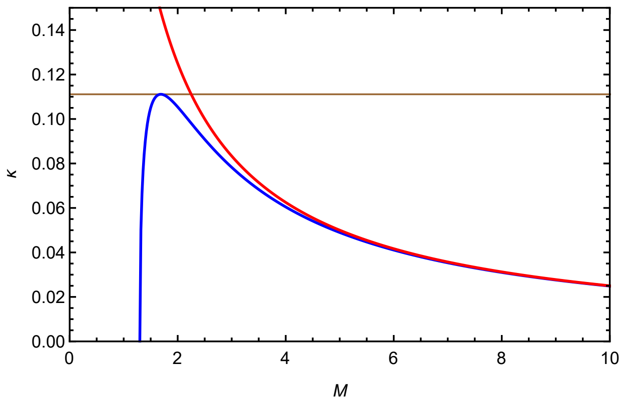

Figure 5.2: The blue line shows the surface gravity κ_+ of the outer horizon for Hayward's BH, with $l = 1$. The curve tends to the Schwarzschild κ (red line) for large M, but it reaches a maximum for $M_{\max} = 27/16$ and drops to zero at $M_\star = 3\sqrt{3}/4$. The horizontal brown line marks the maximum $\kappa_{\max} = 1/9$.

The position of the maximum can be obtained by finding the extrema of $F'(r, M)$ while simultaneously imposing $F(r, M) = 0$ (i.e. that the (r, M) couple corresponds to the BH horizons). Oviously, this is tantamount to find the extrema of the function

$$G\left(r, M\right) = \frac{\partial F(r, M)}{\partial r} + \lambda F\left(r, M\right) \tag{5.7}$$

w.r.t. r, M and λ, where the latter is a Lagrange multiplier (we emphasized that in this case F must be viewed as a function of both r and M). The result is $r_{\max} = 3l$ and $M_{\max} = 27l/16$, with a corresponding maximum surface gravity $\kappa_{\max} = 1/9l$.

Although derived for the special case of Hayward's BH, these features are common to all the models proposed in the literature with $\Phi(r) = 0$: for example they can be observed in the Bardeen BH [227], in the Dynmnikova BH [228] and in regular BHs motivated by asymptotic safety [229] (for a review of regular BH models see [225]). As we will see in Sec.5.3 and in Sec.5.5, the inclusion of a nonvanishing $\Phi(r)$ can alter this generic picture, sometimes in significant ways.

For the purposes of the next Section, it is useful to rewrite the line element (5.1) in advanced null coordinates

$$ds^2 = -e^{-2\Phi(r)}F(r)dv^2 + 2e^{-\Phi(r)}dvdr + r^2\left(r^2 d\theta^a + \sin^2\theta d\varphi^2\right) \tag{5.8}$$

where the advanced null time v is defined as

$$dv = dt + \frac{dr}{e^{-\Phi(r)}F(r)}\,. \tag{5.9}$$

In these coordinates, ingoing and outgoing null rays are described respectively by the equations

$$dv = 0\,, \qquad \text{(ingoing null rays)} \tag{5.10a}$$

$$\frac{dr}{dv} = \frac{e^{-\Phi(r)}F(r)}{2r}\,. \qquad \text{(outgoing null rays)} \tag{5.10b}$$

If we expand (5.10b) around $r_\pm$, using the definition (5.12) of surfce gravity, we obtain that outgoing null rays suffer from an exponential peeling along both the inner and the outer horizons

$$\frac{d(r-r_\pm)}{dv} = \pm|\kappa_\pm|(r-r_\pm) + o(r-r_\pm) \tag{5.11}$$

where

$$\kappa_\pm = \frac{e^{-\Phi(r_\pm)}}{2}F'(r_\pm)\,. \tag{5.12}$$

is the surface gravity at the outer (plus sign) and ineer (minus sign) horizons. The absolute values in (5.11) come from the fact that κ_+ (resp. κ_-) is positive (negative). In particular, while at the outer horizon we have a red shift, rays at the inner horizon experience a blue shift. This blue shift is at the core of the prediction, described in the next Section, the the inner horizon is classically unstable.

5.3 Instability of the inner horizon

The potentially unstable nature of the inner horizon due to the exponential focusing of null rays was previously noticed, and thoroughly studied, in the different but related context of charged and rotating black holes. While still being an active research area (see, for instance, the recent works [230, 231]), the main aspects were settled more than two decades ago [232, 233], though formal proofs of a number of technical aspects were only available later [234].

In brief terms, the central conclusion of these works is that the inner horizon is unstable in the presence of both ingoing and outgoing perturbations.

Both types of rays generically exist in realistic collapse scenarios: indeed, outgoing radiation will be emitted from the star even after the formation of the horizon, while ingoing perturbations can come e.g. from the backscattering of gravitational radiation. Therefore inner horizon instabilities are expected to form in a regular BH scenario. To date, the phenomenon for regular BHs has been analyzed only for the specific case of the "loop black hole" [235]. However, here we see that the analysis can be easily extended to all regular black hole geometries.

The process can be modelled following [236], where outgoing and ingoing perturbations are described in terms of null shells. This simplification allows to perform all necessary calculations analytically, and exploits the Dray–'t Hooft–Redmount (DTR) relation [237, 238] (see Eq.(5.13) below).

The situation is schematically represented in the Penrose diagram of Fig.5.3. The ingoing and outgoing shells meet at the radius r_0 for a given moment of time. Using the null coordinates (u, v), we can study the behavior of the system when this crossing point is displaced along a null outgoing curve: that is, we take a constant value $u = u_0$ (this value is arbitrary as long as it lies inside the outer horizon) and move r_0 along the v direction, such that it spans a curve $r_0(v)|_{u=u_0}$.

If we focus on a local neighbourhood around the crossing point r_0, the ingoing and outgoing shells divide the spacetime in four regions (A, B, C and D) with different geometries and, in particular, different values of the mass parameter $m(r)$ of the corresponding regular BH geometry. It was shown in [237, 238] that the masses of the four regions are related to the masses of the ingoing ond outgoing shells, m_{in} and m_{out}, by the so called DTR relations. These relations are noteworthy independent of the field equations and are formulated on purely geometrical grounds.

For a spherically symmetric geometry of the form (5.1), this relation takes the simple form [236]

$$|F_A(r_0)F_B(r_0)| = |F_C(r_0)F_D(r_0)|. \tag{5.13}$$

as a constraint on the coefficient g^{rr} of the metric. Eq. (5.13) can be manipulated in order to obtain

$$m_A(r_0) = m_B(r_0) + m_{\text{in}}(r_0) + m_{\text{out}}(r_0) - \frac{2m_{\text{out}}(r_0)m_{\text{in}}(r_0)}{r_0 F_B(r_0)}, \tag{5.14}$$

where we have $m_{\text{in}}(r_0) = m_C(r_0) - m_B(r_0)$, and also $m_{\text{out}}(r_0) = m_D(r_0) - m_B(r_0)$. The first three terms on the right-hand side of Eq. (5.14) have a

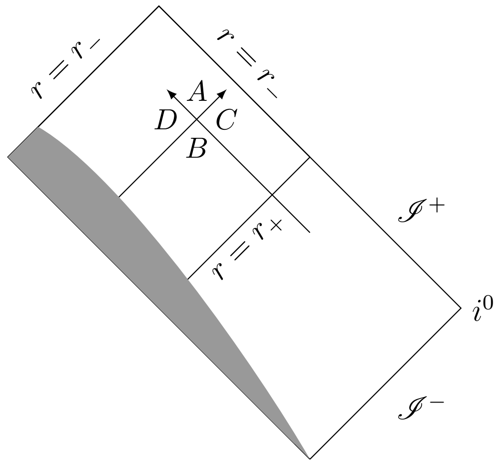

Figure 5.3: Schematic Penrose diagram of a star collapsing to a regular black hole with concentric outgoing and ingoing null shells. The DTR relation is applied to the crossing point r_0 between outgoing and ingoing shells. The corresponding four spacetime regions A, B, C and D are depicted. We only depict the relevant quadrants of the maximally extended Penrose diagram of a regular black hole (see *e.g.* [225] for the full diagram).

clear physical meaning: m_B measures the mass of the region between the ingoing and outgoing shell and, therefore, the original mass of the regular black hole before the ingoing shell is absorbed. This is moreover the region in which the coordinates (u, v) are defined. On the other hand, m_{in} and m_{out} are the masses of the ingoing and outgoing shells. These three contributions are finite, but the last contribution has to be analyzed carefully. The reason is that, as the point $r_0(v)|_{u=u_0}$ gets closer to the location of the inner horizon, $F_B(r_0) \to 0$. This implies that, in order to understand the evolution of the system at late times, we need to understand the behavior with v of $m_{\text{in}}(r_0(v)|_{u=u_0})$ and $F_B(r_0(v)|_{u=u_0})$ (note that m_{out} is constant along $u = u_0$).

