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Moduli of sheaves, quiver moduli, and stability

Ph.D. thesis by Andrea Maiorana

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Abstract

The goal of this thesis is to provide a modern interpretation and an extension of the classical works of the 1970s and 1980s constructing moduli spaces of vector bundles and coherent sheaves on projective spaces by means of "linear data", that is spaces of matrices modulo a linear group action. These works culminated with the description by Drézet and Le Potier of the moduli spaces of Gieseker-semistable sheaves on $\mathbb{P}^2$ as what are called today quiver moduli spaces. We show that this can be naturally understood and generalized using the language of derived categories and stability structures on them. In particular, we obtain analogous explicit constructions for moduli of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, and we investigate these moduli spaces using the theory of quiver moduli.

Contents

Chapter 1

Introduction

A recurring theme in algebraic geometry is the study of moduli spaces, varieties whose points parameterize in a natural way algebro-geometric objects of some kind. In this thesis we will focus on two types of moduli spaces, namely moduli of coherent sheaves on a projective variety and moduli of representations of a quiver, and we will discuss certain relations between them.

1.1 Monads and moduli of sheaves

1.1.1 Moduli spaces of semistable sheaves

Let us consider first moduli of sheaves: the first general construction of moduli spaces of vector bundles (that is, locally free sheaves) on a projective curve was given by Mumford using GIT, and then it was extended by Seshadri, Gieseker, Maruyama and Simpson among others to prove the existence, as projective schemes, of moduli spaces of semistable coherent sheaves on a projective scheme X of any dimension.

In all these works Geometric Invariant Theory (GIT) was used, and the idea of the construction was the following: one shows that all the sheaves under consideration can be written as quotients of a fixed bundle $\mathcal{H}$ on X, and thus correspond to points of a subscheme $R \subset \mathrm{Quot}(\mathcal{H})$ of the "Quot" scheme parameterizing quotient sheaves of $\mathcal{H}$, up to the action of the group $G = \mathrm{Aut}(\mathcal{H})$, which is reductive. Then one needs to construct a suitable linearization $\mathcal{L}$ which identifies R as the GIT-semistable locus in the closure $\overline{R}$, so that the moduli space can be obtained as the GIT quotient $R//_{\mathcal{L}}G$. We will review in §4.2 the main aspects of this theory.

1.1.2 Monads and linear data

By the late 1970s, some people were studying an alternative and much more explicit way to construct moduli spaces of bundles over projective spaces: their approach consisted of parameterizing vector bundles as middle cohomologies $H^0(M^\bullet)$ of certain *monads*, that is complexes

$$M^\bullet : M^{-1} \xrightarrow{a} M^0 \xrightarrow{b} M^1$$

of vector bundles, where a is injective and b is surjective. By fixing the bundles M^{-1}, M^0, M^1 carefully and by varying the maps a, b, one obtains a family of sheaves $H^0(M^\bullet)$, which in many cases turns out to be rich enough to describe moduli spaces. This concept was first used by Horrocks [Hor64], and we will provide more details on it in §4.3.

Barth [Bar77] showed that every stable bundle $\mathcal{E}$ of rank 2, degree 0 and $c_2 = k$ on the complex projective plane $\mathbb{P}^2 = \mathbb{P}_{\mathbb{C}}(Z)$ is isomorphic to the middle cohomology of a monad whose differentials a, b only depend on a certain *Kronecker module* $f \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k \otimes Z^\vee, \mathbb{C}^k)$ constructed from $\mathcal{E}$. Moreover, this construction gives a bijection between such bundles $\mathcal{E}$ up to isomorphism and elements of a subvariety $\tilde{R} \subset \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k \otimes Z^\vee, \mathbb{C}^k)$ up to the action of $\mathrm{GL}_k(\mathbb{C})$. It follows that we have a surjective morphism $\tilde{R} \to \mathrm{M}^{\mathrm{st}}$ identifying the moduli space M^{st} of stable bundles with the given numerical invariants as a $\mathrm{GL}_k(\mathbb{C})$-quotient of $\tilde{R}$. By analyzing the variety $\tilde{R}$, Barth was able to prove rationality and irreducibility of M^{st}. Barth and Hulek extended this construction first to all moduli spaces of rank 2 bundles [BH78, Hul79], and then to moduli of bundles with any rank and zero degree [Hul80]. These works were also fundamental to find explicit constructions of instantons, or anti self-dual Yang-Mills connections, by means of linear data [ADHM78, Don84].

The main limit of these techniques was the difficulty in showing that a bundle (or more generally a coherent sheaf) can be recovered as the middle cohomology of a monad of fixed type. The situation changed after the work of Beilinson [Bei78] describing the bounded derived category of coherent sheaves on projective spaces. This gave in particular a systematic way to approximate any coherent sheaf $\mathcal{E}$ on $\mathbb{P}^2$ by a spectral sequence which reduces to a monad whose cohomology is $\mathcal{E}$ in case this is semistable.[1] In this way, Drézet and Le Potier generalized in [DLP85] the works of Barth and Hulek to all Gieseker-semistable torsion-free sheaves on $\mathbb{P}^2$. They also showed that, after imposing an analogue of Gieseker semistability, the "Kronecker" complexes

$$V_{-1} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow V_0 \otimes \Omega^1_{\mathbb{P}^2}(1) \longrightarrow V_1 \otimes \mathcal{O}_{\mathbb{P}^2} \tag{1.1.1}$$

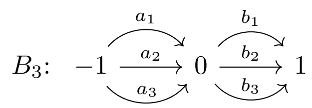
are forced to be monads, and taking their middle cohomology gives Gieseker-semistable sheaves. Moreover, this gives a bijective correspondence between isomorphism classes of semistable Kronecker complexes and isomorphism classes of semistable torsion-free sheaves, having fixed a class $v \in K_0(\mathbb{P}^2)$; thus the moduli space $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2}(v)$ of such sheaves is a quotient of the semistable locus $R^{\mathrm{ss}} \subset R_V$ in the vector space R_V of all Kronecker complexes by the action of $G_V := \prod_i \mathrm{GL}_{\mathbb{C}}(V_i)$.

1.1.3 Encoding monads with semistable quiver representations

Now we can observe that, since

$$\mathrm{Hom}(V_{-1} \otimes \mathcal{O}_{\mathbb{P}^2}(-1), V_0 \otimes \Omega^1_{\mathbb{P}^2}(1)) \simeq \mathrm{Hom}_{\mathbb{C}}(V_{-1}, V_0)^{\oplus 3}\,,$$
$$\mathrm{Hom}(V_0 \otimes \Omega^1_{\mathbb{P}^2}(1), V_1 \otimes \mathcal{O}_{\mathbb{P}^2}) \simeq \mathrm{Hom}_{\mathbb{C}}(V_0, V_1)^{\oplus 3}\,,$$

Kronecker complexes can be seen as representations of the *Beilinson quiver*



constrained by some relations, which we encode with an ideal $J' \subset \mathbb{C}Q$ in the path algebra of Q,[2] forcing the maps in Eq. (1.1.1) to form a complex. In fact, after fixing the dimension vector $d^v := (\dim_{\mathbb{C}} V_{-1}, \dim_{\mathbb{C}} V_0, \dim_{\mathbb{C}} V_1)$ of the Kronecker complexes, the above notion of Gieseker-like stability coincides with the usual concept of θ_v-stability for quiver representations, for some $\theta_v \in \mathbb{Z}^{\{-1,0,1\}}$. The latter was introduced by King [Kin94], who also showed that moduli spaces of d^v-dimensional θ_v-semistable representations, which we denote by $\mathrm{M}^{\mathrm{ss}}_{B_3,J',\theta_v}(d^v)$, can be constructed via GIT (see §4.1.4 for details): the set R^{ss} becomes the semistable locus of a linearization $\mathcal{L}_v$ of the action of G_V, so we recover $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2}(v)$ as the GIT quotient $R^{\mathrm{ss}}/\!/_{\mathcal{L}_v} G_V = \mathrm{M}^{\mathrm{ss}}_{B_3,J',\theta_v}(d^v)$:

$$\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2}(v) \simeq \mathrm{M}^{\mathrm{ss}}_{B_3,J',\theta_v}(d^v)\,. \tag{1.1.2}$$

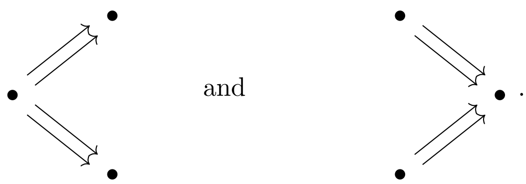
This also proves the existence of $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2}(v)$ as a projective scheme independently from the general theory of Gieseker and Simpson mentioned in §1.1.1.[3]

More recently, an analogous construction was carried out by Kuleshov in [Kul97], where, for certain choices of the numerical invariants, moduli spaces of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, Gieseker-stable with respect to the anticanonical polarization, were constructed as moduli of stable representations of the quivers

[1] For details on *Beilinson spectral sequences* see Ex. 2.1.25. The construction of monads using these spectral sequences is explained e.g. in [OSS80, Ch. 2, §4].

[2] The notation J' is only motivated by consistency with Chapter 5. Quiver representations and relations are reviewed in §4.1.

[3] In fact, another linearization of the action $G_V \curvearrowright R$ providing the interpretation of R^{ss} as a GIT-semistable locus was found in [LP94] without referring to quiver moduli. Remarkably, already in [Hul80] it was observed that the Kronecker modules $f \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k \otimes Z^\vee, \mathbb{C}^k)$ producing rank 2, degree 0 stable bundles can be characterized as GIT-stable points.



In what follows we will see that this is a special case of a general construction working for all moduli of semistable torsion-free sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. Finally, we mention that the techniques of [DLP85] have been used in [NS07, FGIK16] to construct moduli spaces of semistable sheaves on noncommutative projective planes.

1.2 Outline of the thesis

1.2.1 Categorical interpretation

The above-mentioned work [Bei78] was followed by many years of research on the structure of the bounded derived category $D^b(X)$ of a projective variety X. In particular, a theory of *exceptional sequences* of objects of $D^b(X)$ was developed in the seminar [Rud90] for this purpose (we give an introduction to this subject in §2.1). By means of this machinery, it is natural to interpret the abelian category $\mathcal{K}$ of Kronecker complexes (1.1.1), which is isomorphic to the category $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(B_3; J')$ of finite-dimensional representations of (B_3, J'), as the heart of a bounded t-structure (Def. 3.3.1) on $D^b(\mathbb{P}^2)$ induced by an exceptional sequence.

Hence the isomorphism (1.1.2) relates two GIT quotients parameterizing objects in two hearts $\mathcal{C} := \mathbf{Coh}_{\mathcal{O}_{\mathbb{P}^2}}$ and $\mathcal{K}$ of $D^b(\mathbb{P}^2)$ which are semistable with respect to some common (in a sense that will be clarified in due time) stability condition. As on both sides the GIT notions of (semi)stability and S-equivalence have categorical interpretations, it is quite natural to expect that these isomorphisms are manifestations of a coincidence of categories of semistable objects inside the hearts $\mathcal{C}, \mathcal{K} \subset D^b(\mathbb{P}^2)$.

The main goal of this thesis is to understand the constructions of the moduli spaces of sheaves via linear data mentioned in the previous section from this "categorified" point of view, and to develope a machinery to produce isomorphisms like (1.1.2) in a systematic way when we are given an exceptional sequence with good properties.

Chapters 2-4 introduce all the tools necessary to achieve this goal. In particular, Chapter 2 reviews the theory of exceptional sequences on triangulated categories, and the derived equivalences that they induce. Chapter 3 studies the concepts of stability on abelian and triangulated categories that are central in the problem. Finally, moduli spaces of quiver representations and of coherent sheaves are discussed in Chapter 4, together with some techniques needed to study them, such as the theory of monads; emphasis is placed on the analogies between the two, and it is explained how they can be somehow related when seen as moduli of objects in derived categories.

1.2.2 Summary of the main results of Chapter 5

Our aim is to construct some moduli spaces (or stacks) of semistable sheaves as quiver moduli spaces by using the arguments mentioned above. The central idea is the following: take a smooth projective variety X with a full strong exceptional sequence $\mathfrak{E}$ on $D^b(X)$, and let $\mathrm{M}^{\mathrm{ss}}_{X,A}(v)$ be the moduli space of coherent sheaves on X in a numerical class $v \in K_{\mathrm{num}}(X)$ that are Gieseker-semistable with respect to an ample divisor $A \subset X$. The sequence $\mathfrak{E}$ induces a triangulated equivalence Ψ between $D^b(X)$ and the bounded derived category $D^b(Q; J)$ of finite-dimensional representations of a certain quiver Q, usually with relations J. The functor Ψ induces a non-standard bounded t-structure on $D^b(X)$, whose heart $\mathcal{K}$ consists of certain *Kronecker complexes* of sheaves, and Gieseker stability makes sense in a generalized way for objects of $\mathcal{K}$. The key observation is that in some cases the hearts $\mathcal{C}, \mathcal{K}$ are somehow compatible with Gieseker stability, in the following sense: imposing Gieseker semistability forces the objects of suitable classes v in the standard heart $\mathcal{C} \subset D^b(X)$ to be also semistable objects of $\mathcal{K}$, and the same is true with $\mathcal{C}$ and $\mathcal{K}$ exchanged. Moreover, semistable Kronecker complexes in the class v are identified through Ψ with

$\theta_{\mathrm{G},v}$-semistable d^v-dimensional representations of (Q, J), for some dimension vector d^v and some (polynomial) weight $\theta_{\mathrm{G},v}$ (also depending on the polarization A) determined by the isomorphism of Grothendieck groups induced by Ψ. As this identification is compatible with the notions of families of semistable sheaves and semistable quiver representations, it implies that their moduli stacks can be identified through Ψ, and thus in particular the coarse moduli spaces $\mathrm{M}^{\mathrm{ss}}_{X,A}(v)$ and $\mathrm{M}^{\mathrm{ss}}_{Q,J,\theta_{\mathrm{G},v}}(d^v)$ are isomorphic.

The simplest example of this phenomenon is discussed in §5.1 for sheaves on the projective line $\mathbb{P}^1$: in this case, the heart $\mathcal{K}$ can be also obtained by tilting $\mathcal{C}$ using the slope-stability condition; this description is used to give in Corollary 5.1.5 an easy proof of Birkhoff-Grothendieck theorem (the well-known classification of coherent sheaves on $\mathbb{P}^1$) via quiver representations.

When X is a surface, however, this simple argument fails as the hearts $\mathcal{C}, \mathcal{K}$ are no longer related by a tilt. Nevertheless, in §5.2 we show that the above-mentioned compatibility between $\mathcal{C}, \mathcal{K}$ and Gieseker stability holds under some hypotheses on the sequence $\mathfrak{E}$ (namely, when $\mathfrak{E}$ is *monad-friendly*, Def. 5.2.1): we define a subset $\tilde{\mathcal{R}}_{A,\mathfrak{E}} \subset K_{\mathrm{num}}(X)$ depending on the ample divisor A and the sequence $\mathfrak{E}$, and we prove in Corollary 5.2.15 that:

Theorem 1.2.1. *For all $v \in \tilde{\mathcal{R}}_{A,\mathfrak{E}}$ we have isomorphisms $\mathrm{M}^{\mathrm{ss}}_{X,A}(v) \simeq \mathrm{M}^{\mathrm{ss}}_{Q,J,\theta_{\mathrm{G},v}}(d^v)$ and $\mathrm{M}^{\mathrm{st}}_{X,A}(v) \simeq \mathrm{M}^{\mathrm{st}}_{Q,J,\theta_{\mathrm{G},v}}(d^v)$.*

The assumptions on $\mathfrak{E}$ are easily seen to be satisfied by some well-known exceptional sequences on the projective plane $\mathbb{P}^2$ and the smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$. The application of the Theorem to them is treated in Sections 5.3 and 5.4, where the only thing left is to determine the data $d^v, \theta_{\mathrm{G},v}$, and $\tilde{\mathcal{R}}_{A,\mathfrak{E}}$ for the given exceptional sequences. In both cases, the regions $\tilde{\mathcal{R}}_{A,\mathfrak{E}}$ that we obtain are large enough to include, up to twisting $\mathfrak{E}$ by line bundles, any class v of positive rank. So, for example, if we start from the exceptional sequence $\mathfrak{E} = (\mathcal{O}(-1), \Omega^1(1), \mathcal{O})$ on $\mathbb{P}^2$, then we deduce the isomorphism (1.1.2) as a manifestation of an equivalence between abelian categories of Gieseker-semistable sheaves with fixed reduced Hilbert polynomial and King-semistable quiver representations. On $X = \mathbb{P}^1 \times \mathbb{P}^1$ we will get a similar construction of $\mathrm{M}^{\mathrm{ss}}_{X,A}(v)$ for any polarization A and any class v of positive rank, providing thus a complete generalization of the result of [Kul97].

Standard properties of the moduli spaces of sheaves, such as smoothness, dimensions and existence of universal sheaves will be quickly deduced using the theory of quiver moduli. We will also study some examples in which these moduli spaces can be constructed very explicitly, just by using some linear algebra and invariant theory. Finally, in §5.5 we briefly discuss how to extend these results and how to use them to study the invariants of moduli spaces: for example, their Chow rings can be characterized along the lines of [ES93, KW95].

1.2.3 Comparison with some recent works on Bridgeland stability conditions

To conclude this introduction we mention that there is a different way to relate moduli of sheaves and quiver moduli by using Bridgeland stability conditions [Bri07]: on a surface X one can define a family of so-called *geometric* stability conditions (these were introduced in [AB13]), some of which are equivalent to Gieseker stability; on the other hand, a full strong exceptional sequence on X induces *algebraic* stability conditions, for which semistable objects are identified to semistable quiver representations. When $X = \mathbb{P}^2$, Ohkawa [Ohk10] constructed stability conditions which are both geometric and algebraic, obtaining as a consequence the explicit isomorphisms between moduli of sheaves and moduli of representations of the Beilinson quiver, as in our Theorems 5.3.3 and 5.3.9. A similar analysis should in principle be possible also for $\mathbb{P}^1 \times \mathbb{P}^1$, for which algebraic stability conditions were studied in [AM17].[4] The main difference in our approach is essentially that we use a weaker notion of stability structure, which includes Gieseker stability both for sheaves and Kronecker complexes. Then we can directly jump from one moduli space to the other, instead of moving through the manifold of Bridgeland stability conditions. In this way the above-mentioned isomorphisms will be obtained with easy computations as examples of a general result.

[4] I thank Emanuele Macrì for pointing out this reference to me.

1.3 Main conventions and notation used in the thesis

Abelian and derived categories

$\mathcal{A}$ will always denote an abelian category. $A \in \mathcal{A}$ or $A \in \mathrm{Ob}(\mathcal{A})$ means that A is an object of $\mathcal{A}$, and $\mathrm{Ob}^{\times}(\mathcal{A})$ is the set (we ignore set-theoretical issues) of nonzero objects. $D(\mathcal{A})$ denotes the derived category of $\mathcal{A}$, and $D^b(\mathcal{A})$, $D^+(\mathcal{A})$, and $D^-(\mathcal{A})$ are respectively the subcategories of bounded, bounded below, and bounded above complexes. When we want to consider all of them together we write $D^*(\mathcal{A})$, for $* \in \{\_, +, -, b\}$. Similarly, by $K^*(\mathcal{A})$ we denote the homotopy categories. The cohomologies of a complex $B^\bullet \in D(\mathcal{A})$ (often written simply as B) will be usually denoted by $H^i(B^\bullet)$, but in some occasions also the notation $h^i(B^\bullet)$ (when $B^\bullet$ is a complex of sheaves) or $H^i_{\mathcal{A}}(B^\bullet)$ (to emphasize the heart $\mathcal{A} \subset D(\mathcal{A})$) will be used. The (canonical) truncations of a complex $B^\bullet$ are denoted by $\tau_{<k}B^\bullet$ and its obvious variants.

Triangulated categories

$\mathcal{D}$ will denote a triangulated category. Some notation regarding Hom spaces, generators, orthogonality, etc. in $\mathcal{D}$ will be introduced in §2.1.1.

Varieties and schemes

By *algebraic* $\mathbb{K}$*-scheme* $(X, \mathcal{O}_X)$, or just X, we mean a scheme of finite type over a field $\mathbb{K}$. By *point* $x \in X$ we always mean closed point. $(X, \mathcal{O}_X)$ is an *algebraic variety* if it is reduced and separated. $\mathbf{AlgSch}_{\mathbb{K}}$ is the category of algebraic $\mathbb{K}$-schemes.

Coherent sheaves

The abelian category of coherent $\mathcal{O}_X$-modules (or simply *coherent sheaves*) on an algebraic $\mathbb{K}$-scheme $(X, \mathcal{O}_X)$ is denoted by $\mathbf{Coh}_{\mathcal{O}_X}$, and its bounded derived category simply by $D^b(X)$. The skyscraper sheaf at a point $x \in X$ is denoted $\mathcal{O}_x$; the tangent and cotangent sheaves are denoted τ_X and Ω_X, and we also write $\Omega^p_X := \wedge^p \Omega_X$ and $\omega_X := \Omega^{\dim X}_X$ when X is smooth and equidimensional. More notation on coherent sheaves will be introduced in §4.2.1.

Algebras and modules

Any algebra A over a field $\mathbb{K}$ will be associative and with unity. The abelian categories of left and right (unitary) A-modules are denoted by ${}_A\mathbf{Mod}$ and $\mathbf{Mod}_A$ respectively. If A is right Noetherian, then $\mathbf{Mod}^{\mathrm{fg}}_A \subset \mathbf{Mod}_A$ denotes the abelian subcategory of finitely-generated modules. Finally, A^{op} is the opposite algebra of A.

Quiver representations

A quiver is denoted as a couple $Q = (I, \Omega)$, where I and Ω are its sets (always assumed finite) of vertices and arrows respectively. In the path algebra $\mathbb{K}Q$, arrows are composed like functions, that is h_2h_1 denotes the path obtained following two arrows h_1 and h_2 in this order; e_i denotes the trivial path at vertex $i \in I$. Relations on Q will be encoded with an ideal $J \subset \mathbb{K}Q$. A representation of Q is denoted as a couple (V, f), where $V = \oplus_{i \in I} V_i$ is an I-graded vector space and $f = (f_h)_{h \in \Omega}$ a collection of linear maps between the V_i's; however, often we will leave f implicit. $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q; J)$ is the abelian category of finite-dimensional representations of Q over a field $\mathbb{K}$, bound by the relations J, and $D^b(Q; J) := D^b(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q; J))$. More notation will be introduced in §4.1.

Polynomials

Let $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. For $d \in \mathbb{N}$, $R[t]_d$ and $R[t]_{\leq d}$ are, respectively, the R-submodules of the polynomial ring $R[t]$ consisting of homogeneous polynomials of degree d, and of polynomials of degree at most d. When not specified otherwise, the elements of $R[t]$ are ordered lexicographically starting from the leading coefficient ($\leq$ is the symbol for this order relation). $R[t]_+$ denotes the subsemigroup of polynomials with positive leading coefficient, and $R[t]_{+,\leq d} := R[t]_+ \cap R[t]_{\leq d}$.

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Chapter 2

Exceptional sequences

2.1 Exceptional sequences in triangulated categories

We will review the theory of exceptional sequences on a linear triangulated category, developed in the seminar [Rud90], also following the notes [GK04]. Since in the literature there are plenty of different conventions for these topics, many details and proofs will be included to avoid confusion.

2.1.1 Notation and conventions on triangulated categories

Throughout the whole section, $\mathcal{D}$ denotes a triangulated category, linear over a field $\mathbb{K}$, and $K_0(\mathcal{D})$ its Grothendieck group. We assume also that $\mathcal{D}$ is *Ext-finite*, that is $\oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(X, Y[i])$ is a finite-dimensional vector space for all $X, Y \in \mathcal{D}$. This means that $\mathcal{D}$ has a well-defined Euler form $\chi : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \to \mathbb{Z}$ given by

$$\chi([X],[Y]) := \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{K}} \operatorname{Hom}_{\mathcal{D}}(X, Y[i]) .$$

The quotient

$$K_{\mathrm{num}}(\mathcal{D}) := K_0(\mathcal{D})/\ker\chi$$

by the two-sided null space $\ker\chi$ of χ is called the *numerical Grothendieck group* of $\mathcal{D}$.
Often $\mathcal{D}$ will be the bounded derived category $D^b(\mathcal{A})$ of an Abelian category $\mathcal{A}$, in which case the obvious map $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$ is an isomorphism, and χ is the usual Euler form on $\mathcal{A}$.

Now we introduce some notation often used in this section:

- given $X, Y \in \mathcal{D}$, $\operatorname{Hom}^\bullet(X,Y) \in D^b(\mathbf{Vec}^{\mathrm{fd}}_{\mathbb{K}})$ is the complex of finite-dimensional $\mathbb{K}$-vector spaces with $\operatorname{Hom}^i(X,Y) := \operatorname{Hom}_{\mathcal{D}}(X, Y[i])$ for $i \in \mathbb{Z}$, and trivial differentials;
- given a bounded complex $V^\bullet$ of finite-dimensional $\mathbb{K}$-vector spaces and an object $X \in \mathcal{D}$ we write $V^\bullet \otimes X := \oplus_{i\in\mathbb{Z}} H^i(V^\bullet) \otimes X[-i]$; this can be defined via the adjunctions

$$\begin{aligned} \operatorname{Hom}_{\mathcal{D}}(V^\bullet \otimes X, Y) &\cong \operatorname{Hom}_{D^b(\mathbf{Vec}^{\mathrm{fd}}_{\mathbb{K}})}(V^\bullet, \operatorname{Hom}^\bullet(X,Y)) , \\ \operatorname{Hom}_{D^b(\mathbf{Vec}^{\mathrm{fd}}_{\mathbb{K}})}(\operatorname{Hom}^\bullet(Y,X)^\vee, V^\bullet) &\cong \operatorname{Hom}_{\mathcal{D}}(Y, V^\bullet \otimes X) . \end{aligned} \tag{2.1.1}$$

Moreover, if $\mathfrak{E}, \mathfrak{E}'$ are collections of objects or subcategories of $\mathcal{D}$, then:

- $\langle \mathfrak{E} \rangle$ denotes the smallest strictly full triangulated subcategory containing $\mathfrak{E}$;
- $\langle \mathfrak{E} \rangle_{\mathrm{ext}}$ denotes the smallest strictly full subcategory containing $\mathfrak{E}$ and closed under extensions[1] (but not necessarily under shifts);
- we write $\mathfrak{E} \perp\!\!\!\perp \mathfrak{E}'$ if $\operatorname{Hom}_{\mathcal{D}}(E, E'[i]) = 0$ for all $i \in \mathbb{Z}$, all $E \in \mathfrak{E}$ and all $E' \in \mathfrak{E}'$;

[1] This means that if $X \to Y \to Z \xrightarrow{+1}$ is a distinguished triangle and X, Z belong to $\langle \mathfrak{E} \rangle_{\mathrm{ext}}$, then so does Y.

- $\mathfrak{E}^{\perp\!\!\perp}$ denotes the full subcategory whose objects X satisfy $\mathrm{Hom}_{\mathcal{D}}(E, X[i]) = 0$ for all $i \in \mathbb{Z}$ and all $E \in \mathfrak{E}$; ${}^{\perp\!\!\perp}\mathfrak{E}$ is defined analogously, and both are strictly full triangulated subcategories of $\mathcal{D}$;
- $\mathfrak{E}$ is said to *generate* $\mathcal{D}$ if $\mathfrak{E}^{\perp\!\!\perp} = 0$. In particular, an object $T \in \mathcal{D}$ is called a *generator* if $T^{\perp\!\!\perp} = 0$.

2.1.2 Admissible triangulated subcategories and mutations

In this subsection we will briefly recall the concepts and main properties of admissible subcategories of a triangulated category, and of mutation functors induced by them.

Let $\mathcal{S} \subset \mathcal{D}$ be a strictly full triangulated subcategory. We denote by $\iota_{\mathcal{S}} : \mathcal{S} \to \mathcal{D}$ the inclusion functor.

Definition 2.1.1. We say that $\mathcal{S}$ is:

- *left-admissible* if $\iota_{\mathcal{S}}$ has a left adjoint $\iota^*_{\mathcal{S}} : \mathcal{D} \to \mathcal{S}$;
- *right-admissible* if $\iota_{\mathcal{S}}$ has a right adjoint $\iota^!_{\mathcal{S}} : \mathcal{D} \to \mathcal{S}$;
- *admissible* if it is both left- and right-admissible.

Proposition 2.1.2. *[Bon89, §3] Let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{D}$ be strictly full triangulated subcategories such that $\mathcal{S}_2 \perp\!\!\perp \mathcal{S}_1$. The following are equivalent:*

i) $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \mathcal{D}$;

ii) $\mathcal{S}_2$ *is right-admissible and* $\mathcal{S}_1 = \mathcal{S}_2^{\perp\!\!\perp}$;

iii) $\mathcal{S}_1$ *is left-admissible and* $\mathcal{S}_2 = {}^{\perp\!\!\perp}\mathcal{S}_1$;

iv) for all $Y \in \mathcal{D}$ *there is a distinguished triangle* $X_2 \to Y \to X_1 \xrightarrow{+1}$ *such that* $X_2 \in \mathcal{S}_2$ *and* $X_1 \in \mathcal{S}_1$.

When these conditions are verified, the triangle in the last item is in fact unique (up to isomorphisms), and it is given by

$$\iota_{\mathcal{S}_2}\iota^!_{\mathcal{S}_2} Y \to Y \to \iota_{\mathcal{S}_1}\iota^*_{\mathcal{S}_1} Y \xrightarrow{+1} .$$

Definition 2.1.3. We say that $(\mathcal{S}_1, \mathcal{S}_2)$ is a *semi-orthogonal decomposition* of $\mathcal{D}$ when the equivalent conditions of Prop. 2.1.2 are verified.

Suppose now that $\mathcal{S} \subset \mathcal{D}$ is an admissible triangulated subcategory. Then by Prop. 2.1.2 we have semiorthogonal decompositions $(\mathcal{S}^{\perp\!\!\perp}, \mathcal{S})$ and $(\mathcal{S}, {}^{\perp\!\!\perp}\mathcal{S})$, and this means that the inclusion functors $\iota_{\mathcal{S}^{\perp\!\!\perp}}$ and $\iota_{{}^{\perp\!\!\perp}\mathcal{S}}$ have a left adjoint $\iota^*_{\mathcal{S}^{\perp\!\!\perp}}$ and a right adjoint $\iota^!_{{}^{\perp\!\!\perp}\mathcal{S}}$ respectively.

Definition 2.1.4. The triangulated functors

$$L_{\mathcal{S}} := \iota^*_{\mathcal{S}^{\perp\!\!\perp}} : \mathcal{D} \longrightarrow \mathcal{S}^{\perp\!\!\perp} \subset \mathcal{D}\,, \quad R_{\mathcal{S}} := \iota^!_{{}^{\perp\!\!\perp}\mathcal{S}} : \mathcal{D} \longrightarrow {}^{\perp\!\!\perp}\mathcal{S} \subset \mathcal{D}$$

are called respectively the *left mutation* and *right mutation* with respect to $\mathcal{S}$.

For all $Y \in \mathrm{Ob}(\mathcal{D})$, $L_{\mathcal{S}}Y$ and $R_{\mathcal{S}}Y$ are defined, by Prop. 2.1.2, by the distinguished triangles

$$\iota^!_{\mathcal{S}} Y \to Y \to L_{\mathcal{S}} Y \xrightarrow{+1},$$
$$R_{\mathcal{S}} Y \to Y \to \iota^*_{\mathcal{S}} Y \xrightarrow{+1},$$

where $\iota^*_{\mathcal{S}}, \iota^!_{\mathcal{S}}$ denote again the left and right adjoints of the inclusion $\iota_{\mathcal{S}} : \mathcal{S} \to \mathcal{D}$.

By using the definitions and Prop. 2.1.2 it is easy to check the following properties of mutations:

Lemma 2.1.5. *Let $\mathcal{S} \subset \mathcal{D}$ be admissible. Then:*

1. $L_{\mathcal{S}}\restriction_{\mathcal{S}} = R_{\mathcal{S}}\restriction_{\mathcal{S}} = 0$, $L_{\mathcal{S}}\restriction_{\mathcal{S}^{\perp\!\!\perp}} = \mathrm{Id}$, $R_{\mathcal{S}}\restriction_{{}^{\perp\!\!\perp}\mathcal{S}} = \mathrm{Id}$;

2. $L_{\mathcal{S}}\restriction_{^{\perp\!\!\perp}\mathcal{S}} : {}^{\perp\!\!\perp}\mathcal{S} \xrightarrow{\sim} \mathcal{S}^{\perp\!\!\perp}$ *and* $R_{\mathcal{S}}\restriction_{\mathcal{S}^{\perp\!\!\perp}} : \mathcal{S}^{\perp\!\!\perp} \xrightarrow{\sim} {}^{\perp\!\!\perp}\mathcal{S}$ *are equivalences, quasi-inverse to each other;*

3. *we have isomorphisms*

$$\begin{gathered}\mathrm{Hom}_{\mathcal{D}}(X,Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, L_{\mathcal{S}}Y) \cong \mathrm{Hom}_{\mathcal{D}}(L_{\mathcal{S}}X, L_{\mathcal{S}}Y)\,, \\ \mathrm{Hom}_{\mathcal{D}}(Y,Z) \cong \mathrm{Hom}_{\mathcal{D}}(R_{\mathcal{S}}Y, Z) \cong \mathrm{Hom}_{\mathcal{D}}(R_{\mathcal{S}}Y, R_{\mathcal{S}}Z)\end{gathered}$$

   for all $X \in {}^{\perp\!\!\perp}\mathcal{S}$*,* $Y \in \mathcal{D}$ *and* $Z \in \mathcal{S}^{\perp\!\!\perp}$*;*

4. *if* $\mathcal{S}' \subset \mathcal{D}$ *is also an admissible triangulated subcategory and* $\mathcal{S}' \perp\!\!\perp \mathcal{S}$*, then* $\langle \mathcal{S}, \mathcal{S}' \rangle$ *is also admissible, and*

$$L_{\langle \mathcal{S},\mathcal{S}'\rangle} = L_{\mathcal{S}} \circ L_{\mathcal{S}'}\,, \quad R_{\langle \mathcal{S},\mathcal{S}'\rangle} = R_{\mathcal{S}'} \circ R_{\mathcal{S}}\,. \tag{2.1.2}$$

2.1.3 Exceptional objects and exceptional sequences

Definition 2.1.6. An object $E \in \mathrm{Ob}(\mathcal{D})$ is called *exceptional* when, for all $\ell \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathcal{D}}(E, E[\ell]) = \begin{cases} \mathbb{K} & \text{if} \quad \ell = 0\,, \\ 0 & \text{if} \quad \ell \neq 0\,. \end{cases}$$

Definition 2.1.7. An ordered sequence $\mathfrak{E} = (E_0, ..., E_n)$ of exceptional objects $E_0, ..., E_n \in \mathrm{Ob}(\mathcal{D})$ is called an *exceptional sequence* (or an *exceptional collection*) in $\mathcal{D}$ if

$$\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[\ell]) = 0$$

for all $i > j$ and all $\ell \in \mathbb{Z}$. Moreover, an exceptional sequence $(E_0, ..., E_n)$ is said to be:

- *full*, or *complete*, if $\mathcal{D}$ is the smallest triangulated subcategory containing $E_0, ..., E_n$;
- *strong* if $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[\ell]) = 0$ for all i, j and all $\ell \in \mathbb{Z} \setminus \{0\}$.

The key property of exceptional sequences is that they generate admissible subcategories, as stated by the next proposition. Notice that this also means that checking that $\mathfrak{E}$ is full is equivalent to verifying that $\mathfrak{E}^{\perp\!\!\perp} = 0$, or that ${}^{\perp\!\!\perp}\mathfrak{E} = 0$.

Proposition 2.1.8. *[Bon89, Thm 3.2] Let* $\mathfrak{E} = (E_0, ..., E_n)$ *be an exceptional sequence in* $\mathcal{D}$*. Then the subcategory* $\langle E_0, ..., E_n \rangle \subset \mathcal{D}$ *is admissible.*

In particular, for any exceptional object $E \in \mathcal{D}$, the subcategory $\langle E \rangle$ is admissible, and in this case it is easy to see using Eq. (2.1.1) that the adjoints to the inclusion $\iota_{\langle E \rangle}$ are given by

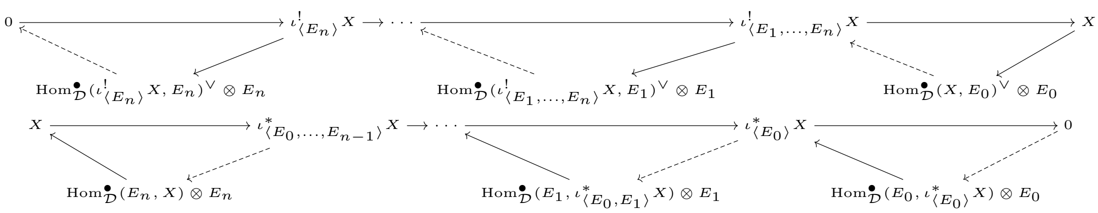
$$\iota^*_{\langle E \rangle} = \mathrm{Hom}^\bullet_{\mathcal{D}}(\,\cdot\,, E)^\vee \otimes_{\mathbb{K}} E\,, \quad \iota^!_{\langle E \rangle} = \mathrm{Hom}^\bullet_{\mathcal{D}}(E, \,\cdot\,) \otimes_{\mathbb{K}} E\,. \tag{2.1.3}$$

In general, the adjoints $\iota^*_{\langle \mathfrak{E} \rangle}, \iota^!_{\langle \mathfrak{E} \rangle}$ are built inductively on the length of $\mathfrak{E}$.

Remark 2.1.9. If the exceptional collection $\mathfrak{E}$ is full, then by using Prop. 2.1.2 we deduce that for any $k \in \{0, ..., n-1\}$ the pair $(\langle E_0, ..., E_k \rangle, \langle E_{k+1}, ..., E_n \rangle)$ is a semi-orthogonal decomposition of $\mathcal{D}$. In particular, for any $X \in \mathcal{D}$ we have distinguished triangles

$$\begin{gathered}\mathrm{Hom}^\bullet_{\mathcal{D}}(E_k, \iota^*_{\langle E_0,...,E_{k+1}\rangle} X) \otimes E_k \to \iota^*_{\langle E_0,...,E_{k+1}\rangle} X \to \iota^*_{\langle E_0,...,E_k\rangle} X \xrightarrow{+1}\,, \\ \iota^!_{\langle E_{k+1},...,E_n\rangle} X \to \iota^!_{\langle E_k,...,E_n\rangle} X \to \mathrm{Hom}^\bullet_{\mathcal{D}}(\iota^!_{\langle E_k,...,E_n\rangle} X, E_k)^\vee \otimes E_k \xrightarrow{+1}\,.\end{gathered}$$

Iterating this we conclude that $\mathfrak{E} = (E_0, ..., E_n)$ induces canonical Postnikov towers (which means that all the triangles appearing are distinguished)



functorial in $X \in \mathcal{D}$. We will see that the Hom spaces appearing in the diagrams can be simplified using dual exceptional sequences (Lemma 2.1.20 and §2.1.6).

Lemma 2.1.10. *If $\mathfrak{E} = (E_0, ..., E_n)$ is a full exceptional sequence, then the classes of the exceptional objects $E_0, ..., E_n$ form a basis of the abelian group $K_0(\mathcal{D})$.*

Thus, in this case, all the full exceptional sequences must have length $n + 1$. Note also that in this basis we have

$$\chi(E_i, E_i) = 1\,, \quad \chi(E_i, E_j) = 0 \text{ for } i > j\,, \tag{2.1.4}$$

which means that χ is nondegenerate, and thus $K_0(\mathcal{D}) = K_{\mathrm{num}}(\mathcal{D}) \simeq \mathbb{Z}^{n+1}$.

Proof. Linear independence of the elements $[E_0], ..., [E_n] \in K_0(\mathcal{D})$ easily follows from Eq. (2.1.4), while they generate $K_0(\mathcal{D})$ because, for any $X \in \mathcal{D}$, the Postnikov systems of Remark 2.1.9 give decompositions

$$[X] = \sum_{k=0}^{n} \chi(\iota^!_{\langle E_k, ..., E_n \rangle} X, E_k)[E_k] = \sum_{k=0}^{n} \chi(E_k, \iota^*_{\langle E_0, ..., E_k \rangle} X)[E_k]\,.$$

□

The following criterion is useful to check fullness of an exceptional sequence on a projective variety:

Lemma 2.1.11. *Let X be a smooth projective variety over $\mathbb{K}$, and let $(E_0, ..., E_n)$ be an exceptional sequence in $D^b(X)$. If for any $x \in X$ the skyscraper sheaf $\mathcal{O}_x$ belongs to $\langle E_0, ..., E_n \rangle$, then the sequence is full.*

Proof. Take a nonzero object $\mathcal{F}^\bullet \in D^b(X)$. Given a point $x \in X$ in the support of the highest nonzero cohomology $H^M(\mathcal{F}^\bullet)$, we have a nonzero morphism

$$\mathcal{F}^\bullet \to H^M(\mathcal{F}^\bullet)[-M] \to \mathcal{O}_x[-M]\,,$$

so $\mathcal{F}^\bullet$ cannot belong to ${}^{\perp\!\!\perp}\langle E_0, ..., E_n \rangle$, which is thus zero. □

Examples 2.1.12.

1 Consider the triangulated category $D^b(\mathbb{P}^n)$. For any $k \in \mathbb{Z}$, the sequence

$$(\mathcal{O}(k), ..., \mathcal{O}(k + n))$$

is a strong exceptional collection, as follows immediately from the usual formulae for the cohomology of line bundles on $\mathbb{P}^n$; it is also full by Lemma 2.1.11, since any skyscraper sheaf $\mathcal{O}_x$ belongs to $\langle \mathcal{O}(k), ..., \mathcal{O}(k + n) \rangle$ (by using the Koszul complex of the section of $\mathcal{O}(1)^{\oplus n}$ vanishing on x). We will show in Ex. 2.1.22.1 that

$$(\Omega^n(k + n), \Omega^{n-1}(k + n - 1), ..., \Omega^1(k + 1), \mathcal{O}(k))\,,$$
$$(\mathcal{O}(k + n), \tau(k + n - 1), \wedge^2\tau(k + n - 2), ..., \wedge^{n-1}\tau(k + 1), \wedge^n\tau(k))\,.$$

are also full strong exceptional collections.

2 Let $X := \mathbb{P}^1 \times \mathbb{P}^1$. Then

$$(\mathcal{O}_X, \mathcal{O}_X(0, 1), \mathcal{O}_X(1, 0), \mathcal{O}_X(1, 1)) \tag{2.1.5}$$

is a strong, full exceptional sequence in $D^b(X)$. Again, the formulae for cohomology of line bundles on $\mathbb{P}^1$ give strong exceptionality of this sequence, while fullness follows from Lemma 2.1.11, as any point $x \in X$ is the zero of a section of $\mathcal{O}_X(1, 0) \oplus \mathcal{O}_X(0, 1)$, whose corresponding Koszul resolution twisted by $\mathcal{O}_X(1, 1)$ is

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(0, 1) \oplus \mathcal{O}_X(1, 0) \to \mathcal{O}_X(1, 1) \to \mathcal{O}_x \to 0\,.$$

More generally, we observe that any smooth projective rational surface (being a blow-up of $\mathbb{P}^2$ or a Hirzebruch surface Σ_e at finitely many points) has a full exceptional collection:

- [KN90] for any $e \in \mathbb{N}$, the eth Hirzebruch surface $\Sigma_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ has a full strong exceptional collection (generalizing Ex. 2.1.12.2)

$$(\mathcal{O}_{\Sigma_e}, \mathcal{O}_{\Sigma_e}(0,1), \mathcal{O}_{\Sigma_e}(1,0), \mathcal{O}_{\Sigma_e}(1,1)),$$

  where $\mathcal{O}_{\Sigma_e}(a,b) := \mathcal{O}_{\Sigma_e}(aH + bF)$, and $H, F \subset \Sigma_e$ are respectively the relative hyperplane divisor such that $H^2 = e$ and a fiber of the projective bundle $\Sigma_e \to \mathbb{P}^1$.

- [HP14, Prop. 2.4] Let S be a smooth projective surface, $p \in S$, and consider the blow-up $\pi : \mathrm{Bl}_p S \to S$ at p and its exceptional divisor E. By applying Orlov's blow-up formula [Orl92] we could show that if S has a full exceptional sequence $\mathfrak{E} = (\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_k)$ of line bundles, then for all $i \in \{1, ..., k\}$, we have a full exceptional sequence on $\mathrm{Bl}_p S$, called the *standard augmentation* of $\mathfrak{E}$ at position i:

$$\mathfrak{E}_i = (\pi^*\mathcal{L}_1(E), \pi^*\mathcal{L}_2(E), ..., \pi^*\mathcal{L}_{i-1}(E), \pi^*\mathcal{L}_i, \pi^*\mathcal{L}_i(E), \pi^*\mathcal{L}_{i+1}, ..., \pi^*\mathcal{L}_k).$$

In fact, it is expected that the only smooth projective surfaces admitting a full exceptional collection are rational (see e.g. [EL15]). This has been proven in [BS17] under the assumption that the collection is strong and consists of line bundles.

2.1.4 Mutations by exceptional objects

We have seen in Prop. 2.1.8 that the subcategory generated by an exceptional sequence is admissible, so it defines left and right mutation functors. In particular, we can consider mutations at a single exceptional object:

Definition 2.1.13. Let $E \in \mathcal{D}$ be an exceptional object. Then we define the *left mutation* and *right mutation* functors with respect to E as

$$L_E := L_{\langle E \rangle} : \mathcal{D} \longrightarrow E^{\perp\!\!\perp} \subset \mathcal{D}, \quad R_E := R_{\langle E \rangle} : \mathcal{D} \longrightarrow {}^{\perp\!\!\perp}E \subset \mathcal{D}.$$

This means that, for all $Y \in \mathcal{D}$, $L_E Y$ and $R_E Y$ are defined by distinguished triangles

$$\mathrm{Hom}_{\mathcal{D}}(E, Y) \otimes_{\mathbb{K}} E \to Y \to L_E Y \xrightarrow{+1}, \quad R_E Y \to Y \to \mathrm{Hom}_{\mathcal{D}}(Y, E)^\vee \otimes_{\mathbb{K}} E \xrightarrow{+1}, \tag{2.1.6}$$

having used the expressions (2.1.3) for the adjoints $\iota^*_{\langle E \rangle}$ and $\iota^!_{\langle E \rangle}$. Often it is useful to use a slightly different definition of these mutations: we let

$$\tilde{L}_E Y := L_E Y[-1], \quad \tilde{R}_E Y := R_E Y[1].$$

According to Eq. (2.1.2), if $(E_0, ..., E_n)$ is an exceptional sequence, then

$$\begin{array}{rcl} L_{\langle E_0, ..., E_n \rangle} & = & L_{E_0} \circ \cdots \circ L_{E_n} \\ & = & \tilde{L}_{E_0} \circ \cdots \circ \tilde{L}_{E_n}[n+1] \\ R_{\langle E_0, ..., E_n \rangle} & = & R_{E_n} \circ \cdots \circ R_{E_0} \\ & = & \tilde{R}_{E_n} \circ \cdots \circ \tilde{R}_{E_0}[-n-1] \end{array}. \tag{2.1.7}$$

Notice that the functors L_E, R_E induce orthogonal (with respect to the Euler form χ) projectors in the Grothendieck group $K_0(\mathcal{D})$, onto $[E]^\perp$ and ${}^\perp[E]$ respectively:

$$[L_E Y] = [Y] - \chi(E, Y)[E], \qquad [R_E Y] = [Y] - \chi(Y, E)[E].$$

The importance of these functors is that they can be used to build new exceptional sequences from old ones, as explained in the following Proposition:

Proposition 2.1.14.

1. *Let $E \in \mathcal{D}$ be an exceptional object and $\mathcal{S} \subset \mathcal{D}$ be an admissible triangulated subcategory. If $E \in {}^{\perp\!\!\perp}\mathcal{S}$, then $L_{\mathcal{S}}E$ is exceptional; If $E \in \mathcal{S}^{\perp\!\!\perp}$, then $R_{\mathcal{S}}E$ is exceptional.*

2. *If (E_0, E_1) is an exceptional sequence, then so are $(L_{E_0}E_1, E_0)$ and $(E_1, R_{E_1}E_0)$; moreover we have*
$$R_{E_0}L_{E_0}E_1 \simeq E_1\,, \quad L_{E_1}R_{E_1}E_0 \simeq E_0 \tag{2.1.8}$$
and
$$\langle L_{E_0}E_1, E_0\rangle = \langle E_0, E_1\rangle = \langle E_1, R_{E_1}E_0\rangle\,.$$

3. *If $\mathfrak{E} = (E_0, ..., E_n)$ is an exceptional sequence, then for $i \in \{1, ..., n\}$ so are*
$$\begin{aligned} \mathtt{L}_i(E_0, ..., E_n) &:= (E_0, ..., E_{i-2}, L_{E_{i-1}}E_i, E_{i-1}, E_{i+1}, ..., E_n)\,,\\ \mathtt{R}_i(E_0, ..., E_n) &:= (E_0, ..., E_{i-2}, E_i, R_{E_i}E_{i-1}, E_{i+1}, ..., E_n)\,, \end{aligned} \tag{2.1.9}$$
and these are full if $\mathfrak{E}$ is.

Proof. By item 2 of Lemma 2.1.5, $L_{\mathcal{S}}$ and $R_{\mathcal{S}}$ are quasi-inverses when restricted to ${}^{\perp\!\!\perp}\mathcal{S}$ and $\mathcal{S}^{\perp\!\!\perp}$ respectively. This fact implies item 1 and also Eq. (2.1.8) by taking $\mathcal{S} = \langle E_0\rangle$ or $\mathcal{S} = \langle E_1\rangle$. The fact that $L_{E_0}E_1$ and $R_{E_1}E_0$ are defined through distinguished triangles as in Eq. (2.1.6) immediately gives
$$\langle L_{E_0}E_1, E_0\rangle \subset \langle E_0, E_1\rangle \supset \langle E_1, R_{E_1}E_0\rangle\,,$$
and then Eq. (2.1.8) gives the opposite inclusions analogously. Finally, item 3 follows immediately from these considerations. □

Definition 2.1.15. The exceptional sequences $\mathtt{L}_i\mathfrak{E}$ and $\mathtt{R}_i\mathfrak{E}$ defined in Eq. (2.1.9) are called respectively i^{th} *left mutation* and i^{th} *right mutation* of $\mathfrak{E}$, for $i \in \{1, ..., n\}$.

Again, we also introduce the alternative notation
$$\begin{aligned} \tilde{\mathtt{L}}_i(E_0, ..., E_n) &:= (E_0, ..., E_{i-2}, \tilde{L}_{E_{i-1}}E_i, E_{i-1}, E_{i+1}, ..., E_n)\,,\\ \tilde{\mathtt{R}}_i(E_0, ..., E_n) &:= (E_0, ..., E_{i-2}, E_i, \tilde{R}_{E_i}E_{i-1}, E_{i+1}, ..., E_n)\,, \end{aligned}$$

So we have defined operations $\mathtt{L}_i, \mathtt{R}_i$ on the set of (full) exceptional sequences of $\mathcal{D}$. We conclude with some properties of these operations:

Proposition 2.1.16.

1. *We have*
$$\begin{aligned} \mathtt{L}_1\mathtt{L}_2\cdots\mathtt{L}_n(E_0, ..., E_n) &= (L_{\langle E_0,...,E_{n-1}\rangle}E_n, E_0, ..., E_{n-1})\,,\\ \mathtt{R}_n\mathtt{R}_{n-1}\cdots\mathtt{R}_1(E_0, ..., E_n) &= (E_1, ..., E_n, R_{\langle E_1,...,E_n\rangle}E_0)\,; \end{aligned} \tag{2.1.10}$$

2. *The transformations $\mathtt{L}_i, \mathtt{R}_i$ define an action of the braid group B_{n+1} on the set of (full) exceptional sequences:*
$$\begin{aligned} \mathtt{L}_i\mathtt{R}_i = \mathtt{R}_i\mathtt{L}_i &= \mathrm{Id} && \text{for all } i \in \{1, ..., n\}\,,\\ \mathtt{L}_i\mathtt{L}_j &= \mathtt{L}_j\mathtt{L}_i && \text{if } |i-j| > 1\,,\\ \mathtt{L}_{i+1}\mathtt{L}_i\mathtt{L}_{i+1} &= \mathtt{L}_i\mathtt{L}_{i+1}\mathtt{L}_i && \text{for all } i \in \{1, ..., n\}\,. \end{aligned}$$

Proof. All the statements are immediate consequences of Prop. 2.1.14 and Eq. (2.1.7): in particular, the third braid relation follows from the observation that
$$L_{L_{E_{i-i}}E_i}L_{E_{i-1}} = L_{\langle L_{E_{i-i}}E_i, E_{i-1}\rangle} = L_{\langle E_{i-1}, E_i\rangle} = L_{E_{i-1}}L_{E_i}\,.$$
□

Remark 2.1.17. If we assume the existence of a full exceptional sequence $\mathfrak{E} = (E_0, ..., E_n)$, then the category $\mathcal{D}$ has a Serre functor $S_{\mathcal{D}}$, and this can be used to simplify the mutations (2.1.10), as for example
$$S_{\mathcal{D}}(E_n) \simeq L_{\langle E_0,...,E_{n-1}\rangle}E_n\,.$$
The details can be found e.g. in [BS10, §2.6].

2.1.5 Dual exceptional sequences

Let $\mathfrak{E} = (E_0, ..., E_n)$ be a full exceptional sequence in $\mathcal{D}$.

Definition 2.1.18. We say that:

- a sequence ${}^\vee\mathfrak{E} = ({}^\vee E_n, ..., {}^\vee E_0)$ such that

$$\mathrm{Hom}_{\mathcal{D}}({}^\vee E_i, E_j[\ell]) = \begin{cases} \mathbb{K} & \text{if } j = i = n - \ell \\ 0 & \text{otherwise} \end{cases}$$

  is *left dual* to $(E_0, ..., E_n)$;

- a sequence $\mathfrak{E}^\vee = (E_n^\vee, ..., E_0^\vee)$ such that

$$\mathrm{Hom}_{\mathcal{D}}(E_i, E_j^\vee[\ell]) = \begin{cases} \mathbb{K} & \text{if } j = i = \ell \\ 0 & \text{otherwise} \end{cases}$$

  is *right dual* to $(E_0, ..., E_n)$.

Explicitly, we are requiring that the only nonzero Homs are

$$\mathrm{Hom}_{\mathcal{D}}({}^\vee E_n, E_n) = \mathrm{Hom}_{\mathcal{D}}({}^\vee E_{n-1}, E_{n-1}[1]) = \cdots = \mathrm{Hom}_{\mathcal{D}}({}^\vee E_0, E_0[n]) = \mathbb{K}\,,$$
$$\mathrm{Hom}_{\mathcal{D}}(E_0, E_0^\vee) = \mathrm{Hom}_{\mathcal{D}}(E_1, E_1^\vee[1]) = \cdots = \mathrm{Hom}_{\mathcal{D}}(E_n, E_n^\vee[n]) = \mathbb{K}\,.$$

Notice that, by definition, the sequence $\mathfrak{E}$ is left dual to $\mathfrak{E}^\vee$ and right dual to ${}^\vee\mathfrak{E}$. In the remainder of this subsection we will prove that:

Proposition 2.1.19. *Both left and right dual sequences to $\mathfrak{E}$ exist, are unique up to isomorphisms, and they are full exceptional sequences.*

Uniqueness follows from the following Lemma, which shows in particular that the elements ${}^\vee E_k$ and $E_k^\vee$ represent a covariant and a contravariant functor respectively, and as such are uniquely determined up to isomorphism:

Lemma 2.1.20. *For all $k \in \{0, ..., n\}$ we have natural isomorphisms*

$$\mathrm{Hom}^\bullet({}^\vee E_k, \cdot) \cong \mathrm{Hom}^\bullet(E_k, \iota^*_{\langle E_0,...,E_k\rangle}(\cdot)[k-n])\,,$$
$$\mathrm{Hom}^\bullet(\cdot, E_k^\vee) \cong \mathrm{Hom}^\bullet(\iota^!_{\langle E_k,...,E_n\rangle}(\cdot)[k], E_k)\,,$$
$$\mathrm{Hom}^\bullet({}^\vee E_k, \cdot)[n] \cong \mathrm{Hom}^\bullet(\cdot, E_k^\vee)^\vee\,.$$

Proof. We have the following isomorphisms, natural in $X \in \mathcal{D}$:

$$\begin{aligned} \mathrm{Hom}^\bullet(E_k, \iota^*_{\langle E_0,...,E_k\rangle}X[k-n]) &\cong \mathrm{Hom}^\bullet(E_k, \iota^*_{\langle E_0,...,E_k\rangle}(X)) \otimes \mathbb{K}[k-n] \\ &\cong \mathrm{Hom}^\bullet(E_k, \iota^*_{\langle E_0,...,E_k\rangle}X) \otimes \mathrm{Hom}^\bullet({}^\vee E_k, E_k) \\ &\cong \mathrm{Hom}^\bullet({}^\vee E_k, \mathrm{Hom}^\bullet(E_k, \iota^*_{\langle E_0,...,E_k\rangle}X) \otimes E_k) \quad . \\ &\cong \mathrm{Hom}^\bullet({}^\vee E_k, \iota^*_{\langle E_0,...,E_k\rangle}X) \\ &\cong \mathrm{Hom}^\bullet({}^\vee E_k, X) \end{aligned}$$

In the last two steps, we have applied the functor $\mathrm{Hom}^\bullet({}^\vee E_k, \cdot)$ to the first triangle of Remark 2.1.9 and to the triangle

$$\iota^!_{\langle E_{k+1},...,E_n\rangle}X \to X \to \iota^*_{\langle E_0,...,E_k\rangle}X \xrightarrow{+1}\,,$$

and we have used that ${}^\vee E_k \in {}^{\perp\!\!\perp}\langle E_0, ..., E_{k-1}\rangle$ and ${}^\vee E_k \in {}^{\perp\!\!\perp}\langle E_{k+1}, ..., E_n\rangle$. This proves the first isomorphism. The proof of the second is analogous, by applying $\mathrm{Hom}(\cdot, E_k^\vee)$ to the second triangle in Remark 2.1.9; the third isomorphism follows easily from applying $\mathrm{Hom}(\cdot, E_k^\vee)$ to the first triangle of the same Remark. ☐

Finally, the following Lemma proves the existence and full exceptionality of $\mathfrak{E}^\vee$ and ${}^\vee\mathfrak{E}$:

Lemma 2.1.21. *The full exceptional sequence*

$$\begin{aligned}&\tilde{L}_n(\tilde{L}_{n-1}\tilde{L}_n)(\tilde{L}_{n-2}\tilde{L}_{n-1}\tilde{L}_n)\cdots(\tilde{L}_1\tilde{L}_2\cdots\tilde{L}_n)\mathfrak{E}\\ &\quad=(\tilde{L}_{E_0}\cdots\tilde{L}_{E_{n-1}}E_n,\tilde{L}_{E_0}\cdots\tilde{L}_{E_{n-2}}E_{n-1},...,\tilde{L}_{E_0}E_1,E_0)\\ &\quad=(L_{\langle E_0,...,E_{n-1}\rangle}E_n[-n],L_{\langle E_0,...,E_{n-2}\rangle}E_{n-1}[-n+1],...,L_{\langle E_0\rangle}E_1[-1],E_0)\end{aligned}$$

is *right* *dual to* $\mathfrak{E}$*, while the full exceptional sequence*

$$\begin{aligned}&\tilde{R}_1(\tilde{R}_2\tilde{R}_1)(\tilde{R}_3\tilde{R}_2\tilde{R}_1)\cdots(\tilde{R}_n\cdots\tilde{R}_2\tilde{R}_1)\mathfrak{E}\\ &\quad=(E_n,\tilde{R}_{E_n}E_{n-1},...,\tilde{R}_{E_n}\cdots\tilde{R}_{E_2}E_1,\tilde{R}_{E_n}\cdots\tilde{R}_{E_1}E_0)\\ &\quad=(E_n,R_{\langle E_n\rangle}E_{n-1}[1],...,R_{\langle E_2,...,E_n\rangle}E_1[n-1],R_{\langle E_1,...,E_n\rangle}E_0[n])\end{aligned}$$

is *left* *dual to* $\mathfrak{E}$*.*

Proof. To check the explicit forms of these iterated mutations we can use Eq. (2.1.10) several times and Eq. (2.1.7). To prove the first statement, let $\mathcal{S}:=\langle E_0,...,E_{j-1}\rangle$. The jth term of the mutated sequence, counting from the right and starting from 0 is then $L_{\mathcal{S}}E_j[-j]$. Since $E_j\in{}^{\perp\!\!\perp}\mathcal{S}$, we have

$$\begin{aligned}\mathrm{Hom}(E_i,L_{\mathcal{S}}E_j[-j][\ell])&=\mathrm{Hom}(R_{\mathcal{S}}E_i,R_{\mathcal{S}}L_{\mathcal{S}}E_j[\ell-j])\\ &=\mathrm{Hom}(R_{\mathcal{S}}E_i,E_j[\ell-j])\\ &=\begin{cases}\mathrm{Hom}(0,E_j[\ell-j])=0 & \text{if } i<j\\ \mathrm{Hom}(E_i,E_j[\ell-j])=\mathbb{K} & \text{if } i=j=\ell\\ \mathrm{Hom}(E_i,E_j[\ell-j])=0 & \text{otherwise}\end{cases},\end{aligned}$$

having used the properties of the mutation functors listed in Lemma 2.1.5. The second statement is proven analogously after taking $\mathcal{S}:=\langle E_{i+1},...,E_n\rangle$. □

Examples 2.1.22.

1 Consider the triangulated category $D^b(\mathbb{P}^n)$. We saw in Ex. 2.1.12.1 that

$$\mathfrak{E}=(\mathcal{O},\mathcal{O}(1),...,\mathcal{O}(n))$$

is a full strong exceptional collection. Now we will compute its right and left duals: we claim that these are[2]

$$\begin{aligned}\mathfrak{E}^\vee&=(\Omega^n(n),\Omega^{n-1}(n-1),...,\Omega^1(1),\mathcal{O})\,,\\ {}^\vee\mathfrak{E}&=(\mathcal{O}(n),\tau(n-1),\wedge^2\tau(n-2),...,\wedge^{n-1}\tau(1),\wedge^n\tau)\,,\end{aligned}$$

and that they are both strong. First of all, recall the Euler exact sequence

$$0\to\Omega^1(1)\to\mathcal{O}\otimes Z^\vee\to\mathcal{O}(1)\to 0\,,$$

having written $\mathbb{P}^n:=\mathbb{P}(Z)$. This induces, by taking exterior powers, the exact sequence

$$0\to\Omega^k(k)\to\wedge^k Z^\vee\otimes\mathcal{O}\to\Omega^{k-1}(k)\to 0\,. \tag{2.1.11}$$

The third arrow can be identified with the evaluation map $\mathrm{Hom}^\bullet(\mathcal{O},\Omega^{k-1}(k))\otimes\mathcal{O}\to\Omega^{k-1}(k)$, whose cone is thus $L_{\mathcal{O}}\Omega^{k-1}(k)\simeq\Omega^k(k)[1]$. This implies that

$$L_{\langle\mathcal{O},...,\mathcal{O}(k-1)\rangle}\mathcal{O}(k)\simeq\Omega^k(k)[k]\,.$$

Indeed, by induction on k,

$$\begin{aligned}L_{\langle\mathcal{O},...,\mathcal{O}(k-1)\rangle}\mathcal{O}(k)&\simeq L_{\mathcal{O}}\left(L_{\langle\mathcal{O},...,\mathcal{O}(k-2)\rangle}\mathcal{O}(k-1)\right)(1)\\ &\simeq L_{\mathcal{O}}\Omega^{k-1}(k)[k-1]\\ &\simeq\Omega^k(k)[k]\end{aligned}$$

[2] One could just check that these are the dual sequences by using the definitions and Bott's formula. Instead, we deduce them by using mutations.

This ends the computation of $\mathfrak{E}^\vee$. To find ${}^\vee\mathfrak{E}$ we can proceed similarly: using the dual of the sequence (2.1.11) we get $R_{\mathcal{O}(n)} \wedge^{n-k-1} \tau(k) \simeq \wedge^{n-k}\tau(k)[-1]$, which by induction on k gives

$$R_{\langle \mathcal{O}(k+1),\dots,\mathcal{O}(n)\rangle}\mathcal{O}(k) \simeq \wedge^{n-k}\tau(k)[k-n]\,.$$

Finally, the fact that the sequence $\mathfrak{E}^\vee$ is strong follows from standard computations: applying $\mathrm{Ext}^\ell(\cdot,\Omega^{k'}(k'))$ to the exact sequence (2.1.11) and using that $H^\ell(\mathbb{P}^n;\Omega^{k'}(k')) = 0$ for all $\ell > 0$ we see that

$$\mathrm{Ext}^\ell(\Omega^k(k),\Omega^{k'}(k')) \simeq \mathrm{Ext}^{\ell+1}(\Omega^{k-1}(k),\Omega^{k'}(k')) \simeq \cdots \simeq H^{\ell+k}(\mathbb{P}^n;\Omega^{k'}(k'-k)) = 0$$

for all $\ell > 0$ and $k, k' \in \{0,\dots,n\}$. The sequence ${}^\vee\mathfrak{E}$ is also strong, being equal to $\mathfrak{E}^\vee$ twisted by $\mathcal{O}(n+1)$.

2 Let $X := \mathbb{P}^1 \times \mathbb{P}^1$. By Ex. 2.1.12.2 we have a full strong exceptional collection

$$\mathfrak{E} = (\mathcal{O},\mathcal{O}(0,1),\mathcal{O}(1,0),\mathcal{O}(1,1))\,.$$

In this case the dual collections are given by

$$\begin{aligned}\mathfrak{E}^\vee &= (\mathcal{O}(-1,-1)[-1],\mathcal{O}(-1,0)[-1],\mathcal{O}(0,-1),\mathcal{O})\,,\\ {}^\vee\mathfrak{E} &= (\mathcal{O}(1,1),\mathcal{O}(1,2),\mathcal{O}(2,1)[1],\mathcal{O}(2,2)[1])\,.\end{aligned}$$

Indeed, in this case the Euler sequence on the first $\mathbb{P}^1 = \mathbb{P}(Z)$ factor induces an exact sequence

$$0 \to \mathcal{O}(-1,0) \to Z^\vee \otimes \mathcal{O} \to \mathcal{O}(1,0) \to 0\,,$$

which as in the previous example implies that

$$L_{\mathcal{O}}\mathcal{O}(1,0) \simeq \mathcal{O}(-1,0)[1]\,.$$

An analogous result holds exchanging the $\mathbb{P}^1$ factors. Using also the fact that $L_{\mathcal{O}(0,1)}\mathcal{O}(1,0) \simeq \mathcal{O}(1,0)$ (since $\mathcal{O}(0,1) \perp\!\!\!\perp \mathcal{O}(1,0)$) we conclude that

$$\begin{aligned}L_{\langle \mathcal{O},\mathcal{O}(0,1)\rangle}\mathcal{O}(1,0) &\simeq L_{\mathcal{O}}\mathcal{O}(1,0) \simeq \mathcal{O}(-1,0)[1]\,,\\ L_{\langle \mathcal{O},\mathcal{O}(0,1),\mathcal{O}(1,0)\rangle}\mathcal{O}(1,1) &\simeq L_{\langle \mathcal{O},\mathcal{O}(0,1)\rangle}\mathcal{O}(1,-1)[1]\\ &\simeq L_{L_{\mathcal{O}}\mathcal{O}(0,1)}L_{\mathcal{O}}\mathcal{O}(1,-1)[1]\\ &\simeq L_{\mathcal{O}(0,-1)[1]}\mathcal{O}(1,-1)[1]\\ &\simeq \mathcal{O}(-1,-1)[2]\,.\end{aligned}$$

The computation of ${}^\vee\mathfrak{E}$ is analogous. Note that the sequences $\mathfrak{E}^\vee$ and ${}^\vee\mathfrak{E}$ are not strong, as for example

$$\mathrm{Hom}^{-1}(\mathcal{O}(-1,-1)[-1],\mathcal{O}(-1,0)) = H^0(X;\mathcal{O}(0,1)) \neq 0\,.$$

3 Let Q be an ordered quiver with relations J, whose vertices are labeled by $0,1,\dots,n$ (this means that there are no arrows from i to j if $j \le i$). Then we have full exceptional collections $\mathfrak{E}, {}^\vee\mathfrak{E}$ on the bounded derived category $D^b(Q;J) := D^b(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q;J))$ made by the objects

$$E_i = S(i)[i-n]\,,\quad {}^\vee E_i = P(i)\,,$$

where $S(i)$ and $P(i)$ denote the standard simple and projective representations associated to each vertex i. Moreover, the collection ${}^\vee\mathfrak{E} = (P(n),\dots,P(0))$ is obviously strong and it is left dual to $\mathfrak{E}$ because of the formula

$$\mathrm{Ext}^\ell(P(i),S(j)) = \begin{cases}\mathbb{C} & \text{if } i=j,\ \ell=0\\ 0 & \text{otherwise}\end{cases}\,.$$

Note also that we have

$$\mathrm{Hom}(P(j),P(i)) \cong e_j\,({}^{\mathbb{K}Q}\!/\!{}_J)\,e_i$$

(e_i denotes the trivial path at vertex $i \in I$) and this vector space consists of the paths from i to j, modulo those belonging to J. Hence

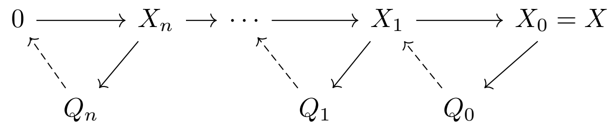
$$({}^{\mathbb{K}Q}\!/\!{}_J)^{\mathrm{op}} \cong \begin{pmatrix}\mathrm{Hom}(P(n),P(n)) & & \\ \vdots & \ddots & \\ \mathrm{Hom}(P(n),P(0)) & \cdots & \mathrm{Hom}(P(0),P(0))\end{pmatrix}\,.$$

The full strong collection made by the projective representations of an ordered quiver considered in this last example is somehow prototypical: we will see in §2.2.2 that (at least when $\mathcal{D}$ is a bounded derived category of coherent sheaves) a full strong exceptional collection determines an ordered quiver with relations (Q, J) and an equivalence $\mathcal{D} \simeq D^b(Q; J)$ mapping the elements of the collection to the standard projective representations.

2.1.6 Generalized Beilinson spectral sequences

Let $\mathfrak{E} = (E_0, ..., E_n)$ be a full exceptional sequence in $\mathcal{D}$.

We have seen in Remark 2.1.9 that any object X induces a canonical Postnikov tower



functorial in $X \in \mathcal{D}$, where the "graded pieces" Q_p can be simplified using the left dual collection ${}^\vee\mathfrak{E} = ({}^\vee E_n, ..., {}^\vee E_0)$ and Lemma 2.1.20:

$$Q_p = \mathrm{Hom}^\bullet_{\mathcal{D}}(\iota^!_{\langle E_p,...,E_n\rangle} X, E_p)^\vee \otimes E_p \cong \mathrm{Hom}^\bullet_{\mathcal{D}}({}^\vee E_p, X[n-p]) \otimes E_p \tag{2.1.12}$$

Recall (see e.g. [GM13, page 262]) that if $H^0 : \mathcal{D} \to \mathcal{A}$ is a cohomological functor valued in some abelian category $\mathcal{A}$, then we can relate the objects $H^k(X) := H^0(X[k])$ and $H^k(Q_p)$ via a spectral sequence

$$E_1^{p,q} = H^q(Q_p[p]) \Rightarrow H^{p+q}(X)\,.$$

Substituting Eq. (2.1.12) and relabeling we get:

Lemma 2.1.23. *For any object $X \in \mathcal{D}$ we have a spectral sequence*

$$E_1^{p,q} = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}^i_{\mathcal{D}}({}^\vee E_{p+n}, X) \otimes H^{q-i}(E_{p+n}) \Rightarrow H^{p+q}(X)\,.$$

Remark 2.1.24. If $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure (see §3.3.1), and the exceptional objects E_p are all contained in $\mathcal{A}$, then the Lemma gives a spectral sequence

$$E_1^{p,q} = \mathrm{Hom}^q_{\mathcal{D}}({}^\vee E_{p+n}, X) \otimes E_{p+n} \Rightarrow H^{p+q}_{\mathcal{A}}(X)\,. \tag{2.1.13}$$

Example 2.1.25. Let $\mathcal{D} = D^b(\mathbb{P}^n)$. First we apply Lemma 2.1.23 to the exceptional collection $\mathfrak{E} = (\mathcal{O}(-n), \mathcal{O}(-n+1), ..., \mathcal{O})$, so that $E_{p+n} = \mathcal{O}(p)$ and ${}^\vee E_{p+n} = \wedge^{-p}\tau(p)$ (see Ex. 2.1.22.1), and we get for any $\mathcal{F}^\bullet \in D^b(\mathbb{P}^n)$ the spectral sequence

$$E_1^{p,q} = \mathbb{H}^q(\mathbb{P}^n; \mathcal{F}^\bullet \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}(p) \Rightarrow H^{p+q}(\mathcal{F}^\bullet)\,.$$

If we use instead the sequence $\mathfrak{E} = (\Omega^n(n), \Omega^{n-1}(n-1), ..., \Omega^1(1), \mathcal{O})$, then $E_{p+n} = \Omega^{-p}(-p)$ and ${}^\vee E_{p+n} = \mathcal{O}(-p)$. Hence we get another spectral sequence,

$$E_1^{p,q} = \mathbb{H}^q(\mathbb{P}^n; \mathcal{F}^\bullet(p)) \otimes \Omega^{-p}(-p) \Rightarrow H^{p+q}(\mathcal{F}^\bullet)\,.$$

These two spectral sequences are commonly referred to as the *Beilinson spectral sequences.*

2.2 Exceptional sequences and derived equivalences

In this section we will recall the famous result, found in [Bae88, Ric89, Bon89], that a derived category of sheaves on a smooth projective variety generated by a full strong exceptional sequence is equivalent to the derived category of representations of a quiver with relations. This is proven in §2.2.2. However, it will be useful to have a relative version of this fact, which is the reason why, following e.g. [TU10, §3], we introduce in §2.2.1 the concept of *tilting generator* on a projective scheme over a finitely generated $\mathbb{K}$-algebra, and we obtain an analogous derived equivalence in this more general setting.

2.2.1 Tilting generators and derived equivalences

Let Y be an algebraic $\mathbb{K}$-scheme, with a projective morphism

$$f : Y \to S$$

to an affine $\mathbb{K}$-scheme $S = \operatorname{Spec} R$ of finite type. The Hom spaces in $\mathbf{Coh}_{\mathcal{O}_Y}$ are finitely generated R-modules because R is Noetherian, and there are well-defined Ext R-modules. For any $T \in \mathbf{Coh}_{\mathcal{O}_Y}$, $\operatorname{End}(T)$ is a finite[3] R-algebra, and the functor $\operatorname{Hom}(T, \cdot)$ takes values in the abelian category of finitely generated right $\operatorname{End}(T)$-modules. Moreover, when T is locally free, this functor has a right derived

$$\Phi_T := R\operatorname{Hom}(T, \cdot) : D^*(Y) \to D^*(\mathbf{Mod}^{\mathrm{fg}}_{\operatorname{End}(T)}) ,$$

for any $* \in \{\_, +, -, b\}$, and a left adjoint $(\cdot) \otimes_{\operatorname{End}(T)} T$, which has a left derived

$$\Xi_L = (\cdot) \otimes^L_{\operatorname{End}(T)} T : D^-(\mathbf{Mod}^{\mathrm{fg}}_{\operatorname{End}(T)}) \to D^-(Y) ,$$

left adjoint to $\Phi_T : D^-(Y) \to D^-(\mathbf{Mod}^{\mathrm{fg}}_R)$.

Definition 2.2.1. A locally free sheaf T on Y is called a *tilting generator* if it is a generator (in the sense of §2.1.1) of $D^-(Y)$, and $\operatorname{Ext}^i(T, T) = 0$ for all $i \geq 1$.[4]

Example 2.2.2. The most interesting situation for us is when S is a point, Y is smooth (so that $D^b(Y)$ is Ext-finite) and $D^b(Y)$ has a full strong exceptional sequence $\mathfrak{E} = (E_0, ..., E_n)$ of vector bundles: in this case $T := \oplus_{i=0}^n E_i$ is a tilting generator and the algebra $\operatorname{End}(T)$ can be naturally described as a bound quiver algebra. This is discussed in more detail in the next subsection.

Theorem 2.2.3. *[TU10, §3]* [5] *If T is a tilting generator, then the triangulated functors*

$$\Phi_T : D^-(Y) \longleftrightarrow D^-(\mathbf{Mod}^{\mathrm{fg}}_{\operatorname{End}(T)}) : \Xi_T$$

are quasi-inverses, and they restrict to an equivalence $D^b(Y) \simeq D^b(\mathbf{Mod}^{\mathrm{fg}}_{\operatorname{End}(T)})$.

Proof. Consider the unit and the counit

$$\operatorname{Id} \overset{\epsilon}{\Rightarrow} \Phi_T \Xi_T , \quad \Xi_T \Phi_T \overset{\eta}{\Rightarrow} \operatorname{Id}$$

of the adjunction $\Xi_T \dashv \Phi_T$. The unit is an isomorphism, as for any $M^\bullet \in D^-(\mathbf{Mod}^{\mathrm{fg}}_{\operatorname{End}(T)})$ we have

$$\begin{aligned} \Phi_T \Xi_T(M^\bullet) &= R\operatorname{Hom}(T, M^\bullet \otimes^L_{\operatorname{End}(T)} T) \\ &\cong M^\bullet \otimes^L_{\operatorname{End}(T)} R\operatorname{Hom}(T, T) \\ &\cong M^\bullet \otimes^L_{\operatorname{End}(T)} \operatorname{End}(T) \cong M^\bullet , \end{aligned}$$

and thus Ξ_T is fully faithful. Now apply the counit η to an object $\mathcal{F}^\bullet \in D^-(Y)$ and take the cone:

$$\Xi_T \Phi_T(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet \to \mathcal{C}^\bullet \overset{+1}{\to} .$$

Then apply Φ_T to this triangle: the first arrow becomes an isomorphism (because the natural transformation $\Phi_T \eta$ is left-inverse to $\epsilon \Phi_T$, which is an isomorphism by what we have just said), so we see that $R\operatorname{Hom}(T, \mathcal{C}^\bullet) = 0$, which by definition of generator implies $\mathcal{C}^\bullet = 0$. Hence also the counit is a natural isomorphism, proving that Φ_T and Ξ_T are quasi-inverses.

Finally, take $M^\bullet \in D^b(\mathbf{Mod}^{\mathrm{fg}}_{\operatorname{End}(T)})$ and apply Φ_T to some truncation

$$\tau_{<m} \Xi_T(M^\bullet) \overset{f}{\to} \Xi_T(M^\bullet) ,$$

[3] That is, $\operatorname{End}(T)$ is finitely generated as an R-module. This implies that it is a Noetherian ring, and thus the category $\mathbf{Mod}^{\mathrm{fg}}_{\operatorname{End}(T)}$ of finitely generated right $\operatorname{End}(T)$-modules is abelian.

[4] We are considering for simplicity only vector bundles, but things work the same for perfect complexes. Notice that our definition does not require anything on the global dimension of the algebra $\operatorname{End}(T)$.

[5] I am grateful to Y. Toda for explaining the content of [TU10] to me.

to get

$$\Phi_T(\tau_{<m}\Xi_T(M^\bullet)) \stackrel{\Phi_T(f)}{\to} \Phi_T\Xi_T(M^\bullet) \simeq M^\bullet \,.$$

$\Phi_T(\tau_{<m}\Xi_T(M^\bullet))$ has only nonvanishing cohomologies in degree $< m+\dim Y$, and thus for $m \ll 0$ it has no nonzero maps to $M^\bullet$. So $\Phi_T(f) = 0$, and thus $\tau_{<m}\Xi_T(M^\bullet) = 0$ because Φ_T is fully faithful. This proves that Ξ_T also restricts to a functor between the bounded derived categories. □

2.2.2 Tilting generators from exceptional sequences

Let X be a smooth projective $\mathbb{K}$-variety and let $\mathfrak{E}$ be a full exceptional sequence in $D^b(X)$, whose left dual ${}^\vee\mathfrak{E} = ({}^\vee E_n, ..., {}^\vee E_0)$ is strong and consists of vector bundles.[6] Consider the vector bundle

$$T := \oplus_{i=0}^n {}^\vee E_i \,.$$

Lemma 2.2.4. *T is a tilting generator.*

Proof. Obviously, $\mathrm{Ext}^i(T,T) = 0$ for all $i > 0$. Let $\mathcal{F}^\bullet \in D^-(X)$ be inside $T^{\perp\!\!\!\perp} = \langle {}^\vee\mathfrak{E}\rangle^{\perp\!\!\!\perp}$. $\langle {}^\vee\mathfrak{E}\rangle = D^b(X)$ contains in particular the powers $\mathcal{O}_X(\ell)$ of an ample line bundle $\mathcal{O}_X(1)$, which means that $R\Gamma(\mathcal{F}^\bullet(\ell)) = 0$ for all $\ell \in \mathbb{Z}$. We claim that this implies $\mathcal{F}^\bullet = 0$. Indeed, by contradiction let $M \in \mathbb{Z}$ be the minimum number such that $\tau_{>M}\mathcal{F}^\bullet = 0$. Then we have a distinguished triangle

$$\tau_{<M}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to h^M(\mathcal{F}^\bullet)[-M] \stackrel{+1}{\to} \,.$$

Applying the functor $R\Gamma(\cdot(\ell))$, the middle term becomes zero, so that

$$R\Gamma(\tau_{<M}\mathcal{F}^\bullet(\ell)) \simeq R\Gamma(h^M(\mathcal{F}^\bullet)(\ell))[-M-1]$$

But for $\ell \gg 0$ the left hand side is concentrated in degree $< M$, while the right-hand side is a nonzero space lying in degree $M+1$, so this isomorphism is absurd. □

Now the $\mathbb{K}$-algebra

$$\mathrm{End}(T) = \begin{pmatrix} \mathrm{Hom}({}^\vee E_n, {}^\vee E_n) & & \\ \vdots & \ddots & \\ \mathrm{Hom}({}^\vee E_n, {}^\vee E_0) & \cdots & \mathrm{Hom}({}^\vee E_0, {}^\vee E_0) \end{pmatrix}$$

is finite-dimensional and basic, which means that it has a complete set of orthogonal idempotents $e_i = \mathrm{Id}_{{}^\vee E_i}$ for $i = 0, ..., n$, and the right modules $e_i \,\mathrm{End}(T) \simeq \mathrm{Hom}(T, {}^\vee E_i)$ are never isomorphic to each other. It follows[7] that we can identify $\mathrm{End}(T)$ with (the opposite of) a bound quiver algebra,

$$\mathrm{End}(T) \cong (\mathbb{K}Q/J)^{\mathrm{op}} \,,$$

where Q is the ordered quiver with vertices $I = \{0, 1, ..., n\}$, and such that the paths between the ith and jth vertices are indexed by a $\mathbb{K}$-basis of the vector space $\mathrm{Hom}({}^\vee E_j, {}^\vee E_i)$ (see Ex. 2.1.22.3).[8] $J \subset \mathbb{K}Q$ is the kernel of the natural map $\mathbb{K}Q \to \mathrm{End}(T)^{\mathrm{op}}$. In particular, we identify right $\mathrm{End}(T)$-modules of finite dimension with representations of (Q, J):

$$\mathbf{Mod}^{\mathrm{fg}}_{\mathrm{End}(T)} \cong \mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q; J) \,.$$

So now Theorem 2.2.3 reads

[6] This useless complication of passing through the left dual ${}^\vee\mathfrak{E}$ is only due to notational consistency with Chapter 5.

[7] See [ASS06, §II.3] for details, but note that the opposite convention for path algebras is used there.

[8] We will always assume to have fixed such a basis, to make the identification $\mathrm{End}(T) \cong (\mathbb{K}Q/J)^{\mathrm{op}}$ canonical. To avoid the choice of a basis, one could consider instead a quiver Q_Z with a single arrow between the ith and jth vertices, but labeled by the vector space $Z_{ij} := \mathrm{Hom}({}^\vee E_j, {}^\vee E_i)$. Then a representations of Q_Z would consist of an I-graded vector space $V = \oplus_{i\in I} V_i$ and a bunch of linear maps $Z_{ij} \to \mathrm{Hom}_{\mathbb{K}}(V_i, V_j)$. However, we will not use this notation for the sake of readability, and we will somehow leave this interpretation as implicit.

Theorem 2.2.5. *We have a triangulated equivalence*

$$\Phi_{^\vee\mathfrak{E}} : D^b(X) \to D^b(Q;J)$$

which maps a complex $\mathcal{F}^\bullet \in D^b(X)$ *to a complex of representations which at the vertex* $i \in \{0,...,n\}$ *of* Q *has the graded vector space* $R\operatorname{Hom}_{\mathcal{D}}({}^\vee E_i, \mathcal{F}^\bullet)$.

Remark 2.2.6. Notice that $\Phi_{^\vee\mathfrak{E}}$ maps each ${}^\vee E_i$ to the projective representation $P(i)$ of Q and each dual E_i to the simple $S(i)[i-n]$ (as we have seen in Ex. 2.1.22.3 that these form the right dual collection of the one made by projectives). Since the standard heart $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q;J) \subset D^b(Q;J)$ is the extension closure of the simple modules $S(i)$, we see that ${}^\vee\mathfrak{E}$ induces a bounded t-structure on $\mathcal{D}$ (see §3.3.1 for the definitions) whose heart is the extension closure of the objects $E_i[n-i]$, $i = 0,...,n$:

$$\mathcal{K} := \Phi^{-1}_{^\vee\mathfrak{E}}(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q;J)) = \langle E_0[n], E_1[n-1], ..., E_n\rangle_{\mathrm{ext}}\,.$$

Now we want to have a version of Theorem 2.2.5 for families of sheaves over a base S. First of all, we need a tilting generator on $X \times S$:

Lemma 2.2.7. *Let* $S = \operatorname{Spec} R$ *be an affine* $\mathbb{K}$*-scheme of finite type and consider the projections*

$$X \overset{\mathrm{pr}_X}{\leftarrow} Y := X \times S \overset{\mathrm{pr}_S}{\rightarrow} S\,.$$

Then the vector bundle

$$T_S := \mathrm{pr}_X^* T = T \boxtimes \mathcal{O}_S$$

is a tilting generator on $X \times S$.

Proof. By Künneth theorem and the fact that $H^i(S;\mathcal{O}_S) = 0$ for $i > 0$ we obtain

$$\operatorname{Ext}^i_{\mathcal{O}_{X\times S}}(T_S, T_S) \cong \operatorname{Ext}^i_{\mathcal{O}_X}(T,T) \otimes_{\mathbb{K}} H^0(S;\mathcal{O}_S) \cong \operatorname{Ext}^i_{\mathcal{O}_X}(T,T) \otimes_{\mathbb{K}} R\,,$$

which is zero for $i > 0$. The fact that $\mathrm{pr}_X^* T$ is a generator of $D^-(X \times S)$ is proven as in Lemma 2.2.4 using the ample line bundle $\mathrm{pr}_X^* \mathcal{O}_X(1) \in \langle {}^\vee E_n \boxtimes \mathcal{O}_S, ..., {}^\vee E_0 \boxtimes \mathcal{O}_S\rangle$. ☐

So we can apply Theorem 2.2.3 to the tilting generator T_S. First observe that

$$\operatorname{End}(T_S) \cong \operatorname{End}(T) \otimes_{\mathbb{K}} R \cong (\mathbb{K}Q/J)^{\mathrm{op}} \otimes_{\mathbb{K}} R\,,$$

and that a finitely generated right $\operatorname{End}(T_S)$-module is the same as a coherent sheaf $\mathcal{V}$ on $S = \operatorname{Spec} R$ with a homomorphism $\mathbb{K}Q/J \to \operatorname{End}(\mathcal{V})$: we denote by

$$\mathbf{Coh}^{Q,J}_{\mathcal{O}_S} \cong \mathbf{Mod}^{\mathrm{fg}}_{\operatorname{End}(T_S)}$$

the category of these objects.[9] Hence Theorem 2.2.3 becomes:

Theorem 2.2.8. *For any affine* $\mathbb{K}$*-scheme* S *of finite type we have an equivalence*

$$R(\mathrm{pr}_S)_*(T_S^\vee \otimes (\cdot)) : D^b(X \times S) \to D^b(\mathbf{Coh}^{Q,J}_{\mathcal{O}_S})$$

(and similarly for the bounded above derived categories).

[9] Notice that if $\mathcal{V}$ is also locally free, then we get a flat family over S of representations of (Q,J), as defined in §4.1.5.

Chapter 3

Stability

This chapter will introduce all the notions of stability structures on abelian and triangulated categories that will be used in the rest of the thesis.

In this introduction we briefly outline how we will use these techniques in the sequel, also to motivate the amount of technicalities of this chapter. The setting of Chapter 5 will be that of a polarized smooth projective variety (X, A) with a full strong exceptional collection ${}^{\vee}\mathfrak{E}$ of vector bundles in its bounded derived category $D^b(X)$. As seen in §2.2.2, in this situation we have an equivalence

$$\Phi_{{}^{\vee}\mathfrak{E}} : D^b(X) \to D^b(Q; J)$$

with the derived category of finite-dimensional representation of a quiver with relations (Q, J), and thus also a non-standard heart $\mathcal{K} \subset D^b(X)$. We will need to extend the notion of Gieseker (semi)stability to objects of $D^b(X)$: to do this, in §3.2 we will first introduce the concept of stability with respect to an alternating form $\sigma : K_0(X) \times K_0(X) \to \mathbb{R}[t]$: we will define a nonzero object $\mathcal{F}$ in a heart $\mathcal{A} \subset D^b(X)$ to be σ-semistable in $\mathcal{A}$ if $\sigma(\mathcal{G}, \mathcal{F}) \leq 0$ for any $0 \neq \mathcal{G} \subsetneq \mathcal{F}$ in $\mathcal{A}$.

In particular, we are interested in the case in which σ is given by

$$\sigma(v, w) := P_{v,A} P'_{w,A} - P_{w,A} P'_{v,A} \,,$$

where $P_{v,A}$ denotes the Hilbert polynomial of a class $v \in K_0(X)$ with respect to the polarization A. If $\mathcal{A}$ is the standard heart $\mathcal{C} := \mathbf{Coh}_{\mathcal{O}_X}$, then we will show in §3.2.2 that this is the usual notion of Gieseker stability of coherent sheaves, which has two key properties (besides the existence of moduli spaces of semistable sheaves, which is the reason it was introduced): the existence of Harder-Narasimhan filtrations and the fact that the heart $\mathcal{C}$ can be tilted with respect to it. Both these properties are consequences of the fact that Gieseker stability can be formulated in terms of an ordering of the nonzero objects of $\mathcal{C}$. Section §3.1 analyzes this concept of order-stability, due to Rudakov [Rud97].

On the other hand, we can apply the above definition of σ-stability to the heart $\mathcal{K}$: if we fix a class $v \in K_0(X) \cong K_0(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q; J))$, then we will see that, for objects of class v in $\mathcal{K} \cong \mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q; J)$, σ-stability coincides with the definition used in [Kin94] to construct moduli spaces of representations of (Q, J).

A recurring problem in this thesis will be to understand how to compare the σ-semistable objects of the two hearts $\mathcal{C}$ and $\mathcal{K}$ lying inside a given class $v \in K_0(X)$. When these are exactly the same, then we will speak of (σ, v)-*compatibility* of the hearts $\mathcal{C}$ and $\mathcal{K}$ (the precise definition is given in §3.3.4). In Chapter 5 we will determine conditions under which this compatibility actually happens, implying in particular that moduli spaces parameterizing semistable coherent sheaves on X and semistable representations of (Q, J) can be identified.

3.1 Order-stabilities on abelian categories

In this section, let $\mathcal{A}$ be an abelian category.

We will review the notion of stability structure on $\mathcal{A}$ introduced in [Rud97], which we will formulate as a *stability phase*, and its main features. This allows to treat in a unified way all the concepts of stability (such as Gieseker stability of coherent sheaves, or slope stability of quiver representations) which are defined in terms of some ordering of the objects of $\mathcal{A}$. The basic definitions and properties are introduced in §3.1.1. In §3.1.3 we will discuss Harder-Narasimhan filtrations and the conditions for their existence. The functorial behaviour of these filtrations is dealt with using the concept of slicing, examined in §3.1.2. Another reference for some of the material of this section is [Joy07, §4].

3.1.1 Rudakov stability phases

We denote by $\mathrm{Ob}^{\times}(\mathcal{A})$ the set of nonzero objects of $\mathcal{A}$.