The quantity m_{in} describes how the ingoing tails decay with v and is determined by Price's law [239, 240, 241, 242, 243]

$$m_{\text{in}}(r_0(v)|_{u=u_0}) \propto v^{-\gamma}, \tag{5.15}$$

where, for the purposes of the present discussion, it is enough to consider the lower bound $\gamma > 0$. The quantity m_{out} vanishes on the inner horizon, but we need to determine how fast it approaches this value when we displace the point r_0 increasing the value of v along ingoing null curves. Along these trajectories Eq. (5.11) applies so that, close to the inner horizon, one has

$$\mathrm{d}v = \frac{2\mathrm{d}r}{e^{-\Phi(r_-)}F'(r_-)(r-r_-)} + o(r-r_H). \tag{5.16}$$

We just need to integrate this equation starting from some value of r greater than, but arbitrarily close to, r_-. Taking into account that $e^{-\Phi(r_-)}F'(r_-) = 2\kappa_- = -2|\kappa_-|$, it follows that

$$F_B(r_0(v)|_{u=u_0}) \propto e^{-|\kappa_-|v}. \tag{5.17}$$

Combining these ingredients, we see that at late times ($v \gg 1/|\kappa_-|$), one has

$$m_A(r_0(v)|_{u=u_0}) \propto v^{-\gamma}e^{|\kappa_-|v}. \tag{5.18}$$

This is the equation that characterizes the phenomenon of mass inflation: the mass parameter in the region A grows exponentially, on a timescale determined by the surface gravity of the inner horizon. In other words, the inner horizon is unstable with a characteristic timescale $1/|\kappa_-|$ measured in the ingoing null coordinate v. This timescale is Planckian for most of the models in the literature (for Hayward's metric this can be easily estimated by observing that, from Fig.5.1, $r_- \approx l$ for all values of the mass). Note that the proportionality constant in Eq. (5.18) is positive, which can be realized by recalling Eq. (5.14) and taking into account the signs of $m_{\rm out}$, $m_{\rm in}$ and $F_B(r_0)$.

5.4 Evaporation time

In this Section we show our main result, that the evaporation time of a non-singular BH is generally infinite. We work under the assumtpions that the only relevant dynamical process driving the evaporation is Hawking radiation, and that the evaporation is adiabatic and quasistatic. By adiabatic we mean that the variation of temperature with time is "slow" during the emission : the precise notion of slowness was formalized by [66, 67] in the condition $|\dot{\kappa}_+/\kappa_+^2| \ll 1$, which was also shown to ensure that the spectrum is Planckian with characteristic temperature $T_H = \kappa_+/2\pi$. By quasistatic we mean that the BH never passes through a phase in which a large fraction of its mass is emitted quickly.

In Sec.5.5 we will critically discuss these assumptions, but here we just consider them as valid through the whole evaporation process. Their combination implies that the mass loss rate is determined by Stefan's law

$$\frac{dM(u)}{du} = -C\sigma_{\rm SB}T_H^4(u)A_H^2(u) = -C'\kappa_+^4(u)r_+^2(u)\,. \tag{5.19}$$

Here C is a positive constant accounting the number of polarizations of the Hawking quanta and for corrections to the effetive absorbtion cross section of the BH, $\sigma_{\rm SB} = \pi^2 k_B^4/60\hbar^3 c^2$ is the Stefan-Boltzmann constant, and $C' = C\sigma_{\rm SB}/4\pi^3$. We emphasized that the temperature and the geometry of the BH depend on the retarded time u at $\mathscr{I}^+$. The precise values of C and $\sigma_{\rm SB}$ are irrelevant for the moment.

Our strategy to prove that the evaporation time is infinite consists into integrating (5.19) in u with $u_\star$ as one of the integration extrema, where $u_\star$ is the time at which $M(u_\star) = M_\star$ and $r_+(u_\star) = r_\star$,[2] and show that this requires $u \to \infty$. Let us assume that the evaporation has proceeded adiabatically and quasi-statically up to a moment when the mass is arbitrarily close to the extremal value, $M = M_\star + \Delta M$, with $\Delta M/M_\star \ll 1$. Then the radius of the outer horizon is

$$r_+ = r_\star + \Delta r = r_\star \left(1 + \epsilon\right) \,, \quad 0 < \epsilon \ll 1 \,. \tag{5.20}$$

Correspondingly, we will have

$$M = M_\star + \Delta M = M_\star \left(1 + \beta \epsilon^\sigma\right) + o(\epsilon^\sigma) \tag{5.21}$$

where we β and σ are two real constants with $\sigma > 0$. The values of β and σ cannot be chosen arbitrarily, but they must be consistent with the defining equation of the outer horizon

$$F(r_+, M) = 0 \tag{5.22}$$

where we explicitated the dependence of F on M. Expanding (5.22) around Δr and ΔM we obtain

$$\begin{aligned} 0 = F(r_\star, M_\star) + \left(\frac{\partial F}{\partial r}\right)_{r_\star, M_\star} \Delta r + \left(\frac{\partial F}{\partial M}\right)_{r_\star, M_\star} \Delta M + \\ + \frac{1}{2}\left(\frac{\partial^2 F}{\partial r^2}\right)_{r_\star, M_\star} \Delta r^2 + \left(\frac{\partial^2 F}{\partial r \partial M}\right)_{r_\star, M_\star} \Delta r \Delta M + \frac{1}{2}\left(\frac{\partial^2 F}{\partial M^2}\right)_{r_\star, M_\star} \Delta M^2 + \dots \end{aligned} \tag{5.23}$$

The first and second terms vanish due to the defining equations (5.5) of the extremal point. Let us first consider the case that $(\partial F/\partial M)_{r_\star, M_\star}$ does not vanish. Then ΔM is of order ϵ^n, where n is the first natural number for which

$$\left(\frac{\partial^n F}{\partial r^n}\right)_{r_\star, M_\star} \neq 0 \,, \tag{5.24}$$

[2]Remember that $M_\star$ and $r_\star$ are the mass and the radius at which the two horizons degenerate into a single one.

so that $\sigma = n$. Then β is given by

$$\beta = -\frac{r_\star^n}{n!\,M_\star}\left(\frac{\partial F}{\partial M}\right)^{-1}_{r_\star,M_\star}\left(\frac{\partial^n F}{\partial r^n}\right)_{r_\star,M_\star}. \tag{5.25}$$

It is important to observe that, from (5.23), n must be at least equal to 2, $\sigma = n \geq 2$. The surface gravity of the outer horizon κ_+ can be also expanded in Δr and ΔM as

$$\kappa_+ = \frac{e^{-\Phi(r_+)}}{2}F'(r_+) = \frac{e^{-\Phi(r_+)}}{2}\sum_{i,j=0}^{\infty}\frac{1}{i!j!}\left(\frac{\partial^{i+j+1}F}{\partial^{i+1}r\partial^j M}\right)_{r_\star,M_\star} r_\star^i\epsilon^i(\Delta M)^j\,. \tag{5.26}$$

From the fact that $\sigma \geq 2$, we see that the leading term in (5.26) gives

$$\kappa_+ = \alpha\epsilon^\gamma + o(\epsilon^\gamma)\,, \tag{5.27}$$

where $\gamma = n-1$ and

$$\alpha = \frac{e^{-\Phi(r_+)}}{2}\frac{1}{n!}\left(\frac{\partial^n F}{\partial r^n}\right)r_\star^{n-1}. \tag{5.28}$$

(We made the educated guess that $\Phi(r_+)$ does not negatively diverge. We will comment about relaxing such assumption in Sec.5.5.)