Definition 3.1.1. Let $(\Xi, \leq)$ be a totally ordered set. A *stability phase* with values in $(\Xi, \leq)$ is a map $\phi : \mathrm{Ob}^{\times}(\mathcal{A}) \to \Xi$ with the following *see-saw property*: for any short exact sequence $0 \to A \to B \to C \to 0$ of nonzero objects in $\mathcal{A}$, we have either $\phi(A) < \phi(B) < \phi(C)$, or $\phi(A) = \phi(B) = \phi(C)$, or $\phi(A) > \phi(B) > \phi(C)$.

This general definition was introduced in [Rud97], where it is equivalently formulated in terms of the induced preorder relation on the set $\mathrm{Ob}^{\times}(\mathcal{A})$. The definition given in [Joy07, §4] also assumes that ϕ takes the same value on objects belonging to the same class in $K_0(\mathcal{A})$.

Examples 3.1.2. For many examples of stability phases we refer to §3.2, where they will be constructed via some other types of stability structures. The basic example to keep in mind is of course the slope

$$\mu : \mathrm{Ob}^{\times}(\mathbf{Coh}_{\mathcal{O}_C}) \to (-\infty, +\infty]$$

of coherent sheaves on a projective curve C, defined by $\mu(\mathcal{E}) := \deg \mathcal{E} / \operatorname{rk} \mathcal{E}$.

Definition 3.1.3. Suppose that a stability phase ϕ on $\mathcal{A}$ is given. A nonzero object $A \in \mathrm{Ob}(\mathcal{A})$ is said to be ϕ-*(semi)stable* when, for any nonzero subobject $0 \neq S \subsetneq A$, we have $\phi(S)\,(\leq)\!<\phi(A)$. Given a phase $\xi \in \Xi$, we define a strictly full subcategory $\mathcal{S}_\phi(\xi) \subset \mathcal{A}$ by

$$\mathrm{Ob}(\mathcal{S}_\phi(\xi)) := \{A \in \mathrm{Ob}^{\times}(\mathcal{A}) \mid A \text{ is } \phi\text{-semistable and } \phi(A) = \xi\} \cup \{\text{zero objects}\}\,.$$

Stability can be characterized also via quotients: a nonzero object A is ϕ-(semi)stable if and only if $\phi(A)\,(\leq)\!<\phi(Q)$ for any nontrivial quotient Q.

Proposition 3.1.4. *Let $\phi : \mathrm{Ob}^{\times}(\mathcal{A}) \to (\Xi, \leq)$ be a stability phase.*

1 Let $0 \to A \to E \to C \to 0$ be a short exact sequence of nonzero objects and let $D \in \mathrm{Ob}^{\times}(\mathcal{A})$: if $\phi(D) \leq \phi(A)$ and $\phi(D) \leq \phi(C)$, then $\phi(D) \leq \phi(E)$ (and similarly with $\leq$ replaced by $\geq$).

2 Let $0 \to A \to E \to C \to 0$ be a short exact sequence of nonzero objects such that $\phi(A) = \phi(E) = \phi(C)$; then A and C are ϕ-semistable if and only if E is ϕ-semistable.

3 Let $f : A \to B$ be a morphism between ϕ-semistable objects in $\mathcal{A}$. Then:

(a) if $\phi(A) > \phi(B)$, then $f = 0$;

(b) if $\phi(A) = \phi(B)$, then $\ker f$, $\operatorname{im} f$, $\operatorname{coker} f$ are ϕ-semistable and

$$\phi(\ker f) = \phi(\operatorname{im} f) = \phi(\operatorname{coker} f) = \phi(A) = \phi(B)\,.$$

4 for all $\xi \in \Xi$, $\mathcal{S}_\phi(\xi)$ is an extension-closed abelian subcategory of $\mathcal{A}$, whose simple objects are the ϕ-stable ones.

Proof. Item 1 is obvious. To prove item 2, suppose that A and C are ϕ-semistable; given $0 \neq S \subsetneq E$, we have a short exact sequence $0 \to S \cap A \to S \to S/S \cap A \to 0$ which embeds in $0 \to A \to E \to C \to 0$. Now $\phi(S \cap A) \leq \phi(A) = \phi(E)$ and $\phi(S/S \cap A) \leq \phi(C) = \phi(E)$, and hence $\phi(S) \leq \phi(E)$ by point 1. The converse direction is easy.
About item 3, it suffices to observe that $\phi(\ker f) = \phi(A) = \phi(\operatorname{im} f) = \phi(B) = \phi(\operatorname{coker} f)$ because of the see-saw property, and item 4 follows from 2 and 3. □

3.1.2 Slicings of abelian categories

The following notion was introduced (in the context of triangulated categories) in [Bri07]; the definition given here is essentially that of *stability data* in [GKR04, Def. 2.4].[1]

Definition 3.1.5. A *slicing* of $\mathcal{A}$ consists of a totally ordered set $(\Xi, \leq)$ and a collection $\mathcal{S} = \{\mathcal{S}(\xi)\}_{\xi\in\Xi}$ of strictly full additive subcategories $\mathcal{S}(\xi) \subset \mathcal{A}$ such that:

i) for all $\xi_1, \xi_2 \in \Xi$ such that $\xi_1 > \xi_2$, for all $A_1 \in \mathrm{Ob}(\mathcal{S}(\xi_1))$ and $A_2 \in \mathrm{Ob}(\mathcal{S}(\xi_2))$ we have $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2) = 0$;

ii) for any nonzero object $A \in \mathrm{Ob}(\mathcal{A})$, there exist $n \in \mathbb{N}$, $\xi_1, ..., \xi_n \in \Xi$ such that $\xi_1 > \cdots > \xi_n$, and a filtration $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$ such that for all $i = 1, ..., n$ we have $A_i/A_{i-1} \in \mathcal{S}(\xi_i)$.

We call such a filtration a $\mathcal{S}$*-filtration* of A, and we will sometimes write it as $0 \subset A_{\xi_1} \subset \cdots \subset A_{\xi_n} = A$; the filtration is *reduced* when all the quotients A_i/A_{i-1} are nonzero, and in this case the uniquely determined (see below) elements $\xi_1, ..., \xi_n$ are called the $\mathcal{S}$*-phases* of A; in particular we set $\phi^{\mathcal{S}}_{\max}(A) := \xi_1$ and $\phi^{\mathcal{S}}_{\min}(A) := \xi_n$. Finally, given an interval $I \subset \Xi$, we consider the extension closure in $\mathcal{A}$

$$\mathcal{S}(I) := \langle \mathcal{S}(\xi),\ \xi \in I \rangle_{\mathrm{ext}} .$$

We will see in Prop. 3.1.8 that the nonzero objects of $\mathcal{S}(I)$ are precisely those whose $\mathcal{S}$-phases are in I.

Examples 3.1.6.

1 Slicings may be thought of as continuous generalizations of the concept of torsion pair (see also Prop. 3.1.8 below): given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ we have a slicing $\mathcal{S} := \{\mathcal{F}, \mathcal{T}\}$ labeled by $\{1, 2\}$; the $\mathcal{S}$-filtration of an object $A \in \mathrm{Ob}^{\times}(\mathcal{A})$ is $0 \subset T \subset A$, where $0 \to T \to A \to F \to 0$ is the unique exact sequence with $T \in \mathrm{Ob}(\mathcal{T})$ and $F \in \mathrm{Ob}(\mathcal{F})$. Generalizing, a slicing $\mathcal{S} := \{\mathcal{S}(k)\}_{k\in\mathbb{N}}$ is the same as a family $\{(\mathcal{T}_k, \mathcal{F}_k)\}_{k\in\mathbb{N}}$ of torsion pairs such that $\mathcal{T}_j \subset \mathcal{T}_i$ for $i < j$. To construct the torsion pairs from $\mathcal{S}$ we take $\mathcal{T}_k := \mathcal{S}([k+1, \infty))$ and $\mathcal{F}_k := \mathcal{S}([0, k])$; conversely, $\mathcal{S}$ is constructed from the torsion pairs by taking $\mathcal{S}(k) := \mathcal{F}_k \cap \mathcal{T}_{k-1}$.

2 We will see in the next subsection that if we have a stability phase ϕ on $\mathcal{A}$ with certain properties, then the categories $\mathcal{S}_\phi(\xi)$ of ϕ-semistable objects of phase ξ form a slicing of $\mathcal{A}$.

Remark 3.1.7. $\mathcal{S}$-filtrations are functorial and (if reduced) unique, which is why the definition of $\mathcal{S}$-phases is well-given. Indeed, consider a morphism $f : A \to B$ between nonzero objects, and take strictly decreasing elements $\xi_1, ..., \xi_n \in \Xi$ and $\mathcal{S}$-filtrations (possibly expanded by adding zero quotients) $0 = A_0 \subset \cdots \subset A_n = A$ and $0 = B_0 \subset \cdots \subset B_n = B$ with $A_i/A_{i-1}, B_i/B_{i-1} \in \mathcal{S}(\xi_i)$ for all $i = 1, ..., n$. Then for all $i \in \{1, ..., n\}$ we have $\mathrm{Hom}_{\mathcal{A}}(A_{i-1}, B_i/B_{i-1}) = 0$, because A_{i-1} is obtained by extensions of objects of phase bigger than ξ_i, and by standard diagram chasing this implies that there is a unique way to complete all the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{i-1} & \longrightarrow & A_i & \longrightarrow & A_i/A_{i-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle h_i} & & \\ 0 & \longrightarrow & B_{i-1} & \longrightarrow & B_i & \longrightarrow & B_i/B_{i-1} & \longrightarrow & 0 \end{array}$$

starting from $f_n = f$. Moreover, by similar arguments, if f is an isomorphisms, then so are all the f_i's and h_i's. The uniqueness of the reduced $\mathcal{S}$-filtration of A then follows from taking $B = A$ and $f = \mathrm{Id}_A$.

Proposition 3.1.8. *Let $\mathcal{S} = \{\mathcal{S}(\xi)\}_{\xi\in\Xi}$ be a slicing of $\mathcal{A}$. Then:*

1 Given $A, B \in \mathrm{Ob}^{\times}(\mathcal{A})$ such that $\phi^{\mathcal{S}}_{\min}(A) > \phi^{\mathcal{S}}_{\max}(B)$, we have $\mathrm{Hom}_{\mathcal{A}}(A, B) = 0$.

[1] In [GKR04] it is also assumed that each subcategory $\mathcal{S}(\xi) \subset \mathcal{A}$ is extension-closed; however this fact follows from the other axioms, see Prop. 3.1.8. Moreover, we also assume that these subcategories are additive (i.e. just that they contain zero objects, as closure under binary sums follows from the other properties).

2 For any interval $J \subset S$ we have

$$\mathrm{Ob}^{\times}(\mathcal{S}(J)) = \{A \in \mathrm{Ob}^{\times}(\mathcal{A}) \text{ with } \mathcal{S}\text{-phases in } J\}\,;$$

in particular, $\mathcal{S}(\xi) = \mathcal{S}(\{\xi\})$ is an extension-closed subcategory for any $\xi \in \Xi$.

3 If (F, T) is a partition of Ξ such that $F < T$, then $(\mathcal{S}(T), \mathcal{S}(F))$ is a torsion pair in $\mathcal{A}$; given $A \in \mathrm{Ob}^{\times}(\mathcal{A})$, the unique short exact sequence $0 \to A_T \to A \to A_F \to 0$ with $A_T \in \mathcal{S}(T)$ and $A_F \in \mathcal{S}(F)$ is obtained by taking the reduced $\mathcal{S}$-filtration of A and letting $A_T := A_{\xi_k}$ and $A_F := A/A_{\xi_k}$, where $k \in \{0, ..., n\}$ is such that $\xi_k \in T$ and $\xi_{k-1} \in F$.

Proof. Item 1 follows from the fact that A and B are obtained as iterated extensions from the quotients A_i/A_{i-1} and B_j/B_{j-1} in their $\mathcal{S}$-filtrations, and there are no maps $A_i/A_{i-1} \to B_j/B_{j-1}$. We prove now item 3: let (F, T) be a partition of Ξ with $F < T$, and define full subcategories $\mathcal{F}, \mathcal{T} \subset \mathcal{A}$ as made by zero objects and all nonzero objects whose $\mathcal{S}$-phases are contained in F and T respectively. Then we have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$ by the previous item, and by appropriately truncating the $\mathcal{S}$-filtration of a nonzero object A we get a short exact sequence $0 \to A_{\xi_k} \to A \to A/A_{\xi_k} \to 0$ with $A_{\xi_k} \in \mathcal{T}$ and $A/A_{\xi_k} \in \mathcal{F}$. Thus $(\mathcal{T}, \mathcal{F})$ is a torsion pair, and in particular $\mathcal{T}, \mathcal{F}$ are extension-closed, so that $\mathcal{T} = \mathcal{S}(T)$ and $\mathcal{F} = \mathcal{S}(F)$.
Finally, given an interval $J \subset \Xi$, consider the partition $\Xi = I \sqcup J \sqcup K$ with $I < J < K$; item 2 follows from applying the previous argument to the partitions $(I \cup J, K)$ and $(I, J \cup K)$. □

3.1.3 Harder-Narasimhan filtrations

Let $(\Xi, \leq)$ be a totally ordered set, and take a stability phase $\phi : \mathrm{Ob}^{\times}(\mathcal{A}) \to \Xi$.

Definition 3.1.9. Let $A \in \mathrm{Ob}(\mathcal{A})$ be a nonzero object. A *maximally destabilizing subobject (MDS)* of A is a nonzero subobject $D \subset A$ such that for any nonzero subobject $S \subset A$ we have $\phi(S) \leq \phi(D)$, and, when $\phi(S) = \phi(D)$, then $S \subset D$.

Clearly, if a MDS exists, then it is unique and is a ϕ-semistable object.

Definition 3.1.10. Let $A \in \mathrm{Ob}(\mathcal{A})$ be a nonzero object. We call *Harder-Narasimhan (HN) filtration* of A with respect to ϕ a filtration

$$0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = A$$

such that for all $i = 1, ..., n$ the quotient A_i/A_{i-1} is ϕ-semistable, and

$$\phi({}^{A_1}/{}_{A_0}) > \phi({}^{A_2}/{}_{A_1}) > \cdots > \phi({}^{A_n}/{}_{A_{n-1}})\,.$$

The elements $\phi({}^{A_1}/{}_{A_0}), ..., \phi({}^{A_n}/{}_{A_{n-1}})$ (which are uniquely determined by Prop. 3.1.14 below) are called the *HN factors* of A, and we write in particular $\phi_{\min}(A) := \phi({}^{A_n}/{}_{A_{n-1}})$ and $\phi_{\max}(A) := \phi({}^{A_1}/{}_{A_0})$. The stability phase ϕ is said to have the *HN property* if every nonzero object in $\mathcal{A}$ has an HN filtration.

Notice that the first nonzero term $A_1 \subset A$ in an HN filtration is the MDS of A.

Remark 3.1.11. If ϕ has the HN property, then $\{\mathcal{S}_{\phi}(\xi)\}_{\xi \in \Xi}$ is a slicing of $\mathcal{A}$, where each $\mathcal{S}_{\phi}(\xi) \subset \mathcal{A}$ (which was defined in Def. 3.1.3 as the category of ϕ-semistable objects of phase ξ) is an extension-closed abelian subcategory by Prop. 3.1.4. In this sense, HN filtrations are functorial.

To find sufficient conditions for the existence of HN filtrations we introduce the following weakenings of the definitions of Artinian and Noetherian objects in $\mathcal{A}$:

Definition 3.1.12. Let $\ltimes$ be one of the relations $\leq, \geq, =$. We say that:[2]

[2]We use a different terminology than [Rud97], where *w-Artinian* means $(\phi, \geq)$-Artinian, *q-Noetherian* means $(\phi, \leq)$-Noetherian and *w-Noetherian* means both $(\phi, \leq)$-Noetherian and $(\phi, \geq)$-Noetherian. See also [Joy07, Def. 4.3].

- a nonzero object $A \in \mathrm{Ob}(\mathcal{A})$ is $(\phi, \ltimes)$-*Artinian* if every descending chain of nonzero subobjects $\cdots \subset A_2 \subset A_1 \subset A$ such that $\cdots \ltimes \phi(A_2) \ltimes \phi(A_1) \ltimes \phi(A)$ stabilizes, i.e. $A_i = A_{i+1}$ for $i \gg 0$; the Abelian category $\mathcal{A}$ is called $(\phi, \ltimes)$-*Artinian* if every nonzero object is $(\phi, \ltimes)$-Artinian;

- a nonzero object $A \in \mathrm{Ob}(\mathcal{A})$ is $(\phi, \ltimes)$-*Noetherian* if every ascending chain of nonzero subobjects $A_1 \subset A_2 \subset \cdots \subset A$ such that $\phi(A_1) \ltimes \phi(A_2) \ltimes \cdots$ stabilizes, i.e. $A_i = A_{i+1}$ for $i \gg 0$; the Abelian category $\mathcal{A}$ is called $(\phi, \ltimes)$-*Noetherian* if every nonzero object is $(\phi, \ltimes)$-Noetherian.

Remark 3.1.13. Clearly, if $\mathcal{A}$ is Artinian (resp. Noetherian), then it is $(\phi, \ltimes)$-Artinian (resp. $(\phi, \ltimes)$-Noetherian) for all $\ltimes$. Notice also that if $\mathcal{A}$ is $(\phi, =)$-Artinian and $(\phi, =)$-Noetherian, then for each $\xi \in \Xi$, the abelian subcategory $\mathcal{S}_\phi(\xi)$ is of finite length.

Proposition 3.1.14. *Let $A \in \mathrm{Ob}^\times(\mathcal{A})$. Then:*

1. *if A is $(\phi, \geq)$-Artinian and $(\phi, \leq)$-Noetherian, then it has a unique MDS $D \subset A$.*

2. *If A has an HN filtration, then it is unique.*

3. *if A is $(\phi, \geq)$-Artinian, $(\phi, \leq)$-Noetherian and $(\phi, \geq)$-Noetherian, then it has an HN filtration.*

Proof. The uniqueness of the MDS is obvious, while for its existence we refer to [Rud97, Prop. 1.9]. We prove item 2: first, suppose to have two HN filtrations $0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = A$ and $0 = A'_0 \subsetneq A'_1 \subsetneq \cdots \subsetneq A'_m = A$. $A_1 = A'_1$ as subobjects of A because they are the MDS; $A_2/A_1 = A'_2/A'_1$ as subobjects of A/A_1, because they are the MDS, and hence also $A_2 = A'_2$ as subobjects of A. Iterating, we see that for all $1 \leq i \leq \min\{m, n\}$ one has $A_i = A'_i$ as subobjects of A. In particular, $m = n$ because after $i = \min\{m, n\}$ there cannot be other subobjects.

Now we prove item 3, i.e. existence of an HN filtration. If A is semistable, then $0 \subsetneq A$ is an HN filtration; otherwise A has a MDS $A_1 \subsetneq A$ because A is $(\phi, \geq)$-Artinian and $(\phi, \leq)$-Noetherian; note that $\phi(A_1) > \phi(A) > \phi(A/A_1)$. So, if A/A_1 is semistable, then $0 = A_0 \subsetneq A_1 \subsetneq A$ is an HN filtration, otherwise we iterate the procedure. Namely, suppose that we have constructed a filtration $0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_k \subsetneq A$ such that for all $i = 1, ..., k$ the quotient $B_i = A_i/A_{i-1}$ is ϕ-semistable, and $\phi(B_1) > \phi(B_2) > \cdots > \phi(B_k) > \phi({}^{A}\!/\!{}_{A_k})$. Then either A/A_k is semistable, i.e. the filtration is an HN filtration, or we can take the MDS $B_{k+1} \subsetneq A/A_k$ and its inverse image $A_{k+1} \subsetneq A$, so that B_{k+1} is semistable, $B_{k+1} \simeq A_{k+1}/A_k$, $A_k \subsetneq A_{k+1} \subsetneq A$ and we have $\phi(B_k) > \phi(B_{k+1}) > \phi({}^{A}\!/\!{}_{A_{k+1}})$: indeed, the inequality $\phi(B_k) > \phi(B_{k+1})$ follows from the see-saw property applied to the short exact sequence

$$0 \to B_k = {}^{A_k}\!/\!{}_{A_{k-1}} \to {}^{A_{k+1}}\!/\!{}_{A_{k-1}} \to {}^{A_{k+1}}\!/\!{}_{A_k} = B_{k+1} \to 0\,,$$

being $\phi(B_k) > \phi({}^{A_{k+1}}\!/\!{}_{A_{k-1}})$ because B_k is the MDS of A/A_{k-1}; the inequality $\phi(B_{k+1}) > \phi({}^{A}\!/\!{}_{A_{k+1}})$ comes from applying the see-saw property to the short exact sequence

$$0 \to B_{k+1} = {}^{A_{k+1}}\!/\!{}_{A_k} \to {}^{A}\!/\!{}_{A_k} \to {}^{A}\!/\!{}_{A_{k+1}} \to 0\,,$$

being $\phi(B_{k+1}) > \phi({}^{A}\!/\!{}_{A_k})$ because B_{k+1} is the MDS of A/A_k. But at some point the procedure must end (i.e. we find a semistable quotient A/A_k), because otherwise we would get an ascending chain $0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A$ with $\phi(A_1) > \phi(A_2) > \cdots$, which is impossible since by hypothesis A is $(\phi, \geq)$-Noetherian. □

3.2 Linear stability structures on abelian categories

Let $\mathcal{A}$ be an abelian category.

In this section we will review some well-known notions of stability defined via linear maps on the Grothendieck group $K_0(\mathcal{A})$. Moreover, in §3.2.2 we will reformulate stability with respect to a polynomial function P (e.g. Gieseker stability) using an alternating form σ_P on $K_0(\mathcal{A})$; this will be useful in the sequel, when we will talk about stability in triangulated categories and will need to make sense of the concept of P-(semi)stable object inside a heart $\mathcal{B} \subset D^b(\mathcal{A})$ of a bounded t-structure other than $\mathcal{A}$ itself.

3.2.1 Stability weights and alternating maps

The simplest notion we will use is that of stability with respect to a *weight*, that is a $\mathbb{Z}$-linear map $\nu : K_0(\mathcal{A}) \to R$ with values in an ordered abelian group $(R, \leq)$ (which will typically be $\mathbb{Z}$, $\mathbb{R}$, or the polynomial ring $\mathbb{R}[t]$ with lexicographical order). This was introduced in [Kin94].

Definition 3.2.1. A nonzero object A in $\mathcal{A}$ is said to be *ν-(semi)stable* if $\nu(A) = 0$ and any strict subobject $0 \neq B \subsetneq A$ satisfies $\nu(B)\,(\geq)\!>0$. A is *ν-polystable* if it is a direct sum of ν-stable objects. We define a strictly full subcategory $\mathcal{S}_\nu \subset \mathcal{A}$ by

$$\mathrm{Ob}(\mathcal{S}_\nu) := \{A \in \mathrm{Ob}^\times(\mathcal{A}) \mid A \text{ is } \nu\text{-semistable}\} \cup \{\text{zero objects}\}\,.$$

Lemma 3.2.2. *$\mathcal{S}_\nu \subset \mathcal{A}$ is a full abelian subcategory, closed under extensions. The (semi)simple objects of this category are the ν-(poly)stable objects.*

The proof is very similar to that of Prop. 3.1.4.

Example 3.2.3. If $Q = (I, \Omega)$ is a quiver and $\theta \in \mathbb{Z}^I$, then the usual notion of θ-stability of a representation of Q is encoded in the weight $\nu_\theta([V]) := \theta \cdot \underline{\dim}\, V$. See §4.1.3 for details.

Second, we take an alternating $\mathbb{Z}$-bilinear form $\sigma : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to R$.

Definition 3.2.4. A nonzero object A in $\mathcal{A}$ is said to be *σ-(semi)stable* if any strict subobject $0 \neq B \subsetneq A$ satisfies $\sigma(B, A)\,(\leq)\!<0$.

Remark 3.2.5. If we fix a class $v \in K_0(\mathcal{A})$, then we can define a weight $\nu_v := \sigma(v, \cdot)$. Notice that an object $A \in \mathcal{A}$ with $[A] = v$ is ν_v-(semi)stable if and only if it is σ-(semi)stable, but this fails if A does not belong to the class v.

We have given these two basic definitions of stability mostly for later notational convenience, and because they will be useful when used on different hearts in a triangulated category (see §3.3). These definitions are very general and do not have particularly interesting properties, mainly because they are too weak to induce an order on the objects of $\mathcal{A}$. However, with σ we can at least order the subobjects of a fixed object A, and we will use the following definition:

Definition 3.2.6. Let A be a nonzero object. A nonzero subobject $S \subset A$ is said to be *σ-maximal* if for any subobject $S' \subset A$ we have $\sigma(S', A) \leq \sigma(S, A)$.

3.2.2 Polynomial stability structures

Take a $\mathbb{Z}$-linear map $P : K_0(\mathcal{A}) \to \mathbb{R}[t]$, and write $P_v(t) = P_A(t)$ for the image of a class $v = [A] \in K_0(\mathcal{A})$. We define an alternating form $\sigma_P : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{R}[t]$ as

$$\sigma_P(v, w) := P_v P'_w - P_w P'_v\,, \tag{3.2.1}$$

where $P'_v(t) := \frac{\mathrm{d}}{\mathrm{d}t} P_v(t)$. Then we can consider σ_P-stability according to Def. 3.2.4:

Definition 3.2.7. A nonzero object A in $\mathcal{A}$ is said to be *P-(semi)stable* if it is σ_P-(semi)stable, that is if $P_B P'_A - P_A P'_B \leq 0$ for any $0 \neq B \subsetneq A$.

As usual, polynomials are ordered lexicographically. This definition does not assume anything on the map P, but it turns out to be much more interesting when P maps the classes of nonzero objects into the set $\mathbb{R}[t]_+ \subset \mathbb{R}[t]$ of polynomials with positive leading coefficient:

Definition 3.2.8. [Rud97, §2] A *polynomial stability structure* is a $\mathbb{Z}$-linear map $P : K_0(\mathcal{A}) \to \mathbb{R}[t]$ such that for any nonzero object $A \in \mathrm{Ob}(\mathcal{A})$ we have $P_A(t) > 0$.

Indeed, we can give $\mathbb{R}[t]_+$ an alternative total preorder:

Definition 3.2.9. We define a total preorder $\preceq_{\mathrm{G}}$ on $\mathbb{R}[t]_+$, called *Gieseker preorder*, by setting

$$p \preceq_{\mathrm{G}} q \iff pq' - p'q \leq 0 \tag{3.2.2}$$

for $p, q \in \mathbb{R}[t]_+$. We also write $p \equiv_{\mathrm{G}} q$ when $p \preceq_{\mathrm{G}} q$ and $q \preceq_{\mathrm{G}} p$, and $p \prec_{\mathrm{G}} q$ when $p \preceq_{\mathrm{G}} q$ and $q \not\preceq_{\mathrm{G}} p$.

We have the following equivalent characterizations of $\preceq_{\mathrm{G}}$, which show that it is indeed a preorder (that is, a total, reflexive and transitive relation) and that it coincides with the preorder considered in [Rud97, §2]:

Lemma 3.2.10. *Take two polynomials $p, q \in \mathbb{R}[t]_+$ and write them as $p(t) = \sum_{i=0}^{\deg p} a_i t^i$ and $q(t) = \sum_{j=0}^{\deg q} b_j t^j$. Then the following statements are equivalent:*

i) $p\,(\preceq_{\mathrm{G}})\!\prec_{\mathrm{G}} q$;

ii) we have

$$\deg p > \deg q \quad \text{or} \quad \begin{cases} \deg p = \deg q =: d \\ \frac{p(t)}{a_d}\,(\leq)\!< \frac{q(t)}{b_d} \end{cases} .$$

If $\deg p \leq \deg q$, then they are also equivalent to

iii) $b_{\deg q}\, p(t)\,(\leq)\!< a_{\deg q}\, q(t)$.

Moreover, we have $p \equiv_{\mathrm{G}} q$ if and only if p and q are proportional.

Proof. Let $D(t) := (pq' - p'q)(t) = \sum_{i=0}^{\deg p} \sum_{j=0}^{\deg q} (j-i) a_i b_j t^{i+j-1}$.

First, suppose that $\deg p \neq \deg q$; then the leading coefficient of $D(t)$ is $(\deg q - \deg p) a_{\deg p} b_{\deg q}$. Hence $p \prec_{\mathrm{G}} q \iff D(t) < 0 \iff \deg q < \deg p$. On the other hand, the statement $(ii \iff iii)$ is vacuously true under the current assumption that $\deg p < \deg q$.

Now suppose instead that $\deg p = \deg q =: d$. Then the equivalence $(ii \iff iii)$ is obvious, as we can divide by the positive numbers a_d and b_d. The equivalence $(i \iff ii)$ follows by looking carefully at the coefficients of $D(t)$: fix $k \in \{0, ..., d-1\}$, and note that the coefficient of degree $d+k-1$ is

$$D_{d+k-1} = \sum_{i+j=d+k} (j-i) a_i b_j = \sum_{i+j=d+k} j(a_i b_j - a_j b_i)\,, \tag{3.2.3}$$

where the indices i, j range from 0 to d; moreover, the coefficients higher than D_{2d-2} are always zero. Now we claim that if $D_{d+k'-1} = 0$ for all $k' > k$, then $D_{d+k-1} = (d-k)(a_k b_d - a_d b_k)$. This can be proved by descending induction on k: for $k = d-1$ the statement is true, as $D_{2d-2} = a_{d-1} b_d - a_d b_{d-1}$. Suppose now that the statement is true for all $h > k$: by hypothesis, we have thus that, for all $h > k$, $0 = D_{d+h-1} = (d-h)(a_h b_d - a_d b_h)$, that is (a_h, b_h) is proportional to (a_d, b_d); but then all these vectors (a_h, b_h) are linearly dependent, that is $a_h b_{h'} - a_{h'} b_h = 0$ for all $h, h' > k$. Hence, in the sum (3.2.3) the only possibly nonzero terms are for $(i, j) = (d, k)$ and $(i, j) = (k, d)$, so that $D_{d+k-1} = (d-k)(a_k b_d - a_d b_k)$, as claimed. So we conclude that if $D_{d+k-1} = 0$ for all $k \in \{0, ..., d-1\}$, then $p(t)/a_d = q(t)/b_d$. If, instead, $D_{d+k-1} \neq 0$ is the leading coefficient of $D(t)$ for some $k \in \{0, ..., d-1\}$, then it is equal to $(d-k)(a_k b_d - a_d b_k)$, and we have $a_h b_d = a_d b_h$ for all $h > k$; this means that

$$p \prec_{\mathrm{G}} q \iff D(t) < 0 \iff \frac{a_k}{a_d} < \frac{b_k}{b_d} \iff \frac{p(t)}{a_d} < \frac{q(t)}{b_d}\,.$$

Finally, the last statement is obvious from (ii). □

Remark 3.2.11. Notice that the last statement of the Lemma extends to any nonzero polynomials $p, q \in \mathbb{R}[t]$: if $pq' - p'q = 0$, then p and q are proportional. Indeed, we can replace p and q by their opposites if necessary and then apply the Lemma to them.

Example 3.2.12. The following are examples of Gieseker-ordered polynomials:

$$2 \equiv_{\mathrm{G}} 5 \succ_{\mathrm{G}} 4t + 10 \equiv_{\mathrm{G}} 2t + 5 \succ_{\mathrm{G}} 4t + 6 \succ_{\mathrm{G}} 3t^2 + 5 \succ_{\mathrm{G}} 6t^2 + 9\,.$$

So we see that $\preceq_{\mathrm{G}}$ descends to a total order on ${}^{\mathbb{R}[t]_+}/_{\equiv_{\mathrm{G}}} = \mathbb{P}_{\mathbb{R}}(\mathbb{R}[t])$. If $P : K_0(\mathcal{A}) \to \mathbb{R}[t]$ is a polynomial stability structure, then we can compose with the quotient map to get a map

$$\phi_P : \mathrm{Ob}^{\times}(\mathcal{A}) \to {}^{\mathbb{R}[t]_+}/_{\equiv_{\mathrm{G}}}\,.$$

Lemma 3.2.13. *ϕ_P is a stability phase (Def. 3.1.1), where ${}^{\mathbb{R}[t]_+}/_{\equiv_{\mathrm{G}}}$ is totally ordered by $\preceq_{\mathrm{G}}$.*

Proof. We have to check that ϕ_P has the see-saw property: take a short exact sequence $0 \to A \to E \to C \to 0$ of nonzero objects. If $\phi_P(A) \equiv_{\mathrm{G}} \phi_P(C)$, then P_A, P_C and $P_E = P_A + P_C$ are proportional, and thus $\phi_P(A) \equiv_{\mathrm{G}} \phi_P(E) \equiv_{\mathrm{G}} \phi_P(C)$.

Suppose now that $\phi_P(A) \prec_{\mathrm{G}} \phi_P(C)$. By Lemma 3.2.10, we have two possibilities: one is that $\deg P_A > \deg P_C$, in which case $\deg P_E = \deg P_A$ and thus $P_E \prec_{\mathrm{G}} P_C$, and $P_A \prec_{\mathrm{G}} P_E$ because P_A and P_E have the same leading coefficient but $P_A < P_A + P_C = P_E$. The other possibility is that $\deg P_A = \deg P_C =: d$ (then P_E has also degree d) and, writing a_d, c_d, e_d for the leading coefficients of these polynomials, we have $c_d P_A < a_d P_C$; hence, $e_d P_A = a_d P_A + c_d P_A < a_d P_A + a_d P_C = a_d P_E$, i.e. $P_A \prec_{\mathrm{G}} P_E$, and $c_d P_E = c_d P_A + c_d P_C < a_d P_C + c_d P_C = e_d P_C$, i.e. $P_E \prec_{\mathrm{G}} P_C$. In both cases we have proven that $\phi_P(A) \prec_{\mathrm{G}} \phi_P(E) \prec_{\mathrm{G}} \phi_P(C)$.

In a similar way one proves that if $\phi_P(A) \succ_{\mathrm{G}} \phi_P(C)$, then $\phi_P(A) \succ_{\mathrm{G}} \phi_P(E) \succ_{\mathrm{G}} \phi_P(C)$. □

By definition, P-(semi)stability is equivalent to ϕ_P-(semi)stability, and by Lemma 3.2.10 these agree with the definition of [Rud97, §2]. The subcategory

$$\mathcal{S}_P(p) := \mathcal{S}_{\phi_P}([p])$$

of P-semistable objects A with $P_A \equiv_{\mathrm{G}} p$ and zero objects (see Def. 3.1.3) is an extension-closed abelian subcategory by Prop 3.1.4.

Examples 3.2.14. Polynomial stabilities of degree 0 and 1.

1 A polynomial stability of degree 0 is just a $\mathbb{Z}$-linear map $r : K_0(\mathcal{A}) \to \mathbb{R}$ such that $r(A) > 0$ for any nonzero object A. In this case all the nonzero objects are semistable with the same phase, and the stable objects are the simple ones. The existence of such an r puts a strong condition on $\mathcal{A}$, for example it is not hard to see that it cannot exist on the category of coherent sheaves on a projective variety X with $\dim X > 0$. Notice also that if the range of r is discrete, then $\mathcal{A}$ is forced to be of finite length: indeed, for any $A \in \mathcal{A}$, r only takes finitely many values between 0 and $r(A)$, so infinite chains of subobjects of A (on which r is strictly monotone) cannot exist.

2 Let $P : K_0(\mathcal{A}) \to \mathbb{R}[t]$ be given by $P_A(t) = r(A)t + d(A)$ for some $\mathbb{Z}$-linear functions $r, d : K_0(\mathcal{A}) \to \mathbb{Z}$. Define also $Z : K_0(\mathcal{A}) \to \mathbb{C}$ by $Z = -d + ir$. Then P is a polynomial stability structure (i.e. $P_A > 0$ for all $A \in \mathrm{Ob}^\times(\mathcal{A})$) if and only if, for any $A \in \mathrm{Ob}^\times(\mathcal{A})$, we have that $r(A) \geq 0$ or that $r(A) = 0$ and $d(A) > 0$; this is equivalent to asking that Z is a central charge (see next subsection). In this case, the "phase" $\phi_Z := \arg(-d + ir)/\pi$ of Z can be identified with the phase ϕ_P of P: we have an isomorphism of ordered sets,

$$h : [rt + d] \in {}^{\mathbb{R}[t]_{+,\leq 1}}/_{\equiv_{\mathrm{G}}} \longmapsto \frac{1}{\pi} \arg(-d + ir) \in (0, 1]\,,$$

such that $h \circ \phi_P = \phi_Z$.

Finally, we study existence of HN filtrations for polynomial stability structures:

Lemma 3.2.15. *If P takes values in numerical polynomials,*[3] *then $\mathcal{A}$ is $(\phi_P, \succeq_{\mathrm{G}})$-Artinian (Def. 3.1.12).*

Proof. Take a chain $\cdots \subset A_1 \subset A_0$ in $\mathcal{A}$ such that that $\cdots \succeq_{\mathrm{G}} P_{A_1} \succeq_{\mathrm{G}} P_{A_0}$, and notice that we must also have $\cdots \leq P_{A_1} \leq P_{A_0}$, since $P_{A_i} - P_{A_{i+1}} = P_{A_i/A_{i+1}} \geq 0$ for all i. Thus for $i \gg 0$ all the polynomials P_{A_i} must have the same degree d and their leading coefficients $a_d^{(i)}$ satisfy $\cdots \leq a_d^{(i+1)} \leq a_d^{(i)} \leq \cdots$. But since the polynomials are numerical, their leading coefficients must be positive multiples of $1/d!$, and hence they cannot decrease indefinitely, so all the $a_d^{(i)}$ are the same for $i \gg 0$. So, by Lemma 3.2.10, for $i \gg 0$ we also have $\cdots \geq P_{A_{i+1}} \geq P_{A_i} \geq \cdots$, which means the sequence of the P_{A_i}'s stabilizes: for $i \gg 0$ we have $P_{A_i/A_{i+1}} = P_{A_i} - P_{A_{i+1}} = 0$, and hence $A_i/A_{i+1} = 0$. Thus the chain $\cdots \subset A_1 \subset A_0$ stabilizes as well. □

[3] Recall that a polynomial $p \in \mathbb{Q}[t]$ is *numerical* if $p(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. Numerical polynomials of degree at most d form a lattice of rank $d + 1$ in $\mathbb{Q}[t]$.

Hence, Prop. 3.1.14 immediately implies:

Proposition 3.2.16. *If P takes values in numerical polynomials and $\mathcal{A}$ is Noetherian, then ϕ_P has the HN property: any $A \in \mathrm{Ob}^{\times}(\mathcal{A})$ has a unique HN filtration with respect to ϕ_P, i.e. a filtration*

$$0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = A$$

such that for all $i = 1, ..., n$ the quotient A_i/A_{i-1} is P-semistable, and

$$P_{A_1/A_0} \succ_{\mathrm{G}} P_{A_2/A_1} \succ_{\mathrm{G}} \cdots \succ_{\mathrm{G}} P_{A_n/A_{n-1}} \, ;$$

Moreover, we have a slicing $\{\mathcal{S}_P(p)\}_{[p]\in\mathbb{R}[t]_+/\equiv_{\mathrm{G}}}$ of $\mathcal{A}$ consisting of abelian subcategories of finite length.

The main motivation for this proposition is of course the existence of HN filtrations of coherent sheaves by Gieseker-semistable ones:

Examples 3.2.17.

1 Let X be a smooth projective variety of dimension n and H an ample divisor on it. Taking the Hilbert polynomial $P_{\mathcal{E},H}$ of a coherent sheaf $\mathcal{E}$ with respect to H gives a polynomial stability

$$P_{\cdot,H} : K_0(\mathbf{Coh}_{\mathcal{O}_X}) \to \mathbb{R}[t]_{\leq n}$$

reproducing the usual notion of Gieseker stability (see §4.2.2 for details). Since $\mathbf{Coh}_{\mathcal{O}_X}$ is a Noetherian abelian category and each $P_{\mathcal{E},H}$ is a numerical polynomial, Proposition 3.2.16 guarantees that $P_{\cdot,H}$ has the HN property. Notice also that $P_{\cdot,H}$ factors through the numerical Grothendieck group $K_{\mathrm{num}}(\mathbf{Coh}_{\mathcal{O}_X})$ (see §4.2.1).

2 If (Q, J) is a quiver with relations and $\zeta \in \mathbb{R}[t]_+^I$ is an array of positive polynomials, then we have a polynomial stability

$$P_{\cdot,\zeta} : K_0(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q, J)) \to \mathbb{R}[t]$$

given by $P_{v,\zeta} := \zeta \cdot \underline{\dim}\, v = \sum_{i\in I} \zeta^i (\underline{\dim}\, v)_i$, where $\underline{\dim}\, v$ is the dimension vector of the class $v \in K_0(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q, J))$ (see §4.1.1 for details). Since $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q, J)$ has finite length, $P_{\cdot,\zeta}$ has the HN property directly from Prop. 3.1.14. Choosing an array of the form $\zeta = (1, ..., 1)t + \theta$ for some $\theta \in \mathbb{R}^I$, this reduces to the more common notion of stability with respect to a slope μ_θ (see Ex. 3.2.20.2).

3.2.3 Central charges

Finally we recall the most common notion of linear stability, that of a central charge. Consider a $\mathbb{Z}$-linear map $Z : K_0(\mathcal{A}) \to \mathbb{C}$, and construct from it a bilinear form $\sigma_Z : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{R}$ by

$$\sigma_Z(v, w) := -\Re Z(v)\Im Z(w) + \Re Z(w)\Im Z(v) \, . \tag{3.2.4}$$

Definition 3.2.18. A nonzero object A in $\mathcal{A}$ is said to be *Z-(semi)stable* if it is σ_Z-(semi)stable.

Equivalently, we are asking that A is (semi)stable with respect to the polynomial map

$$P_v(t) := t\Im Z(v) - \Re Z(v)$$

on $K_0(\mathcal{A})$, according to Def. 3.2.7. Again, this notion of stability is most useful when the cone of effective classes in $K_0(\mathcal{A})$ is mapped to a proper subcone of $\mathbb{C}$, as this allows to order the objects of $\mathcal{A}$ according to the phases of their images under Z. Commonly, one requires that the positive cone is mapped by Z inside the semi-closed upper half-plane $\mathbb{H} \cup \mathbb{R}_{<0}$, which is as saying that P is a polynomial stability of degree 1 (see Ex. 3.2.14.2):

Definition 3.2.19. [Bri07] Z is called a *stability function*, or *central charge*, when for any nonzero object A we have $\Im Z(A) \geq 0$, and we have $\Im Z(A) = 0$ only if $\Re Z(A) < 0$. Z has the *HN property* if the polynomial stability P has.

In this case we denote by

$$\phi_Z(A) := \frac{1}{\pi} \arg Z(A) \in (0,1]$$

the *phase* of a nonzero object A. Note that (again by Ex. 3.2.14.2) now we have $P_A \preceq_{\mathrm{G}} P_B$ if and only if $\phi_Z(A) \leq \phi_Z(B)$, and similarly if we replace phases by *slopes*

$$\mu_Z(A) := -\cot \phi_Z(A) = -\frac{\Re Z(A)}{\Im Z(A)} \in (-\infty, +\infty] \,.$$

Thus, objects are ordered by their slopes, and the above definitions of stability and HN filtrations take now the usual forms. We write now

$$\mathcal{S}_Z(\phi) \subset \mathcal{A}$$

for the extension-closed abelian subcategory of Z-semistable objects of fixed phase ϕ.

Examples 3.2.20.

1 If C is a projective curve, then we have a central charge $Z : K_0(\mathbf{Coh}_{\mathcal{O}_C}) \to \mathbb{C}$ given by $Z := -\deg + i \operatorname{rk}$, inducing the usual slope $\mu(\mathcal{E}) := \deg \mathcal{E} / \operatorname{rk} \mathcal{E}$. Since $\mathbf{Coh}_{\mathcal{O}_C}$ is Noetherian and Z is valued in Gaussian integers, it has the HN property by Prop. 3.2.16 (in fact this is a special case of Example 3.2.17.1). Notice also that Z factors through the numerical Grothendieck group $K_{\mathrm{num}}(\mathbf{Coh}_{\mathcal{O}_C}) \simeq \mathbb{Z}^2$ (Eq. (4.2.2)).

2 If $Q = (I, \Omega)$ is a quiver and $\theta \in \mathbb{R}^I$, then we have a central charge $Z : K_0(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q)) \to \mathbb{C}$ given by $Z(v) = (-\theta + i(1, ..., 1)) \cdot \underline{\dim}\, v$, inducing the slope

$$\mu_\theta(v) = \frac{\theta \cdot \underline{\dim}\, v}{\sum_{i \in I} (\underline{\dim}\, v)_i} \,.$$

Since the category $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q)$ has finite length, Z has the HN property by Prop. 3.1.14.

3.3 Stability in triangulated categories

In this section we will extend the definitions of stability introduced in the previous sections to a triangulated category. As in the well-known case of central charges, first considered in [Bri07], this is simply done by defining a stability structure on a triangulated category as a stability structure on the heart of a bounded t-structure, a notion that will be recalled in §3.3.1. In §3.3.3 we will review the process of tilting with respect to stability structures, while in §3.3.4 we will introduce a concept of compatibility of stability structures on different hearts.

Throughout the whole section, $\mathcal{D}$ will denote a triangulated category.

3.3.1 t-structures

In this paragraph we recall the concept of t-structure on $\mathcal{D}$. All the details can be found in [BBD82].

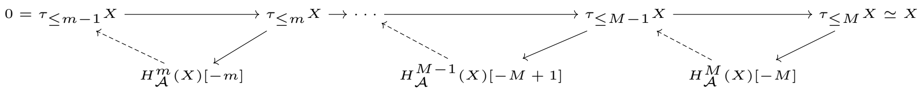
Definition 3.3.1. A *t-structure* on $\mathcal{D}$ consists of a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full subcategories of $\mathcal{D}$ such that, writing $\mathcal{D}^{\leq \ell} := \mathcal{D}^{\leq 0}[-\ell]$ and $\mathcal{D}^{\geq \ell} := \mathcal{D}^{\geq 0}[-\ell]$ for $\ell \in \mathbb{Z}$, we have:

1. $\operatorname{Hom}_{\mathcal{D}}(X, Y) = 0 \ \forall X \in \mathcal{D}^{\leq 0}, \forall Y \in \mathcal{D}^{\geq 1}$;
2. $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}[-1]$ and $\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\geq 0}[1]$;
3. for all $E \in \mathcal{D}$ there is a distinguished triangle $X \to E \to Y \xrightarrow{+1}$ for some $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$.

The intersection $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the *heart* of the t-structure. Finally, we also write $\mathcal{D}^{[m,M]} := \mathcal{D}^{\geq m} \cap \mathcal{D}^{\leq M}$ for $m \leq M$, and we say that the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is *bounded* when for all $X \in \mathcal{D}$ there exists $\ell \in \mathbb{N}$ such that $X \in \mathcal{D}^{[-\ell,\ell]}$.

Proposition 3.3.2. *[BBD82, §1.3] Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure.*

1. *The heart $\mathcal{A}$ is an extension-closed Abelian category.*
2. *A sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$ is exact if and only if it can be completed to a distinguished triangle in $\mathcal{D}$.*
3. *The inclusions $\mathcal{D}^{\leq \ell} \hookrightarrow \mathcal{D}, \mathcal{D}^{\geq \ell} \hookrightarrow \mathcal{D}$ have a right adjoint $\tau_{\leq \ell}$ and a left adjoint $\tau_{\geq \ell}$ respectively, and the functors*
$$H^{\ell}_{\mathcal{A}} := \tau_{\geq 0} \circ \tau_{\leq 0}[\ell] : \mathcal{D} \longrightarrow \mathcal{A}$$
are cohomological.
4. *if $m \leq M$, then for any $X \in \mathcal{D}^{[m,M]}$ we have a Postnikov tower (i.e. triangles are distinguished)*



In particular, $X \in \mathcal{A}$ if and only if $H^i_{\mathcal{A}}(X) = 0$ for all $i \neq 0$.

These facts also easily imply that:

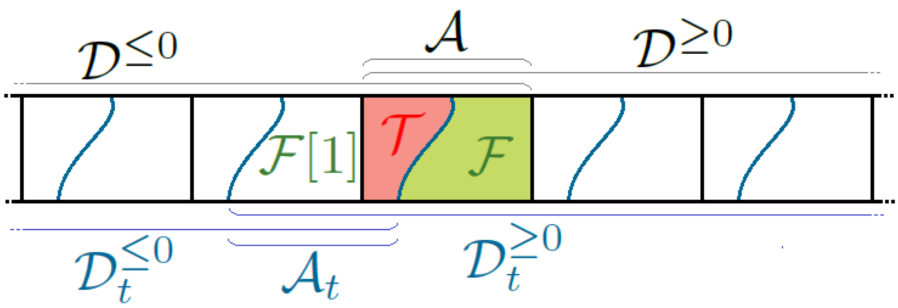
Corollary 3.3.3. *If the t-structure is bounded, then the inclusion $\mathcal{A} \hookrightarrow \mathcal{D}$ gives an isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{D})$ between the Grothendieck groups, whose inverse is*
$$[X] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [H^i_{\mathcal{A}}(X)] \,.$$

Examples 3.3.4.

1 If $\mathcal{A}$ is an Abelian category, then the bounded derived category $D^b(\mathcal{A})$ has a *standard* bounded t-structure whose heart is $\mathcal{A}$.

2 If $\Psi : \mathcal{D}_1 \to \mathcal{D}_2$ is an equivalence of triangulated categories, any t-structure on $\mathcal{D}_1$ induces a t-structure on $\mathcal{D}_2$ in the obvious way; in particular, when we are dealing with derived categories, the standard t-structures may be mapped to non-standard ones.

Besides these two examples, the most common way of constructing t-structures is by *tilting*: if $(\mathcal{T}, \mathcal{F})$ is a torsion pair in the heart $\mathcal{A} \subset \mathcal{D}$ of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, then we can define a new t-structure $(\mathcal{D}^{\leq 0}_t, \mathcal{D}^{\geq 0}_t)$ on $\mathcal{D}$ via a *tilt*, i.e. by taking
$$\mathrm{Ob}(\mathcal{D}^{\leq 0}_t) := \{X \in \mathcal{D} \mid H^0_{\mathcal{A}}(X) \in \mathcal{T},\ H^{\ell}_{\mathcal{A}}(X) = 0\ \forall \ell > 0\}\,,$$
$$\mathrm{Ob}(\mathcal{D}^{\geq 0}_t) := \{X \in \mathcal{D} \mid H^{-1}_{\mathcal{A}}(X) \in \mathcal{F},\ H^{\ell}_{\mathcal{A}}(X) = 0\ \forall \ell < -1\}\,.$$
The heart $\mathcal{A}_t = \mathcal{D}^{\leq 0}_t \cap \mathcal{D}^{\geq 0}_t$ of this new t-structure is the extension closure
$$\mathcal{A}_t = \langle \mathcal{T}, \mathcal{F}[1] \rangle_{\mathrm{ext}}\,.$$



Lemma 3.3.5. *(See e.g. [Pol07, §1.1]) The tilted t-structure satisfies*
$$\mathcal{D}^{\leq 0}_t \subset \mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}_t\,. \tag{3.3.1}$$
Conversely, a t-structure with this property is obtained by tilting $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ at the torsion pair $(\mathcal{A} \cap \mathcal{A}_t, \mathcal{A} \cap \mathcal{A}_t[-1])$.

3.3.2 Stability structures on hearts

We extend the previous notions of stability to a triangulated category giving the following definition:

Definition 3.3.6. By a stability structure (of any of the types considered in the previous two sections) on $\mathcal{D}$ we mean a stability structure on the heart $\mathcal{A} \subset \mathcal{D}$ of a bounded t-structure.

The following observations show in particular why it is worth to consider stabilities on hearts, rather than on arbitrary abelian subcategories.

Remarks 3.3.7.

1 Linear stability structures as in §3.2 on any heart $\mathcal{A} \subset \mathcal{D}$ can be seen as (bi)linear maps on a common group $K_0(\mathcal{D})$, since $K_0(\mathcal{A}) \cong K_0(\mathcal{D})$ by Corollary 3.3.3.

2 Fixing e.g. an alternating form $\sigma : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \to R$ gives a stability structure on any heart in $\mathcal{D}$; when an object $D \in \mathcal{D}$ lies in different hearts, it is necessary to specify with respect to which of them we are considering it being (semi)stable or not. This is because being a subobject is a notion that depends on the heart, although it can be characterized in terms of distinguished triangles in $\mathcal{D}$: by Prop. 3.3.2, a morphism $S \to A$ in a heart $\mathcal{A}$ is injective if and only if, when completed to a distinguished triangle $S \to A \to C \stackrel{+1}{\to}$, we have $C \in \mathcal{A}$.

3 In the cases where the stability structures give an ordering (i.e. they induce a stability phase) to the objects of the heart $\mathcal{A}$, there is an alternative way of characterizing them by using slicings on $\mathcal{D}$, as first noticed in [Bri07] for central charges. The notion of slicing on a triangulated category is very similar to Def. 3.1.5, where filtrations are replaced by Postnikov towers. This definition generalizes the concept of a t-structure, just as a slicing on an Abelian category extended that of a torsion pair. The upshot is that a slicing on $\mathcal{D}$ determines a family of hearts, and is the same as a slicing on one of them. Since we will not need this formalism, we just refer to [GKR04] for details.

3.3.3 Stability structures and tilting

As anticipated in Remark 3.3.7.2, when we have a stability structure making sense on different hearts in $\mathcal{D}$, there is no general way of relating semistable objects in such hearts. In this subsection we illustrate that the semistable objects with respect to a polynomial stability P change in a predictable way when we switch to a new heart constructed from the old one via a tilting induced by P.

Consider first a stability phase $\phi : \mathrm{Ob}^\times(\mathcal{A}) \to (\Xi, \leq)$ on the heart $\mathcal{A} \subset \mathcal{D}$ of a bounded t-structure. Suppose that ϕ has the HN property, so that it induces a slicing $\mathcal{S}_\phi = \{\mathcal{S}_\phi(\xi)\}_{\xi\in\Xi}$ of $\mathcal{A}$. As we saw in Prop. 3.1.8, any partition $\Xi = F \sqcup T$ with $F < T$, induces a torsion pair $(\mathcal{S}_\phi(T), \mathcal{S}_\phi(F))$ in $\mathcal{A}$. On the tilted heart

$$\mathcal{A}_t = \langle \mathcal{S}_\phi(T), \mathcal{S}_\phi(F)[1]\rangle_{\mathrm{ext}}$$

we have an induced slicing: define a total order $\leq_t$ on $\Xi_t := T \sqcup F$ by setting $T <_t F$ and keeping the previous order $\leq$ inside F and T.

Lemma 3.3.8. *Define $\mathcal{S}_{\phi,t}(\xi) := \mathcal{S}_\phi(\xi)$ for $\xi \in T$ and $\mathcal{S}_{\phi,t}(\xi) := \mathcal{S}_\phi(\xi)[1]$ for $\xi \in F$. Then $\{\mathcal{S}_{\phi,t}(\xi)\}_{\xi\in\Xi_t}$ is a slicing of the tilted heart $\mathcal{A}_t$.*

Proof. The condition on the vanishing of Hom spaces is clear, as $\mathrm{Hom}_\mathcal{D}(A[1], B) = 0$ for all $A, B \in \mathcal{A}_t$. Take $A \in \mathrm{Ob}^\times(\mathcal{A}_t)$. Since $(\mathcal{S}_\phi(F)[1], \mathcal{S}_\phi(T))$ is a torsion pair in $\mathcal{A}_t$, we have a unique short exact sequence

$$0 \to A_F[1] \to A \to A_T \to 0$$

in $\mathcal{A}_t$ with $A_F \in \mathcal{S}_\phi(F)$ and $A_T \in \mathcal{S}_\phi(T)$, and the slicing $\mathcal{S}_\phi$ gives filtrations of $A_F[1]$ and A_T with quotients in the categories $\mathcal{S}_{\phi,t}(\xi)$; together, these give the required $\mathcal{S}_{\phi,t}$-filtration of A. $\square$

In general we do not expect to be able to recover this new slicing $\mathcal{S}_{\phi,t}$ from a stability phase on $\mathcal{A}_t$. However, this can be done in the case when ϕ is the stability phase ϕ_P associated to some polynomial stability P, say of degree d, on $\mathcal{A}$. Recall that in this case we denote the induced slicing of $\mathcal{A}$ by

$$\{\mathcal{S}_P(p)\}_{[p]\in\Xi}\,, \quad \text{with } \Xi = {}^{\mathbb{R}[t]_{+,\leq d}}/_{\equiv_{\mathrm{G}}}\,.$$

Now, if $\Xi = F \sqcup T$ as above, then (essentially by the arguments in the proof of Lemma 3.2.13) T and F are images of disjoint convex cones $\tilde{T}, \tilde{F}$ whose union is $\mathbb{R}[t]_{+,\leq d}$. Moreover, P maps the nonzero objects of the categories $\mathcal{S}_{\phi_P}(T)$ and $\mathcal{S}_{\phi_P}(F)$ respectively to $\tilde{T}$ and $\tilde{F}$, which means that the nonzero objects of the tilted heart $\mathcal{A}_t$ are mapped by P in the convex cone $C_t := \tilde{T} \sqcup (-\tilde{F})$. So we can define

$$\phi_{P,t} : \mathrm{Ob}(\mathcal{A}_t) \to \mathbb{P}(C_t)$$

again by composing P with the quotient map $C_t \to \mathbb{P}(C_t)$. To order the elements of $\mathbb{P}(C_t)$ we can just extend the definition (3.2.2) of the Gieseker preorder $\preceq_{\mathrm{G}}$ to any nonzero $p, q \in \mathbb{R}[t]_{\leq d}$ (although of course $\preceq_{\mathrm{G}}$ is not a preorder relation on the whole $\mathbb{R}[t]_{\leq d}$):

Lemma 3.3.9. *$\preceq_{\mathrm{G}}$ is a preorder on C_t which descends to a total order $\preceq_{\mathrm{G}}$ on $\mathbb{P}(C_t) = C_t/\equiv_{\mathrm{G}}$. Moreover, $\phi_{P,t} : \mathrm{Ob}^\times(\mathcal{A}_t) \to (\mathbb{P}(C_t), \preceq_{\mathrm{G}})$ is a stability phase with the HN property, and its induced slicing of $\mathcal{A}_t$ is the one constructed from $\mathcal{S}_P$ as in Lemma 3.3.8, that is*

$$\mathcal{S}_{\phi_{P,t}}([p]) = \mathcal{S}_P(p) \text{ for } [p] \in T\,, \quad \mathcal{S}_{\phi_{P,t}}([p]) = \mathcal{S}_P(p)[1] \text{ for } [p] \in F\,.$$

Proof. Notice that $p \preceq_{\mathrm{G}} q \iff -p \preceq_{\mathrm{G}} -q \iff p \succeq_{\mathrm{G}} -q$, and that every $p \in C_t$ is either in the cone $\mathbb{R}[t]_{+,\leq d}$ or in its opposite. Thus we have basically inverted the order of $\tilde{T}, \tilde{F}$ in defining the new preorder; like before, $p \equiv_{\mathrm{G}} q$ if and only if p and q are proportional, so this preorder descends to a total order on $\mathbb{P}(C_t) = C_t/\equiv_{\mathrm{G}}$.

Now we have to check that $\phi_{P,t}$ has the see-saw property, so let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}_t = \langle \mathcal{S}_{\phi_P}(T), \mathcal{S}_{\phi_P}(F)[1]\rangle_{\mathrm{ext}}$. If P_A and P_C belong to $\tilde{T}$, then so does P_B by convexity, and on these objects ϕ_P and $\phi_{P,t}$ have the same values, so we can use Lemma 3.2.13; the same argument applies if P_A and P_C are both in $-\tilde{F}$. So we only have to consider the case in which P_A and P_C belong to different halves, say $P_A \in \tilde{T}$ and $P_C \in -\tilde{F}$, so that $P_C \succ_{\mathrm{G}} P_A$; if $P_B \in \tilde{T}$, then the relation $P_A = P_B + (-P_C)$ in $\mathbb{R}[t]_{+,\leq d}$ gives $-P_C \prec_{\mathrm{G}} P_A \prec_{\mathrm{G}} P_B$ as in Lemma 3.2.13, hence $P_A \prec_{\mathrm{G}} P_B \prec_{\mathrm{G}} P_C$. If on the other hand $P_B \in -\tilde{F}$, then we have $P_A + (-P_B) = (-P_C)$ in $\mathbb{R}[t]_{+,\leq d}$, which gives $P_A \succ_{\mathrm{G}} -P_C \succ_{\mathrm{G}} -P_B$ and thus again $P_A \prec_{\mathrm{G}} P_B \prec_{\mathrm{G}} P_C$. The case in which $P_C \in \tilde{T}$ and $P_A \in -\tilde{F}$ is dealt with in the same way.

Finally, the $\phi_{P,t}$-semistable objects are the same as ϕ_P-semistable objects, up to a possible shift: given $A \in \mathrm{Ob}^\times(\mathcal{A}_t)$, consider the short exact sequence $0 \to A_F[1] \to A \to A_T \to 0$ with $A_F \in \mathcal{S}_{\phi_P}(F)$ and $A_T \in \mathcal{S}_{\phi_P}(T)$. By the see-saw property, this sequence is destabilizing unless $A_F = 0$ or $A_T = 0$. By using also the HN filtrations of A_F and A_T with respect to ϕ_P we see that if A is $\phi_{P,t}$-semistable, then it either lies in $\mathcal{S}_{\phi_P}(T)$ and is ϕ_P-semistable, or it is in $\mathcal{S}_{\phi_P}(F)[1]$, in which case $A[-1]$ is ϕ_P-semistable. Conversely, we can proceed analogously using $\mathcal{S}_{\phi_P,t}$-filtrations and conclude that ϕ_P-semistable objects are, possibly after a shift, also $\phi_{P,t}$-semistable objects. So the categories of $\phi_{P,t}$-semistable objects form a slicing, which means that $\phi_{P,t}$ has the HN property. □

We end this subsection by noticing that in general we do not have $\phi_{P,t} = \phi_{P_t}$ for some polynomial stability P_t on $\mathcal{A}_t$ (as will be clear in Example 3.3.10.2), except when $d = 1$, that is when $P = -\Re Z + t\Im Z$ for a central charge Z: if $\Xi = (0, 1]$ is split as $(0, 1] = (0, \xi_0] \cup (\xi_0, 1]$, then the objects of the tilted heart

$$\mathcal{A}_t = \langle \mathcal{S}_Z((\xi_0, 1]), \mathcal{S}_Z((0, \xi_0])[1]\rangle_{\mathrm{ext}} \tag{3.3.2}$$

are mapped by Z in the upper-half plane rotated by an angle $\pi\xi_0$. Hence we can just rotate Z accordingly to get a central charge Z_t on $\mathcal{A}_t$:

$$Z_t := e^{-i\pi\xi_0} Z\,.$$

Clearly the stability phase ϕ_{Z_t} constructed from Z_t is the one studied in Lemma 3.3.9, so we see as a special case that for objects in the intersection $\mathcal{A} \cap \mathcal{A}_t = \mathcal{S}_Z((\xi_0, 1])$ the notions of (semi)stability

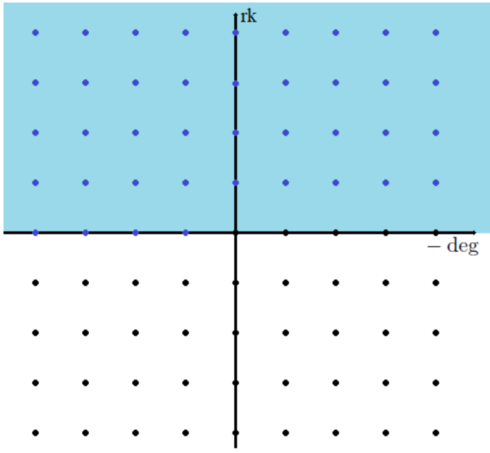


Figure 3.1: The blue dots are the classes of nonzero sheaves. No open half plane can contain all of them.

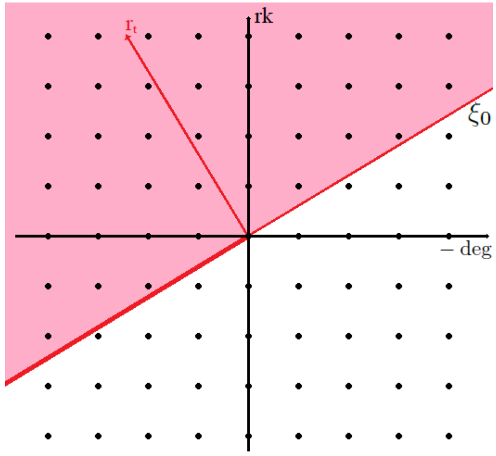


Figure 3.2: The image in $K_{\mathrm{num}}(C)$ of the tilted heart $\mathcal{C}_t$. If ξ_0 is suitably chosen, then all the classes of nonzero objects are contained in the open half-plane $r_t > 0$.

with respect to Z or Z_t coincide. This is maybe the main reason why central charges are best-behaved among all the types of stability structures introduced so far, when extended to triangulated categories.

For polynomial stabilities of degree $d \geq 2$ this rotation procedure is impossible, but we can still construct *some other* polynomial stability on $\mathcal{A}_t$. If the tilt is chosen appropriately, then we can also construct on $\mathcal{A}_t$ a polynomial stability of degree less than d, as outlined in the following examples.

Examples 3.3.10.

1 We construct degree zero stability structures on a curve. Let C be a smooth projective curve. We have seen (Ex. 3.2.20.1) that on $\mathcal{C} = \mathbf{Coh}_{\mathcal{O}_C}$ we have a "numerical" central charge $Z = -\deg + i\,\mathrm{rk} : K_{\mathrm{num}}(C) \to \mathbb{C}$, which is the same as a degree 1 polynomial stability $P = t\,\mathrm{rk} + \deg$. This is the best we can achieve on the standard heart $\mathcal{C}$, in the sense that there is no $\mathbb{Z}$-linear function $r : K_{\mathrm{num}}(C) \to \mathbb{R}$ positively valued on nonzero coherent sheaves, since it is clear that the subsemigroup in $K_{\mathrm{num}}(C) \simeq \mathbb{Z}^2$ that they span cannot be contained in any open half-plane (see Figure 3.1).

As just explained, we can choose a phase $\xi_0 \in (0, 1)$ (or a slope $\mu_0 = -\cot \xi_0$) and get a central charge $Z_t = -d_t + ir_t := e^{-i\pi\xi_0} Z$ on the tilted heart $\mathcal{C}_t$ of Eq. 3.3.2. Now suppose that we have chosen ξ_0 so that $\mathcal{S}_Z(\xi_0) = 0$ (i.e. there are no semistable objects of phase ξ_0). Then the phase measured by Z of all the nonzero objects of $\mathcal{C}_t$ is contained in $(\xi_0, \xi_0 + \pi)$, or in other words $r_t(X) = \Im Z_t(X) > 0$ for all $X \in \mathrm{Ob}^{\times}(\mathcal{C}_t)$, which means that $r_t : K_{\mathrm{num}}(C) \to \mathbb{R}$ is a polynomial stability of degree zero. Of course the condition $\mathcal{S}_Z(\xi_0) = 0$ is met if μ_0 is irrational. If $C = \mathbb{P}^1$, the Z-semistable objects can only have integral slopes (see Figure 5.4), so we can also choose any ξ_0 different from $\arg(k + i)$ for all $k \in \mathbb{Z}$.

Notice also that if we tilt at a rational slope μ_0, then r_t has discrete range, and thus $\mathcal{C}_t$ is an abelian category of finite length, as seen in Ex. 3.2.14.1.

2 Degree one stability structures (i.e. central charges) on surfaces. Let (X, H) be a polarized smooth projective surface. Taking the Hilbert polynomial gives a polynomial stability $P_{.,H} : K_{\mathrm{num}}(X) \to \mathbb{R}[t]_{\leq 2}$ of degree 2 on $\mathbf{Coh}_{\mathcal{O}_X}$ with the HN property. It is well-known (see e.g. [BBHR09, Prop. D.23]) that the standard heart $\mathcal{C} := \mathbf{Coh}_{\mathcal{O}_X}$ does not admit (numerical) polynomial stabilities of degree 1, that is central charges, but these may be constructed on other hearts in $D^b(X)$. As in the previous example, we can easily construct such hearts by tilting $\mathcal{C} := \mathbf{Coh}_{\mathcal{O}_X}$ with respect to $P_{.,H}$: fix a polynomial $p_0 \in \mathbb{R}[t]_{+,\leq 2}$ and tilt $\mathcal{C}$ with

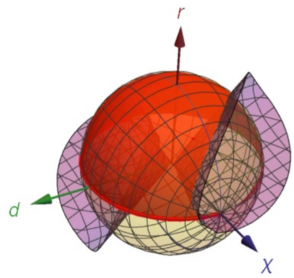


Figure 3.3: The target space $\mathbb{R}[t]_{\leq 2} \simeq \mathbb{R}^3$ of $P_{\cdot,H} = rt^2 + dt + \chi$. The classes of nonzero sheaves are in the orange region (the purple cone is the locus where $\Delta < 0$).

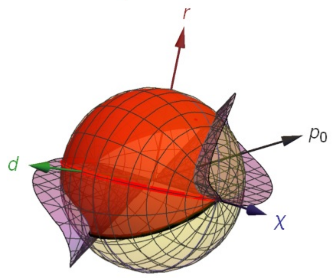


Figure 3.4: The nonzero objects of the tilted heart $\mathcal{C}_t$ are mapped in the convex hull of the new orange region.

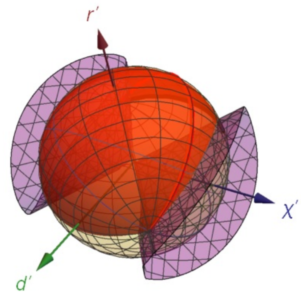


Figure 3.5: After switching to rotated coordinates r', d', χ', the nonzero objects of the tilted heart $\mathcal{C}_t$ are mapped in the region $\{r' > 0\} \cup \{r' = 0 \text{ and } d' > 0\}$.

respect to the partition

$$\mathbb{R}[t]_{+,\leq 2}/\equiv_{\mathrm{G}} = F \sqcup T := (-\infty, [p_0]] \sqcup ([p_0], [1]]\,.$$

Now the nonzero objects of the tilted heart

$$\mathcal{C}_t = \langle \mathcal{S}_{P_{\cdot,H}}(([p_0], 1]), \mathcal{S}_{P_{\cdot,H}}((-\infty, [p_0]])[1] \rangle_{\mathrm{ext}}$$

are mapped by $P_{\cdot,H}$ in the cone $T \sqcup (-F)$, which is easily seen to be obtained by rotating the cone $\mathbb{R}[t]_{+,\leq 2}$ inside $\mathbb{R}[t]_{\leq 2} \cong \mathbb{R}^3$.

Thus a polynomial stability structure $P_t = at^2 + bt + c$ on the tilted heart $\mathcal{C}_t$ can be constructed by rotating $P_{\cdot,H}$ appropriately, so that $[-p_0]$ becomes the greatest phase. But again, if there are no semistable sheaves whose Hilbert polynomial is proportional to p_0, then P_t never reaches this maximal phase. This means that $\tilde{P} := at + b$ is also positively-valued on $\mathrm{Ob}^{\times}(\mathcal{C}_t)$, that is $Z := -b + ia$ is a central charge on $\mathcal{C}_t$. Also in this case we can choose an irrational direction $[p_0]$ to make this work. But we can also choose any p_0 inside the cone described by the inequality $\Delta < 0$ (where Δ is the discriminant), as there are no semistable sheaves with Hilbert polynomial in this cone (see Rmk 4.2.10.1 and the figures).

As a side remark, it would be interesting to check if the central charges of Example 3.3.10.2 are Bridgeland stability conditions (i.e. if they have the HN property and the so-called *support property*) and compare them with those constructed in [AB13] on tilted hearts with respect to slope-stability.

3.3.4 Compatibility of hearts under a stability structure

Take a triangulated category $\mathcal{D}$, an alternating $\mathbb{Z}$-bilinear form $\sigma : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \to \mathbb{R}[t]$, the hearts $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$ of two bounded t-structures, and $v \in K_0(\mathcal{D})$.

To relate σ-(semi)stable objects in the two hearts, we would like the following compatibility conditions to be satisfied. First, we want the σ-semistable objects in one heart to belong also to the other:

(C1) For any object $D \in \mathcal{D}$ belonging to the class v, the following conditions hold:

- (a) if D is a σ-semistable object of $\mathcal{A}$, then it also belongs to $\mathcal{B}$;
- (b) if D is a σ-semistable object of $\mathcal{B}$, then it also belongs to $\mathcal{A}$.

Second, we want that σ-(semi)stability can be equivalently checked in one heart or the other:

(C2) For any object $D \in \mathcal{A} \cap \mathcal{B}$ belonging to the class v, we have that D is σ-(semi)stable in $\mathcal{A}$ if and only if it is σ-(semi)stable in $\mathcal{B}$.

Definition 3.3.11. We say that the hearts $\mathcal{A}$ and $\mathcal{B}$ are (σ, v)-*compatible* when they satisfy the above conditions (C1) and (C2).

Remark 3.3.12. Denote now by $\mathcal{A}^{\mathrm{st}}_{\sigma,v} \subset \mathcal{A}^{\mathrm{ss}}_{\sigma,v} \subset \mathcal{A}$ the subcategories of σ-stable and σ-semistable objects in $\mathcal{A}$ of class v, and similarly with $\mathcal{A}$ replaced by $\mathcal{B}$. Then $\mathcal{A}$ and $\mathcal{B}$ are (σ, v)-compatible if and only if

$$\mathcal{A}^{\mathrm{ss}}_{\sigma,v} = \mathcal{B}^{\mathrm{ss}}_{\sigma,v} \quad \text{and} \quad \mathcal{A}^{\mathrm{st}}_{\sigma,v} = \mathcal{B}^{\mathrm{st}}_{\sigma,v} \,.$$

In particular, notice that (σ, v)-compatibility is an equivalence relation between hearts of bounded t-structures in $\mathcal{D}$.

Consider for example the case of the alternating form σ_P induced by a polynomial function $P : K_0(\mathcal{D}) \to \mathbb{R}[t]_{\leq d}$ as in Eq. (3.2.1):

Lemma 3.3.13. *Suppose that P is a polynomial stability with the HN property on the heart $\mathcal{A}$, and take a partition*

$$\mathbb{R}[t]_{+,\leq d}/\equiv_{\mathrm{G}} =: \Xi = F \sqcup T$$

with $F \prec_{\mathrm{G}} T$ and the associated tilted heart $\mathcal{A}_t$ as in §3.3.3. Then $\mathcal{A}$ and $\mathcal{A}_t$ are (σ_P, v) compatible for any $v \in K_0(\mathcal{D})$ such that $[P_v] \in T$.

Proof. This follows immediately from Lemma 3.3.9, where we have shown that the abelian categories of σ_P-semistable objects in $\mathcal{A}$ and $\mathcal{A}_t$ with a fixed phase $[p] \in T$ coincide. In particular, σ_P-stable objects also coincide, as they are the simple objects in these abelian categories. □

Remark 3.3.14. Typically (e.g. this is the case when $\mathcal{D} = D^b(X)$, as discussed in §4.2.4) there is some notion of families of objects in the hearts of $\mathcal{D}$, so that we have moduli stacks (or even moduli spaces) $\mathfrak{M}_{\mathcal{A},\sigma}(v)$, $\mathfrak{M}_{\mathcal{B},\sigma}(v)$ of σ-(semi)stable objects in $\mathcal{A}, \mathcal{B}$ respectively, and belonging to the class v. Then we have $\mathfrak{M}_{\mathcal{A},\sigma}(v) = \mathfrak{M}_{\mathcal{B},\sigma}(v)$ if $\mathcal{A}$ and $\mathcal{B}$ are (σ, v)-compatible.

Chapter 4

Moduli

In this chapter we provide all the facts about quiver representations, coherent sheaves, and their moduli spaces that will be used in the rest of the thesis.

In the whole chapter $\mathbb{K}$ will denote an algebraically closed field of characteristic 0.

4.1 Quiver moduli

This section is an introduction to moduli spaces of representations of quivers, or *quiver moduli*, a topic that was introduced by King in [Kin94]. The material in subsections 4.1.1-4.1.7 is more or less standard, and it is mostly based on King's paper and on the notes [Rei08].

The last two subsections are devoted to a detailed study of moduli of representations of (generalized) Kronecker quivers. Most of the results there are due to Drézet [Dré87], although sometimes we reprove them using a different approach.

4.1.1 Quiver representations

In this subsection we set up the notation about the representation theory of quivers.

A *quiver* Q is an oriented graph. Formally, it is a couple $Q = (I, \Omega)$, consisting of a set I of vertices, a collection Ω of arrows between them and source and target maps $s, t : \Omega \to I$. We only consider finite quivers here.

A *representation* of Q over a field $\mathbb{K}$ consists of an I-graded $\mathbb{K}$-vector space $V = \oplus_{i\in I} V_i$ and an element $f = (f_h)_{h\in\Omega}$ of the vector space

$$R_V := \oplus_{h\in\Omega} \operatorname{Hom}_{\mathbb{K}}(V_{s(h)}, V_{t(h)})\,.$$

With the obvious definition of morphism between representations, the finite-dimensional representations of Q make an abelian category, which we denote by $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q)$. This is identified with the category of *left*[1] modules over the path algebra $\mathbb{K}Q$ which are finite-dimensional over $\mathbb{K}$. The dimension vector of a representation on V is the array $\underline{\dim}\, V = (\dim_{\mathbb{K}} V_i)_{i\in I} \in \mathbb{N}^I$; this is additive on short exact sequences, and thus it descends to a group homomorphism $\underline{\dim} : K_0(Q) \to \mathbb{Z}^I$ on the Grothendieck group $K_0(Q) := K_0(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q))$. The Euler form χ on $K_0(Q)$ can be written via $\underline{\dim}$: we can define a bilinear form on $\mathbb{Z}^I$, also denoted by χ, as

$$\chi(d, d') := \sum_{i\in I} d_i d'_i - \sum_{h\in\Omega} d_{s(i)} d'_{t(i)}\,, \tag{4.1.1}$$

and then $\chi(v, w) = \chi(\underline{\dim}\, v, \underline{\dim}\, w)$ for all $v, w \in K_0(Q)$.

The category $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q)$ is of finite length and hereditary, i.e. of homological dimension at most 1. If Q is acyclic (i.e. it does not contain oriented cycles), then its simple objects are the representations $S(i)$ with $\mathbb{K}$ at the ith vertex and zeroes elsewhere; in particular, the classes of these objects form a basis of the Grothendieck group $K_0(Q)$, which is then identified with the lattice $\mathbb{Z}^I$ via the isomorphism $\underline{\dim} : K_0(Q) \to \mathbb{Z}^I$.

[1] As anticipated in §1.3, we adopt the convention in which arrows of Q are composed in $\mathbb{K}Q$ like functions.

4.1.2 Relations

Take a quiver $Q = (I, \Omega)$. Often we will consider representations of Q subject to certain relations: a *relation* on Q is a $\mathbb{K}$-linear combination

$$\rho = \sum_{\ell=1}^{m} \alpha_\ell h_{\ell,1} \cdots h_{\ell,k_\ell} \in \mathbb{K}Q$$

of paths of length $k_\ell \geq 2$ with common source $s(\rho) = s(h_{\ell,k_\ell})$ and target $t(\rho) = t(h_{\ell,1})$. A representation (V, f) is *bounded by* ρ if the linear map

$$f_\rho := \sum_{\ell=1}^{m} \alpha_\ell f_{h_{\ell,1}} \circ \cdots \circ f_{h_{\ell,k_\ell}} : V_{s(\rho)} \to V_{t(\rho)}$$

is zero. A set of relations generates an ideal $J \subset \mathbb{K}Q$ containing only paths of length at least two, and any such ideal is generated by relations. We will call the couple (Q, J) a *quiver with relations.*

Representations of Q bounded by all the relations in J will be simply referred to as *representations of* (Q, J). They form a Serre subcategory[2]

$$\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q; J) \subset \mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q)$$

equivalent to left $\mathbb{K}Q/J$-modules of finite dimension.

If we restrict to representations on a fixed graded vector space $V = \oplus_{i \in I} V_i$, those bounded by the relations in J form a closed subscheme[3]

$$X_{V,J} \subset R_V \ ,$$

namely the zero locus of the maps $R_V \to \mathrm{Hom}_{\mathbb{K}}(V_{s(\rho)}, V_{t(\rho)})$ sending a representation f to its evaluation f_ρ, for all relations $\rho \in J$.

4.1.3 Stability for quiver representations

Take a quiver $Q = (I, \Omega)$. The first definition of stable representation of Q was given by King using a $\mathbb{Z}$-valued or $\mathbb{R}$-valued weight as in Def. 3.2.1. We consider more generally weights with values in the polynomial ring $\mathbb{R}[t]$. Take an array $\theta \in \mathbb{R}[t]^I$: this defines $\nu_\theta : K_0(Q) \to \mathbb{R}[t]$ by

$$\nu_\theta(v) := \theta \cdot \underline{\dim}\, v = \sum_{i \in I} \theta^i (\underline{\dim}\, v)_i \tag{4.1.2}$$

(when Q is acyclic, we have seen that $K_0(Q) \simeq \mathbb{Z}^I$, thus every polynomial weight on $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q)$ has this form).

Definition 4.1.1. We call a representation (V, f) *θ-(semi)stable* when it is ν_θ-(semi)stable according to Def. 3.2.1, namely when we have $\theta \cdot \underline{\dim}\, V = 0$ and $\theta \cdot \underline{\dim}\, W$ $(\geq)\!>0$ for any subrepresentation $0 \neq W \subsetneq V$.[4] We denote by

$$R^{\mathrm{st}}_{V,\theta} \subset R^{\mathrm{ss}}_{V,\theta} \subset R_V$$

the subsets of θ-stable and θ-semistable representations on V, and by

$$\mathcal{S}_\theta \subset \mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q) \tag{4.1.3}$$

the subcategory of θ-semistable representations of Q (of any dimension), including the zero representation.

[2]That is, an abelian subcategory closed under taking subobjects, quotient objects, and extensions.

[3]Usually people ignore the possible non-reducedness of $X_{V,J}$. However, this is necessary for consistency with the discussion of §4.1.5, as our moduli functors are defined on the category $\mathbf{AlgSch}_{\mathbb{K}}$ of possibly non reduced algebraic schemes. Anyway, this would not make a significant difference in the rest of the thesis, as in the cases of our interest the scheme $X_{V,J}$ will always be reduced, at least when intersected with the locus of semistable representations.

[4]This is the definition used in [Kin94]; [Rei08] uses instead slope-stability as in Ex. 3.2.20.2.

Remarks 4.1.2.

1 By Lemma 3.2.2, $\mathcal{S}_\theta$ is an extension-closed abelian subcategory, and the θ-stable representations are exactly the simple objects of $\mathcal{S}_\theta$. In particular, the endomorphism algebra of a θ-stable representation is a finite-dimensional division algebra over the algebraically closed field $\mathbb{K}$, and hence only contains multiples of the identity.

2 Suppose that a dimension vector $d \in \mathbb{N}^I$ such that $\theta \cdot d = 0$ is *θ-coprime*, meaning that we have $\theta \cdot d' \neq 0$ for any $0 \neq d' < d$.[5] Then a d-dimensional representation cannot be strictly θ-semistable. Note also that if d is θ-coprime, then it is a primitive vector of $\mathbb{Z}^I$. Conversely, if d is a primitive vector, $\theta \cdot d = 0$ and the coefficients $\theta^0, ..., \theta^k \in \mathbb{R}^I$ of θ span a subspace of dimension at least equal to $\#I - 1$, then d is θ-coprime.

As $\mathcal{S}_\theta$ has finite length, any θ-semistable representation admits a composition series with θ-stable quotients, which are unique up to isomorphisms and permutations:

Definition 4.1.3. Two θ-semistable representations are called *S-equivalent* if they have the same composition factors in $\mathcal{S}_\theta$.

The reason of this definition is of course that it is related to S-equivalence in the sense of GIT (see Prop. 4.1.5 in the next subsection).

4.1.4 Moduli spaces of quiver representations

Fix a quiver $Q = (I, \Omega)$ and a dimension vector $d \in \mathbb{N}^I$, and consider representations of Q on $V := \mathbb{K}^d := \oplus_{i \in I} \mathbb{K}^{d_i}$. The group $G_d := \prod_{i \in I} \mathrm{GL}_{\mathbb{K}}(d_i)$ acts on the representation space $R_d = R_V$ by simultaneous conjugation:

$$(g_i)_{i \in I} \cdot (f_h)_{h \in \Omega} := (g_{t(h)} f_h g_{s(h)}^{-1})_{h \in \Omega} .$$

Clearly the isomorphism classes of representations on V are the orbits of this action, that is the elements of the quotient set R_d/G_d. The subgroup $\Delta := \{(\lambda \, \mathrm{Id}_i)_{i \in I} \, , \; \lambda \in \mathbb{K}^\times\}$ acts trivially, so the action descends to $PG_d := G_d/\Delta$.

Now we want to consider quotient *spaces* of R_d by G_d to have varieties parameterizing quiver representations. The set-theoretical quotient $R_d/G_d = R_d/PG_d$ often is not a variety, but as PG_d is a reductive group we can consider the classical invariant theory quotient

$$\mathrm{M}_{Q,0}(d) := R_d /\!/ PG_d = \operatorname{Spec} \mathbb{K}[R_d]^{G_d}$$

Generators of the invariant ring $\mathbb{K}[R_d]^{G_d}$ are given by a result of Le Bruyn and Procesi:

Proposition 4.1.4. *[LBP90, Thm 1] The $\mathbb{K}$-algebra $\mathbb{K}[R_d]^{G_d}$ is generated by traces of oriented cycles in Q, that is the polynomial functions*

$$T_w(f) := \mathrm{tr}(f_{h_1} \circ \cdots \circ f_{h_n}) \, ,$$

for $w = h_1 \cdots h_n$ an oriented cycle in Q. In fact, it is sufficient to consider only the oriented cycles of length $n \leq (\sum_{i \in I} d_i)^2$.

When Q is acyclic, thus, $\mathrm{M}_{Q,0}(d)$ is just a point. So it is better to consider instead a GIT quotient with respect to a linearization: given an integral array $\theta \in \mathbb{Z}^I$ such that $d \cdot \theta = 0$, we construct a character $\chi_\theta : PG_d \to \mathbb{K}^\times$ by $\chi_\theta(g) := \prod_{i \in I} (\det g_i)^{\theta_i}$; this induces a linearization of the trivial line bundle on R_d and then a notion of (semi)stability.

The key observation is now that this notion is the same as θ-(semi)stability:

Proposition 4.1.5. *[Kin94, Prop. 3.1 and 3.2] The semistable and stable loci with respect to the linearization induced by χ_θ are precisely $R^{\mathrm{ss}}_{d,\theta}$ and $R^{\mathrm{st}}_{d,\theta}$. Moreover, two θ-semistable representations are S-equivalent (in the sense of Def. 4.1.3) if and only if the corresponding points in $R^{\mathrm{ss}}_{d,\theta}$ are S-equivalent with respect to this linearization.*[6]

[5]The notation $0 \neq d' < d$ means that $0 \neq d' \neq d$ and $d'_i \leq d_i$ for all $i \in I$.

[6]Recall that this means that the closures of their orbits inside $R^{\mathrm{ss}}_{d,\theta}$ intersect each other.

So we can now take the GIT quotients with respect to the above linearizations, which we denote by

$$\mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d) := R^{\mathrm{ss}}_{d,\theta}/\!/_{\chi_\theta} PG_d\,, \quad \mathrm{M}^{\mathrm{st}}_{Q,\theta}(d) := R^{\mathrm{st}}_{d,\theta}/\!/_{\chi_\theta} PG_d\,.$$

Notice that $\mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$ comes with a projective morphism to the affine quotient $\mathrm{M}_{Q,0}(d)$ (and hence it is a projective variety when Q is acyclic), and $\mathrm{M}^{\mathrm{st}}_{Q,\theta}(d) \subset \mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$ is an open subset.

We call these varieties the *moduli spaces* of θ-(semi)stable representations of Q on V. This terminology is obviously motivated by the fact that $\mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$ and $\mathrm{M}^{\mathrm{st}}_{Q,\theta}(d)$ are varieties which parameterize θ-semistable representations up to S-equivalence and θ-stable representations up to isomorphism. In §4.1.5 we will see that in fact these are coarse moduli spaces of certain moduli functors.

Notice also that so far we have defined these moduli spaces only for $\theta \in \mathbb{Z}^I$. However, they make perfect sense also for $\theta \in \mathbb{R}[t]^I$, as explained in §4.1.6.

Finally, we can include relations in the picture: if (Q, J) is a quiver with relations, then the closed subscheme $X_{d,J} \subset R_d$ cut by the relations is clearly PG_d-invariant, and by taking the GIT quotient we get *moduli spaces*

$$\mathrm{M}^{\mathrm{ss/st}}_{Q,\theta}(d) := (R^{\mathrm{ss/st}}_{d,\theta} \cap X_{d,J})/\!/_{\chi_\theta} PG_d$$

of θ-semistable representations of (Q, J), which are closed subschemes of the varieties $\mathrm{M}^{\mathrm{ss/st}}_{Q,\theta}(d)$.

4.1.5 Modular interpretations of quiver moduli

Take an algebraic $\mathbb{K}$-scheme S.

Definition 4.1.6. A *(flat) family of representations* of $Q = (I, \Omega)$ over S consists of an I-graded vector bundle $\mathcal{V} = \oplus_{i\in I}\mathcal{V}_i$ over S and a collection $f = (f_h)_{h\in\Omega}$ of bundle maps belonging to the vector space

$$R_{\mathcal{V}} := \oplus_{h\in\Omega} \mathrm{Hom}_{\mathbb{K}}(\mathcal{V}_{s(h)}, \mathcal{V}_{t(h)})\,.$$

Together with the obvious definition of morphisms, flat families of d-dimensional representations and their isomorphisms form a groupoid fibration $\mathfrak{M}_Q(d) \to \mathbf{AlgSch}_{\mathbb{K}}$; this is a stack by standard arguments, as bundles and bundle maps glue appropriately over open covers. In fact, $\mathfrak{M}_Q(d)$ is an algebraic $\mathbb{K}$-stack, as we will see that it is isomorphic to the stack quotient $[R_d/G_d]$.

When we also have relations $J \subset \mathbb{K}Q$, we say that the family $(\mathcal{V}, f)$ is *bounded* by J, and that it is a family of representations of (Q, J), when for any relation $\rho \in J$ the evaluation map $R_{\mathcal{V}} \to \mathrm{Hom}(\mathcal{V}_{s(\rho)}, \mathcal{V}_{t(\rho)})$ at ρ evaluates f as $f_\rho = 0$.

Remark 4.1.7. Just as representations of (Q, J) correspond to left $\mathbb{K}Q/J$-modules, a flat family of representations over S is the same as a locally-free $\mathcal{O}_S$-module $\mathcal{V}$ with a homomorphism $\mathbb{K}Q/J \to \mathrm{End}(\mathcal{V})$ making $\mathcal{V}$ into a sheaf of left $(\mathbb{K}Q/J) \otimes_{\mathbb{K}} \mathcal{O}_S$-modules.

Such families form a closed substack $\mathfrak{M}_{Q,J}(d) \subset \mathfrak{M}_Q(d)$, isomorphic to the quotient $[X_{d,J}/G_d]$. In the remainder of this subsection we will give a proof of this isomorphism, which although intuitive and well-known seems to be hard to find in the literature.

First of all, consider the tautological family $\mathcal{U}$ of d-dimensional representations on $X_{d,J}$, namely the trivial bundle $\mathcal{U} = X_{d,J} \times \mathbb{K}^d \to X_{d,J}$ with the bundle maps F_h whose fibers over a point $f = (f_h) \in X_{d,J}$ are $F_h(f) = f_h$.