We are now ready to show that the evaporation time is infinite. Using (5.19) in the limit $\epsilon \to 0$, the evaporation time Δu is equal to

$$\Delta u = \left(\frac{M_\star\beta\sigma}{C'\alpha^4 r_\star}\right)\int_{\epsilon_0}^{0} d\epsilon\,\epsilon^{\sigma-4\gamma-1} \tag{5.29}$$

where ϵ_0 is a generic starting point arbitrarily close to 0. It follows that the evaporation time is finite if and only if

$$\sigma - 4\gamma > 0\,. \tag{5.30}$$

However this condition is manifestly violated, since $\sigma = n \geq 2$ and $\gamma = n-1$ imply $\sigma - 4\gamma = 4 - 3n \leq -2$. $\square$

We now generalize the result to the case $(\partial F/\partial M)_{r_\star,M_\star} = 0$. Since this is just a technical issue, the reader can go directly to the next Section if she wishes. We distinguish three cases, depending if σ in (5.21) is equal, smaller or equal to one.

- Case $\sigma = 1$. If $\sigma = 1$, then from (5.26) and from $\Delta M \propto \epsilon^\sigma = \epsilon$, the exponent γ in (5.27) is a natural number bigger than one. But then $\sigma - 4\gamma$ must be negative. $\square$

- Case $\sigma < 1$. If $\gamma \leq 1$, $\sigma - 4\gamma$ is trivially negative. Therfore we consider only the case $\gamma < 1$. In this case, (5.26) implies $\gamma = I + J\sigma$, where I and J are integers such that $I \geq 0$ and $J \geq 1$. Then $\sigma - 4\gamma = -4I - (4J-1)\,\sigma$ and we see that the term in round brackets is positive, hence the conclusion. □

- Case $\sigma > 1$. If $\sigma > 1$, (5.26) implies that $\gamma = n - 1$, where n is defined as in (5.24). Moreover, from (5.22), it follows that the terms proportional to $\Delta r^n \propto \epsilon^n$ can be compensated only by terms of the type $(\Delta r)^I (\Delta M)^J \propto \epsilon^{I+J\sigma}$, where I and J are integers such that $I + J\sigma = n$ and $I, J \geq 1$. It then follows that $\sigma \leq n$ which, combined with $\gamma = n - 1$, shows that $\sigma - 4\gamma < 0$. □

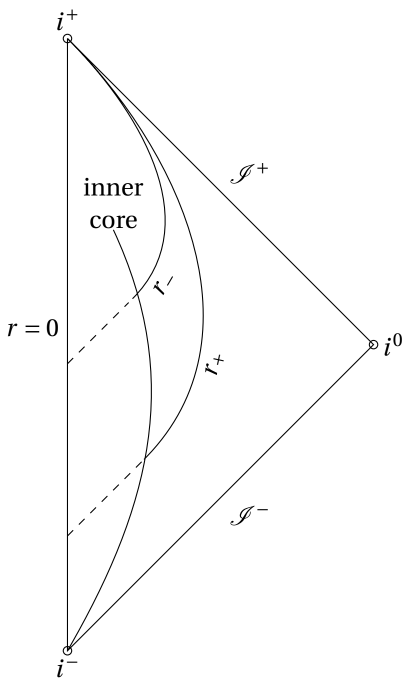

Figure 5.4: The figure shows the evaporation of a BH with a regular core, taking an infinite time for the inner and outer horizons to meet.

This completes our derivation. The Penrose diagram in Fig.5.4 depicts an evaporating regular BH, for which the inner and outer horizon meet in an infinite retarded time. In the next Section we critically discuss the main assumptions behind our result.

5.5 Adiabatacity and quasistaticity

It was argued in [66, 67] that Hawking radiation has a Planckian spectrum if the following adiabaticity criterion is satisfied

$$\left|\frac{\dot{\kappa}_+}{\kappa_+^2}\right| \ll 1 \tag{5.31}$$

where a dot indicates a derivative w.r.t. the retarded time u. Physically the criterion corresponds to the fact that the emission of a quantum with energy corresponding to the Planckian peak of the spectrum should not see a large fractional change of the temperature. For regular BHs, this condition can fail only in the last stages of the evaporation, when both $\dot{\kappa}_+$ and κ_+ go to zero. Let us show that this is not the case and (5.31) is satisfied even at the final moments. From (5.27)

$$\dot{\kappa}_+ \propto \epsilon^{\gamma-1}\dot{\epsilon} \tag{5.32}$$

while, from (5.21) and (5.19),

$$\epsilon^{\sigma-1}\dot{\epsilon} \propto \dot{M} \propto \epsilon^{4\gamma} \implies \dot{\epsilon} \propto \epsilon^{4\gamma-\sigma+1}\,. \tag{5.33}$$

Therefore $\dot{\kappa}_+ \propto \epsilon^{5\gamma-\sigma}$ and the criterion (5.31) becomes

$$\epsilon^{3\gamma-\sigma} \ll 1 \tag{5.34}$$

which is satisfied if and only if $3\gamma - \sigma > 0$. By repeating the same analysis as in the end of the previous section, one can show that indeed $3\gamma - \sigma$ is a positive quantity, and the adiabaticity criterion is respected.

We now turn to quasistaticity. In our derivation of the evaporation time we assumed that the BH, starting from a large macroscopic mass, smoothly reaches a configuration in which the mass has shrunk down close to $M_\star$. However we have seen that, viewed as a function of M, the Hawking temperature $\kappa_+/2\pi$ has a maximum. Therefore the evaporation can exit the quasistatic regime if the maximum temperature is high enough that a large fraction of the BH mass is emitted in a short time. To see if this is the case, let us first consider the analytically tractable case of Hayward's metric. As we saw in Sec.5.2, the maximum of the surface gravity occurs for $r_{\rm max} = 3l$, $M_{\rm max} = 27l/16$ and $\kappa_{\rm max} = 1/9l$. Therefore

$$T_{\rm max} = \frac{1}{18\pi\, l} \approx \frac{10^{-2}}{l}\,. \tag{5.35}$$

Since we identify l with the Planck lenght, the temperature at the maximum is just two orders of magnitude smaller than the Planckian temperature. On the other hand Stefan's law (5.19) at the maximum, assuming that $C = \mathcal{O}(1)$, gives

$$\dot{M}_{\rm max} = -\frac{1}{3^7 2^4 5\pi}\,. \tag{5.36}$$

Since the only relevant scale of the problem is l, we can estimate the typical mass emitted at the maximum as $\Delta M_{\rm max} \approx |\dot{M}_{\rm max}| l$, where in natural units l is the Planck time. Therefore $\Delta M_{\rm max}/M_{\rm mass} \approx 10^{-6}$ and we see that the quasistatic condition is not violated. We interpret this result as an indication that the system is not much perturbed even at the maximum of the temperature emission.

We have numerically repeated the same calculation for the regular BHs proposed by Bardeen and Dynmikova, finding that the fractional mass emitted at the maximum is, respectively, 10^{-7} and 10^{-5}. A similar conclusion was reached in [229] in the context of BHs regularized by asymptotic safety.

In all these examples, the function $\Phi(r)$ in the metric (5.1) is identically zero. It is certainly possible to increase the fractional mass emitted at the maximum, going beyond the quasistatic regime, in a model where $\Phi(r) \neq 0$ is very large at the maximum, although we are not aware of any specific regular metric with this property.