Lemma 4.1.8. *$(\mathcal{U}, (F_h)_{h\in\Omega})$ is a versal family for $\mathfrak{M}_{Q,J}(d)$ whose symmetry groupoid is the action groupoid $G_d \times X_{d,J} \rightrightarrows X_{d,J}$ (Def. 4.A.3).*

Proof. Given a family $(\mathcal{V}, (f_h))$ over a scheme S, we can take a Zariski cover $S = \cup_\alpha U_\alpha$ and trivializations $\mathcal{V}_i\restriction_{U_\alpha} \overset{\simeq}{\to} \mathbb{K}^{d_i} \times U_\alpha$. Thus, for any α, the restrictions to U_α of the bundle maps f_h can be seen as morphisms $U_\alpha \to \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{d_{s(h)}}, \mathbb{K}^{d_{t(h)}})$, which together give a morphism $u : U_\alpha \to X_{d,J}$ such that $u^*(\mathcal{U}, (F_h)) \simeq (\mathcal{V}\restriction_{U_\alpha}, (f_h\restriction_{U_\alpha}))$.
Now we have to verify that we have a fiber product

$$\begin{array}{ccc} \underline{G_d \times X_{d,J}} & \xrightarrow{\lambda} & \underline{X_{d,J}} \\ {\scriptstyle \mathrm{pr}_{X_{d,J}}}\downarrow & \overset{\eta}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle \phi_{\mathcal{U}}} \\ \underline{X_{d,J}} & \xrightarrow{\phi_{\mathcal{U}}} & \mathfrak{M}_{Q,J}(d) \end{array}$$

where $\phi_{\mathcal{U}} : \underline{X_{d,J}} \to \mathfrak{M}_{Q,J}(d)$ is the morphism associated to the family $(\mathcal{U}, (F_h))$. First of all, notice that this family is G_d-equivariant: we have a tautological isomorphism $\phi : \mathrm{pr}^*_{X_{d,J}} \mathcal{U} \overset{\simeq}{\to} \lambda^* \mathcal{U}$ which, over a point $(g, f) \in G_d \times X_{d,J}$, is given by the isomorphism $g = (g_i)_{i \in I} : \mathbb{K}^d \to \mathbb{K}^d$ between the representations f and $g \cdot f$. Thus we have a natural isomorphism η as in the diagram above, given on an element $S \overset{(g,f)}{\to} G_d \times X_{d,J}$ of $\underline{G_d \times X_{d,J}}(S)$ by

$$\eta_{(g,f)} := (g, f)^* \phi : f^* \mathcal{U} \overset{\simeq}{\to} (g \cdot f)^* \mathcal{U} \,,$$

making the diagram 2-commutative. So we have an induced morphism

$$\underline{G_d \times X_{d,J}} \to \underline{X_{d,J}} \times_{\mathfrak{M}_{Q,J}(d)} \underline{X_{d,J}}$$

sending $S \overset{(g,f)}{\to} G_d \times X_{d,J}$ to $(f, g \cdot f, \eta_{(g,f)})$. But this is an isomorphism, as we can construct an inverse: given an element $(f, f', f^* \mathcal{U} \overset{\alpha}{\to} f'^* \mathcal{U})$ over S of the fiber product, the isomorphism α can be seen naturally as a map $g : S \to G_d$ such that $g \cdot f = f'$, so we just map (f, f', α) to (g, f). □

The previous Lemma, together with Prop. 4.A.4, immediately implies[7]

Proposition 4.1.9. $\mathfrak{M}_{Q,J}(d) \simeq [X_{d,J}/G_d]$.

Finally, given $\theta \in \mathbb{R}[t]^I$ we can consider the open substacks

$$\mathfrak{M}^{\mathrm{st}}_{Q,J,\theta}(d) \subset \mathfrak{M}^{\mathrm{ss}}_{Q,J,\theta}(d) \subset \mathfrak{M}_{Q,J}(d)$$

of θ-stable and θ-semistable representations. These are isomorphic to quotient stacks $[R^{\mathrm{st/ss}}_{d,\theta} \cap X_{d,J}/G_d]$, and thus they are corepresented by the categorical quotients $\mathrm{M}^{\mathrm{st/ss}}_{Q,J,\theta}(d)$ introduced in the previous subsection. Hence $\mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$ is a coarse moduli space for S-equivalence classes of d-dimensional θ-semistable representations, while the points of $\mathrm{M}^{\mathrm{st}}_{Q,\theta}(d)$ correspond to isomorphism classes of θ-stable representations.

We end by mentioning that a sufficient condition for the versal family $\mathcal{U}$ to descend to the quotient $\mathrm{M}^{\mathrm{st}}_{Q,\theta}(d)$ is that the vector $d \in \mathbb{Z}^I$ is primitive:

Lemma 4.1.10. *[Kin94, Prop. 5.3] If d is primitive (that is, $\gcd(d_i)_{i \in I} = 1$), then $\mathrm{M}^{\mathrm{st}}_{Q,\theta}(d)$ admits a universal family.*

4.1.6 Walls and chambers

Fix a quiver $Q = (I, \Omega)$ and a dimension vector $d \in \mathbb{Z}^I$. Now we will partition the hyperplane $d^\perp \subset \mathbb{R}^I$ into finitely many locally closed subsets where different θ give the same θ-(semi)stable representations.

Definition 4.1.11. We call $\theta_1, \theta_2 \in d^\perp$ *numerically equivalent* when, for any $d' \leq d$ (which means that $d'_i \leq d_i$ for all $i \in I$), $\theta_1 \cdot d'$ and $\theta_2 \cdot d'$ have the same sign (positive, negative or zero).

When this is the case, then a d-dimensional representation of Q is θ_1-(semi)stable if and only if it is θ_2-(semi)stable:

$$R^{\mathrm{ss}}_{d,\theta_1} = R^{\mathrm{ss}}_{d,\theta_2}, \quad R^{\mathrm{st}}_{d,\theta_1} = R^{\mathrm{st}}_{d,\theta_2} \,.$$

We have a finite collection $\{W(d')\}_{d' \in J(d)}$ of rational hyperplanes in $d^\perp$, called *(numerical) walls*, of the form

$$W(d') = \{\theta \in d^\perp \mid \theta \cdot d' = 0\} \,,$$

and where

$$J(d) := \{d' \in \mathbb{N}^I \mid d' \leq d \text{ but } d \text{ is not a multiple of } d'\} \,.$$

[7] In fact, it is also not hard to write an explicit equivalence $\mathfrak{M}_{Q,J}(d) \leftrightarrow [X_{d,J}/G_d]$.

Lemma 4.1.12. *The numerical equivalence classes in $d^\perp$ are the connected components of the locally closed subsets*

$$\cap_{d_1\in J_1} W(d_1) \setminus \cup_{d_2\in J_2} W(d_2)$$

indexed by (possibly trivial) partitions $J(d) = J_1 \sqcup J_2$.

Proof. If $\theta_1, \theta_2 \in d^\perp$ are numerically equivalent, then the set J_1 of walls they belong to is the same, while for any $d_2 \in J_2 := J(d) \setminus J_1$ they both belong to the same half-space, say $H(d_2)$, cut by the wall $W(d_2)$. Thus both θ_1, θ_2 belong to the convex set $(\cap_{d_1\in J_1} W(d_1)) \cap (\cap_{d_2\in J_2} H(d_2))$.

Viceversa, if $\theta_1, \theta_2 \in d^\perp$ are in the same connected component of some $\cap_{d_1\in J_1} W(d_1) \setminus \cup_{d_2\in J_2} W(d_2)$, then they are orthogonal to each $d_1 \in J_1$; moreover, each $W(d_2)$ divides $\cap_{d_1\in J_1} W(d_1)$ in two parts, and θ_1, θ_2 must lie in the same. □

Remark 4.1.13. The partition $J(d) = \emptyset \sqcup J(d)$ gives open dense equivalence classes in $d^\perp$, called *(numerical) chambers*. Notice that if d is primitive (i.e. $\gcd(d_i)_{i\in I} = 1$) then there are no strictly θ-semistable representations for θ in such a chamber, so $\mathrm{M}^{\mathrm{st}}_{Q,\theta}(d) = \mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$ is smooth (and projective, if Q is acyclic).

By construction, the subsets $R^{\mathrm{ss}}_{d,\theta}$ and $R^{\mathrm{st}}_{d,\theta}$ do not change when the array θ moves inside a numerical equivalence class, and any such class contains integral arrays, because it is a cone and the walls are rational. This means that also for a real array θ orthogonal to d the moduli spaces $\mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$ and $\mathrm{M}^{\mathrm{st}}_{Q,\theta}(d)$ make sense and are constructed as GIT quotients after choosing a numerically equivalent integral weight $\theta' \in \mathbb{Z}^I$. In fact, the following argument shows that the same is true if, more generally, θ is a polynomial array:

Remark 4.1.14. Consider a polynomial array $\theta = \sum_{\ell=0}^n \theta_\ell t^\ell \in \mathbb{R}[t]^I$ such that $\theta \cdot d = 0$. We outline a procedure to construct an array $\tilde\theta \in \mathbb{R}^I$ numerically equivalent to θ (the notion of numerical equivalence of Def. 4.1.11 is extended to polynomial arrays ordered lexicographically): if the leading coefficient θ_n does not belong to any wall, then θ_n is numerically equivalent to θ. Otherwise (in fact, in any case), for $\epsilon_{n-1} > 0$ small enough, $(\theta_n + \epsilon_{n-1}\theta_{n-1}) \cdot d'$ has, whenever nonzero, the same sign as $\theta \cdot d'$; in particular, if $\theta_n + \epsilon_{n-1}\theta_{n-1}$ is not on a wall, then it is numerically equivalent to θ; otherwise, we continue adding other coefficients until we get an array numerically equivalent to θ (in the worst case, if there is a wall $W(d')$ containing every coefficient θ_ℓ, then we end up with $\tilde\theta = \theta_n + \epsilon_{n-1}\theta_{n-1} + \cdots + \epsilon_0\theta_0$ lying on $W(d')$).

So, for example, if $\theta = t\theta_1 + \theta_0 \in \mathbb{R}[t]^I$ then we can choose $\epsilon > 0$ small enough so that

$$R^{\mathrm{ss}}_{d,\theta} = R^{\mathrm{ss}}_{d,\theta'}, \quad R^{\mathrm{st}}_{d,\theta} = R^{\mathrm{st}}_{d,\theta'},$$

where $\theta' \in \mathbb{Z}^I$ is some integral array lying in the same numerical equivalence class as $\theta_1 + \epsilon\theta_0$.

Finally, note that some numerical walls may not be "real" walls, in the sense that θ-(semi)stable representations may not change when θ moves across them. However, finding which numerical walls are real walls requires the knowledge of the θ-(semi)stable representations, which is more than a numerical task. For our purposes, knowing the numerical walls will always be enough.

We also refer to [HdlP02, §3.4] for more details about the wall system.

4.1.7 Some geometric properties of quiver moduli

In the following remarks we list a few properties of quiver moduli spaces that will be useful in the sequel. $Q = (I, \Omega)$ denotes a quiver, $d \in \mathbb{N}^I$ a dimension vector and $\theta \in \mathbb{R}^I$ an array orthogonal to d.

Remarks 4.1.15.

1 The action of PG_d on $R^{\mathrm{st}}_{d,\theta}$ is free by Remark 4.1.2.1 and thus $\mathrm{M}^{\mathrm{st}}_{Q,\theta}(d)$ is a smooth variety, and its dimension is $\dim R_d - \dim PG_d$, that is

$$\dim \mathrm{M}^{\mathrm{st}}_{Q,\theta}(d) = \sum_{h\in\Omega} d_{s(h)} d_{t(h)} - \sum_{i\in I} d_i^2 + 1 = 1 - \chi(d,d)\,, \tag{4.1.4}$$

where χ is the Euler form of Eq. (4.1.1). Similarly, the quotient stacks $\mathfrak{M}^{\mathrm{ss}}_{Q,\theta}(d)$ and $\mathfrak{M}^{\mathrm{st}}_{Q,\theta}(d)$ are smooth of dimension $-\chi(d,d)$.

2 If d is θ-coprime (see Remark 4.1.2.2) and Q is acyclic, then there are no strictly semistable representations, so $\mathrm{M} := \mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d) = \mathrm{M}^{\mathrm{st}}_{Q,\theta}(d)$ is smooth and projective, and it admits a universal family $\mathcal{U} = \oplus_{i\in I}\mathcal{U}_i$ by Lemma 4.1.10. Moreover, its Chow ring $A^*(\mathrm{M})$ has the following properties [KW95, Theorem 3]:

(a) as an abelian group, $A^*(\mathrm{M})$ is free, and numerical and algebraic equivalence coincide;

(b) for $\mathbb{K} = \mathbb{C}$, the cycle map $A^*(\mathrm{M}) \to H^{2*}(\mathrm{M}, \mathbb{Z})$ is an isomorphism; in particular, M has no odd cohomology and no torsion in even cohomology;

(c) as a $\mathbb{Z}$-algebra, $A^*(\mathrm{M})$ is generated by the Chern classes of the bundles $\mathcal{U}_i$.

3 Take $\mathbb{K} = \mathbb{C}$, and suppose that Q is an acyclic quiver. In [Rei03], Reineke computed the Betti numbers of the moduli spaces $\mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$ in the coprime case by using the stratification of the spaces R_d by Harder-Narasimhan types to produce relations in the Hall algebras of the finitary categories $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{F}_q}(Q)$; he used these to determine the number of points over $\mathbb{F}_q$ of $\mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$, and then their Betti numbers as complex manifolds, via Weil conjectures. With a similar argument using instead motivic Hall algebras, Joyce [Joy08] computed the classes of $\mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$ in the Grothendieck ring $K_0(\mathbf{Var}_{\mathbb{C}})$ of varieties. We briefly recall this result (see e.g. [Bri12] for the concepts used here): first, it is convenient to slightly change for a moment our notion of stability for quiver representations, using instead the slope-stability introduced in Ex. 3.2.20.2: now, given *any* $\theta \in \mathbb{Z}^I$, we have a stability phase (Def. 3.1.1)

$$\mathrm{Ob}^{\times}(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(Q)) \to K_0(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(Q)) \cong \mathbb{Z}^I \xrightarrow{\mu_\theta} \mathbb{R}$$

given by $\mu_\theta(d) := (\theta \cdot d)/(\sum_{i\in I} d_i)$.[8] As in §4.1.5, these give us open substacks

$$\mathfrak{M}^{\mathrm{st}}_{Q,\mu_\theta}(d) \subset \mathfrak{M}^{\mathrm{ss}}_{Q,\mu_\theta}(d) \subset \mathfrak{M}_Q(d) = [R_d/G_d]$$

of μ_θ-(semi)stable representations. The three stacks appearing in the last equation determine elements

$$\delta^d_{\mathrm{st}}(\mu_\theta) := [\mathfrak{M}^{\mathrm{st}}_{Q,\mu_\theta}(d) \hookrightarrow \mathfrak{M}_Q]\,, \quad \delta^d_{\mathrm{ss}}(\mu_\theta) := [\mathfrak{M}^{\mathrm{ss}}_{Q,\mu_\theta}(d) \hookrightarrow \mathfrak{M}_Q]\,, \quad \delta^d = [\mathfrak{M}_Q(d) \hookrightarrow \mathfrak{M}_Q]$$

in the motivic Hall algebra $\mathcal{H}(Q) = K_0(\mathbf{St}_{\mathfrak{M}_Q})$, where $\mathfrak{M}_Q = \sqcup_{d\in\mathbb{N}^I}\mathfrak{M}_Q(d)$, and Joyce proved in [Joy08, Thm 5.2] that

$$\delta^d_{\mathrm{ss}}(\mu_\theta) = \sum_{s\geq 1}\sum_{d^1,\dots,d^s} (-1)^{s-1}\delta^{d^1} * \cdots * \delta^{d^s}\,, \tag{4.1.5}$$

where the sum is taken over all nonzero $d^1, ..., d^s \in \mathbb{N}^I$ such that $\sum_j d^j = d$ and $\mu_\theta(d^1 + \cdots + d^j) > \mu_\theta(d^{j+1} + \cdots + d^s)$ for all j. Since $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(Q)$ is hereditary, we have an integration map which applied to (4.1.5) gives the formula

$$[\mathfrak{M}^{\mathrm{ss}}_{Q,\mu_\theta}(d)] = \sum_{s\geq 1}\sum_{d^1,\dots,d^s} (-1)^{s-1}\mathbb{L}^{-\sum_{j<k}\chi(d^k,d^j)}[\mathfrak{M}_Q(d^1)]\cdots[\mathfrak{M}_Q(d^s)] \tag{4.1.6}$$

in the Grothendieck ring $K_0(\mathbf{St}_{\mathbb{C}})$ of stacks, where $\mathbb{L} = [\mathbb{C}]$ is the Lefschetz class and the sum is as in Eq. 4.1.5. Here χ is the Euler form of Q defined in Eq. (4.1.1), and the classes in the right-hand side are easily computed as

$$[\mathfrak{M}_Q(d)] = \frac{[R_d]}{[G_d]} = \frac{[R_d]}{\prod_{i\in I}[\mathrm{GL}(d_i,\mathbb{C})]} = \frac{\mathbb{L}^{\sum_{h\in\Omega} d_{s(h)}d_{t(h)}}}{\prod_{i\in I}\prod_{\ell=1}^{d_i}(\mathbb{L}^{d_i} - \mathbb{L}^{\ell-1})}\,.$$

When $\mathrm{M}^{\mathrm{ss}}_{Q,\mu_\theta}(d) = \mathrm{M}^{\mathrm{st}}_{Q,\mu_\theta}(d)$, then the equality

$$[\mathrm{M}^{\mathrm{ss}}_{Q,\mu_\theta}(d)] = (\mathbb{L}-1)[\mathfrak{M}^{\mathrm{ss}}_{Q,\mu_\theta}(d)]$$

[8] Thus a representation V is called *μ_θ-(semi)stable* when for any subrepresentation $0 \neq W \subsetneq V$ we have $\mu_\theta(W)\,(\leq)\!<\mu_\theta(V)$. Notice that a d-dimensional representation is θ-(semi)stable (according to Def. 4.1.1) if and only if it is $\mu_{-\theta}$-(semi)stable and $\mu_\theta(d) = 0$. In fact, for a fixed d, μ_θ-stability is always the same as θ'-stability for a suitable $\theta' \in d^\perp$, so this definition does not give anything new, except that it allows semistable objects of different slopes.

allows us to compute any (stacky) motivic invariant of $\mathrm{M}^{\mathrm{ss}}_{Q,\mu_\theta}(d)$, such as Betti and Hodge numbers.

In the presence of relations $J \subset \mathbb{C}Q$, the Serre subcategory $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(Q;J) \subset \mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(Q)$ is not hereditary but induces a subalgebra $\mathcal{H}(Q,J) \subset \mathcal{H}(Q)$ in which the analogue of Eq. (4.1.5) holds, and applying to it the integration map of $\mathcal{H}(Q)$ we deduce that

$$[\mathfrak{M}^{\mathrm{ss}}_{Q,J,\mu_\theta}(d)] = \sum_{s\geq 1} \sum_{d^1,\ldots,d^s} (-1)^{s-1} \mathbb{L}^{-\sum_{j<k}\chi(d^k,d^j)} \frac{[X_{d^1,J}]}{[G_{d^1}]} \cdots \frac{[X_{d^s,J}]}{[G_{d^s}]}\,, \tag{4.1.7}$$

where $X_{d,J} \subset R_d$ denotes the affine variety cut by the relations J (see §4.1.2) and again χ is the Euler form of Q (and not the Euler form of the category $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(Q;J)$).

4 [Rei17] If Q is an acyclic quiver and $d \in \mathbb{N}^I$ is primitive, then it is possible to find $\theta' \in d^\perp$ close to θ such that d is θ'-coprime, and we have a small desingularization $\mathrm{M}^{\mathrm{st}}_{Q,\theta'}(d) \to \mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$, which together with Eq. (4.1.6) computes the intersection cohomology of $\mathrm{M}^{\mathrm{ss}}_{Q,\theta}(d)$.

4.1.8 Kronecker moduli spaces

In this thesis we will often consider moduli of representations of the (generalized) *Kronecker quiver*

$$K_n\colon \quad -1 \;\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}\; 0$$

with $n \geq 2$ arrows. Its representations, or *Kronecker modules*, can be seen as linear maps $f: V_{-1} \otimes Z \to V_0$, where Z is a n-dimensional vector space with a fixed basis $\{e_0, \ldots, e_{n-1}\}$, or as left modules over the *Kronecker algebra*

$$\mathbb{K}K_n = \left(\begin{smallmatrix} \mathbb{K} & Z \\ 0 & \mathbb{K}\end{smallmatrix}\right)\,.$$

The representation theory of the quiver K_2 is the classical *Kronecker problem* of classifying pairs $\mathbb{K}^{d_{-1}} \rightrightarrows \mathbb{K}^{d_0}$ of linear maps up to change of basis in the source and target spaces. The solution is the following (see also e.g. [Ben98, Theorem 4.3.2] for a modern treatment):

Proposition 4.1.16. *[Kro90] All the indecomposable representations of K_2 up to isomorphism are listed below, for $k \geq 0$:*

$$\mathbb{K}^k \underset{J_k(\lambda)}{\overset{\mathbb{I}_k}{\rightrightarrows}} \mathbb{K}^k\,, \qquad \mathbb{K}^k \underset{\mathbb{I}_k}{\overset{J_k(0)^t}{\rightrightarrows}} \mathbb{K}^k\,, \qquad \mathbb{K}^k \underset{(0\ \mathbb{I}_k)^t}{\overset{(\mathbb{I}_k\ 0)^t}{\rightrightarrows}} \mathbb{K}^{k+1}\,, \qquad \mathbb{K}^{k+1} \underset{(0\ \mathbb{I}_k)}{\overset{(\mathbb{I}_k\ 0)}{\rightrightarrows}} \mathbb{K}^k\,,$$

where $J_k(\lambda)$ is the k-dimensional Jordan matrix with eigenvalue $\lambda \in \mathbb{K}$,

$$J_k(\lambda) = \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}.$$

On the other hand, for $n \geq 3$ the quiver K_n is *wild*, that is its representations cannot be listed in 1-parameter families. We then turn to the geometric approach to the representation theory of these quivers, namely to studying moduli of their semistable representations. First of all, notice that the only arrays $\theta \in \mathbb{Z}^{\{-1,0\}}$ giving nontrivial stability weights ν_θ on the representations of K_n are those with $\theta^0 > 0$. These are all in the same chamber, and give the usual notion of (semi)stability for Kronecker modules (see e.g. [Dré87, Prop. 15]):

Definition 4.1.17. A Kronecker module f with $V_{-1} \neq 0$ is (semi)stable if and only if for any subrepresentation $W \subsetneq V$ with $W_{-1} \neq 0$ we have

$$\frac{\dim W_0}{\dim W_{-1}} \;(\geq)\!>\; \frac{\dim V_0}{\dim V_{-1}}\,.$$

We denote by

$$K(n; d_{-1}, d_0) := \mathrm{M}^{\mathrm{ss}}_{K_n,(-d_0,d_{-1})}(d)$$

the moduli space of semistable Kronecker modules of dimension vector d, and by $K_{\mathrm{st}}(n; d_{-1}, d_0) \subset K(n; d_{-1}, d_0)$ the stable locus.

Now we study the properties of these moduli spaces, following [Dré87]. Given a Kronecker module $f \in \mathrm{Hom}_{\mathbb{K}}(V_{-1} \otimes Z, V_0)$, we will often use the notation

$$f_z := f(\cdot \otimes z) \in \mathrm{Hom}_{\mathbb{K}}(V_{-1}, V_0)$$

for $z \in Z$, and $f_j := f_{e_j}$ for $j = 0, ..., n-1$ (here $\{e_0, ..., e_{n-1}\}$ is the basis of Z that we fixed from the beginning); the index j will be tacitly summed when repeated, e.g. for $z \in Z$ we write $z = z^j e_j$ and $f_z = z^j f_j$.

Proposition 4.1.18. *Let $d_{-1} \neq 0$ and $d_0 \neq 0$.*

1. *If $nd_{-1} < d_0$ or $d_{-1} > nd_0$, then $K(n; d_{-1}, d_0) = \emptyset$;*
2. *$K(n; d_{-1}, nd_{-1}) = \mathrm{pt}$, and $K_{\mathrm{st}}(n; 1, n) = \mathrm{pt}$, while $K_{\mathrm{st}}(n; d_{-1}, nd_{-1}) = \emptyset$ if $d_{-1} > 1$;*
3. *$K(n; nd_0, d_0) = \mathrm{pt}$, and $K_{\mathrm{st}}(n; n, 1) = \mathrm{pt}$, while $K_{\mathrm{st}}(n; nd_0, d_0) = \emptyset$ if $d_0 > 1$;*
4. *$\dim K_{\mathrm{st}}(n; d_{-1}, d_0) = nd_{-1}d_0 + 1 - d_{-1}^2 - d_0^2$.*

Proof.

1. For $nd_{-1} < d_0$, any representation (V_{-1}, V_0) is destabilized by the proper subrepresentation $(V_{-1}, \sum_j \mathrm{im}\, f_j)$; similarly, when $d_{-1} > nd_0$ the subrepresentation $(\cap_j \ker f_j, 0)$ is nonzero and destabilizing.
2. If $d_0 = nd_{-1}$, then the only semistable representations are those with $\oplus_j \mathrm{im}\, f_j = V_1$, i.e. those such that every map $f_j : V_{-1} \to V_0$ is injective and the images $\mathrm{im}\, f_j$ have trivial intersection. Moreover, f is completely determined by the images $f_j(v_\ell)$ of a basis $\{v_\ell\}$ of V_{-1}, and these form a basis of V_0; thus every two semistable representations are connected by a change of basis in V_0, and hence they are isomorphic. If $d_{-1} > 1$, then none of these representations is stable, as it is semidestabilized by some subrepresentation $(\mathbb{K}, \mathbb{K}^n)$; otherwise, every semistable representation is also stable.
3. This is proven like the previous item, or it just follows from it by duality (see Prop. 4.1.19 below)
4. This is a special case of the dimension formula (4.1.4).

□

Proposition 4.1.19. *[Dré87, Prop. 21-22] We have isomorphisms*

$$K(n; d_{-1}, d_0) \simeq K(n; d_0, d_{-1}) \,,$$
$$K(n; d_{-1}, d_0) \simeq K(n; nd_{-1} - d_0, d_{-1}) \simeq K(n; d_0, nd_0 - d_{-1}) \,,$$

all restricting to isomorphisms of the stable loci.

Proof. The first isomorphism is just obtained by taking duals: θ-(semi)stable representations of a quiver Q correspond to $-\theta$-semistable representations of its opposite Q^{op} (i.e. Q with arrow reversed). In this case the opposite of K_n is identifiable with K_n itself, and the correct relabeling of vertices gives the original notion of stability.

For the second isomorphism we refer to [Dré87]; the third isomorphism is just the inverse of the second. □

Example 4.1.20. For $0 \leq k \leq n$, $K(n; 1, k) \simeq K(n; k, 1)$ coincide with their stable loci, and they are isomorphic to the Grassmannian $\mathrm{G}_k(n)$ of k-dimensional subspaces of $\mathbb{K}^n$. Indeed, we can identify the $PG_{(1,k)}$-space $R_{(1,k)}$ with the space $M_{k\times n}(\mathbb{K})$ acted by $PG_{(1,k)} \simeq \mathrm{GL}(k, \mathbb{K})$ by left multiplication, and the stable (or semistable) locus consists of matrices of rank k, whose $\mathrm{GL}(k, \mathbb{K})$-orbits are identified, by taking spans of the rows, with k-dimensional subspaces of $\mathbb{K}^n$.

4.1.9 Diagonal Kronecker moduli and determinantal hypersurfaces

In this subsection, which uses the same notation as the previous one, we will analyze in more detail the Kronecker moduli spaces with $d_{-1} = d_0 =: k$. In particular, we will prove that in some cases they are isomorphic to projective spaces.

We recall that Z is a $\mathbb{K}$-vector space of dimension $n \geq 2$. Let $\mathbb{K}[Z]_k \cong \mathrm{Sym}^k Z^\vee$ denote the space of homogeneous polynomial functions $Z \to \mathbb{K}$ of degree k, and consider the set

$$U_{Z,k} = \left\{ f \in R_{(k,k)} \;\middle|\; \max_{z \in Z} \mathrm{rk}\, f_z = k \right\}$$

in $R_{(k,k)} = \mathrm{Hom}_{\mathbb{K}}(Z \otimes \mathbb{K}^k, \mathbb{K}^k)$. This is clearly open and invariant under the action of $G_{(k,k)} = \mathrm{GL}_k(\mathbb{K}) \times \mathrm{GL}_k(\mathbb{K})$. We define a morphism

$$\varphi : U_{Z,k} \to \mathbb{P}(\mathbb{K}[Z]_k)$$

sending a Kronecker module $f \in U_{Z,k}$ to the class of the polynomial function $z \mapsto \det(f_z)$.

Now observe that $U_{Z,k}$ is contained in the semistable locus $R^{\mathrm{ss}}_{(k,k)} \subset R_{(k,k)}$; indeed, a $f \in U_{Z,k}$ cannot have a subrepresentation (W_{-1}, W_0) with $\dim W_{-1} > \dim W_0$, because f_z is an isomorphism for generic $z \in Z$. So it is mapped onto an open subset

$$\tilde{U}_{Z,k} \subset K(n; k, k)\,,$$

in the GIT quotient, and the morphism φ, which is clearly $G_{(k,k)}$-invariant, descends to a morphism

$$\tilde{\varphi} : \tilde{U}_{Z,k} \to \mathbb{P}(\mathbb{K}[Z]_k)\,.$$

Lemma 4.1.21. *If $n = 2$ or $k = 1, 2$, then $\tilde{U}_{Z,k} = K(n; k, k)$. Moreover, $R^{\mathrm{ss}}_{(1,1)} = R^{\mathrm{st}}_{(1,1)} = R_{(1,1)} \setminus \{0\}$ for $n \geq 2$, and $R^{\mathrm{st}}_{(k,k)} = \emptyset$ for $n = 2$ and $k \geq 2$.*

Proof. We want to prove the inclusion $R^{\mathrm{ss}}_{(k,k)} \subset U_{Z,k}$. First we consider the case $n = 2$: it is clear that $R^{\mathrm{ss}}_{(1,1)} = R^{\mathrm{st}}_{(1,1)}$, while for $k \geq 2$ no Kronecker module f can be stable: if $f_0, f_1 \in \mathrm{M}_d(\mathbb{K})$ are both invertible, then let $v \in \mathbb{K}^d$ be an eigenvector of $f_1^{-1} f_0$, so that $f_0(v)$ and $f_1(v)$ are proportional, and $(\mathrm{Span}_{\mathbb{K}}(v), \mathrm{Span}_{\mathbb{K}}(f_0(v))$ is a semidestabilizing subrepresentation; on the other hand, if f_0, say, is not invertible, then we can take a nonzero $v \in \ker f_0$ to form a semidestabilizing subrepresentation $(\mathrm{Span}_{\mathbb{K}}(v), \mathrm{Span}_{\mathbb{K}}(f_1(v))$. So any polystable representation f can be written, up to isomorphism, as $f_j = \mathrm{diag}(a^1_j, \cdots, a^m_j)$ for $[a^1], ..., [a^m] \in \mathbb{P}^1_{\mathbb{K}}$ (unique up to permutations), and thus $f_z = z^j f_j$ has nonvanishing determinant for general $z \in Z$. Then $U_{Z,k}$ is open, invariant and contains the polystable locus R^{ps}, which implies that $R^{\mathrm{ss}} \subset U_{Z,k}$.[9]

Now let $n \geq 2$ and $k = 2$ (for $k = 1$ the statement is obvious), and take $f \in R_{(2,2)} \setminus U_{Z,2}$: we claim that f is unstable. By assumption, for all $z \in Z$ we have $\det f_z = 0$, i.e.

$$(f_z)_{11}(f_z)_{22} = (f_z)_{12}(f_z)_{12}\,,$$

which by uniqueness of factorizations in $\mathbb{K}[Z]$ means that either the rows or the columns of f_z are proportional with coefficients independent of z; in the first case we have thus $\sum_{z \in Z} f_z \neq \mathbb{K}^k$, and in the second $\cap_{z \in Z} \ker f_z \neq 0$. In both cases f is unstable. □

Remark 4.1.22. The equality $U_{Z,k} = R^{\mathrm{ss}}_{(k,k)}$ is false in general: for example, in the case $n = k = 3$, the Kronecker module

$$f_z := \begin{pmatrix} 0 & z_0 & z_1 \\ -z_0 & 0 & z_2 \\ -z_1 & -z_2 & 0 \end{pmatrix}$$

is semistable but does not belong to $U_{Z,3}$.[10]

[9] Indeed, for any $f \in R^{\mathrm{ss}}$ the closure $\overline{O(f)}$ of its $G_{(k,k)}$-orbit contains polystable points, so it intersect $U_{Z,k}$. Hence $O(f)$ intersects $U_{Z,k}$ as well, and thus it is contained in it since $U_{Z,k}$ is invariant.

[10] I want to thank Khazhgali Kozhasov for suggesting this example.

The image of $\tilde{\varphi}$ is the locus of degree k determinantal hypersurfaces in $\mathbb{P}(Z)$, and the previous lemma shows that this is closed if $k = 2$. Note that in general this locus is much smaller than $\mathbb{P}(\mathbb{K}[Z]_k)$, as for fixed k we have $\dim U_{Z,k} \sim n$ and $\dim \mathbb{P}(\mathbb{K}[Z]_k) \sim n^k$ as $n \to +\infty$. In fact, it is known from [Dic21] that the generic element of $\mathbb{K}[Z]_k$ is determinantal if and only if $n = 2$, $n = 3$ or $n = 4$, $k \leq 3$.

Proposition 4.1.23. *In the following cases $\tilde{\varphi}$ is an isomorphism:*

1. $k = 1$*; so* $K(n; 1, 1) = K_{\mathrm{st}}(n; 1, 1) \simeq \mathbb{P}^{n-1}$*;*
2. $n = 2$*; so* $K(2; k, k) \simeq \mathbb{P}^k$*, while* $K_{\mathrm{st}}(2; k, k) = \emptyset$ *for* $k \geq 2$*;*
3. $n = 3$ *and* $k = 2$*; so* $K(3; 2, 2) \simeq \mathbb{P}^5$*, and stable Kronecker modules correspond to irreducible quadratic forms on* Z*, i.e. to the points in the complement of the cubic symmetroid in* $\mathbb{P}^5$*.*[11]

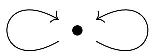
The isomorphism $K(3; 2, 2) \simeq \mathbb{P}^5$ was found in [Dré87, Lemme 25] by different methods. Our proof only uses invariant theory.

Proof. For $k = 1$ there is nothing to prove. For $n = 2$, the morphism $\varphi : R^{\mathrm{ss}}_{(k,k)} \to \mathbb{P}_{\mathbb{K}}(\mathbb{K}[Z]_k)$ sends the polystable representation $f_j = \mathrm{diag}(a_j^1, \cdots, a_j^m)$ to the class $[\prod_{\ell=1}^k a_j^\ell e^{*j}]$, where $\{e^{*0}, e^{*1}\}$ is the dual basis of $\{e_0, e_1\}$; this shows that φ maps non-isomorphic polystable representations to distinct classes and that φ is surjective, because every element $h \in \mathbb{K}[Z]_k$ can be factored as $h = \prod_{\ell=1}^k a_j^\ell e^{*j}$. This is enough to conclude that φ is the categorical quotient map: as the induced map $\tilde{\varphi}$ is a bijective morphism from an irreducible variety to a smooth variety, and hence an isomorphism.

Now let $n = 3$ and $k = 2$. $\tilde{\varphi} : K(3; 2, 2) \to \mathbb{P}(\mathbb{K}[Z]_2)$ has closed image, and it is surjective as any quadric in $\mathbb{P}^2$ is determinantal, being the irreducible ones projectively equivalent to $z_0 z_1 - z_2^2 = 0$. So we only have to prove that φ separates polystable orbits: let

$$f, f' \in R^{\mathrm{ss}}_{(2,2)} = U_{Z,2} \,.$$

We can fix a basis $\{e_0, e_1, e_2\}$ of Z so that the matrices f_0, f_0' are both invertible, and up to replacing f, f' by isomorphic (that is, in the same $G_{(2,2)}$-orbit) Kronecker modules, we may assume that $f_0 = f_0' = \mathbb{I}_2$, and consider the couples (f_1, f_2) and (f_1', f_2') as representations of the two-loop quiver

$L_2 :$  .

At this point we observe that:

- f is a (poly)stable Kronecker module if and only if (f_1, f_2) is a (semi)simple representation of L_2;
- f, f' are isomorphic Kronecker modules if and only if (f_1, f_2) and (f_1', f_2') are isomorphic representations of L_2, that is they are in the same orbit under joint conjugations.

So we are left to check that the map $R_2(L_2) \to \mathbb{K}[Z]_2$, sending a 2-dimensional representation (A, B) of L_2 (i.e. a couple of 2×2 matrices) to the homogeneous polynomial $h_{A,B}(z_0, z_1, z_2) := \det(z_0 \mathbb{I}_2 + z_1 A + z_2 B)$ (whose z_0^2 coefficient is always 1), separates closed $\mathrm{GL}_2(\mathbb{K})$-orbits. To do this we can use classical results in invariant theory: the invariant $\mathbb{K}$-algebra $\mathbb{K}[R_2(L_2)]^{\mathrm{GL}_2(\mathbb{K})}$ is (freely) generated by the five polynomial functions sending a couple (A, B) to $\mathrm{tr}\, A, \mathrm{tr}\, A^2, \mathrm{tr}\, B, \mathrm{tr}\, B^2$ and $\mathrm{tr}\, AB$ respectively (see [KP00, page 21]). But these polynomial functions can all be written as linear combinations of coefficients of $h_{A,B}(z_0, z_1, z_2)$: indeed, $\mathrm{tr}\, A$ and $\mathrm{tr}\, A^2 = (\mathrm{tr}\, A)^2 - 2 \det A$ are recovered from the coefficients of $h_{A,B}(t, -1, 0)$, which is the characteristic polynomial of A, and the same argument works for B; similarly, $\mathrm{tr}(A+B)^2 = \mathrm{tr}\, A^2 + \mathrm{tr}\, B^2 + 2\, \mathrm{tr}\, AB$ is recovered from the

[11] Recall that for $n = 3$ a quadratic form $q \in \mathbb{K}[Z]_2$ is irreducible if and only if, when written as $q(z) = z^i A_{ij} z^j$ for $A \in \mathrm{Sym}_3(\mathbb{K})$, we have $\det A \neq 0$. The hypersurface made of the elements $[A] \in \mathbb{P}_{\mathbb{K}}(\mathrm{Sym}_3(\mathbb{K})) \simeq \mathbb{P}^5$ such that $\det A = 0$ is called the *cubic symmetroid*.

characteristic polynomial $h_{A,B}(t,-1,-1)$ of $A+B$. This ends the proof that $\tilde{\varphi}$ is an isomorphism. Finally, observe that the strictly semistable representations in $R^{\mathrm{ss}}_{(2,2)}$ are those S-equivalent to one of the form

$$\begin{pmatrix} (f_z)_{11} & 0 \\ 0 & (f_z)_{22} \end{pmatrix},$$

and hence precisely those mapped by φ to a reducible quadratic form. □

4.2 Moduli of semistable sheaves

In this section, X denotes a smooth projective irreducible variety over $\mathbb{K}$ (an algebraically closed field of characteristic 0), and A is an ample divisor on X.

In the first three subsections, we will review the essential facts on semistable coherent sheaves on X and their moduli, mainly following [HL10]; this is standard material, except for our formulation in §4.2.2 of Gieseker stability in terms of an alternating form on $K_0(X)$. Finally, in §4.2.4 we will start analyzing families of objects in different hearts of $D^b(X)$, collecting some already known facts which will be used in Chapter 5.

4.2.1 Coherent sheaves and numerical invariants

$\mathbf{Coh}_{\mathcal{O}_X}$ denotes the Abelian category of coherent $\mathcal{O}_X$-modules, and $K_0(X) = K_0(\mathbf{Coh}_{\mathcal{O}_X})$ is its Grothendieck group. In this subsection we will consider several invariants of coherent sheaves which are additive on short exact sequences, and hence descend to homomorphisms from $K_0(X)$.

Remark 4.2.1. Since X is smooth, any coherent sheaf has a locally free resolution, and this implies that $K_0(X)$ can be naturally identified with the Grothendieck group $K^0(X)$ of the exact subcategory of locally free sheaves. In particular, additive invariants of locally free sheaves extend to all coherent sheaves, and the tensor product of vector bundles makes $K_0(X)$ into a commutative ring whose unity is $[\mathcal{O}_X]$.

Given a sheaf $\mathcal{E} \in \mathbf{Coh}_{\mathcal{O}_X}$, we will often consider the following invariants, which are easily checked to be additive on short exact sequences and hence only depending on the class $[\mathcal{E}] \in K_0(X)$:

- the rank $\operatorname{rk}\mathcal{E}$, i.e. the only integer such that $\mathcal{E}\restriction_U \simeq \mathcal{O}_U^{\oplus \operatorname{rk}\mathcal{E}}$, for some nonempty Zariski open $U \subset X$;
- the Euler characteristic $\chi(\mathcal{E}) := \sum_i \dim H^i(X;\mathcal{E})$;
- the Hilbert polynomial

$$P_{\mathcal{E},A}(t) = \sum_{i=0}^{\dim \mathcal{E}} \frac{\alpha_i(\mathcal{E})}{i!} t^i := \chi(X;\mathcal{E}(tA)), \tag{4.2.1}$$

  where $\dim\mathcal{E}$ is the dimension of the support of $\mathcal{E}$; we have $\operatorname{rk}\mathcal{E} = \alpha_{\dim X}(\mathcal{E})/A^{\dim X}$ and $\chi(\mathcal{E}) = P_{\mathcal{E},A}(0) = \alpha_0(\mathcal{E})$;
- the determinant $\det\mathcal{E} \in \operatorname{Pic} X$, defined as $\det\mathcal{E} := \wedge^{\operatorname{rk}\mathcal{E}}\mathcal{E}$ when $\mathcal{E}$ is locally free and as $\det\mathcal{E} := \prod_i (\det\mathcal{E}_i)^{(-1)^i}$ when $0 \to \mathcal{E}_k \to \cdots \to \mathcal{E}_0 \to \mathcal{E} \to 0$ is a locally free resolution;
- the Chern character $\operatorname{ch}\mathcal{E} \in A(X)_{\mathbb{Q}}$, defined in [Ful12, page 56] for locally free sheaves, and given by $\operatorname{ch}\mathcal{E} := \sum_i (-1)^i \operatorname{ch}\mathcal{E}_i$ when we are given a locally free resolution as above;
- the degree $\deg_A \mathcal{E} := A^{\dim X - 1} \cdot \operatorname{ch}_1 \mathcal{E}$ with respect to A.

Other (not additive) invariants of $\mathcal{E}$ are the reduced Hilbert polynomial $p_{\mathcal{E},A} := P_{\mathcal{E},A}/\alpha_{\dim\mathcal{E}}$, the slope $\mu_A(\mathcal{E}) := \deg_A\mathcal{E}/\operatorname{rk}\mathcal{E}$ (when $\operatorname{rk}\mathcal{E} \neq 0$) and the Chern class $c(\mathcal{E}) \in A^*(X)$.

Moreover, recall that the Euler characteristic $\chi(\mathcal{E},\mathcal{F}) := \sum_i \dim \operatorname{Ext}^i(\mathcal{E},\mathcal{F})$ of a pair of sheaves descends to a bilinear form $\chi : K_0(X) \times K_0(X) \to \mathbb{Z}$, called *Euler form*, and the quotient

$$K_{\mathrm{num}}(X) := {}^{K_0(X)}\!/\!_{\ker\chi} \tag{4.2.2}$$

is called the *numerical Grothendieck group* of X. In this case, Serre duality gives $\chi(v,w) = (-1)^n\chi(w,[\omega_X]v)$.

Finally, the Hilbert polynomial and the Euler form can be computed by the Hirzebruch-Riemann-Roch Theorem:

$$P_{v,A}(t) = \int_X e^{t[A]}\operatorname{ch} v \operatorname{td} X\,,\quad \chi(v,w) = \int_X (\operatorname{ch} v)^\vee \operatorname{ch} w \operatorname{td} X\,.$$

Thus we see that rank, degree, Chern character and Hilbert polynomial are *numerical* invariants, that is they descend to homomorphisms on $K_{\mathrm{num}}(X)$. In particular, it makes sense to write $\operatorname{rk} v, \deg_A v, \operatorname{ch} v, P_{v,A}$ for $v \in K_{\mathrm{num}}(X)$. We write explicitly the above formula for curves and surfaces:

1. if $\dim X = 1$ and $g(X)$ is the genus of X, then
$$P_{v,A}(t) = t \operatorname{rk} v \deg(A) + \deg v + \operatorname{rk} v(1-g(X))\,; \tag{4.2.3}$$
2. if $\dim X = 2$, then
$$P_{v,A}(t) = t^2\frac{\operatorname{rk} v\, A^2}{2} \;+\; t\left(\deg_A v - \operatorname{rk} v\,\frac{A\cdot K_X}{2}\right) + \chi(v)\,, \tag{4.2.4}$$
where $\chi(v) = \operatorname{rk} v\,\chi(X;\mathcal{O}_X) + (\operatorname{ch}_2 v + c_1(v)c_1(X)/2)$.

Examples 4.2.2.

1 Let $X := \mathbb{P}^2$. Recall that $A^*(\mathbb{P}^2) = {}^{\mathbb{Z}[x]}/_{x^3}$, where $x = c_1(\mathcal{O}(1)) = [H]$, $H \subset \mathbb{P}^2$ being a line, and $x^2 = [\mathrm{pt}]$. Using that $\chi(\mathcal{O}_{\mathbb{P}^2}) = 1$, $K_{\mathbb{P}^2} = -3H$ and $\operatorname{td}(\mathbb{P}^2) = 1 + \frac{3}{2}x + x^2$, the HRR formula reduces to
$$\begin{aligned} P_{\mathcal{E},H}(t) &= (t+1)(t+2)\tfrac{\operatorname{rk}\mathcal{E}}{2} + (2t+3)\tfrac{\deg\mathcal{E}}{2} + \textstyle\int_{\mathbb{P}^2}\operatorname{ch}_2\mathcal{E} \\ &= (t^2+3t)\tfrac{\operatorname{rk}\mathcal{E}}{2} + t\deg\mathcal{E} + \chi(\mathcal{E})\,, \end{aligned}$$
and thus $\chi(\mathcal{E}) = P_{\mathcal{E},H}(0) = \operatorname{rk}\mathcal{E} + \frac{3}{2}\deg\mathcal{E} + \int_{\mathbb{P}^2}\operatorname{ch}_2\mathcal{E}$.

The following table contains some examples of numerical invariants of coherent sheaves on $\mathbb{P}^2$ (here $C_d \subset \mathbb{P}^2$ is a curve of degree d).