It is much less trivial to go beyond the adiabatic regime to make the evaporation time finite. In this case, we need to violate the assumption that $\Phi(r)$ is regular at $M = M_\star$ and $r = r_\star$. If we take

$$e^{-\Phi_\star} \propto \epsilon^{-\delta}\,, \quad \delta > 0 \tag{5.37}$$

then it is sufficient that δ is in the range

$$(n-1) - \frac{\sigma}{4} < \delta \leq (n-1) \tag{5.38}$$

in order to have a finite evaporation time (lower bound) and a non-divergent surface gravity (upper bound). [The factors of $(n-1)$ come from the fact that we are assuming from simplicity that $(\partial F/\partial M)_{r_\star, M_\star} \neq 0$, see the discussion before Eq.(5.24).] Notice that the lower bound automatically leads also to a violation of the adiabaticity condition $3\gamma - \sigma > 0$. When the upper bound is saturated, we are in the special case in which the surface gravity at extremality does not vanish and it is discontinuous, $\lim_{r_+ \to r_\star} \kappa_+ > 0$ and $\lim_{r_- \to r_\star} \kappa_- < 0$.

There are other properties, besides the bound (5.38), that $\Phi(r)$ must satisfy in order to give a consistent regular BH evaporation picture. Regularity at the origin requires that $\Phi'(r)$ vanishes at least linearly for $r \to 0$. Moreover, we also want the instability timescale to be much larger than the evaporation one. This requires $\exp\left[-\Phi(r_-)\right] \ll 1$ at the inner horizon. Strictly speaking this is necessary only as long as $M \gtrsim M_\star$, because $F'(r_-) \to 0$ as $M \to M_\star$; then, from Eq.(5.12), κ_- can still be small even if $\exp\left[-\Phi(r_-)\right]$ is not.

Although it cannot be excluded in principle, we do not expect to be easy to find a regular BH metric satisfying all the above requirements at the same time.

5.6 Discussion and outlook

We have analyzed two aspects of regular black holes, namely the instability of the inner horizon and the evaporation time. We have shown that the inner horizon is unstable on a finite time scale and that this instabilty is unavoidable in complete generality. Moreover, we provided a self-consistent computation of the evaporation time, concluding that it is infinite in all the models of regular BHs, unless $\Phi(r) \neq 0$ and satisfies very special conditions. In conjuction, these two findings determine an inconsistency of nonsingular BH models, since instabilities manifest as dynamically generated singularities.

We have discussed that, in order for the evaporation to occur in a finite time, adiabaticity and/or quasistaticity must be violated at some stage of the process. In other words, one must go beyond the conventional Hawking picture. This results in a theoretical ambiguity that cannot be resolved internally to the model in a "natural way", but it must be addressed by invoking mechanisms from outside reasonings.

It must be noted that our results are not the first ones to highlight possible inconsistencies of regular BHs. In particular, there is evidence in the literature that, even if we suppose that some unknown semiclassical mechanism makes the evaporation time finite, an outburst of negative energy will be emitted, which can exceed in absolute magnitude the oringinal BH mass in violation of energy conservation [224, 244, 245, 246]. As for the mass inflation instability, this negative emission is rooted in the exponential blue-shift of the inner horizon.

Although we cannot claim it in an *absolutely* conclusive manner (see the discussion of Sec.5.5), we interpret our results as additional evidences against the viability of regular BHs in their realizations so far, and view these models much less appealing from a theoretical point of view.

6 General conclusions

In this Thesis we have investigated the theoretical physics of various black hole models alternative to the standard black holes of GR. These models originated from different motivations: black holes with universal horizons are found in modified gravity theories which break local Lorentz symmetry; Einstein-Maxwell-dilaton black holes originate in string theory and in lower dimensional compactifications, but they also serve as proxies for black holes in theories propagating additional degrees of freedom; regular black holes are motivated by the efforts to understand how quantum gravity solves the classical singularities.

In each of the above cases, we have obtained results which appear to be relevant for the follow up research in their respective fields. Let us summarise them, along with possible future research directions.

- In the case of Lorentz-violating black holes, we provided a systematic way of deriving a Smarr formula, which can be useful to characterize their stationary properties. Moreover, we presented convincing arguments that these black holes do not always obey a standard first law of mechanics, in the sense of a law connecting the asymptotic conserved charges with the geometry of the event horizon. To be more conservative, we have argued that such a law cannot be a general property of Lorentz violating theories, but we do not excluded that it can be valid in specific subsets, such as aymptotically flat four dimensional black holes or asymptotically Lifshitz lower dimensional black holes. Given that the latter cases are physically motivated, we find that exploring them more deeply would constitute the natural continuation of our research.
- In the case of black holes in Einstein-Maxwell-dilaton theory, we have

studied two complementary aspects. On the one hand, we have shown that their monopole scalar charge is predictable even without knowing the explicit form of the solution, expecially in the (physically motivated) weak electric charge limit. On the other hand, we have characterized how the parameters of the theory can be constrained from the detection of the ringdown modes of the black holes. Einstein-Maxwell-dilaton is a special, and perhaps the most simple, example of Einstein-Maxwell-scalar theories, in which a scalar field couples nonminimally with the electromagnetic field. For certain choices of the scalar coupling, these theories can give rise to interesting phenomena like spontaneous scalarization, whose study has raised attention only recently [247]. Therefore we regard our work as a first step towards exploring the phenomenology of these theories. It would be certainly interesting to investigate how our findings in Einstein-Maxwell-dilaton generalize or can be extended to the more general class of Einstein-Maxwell-scalar, with more complex nonminimal couplings between the scalar and the electromagnetic fields.

- In the case of regular black holes, we investigated their internal consistency against Hawking evaporation and inner horizon instability. We found that, for generic nonsingular metrics, the models are inconsistent because they predict the dynamical emergence of singularities. We noticed that, although there can still be some viable classes of regular black holes, the conditions that they must obey do not seem to be easily satisfiable. We therefore interpret our results, in conjuction with previous ones in the literature, as strongly indicating that more radical solutions to the problem of singularities must be considered.

We also want to emphasize that, besides the contextual significance of the above results, we also developed thechniques for addressing the respective problems, which can be useful well beyond the specific cases considered in this Thesis. For example, the prescription for deriving a Smarr formula can be applied in principle to any stationary black hole, and it can be flexible enough to encopass special cases in which the stationary symmetries are only partially respected: we already saw an example of such flexibility in the case of khronometric theory, in which the khronon field cannot be assumed to be static.

In the study of the ringdown of Einstein-Maxwell-dilaton black holes, we developed an approximation scheme which is a generalization of the Dudley-Finley scheme, originally proposed in the simpler case of electrovacuum black holes. Given that quasinormal modes are notoriously difficult to compute exactly, it is crucial to implement reliable simplification techniques. Having shown that the Dudley-Finley scheme can be adapted to Einstein-Maxwell-dilaton can stimulate similar extensions in other modified theories.

A

Proof of equation (3.14)

We start by describing the essential steps for the proof of Eq.(3.14). First of all, rewrite the line element (3.13) in cartesian spatial coordinates as

$$ds^2 = g_{ab}dx^a dx^b = -f(r)dt^2 + h_{IJ}dX^I dX^J \tag{A.1}$$

where X^I are coordinates such that $X^I X^J \delta_{IJ} = r^2$ and $\delta_{IJ}dX^I dX^J = dr^2 + r^2 d\Omega^2_{D-2}$. The spatial indices I, J spans $I, J = 2 \dots D$. The spatial metric h_{IJ} is given by

$$h_{IJ} = \delta_{IJ} + \frac{(1-f)}{f}\frac{X_I X_J}{r^2} \tag{A.2}$$

with inverse

$$h^{IJ} = \delta^{IJ} - (1-f)\frac{X^I X^J}{r^2}\,. \tag{A.3}$$

In (A.2) it is understood that the indices of the X^I's are lowered with the delta δ_{IJ}, a convention that we adopt throughout all this Section.

In the coordinates (t, X^I), the only nonvanishing components of the Christoffel symbol are

$$\Gamma^{t}_{t\,I} = \frac{f'}{f}\frac{X_I}{2r}\,, \quad \Gamma^{I}_{t\,t} = f\,f'\frac{X^I}{2r}\,, \tag{A.4a}$$

$$\Gamma^{I}_{J\,K} = \frac{f}{2}\left[\left(\frac{1-f}{fr^2}\right)'\frac{X^I X_J X_K}{r} + 2\left(\frac{1-f}{fr^2}\right)X^I\delta_{JK}\right]\,, \tag{A.4b}$$

where a prime denotes derivative w.r.t. the radial coordinate r. Correspondingly, the only nonvanishing components of the Riemann tensor are

$$R_{tI}{}^{tJ} = -\frac{f'}{2r}\delta^J_I - \frac{f''}{2r^2}X_I X^J + \frac{f'}{2r^3}X_I X^J\,, \tag{A.5a}$$

$$R_{IJ}{}^{KL} = \frac{2(1-f)}{r^2}\delta^K_{[I}\delta^L_{J]} - \frac{2f'}{r^3}\delta^{[K}_{[I}X_{J]}X^{L]} - \frac{4(1-f)}{r^4}\delta^{[K}_{[I}X_{J]}X^{L]}\,. \tag{A.5b}$$

Now, from the definition (3.8b) of $\mathcal{L}^{(k)}$ and from the expressions (A.5), we have

$$\begin{aligned}\mathcal{L}^{(k)} = \frac{1}{2^k}\delta^{a_1b_1\ldots a_kb_k}_{c_1d_1\ldots c_kd_k}R_{a_1b_1}{}^{c_1d_1}\ldots R_{a_kb_k}{}^{c_kd_k} = \\ = \underbrace{\frac{1}{2^k}\delta^{I_1J_1\ldots I_kJ_k}_{K_1L_1\ldots K_kL_k}R_{I_1J_1}{}^{K_1L_1}\ldots R_{I_kJ_k}{}^{K_kL_k}}_{\mathcal{H}^{(k)}} + \\ + \underbrace{\frac{k}{2^{k-2}}\delta^{J_1\ldots I_kJ_k}_{L_1\ldots K_kL_k}R_{tJ_1}{}^{tL_1}\ldots R_{I_kJ_k}{}^{K_kL_k}}_{\mathcal{T}^{(k)}} .\end{aligned} \tag{A.6}$$

Let us compute separately the "spatial" contribution $\mathcal{H}^{(k)}$ and the "temporal" contribution $\mathcal{T}^{(k)}$. The spatial contribution is equal to

$$\mathcal{H}^{(k)} = \frac{\sigma_k}{r^{D-2}}\frac{d}{dr}\left[(1-f)^k r^{D-2k-1}\right] , \quad \sigma_k = \frac{(D-2)!}{(D-2k+1)!} . \tag{A.7}$$

Proof. From (A.5) we have

$$\begin{aligned}\mathcal{H}^{(k)} =& \frac{(1-f)^k}{r^{2k}}\delta^{I_1J_1\ldots I_kJ_k}_{I_1J_1\ldots I_kJ_k} - 2k\frac{(1-f)^k}{r^{2k+2}}\delta^{AJ_1\ldots I_kJ_k}_{BJ_1\ldots I_kJ_k}X_AX^B + \\ & - k\frac{(1-f)^{k-1}f'}{r^{2k+1}}\delta^{AJ_1\ldots I_kJ_k}_{BJ_1\ldots I_kJ_k}X_AX^B .\end{aligned} \tag{A.8}$$

The reader can verify, by induction over k, that

$$\delta^{I_1J_1\ldots I_kJ_k}_{I_1J_1\ldots I_kJ_k} = \frac{d!}{(d-2k)!} = \frac{(D-1)!}{(D-2k-1)!} , \tag{A.9a}$$

$$\delta^{AJ_1\ldots I_kJ_k}_{BJ_1\ldots I_kJ_k}X_AX^B = \frac{(d-1)!}{(d-2k)!}r^2 = \frac{(D-2)!}{(D-2k-1)!}r^2 \tag{A.9b}$$

where $d = D-1$ is the dimensionality of the spatial slices. Then Eq.(A.7) follows immediately. □

With similar manipulations, one can show that the temporal contribution is equal to

$$\mathcal{T}^{(k)} = -\frac{\gamma_k k}{r^{D-2}}\frac{d}{dr}\left[(1-f)^{k-1}f'r^{D-2k}\right] , \quad \gamma_k = \frac{(D-2)!}{(D-2k)!} . \tag{A.10}$$

As a last step, observe that $\mathcal{H}^{(k)}$ and $\mathcal{T}^{(k)}$ can be rewritten as

$$\mathcal{H}^{(k)} = \frac{\gamma_k}{r^{D-2}}\frac{d}{dr}\left[(1-f)^k\frac{dr^{D-2k}}{dr}\right] , \tag{A.11a}$$

$$\mathcal{T}^{(k)} = \frac{\gamma_k}{r^{D-2}}\frac{d}{dr}\left[\frac{d(1-f)^k}{dr}r^{D-2k}\right] , \tag{A.11b}$$

from which (3.14) eventually follows. This completes our proof of Eq.(3.14).
□

Bibliography

[1] E. Di Casola, S. Liberati, and S. Sonego, American Journal of Physics **83**, 39 (2015).

[2] R. Penrose, Phys. Rev. Lett. **14**, 57 (1965).

[3] S. Hawking and G. Ellis, *The large scale structure of space-time* (CUP, Cambridge, 1973).

[4] J. M. M. Senovilla, General Relativity and Gravitation **30**, 701 (1998).

[5] R. Percacci, *An Introduction to Covariant Quantum Gravity and Asymptotic Safety* (World Scientific, 2017).

[6] T. Jacobson, Phys. Rev. D **81**, 101502 (2010).

[7] F. Müller-Hoissen, Physics Letters B **201**, 325 (1988).

[8] T. Baker, D. Psaltis, and C. Skordis, The Astrophysical Journal **802**, 63 (2015).

[9] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. **116**, 061102 (2016).

[10] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. **116**, 241103 (2016).

[11] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaboration), Phys. Rev. Lett. **118**, 221101 (2017).

[12] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), The Astrophysical Journal Letters **851**, L35 (2017).

[13] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. **119**, 141101 (2017).

[14] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. **119**, 161101 (2017).

[15] N. Yunes, K. Yagi, and F. Pretorius, Phys. Rev. D **94**, 084002 (2016).

[16] B. P. Abbott *et al.*, The Astrophysical Journal Letters **848**, L12 (2017).

[17] P. Creminelli and F. Vernizzi, Phys. Rev. Lett. **119**, 251302 (2017).

[18] J. M. Ezquiaga and M. Zumalacárregui, Phys. Rev. Lett. **119**, 251304 (2017).

[19] T. Baker, E. Bellini, P. G. Ferreira, M. Lagos, J. Noller, and I. Sawicki, Phys. Rev. Lett. **119**, 251301 (2017).

[20] E. Berti, K. Yagi, and N. Yunes, General Relativity and Gravitation **50**, 46 (2018).

[21] E. Berti, K. Yagi, H. Yang, and N. Yunes, General Relativity and Gravitation **50**, 49 (2018).

[22] M. Heusler, *Black hole uniqueness theorems* (Cambridge University Press, 1996).

[23] G. W. Gibbons, Communications in Mathematical Physics **44**, 245 (1975).

[24] V. Cardoso, C. F. Macedo, P. Pani, and V. Ferrari, Journal of Cosmology and Astroparticle Physics **2016**, 054 (2016).

[25] T. P. Sotiriou, Classical and Quantum Gravity **32**, 214002 (2015).

[26] E. Berti *et al.*, Classical and Quantum Gravity **32**, 243001 (2015).

[27] M. S. Volkov, arXiv preprint arXiv:1601.08230 .

[28] V. Cardoso and L. Gualtieri, Classical and Quantum Gravity **33**, 174001 (2016).