Sheaf	rk	$\deg_H$	χ	c	ch	$P_{\cdot,H}(t)$
$\mathcal{O}_{\mathbb{P}^2}(d)$	1	d	$1+\frac{d}{2}(d+3)$	$1+dx$	$1+dx+\frac{d^2}{2}x^2$	$\frac{t^2}{2}+t(d+\frac{3}{2})+1+\frac{d}{2}(d+3)$
$i_*\mathcal{O}_H$	0	1	1	$1+x+x^2$	$x-\frac{1}{2}x^2$	$t+1$
$i_*\mathcal{O}_{C_d}$	0	d	$\frac{d}{2}(3-d)$	$1+dx+d^2x^2$	$dx-\frac{d^2}{2}x^2$	$td+\frac{d}{2}(3-d)$
$i_*\mathcal{O}_{\{x_1,\dots,x_n\}}$	0	0	n	$1-nx^2$	nx^2	n
$\mathcal{I}_{\{x_1,\dots,x_n\}}$	1	0	$1-n$	$1+nx^2$	$1-nx^2$	$\frac{t^2}{2}+t\frac{3}{2}+1-n$
$\tau_{\mathbb{P}^2}$	2	3	8	$1+3x+3x^2$	$1+3x+\frac{3}{2}x^2$	t^2+6t+8

2 Let $X := \mathbb{P}^1\times\mathbb{P}^1$. We have $A^*(X) = {}^{\mathbb{Z}[x,y]}/_{(x^2,y^2)}$, where
$$x := c_1(\mathcal{O}_X(1,0)) = [H]\,,\quad y := c_1(\mathcal{O}_X(0,1)) = [F]\,,\quad xy = [\mathrm{pt}]$$
and $H := \{\mathrm{pt}\}\times\mathbb{P}^1$ and $F := \mathbb{P}^1\times\{\mathrm{pt}\}$. Since $\chi(X) = 1$, $K_X = -2H-2F$ and $\operatorname{td}(\mathbb{P}^1\times\mathbb{P}^1) = 1+x+y+xy$, we get, for $A = aH+bF$, the formula
$$\begin{aligned} P_{\mathcal{E},A}(t) \;=\; & t^2(ab\operatorname{rk}\mathcal{E}) \;+\; t\,(a\deg_H\mathcal{E} + b\deg_F\mathcal{E} + \operatorname{rk}\mathcal{E}\,(a+b)) + \\ & + \operatorname{rk}\mathcal{E} + \deg_H\mathcal{E} + \deg_F\mathcal{E} + \textstyle\int_X\operatorname{ch}_2\mathcal{E}\,. \end{aligned}$$

Some examples of numerical invariants:

Sheaf	rk	$\deg_H$	$\deg_F$	χ	c	ch	$P_{\cdot,A}(t)$
$\mathcal{O}_S(h,f)$	1	h	f	$1+h+f+hf$	$1+hx+fy$	$1+hx+fy+hfxy$	$t^2(ab)+t(ah+bf+a+b)+1+h+f+hf$
$i_*\mathcal{O}_H$	0	1	0	1	$1+x$	$1+x$	$ta+1$
$i_*\mathcal{O}_F$	0	0	1	1	$1+y$	$1+y$	$tb+1$
$i_*\mathcal{O}_{\{x_1,\dots,x_n\}}$	0	0	0	n	$1-nx^2$	nx^2	n
$\mathcal{I}_{\{x_1,\dots,x_n\}}$	1	0	0	$1-n$	$1+nx^2$	$1-nx^2$	$t^2(ab)+t(a+b)+1-n$
τ_X	2	2	2	6	$1+2x+2y+4xy$	$1+2x+2y$	$t^2(2ab)+t(2a+2b)+6$

4.2.2 Gieseker stability

Definition 4.2.3. $\mathcal{E} \in \mathbf{Coh}_{\mathcal{O}_X}$ is said to be *Gieseker-(semi)stable* with respect to A if it is $P_{\cdot,A}$-(semi)stable according to Def. 3.2.7.

Here we are seeing the Hilbert polynomial as a polynomial stability structure $P_{\cdot,A} : K_0(X) \to \mathbb{R}[t]$ (Def. 3.2.8). So $\mathcal{E}$ is Gieseker-(semi)stable if and only if, for any coherent subsheaf $0 \neq \mathcal{F} \subsetneq \mathcal{E}$, we have the inequality $P_{\mathcal{F},A} \preceq_{\mathrm{G}} P_{\mathcal{E},A}$, where $\preceq_{\mathrm{G}}$ is the *Gieseker preorder* introduced in Eq. (3.2.2). Lemma 3.2.10 says that this inequality is equivalent to

$$\alpha_{\dim \mathcal{E}}(\mathcal{E}) P_{\mathcal{F},A}(t) \,(\leq)\!< \alpha_{\dim \mathcal{E}}(\mathcal{F}) P_{\mathcal{E},A}(t)$$

(where the notation is as in Eq. (4.2.1), and as usual $\leq$ is the lexicographical order), so our definition agrees with the standard one given in [HL10, §1.2]. As discussed in Chapter 3, this reformulation will be useful when extending the definition to other objects of the derived category $D^b(X)$.

By Prop. 3.2.16, Gieseker-semistable sheaves with Hilbert polynomial proportional to $p \in \mathbb{Q}[t]$[12] form an abelian subcategory, denoted now simply

$$\mathcal{S}_A(p) \subset \mathbf{Coh}_{\mathcal{O}_X} \,, \tag{4.2.5}$$

which is of finite length and closed under extensions, and whose simple objects are the stable sheaves. Again, motivated by GIT (Theorem 4.2.9) we will say that:

Definition 4.2.4. Two sheaves in $\mathcal{S}_A(p)$ are called *S-equivalent* if they have the same composition factors.

Recall that we also have the notion of slope-stability:

Definition 4.2.5. A torsion-free sheaf $\mathcal{E} \in \mathbf{Coh}_{\mathcal{O}_X}$ is *slope-(semi)stable* if for any coherent subsheaf $\mathcal{F} \subsetneq \mathcal{E}$ with $0 < \operatorname{rk} \mathcal{F} < \operatorname{rk} \mathcal{E}$ we have $\mu_A(\mathcal{F}) \,(\leq)\!< \mu_A(\mathcal{E})$.

However, if not specified otherwise, (semi)stability for a sheaf will always mean Gieseker (semi)stability.

Remarks 4.2.6. Some remarks on the notion of Gieseker (semi)stability:

1 If $\mathcal{E}$ is Gieseker-semistable, then it is automatically pure (that is, all its subsheaves have the same dimension), and in particular it is torsion-free if and only if $\dim \mathcal{E} = \dim X$.

2 As Gieseker-stable sheaves with Hilbert polynomial proportional to p are the simple objects in the category $\mathcal{S}_A(p)$, their only endomorphisms are multiples of the identity, by the same arguments of Remark 4.1.2.1.

3 The category $\mathbf{Coh}_{\mathcal{O}_X}$ is Noetherian and Hilbert polynomials are numerical; then, by Prop. 3.2.16, any coherent sheaf $\mathcal{E}$ has a unique Harder-Narasimhan filtration

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_\ell = \mathcal{E}$$

with Gieseker-semistable quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ of $\preceq_{\mathrm{G}}$-decreasing Hilbert polynomials (when $\mathcal{E}$ is pure this simply means that $p_{\mathcal{E}_1,A} > p_{\mathcal{E}_2/\mathcal{E}_1,A} > \cdots > p_{\mathcal{E}/\mathcal{E}_{\ell-1},A}$). We write

$$P_{\mathcal{E},A,\max} := P_{\mathcal{E}_1,A} \,, \quad P_{\mathcal{E},A,\min} := P_{\mathcal{E}/\mathcal{E}_{\ell-1},A} \,. \tag{4.2.6}$$

4 Suppose that $\dim X = 1$: any $\mathcal{E} \in \mathbf{Coh}_{\mathcal{O}_X}$ is the direct sum of its torsion-free and torsion parts, so it is pure if and only if they are not both nonzero; a torsion-free $\mathcal{E}$ (which is also a vector bundle) is Gieseker-(semi)stable if and only if it is slope-(semi)stable, and the slope condition may be checked on vector subbundles only; on the other hand, any torsion sheaf is Gieseker-semistable, and it is Gieseker-stable if and only if it is a simple object in $\mathbf{Coh}_{\mathcal{O}_X}$, i.e. a skyscraper sheaf.

[12]Equivalently, we are considering sheaves with *reduced* Hilbert polynomial equal to p, if this is suitably normalized.

Finally, recall from §3.2.2 that we can also express Gieseker stability in terms of a polynomial-valued alternating form $\sigma_{P_{\cdot,A}}$ on $K_0(X)$. However, to simplify the computations in Chapter 5 we also introduce the simpler alternating forms $\sigma_{\mathrm{M}}, \sigma_\chi : K_0(X) \times K_0(X) \to \mathbb{Z}$ given by

$$\sigma_{\mathrm{M}}(v,w) := \deg_A v \operatorname{rk} w - \deg_A w \operatorname{rk} v\,, \quad \sigma_\chi(v,w) := \chi(v) \operatorname{rk} w - \chi(w) \operatorname{rk} v \tag{4.2.7}$$

and also the $\mathbb{Z}[t]$-valued form

$$\sigma_{\mathrm{G}} := t\sigma_{\mathrm{M}} + \sigma_\chi\,. \tag{4.2.8}$$

Now we can express Gieseker stability on curves and surfaces as stability with respect to these forms, in the sense of Def. 3.2.4:

Lemma 4.2.7.

1. *If* $\dim X = 1$*, then* $\sigma_{\mathrm{M}} = \sigma_\chi$*, and Gieseker (semi)stability and* σ_{M}*-(semi)stability of sheaves are equivalent; for sheaves of positive rank these are also equivalent to slope-stability;*
2. *if* $\dim X = 2$*, then for sheaves of positive rank Gieseker (semi)stability is equivalent to* σ_{G}*-(semi)stability; for torsion-free sheaves, slope semistability is equivalent to* σ_{M}*-semistability.*

For $\dim X = 2$, the restriction to positive rank is necessary as sheaves supported on points are in the kernel of σ_{G}. Note also that $\mathcal{O}_X$ is slope-stable but not σ_{M}-stable, as $\sigma_{\mathrm{M}}(\mathcal{I}_x, \mathcal{O}_X) = 0$, where $\mathcal{I}_x \subset \mathcal{O}_X$ is the ideal sheaf of a point.

Proof.

1. The first statement is just the observation that the alternating form induced by the Hilbert polynomial as in Eq. (3.2.1) is $\sigma_{P_{\cdot,A}} = \deg A\, \sigma_{\mathrm{M}} = \deg A\, \sigma_\chi$. The second statement is also obvious.
2. In this case we have

$$\sigma_{P_{\cdot,A}} = \frac{t^2}{2} A^2 \sigma_{\mathrm{M}} + \left(t\, A^2 - \frac{A \cdot K_X}{2} \right) \sigma_\chi + \sigma_0\,,$$

where $\sigma_0(v,w) := \chi(v) \deg w - \chi(w) \deg v$. But if $\operatorname{rk} w \neq 0$, then σ_0 is irrelevant as $\sigma_{\mathrm{M}}(v,w) = \sigma_\chi(v,w) = 0$ implies $\sigma_0(v,w) = 0$, so $\sigma_{P_{\cdot,A}}$ can be replaced by $\sigma_{\mathrm{G}} = t\sigma_{\mathrm{M}} + \sigma_\chi$. The final claim follows from the equality

$$\mu_A(v) - \mu_A(w) = \frac{1}{\operatorname{rk} v \operatorname{rk} w} \sigma_{\mathrm{M}}(v,w)$$

and from the fact that any coherent subsheaf $\mathcal{F} \subsetneq \mathcal{E}$ with $\operatorname{rk} \mathcal{F} = \operatorname{rk} \mathcal{E}$ gives $\sigma_{\mathrm{M}}(\mathcal{F}, \mathcal{E}) = -\deg(\mathcal{E}/\mathcal{F}) \operatorname{rk} \mathcal{E} \leq 0$.

□

Remark 4.2.8. In fact, the same arguments apply to any heart $\mathcal{A} \subset D^b(X)$ of a bounded t-structure: if $\dim X = 1$, then $P_{\cdot,A}$-(semi)stability and σ_{M}-(semi)stability in $\mathcal{A}$ are equivalent; if $\dim X = 2$, then $P_{\cdot,A}$-(semi)stability and σ_{G}-(semi)stability are equivalent for objects of nonzero rank in $\mathcal{A}$.

4.2.3 Moduli of semistable sheaves

The main reason why Gieseker stability was introduced was that it allowed to construct moduli spaces of semistable sheaves. In this subsection we will briefly review this construction.

Fix a numerical class $v \in K_{\mathrm{num}}(X)$. By *(flat) family* of coherent sheaves of class v on X over an algebraic $\mathbb{K}$-scheme S we mean an element $\mathcal{F} \in \mathbf{Coh}_{\mathcal{O}_{X \times S}}$, flat over S, where for each $s \in S$, the slice $\mathcal{F}_s := \iota_s^* \mathcal{F} \in \mathbf{Coh}_{\mathcal{O}_X}$ (where $\iota_s : X \to X \times S$ maps x to (x,s)) belongs to v; an isomorphism of families is an isomorphism in $\mathbf{Coh}_{\mathcal{O}_{X \times S}}$. This defines a groupoid fibration $\mathfrak{M}_X(v)$ over $\mathbf{AlgSch}_{\mathbb{K}}$, which by routine arguments is in fact a stack [LMB00, §2.4.4 and §3.4.4]. Restricting to (semi)stable sheaves with respect to A we define similarly the stacks $\mathfrak{M}^{\mathrm{ss}}_{X,A}(v)$ and $\mathfrak{M}^{\mathrm{st}}_{X,A}(v)$.

Theorem 4.2.9. *There exists a projective* $\mathbb{K}$*-scheme* $\mathrm{M}^{\mathrm{ss}}_{X,A}(v)$ *which corepresents* $\mathfrak{M}^{\mathrm{ss}}_{X,A}(v)$*, and which is a coarse moduli space for S-equivalence classes (in the sense of Def. 4.2.4) of Gieseker-semistable coherent* $\mathcal{O}_X$*-modules in* v*. It also has an open subscheme* $\mathrm{M}^{\mathrm{st}}_{X,A}(v)$ *corepresenting* $\mathfrak{M}^{\mathrm{st}}_{X,A}(v)$ *and parameterizing isomorphism classes of Gieseker-stable sheaves.*

Now we briefly sketch the main points of the proof of this result:

Proof. The proof can be divided in two main parts: first ("rigidification") we present $\mathfrak{M}^{\mathrm{ss}}_{X,A}(v)$ as a quotient stack $[R/G]$, and then ("linearization") we construct a categorical quotient by interpreting R as the GIT-semistable locus in a G-space.

Having fixed the class v (hence the Hilbert polynomial $P_{v,H}$), there is $n_0 \in \mathbb{N}$ such that every semistable sheaf $\mathcal{E}$ in v is n_0-regular, which in particular means that $\mathcal{E}(n_0)$ has vanishing higher cohomologies, so that $\dim_{\mathbb{K}} H^0(X;\mathcal{E}(n_0)) = P(n_0)$, and that it is globally generated, that is the evaluation map

$$\mathbb{K}^{P(n_0)} \otimes_{\mathbb{K}} \mathcal{O}(-n_0) \simeq H^0(X;\mathcal{E}(n_0)) \otimes_{\mathbb{K}} \mathcal{O}(-n_0) \to \mathcal{E} \tag{4.2.9}$$

is surjective. The first general proof of this theorem, due to Simpson [Sim94], uses this fact to parameterize all semistable sheaves in v by an open set R in the Quot scheme $\mathrm{Quot}(\mathcal{H}, v)$ of quotients of $\mathcal{H} := \mathcal{O}(-n_0)^{P(n_0)}$. The natural action of $\mathrm{GL}(P(n_0))$ on $\mathrm{Quot}(\mathcal{H}, v)$ fixes R, its orbits correspond to isomorphism classes of semistable sheaves, and stabilizers are automorphism groups of the sheaves; more precisely we have $\mathfrak{M}^{\mathrm{ss}}_{X,A}(v) \simeq [R/\mathrm{GL}(P(n_0))]$, as one can show finding an explicit equivalence (see e.g. [Góm01, Prop. 3.3] or [HL10, Lemmas 4.3.1-4.3.2]) or noticing that the universal family of the Quot scheme restricts to a versal family $R \to \mathfrak{M}^{\mathrm{ss}}_{X,A}(v)$ with symmetry group $\mathrm{GL}(P(n_0))$, and then using Prop. 4.A.4. The linearization procedure is then obtained via the Grothendieck's embedding of $\mathrm{Quot}(\mathcal{H}, v)$ into the Grassmannian $\mathrm{G}(\mathbb{K}^{P(n_0)} \otimes H, P(n_1))$, where $n_1 > n_0$ is a sufficiently big integer and $H := H^0(X;\mathcal{O}_X(n_1 - n_0))$, i.e. by mapping the quotient sheaf (4.2.9) to the quotient space

$$\mathbb{K}^{P(n_0)} \otimes H \simeq H^0(X;\mathcal{E}(n_0)) \otimes H \to H^0(X;\mathcal{E}(n_1))\,.$$

This embedding endows $\mathrm{Quot}(\mathcal{H}, v)$ with a $\mathrm{GL}(P(n_0))$-linearized ample line bundle, for which it turns out that the GIT-semistable locus in $\overline{R}$ is precisely R, the stable locus is the open set parameterizing stable sheaves, and orbits in R whose closures intersect correspond to S-equivalent sheaves. This means that we have a categorical quotient $R \to \mathrm{M}^{\mathrm{ss}}_{X,A}(v)$ corepresenting $\mathfrak{M}^{\mathrm{ss}}_{X,A}(v)$.

A modification of this proof, due to Álvarez-Cónsul and King [ÁCK07] and more in the spirit of this thesis, avoids the choice of an isomorphism $\mathbb{K}^{P(n_0)} \simeq H^0(X;\mathcal{E}(n_0))$ leading to the above embedding of R in $\mathrm{G}(\mathbb{K}^{P(n_0)} \otimes H, P(n_1))$, to get instead a functor $\Phi : \mathbf{Coh}_X \to \mathbf{Rep}^{\mathrm{fd}}_{K_H}$ sending a sheaf $\mathcal{E}$ to the Kronecker module (i.e. a representation of the Kronecker quiver K_H with $\dim_{\mathbb{K}} H$ arrows, see §4.1.8)

$$H^0(X;\mathcal{E}(n_0)) \otimes H \to H^0(X;\mathcal{E}(n_1))$$

of dimension vector $d = (P(n_0), P(n_1))$. This construction can be done familywise: a flat family of coherent sheaves is sent to a flat family of Kronecker modules in the sense of §4.1.5, and since Φ is faithful on n_0-regular sheaves, one obtains a locally closed embedding

$$\mathfrak{M}^{\mathrm{reg}}_X(v) \hookrightarrow \mathfrak{K}(H; P(n_0), P(n_1)) := [R_d/G_d]$$

identifying the substack $\mathfrak{M}^{\mathrm{reg}}_X(v) \subset \mathfrak{M}_X(v)$ of n_0-regular sheaves as a quotient stack $[Q/G_d]$, for some subscheme $Q \subset R_d$. Moreover, for $n_1 \gg n_0 \gg 0$, Φ maps the (semi)stable sheaves in v onto the (semi)stable Kronecker modules inside Q, so that

$$\mathfrak{M}^{\mathrm{ss/st}}_{X,A}(v) \simeq [Q^{\mathrm{ss/st}}/G_d]\,,$$

where $Q^{\mathrm{ss/st}} = Q \cap R_d^{\mathrm{ss/st}}$. But now the linearization part has been already taken care of, as we saw in Prop. 4.1.5 that $R_d^{\mathrm{ss/st}}$ are GIT-(semi)stable loci. So we get a categorical quotient $\mathrm{M}^{\mathrm{ss}}_{X,A}(v)$ as a subscheme of the GIT quotient $K(H; P(n_0), P(n_1)) = R_d^{\mathrm{ss}}//PG_d$, and an open subscheme $\mathrm{M}^{\mathrm{st}}_{X,A}(v) \subset \mathrm{M}^{\mathrm{ss}}_{X,A}(v)$. These are thus quasi-projective schemes, and by using the valuative criterion of properness it can be shown that $\mathrm{M}^{\mathrm{ss}}_{X,A}(v)$ is in fact projective. □

Remarks 4.2.10. We recall some properties of the moduli spaces $\mathrm{M}^{\mathrm{ss}}_{X,A}(v)$ when $\dim X = 2$.

1 [HL10, Theorem 3.4.1] If $\mathrm{M}^{\mathrm{ss}}_{X,A}(v) \neq \emptyset$, then the *Bogomolov inequality* holds:

$$\Delta(v) := c_1(v)^2 - 2\,\mathrm{rk}\,v\ \mathrm{ch}_2(v) \geq 0\,. \tag{4.2.10}$$

2 If $\deg_A \omega_X < 0$, then for any stable $\mathcal{F}$ the obstruction space $\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \simeq \mathrm{Hom}(\mathcal{F},\mathcal{F}\otimes\omega_X)^\vee$ vanishes and $\mathrm{End}(\mathcal{F}) \simeq \mathbb{K}$, thus the tangent space $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ has dimension $1-\chi(\mathcal{F},\mathcal{F})$. Hence, by [HL10, Corollary 4.5.2], $\mathrm{M}^{\mathrm{st}}_{X,A}(v)$ is smooth of dimension

$$\dim \mathrm{M}^{\mathrm{st}}_{X,A}(v) = 1 - \chi(v,v) = 1 - (\mathrm{rk}\,v)^2\chi(\mathcal{O}_X) + \Delta(v)\,. \tag{4.2.11}$$

3 [HL10, Corollary 4.6.7] if $\gcd(\mathrm{rk}\,v, \deg_A v, \chi(v)) = 1$, then $\mathrm{M}^{\mathrm{ss}}_{X,A}(v)$ is equal to $\mathrm{M}^{\mathrm{st}}_{X,A}(v)$ and it has a universal family.

4.2.4 Families of objects in the derived category

Take an algebraic $\mathbb{K}$-scheme S. To generalize the notion of flat families of sheaves to other objects in the heart $\mathcal{A} \subset D^b(X)$ of a bounded t-structure, Bridgeland introduced the following definition [Bri02, Def. 3.7]:

Definition 4.2.11. A *family* over S of objects of $\mathcal{A}$ having a common property P is an object $\mathcal{F}$ of $D^b(X\times S)$ such that, for any point $s \in S$, the object $\mathcal{F}_s := L\iota_s^*\mathcal{F}$ is in $\mathcal{A}$ and has the property P, where $\iota_s : X \to X\times S$ maps x to (x,s). In particular, denote by $\mathfrak{M}_{\mathcal{A}}(v)$ the stack[13] of families of objects of $\mathcal{A}$ within a fixed class $v \in K_0(X)$.

When $\mathcal{A}$ is the standard heart $\mathcal{C} := \mathbf{Coh}_{\mathcal{O}_X}$, this is indeed the usual definition of flat family of sheaves, so we recover the stack $\mathfrak{M}_X(v)$ of coherent sheaves (see §4.2.3):

Lemma 4.2.12. *[Huy06, Lemma 3.31] $\mathcal{F} \in D^b(X\times S)$ is isomorphic to a sheaf on $X\times S$ flat over S if and only if, for all $s\in S$, the object $\mathcal{F}_s$ is isomorphic to a sheaf on X. Thus $\mathfrak{M}_{\mathcal{C}}(v) \simeq \mathfrak{M}_X(v)$.*

Suppose that $D^b(X)$ has a full strong exceptional collection ${}^\vee\mathfrak{E}$ of vector bundles, and consider the equivalence $\Phi_{{}^\vee\mathfrak{E}} : D^b(X) \to D^b(Q;J)$ of Theorem 2.2.5 together with the induced isomorphism $\psi : K_0(X) \to K_0(Q;J)$ and the heart $\mathcal{K} := \Phi_{{}^\vee\mathfrak{E}}^{-1}(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q;J))$ (as in Remark 2.2.6).

Now, as explained in [Ohk10, §4.2], we will identify families of objects in $\mathcal{K}$ with flat families of representations of (Q,J), whose moduli stacks were denoted by $\mathfrak{M}_{Q,J}(d)$ (see §4.1.5). To do this we will use the relative version of the equivalence $\Phi_{{}^\vee\mathfrak{E}}$: recall that if S is affine, then by Theorem 2.2.8 we have an equivalence

$$R(\mathrm{pr}_S)_*(T_S^\vee \otimes (\cdot)) : D^*(X\times S) \to D^*(\mathbf{Coh}^{Q,J}_{\mathcal{O}_S}) \tag{4.2.12}$$

for $* \in \{-,b\}$, and where $T_S := T \boxtimes \mathcal{O}_S$ is the relative tilting bundle constructed from ${}^\vee\mathfrak{E}$ in §2.2.2. For any base-change

$$\begin{array}{ccc} X\times S' & \xrightarrow{\mathrm{Id}_X\times f} & X\times S \\ {\scriptstyle \mathrm{pr}_{S'}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_S} \\ S' & \xrightarrow{f} & S \end{array}$$

we have the 2-commutative diagram (see e.g. [Sta18, tag 0E1V])

$$\begin{array}{ccc} D^-(X\times S') & \xleftarrow[L(\mathrm{Id}_X\times f)^*]{} & D^-(X\times S) \\ {\scriptstyle R(\mathrm{pr}_{S'})_*(T_{S'}^\vee\otimes(\cdot))}\downarrow & & \downarrow{\scriptstyle R(\mathrm{pr}_S)_*(T_S^\vee\otimes(\cdot))} \\ D^-(\mathbf{Coh}^{Q,J}_{\mathcal{O}_{S'}}) & \xleftarrow[Lf^*]{} & D^-(\mathbf{Coh}^{Q,J}_{\mathcal{O}_S}) \end{array}\,.$$

[13]This is in fact an algebraic stack, studied in a broader context in [Lie06].

Lemma 4.2.13. *The images under the equivalence* (4.2.12) *of families of objects of* $\mathcal{K}$ *are isomorphic to flat families of representations of* (Q, J)*. Thus* $\mathfrak{M}_{\mathcal{K}}(v) \simeq \mathfrak{M}_{Q,J}(d^v)$*, where* $d^v := \underline{\dim}\,\psi(v)$*.*

Proof. In the last diagram, take in particular f to be the embedding $\{s\} \hookrightarrow S$ of a point. Using Lemma 4.2.12 applied to the case $X = \mathrm{pt}$, we see that a family over S of objects of $\mathcal{K}$ is sent to an object isomorphic to a locally free sheaf, i.e. a flat family of representations of (Q, J). As the equivalence (4.2.12) is compatible with pullbacks, we get an isomorphism between the restrictions of $\mathfrak{M}_{\mathcal{K}}(v)$ and $\mathfrak{M}_{Q,J}(d^v)$ to the category of affine algebraic $\mathbb{K}$-schemes. Since these are stacks, this is enough to conclude that $\mathfrak{M}_{\mathcal{K}}(v) \simeq \mathfrak{M}_{Q,J}(d^v)$. □

4.3 Monads

Let X be a smooth projective variety over $\mathbb{K}$ (an algebraically closed field of characteristic 0).

In this section we collect some facts about the theory of monads on X which are somewhat scattered in the literature. Here we will denote the cohomologies of a cochain complex $M^\bullet$ of sheaves on X by $h^i(M^\bullet)$ to avoid confusion with the sheaf cohomologies $H^i(X; M^j)$.

4.3.1 Basics of monads

Definition 4.3.1. A *monad* over X is a three-term cochain complex

$$M^\bullet : M^{-1} \xrightarrow{a} M^0 \xrightarrow{b} M^1$$

of finite-rank locally free $\mathcal{O}_X$-modules such that a is injective and b is surjective. A *morphism* between two monads is a morphism of complexes. The middle cohomology sheaf

$$h^0(M^\bullet) = {}^{\ker b}\!/\!_{\operatorname{im} a}$$

is called the *cohomology sheaf* of the monad.

We call *display* of a monad $M^\bullet$ with cohomology sheaf $\mathcal{E} = h^0(M^\bullet)$ the following diagram, whose rows and columns are exact sequences:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^{-1} & \xrightarrow{a} & \ker b & \xrightarrow{q} & \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^{-1} & \xrightarrow{a} & M^0 & \xrightarrow{q} & \operatorname{coker} a & \longrightarrow & 0\\
 & & & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
 & & & & M^1 & \xrightarrow{\mathrm{Id}} & M^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Now we focus on morphisms between two monads and their relations to the induced morphisms between their cohomologies. The key is the following fact:

Lemma 4.3.2. *[GM13, IV.2.2][DLP85, Prop. 2.3] Let* $M^\bullet, N^\bullet$ *be bounded complexes in* $\mathbf{Coh}_{\mathcal{O}_X}$ *such that*

$$\operatorname{Ext}^\ell(M^i, N^j) = 0 \text{ for all } i, j \text{ and } \ell \geq 1\,.$$

Then for any $\ell \in \mathbb{Z}$ *the canonical map*

$$h^\ell(\operatorname{Hom}^\bullet(M^\bullet, N^\bullet)) \cong \operatorname{Hom}_{K^b(X)}(M^\bullet, N^\bullet[\ell]) \to \operatorname{Hom}_{D^b(X)}(M^\bullet, N^\bullet[\ell]) \qquad (4.3.1)$$

is an isomorphism.

Remark 4.3.3. In particular, if in the previous Lemma $M^\bullet$ and $N^\bullet$ are monads, then[14]

$$h^\ell(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)) \cong \mathrm{Ext}^\ell(h^0(M^\bullet), h^0(N^\bullet)) .$$

If, moreover, we assume that $\mathrm{Hom}(M^i, N^j) = 0$ for $i > j$, then for $\ell = 0$ the left hand side reduces to $\mathrm{Hom}(M^\bullet, N^\bullet)$, and thus Eq. (4.3.1) reduces to an isomorphism

$$\mathrm{Hom}(M^\bullet, N^\bullet) \to \mathrm{Hom}(h^0(M^\bullet), h^0(N^\bullet)) .$$

Finally, the following Lemma will be useful when we will consider monads constructed from strong exceptional collections:

Lemma 4.3.4. *Let M_p^i, $i = -1, 0, 1$, $p = 0, ..., n$ be coherent sheaves such that $\mathrm{Ext}^\ell(M_p^i, M_q^j) = 0$ and $\mathrm{Hom}(M_p^{i+1}, M_q^i) = 0$ for all $\ell \geq 1$ and all i, j, p, q. Then the category of cochain complexes of the form*

$$\oplus_{p=1}^n (M_p^{-1})^{d_p^{-1}} \to \oplus_{p=1}^n (M_p^0)^{d_p^0} \to \oplus_{p=1}^n (M_p^1)^{d_p^1} \tag{4.3.2}$$

(concentrated in degrees -1,0,1), for $d_p^{-1}, d_p^0, d_p^1 \in \mathbb{N}$, and cochain maps between them is equivalent (in the natural way) to the extension-closure

$$\langle M_p^i[-i] \, , i = -1, 0, 1, \ p = 0, ..., n \rangle_{\mathrm{ext}} \subset D^b(X) .$$

Proof. For two complexes $N_1^\bullet, N_2^\bullet$ as in Eq. (4.3.2) we have, by Lemma 4.3.2,

$$\mathrm{Hom}_{K^b(X)}(N_1^\bullet, N_2^\bullet[\ell]) \cong \mathrm{Hom}_{D^b(X)}(N_1^\bullet, N_2^\bullet[\ell]) ,$$

for all $\ell \in \mathbb{Z}$. Moreover, by hypothesis no nontrivial cochain homotopies can exist between maps $N_1^\bullet \to N_2^\bullet$, so that

$$\mathrm{Hom}(N_1^\bullet, N_2^\bullet) \cong \mathrm{Hom}_{K^b(X)}(N_1^\bullet, N_2^\bullet).$$

Finally, the cone of a map $N_1^\bullet[-1] \to N_2^\bullet$ is clearly a complex of the same form (4.3.2), and any such complex can be built as a two-step extension from the objects

$$\oplus_{p=1}^n (M_p^{-1})^{d_p^{-1}}[1] , \quad \oplus_{p=1}^n (M_p^0)^{d_p^0} , \quad \text{and} \quad \oplus_{p=1}^n (M_p^1)^{d_p^1}[-1] .$$

□

4.3.2 Families of sheaves from monads

Let us fix three locally free sheaves α, β, γ such that $\mathrm{Hom}(\beta, \alpha) = \mathrm{Hom}(\gamma, \beta) = \mathrm{Hom}(\gamma, \alpha) = 0$, and such that all higher Ext spaces between them vanish. Typically, these sheaves are obtained as direct sums of elements forming a strong exceptional collection on $D^b(X)$.

We consider monads of the form

$$M_{a,b}^\bullet : \alpha \xrightarrow{a} \beta \xrightarrow{b} \gamma ,$$

for (a, b) in the quasi-affine subscheme

$$Q := \left\{ (a, b) \in V \;\middle|\; \begin{matrix} b \circ a = 0 \\ a \text{ injective} \\ b \text{ surjective} \end{matrix} \right\}$$

of $V := \mathrm{Hom}_{\mathcal{O}_X}(\alpha, \beta) \oplus \mathrm{Hom}_{\mathcal{O}_X}(\beta, \gamma)$. We have a tautological monad[15]

$$M^\bullet : \alpha \boxtimes \mathcal{O}_Q \to \beta \boxtimes \mathcal{O}_Q \to \gamma \boxtimes \mathcal{O}_Q$$

over $X \times Q$ whose restriction to each slice $X \times \{(a, b)\} \cong X$ is $M_{a,b}^\bullet$.

[14]Notice that in fact less vanishings are needed to obtain such isomorphisms in the case of monads, as one sees by using the spectral sequence

$$E_1^{p,q} = \oplus_{j-i=p} \mathrm{Ext}^q(M^i, N^j) \Rightarrow \mathrm{Hom}_{D^b(X)}(M^\bullet, N^\bullet[p+q]) .$$

[15]This is indeed a monad because its restriction to any $(a, b) \in Q$ is a monad (see e.g. [BBR15, Lemma 2.1]).

Lemma 4.3.5. *The cohomology $h^0(M^\bullet)$ is flat over Q, and its restriction to $X \times \{(a,b)\}$ is*

$$h^0(M^\bullet)_{(a,b)} \simeq h^0(M^\bullet_{a,b})\,.$$

Proof. For all $(a,b) \in Q$, let $\iota_{(a,b)} : X \hookrightarrow X \times Q$ be the map sending x to $(x,(a,b))$, and consider the pullback $L\iota^*_{(a,b)} : D^-(X \times Q) \to D^-(X)$. Then we have

$$L\iota^*_{(a,b)} M^\bullet = M^\bullet_{a,b} \simeq h^0(M^\bullet_{a,b})$$

in $D^b(X)$. By Lemma 4.2.12, this implies that $M^\bullet$ is isomorphic in $D^b(X \times Q)$ to a sheaf which is flat over Q, and this is $h^0(M^\bullet)$. In particular,

$$L\iota^*_{(a,b)} M^\bullet \simeq L\iota^*_{(a,b)} h^0(M^\bullet) = h^0(M^\bullet)_{(a,b)}\,.$$

□

Hence we have built a family parameterized by Q of coherent sheaves on X, all within the same class $[\beta] - [\alpha] - [\gamma] \in K_0(X)$.

To study the locus Q, we first observe that it is nothing but the intersection between the open set where a is injective and b is surjective, and the closed set defined as the vanishing locus of the composition map

$$C : V \to \mathrm{Hom}(\alpha, \gamma)$$

sending $(a,b) \mapsto b \circ a$. The differential $\mathrm{d}C_{(a,b)}$ at a point $(a,b) \in Q$ is the linear map

$$\mathrm{d}C_{(a,b)} : \mathrm{Hom}(\alpha,\beta) \oplus \mathrm{Hom}(\beta,\gamma) \to \mathrm{Hom}(\alpha,\gamma)$$

mapping a couple (a',b') to $b \circ a' + b' \circ a$. This is precisely the differential in degree 1 of the Hom-complex $\mathrm{Hom}^\bullet(M^\bullet_{a,b}, M^\bullet_{a,b})$. Thus by Lemma 4.3.2 (and the subsequent Remark) we obtain that

$$\mathrm{coker}\, \mathrm{d}C_{(a,b)} \simeq \mathrm{Ext}^2(\mathcal{E}_{a,b}, \mathcal{E}_{a,b})\,.$$

where $\mathcal{E}_{a,b} := h^0(M^\bullet_{a,b})$. In particular, we conclude that:

Corollary 4.3.6. *At points $(a,b) \in Q$ where $\mathrm{Ext}^2(\mathcal{E}_{a,b}, \mathcal{E}_{a,b}) = 0$ the scheme Q is a smooth complete intersection, of dimension*

$$\dim_{\mathbb{K}} \mathrm{Hom}(\alpha,\beta) + \dim_{\mathbb{K}} \mathrm{Hom}(\beta,\gamma) - \dim_{\mathbb{K}} \mathrm{Hom}(\alpha,\gamma)\,.$$

Remark 4.3.7. In the applications of this result, we will consider the open subsets $Q^{\mathrm{st}} \subset Q^{\mathrm{ss}} \subset Q$ where the cohomology sheaves $\mathcal{E}_{a,b}$ are Gieseker stable and semistable. If X is a Del Pezzo surface, then for these sheaves we will have $\mathrm{Ext}^2(\mathcal{E}_{a,b}, \mathcal{E}_{a,b}) = 0$ by Serre duality and positivity of the anticanonical divisor, so that Q^{st} and Q^{ss} are smooth.

4.A Appendix: algebraic stacks

In this appendix we briefly recall how to realize a stack as a quotient stack $[X/G]$ by identifying a versal family over X whose symmetries are encoded in an action $G \curvearrowright X$. All the moduli stacks considered in this thesis admit this description.

Let $\mathfrak{M}$ be a category fibered in groupoids (CFG) over the category $\mathbf{AlgSch}_{\mathbb{K}}$ of algebraic $\mathbb{K}$-schemes: we have thus a functor $\pi : \mathfrak{M} \to \mathbf{AlgSch}_{\mathbb{K}}$, and we denote by $\mathfrak{M}(S)$ the groupoid of *families* over a scheme S, i.e. the objects of $\mathfrak{M}$ mapping to S. For simplicity we assume to have chosen a cleavage, i.e. a preferred pullback $f^*\mathcal{F}$ for any morphism $f : S' \to S$ of schemes and any family $\mathcal{F} \in \mathfrak{M}(S)$. Recall that $\mathfrak{M}$ is a *stack* when, roughly, families on S and morphisms between them can be defined on open covers of S and glued together. If T is a scheme, then $\underline{T}$ denotes the stack canonically associated to it (its families over S are maps $S \to T$). We refer to [LMB00, BCE+12] for the precise definitions and for the concept of representability used in the next definition:

Definition 4.A.1. A stack $\mathfrak{M}$ is said to be *algebraic* when its diagonal morphism $\Delta : \mathfrak{M} \to \mathfrak{M} \times \mathfrak{M}$ is representable by schemes and there are $U \in \mathbf{AlgSch}_{\mathbb{K}}$ and a smooth surjective morphism $\underline{U} \to \mathfrak{M}$.

Another relevant notion is that of an *algebraic groupoid*, by which we mean a groupoid $R \rightrightarrows U$ internal to $\mathbf{AlgSch}_{\mathbb{K}}$. There is a standard way to associate to $R \rightrightarrows U$ a CFG consisting of the torsors over it: we recall this construction in the simple case of an action groupoid, which is the only one relevant to us: given an affine algebraic group G and an algebraic action $\lambda : G \times X \to X$, we have an algebraic groupoid $G \times X \rightrightarrows X$ (where the source and target maps are λ and the projection pr_X onto X), and a $(G \times X \rightrightarrows X)$-torsor over a scheme S is defined to be a left principal G-bundle $P \to S$ (locally trivial in the étale topology) together with a G-equivariant map $P \to X$. With the obvious notion of morphisms between torsors, these make a CFG, denoted by $[X/G]$. In this case, this is actually an algebraic stack, called the *stack quotient* of X by G.

Under mild conditions, any algebraic stack is isomorphic to the stack of torsors of an algebraic groupoid, which can be found as the symmetry groupoid of some special family in $\mathfrak{M}$, called *versal*; in fact, identifying such a family is also useful to prove that a stack is algebraic.

Definition 4.A.2. An algebraic groupoid $R \rightrightarrows U$ is said to be the *symmetry groupoid* of a family $\mathcal{F} \in \mathfrak{M}(U)$ if the diagram

$$\begin{array}{ccc} \underline{R} & \xrightarrow{s} & \underline{U} \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \phi_{\mathcal{F}}} \\ \underline{U} & \xrightarrow{\phi_{\mathcal{F}}} & \mathfrak{M} \end{array}$$

is a fiber product of CFGs, where $\phi_{\mathcal{F}}$ is the morphism of CFGs naturally associated to $\mathcal{F}$ (i.e. it sends a map $f : S \to U$ to the family $f^*\mathcal{F}$).

Definition 4.A.3. [Beh14, Def. 1.138] We call $\mathcal{U} \in \mathfrak{M}(U)$ a *versal family* (in the étale topology) for $\mathfrak{M}$ if:

- for any family $\mathcal{F} \in \mathfrak{M}(S)$ there are an étale covering $\{S_\alpha \to S\}$ and morphisms $f_\alpha : S_\alpha \to U$ such that $f_\alpha^*\mathcal{U} \simeq \mathcal{F}\restriction_{S_\alpha}$;
- $\mathcal{U}$ has a symmetry groupoid.

If, besides finding a versal family $\mathcal{U} \in \mathfrak{M}(U)$, we are able to describe its symmetries with an affine group acting on U, then we have a complete description of our stack:

Proposition 4.A.4. *[Beh14, §1.3.3],[BCE⁺12, Thm 4.35] If $\mathfrak{M}$ is a stack with a versal family $\mathcal{U}$ whose symmetry groupoid is the action groupoid $G \times X \rightrightarrows X$ for some action λ of an affine group G, then $\mathfrak{M} \simeq [X/G]$. In particular, $\mathfrak{M}$ is algebraic.*

Chapter 5

Moduli of semistable sheaves as quiver moduli

This chapter, whose content was anticipated in §1.2.2, is a slightly expanded version of the paper [Mai17].

In Section 5.2 we will prove Theorem 1.2.1 in the Introduction: on a surface X, a suitable exceptional collection $\mathfrak{E}$ induces isomorphisms between moduli spaces of semistable sheaves on X and moduli of representations of the quiver with relations (Q, J) associated to $\mathfrak{E}$. This will arise as a consequence of a natural equivalence between categories of the objects parameterized by these moduli spaces.

Before doing that, we consider the analogous problem on $\mathbb{P}^1$, whose derived category is equivalent to $D^b(K_2)$, for K_2 the Kronecker quiver. This case is particularly simple because we can describe the heart $\mathcal{K} \subset D^b(\mathbb{P}^1)$ of Kronecker complexes as a tilt of the standard heart $\mathcal{C} = \mathbf{Coh}_{\mathcal{O}_{\mathbb{P}^1}}$ with respect to slope-stability, and this means that the semistable objects in $\mathcal{K}$ and $\mathcal{C}$ are easily related. In particular, it follows that we can describe the indecomposable sheaves on $\mathbb{P}^1$ (i.e. prove Birkhoff-Grothendieck theorem) in terms of indecomposable representations of K_2, whose classification is a classical problem in linear algebra.

In Sections 5.3 and 5.4 we apply the main theorem to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, which is now just a matter of computations, and we recover the results of [DLP85, Kul97] as special cases. Then we discuss how to reduce the study of moduli of sheaves on these surfaces to the properties of quiver moduli, and we consider some examples. Finally, Section 5.5 is devoted to the discussion of some further applications of this work. In particular, we explain how invariants of moduli spaces of sheaves can be studied using the descriptions of the previous sections.

In this chapter $\mathbb{K}$ will denote an algebraically closed field of characteristic 0.

5.1 Sheaves on $\mathbb{P}^1$ and Kronecker modules

In this section the well-known classification of coherent sheaves on $\mathbb{P}^1$ is deduced via the representation theory of the Kronecker quiver K_2, as an easy anticipation of the ideas introduced in the next sections.

5.1.1 Representations of K_2 and Kronecker complexes on $\mathbb{P}^1$

Let Z be a 2-dimensional $\mathbb{K}$-vector space with a basis $\{e_0, e_1\}$, and consider the complex projective line $\mathbb{P}^1 := \mathbb{P}_{\mathbb{K}}(Z)$. Fix also an integer $k \in \mathbb{Z}$.

We are interested in the finite-dimensional representations of the Kronecker quiver

$$K_2\colon\quad -1 \rightrightarrows 0\ ,$$

that is Kronecker modules $f \in \mathrm{Hom}_{\mathbb{K}}(V_{-1} \otimes Z, V_0)$ (see §4.1.8), and their relations with sheaves on $\mathbb{P}^1$.

As seen in Example 2.1.12.1, the couple $\mathfrak{E}_k = (E_{-1}, E_0) := (\mathcal{O}_{\mathbb{P}^1}(k-1), \mathcal{O}_{\mathbb{P}^1}(k))$ is a full strong exceptional sequence in $D^b(\mathbb{P}^1)$, and so is its left dual collection, which is given by ${}^\vee\mathfrak{E}_k = ({}^\vee E_0, {}^\vee E_{-1}) := (\mathcal{O}_{\mathbb{P}^1}(k), \tau_{\mathbb{P}^1}(k-1))$.[1] Here note that $\tau_{\mathbb{P}^1}(k-1) \simeq \mathcal{O}_{\mathbb{P}^1}(k+1)$. Hence, the tilting generator

$$T_k := \oplus_{i=-1}^{0} {}^\vee E_i = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \tau_{\mathbb{P}^1}(k-1)$$

induces by Theorem 2.2.5 a triangulated equivalence

$$\Psi_k := \Phi_{{}^\vee\mathfrak{E}_k} : D^b(\mathbb{P}^1) \to D^b(K_2)\,,$$

as $\mathrm{End}_{\mathcal{O}_{\mathbb{P}^1}}(T_k)$ may be identified with $\mathbb{K}K_2^{\mathrm{op}}$ via the isomorphism $H^0(\mathbb{P}^1; \tau_{\mathbb{P}^1}(-1)) \cong Z$. Ψ_k sends a complex $\mathcal{F}^\bullet$ of coherent sheaves to the complex of representations

$$R\,\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\tau_{\mathbb{P}^1}(k-1), \mathcal{F}^\bullet) \rightrightarrows R\,\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{F}^\bullet)\,. \tag{5.1.1}$$

As usual we denote by $\mathcal{C} := \mathbf{Coh}_{\mathcal{O}_{\mathbb{P}^1}} \subset D^b(\mathbb{P}^1)$ the heart of the standard t-structure and by $\mathcal{K}_k \subset D^b(\mathbb{P}^1)$ the heart of the t-structure induced from the standard one in $D^b(K_2)$ via the equivalence Ψ_k.

Lemma 5.1.1. *The objects of $\mathcal{K}_k$ are, up to isomorphism in $D^b(\mathbb{P}^1)$, the* Kronecker complexes

$$V_{-1} \otimes \mathcal{O}_{\mathbb{P}^1}(k-1) \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{P}^1}(k) \tag{5.1.2}$$

(concentrated in degrees -1,0), and morphisms between them are cochain maps.