[29] C. A. R. Herdeiro and E. Radu, International Journal of Modern Physics D **24**, 1542014 (2015).

[30] H. Witek, V. Cardoso, A. Ishibashi, and U. Sperhake, Phys. Rev. D **87**, 043513 (2013).

[31] H. Okawa, H. Witek, and V. Cardoso, Phys. Rev. D **89**, 104032 (2014).

[32] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991).

[33] J. H. Horne and G. T. Horowitz, Physical Review D **46**, 1340 (1992).

[34] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago and London, 1984).

[35] E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, 2004).

[36] R. Brito, V. Cardoso, and P. Pani, Lect. Notes Phys **906**, 18 (2015).

[37] D. Christodoulou, Phys. Rev. Lett. **25**, 1596 (1970).

[38] G. W. Gibbons and R. E. Kallosh, Phys. Rev. D **51**, 2839 (1995).

[39] B. Cropp, S. Liberati, and M. Visser, Classical and Quantum Gravity **30**, 125001 (2013).

[40] J. M. Bardeen, B. Carter, and S. W. Hawking, Communications in Mathematical Physics **31**, 161 (1973).

[41] R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics* (University of Chicago Press, 1994).

[42] J. M. Bardeen, B. Carter, and S. W. Hawking, Communications in Mathematical Physics **31**, 161 (1973).

[43] B. Carter, *Black hole equilibrium states* (1973) pp. 57–214.

[44] I. Rácz and R. M. Wald, Classical and Quantum Gravity **13**, 539 (1996).

[45] I. Racz and R. M. Wald, Classical and Quantum Gravity **9**, 2643 (1992).

[46] R. M. Wald, Phys. Rev. D **48**, R3427 (1993).

[47] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).

[48] S. Sarkar, in *Gravity and the Quantum* (Springer, 2017) pp. 363–374.

[49] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).

[50] W. Israel, Phys. Rev. Lett. **57**, 397 (1986).

[51] R. Wald, Annals of Physics **82**, 548 (1974).

[52] B. Sullivan and W. Israel, Physics Letters A **79**, 371 (1980).

[53] I. Rácz, Classical and Quantum Gravity **17**, 4353 (2000).

[54] R. M. Wald, Journal of Mathematical Physics **31**, 2378 (1990), https://doi.org/10.1063/1.528839 .

[55] J. Lee and R. M. Wald, Journal of Mathematical Physics **31**, 725 (1990), https://doi.org/10.1063/1.528801 .

[56] A. Ashtekar, S. Fairhurst, and B. Krishnan, Phys. Rev. D **62**, 104025 (2000).

[57] A. Ashtekar, C. Beetle, and J. Lewandowski, Phys. Rev. D **64**, 044016 (2001).

[58] K. Prabhu, Classical and Quantum Gravity **34**, 035011 (2017).

[59] T. Regge and C. Teitelboim, Annals of Physics **88**, 286 (1974).

[60] D. Sudarsky and R. M. Wald, Phys. Rev. D **46**, 1453 (1992).

[61] S. W. Hawking, Nature **248**, 30 (1974).

[62] S. W. Hawking, Communications in Mathematical Physics **43**, 199 (1975).

[63] R. M. Wald, Communications in Mathematical Physics **45**, 9 (1975).

[64] L. Parker, Phys. Rev. D **12**, 1519 (1975).

[65] S. W. Hawking, Phys. Rev. D **14**, 2460 (1976).

[66] C. Barceló, S. Liberati, S. Sonego, and M. Visser, Phys. Rev. D **83**, 041501 (2011).

[67] C. Barceló, S. Liberati, S. Sonego, and M. Visser, Journal of High Energy Physics **2011**, 1 (2011).

[68] W. G. Unruh and R. M. Wald, Phys. Rev. D **25**, 942 (1982).

[69] V. P. Frolov and D. N. Page, Phys. Rev. Lett. **71**, 3902 (1993).

[70] A. C. Wall, Phys. Rev. D **85**, 104049 (2012).

[71] S. D. Mathur, Classical and Quantum Gravity **26**, 224001 (2009).

[72] D. N. Page, Phys. Rev. Lett. **71**, 3743 (1993).

[73] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, Journal of High Energy Physics **2013**, 62 (2013).

[74] S. W. Hawking, M. J. Perry, and A. Strominger, Phys. Rev. Lett. **116**, 231301 (2016).

[75] W. G. Unruh and R. M. Wald, Reports on Progress in Physics **80**, 092002 (2017).

[76] S. Sarkar and A. C. Wall, Phys. Rev. D **88**, 044017 (2013).

[77] A. C. Wall, International Journal of Modern Physics D **24**, 1544014 (2015).

[78] X. Dong, Journal of High Energy Physics **2014**, 44 (2014).

[79] M. Zilhão, V. Cardoso, C. Herdeiro, L. Lehner, and U. Sperhake, Phys. Rev. D **90**, 124088 (2014).

[80] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[81] F. J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970).

[82] F. J. Zerilli, Phys. Rev. D **2**, 2141 (1970).

[83] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, 1983).

[84] K. D. Kokkotas and B. G. Schmidt, Living Reviews in Relativity **2**, 2 (1999).

[85] H.-P. Nollert, Classical and Quantum Gravity **16**, R159 (1999).

[86] E. Berti, V. Cardoso, and A. O. Starinets, Classical and Quantum Gravity **26**, 163001 (2009).

[87] P. Pani, International Journal of Modern Physics A **28**, 1340018 (2013).

[88] F. J. Zerilli, Phys. Rev. D **9**, 860 (1974).

[89] E. Witten, Nuclear Physics B **188**, 513 (1981).

[90] F. Cooper, A. Khare, and U. Sukhatme, Physics Reports **251**, 267 (1995).

[91] P. T. Leung, A. Maassen van den Brink, W. M. Suen, C. W. Wong, and K. Young, Journal of Mathematical Physics **42**, 4802 (2001), https://doi.org/10.1063/1.1388900 .

[92] E. W. Leaver, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **402**, 285 (1985).

[93] E. W. Leaver, Phys. Rev. D **41**, 2986 (1990).

[94] H.-P. Nollert, Phys. Rev. D **47**, 5253 (1993).

[95] S. A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972).

[96] B. F. Whiting, Journal of Mathematical Physics **30**, 1301 (1989), https://doi.org/10.1063/1.528308 .

[97] P. Pani, E. Berti, and L. Gualtieri, Phys. Rev. Lett. **110**, 241103 (2013).

[98] P. Pani, E. Berti, and L. Gualtieri, Phys. Rev. D **88**, 064048 (2013).

[99] Z. Mark, H. Yang, A. Zimmerman, and Y. Chen, Phys. Rev. D **91**, 044025 (2015).

[100] O. J. C. Dias, M. Godazgar, and J. E. Santos, Phys. Rev. Lett. **114**, 151101 (2015).

[101] R. Brito, S. Ghosh, E. Barausse, E. Berti, V. Cardoso, I. Dvorkin, A. Klein, and P. Pani, Phys. Rev. Lett. **119**, 131101 (2017).

[102] R. Brito, S. Ghosh, E. Barausse, E. Berti, V. Cardoso, I. Dvorkin, A. Klein, and P. Pani, Phys. Rev. D **96**, 064050 (2017).

[103] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. **112**, 221101 (2014).

[104] C. Herdeiro, E. Radu, and H. Rúnarsson, Classical and Quantum Gravity **33**, 154001 (2016).

[105] J. C. Degollado, C. A. Herdeiro, and E. Radu, Physics Letters B **781**, 651 (2018).

[106] L. Barack, V. Cardoso, S. Nissanke, T. P. Sotiriou, A. Askar, C. Belczynski, G. Bertone, E. Bon, D. Blas, R. Brito, *et al.*, arXiv preprint arXiv:1806.05195 (2018).

[107] C. Goebel, The Astrophysical Journal **172**, L95 (1972).

[108] V. Ferrari and B. Mashhoon, Phys. Rev. D **30**, 295 (1984).

[109] B. Mashhoon, Phys. Rev. D **31**, 290 (1985).

[110] V. Cardoso, A. S. Miranda, E. Berti, H. Witek, and V. T. Zanchin, Phys. Rev. D **79**, 064016 (2009).

[111] S. R. Dolan, Phys. Rev. D **82**, 104003 (2010).

[112] H. Yang, D. A. Nichols, F. Zhang, A. Zimmerman, Z. Zhang, and Y. Chen, Phys. Rev. D **86**, 104006 (2012).

[113] H. Yang, F. Zhang, A. Zimmerman, D. A. Nichols, E. Berti, and Y. Chen, Phys. Rev. D **87**, 041502 (2013).

[114] H. Yang, A. Zimmerman, A. i. e. i. f. Zenginoğlu, F. Zhang, E. Berti, and Y. Chen, Phys. Rev. D **88**, 044047 (2013).

[115] S. Detweiler, The Astrophysical Journal **239**, 292 (1980).

[116] E. Berti, V. Cardoso, J. A. Gonzalez, U. Sperhake, M. Hannam, S. Husa, and B. Brügmann, Phys. Rev. D **76**, 064034 (2007).

[117] O. Dreyer, B. Kelly, B. Krishnan, L. S. Finn, D. Garrison, and R. Lopez-Aleman, Classical and Quantum Gravity **21**, 787 (2004).

[118] E. Berti, A. Sesana, E. Barausse, V. Cardoso, and K. Belczynski, Phys. Rev. Lett. **117**, 101102 (2016).

[119] O. J. Tattersall, P. G. Ferreira, and M. Lagos, Phys. Rev. D **97**, 044021 (2018).

[120] V. Cardoso and L. Gualtieri, Phys. Rev. D **80**, 064008 (2009).

[121] V. Cardoso and L. Gualtieri, Phys. Rev. D **80**, 064008 (2009).

[122] V. Cardoso and L. Gualtieri, Phys. Rev. D **81**, 089903 (2010).

[123] C. Molina, P. Pani, V. Cardoso, and L. Gualtieri, Phys. Rev. D **81**, 124021 (2010).

[124] P. Pani and V. Cardoso, Phys. Rev. D **79**, 084031 (2009).

[125] J. L. Blázquez-Salcedo, C. F. B. Macedo, V. Cardoso, V. Ferrari, L. Gualtieri, F. S. Khoo, J. Kunz, and P. Pani, Phys. Rev. D **94**, 104024 (2016).

[126] J. L. Blázquez-Salcedo, F. S. Khoo, and J. Kunz, Phys. Rev. D **96**, 064008 (2017).

[127] T. Kobayashi, H. Motohashi, and T. Suyama, Phys. Rev. D **85**, 084025 (2012).

[128] T. Kobayashi, H. Motohashi, and T. Suyama, Phys. Rev. D **89**, 084042 (2014).

[129] O. J. Tattersall and P. G. Ferreira, Phys. Rev. D **97**, 104047 (2018).

[130] K. Glampedakis, G. Pappas, H. O. Silva, and E. Berti, Phys. Rev. D **96**, 064054 (2017).

[131] G. W. Horndeski, International Journal of Theoretical Physics **10**, 363 (1974).

[132] C. Deffayet, X. Gao, D. A. Steer, and G. Zahariade, Phys. Rev. D **84**, 064039 (2011).

[133] P. G. Bergmann, International Journal of Theoretical Physics **1**, 25 (1968).

[134] R. V. Wagoner, Phys. Rev. D **1**, 3209 (1970).

[135] S. W. Hawking, Communications in Mathematical Physics **25**, 167 (1972).

[136] T. P. Sotiriou and V. Faraoni, Phys. Rev. Lett. **108**, 081103 (2012).

[137] L. Hui and A. Nicolis, Phys. Rev. Lett. **110**, 241104 (2013).

[138] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. Lett. **112**, 251102 (2014).

[139] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. D **90**, 124063 (2014).

[140] R. Benkel, T. P. Sotiriou, and H. Witek, Phys. Rev. D **94**, 121503 (2016).

[141] R. Benkel, T. P. Sotiriou, and H. Witek, Classical and Quantum Gravity **34**, 064001 (2017).

[142] K. Yagi, L. C. Stein, N. Yunes, and T. Tanaka, Phys. Rev. D **85**, 064022 (2012).

[143] K. Prabhu and L. C. Stein, arXiv preprint arXiv:1805.02668 (2018).

[144] K. Yagi, L. C. Stein, and N. Yunes, Phys. Rev. D **93**, 024010 (2016).

[145] A. Lehébel, E. Babichev, and C. Charmousis, Journal of Cosmology and Astroparticle Physics **2017**, 037 (2017).

[146] J. L. Blázquez-Salcedo *et al.*, Proceedings of the International Astronomical Union **12**, 265–272 (2016).

[147] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, Phys. Rev. Lett. **120**, 131104 (2018).

[148] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. **120**, 131103 (2018).

[149] V. P. Frolov, A. Zelnikov, and U. Bleyer, Annalen der Physik **499**, 371 (1987).

[150] J. Preskill, P. Schwarz, A. Shapere, S. Trivedi, and F. Wilczek, Modern Physics Letters A **6**, 2353 (1991).

[151] S. Yazadjiev, Phys. Rev. D **82**, 124050 (2010).

[152] D. Mattingly, Living Reviews in Relativity **8**, 5 (2005).

[153] V. A. Kostelecký and N. Russell, Rev. Mod. Phys. **83**, 11 (2011).

[154] S. Liberati, Classical and Quantum Gravity **30**, 133001 (2013).

[155] P. Hořava, Phys. Rev. D **79**, 084008 (2009).

[156] D. Blas, O. Pujolàs, and S. Sibiryakov, Phys. Rev. Lett. **104**, 181302 (2010).

[157] T. P. Sotiriou, Journal of Physics: Conference Series **283**, 012034 (2011).

[158] T. Jacobson, arXiv preprint arXiv:0801.1547 (2008).

[159] A. Emir Gümrükçüoğlu, M. Saravani, and T. P. Sotiriou, Phys. Rev. D **97**, 024032 (2018).

[160] J. Oost, S. Mukohyama, and A. Wang, Phys. Rev. D **97**, 124023 (2018).

[161] D. Blas and S. Sibiryakov, Phys. Rev. D **84**, 124043 (2011).

[162] E. Barausse, T. Jacobson, and T. P. Sotiriou, Phys. Rev. D **83**, 124043 (2011).

[163] P. Berglund, J. Bhattacharyya, and D. Mattingly, Phys. Rev. D **85**, 124019 (2012).

[164] J. Bhattacharyya and D. Mattingly, International Journal of Modern Physics D **23**, 1443005 (2014).

[165] E. Barausse and T. P. Sotiriou, Classical and Quantum Gravity **30**, 244010 (2013).

[166] T. P. Sotiriou, I. Vega, and D. Vernieri, Phys. Rev. D **90**, 044046 (2014).

[167] S. Basu, J. Bhattacharyya, D. Mattingly, and M. Roberson, Phys. Rev. D **93**, 064072 (2016).

[168] N. Franchini, M. Saravani, and T. P. Sotiriou, Phys. Rev. D **96**, 104044 (2017).

[169] J. Bhattacharyya, M. Colombo, and T. P. Sotiriou, Classical and Quantum Gravity **33**, 235003 (2016).

[170] E. Barausse, T. P. Sotiriou, and I. Vega, Phys. Rev. D **93**, 044044 (2016).

[171] B. Z. Foster, Phys. Rev. D **73**, 024005 (2006).

[172] D. Lovelock, Journal of Mathematical Physics **12**, 498 (1971).

[173] B. Zumino, Physics Reports **137**, 109 (1986).

[174] C. Charmousis, "Higher order gravity theories and their black hole solutions," in *Physics of Black Holes: A Guided Tour*, edited by E. Papantonopoulos (Springer Berlin Heidelberg, Berlin, Heidelberg, 2009) pp. 299–346.