Proof. Let $A := \mathrm{End}_{\mathcal{O}_{\mathbb{P}^1}}(T_k)$. Ψ_k maps the exceptional objects ${}^\vee E_i$, $i = 0, -1$, to the standard projective right A-modules $\mathrm{Id}_{{}^\vee E_i} A$, which correspond to the Kronecker modules

$$P_0 = (0 \otimes Z \to \mathbb{K})\,, \quad P_{-1} = (\mathbb{K} \otimes Z \xrightarrow{\mathrm{Id}} Z)\,.$$

Now the heart $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(K_2)$, which is the extension closure of the simple modules S_{-1}, S_0, is mapped to the extension closure $\mathcal{K}_k$ of $E_{-1}[1], E_0$ (see Remark 2.2.6). Thus the claim follows from Lemma 4.3.4. □

Remark 5.1.2. Notice that we also have a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{K}}(V_{-1} \otimes Z, V_1) \cong \mathrm{Hom}(V_{-1} \otimes \mathcal{O}_{\mathbb{P}^1}(k-1), V_0 \otimes \mathcal{O}_{\mathbb{P}^1}(k))\,, \tag{5.1.3}$$

which means that we have a natural way to identify Kronecker modules and Kronecker complexes. In fact, the functor Ψ_k acts according to this identification. To see this,[2] we can "resolve" a Kronecker module $f \in \mathrm{Hom}_{\mathbb{K}}(V_{-1} \otimes Z, V_1)$ with the projectives P_0, P_{-1}, via the canonical distinguished triangle

$$\mathrm{hoker} f \otimes_{\mathbb{K}} P_0 \to V_{-1} \otimes_{\mathbb{K}} P_{-1} \to (V_\bullet, f) \xrightarrow{+1}$$

in $D^b(K_2)$, where $\mathrm{hoker} f \in D^b(\mathbf{Vec}_{\mathbb{K}})$ is the complex $V_{-1} \otimes Z \xrightarrow{f} V_0$ located in degrees 0 and 1. Now, under Ψ_k^{-1} the first arrow in the triangle is sent to the canonical cochain map $\mathrm{hoker} f \otimes \mathcal{O}(k) \to V_{-1} \otimes \tau_{\mathbb{P}^1}(k-1)$, which is easily seen to be surjective and to have kernel given by a shifted Kronecker complex $K^\bullet[-1]$, so that we have a distinguished triangle

$$K^\bullet[-1] \to \mathrm{hoker} f \otimes_{\mathbb{K}} \mathcal{O}(k) \to V_{-1} \otimes_{\mathbb{K}} \tau_{\mathbb{P}^1}(k-1) \xrightarrow{+1}\,,$$

and thus $\Psi_k(K^\bullet) \simeq (V_\bullet, f)$. But now $K^\bullet$ is precisely the complex canonically identified with f, as follows from recalling that the usual isomorphism $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(k-1), \mathcal{O}_{\mathbb{P}^1}(k)) \cong Z^\vee$ used in the identification (5.1.3) is the one obtained from $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(k) \otimes Z, \mathcal{O}_{\mathbb{P}^1}(k)) \cong Z^\vee$ by restriction along $\mathcal{O}_{\mathbb{P}^1}(k-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(k) \otimes Z$.

[1] Notice that we are using a different convention than Chapter 2 in the labeling of the objects of these exceptional sequences. The reason is that it will be better to have the simple mnemonic rule $E_i = \mathcal{O}(k+i)$ when doing computations. The labeling of the vertices of K_2 has been chosen accordingly. The same principle will be followed with the other collections and quivers considered in this chapter.

[2] I am very grateful to A. Kuznetsov for carefully explaining this argument to me.

Ψ_k induces an isomorphism $\psi_k : K_0(\mathbb{P}^1) \to K_0(K_2)$ between the Grothendieck groups, which are free of rank 2. Hence, coordinates of an element $v \in K_0(\mathbb{P}^1)$ are provided either by the couple $(\operatorname{rk} v, \deg v)$ or by the dimension vector

$$d^v = (d^v_{-1}, d^v_0) := \underline{\dim}(\psi_k(v)) .$$

The simple representations $S(-1)$ and $S(0)$, whose dimension vectors are $(1,0)$ and $(0,1)$ respectively, correspond to the complexes $\mathcal{O}_{\mathbb{P}^1}(k-1)[1]$, with $(\operatorname{rk}, \deg) = (-1, 1-k)$, and $\mathcal{O}_{\mathbb{P}^1}(k)$, with $(\operatorname{rk}, \deg) = (1, k)$. So we deduce that the linear transformation between the two sets of coordinates is given by

$$\begin{pmatrix} \operatorname{rk} v \\ \deg v \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1-k & k \end{pmatrix} \begin{pmatrix} d^v_{-1} \\ d^v_0 \end{pmatrix} , \quad \begin{pmatrix} d^v_{-1} \\ d^v_0 \end{pmatrix} = \begin{pmatrix} -k & 1 \\ 1-k & 1 \end{pmatrix} \begin{pmatrix} \operatorname{rk} v \\ \deg v \end{pmatrix} . \tag{5.1.4}$$

5.1.2 Semistable sheaves and Kronecker complexes

As in Eq. (4.2.7), we consider the alternating form $\sigma_{\mathrm{M}} : K_0(\mathbb{P}^1) \times K_0(\mathbb{P}^1) \to \mathbb{Z}$ given by

$$\sigma_{\mathrm{M}}(v, w) := \deg v \operatorname{rk} w - \deg w \operatorname{rk} v .$$

This is also the alternating form σ_Z induced by the central charge $Z = -\deg + i \operatorname{rk}$ as in Equation (3.2.4), and coincides with the antisymmetrization of the Euler form χ. We have seen in Lemma 4.2.7 that, on the standard heart $\mathcal{C} = \mathbf{Coh}_{\mathcal{O}_{\mathbb{P}^1}}$, σ_{M} reproduces Gieseker stability. Now we also consider σ_{M}-stability on the heart $\mathcal{K}_k$: first of all, for any $v \in K_0(\mathbb{P}^1)$ we can write

$$\nu_{\mathrm{M},v}(w) := \sigma_{\mathrm{M}}(v, w) = -d^v_0 d^w_{-1} + d^v_{-1} d^w_0 = \theta_{\mathrm{M},v} \cdot d^w ,$$

where the dot is the standard scalar product in $\mathbb{Z}^{\{-1,0\}}$ and

$$\theta_{\mathrm{M},v} := \begin{pmatrix} -d^v_0 \\ d^v_{-1} \end{pmatrix} = \begin{pmatrix} (k-1) \operatorname{rk} v - \deg v \\ -k \operatorname{rk} v + \deg v \end{pmatrix} .$$

Remark 5.1.3. A Kronecker complex $K_V \in \mathcal{K}_k$ of class $[K_V] = v$ is a σ_{M}-(semi)stable object of $\mathcal{K}_k$ (according to Def. 3.2.4), if and only if the Kronecker module V corresponding to it via Ψ_k is a $\theta_{\mathrm{M},v}$-(semi)stable representation of K_2. For $\theta^0_{\mathrm{M},v} = d^v_{-1}$ positive, this is the usual definition of (semi)stable Kronecker module (Def. 4.1.17).

Consider an object in the intersection of the hearts $\mathcal{K}_k$ and $\mathcal{C}$ in $D^b(\mathbb{P}^1)$: this can be seen either as an injective Kronecker complex or as the sheaf given by its cokernel. The following observation shows that for such an object the two notions of stability coincide:

Proposition 5.1.4. *$\mathcal{K}_k$ is the heart obtained by tilting the standard heart $\mathcal{C}$ at the torsion pair*

$$(\mathcal{T}^Z_{\geq\phi_k}, \mathcal{F}^Z_{<\phi_k}) := (\mathcal{S}_Z([\phi_k, 1]), \mathcal{S}_Z((0, \phi_k)))$$

induced as in §3.3.3 by the central charge Z and the partition $(0,1] = (0,\phi_k) \sqcup [\phi_k, 1]$, where $\phi_k := \arg(-k+i)/\pi$ is the phase of the sheaf $\mathcal{O}_{\mathbb{P}^1}(k)$. In particular, for any $\phi \in [\phi_k, 1]$ the categories of Z-semistable objects with phase ϕ in the two hearts coincide: $\mathcal{S}^{(\mathcal{C})}_Z(\phi) = \mathcal{S}^{(\mathcal{K}_k)}_Z(\phi)$.

We denote by $\mathcal{R}_k \subset K_0(\mathbb{P}^1)$ the cone spanned by the objects of $\mathcal{C} \cap \mathcal{K}_k$, that is

$$\begin{aligned} \mathcal{R}_k &:= \{v \in K_0(\mathbb{P}^1) \mid \operatorname{rk} v \geq 0 \text{ and } \deg v \geq k \operatorname{rk} v\} \\ &= \{v \in K_0(\mathbb{P}^1) \mid d^v_0 \geq d^v_{-1} \geq 0\} \end{aligned} \tag{5.1.5}$$

The Proposition implies (as a special case of Lemma 3.3.13) that for any class $v \in \mathcal{R}_k$ the hearts $\mathcal{C}, \mathcal{K}$ are (σ_{G}, v)-compatible (Def. 3.3.11). Namely, we have:

(C1) a slope-(semi)stable sheaf $\mathcal{F} \in \mathcal{C}$ with $[\mathcal{F}] = v$ belongs to $\mathcal{K}_k$, that is, it is isomorphic to the cokernel of an injective Kronecker complex $K_V \in \mathcal{K}_k$; similarly, a (semi)stable Kronecker complex $K_V \in \mathcal{K}_k$ with $[K_V] = v$ belongs to $\mathcal{C}$, which means that it is injective;

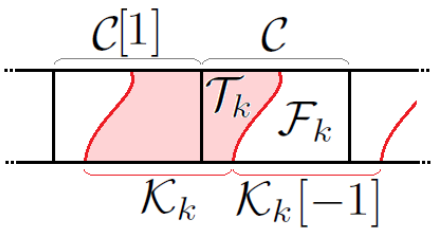


Figure 5.1: The hearts $\mathcal{C}, \mathcal{K}_k \subset D^b(\mathbb{P}^1)$

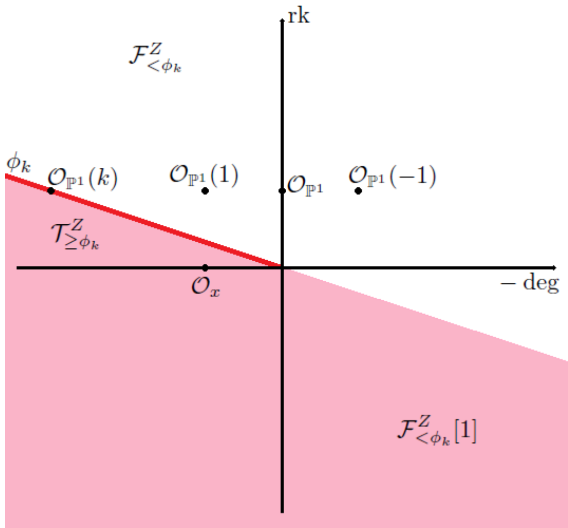


Figure 5.2: The Grothendieck group $K_0(\mathbb{P}^1)$

(C2) an object $K_V \simeq \mathcal{F}$ of class v in $\mathcal{C} \cap \mathcal{K}_k$ is (semi)-stable as a Kronecker complex if and only if it is (semi)-stable as a sheaf.

Proof. The heart $\mathcal{K}_k$ lies in $\langle \mathcal{C}, \mathcal{C}[1] \rangle_{\text{ext}}$, and then by Lemma 3.3.5 it is obtained by tilting $\mathcal{C}$ at the torsion pair $(\mathcal{T}_k, \mathcal{F}_k)$ given by $\mathcal{T}_k := \mathcal{C} \cap \mathcal{K}_k$ and $\mathcal{F}_k := \mathcal{C} \cap \mathcal{K}_k[-1]$. Consider also the above torsion pair $(\mathcal{T}^Z_{\geq\phi_k}, \mathcal{F}^Z_{<\phi_k})$. Now, using the explicit form (5.1.1) of Ψ_k, we will see that $\mathcal{T}^Z_{\geq\phi_k} \subset \mathcal{T}_k$ and $\mathcal{F}^Z_{<\phi_k} \subset \mathcal{F}_k$, which implies that the two torsion pairs must coincide: a sheaf $\mathcal{G} \in \mathcal{T}^Z_{\geq\phi_k}$ satisfies $\text{Ext}^1(\tau_{\mathbb{P}^1}(k-1), \mathcal{G}) = \text{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{G}) = 0$ by Serre duality, and thus it belongs to $\mathcal{K}_k$, and hence to $\mathcal{T}_k$. On the other hand, for a sheaf $\mathcal{F} \in \mathcal{F}^Z_{<\phi_k}$ we have $\text{Hom}(\tau_{\mathbb{P}^1}(k-1), \mathcal{F}) = \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{F}) = 0$, which means that it belongs to $\mathcal{K}_k[-1]$, and hence to $\mathcal{F}_k$. □

Corollary 5.1.5. *(Birkhoff-Grothendieck Theorem) Every coherent sheaf $\mathcal{F} \in \mathbf{Coh}_{\mathcal{O}_{\mathbb{P}^1}}$ is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(\ell)$ and structure sheaves of fat points.*

Proof. For an object in $\mathcal{T}^Z_{\geq\phi_k} = \mathcal{C} \cap \mathcal{K}_k$, being indecomposable is the same when considered in $\mathcal{C}$ or $\mathcal{K}_k$. We have seen in Prop. 4.1.16 that the indecomposable representations of K_2 are

$$\mathbb{K}^n \underset{J_n(\lambda)}{\overset{\mathbb{I}_n}{\rightrightarrows}} \mathbb{K}^n, \quad \mathbb{K}^n \underset{\mathbb{I}_n}{\overset{J_n(0)^t}{\rightrightarrows}} \mathbb{K}^n, \quad \mathbb{K}^n \underset{(0\ \mathbb{I}_n)^t}{\overset{(\mathbb{I}_n\ 0)^t}{\rightrightarrows}} \mathbb{K}^{n+1}, \quad \mathbb{K}^{n+1} \underset{(0\ \mathbb{I}_n)}{\overset{(\mathbb{I}_n\ 0)}{\rightrightarrows}} \mathbb{K}^n,$$

where $J_n(\lambda)$ is the n-dimensional Jordan matrix with eigenvalue $\lambda \in \mathbb{K}$. The first three representations correspond to injective Kronecker complexes whose cokernels are, respectively, a torsion sheaf with length n support at the point $[-\lambda : 1]$, a torsion sheaf with length n support at $[1 : 0]$ and the line bundle $\mathcal{O}_{\mathbb{P}^1}(k+n)$. Indeed, for example the first Kronecker module is mapped to a Kronecker complex $\mathcal{O}_{\mathbb{P}^1}(k-1)^{\oplus n} \xrightarrow{\eta} \mathcal{O}_{\mathbb{P}^1}(k)^{\oplus n}$ in which the bundle map η has the fiber over a point $[z_0 : z_1] \in \mathbb{P}^1$ represented by the matrix $\eta(z) = z^0\mathbb{I}_n + z^1 J_n(\lambda)$ (recall Remark 5.1.2). We have $\det \eta(z) = (z^0 + z^1\lambda)^n$, which means that η is generically nonsingular, but degenerates with order n at $[z_0 : z_1] = [-\lambda : 1]$. Hence η is injective and its cokernel has degree n and is supported at the fat point $[-\lambda : 1]$ of length n. The last representation of the list gives a Kronecker complex which has nontrivial cohomology in degree -1, and hence is not in $\mathcal{C}$.

Now take any $\mathcal{F} \in \mathbf{Coh}_{\mathcal{O}_{\mathbb{P}^1}}$ and choose $k \in \mathbb{Z}$ such that the minimum HN phase of $\mathcal{F}$ is at least $\phi_k = \arg(-k+i)/\pi$. If $\mathcal{F} = \oplus_i \mathcal{F}_i$ is the decomposition of $\mathcal{F}$ in indecomposables, then every $\mathcal{F}_i$ has HN phases $\geq \phi_k$, so $\mathcal{F}_i \in \mathcal{T}^Z_{\geq \phi_k}$, and then it is also an indecomposable object in $\mathcal{K}_k$, which means that it is isomorphic to one of the three sheaves listed above. □

5.1.3 Moduli spaces

Fix $k \in \mathbb{Z}$ and a class $v \in \mathcal{R}_k$ (see Eq. (5.1.5)). By Proposition 5.1.4 and the discussion of §4.2.4, we can identify the moduli stacks $\mathfrak{M}^{\mathrm{ss}}_{\mathbb{P}^1}(v)$ and $\mathfrak{M}^{\mathrm{ss}}_{K_2,\theta_{\mathrm{M},v}}(d^v)$, as well as the substacks of stable objects. Hence their coarse moduli spaces are isomorphic (notation as in Def. 4.1.17):

$$\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^1}(v) \simeq K(2; d^v_{-1}, d^v_0)\,, \quad \mathrm{M}^{\mathrm{st}}_{\mathbb{P}^1}(v) \simeq K_{\mathrm{st}}(2; d^v_{-1}, d^v_0)\,.$$

Collecting the results of §4.1.8 and §4.1.9, we can now describe explicitly all the moduli spaces $K(2; d^v_{-1}, d^v_0)$ and $K_{\mathrm{st}}(2; d^v_{-1}, d^v_0)$: indeed, consider the linear transformation

$$M = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

acting in the (d_{-1}, d_0) plane; the orbits of M are on lines of slope 1 (see Figure 5.3). The region $\mathcal{R} = \{d_0 \geq d_{-1} \geq 0\}$ and the diagonal $d_{-1} = d_0$ are invariant under M. Then Proposition 4.1.19 says that integral points in $\mathcal{R}$ lying in the same M-orbit, as well as symmetric points with respect to the diagonal $d_{-1} = d_0$, give isomorphic moduli spaces.

Thus it is enough to consider the wedge $d_0 \geq 2d_{-1}$ and the diagonal $d_{-1} = d_0$, which are described respectively in Prop. 4.1.18 and Prop. 4.1.23.

For $p, q \in \mathbb{Z}$, we call ℓ_p, r_q the lines in the (d_{-1}, d_0)-plane drawn in Figure 5.3: ℓ_p is the oblique line $\{pd_0 = (p+1)d_{-1}\}$ if $p > 0$, the diagonal $\{d_0 = d_{-1}\}$ for $p = 0$ and the line $\{pd_{-1} = (p-1)d_0\}$ for $p < 0$, while $r_q := \{d_0 = d_{-1} + q\}$. Putting the above-mentioned results together, we get:

Theorem 5.1.6. *We assume that $d_{-1} > 0$ and $d_0 > 0$.*

1. *$K(2; d_{-1}, d_0)$ is nonempty if and only if (d_{-1}, d_0) lies on a line ℓ_p;*
2. *if $(d_{-1}, d_0) \in \ell_p \cap r_q$ for some $p, q \in \mathbb{Z}$ with $q \neq 0, \pm 1$, then $K(2; d_{-1}, d_0) = \mathrm{pt}$, while $K_{\mathrm{st}}(2; d_{-1}, d_0) = \emptyset$;*
3. *if there is some $p \in \mathbb{Z}$ such that $(d_{-1}, d_0) \in \ell_p \cap r_1$ or $(d_{-1}, d_0) \in \ell_p \cap r_{-1}$, then $K(2; d_{-1}, d_0) = K_{\mathrm{st}}(2; d_{-1}, d_0) = \mathrm{pt}$;*
4. *if $(d_{-1}, d_0) \in \ell_0 = r_0$, then $K(2; d_{-1}, d_0) \simeq \mathbb{P}^{d_0}$; moreover $K_{\mathrm{st}}(2; 1, 1) \simeq \mathbb{P}^1$, while $K_{\mathrm{st}}(2; m, m) = \emptyset$ for $m \geq 2$.*

Now we can translate this into a classification of moduli of sheaves on $\mathbb{P}^1$ (depicted in Figure 5.4):

Corollary 5.1.7. *Fix $v \in K_0(\mathbb{P}^1)$.*

1. *Suppose that $\operatorname{rk} v > 0$ and $\deg v$ is a multiple of $\operatorname{rk} v$ (i.e. $\mu(v) \in \mathbb{Z}$); then $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^1}(v)$ is a point, while $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^1}(v)$ is a point if $\operatorname{rk} v = 1$ and empty otherwise;*
2. *if $\operatorname{rk} v = 0$ and $\deg v \geq 0$, then $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^1}(v) \simeq \mathbb{P}^{\deg v}$; moreover $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^1}(v) \simeq \mathbb{P}^1$ for $\deg v = 1$, while $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^1}(v) = \emptyset$ if $\deg v \geq 2$;*
3. *in all the other cases $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^1}(v)$ is empty.*

Proof. Choose $k \in \mathbb{Z}$ so that $v \in \mathcal{R}_k$. The statements immediately follow from the Theorem, by noticing that the transformation (5.1.4) maps the lines ℓ_p with $p > 0$ and the lines r_q respectively to the lines $(p+k)\operatorname{rk} v = \deg v$ and the horizontal lines $\operatorname{rk} v = q$ in the $(-\deg v, \operatorname{rk} v)$ plane. □

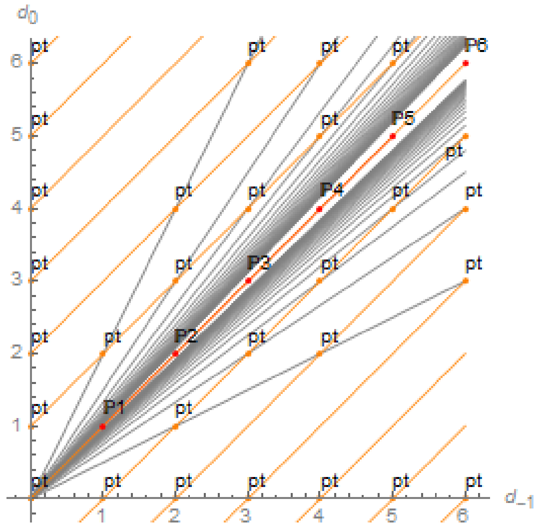


Figure 5.3: The moduli spaces $K(2; d_{-1}, d_0)$ for all values of $d_{-1}, d_0 \in \mathbb{N}$.

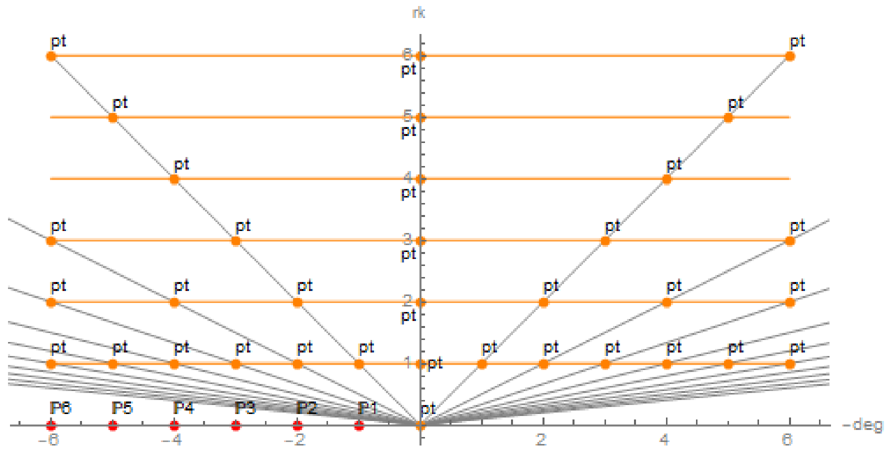


Figure 5.4: The moduli spaces $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^1}(v)$ for all values of $v \in K_0(\mathbb{P}^1)$ with $\operatorname{rk} v \geq 0$.

Remark 5.1.8. The statements of the Corollary can be easily explained in sheaf-theoretic terms via Birkhoff-Grothendieck Theorem:

1. A semistable sheaf of rank $r > 0$ must be a direct sum of r copies of the same line bundle $\mathcal{O}_{\mathbb{P}^1}(\ell)$, so it has degree $r\ell$; it is stable if and only if $r = 1$.

2. The polystable sheaves of rank 0 and degree d are direct sums $\mathcal{O}_{x_1} \oplus \cdots \oplus \mathcal{O}_{x_d}$ of skyscraper sheaves and as such they are in 1-1 correspondence with points of the dth symmetric product $\mathbb{P}^d$ of $\mathbb{P}^1$; in particular, they can be stable if and only if $d = 1$. The structure sheaf of a fat point is also semistable, but not polystable, and it degenerates (i.e. is S-equivalent, Def. 4.2.4) to a multiple of the skyscraper sheaf on the reduced point where it is supported.

5.2 Gieseker stability and quiver stability on surfaces

In this section we discuss how to relate Gieseker-semistable sheaves on a surface X with a nice exceptional sequence to semistable representations of the associated quiver. The idea is analogous to what we did for $\mathbb{P}^1$ in the previous section, but the situation becomes now more involved and requires a different analysis.

Let X be a smooth irreducible projective complex surface with an ample divisor $A \subset X$.

5.2.1 Preliminary considerations

First of all we assume that X has a strong full exceptional collection ${}^\vee\mathfrak{E} = ({}^\vee E_n, ..., {}^\vee E_0)$ of vector bundles, so that by Theorem 2.2.5 we get an equivalence (for convenience we include now a shift)

$$\Psi := \Phi_{{}^\vee\mathfrak{E}}[1] : D^b(X) \longrightarrow D^b(Q; J)\,. \tag{5.2.1}$$

Recall that Ψ maps a complex $\mathcal{F}^\bullet$ in $D^b(X)$ to a complex of representations of Q given, at a vertex $i \in \{0, ..., n\}$ of Q, by the graded vector space

$$R\,\mathrm{Hom}({}^\vee E_i, \mathcal{F}^\bullet)[1]\,. \tag{5.2.2}$$

Ψ induces in particular an isomorphism $\psi : K_0(X) \to K_0(Q; J)$, and a t-structure on $D^b(X)$ whose heart is denoted by $\mathcal{K} := \Psi^{-1}(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q; J))$ and equals the extension closure of the objects $E_i[n-i-1]$, where $\mathfrak{E} = (E_0, ..., E_n)$ is the right dual collection to ${}^\vee\mathfrak{E}$ (see Remark 2.2.6, but recall that now $\mathcal{K}$ is also shifted by one place to the right).

The polynomial-valued alternating form $\sigma_{\mathrm{G}} = t\sigma_{\mathrm{M}} + \sigma_\chi$ on $K_0(X)$, defined in Eq. (4.2.8), reproduces Gieseker stability when regarded as a stability structure on the standard heart $\mathcal{C} = \mathbf{Coh}_{\mathcal{O}_X}$ (Lemma 4.2.7). On the other hand, if we see σ_{G} as a stability structure on $\mathcal{K}$, then an object $K_V \in \mathcal{K}$ in a class $v \in K_0(X)$ and corresponding via Ψ to a representation $V \in \mathbf{Rep}^{\mathrm{fd}}_{\mathbb{K}}(Q; J)$ is σ_{G}-(semi)stable if and only if V is $\theta_{\mathrm{G},v}$-(semi)stable in the sense of Def. 4.1.1, where the polynomial array

$$\theta_{\mathrm{G},v} = t\theta_{\mathrm{M},v} + \theta_{\chi,v} \in \mathbb{Z}[t]^I$$

is defined by (the dot denotes the standard scalar product in $\mathbb{Z}^I$)

$$\nu_{\mathrm{M},v}(w) = \sigma_{\mathrm{M}}(v, w) = \theta_{\mathrm{M},v} \cdot \underline{\dim}\,\psi(w)\,, \quad \nu_{\chi,v}(w) = \sigma_\chi(v, w) = \theta_{\chi,v} \cdot \underline{\dim}\,\psi(w)\,. \tag{5.2.3}$$

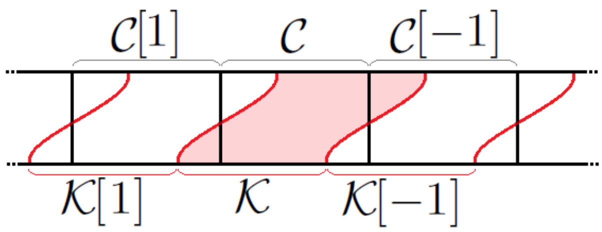


Figure 5.5: The hearts $\mathcal{C}, \mathcal{K} \subset D^b(X)$

Unlike what happened for $\mathbb{P}^1$, now $\mathcal{K}$ is not obtained as a tilt of the standard heart $\mathcal{C}$ with respect to a stability condition (it never satisfies Eq. (3.3.1) because it intersects three shifts of $\mathcal{C}$, see Figure 5.5). So there seems to be no reason to expect a priori any relation between stability on one heart and on the other. Nevertheless, we will see that under certain hypotheses this kind of compatibility exists; more precisely, we discuss when the hearts $\mathcal{C}, \mathcal{K}$ are (σ_{G}, v)-compatible in the sense of Def. 3.3.11. Doing this requires the following extra hypotheses on the collections $\mathfrak{E}, {}^\vee\mathfrak{E}$, which will be always assumed in this section:

Definition 5.2.1. The strong exceptional sequence $\mathfrak{E}$ will be called *monad-friendly* (with respect to the ample divisor A) if the following assumptions are satisfied:

(A1) the objects ${}^\vee E_i$ are locally free sheaves which are Gieseker-semistable with respect to A;

(A2) every element of $\mathcal{K}$ is isomorphic to a complex K_V of locally free sheaves concentrated in degrees $-1, 0, 1$;

By analogy with the situation discussed in §1.1.2, we will call such complexes K_V *Kronecker complexes.*

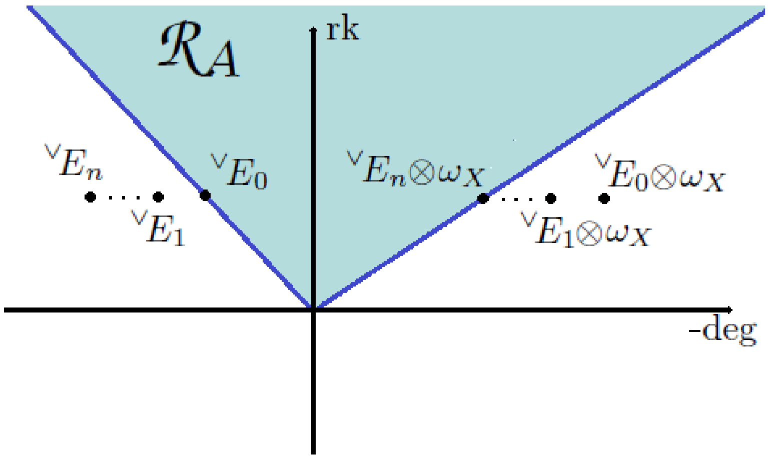


Figure 5.6: The region $\mathcal{R}_A$.

Remark 5.2.2. In the specific cases that will be examined in the next sections, assumption (A2) will follow from the fact that the objects $E_i[n-i-1]$ generating $\mathcal{K}$ turn out to group into three *blocks*, where the objects in each block are orthogonal to each other, and they are all isomorphic to vector bundles $\tilde{E}_i$ shifted to degree $-1, 0$ or 1, depending on the block, and with vanishing positive Ext spaces between them. Because of this and Lemma 4.3.4, the complex $K_V \in \mathcal{K}$ corresponding to some representation V of (Q, J) consists, in each degree $\ell = -1, 0, 1$, of a direct sum of vector bundles of the form $V_i \otimes \tilde{E}_i$. This means in particular that we can write down explicitly the cohomological functors $H^\ell_{\mathcal{K}}$ of the non-standard t-structure as functors mapping a complex $\mathcal{F}^\bullet \in D^b(X)$ to a complex $K_V \in \mathcal{K}$ with

$$V_i = R^{\ell+1} \operatorname{Hom}({}^\vee E_i, \mathcal{F}^\bullet)\,. \tag{5.2.4}$$

5.2.2 Condition (C1)

In this subsection we will study condition (C1) of §3.3.4. We keep the assumptions and notation of the previous subsection; in particular, we have a monad-friendly strong exceptional sequence $\mathfrak{E}$ on X. First of all, we want to show that a semistable sheaf $\mathcal{F}$ in a class v also belongs to the heart $\mathcal{K}$, that is, it is isomorphic to the middle cohomology of a certain monad K_V (recall from §4.3 that a Kronecker complex K_V is a *monad* when it has zero cohomology in degrees $\ell \neq 0$). This amounts to checking the vanishing on $\mathcal{F}$ of the cohomological functors $H^\ell_{\mathcal{K}}$ for $\ell \neq 0$, which in turn reduces, by Eq. (5.2.2), to verifying the vanishing of some Ext spaces. For this to work we need to choose v appropriately: we denote by $\mathcal{R}_A \subset \mathcal{R}^{\mathrm{G}}_A \subset K_0(X)$ the regions

$$\begin{aligned}
\mathcal{R}_A &:= \{v \in K_0(X) \mid \operatorname{rk} v > 0\,,\ \max_i \mu_A({}^\vee E_i \otimes \omega_X) < \mu_A(v) < \min_i \mu_A({}^\vee E_i)\}\,,\\
\mathcal{R}^{\mathrm{G}}_A &:= \{v \in K_0(X) \mid \operatorname{rk} v > 0\,,\ \max^{\mathrm{G}}_i P_{{}^\vee E_i \otimes \omega_X, A} \prec_{\mathrm{G}} P_{v,A} \prec_{\mathrm{G}} \min^{\mathrm{G}}_i P_{{}^\vee E_i, A}\}\\
&= \{v \in K_0(X) \mid \operatorname{rk} v > 0\,,\ \max_i p_{{}^\vee E_i \otimes \omega_X, A} < p_{v,A} < \min_i p_{{}^\vee E_i, A}\}
\end{aligned} \tag{5.2.5}$$

(recall that $p_{v,A}$ denotes the *reduced* Hilbert polynomial, §4.2.1). For these regions to be nonempty, the exceptional sheaves ${}^\vee E_i$ must have their slopes concentrated in a sufficiently narrow region, and the anticanonical bundle $\omega_X^\vee$ must be sufficiently positive (Figure 5.6).

Remark 5.2.3. We could also twist the collection $\mathfrak{E}$ by a line bundle, to shift the regions $\mathcal{R}_A, \mathcal{R}^{\mathrm{G}}_A$ accordingly: if these are wide enough and the line bundle has small but nonzero degree, then with such twists we can cover the whole region $\operatorname{rk} v > 0$. When this is the case, like in the examples that we will consider, we are thus free to start with any class $v \in K_0(X)$ with positive rank, provided that we choose $\mathfrak{E}$ appropriately.

Lemma 5.2.4. *Suppose that $v \in \mathcal{R}^{\mathrm{G}}_A$ (resp. $v \in \mathcal{R}_A$). Then any Gieseker-semistable (resp. slope-semistable) sheaf $\mathcal{F} \in v$ belongs to the heart $\mathcal{K}$.*

Proof. Since each ${}^\vee E_i$ is Gieseker-semistable by assumption $(A1)$, we have $\mathrm{Hom}({}^\vee E_i, \mathcal{F}) = 0$ because of the inequality $p_{\mathcal{F},A} < p_{{}^\vee E_i,A}$; on the other hand, the inequality $p_{{}^\vee E_i \otimes \omega_X,A} < p_{\mathcal{F},A}$ and Serre duality give $\mathrm{Ext}^2({}^\vee E_i, \mathcal{F}) = \mathrm{Hom}(\mathcal{F}, {}^\vee E_i \otimes \omega_X) = 0$. So $H^{-1}_{\mathcal{K}}(\mathcal{F}) = H^1_{\mathcal{K}}(\mathcal{F}) = 0$ by Eq. (5.2.2). For the case of slope semistability the proof is the same. □

Now we deal with the same problem with the two hearts $\mathcal{C}, \mathcal{K}$ exchanged: we want a σ_{G}-semistable Kronecker complex $K_V \in \mathcal{K}$ of class v to be a monad, that is to belong to $\mathcal{C}$. To obtain this, we observe that when K_V is not a monad, we can construct a destabilizing subcomplex or quotient complex using the following idea from [FGIK16, §2]:[3] consider the skyscraper sheaf $\mathcal{O}_x$ over some point $x \in X$. Clearly $H^\ell_{\mathcal{K}}(\mathcal{O}_x) = 0$ for all $\ell \neq -1$, which means that there is a Kronecker complex $K_x \in \mathcal{K}$ which has cohomology $\mathcal{O}_x$ in degree 1, and zero elsewhere, that is to say that $K_x \simeq \mathcal{O}_x[-1]$ in $D^b(X)$. Observe that this complex is self-dual: we have

$$K_x^\vee \simeq \mathcal{O}_x^\vee[1] \simeq \mathcal{O}_x[-1] \simeq K_x \,,$$

in $D^b(X)$, where $\mathcal{O}_x^\vee \simeq \mathcal{O}_x[-2]$ is the derived dual of $\mathcal{O}_x$.

Proposition 5.2.5. *If the second map in a Kronecker complex K_V is not surjective at some point $x \in X$, then there is a nonzero morphism $K_V \to K_x$. If the first map in K_V is not injective at x, then there is a nonzero morphism $K_x \to K_V$.*

Proof. Suppose that the second map $b : K_V^0 \to K_V^1$ in K_V is not surjective at some $x \in X$: we have then a surjective morphism $c : K_V^1 \to \mathcal{O}_x$ such that $c \circ b = 0$, and this gives a cochain map $K_V \to \mathcal{O}_x[-1]$, and thus a nonzero morphism $K_V \to K_x$ in $\mathcal{K}$.
Now suppose that the first map is not injective at x: then we can apply the previous argument to the complex $K_V^\vee$ to get a nonzero map $K_V^\vee \to K_x$, hence a nonzero $K_x \simeq K_x^\vee \to K_V$. □

In following two lemmas we prove that for any Kronecker complex $K_V \in \mathcal{K}$ of class v and any σ_{G}-maximal subobject $K_W \subset K_V$ in $\mathcal{K}$ (see Def. 3.2.6), we have some vanishings in the cohomologies of K_W and K_V/K_W, provided that the class $v \in K_0(X)$ chosen satisfies some constraints imposed by the complex K_x. Notice that when K_V is σ_{G}-semistable, then it is a σ_{G}-maximal subobject of itself, and thus K_V will turn out in Cor. 5.2.9 to be a monad.

Lemma 5.2.6. *Take $K_V \in \mathcal{K}$ of class v. Suppose that for any $x \in X$ and any nonzero subobject $S \subset K_x$ in $\mathcal{K}$ we have $\nu_{\mathrm{G},v}(S) := \sigma_{\mathrm{G}}(v, S) > 0$. Then any σ_{G}-maximal subobject $K_W \subset K_V$ satisfies $H^1_{\mathcal{C}}(K_W) = 0$.*

Proof. If $H^1_{\mathcal{C}}(K_W) \neq 0$, which means that the second map in K_W is not surjective at some point $x \in X$, then there is a nonzero morphism $f : K_W \to K_x$ by Prop. 5.2.5. So we have $\nu_{\mathrm{G},v}(K_W) = \nu_{\mathrm{G},v}(\ker f) + \nu_{\mathrm{G},v}(\mathrm{im}\, f)$ and, by hypothesis, $\nu_{\mathrm{G},v}(\mathrm{im}\, f) > 0$. If $\ker f = 0$ then $\nu_{\mathrm{G},v}(K_W) > 0$, while if $\ker f \neq 0$ then $\nu_{\mathrm{G},v}(\ker f) = \nu_{\mathrm{G},v}(K_W) - \nu_{\mathrm{G},v}(\mathrm{im}\, f) < \nu_{\mathrm{G},v}(K_W)$; in both cases, $\sigma_{\mathrm{G}}(K_W, K_V) = -\nu_{\mathrm{G},v}(K_W)$ is not maximal. □

Lemma 5.2.7. *Take $K_V \in \mathcal{K}$ of class v. Suppose that, for any $x \in X$, K_x is $\nu_{\mathrm{M},v}$-semistable and every quotient Q of K_x with $\nu_{\mathrm{M},v}(Q) = 0$ satisfies $H^{-1}_{\mathcal{C}}(Q) = 0$. Then for any σ_{M}-maximal subobject $K_W \subset K_V$ we have $H^{-1}_{\mathcal{C}}(K_V/K_W) = 0$.*

Notice that if $K_W \subset K_V$ is σ_{G}-maximal then it is also σ_{M}-maximal. It is also worth mentioning here that $\nu_{\mathrm{M},v}(K_x) = 0$ and $\nu_{\mathrm{G},v}(K_x) = \nu_{\chi,v}(K_x) = \mathrm{rk}\, v$.

Proof. Let $K_W \subset K_V$ be a σ_{M}-maximal subobject, which means that the quotient $K_U := K_V/K_W$ maximizes $\nu_{\mathrm{M},v} = \sigma_{\mathrm{M}}(v, \cdot)$. We have to prove that the first map in K_U is injective. This is clearly true if such a map is injective at every point of X; thus suppose now that it is not injective at some point $x \in X$, so that we have a nonzero morphism $g : K_x \to K_U$ by Prop. 5.2.5. Now $\nu_{\mathrm{M},v}(K_U) = \nu_{\mathrm{M},v}(K_x/\ker g) + \nu_{\mathrm{M},v}(\mathrm{coker}\, g)$, and $\nu_{\mathrm{M},v}(K_x/\ker g) \leq 0$ by hypothesis.

If $\mathrm{coker}\, g = 0$, then $0 \leq \nu_{\mathrm{M},v}(K_U) = \nu_{\mathrm{M},v}(K_x/\ker g) \leq 0$, implying $H^{-1}_{\mathcal{C}}(K_U) = 0$. On the other hand, if the cokernel

$$K_U \overset{c^{(0)}}{\to} K_U^{(1)} := \mathrm{coker}\, g$$

[3] I want to thank Alexander Kuznetsov for pointing out the reference [FGIK16] to me.

is nonzero, then looking at the exact sequence

$$0 \to \ker g \to K_x \xrightarrow{g} K_U \xrightarrow{c^{(0)}} K_U^{(1)} \to 0$$

we see that $\nu_{\mathrm{M},v}(K_x/\ker g) = 0$ (otherwise $\nu_{\mathrm{M},v}(K_U) < \nu_{\mathrm{M},v}(\operatorname{coker} g)$, contradicting maximality of $\nu_{\mathrm{M},v}(K_U)$), so that $\nu_{\mathrm{M},v}(K_U^{(1)}) = \nu_{\mathrm{M},v}(K_U)$. Hence $K_U^{(1)}$ is also a quotient of K_V of maximal $\nu_{\mathrm{M},v}$, and $H_{\mathcal{C}}^{-1}(\ker c^{(0)}) = H_{\mathcal{C}}^{-1}(K_x/\ker g) = 0$.

By applying the same argument to $K_U^{(1)}$ we see that either we can immediately conclude that $H_{\mathcal{C}}^{-1}(K_U^{(1)}) = 0$, in which case we stop here, or we can construct a further quotient

$$K_U^{(1)} \xrightarrow{c^{(1)}} K_U^{(2)}$$

with maximal $\nu_{\mathrm{M},v}$ and such that $H_{\mathcal{C}}^{-1}(\ker c^{(1)}) = 0$. After finitely many steps ($\mathcal{K}$ has finite lenght) we end up with a chain

$$K_U = K_U^{(0)} \xrightarrow{c^{(0)}} K_U^{(1)} \xrightarrow{c^{(1)}} K_U^{(2)} \xrightarrow{c^{(2)}} \cdots \xrightarrow{c^{(\ell-1)}} K_U^{(\ell)}$$

of surjections with $H_{\mathcal{C}}^{-1}(\ker c^{(i)}) = 0$ for all $i \geq 0$ and $H_{\mathcal{C}}^{-1}(K_U^{(\ell)}) = 0$. This implies that $H_{\mathcal{C}}^{-1}(K_U) = 0$. □

Remark 5.2.8. Notice that the hypotheses of Lemmas 5.2.6 and 5.2.7 are verified under the stronger assumptions that $\operatorname{rk} v > 0$ and for all $x \in X$, K_x is $\nu_{\mathrm{M},v}$-stable.[4]

It is convenient to gather the conditions on v imposed by the hypotheses of Lemmas 5.2.6 and 5.2.7 or by Remark 5.2.8 in the definition of two regions $\mathcal{S}_A^\circ \subset \mathcal{S}_A \subset K_0(X)$:

$$\mathcal{S}_A := \left\{ v \in K_0(X) \;\middle|\; \begin{array}{l} \text{for any } x \in X \text{ we have } \nu_{\mathrm{G},v}(S) > 0 \text{ for any } 0 \neq S \subset K_x\,, \text{ and} \\ H_{\mathcal{C}}^{-1}(Q) = 0 \text{ for any quotient } K_x \to Q \text{ with } \nu_{\mathrm{M},v}(Q) = 0 \end{array} \right\},$$

$$\mathcal{S}_A^\circ := \{ v \in K_0(X) \mid \operatorname{rk} v > 0 \text{ and } K_x \text{ is } \nu_{\mathrm{M},v}\text{-stable for all } x \in X \}\,. \tag{5.2.6}$$

Again, in the examples it will be enough to twist the collection $\mathfrak{E}$ by a line bundle to have any $v \in K_0(X)$ of positive rank inside such a region.

Corollary 5.2.9. *Take $K_V \in \mathcal{K}$ of class $v \in \mathcal{S}_A$. If K_V is σ_{G}-semistable, then it is a monad, that is $K_V \in \mathcal{C}$.*

Proof. If a nonzero K_V is σ_{G}-semistable (hence $\nu_{\mathrm{G},v}$-semistable), then it has minimal $\nu_{\mathrm{G},v}$ between its subobjects, and maximal $\nu_{\mathrm{M},v}$ between its quotients. So we can apply Lemmas 5.2.6 and 5.2.7 to deduce that $H_{\mathcal{C}}^{-1}(K_V) = H_{\mathcal{C}}^{1}(K_V) = 0$. □

Summing up, Lemma 5.2.4 and Corollary 5.2.9 tell us that:

Proposition 5.2.10. *Assume that $\mathfrak{E}$ is monad-friendly (Def. 5.2.1). Then condition (C1) of §3.3.4 is verified for Gieseker stability σ_{G}, the hearts $\mathcal{C}, \mathcal{K}$ and for all $v \in \mathcal{R}_A^{\mathrm{G}} \cap \mathcal{S}_A$.*

5.2.3 Condition (C2)

Now we turn to the analysis of condition (C2) of §3.3.4: we want to show that a monad $K_V \in \mathcal{K}$ of class v is σ_{G}-(semi)stable as an object of $\mathcal{K}$ if and only if its middle cohomology is σ_{G}-(semi)stable as an object of $\mathcal{C}$, that is, a Gieseker-(semi)stable sheaf. Again, notation and assumptions are as in §5.2.1. First we prove the "only if" direction:

Lemma 5.2.11. *Suppose that $v \in \mathcal{R}_A^{\mathrm{G}}$, and let $K_V \in v$ be monad which is a σ_{G}-(semi)stable object of $\mathcal{K}$. Then its middle cohomology $H_{\mathcal{C}}^0(K_V)$ is a Gieseker-(semi)stable sheaf.*

[4] As for the hypothesis of Lemma 5.2.6, note that $\nu_{\mathrm{G},v}(K_x) = \operatorname{rk} v > 0$.