[175] J. T. Wheeler, Nuclear Physics B **273**, 732 (1986).

[176] R. C. Myers and J. Z. Simon, Phys. Rev. D **38**, 2434 (1988).

[177] R.-G. Cai, Physics Letters B **582**, 237 (2004).

[178] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).

[179] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).

[180] D. Kastor, S. Ray, and J. Traschen, Classical and Quantum Gravity **26**, 195011 (2009).

[181] D. Kastor, S. Ray, and J. Traschen, Classical and Quantum Gravity **27**, 235014 (2010).

[182] D. Kastor, S. Ray, and J. Traschen, Classical and Quantum Gravity **28**, 195022 (2011).

[183] T. Regge and C. Teitelboim, Annals of Physics **88**, 286 (1974).

[184] T. Jacobson and R. C. Myers, Phys. Rev. Lett. **70**, 3684 (1993).

[185] D. Kastor, Classical and Quantum Gravity **25**, 175007 (2008).

[186] C. Eling, Phys. Rev. D **73**, 084026 (2006).

[187] B. Z. Foster, Phys. Rev. D **73**, 024005 (2006).

[188] B. Z. Foster, Phys. Rev. D **72**, 044017 (2005).

[189] J. Bhattacharyya, (2013).

[190] P. Berglund, J. Bhattacharyya, and D. Mattingly, Phys. Rev. Lett. **110**, 071301 (2013).

[191] C. Ding, A. Wang, X. Wang, and T. Zhu, Nuclear Physics B **913**, 694 (2016).

[192] F. Michel and R. Parentani, Phys. Rev. D **91**, 124049 (2015).

[193] B. Cropp, S. Liberati, A. Mohd, and M. Visser, Phys. Rev. D **89**, 064061 (2014).

[194] B. Cropp, Ph.D. thesis, SISSA (2013).

[195] K. Lin, O. Goldoni, M. F. da Silva, and A. Wang, Phys. Rev. D **91**, 024047 (2015).

[196] T. Griffin, P. Hořava, and C. M. Melby-Thompson, Phys. Rev. Lett. **110**, 081602 (2013).

[197] J. Cheyne and D. Mattingly, Phys. Rev. D **97**, 066024 (2018).

[198] S. Basu, J. Bhattacharyya, D. Mattingly, and M. Roberson, Phys. Rev. D **93**, 064072 (2016).

[199] P. Jai-akson, A. Chatrabhuti, O. Evnin, and L. Lehner, Physical Review D **96**, 044031 (2017).

[200] E. W. Hirschmann, L. Lehner, S. L. Liebling, and C. Palenzuela, Phys. Rev. D **97**, 064032 (2018).

[201] F.-L. Julié, Journal of Cosmology and Astroparticle Physics **2018**, 026 (2018).

[202] F. McCarthy, D. Kubizňák, and R. B. Mann, Phys. Rev. D **97**, 104025 (2018).

[203] Y. Mizuno, Z. Younsi, C. M. Fromm, O. Porth, M. De Laurentis, H. Olivares, H. Falcke, M. Kramer, and L. Rezzolla, Nature Astronomy , 1 (2018).

[204] C. M. Will, Living Reviews in Relativity **17**, 4 (2014).

[205] V. Ferrari, M. Pauri, and F. Piazza, Physical Review D **63**, 064009 (2001).

[206] C. F. E. Holzhey and F. Wilczek, Nucl. Phys. **B380**, 447 (1992), arXiv:hep-th/9202014 [hep-th] .

[207] R. A. Konoplya, Gen. Rel. Grav. **34**, 329 (2002), arXiv:gr-qc/0109096 [gr-qc] .

[208] `https://github.com/richbrito/QNM_EMD/`.

[209] M. Zilhão, V. Cardoso, C. Herdeiro, L. Lehner, and U. Sperhake, Phys. Rev. D **85**, 124062 (2012).

[210] M. Zilhão, V. Cardoso, C. Herdeiro, L. Lehner, and U. Sperhake, Phys. Rev. D **89**, 044008 (2014).

[211] S. L. Liebling and C. Palenzuela, Phys. Rev. D **94**, 064046 (2016).

[212] A. L. Dudley and J. D. Finley, Phys. Rev. Lett. **38**, 1505 (1977).

[213] A. L. Dudley and J. D. F. III, Journal of Mathematical Physics **20**, 311 (1979), https://doi.org/10.1063/1.524064 .

[214] P. Pani, E. Berti, and L. Gualtieri, Physical Review D **88**, 064048 (2013).

[215] E. Berti and K. D. Kokkotas, Phys. Rev. D **71**, 124008 (2005).

[216] P. Pani, V. Cardoso, L. Gualtieri, E. Berti, and A. Ishibashi, Phys. Rev. D **86**, 104017 (2012).

[217] R. Brito, V. Cardoso, and P. Pani, Phys. Rev. D **88**, 023514 (2013).

[218] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, The Astrophysical Journal **178**, 347 (1972).

[219] P. Pani, E. Berti, and L. Gualtieri, Phys. Rev. Lett. **110**, 241103 (2013).

[220] V. Cardoso, I. P. Carucci, P. Pani, and T. P. Sotiriou, Phys. Rev. Lett. **111**, 111101 (2013).

[221] W. F. Buell and B. A. Shadwick, American Journal of Physics **63**, 256 (1995).

[222] A. Ashtekar and M. Bojowald, Classical and Quantum Gravity **22**, 3349 (2005).

[223] S. A. Hayward, Phys. Rev. Lett. **96**, 031103 (2006).

[224] V. P. Frolov, Journal of High Energy Physics **2014**, 49 (2014).

[225] S. Ansoldi, arXiv: 0802.0330 .

[226] I. Dymnikova, Classical and Quantum Gravity **19**, 725 (2002).

[227] A. Borde, Phys. Rev. D **55**, 7615 (1997).

[228] I. Dymnikova, General Relativity and Gravitation **24**, 235 (1992).

[229] K. Falls and D. F. Litim, Phys. Rev. D **89**, 084002 (2014).

[230] V. Cardoso, J. a. L. Costa, K. Destounis, P. Hintz, and A. Jansen, Phys. Rev. Lett. **120**, 031103 (2018).

[231] O. J. C. Dias, F. C. Eperon, H. S. Reall, and J. E. Santos, Phys. Rev. D **97**, 104060 (2018).

[232] E. Poisson and W. Israel, Phys. Rev. Lett. **63**, 1663 (1989).

[233] E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990).

[234] M. Dafermos, Communications on Pure and Applied Mathematics **58**, 445.

[235] E. G. Brown, R. Mann, and L. Modesto, Phys. Rev. D **84**, 104041 (2011).

[236] C. Barrabes, W. Israel, and E. Poisson, Classical and Quantum Gravity **7**, L273 (1990).

[237] T. Dray and G. 't Hooft, Communications in Mathematical Physics **99**, 613 (1985).

[238] I. H. Redmount, Progress of Theoretical Physics **73**, 1401 (1985).

[239] R. H. Price, Phys. Rev. D **5**, 2419 (1972).

[240] E. S. C. Ching, P. T. Leung, W. M. Suen, and K. Young, Phys. Rev. D **52**, 2118 (1995).

[241] C. Gundlach, R. H. Price, and J. Pullin, Phys. Rev. D **49**, 883 (1994).

[242] C. Gundlach, R. H. Price, and J. Pullin, Phys. Rev. D **49**, 890 (1994).

[243] M. Dafermos and I. Rodnianski, Inventiones mathematicae **162**, 381 (2005).

[244] E. Bianchi, T. De Lorenzo, and M. Smerlak, Journal of High Energy Physics **2015**, 180 (2015).

[245] T. De Lorenzo, Ph.D. thesis, University of Pisa (2013).

[246] V. P. Frolov and A. Zelnikov, Phys. Rev. D **95**, 124028 (2017).

[247] C. A. Herdeiro, E. Radu, N. Sanchis-Gual, and J. A. Font, arXiv preprint arXiv:1806.05190 (2018).