Proof. Suppose that $\mathcal{F} := H^0_{\mathcal{C}}(K_V)$ is not Gieseker-semistable. Let $\mathcal{F}_1 \subset \mathcal{F}$ be the maximally destabilizing subsheaf (i.e. the first nonzero term in the HN filtration of $\mathcal{F}$, see §3.1.3), which is semistable and satisfies $P_{\mathcal{F}_1,A} \succ_{\mathrm{G}} P_{\mathcal{F},A} \succeq_{\mathrm{G}} P_{\mathcal{F}/\mathcal{F}_1,A,\max}$ (notation as in Eq. (4.2.6)). Then, as in the proof of Lemma 5.2.4, we deduce that

$$\mathrm{Hom}({}^{\vee}E_i, \mathcal{F}/\mathcal{F}_1) = 0\,, \quad \mathrm{Ext}^2({}^{\vee}E_i, \mathcal{F}_1) = \mathrm{Hom}(\mathcal{F}_1, {}^{\vee}E_i \otimes \omega_X) = 0$$

for all i. These vanishings mean that $H^\ell_{\mathcal{K}}(\mathcal{F}/\mathcal{F}_1) = 0$ for all $\ell \neq 0, 1$ and $H^\ell_{\mathcal{K}}(\mathcal{F}_1) = 0$ for all $\ell \neq -1, 0$, so we get a long exact sequence

$$0 \to H^{-1}_{\mathcal{K}}(\mathcal{F}_1) \to 0 \to 0 \to H^0_{\mathcal{K}}(\mathcal{F}_1) \to H^0_{\mathcal{K}}(\mathcal{F}) \to H^0_{\mathcal{K}}(\mathcal{F}/\mathcal{F}_1) \to 0 \to 0 \to H^1_{\mathcal{K}}(\mathcal{F}/\mathcal{F}_1) \to 0\,, \tag{5.2.7}$$

showing that $H^{-1}_{\mathcal{K}}(\mathcal{F}_1) = H^1_{\mathcal{K}}(\mathcal{F}/\mathcal{F}_1) = 0$, that is $\mathcal{F}, \mathcal{F}/\mathcal{F}_1 \in \mathcal{K}$, and the remaining short exact sequence means that $K_V = H^0_{\mathcal{K}}(\mathcal{F})$ is not σ_{G}-semistable.

Finally, if $\mathcal{F}$ is strictly σ_{G}-semistable, then we take $\mathcal{F}_1 \subsetneq \mathcal{F}$ with $P_{\mathcal{F}_1,A} \equiv_{\mathrm{G}} P_{\mathcal{F},A} \equiv_{\mathrm{G}} P_{\mathcal{F}/\mathcal{F}_1,A}$ (hence $\mathcal{F}_1$ and $\mathcal{F}/\mathcal{F}_1$ are semistable) and again we get a short exact sequence as in Eq. (5.2.7), showing that K_V is not σ_{G}-stable. □

Now we prove the "if" direction with a specular argument:

Lemma 5.2.12. *Suppose that $K_V \in \mathcal{K}$ is a monad of class $v \in \mathcal{S}_A$ whose middle cohomology $H^0_{\mathcal{C}}(K_V)$ is a Gieseker-(semi)stable sheaf. Then K_V is σ_{G}-(semi)stable as an object of $\mathcal{K}$.*

Proof. Suppose that K_V is σ_{G}-unstable, take a σ_{G}-maximal subobject $0 \neq K_W \subsetneq K_V$ in $\mathcal{K}$ (this exists as the subobjects of K_V can only belong to finitely many classes in $K_0(X)$) and apply Lemmas 5.2.6 and 5.2.7, to get the vanishings $H^1_{\mathcal{C}}(K_W) = H^{-1}_{\mathcal{C}}(K_V/K_W) = 0$ and then an exact sequence

$$0 \to H^{-1}_{\mathcal{C}}(K_W) \to 0 \to 0 \to H^0_{\mathcal{C}}(K_W) \to H^0_{\mathcal{C}}(K_V) \to H^0_{\mathcal{C}}({}^{K_V}\!/_{K_W}) \to 0 \to 0 \to H^1_{\mathcal{C}}({}^{K_V}\!/_{K_W}) \to 0 \tag{5.2.8}$$

showing that $K_W, K_V/K_W \in \mathcal{C}$ and that $\mathcal{F} := H^0_{\mathcal{C}}(K_V)$ is also σ_{G}-unstable as an object of $\mathcal{C}$. Now suppose that K_V is strictly σ_{G}-semistable: we have again a $0 \neq K_W \subsetneq K_V$ maximizing $\nu_{\mathrm{G},v}$, so that the lemmas apply and we end up with a short exact sequence as in (5.2.8), showing that $\mathcal{F}$ is not σ_{G}-stable. □

So we can conclude that:

Proposition 5.2.13. *Assume that $\mathfrak{E}$ is monad-friendly (Def. 5.2.1). Then condition (C2) of §3.3.4 is verified for Gieseker stability σ_{G}, the hearts $\mathcal{C}, \mathcal{K}$ and for all $v \in \mathscr{R}^{\mathrm{G}}_A \cap \mathcal{S}_A$.*

5.2.4 Conclusions

We summarize the results of §5.2.2 and §5.2.3. We recall that X is a smooth projective surface, A is an ample divisor, and we are supposing that $D^b(X)$ has a full strong exceptional collection ${}^{\vee}\mathfrak{E}$ which is monad-friendly with respect to A (Def. 5.2.1). Recall also that ${}^{\vee}\mathfrak{E}$ determines a quiver Q with relations J, together with an equivalence $\Psi : D^b(X) \to D^b(Q; J)$ (Eq. (5.2.1)), a heart $\mathcal{K} \subset D^b(X)$ and an isomorphism $\psi : K_0(X) \to K_0(Q; J)$. For any class $v \in K_0(X)$ we denote by $d^v := \underline{\dim}\, \psi(v) \in \mathbb{Z}^I$ the corresponding dimension vector, and by $\theta_{\mathrm{G},v} = t\theta_{\mathrm{M},v} + \theta_{\chi,v} \in \mathbb{Z}[t]^I$ the array of polynomials defined in Eq. (5.2.3).

Now consider the conical region

$$\tilde{\mathscr{R}}_{A,\mathfrak{E}} := \mathscr{R}^{\mathrm{G}}_A \cap \mathcal{S}_A \subset K_0(X)$$

defined as the intersection of the regions in Equations (5.2.5) and (5.2.6).

Theorem 5.2.14. *For all $v \in \tilde{\mathscr{R}}_{A,\mathfrak{E}}$, the hearts $\mathcal{C}$ and $\mathcal{K}$ are (σ_{G}, v)-compatible (Def. 3.3.11). Thus, Ψ restricts to an equivalence between the category of Gieseker-(semi)stable sheaves of class v on X and the category of d_v-dimensional $\theta_{\mathrm{G},v}$-(semi)stable representations of (Q, J).*

As already observed at the end of §3.3.4, this theorem implies that the moduli stack $\mathfrak{M}^{\mathrm{ss}}_{X,A}(v)$ of σ_G-semistable objects in $\mathcal{C}$ with class v coincides with the moduli stack of σ_G-semistable objects in $\mathcal{K}$ with class v, which (recall the discussion of §4.2.4) is isomorphic to the quiver moduli stack $\mathfrak{M}^{\mathrm{ss}}_{Q,J,\theta_{\mathrm{G},v}}(d^v)$. Similar arguments apply to the stable loci. Hence:

Corollary 5.2.15. *For all $v \in \tilde{\mathcal{R}}_{A,\mathfrak{E}}$ we have isomorphisms*

$$\mathfrak{M}^{\mathrm{ss}}_{X,A}(v) \simeq \mathfrak{M}^{\mathrm{ss}}_{Q,J,\theta_{\mathrm{G},v}}(d^v) \quad \text{and} \quad \mathfrak{M}^{\mathrm{st}}_{X,A}(v) \simeq \mathfrak{M}^{\mathrm{st}}_{Q,J,\theta_{\mathrm{G},v}}(d^v)\,.$$

In particular, we have isomorphisms

$$\mathrm{M}^{\mathrm{ss}}_{X,A}(v) \simeq \mathrm{M}^{\mathrm{ss}}_{Q,J,\theta_{\mathrm{G},v}}(d^v) \quad \text{and} \quad \mathrm{M}^{\mathrm{st}}_{X,A}(v) \simeq \mathrm{M}^{\mathrm{st}}_{Q,J,\theta_{\mathrm{G},v}}(d^v)$$

between the coarse moduli spaces.

Recall that the construction of the moduli space $\mathrm{M}^{\mathrm{ss}}_{Q,J,\theta_{\mathrm{G},v}}(d^v)$ for a polynomial array $\theta_{\mathrm{G},v} \in \mathbb{Z}[t]^I$ was explained in §4.1.6.

5.3 Application to $\mathbb{P}^2$

In this section we apply the previous results taking X to be the complex projective plane $\mathbb{P}^2 = \mathbb{P}_{\mathbb{K}}(Z)$, where Z is a 3-dimensional $\mathbb{K}$-vector space. We choose the ample divisor as $A = H$, the divisor of a line, and we write $\deg := \deg_H$ for simplicity. For the computations with numerical invariants of sheaves we will use the formulas of Example 4.2.2.1.

5.3.1 The first equivalence

Take, as in Ex. 2.1.12.1, the full strong collections

$$\mathfrak{E} = (E_{-1}, E_0, E_1) = (\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))\,,$$
$$^{\vee}\mathfrak{E} = ({}^{\vee}E_1, {}^{\vee}E_0, {}^{\vee}E_{-1}) = (\mathcal{O}_{\mathbb{P}^2}(1), \tau_{\mathbb{P}^2}, \wedge^2\tau_{\mathbb{P}^2}(-1))$$

(note that $\wedge^2\tau_{\mathbb{P}^2}(-1) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$). We apply Theorem 2.2.5 to the collection ${}^{\vee}\mathfrak{E}$: the tilting sheaf $T = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \tau_{\mathbb{P}^2} \oplus \wedge^2\tau_{\mathbb{P}^2}(-1)$ has endomorphism algebra

$$\mathrm{End}_{\mathcal{O}_{\mathbb{P}^2}}(T) = \begin{pmatrix} \mathbb{K} & & \\ Z & \mathbb{K} & \\ \wedge^2 Z & Z & \mathbb{K} \end{pmatrix}$$

which is identified, after fixing a $\mathbb{K}$-basis e_0, e_1, e_2 of Z, with the opposite of the bound quiver algebra $\mathbb{K}B_3/J$ of the *Beilinson quiver*

$$B_3\colon \quad -1 \overset{a_1}{\underset{a_3}{\xrightarrow{a_2}}} 0 \overset{b_1}{\underset{b_3}{\xrightarrow{b_2}}} 1$$

with quadratic relations $J = (b_i a_j + b_j a_i,\ i,j = 1,2,3)$. So we get a triangulated equivalence

$$\Psi := \Phi_{{}^{\vee}\mathfrak{E}}[1] : D^b(\mathbb{P}^2) \to D^b(B_3; J)\,.$$

This maps a complex $\mathcal{F}^\bullet \in D^b(\mathbb{P}^2)$ to the complex of representations

$$R\,\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\wedge^2\tau_{\mathbb{P}^2}(-1), \mathcal{F}^\bullet)[1] \substack{\to\\ \to\\ \to} R\,\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\tau_{\mathbb{P}^2}, \mathcal{F}^\bullet)[1] \substack{\to\\ \to\\ \to} R\,\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{F}^\bullet)[1]\,.$$

The standard heart of $D^b(B_3; J)$ is sent to the heart

$$\mathcal{K} := \langle \mathcal{O}_{\mathbb{P}^2}(-1)[1], \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)[-1] \rangle_{\mathrm{ext}}$$

whose objects are (by Lemma 4.3.4) *Kronecker complexes*

$$K_V : V_{-1} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow V_1 \otimes \mathcal{O}_{\mathbb{P}^2}(1)\,,$$

where the middle sheaf is in degree 0. Moreover, the objects of ${}^{\vee}\mathfrak{E}$ are semistable bundles. Thus the assumptions the sequence $\mathfrak{E}$ is monad-friendly with respect to H (Def. 5.2.1).

The equivalence Ψ also gives an isomorphism $\psi : K_0(\mathbb{P}^2) \to K_0(B_3; J)$; coordinates on the Grothendieck groups are provided by the isomorphisms

$$K_0(\mathbb{P}^2) \overset{(\mathrm{rk},\deg,\chi)}{\longrightarrow} \mathbb{Z}^3\,, \quad K_0(B_3; J) \overset{\underline{\dim}}{\longrightarrow} \mathbb{Z}^3\,,$$

and we denote by

$$(d^v_{-1}, d^v_0, d^v_1) = d^v := \underline{\dim}\,\psi(v)$$

the coordinates of $\psi(v) \in K_0(B_3; J)$ with respect to the basis of simple representations $S(-1), S(0), S(1)$; using the fact that these are mapped to $\mathcal{O}_{\mathbb{P}^2}(-1)[1], \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)[-1]$, we find that the base-change matrices between the two coordinate sets are

$$\begin{pmatrix} d^v_{-1} \\ d^v_0 \\ d^v_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 3 & -2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{rk}\, v \\ \deg v \\ \chi(v) \end{pmatrix}, \quad \begin{pmatrix} \mathrm{rk}\, v \\ \deg v \\ \chi(v) \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} d^v_{-1} \\ d^v_0 \\ d^v_1 \end{pmatrix}. \tag{5.3.1}$$

So, given $v \in K_0(\mathbb{P}^2)$, the arrays $\theta_{\mathrm{M},v}, \theta_{\chi,v} \in \mathbb{Z}^{\{-1,0,1\}}$ associated to the alternating forms $\sigma_{\mathrm{M}}, \sigma_\chi$ as in equation (5.2.3) are given by

$$\theta_{\mathrm{M},v} = \begin{pmatrix} -\mathrm{rk}\, v - \deg v \\ \deg v \\ \mathrm{rk}\, v - \deg v \end{pmatrix} = \begin{pmatrix} -d^v_0 + 2d^v_1 \\ d^v_{-1} - d^v_1 \\ -2d^v_{-1} + d^v_0 \end{pmatrix},$$
$$\theta_{\chi,v} = \begin{pmatrix} -\chi(v) \\ -\mathrm{rk}\, v + \chi(v) \\ 3\,\mathrm{rk}\, v - \chi(v) \end{pmatrix} = \begin{pmatrix} -d^v_0 + 3d^v_1 \\ d^v_{-1} - 2d^v_1 \\ -3d^v_{-1} + 2d^v_0 \end{pmatrix}.$$

Before applying the results of §5.2, we compute explicitly the regions $\mathcal{R}_H, \mathcal{S}^\circ_H \subset K_0(\mathbb{P}^2)$ of equations (5.2.5) and (5.2.6):

Lemma 5.3.1. $\mathcal{R}_H = \mathcal{S}^\circ_H = \{|\deg v| < \mathrm{rk}\, v\} = \{d^v_0 > 2d^v_{-1} \text{ and } d^v_0 > 2d^v_1\}$.

Proof. The slopes of $\mathcal{O}_{\mathbb{P}^2}(1), \mathrm{T}_{\mathbb{P}^2}, \wedge^2\mathrm{T}_{\mathbb{P}^2}(-1) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ and their twists by $\omega_{\mathbb{P}^2}$ are respectively $1, 3/2, 2$ and $-2, -3/2, -1$, so

$$\begin{aligned} \mathcal{R}_H &= \{-1 < \mu(v) < 1\} = \{|\deg v| < \mathrm{rk}\, v\} = \{|d^v_{-1} - d^v_1| < -d^v_{-1} + d^v_0 - d^v_1\} \\ &= \{d^v_0 > 2d^v_{-1} \text{ and } d^v_0 > 2d^v_1\}\,. \end{aligned}$$

Now take $x \in \mathbb{P}^2$ and let $p, q \in Z^\vee$ be linear forms whose common zero is x, and notice that the Kronecker complex

$$K_x : \mathcal{O}_{\mathbb{P}^2}(-1) \overset{\binom{p}{q}}{\longrightarrow} \mathbb{K}^2 \otimes \mathcal{O}_{\mathbb{P}^2} \overset{(q\ -p)}{\longrightarrow} \mathcal{O}_{\mathbb{P}^2}(1)$$

is quasi-isomorphic to $\mathcal{O}_x[-1]$. We want K_x to be $\nu_{\mathrm{M},v}$-stable: its only nontrivial subcomplexes have dimension vectors d' equal to $(0,2,1), (0,1,1)$ and $(0,0,1)$; the inequalities $\theta_{\mathrm{M},v} \cdot d' > 0$ read

$$\mathrm{rk}\, v - \deg v > 0\,, \quad \mathrm{rk}\, v > 0\,, \quad \mathrm{rk}\, v + \deg v > 0\,,$$

so that $\mathcal{S}^\circ_H = \mathcal{R}_H$. □

Remark 5.3.2. Notice that, after twisting by a line bundle, every sheaf of positive rank can be brought inside the region $\mathcal{R}_H$. Hence it is enough to consider this region to describe all the moduli spaces $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v)$ with $\mathrm{rk}\, v > 0$.

We can now apply Corollary 5.2.15:

Theorem 5.3.3. *For any $v \in \mathcal{R}_H$ we have isomorphisms*

$$\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v) \simeq \mathrm{M}^{\mathrm{ss}}_{B_3,J,\theta_{\mathrm{G},v}}(d^v) \quad \text{and} \quad \mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2,H}(v) \simeq \mathrm{M}^{\mathrm{st}}_{B_3,J,\theta_{\mathrm{G},v}}(d^v)\,.$$

Remarks 5.3.4. Many of the known properties of $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v)$ can be recovered from the isomorphisms of Theorem 5.3.3:

1 $v \in \mathcal{R}_H$ is primitive if and only if $\gcd(\mathrm{rk}\, v, \deg v, \chi(v)) = 1$. In this case we have that $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v) = \mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2,H}(v)$ and there is a universal family, either by Remark 4.2.10.2 or Remark 4.1.15.2.[5]

2 By Cor. 4.3.6 (note that a semistable sheaf $\mathcal{E}$ has $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) = 0$ by Serre duality), the variety $X_J \subset R_{d^v}(B_3)$ cut by the relations J intersects the semistable locus $R^{\mathrm{ss}}_{d^v,\theta_{\mathrm{G},v}}(B_3)$ in a smooth complete intersection. In particular $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2,H}(v) \simeq \mathrm{M}^{\mathrm{st}}_{B_3,J,\theta_{\mathrm{G},v}}(d^v)$ is smooth and we can compute its dimension as the dimension of the quotient $R^{\mathrm{st}}_{d^v,\theta_{\mathrm{G},v}}(B_3)/PG_{d^v}$ minus the number $6d^v_{-1}d^v_1$ of relations imposed; the result is

$$\dim \mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2,H}(v) = 1 - \mathrm{rk}\, v^2 + \Delta(v)\,,$$

in agreement with Eq. (4.2.11).

3 If $\theta^{-1}_{\mathrm{M},v} > 0$ or $\theta^{1}_{\mathrm{M},v} < 0$ then every d^v-dimensional representation is $\theta_{\mathrm{G},v}$-unstable, so $\mathrm{M}^{\mathrm{ss}}_{B_3,J,\theta_{\mathrm{G},v}}(d^v)$ is empty. But for all $v \in \mathcal{R}_H$ we have

$$\theta^{-1}_{\mathrm{M},v} = -\,\mathrm{rk}\, v - \deg v < 0 \quad \text{and} \quad \theta^{1}_{\mathrm{M},v} = \mathrm{rk}\, v - \deg v > 0\,.$$

Notice also that the existence of a semistable sheaf $\mathcal{F}$ in $v \in \mathcal{R}_H$ implies that all the components of the array $\underline{\dim}\, v$ are nonnegative. Thus for example $2\,\mathrm{ch}_2\, v = -d^v_{-1} - d^v_1 \leq 0$, with the equality holding only when $\mathcal{F}$ is trivial. From this simple observation we can easily deduce the Bogomolov inequality (4.2.10):

Proposition 5.3.5. *If $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v) \neq \emptyset$ for some $v \in K_0(\mathbb{P}^2)$, then*

$$\Delta(v) := (\deg v)^2 - 2\,\mathrm{rk}\, v\, \mathrm{ch}_2(v) \geq 0\,.$$

Proof. For $\mathrm{rk}\, v = 0$ the statement is obvious. If $\mathrm{rk}\, v > 0$, then after twisting by a line bundle (which does not change the discriminant Δ) we can reduce to the case $v \in \mathcal{R}_H$: for such v we have just observed that $\mathrm{ch}_2\, v \leq 0$, and hence $\Delta(v) \geq 0$. □

Finally, observe that whether a class $v \in K_0(\mathbb{P}^2)$ belongs to the region $\mathcal{R}_H$ only depends on the ray generated by the Hilbert polynomial $P_{v,H}$ in $\mathbb{R}[t]_{\leq 2}$. Thus we can extend the equivalence of Thm 5.2.14 to whole abelian categories of semistable sheaves with fixed reduced Hilbert polynomial:

Theorem 5.3.6. *If $p \in \mathbb{R}[t]$ is the Hilbert polynomial of a class $v \in \mathcal{R}_H$, then Ψ restricts to an equivalence between the abelian categories $\mathcal{S}_H(p)$ and $\mathcal{S}_{\theta_{\mathrm{G},v}}$ (defined in Equations* (4.2.5) *and* (4.1.3)*).*

Proof. Identify $\mathcal{S}_{\theta_{\mathrm{G},v}}$ with the category of $\nu_{\mathrm{G},v}$-semistable Kronecker complexes, via Ψ. The inclusion $\mathcal{S}_H(p) \subset \mathcal{S}_{\theta_{\mathrm{G},v}}$ is clear, as any nonzero $\mathcal{F} \in \mathcal{S}_H(p)$ has class $[\mathcal{F}] \in \mathcal{R}_H$. For the converse, take a $\theta_{\mathrm{G},v}$-semistable representation (V,f) of (B_3,J), and let $w := \psi^{-1}[V,f] \in K_0(\mathbb{P}^2)$. By definition of $\theta_{\mathrm{G},v}$-stability we have $\sigma_{\mathrm{G}}(v,w) = 0$ and then $pP'_{w,H} - p'P_{w,H} = 0$, which by Remark 3.2.11 implies that $P_{w,H} = \alpha p$ for some $\alpha \in \mathbb{R}$. In fact, $\alpha \neq 0$ since $P_{w,H} \neq 0$ by Eq. (5.3.1), and α cannot be negative because otherwise we would have $-P_{w,H} \in \mathcal{R}_H$, and then $\theta^{-1}_{\mathrm{M},w} > 0$ and $\theta^{1}_{\mathrm{M},w} < 0$, which (as observed in Remark 5.3.4.3) would contradict the existence of the semistable representation (V,f) in w. Thus $w \in \mathcal{R}_H$, and hence $\Psi^{-1}(V,f) \in \mathcal{S}_H(p)$ by Thm 5.2.14. This concludes the proof that $\mathcal{S}_H(p) \supset \mathcal{S}_{\theta_{\mathrm{G},v}}$. □

[5] Notice that for $v \in \mathcal{R}_H$ the arrays $\theta_{\mathrm{M},v}, \theta_{\chi,v}$ are linearly independent, so d^v is $\theta_{\mathrm{G},v}$-coprime by Remark 4.1.2.2.

5.3.2 The second equivalence

Now we will use instead the full strong collections

$$\mathfrak{E}' = (E'_{-1}, E'_0, E'_1) = (\Omega^2_{\mathbb{P}^2}(2), \Omega^1_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}),$$
$$^\vee\mathfrak{E}' = (^\vee E'_1, {}^\vee E'_0, {}^\vee E'_{-1}) = (\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$$

(note that $\Omega^2_{\mathbb{P}^2}(2) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$). The tilting sheaf $T' = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ has endomorphism algebra

$$\operatorname{End}_{\mathcal{O}_{\mathbb{P}^2}}(T') = \begin{pmatrix} \mathbb{K} & & \\ Z^\vee & \mathbb{K} & \\ S^2 Z^\vee & Z^\vee & \mathbb{K} \end{pmatrix}$$

which is identified, after fixing a $\mathbb{K}$-basis e_0, e_1, e_2 of Z, to the opposite of the bound quiver algebra $\mathbb{K}B_3/J'$, where now $J' = (b_i a_j - b_j a_i,\ i,j = 1,2,3)$. The new equivalence

$$\Psi' := \Phi_{^\vee\mathfrak{E}'}[1] : D^b(\mathbb{P}^2) \to D^b(B_3; J')$$

sends a complex $\mathcal{F}^\bullet \in D^b(\mathbb{P}^2)$ to the complex of representations

$$R\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{F}^\bullet)[1] \substack{\to\\\to\\\to} R\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{F}^\bullet)[1] \substack{\to\\\to\\\to} R\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{F}^\bullet)[1]$$

and the standard heart of $D^b(B_3; J')$ is now sent to the heart

$$\mathcal{K}' := \langle \Omega^2_{\mathbb{P}^2}(2)[1], \Omega^1_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}[-1] \rangle_{\text{ext}}$$

whose objects are complexes

$$K'_V : V_{-1} \otimes \Omega^2_{\mathbb{P}^2}(2) \longrightarrow V_0 \otimes \Omega^1_{\mathbb{P}^2}(1) \longrightarrow V_1 \otimes \mathcal{O}_{\mathbb{P}^2}$$

with the middle term in degree 0. These are the Kronecker complexes originally used in [DLP85], and we see again that $\mathfrak{E}'$ is monad-friendly with respect to A (Def. 5.2.1).

Ψ' induces a different isomorphism $\psi' : K_0(\mathbb{P}^2) \to K_0(B_3; J')$. Given $v \in K_0(\mathbb{P}^2)$, we write now

$$({d'}^v_{-1}, {d'}^v_0, {d'}^v_1) = {d'}^v := \underline{\dim}\, \psi'(v)$$

for the coordinates with respect to the basis of simple representations $S(-1), S(0), S(1)$; these are mapped to the objects $\Omega^2_{\mathbb{P}^2}(2)[1] \simeq \mathcal{O}_{\mathbb{P}^2}(-1)[1], \Omega^1_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}[-1]$, for which the triple $(\operatorname{rk}, \deg, \chi)$ is equal to $(-1, 1, 0)$, $(2, -1, 0)$ and $(-1, 0, -1)$ respectively. This gives the linear transformations

$$\begin{pmatrix} {d'}^v_{-1} \\ {d'}^v_0 \\ {d'}^v_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \operatorname{rk} v \\ \deg v \\ \chi(v) \end{pmatrix}, \quad \begin{pmatrix} \operatorname{rk} v \\ \deg v \\ \chi(v) \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} {d'}^v_{-1} \\ {d'}^v_0 \\ {d'}^v_1 \end{pmatrix}. \tag{5.3.2}$$

For $v \in K_0(\mathbb{P}^2)$ define now $\theta'_{\mathrm{G},v} = t\theta'_{\mathrm{M},v} + \theta'_{\chi,v}$ by

$$\nu_{\mathrm{M},v}(w) = \sigma_{\mathrm{M}}(v,w) = \theta'_{\mathrm{M},v} \cdot {d'}^w, \quad \nu_{\chi,v}(w) = \sigma_\chi(v,w) = \theta'_{\chi,v} \cdot {d'}^w,$$

where the new arrays $\theta'_{\mathrm{M},v}, \theta'_{\chi,v} \in \mathbb{Z}^{\{-1,0,1\}}$ are given by

$$\begin{aligned} \theta'_{\mathrm{M},v} &= \begin{pmatrix} -\operatorname{rk} v - \deg v \\ 2\deg v + \operatorname{rk} v \\ -\deg v \end{pmatrix} = \begin{pmatrix} -{d'}^v_0 + {d'}^v_1 \\ {d'}^v_{-1} - {d'}^v_1 \\ -{d'}^v_{-1} + {d'}^v_0 \end{pmatrix}, \\ \theta'_{\chi,v} &= \begin{pmatrix} -\chi(v) \\ 2\chi(v) \\ \operatorname{rk} v - \chi(v) \end{pmatrix} = \begin{pmatrix} {d'}^v_1 \\ -2{d'}^v_1 \\ -{d'}^v_{-1} + 2{d'}^v_0 \end{pmatrix}. \end{aligned} \tag{5.3.3}$$

Lemma 5.3.7. *The regions of interest are now (the inequalities are between polynomials in t, lexicographically ordered as usual)*

$$\mathcal{R}_H{}' = \mathcal{S}^\circ_H{}' = \{v \in K_0(\mathbb{P}^2) \mid 0 < -\deg v < \operatorname{rk} v\} = \{v \in K_0(\mathbb{P}^2) \mid {d'}^v_0 > {d'}^v_{-1} \text{ and } {d'}^v_0 > {d'}^v_1\},$$
$$\mathcal{R}^{\mathrm{G}}_H{}' = \{v \in K_0(\mathbb{P}^2) \mid -t\operatorname{rk} v < t\deg v + \chi(v) < \operatorname{rk} v\},$$
$$\mathcal{S}_H{}' = \{v \in K_0(\mathbb{P}^2) \mid -(t+1)\operatorname{rk} v < t\deg v + \chi(v) < \operatorname{rk} v, \quad \deg v \neq -\operatorname{rk} v\}.$$

Proof. First note that for classes $v, v' \in K_0(X)$ of positive rank, the inequality $p_{H,v} < p_{H,v'}$ is equivalent to $\mu_H(v) < \mu_H(v')$ or $\mu_H(v) = \mu_H(v')$ and $\chi(v) \operatorname{rk} v' < \chi(v') \operatorname{rk} v$. This time we get

$$\mathcal{R}_H{}' = \{-1 < \mu_H(v) < 1\} = \{0 < -\deg v < \operatorname{rk} v\},$$
$$\mathcal{R}_H^{\mathrm{G}}{}' = \{-t < t\mu_H(v) + \tfrac{\chi(v)}{\operatorname{rk} v} < 0t + 1\} = \{-t \operatorname{rk} v < t \deg v + \chi(v) < \operatorname{rk} v\}.$$

To find the expressions for $\mathcal{S}_H^{\circ}{}'$ and $\mathcal{S}_H{}'$ we observe that for any $x \in \mathbb{P}^2$ we can take a section $s \in H^0(\tau_{\mathbb{P}^2}(-1))$ whose zero locus is x, and define the Kronecker complex

$$K'_x : \Omega^2_{\mathbb{P}^2}(2) \xrightarrow{\iota_s} \Omega^1_{\mathbb{P}^2}(1) \xrightarrow{\iota_s} \mathcal{O}_{\mathbb{P}^2},$$

so that $K'_x \simeq \mathcal{O}_x[-1]$. The nonzero subcomplexes of K'_x have dimension vectors $d' = (0,0,1), (0,1,1), (1,1,1)$, and the inequalities $\theta'_{\mathrm{G},v} \cdot d' > 0$ read

$$t(-\deg v) + (\operatorname{rk} v - \chi(v)) > 0, \quad t(\operatorname{rk} v + \deg v) + \operatorname{rk} v + \chi(v) > 0, \quad \operatorname{rk} v > 0.$$

The nontrivial quotients Q of K'_x have dimensions $(1,1,0)$ and $(1,0,0)$, and only for the latter we have $H^{-1}_{\mathcal{C}}(Q) \neq 0$, so we only have to require that it has

$$0 \neq \nu'_{\mathrm{M},v}(Q) = \theta'_{\mathrm{M},v} \cdot (1,0,0) = -\operatorname{rk} v - \deg v.$$

□

Remark 5.3.8. This time the cone $\mathcal{R}_H{}'$ is not wide enough to describe all moduli spaces for positive rank: if a torsion-free sheaf $\mathcal{F}$ has $\mu_H(\mathcal{F}) \in \mathbb{Z}$, then no twist of it is in $\mathcal{R}_H{}'$. However, if $\mathcal{F}$ is non-trivial and Gieseker-semistable, then it has a twist $\mathcal{F}(k)$ of zero slope and $\chi(\mathcal{F}(k)) = \operatorname{rk} \mathcal{F} + 3 \deg_H \mathcal{F}/2 + \operatorname{ch}_2 \mathcal{F} < \operatorname{rk} \mathcal{F}$ (because $\operatorname{ch}_2 \mathcal{F} < 0$, as observed just before Prop. 5.3.5), so that it is contained in

$$\tilde{\mathcal{R}}_{H,\mathfrak{E}'} = \mathcal{R}_H^{\mathrm{G}}{}' \cap \mathcal{S}_H{}' = \{v \in K_0(\mathbb{P}^2) \mid -t \operatorname{rk} v < t \deg v + \chi(v) < \operatorname{rk} v, \quad \deg v \neq -\operatorname{rk} v\}.$$

So we can apply Corollary 5.2.15 to the collection $\mathfrak{E}'$:

Theorem 5.3.9. *Let $v \in \tilde{\mathcal{R}}_{H,\mathfrak{E}'}$. Then we have isomorphisms*

$$\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v) \simeq \mathrm{M}^{\mathrm{ss}}_{B_3,J',\theta'_{\mathrm{G},v}}(d'^v) \quad \text{and} \quad \mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2,H}(v) \simeq \mathrm{M}^{\mathrm{st}}_{B_3,J',\theta'_{\mathrm{G},v}}(d'^v).$$

Moreover, remarks analogous to those at the end of the previous subsection apply to this situation, and similarly we also deduce an equivalence of abelian categories of semistable objects:

Theorem 5.3.10. *If $p \in \mathbb{R}[t]$ is the Hilbert polynomial of a class $v \in \tilde{\mathcal{R}}_{H,\mathfrak{E}'}$, then Ψ restricts to an equivalence between the abelian categories $\mathcal{S}_H(p)$ and $\mathcal{S}_{\theta'_{\mathrm{G},v}}$ (defined in Equations* (4.2.5) *and* (4.1.3)*).*

5.3.3 Examples

Now we will see some examples in which $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v)$ can be determined more or less explicitly using the isomorphisms of Theorems 5.3.3 and 5.3.9.

The first observation is that we can choose $v \in K_0(\mathbb{P}^2)$ so that at least one of the invariants $d^v_{-1}, d^v_1, d'^v_{-1}, d'^v_1$ vanish (and via equations (5.3.1) and (5.3.2) each of these conditions turns into a linear relation on $\operatorname{rk} v$, $\deg v$ and $\chi(v)$). In these cases, the representations of B_3 under consideration reduce to representations of the Kronecker quiver K_3, the relations J and J' are trivially satisfied and in any case the stability conditions reduce to the standard one for Kronecker modules (Def. 4.1.17). This means that $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v)$ is isomorphic to some Kronecker moduli space $K(3; m, n)$, for which we can use the properties described in §4.1.8 and §4.1.9. More precisely, as special cases of Theorems 5.3.3 and 5.3.9 we have:

Corollary 5.3.11. *First, let $v \in \mathcal{R}_H$:*

1. *if $d^v_{-1} = \operatorname{rk} v - 2\deg v - \chi(v) = 0$, then $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v) \simeq K(3; d^v_0, d^v_1)$;*

2. *if* $d_1^v = \operatorname{rk} v + \deg v + \chi(v) = 0$, *then* $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v) \simeq K(3; d_{-1}^v, d_0^v)$.

Now let $v \in \tilde{\mathcal{R}}_{H,\mathfrak{E}'}$*:*

3. *if* ${d'}_{-1}^v = \operatorname{rk} v + 2\deg v - \chi(v) = 0$, *then* $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v) \simeq K(3; {d'}_0^v, {d'}_1^v)$;

4. *if* ${d'}_1^v = -\chi(v) = 0$, *then* $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v) \simeq K(3; {d'}_{-1}^v, {d'}_0^v)$.

Similar isomorphisms hold for the stable loci.

Recall also that twisting by $\mathcal{O}_{\mathbb{P}^2}(1)$ gives isomorphic moduli spaces. In the examples below we will only consider classes v normalized as before, that is belonging to the regions $\mathcal{R}_H$ or $\tilde{\mathcal{R}}_{H,\mathfrak{E}'}$.

Since we have an isomorphism $K_0(\mathbb{P}^2) \xrightarrow{(\mathrm{rk},\deg,\chi)} \mathbb{Z}^3$, we will often write

$$\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(\operatorname{rk} v, \deg v, \chi(v)) \quad \text{and} \quad \mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2,H}(\operatorname{rk} v, \deg v, \chi(v))$$

to indicate $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(v)$ and $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2,H}(v)$.

Examples 5.3.12.

1 Let $r > 0$ and $(d_{-1}^v, d_0^v, d_1^v) = (0, r, 0)$, so that $(\operatorname{rk} v, \deg v, \chi(v)) = (r, 0, r)$. For this choice there is a unique representation of B_3, which is always semistable, and stable only for $r = 1$. So $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(r, 0, r)$ is a point, and $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2,H}(r, 0, r)$ is a point for $r = 1$ and empty for $r > 1$. The only Gieseker-semistable sheaf with these invariants is the trivial bundle $\mathcal{O}_{\mathbb{P}^2}^{\oplus r}$.

2 Let $m > 0$ and $({d'}_{-1}^v, {d'}_0^v, {d'}_1^v) = (0, m, 0)$, so that $(\operatorname{rk} v, \deg v, \chi(v)) = (2m, -m, 0)$. Again, $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(2m, -m, 0)$ is a point, and $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2,H}(2m, -m, 0)$ is a point for $m = 1$ and empty for $m > 1$: the only Gieseker-semistable sheaf with these invariants is $\Omega^1_{\mathbb{P}^2}(1)^{\oplus m}$.

3 Let m be a positive integer. We have $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(5m, -2m, 0) \simeq K(3; m, 3m) \simeq \mathrm{pt}$, with empty stable locus for $m > 1$: the only Gieseker-semistable sheaf with these invariants is the right mutation $R_{\Omega^1_{\mathbb{P}^2}(1)} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus m}[1]$.

4 By Cor. 5.3.11 and Prop. 4.1.23 we have $\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(2, 0, 0) \simeq K(3; 2, 2) \simeq \mathbb{P}^5$, and the stable locus is the complement of the cubic symmetroid in $\mathbb{P}^5$. See also [OSS80, Ch.2, §4.3] for a sheaf-theoretical proof of this isomorphism.

5 Since $\mathrm{Pic}^0(\mathbb{P}^2)$ is trivial, by sending a 0-dimensional subscheme $Z \subset X$ of length ℓ to its ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ we get isomorphisms

$$\begin{aligned} \mathrm{Hilb}^\ell(\mathbb{P}^2) &\simeq \mathrm{M}^{\mathrm{ss/st}}_{\mathbb{P}^2,H}(1, 0, 1-\ell) \\ &\simeq \mathrm{M}^{\mathrm{ss/st}}_{B_3,J,(-t+\ell-1,-\ell,t+2+\ell)}(\ell, 2\ell+1, \ell) \simeq \mathrm{M}^{\mathrm{ss/st}}_{B_3,J',(-t+\ell-1,t+2-2\ell,\ell)}(\ell, \ell, \ell-1)\,, \end{aligned}$$

where $\mathrm{Hilb}^\ell(\mathbb{P}^2)$ is the Hilbert scheme of ℓ points in $\mathbb{P}^2$. In particular, $\mathrm{Hilb}^1(\mathbb{P}^2) \simeq \mathbb{P}^2$ must be isomorphic to the moduli spaces $\mathrm{M}^{\mathrm{ss}}_{B_3,J,(-t,-1,t+3)}(1, 3, 1)$ and $\mathrm{M}^{\mathrm{ss}}_{B_3,J',(-t,t,1)}(1, 1, 0)$, and also to their stable loci.

We can obtain these isomorphisms directly from the representation theory of B_3: for the second isomorphism we just observe that

$$\mathrm{M}^{\mathrm{ss}}_{B_3,J',(-t,t,1)}(1, 1, 0) = K(3; 1, 1) = K_{\mathrm{st}}(3; 1, 1) \simeq \mathrm{G}_1(3) = \mathbb{P}^2\,.$$

We can also check the isomorphism $\mathrm{M}^{\mathrm{ss}}_{B_3,J,(t+3,-2,-t+3)}(1, 3, 1) \simeq \mathbb{P}^2$ directly:[6] first note that the arrays

$$\theta_{\mathrm{G},v} = (-t, -1, t+3) \quad \text{and} \quad \tilde{\theta} = (-1, -1, 4)$$

are numerically equivalent (Def. 4.1.11) by looking at the walls in $(1, 3, 1)^\perp$ (see Figure 5.7). Then, by the symmetry $B_3 \simeq B_3^{\mathrm{op}}$ we also see that $\mathrm{M}^{\mathrm{ss}}_{B_3,J,(-1,-1,4)}(1, 3, 1) \simeq \mathrm{M}^{\mathrm{ss}}_{B_3,J,(-4,1,1)}(1, 3, 1)$. So we are interested in understanding $(-4, 1, 1)$-stability for representations

[6] I thank Markus Reineke for suggesting me this approach.

$$\mathbb{K} \overset{a_1,\,a_2,\,a_3}{\longrightarrow} \mathbb{K}^3 \overset{b_1,\,b_2,\,b_3}{\longrightarrow} \mathbb{K} \;.$$

We also write $a = (a_1, a_2, a_3), b^t = (b_1^t, b_2^t, b_3^t) \in \mathrm{M}_3(\mathbb{K})$. Such a representation (a, b) is $(-4, 1, 1)$-unstable if and only if it admits a subrepresentation of dimension $(1, 2, 1)$ or $(1, w_0, 0)$ for some $w_0 \in \{0, 1, 2, 3\}$, and this happens if and only if $\mathrm{rk}\, a \leq 2$ or there is a w_0-dimensional subspace $W_0 \subset \mathbb{K}^3$ such that $\mathrm{im}\, a \subset W_0 \subset \ker b$. Hence (note also that $(1, 3, 1)$ is $(-4, 1, 1)$-coprime) the $(-4, 1, 1)$-(semi)stable locus in $R := R_{(1,3,1)}(B_3) \cong \mathrm{M}_3(\mathbb{K})^{\oplus 2}$ is

$$R^{\mathrm{ss}} = R^{\mathrm{st}} = \{(a, b) \in \mathrm{M}_3(\mathbb{K})^{\oplus 2} \mid \mathrm{rk}\, a = 3 \text{ and } b \neq 0\} .$$

The map $R^{\mathrm{ss}} \to \mathbb{P}(\mathrm{M}_3(\mathbb{K}))$ sending a couple (a, b) to the class of the matrix $ba = (b_j a_i)_{i,j=1,2,3}$ is clearly $G_{(1,3,1)}$-invariant, surjective, and it separates orbits, since if two couples $(a, b), (a', b')$ are such that $ba = \lambda b' a'$ for some $\lambda \in \mathbb{K}^\times$, then $(ga\lambda^{-1}, bg^{-1}) = (a', b')$ for $g := \lambda a' a^{-1}$. Hence it descends to an isomorphism

$$\mathrm{M}^{\mathrm{ss}}_{B_3,(-4,1,1)}(1, 3, 1) = R^{\mathrm{ss}}/PG_{(1,3,1)} \to \mathbb{P}(\mathrm{M}_3(\mathbb{K})) \simeq \mathbb{P}^8 . \tag{5.3.4}$$

Finally, the relations J cut down the subvariety $X_{(1,3,1),J} = \{(a, b) \in R \mid a_i b_j + a_j b_i = 0\}$, thus the previous isomorphism restricts to

$$\mathrm{M}^{\mathrm{ss}}_{B_3,J,(-4,1,1)}(1, 3, 1) = (X_{(1,3,1),J} \cap R^{\mathrm{ss}})/PG_{(1,3,1)} \simeq \mathbb{P}(\mathrm{Ant}_3(\mathbb{K})) \simeq \mathbb{P}^2 ,$$

where $\mathrm{Ant}_3(\mathbb{K}) \subset \mathrm{M}_3(\mathbb{K})$ is the subspace of antisymmetric matrices.

6 For $(d'^v_{-1}, d'^v_0, d'^v_1) = (1, 3, 1)$ we have $\theta'_{\mathrm{G},v} = (-2t + 1, -2, 2t + 5)$, which is also equivalent to $\tilde{\theta} = (-1, -1, 4)$ (see Figure 5.7). Imposing the symmetric relations J' instead, the isomorphism (5.3.4) restricts to $\mathrm{M}^{\mathrm{ss}}_{B_3,J',(-4,1,1)}(1, 3, 1) \simeq \mathbb{P}(\mathrm{Sym}_3(\mathbb{K})) \simeq \mathbb{P}^5$, where $\mathrm{Sym}_3(\mathbb{K}) \subset \mathrm{M}_3(\mathbb{K})$ is the subspace of symmetric matrices. Hence

$$\mathrm{M}^{\mathrm{ss}}_{\mathbb{P}^2,H}(4, -5, 1) \simeq \mathrm{M}^{\mathrm{ss}}_{B_3,J',(-2t+1,-2,2t+5)}(1, 3, 1) \simeq \mathbb{P}^5 .$$

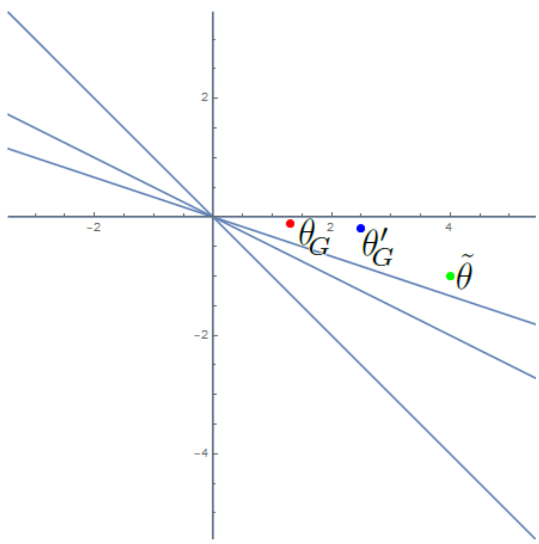


Figure 5.7: The plane $(1, 3, 1)^\perp$ in $K_0(B_3) \simeq \mathbb{Z}^3$, represented with respect to the basis $\{(-1, 0, 1), (-3, 1, 0)\}$. The lines are the numerical walls, while the dots are the points $\theta_{G,(1,3,1)} = (-1, 0, 1) + \epsilon(3, -2, 3)$, $\theta'_{G,(1,3,1)} = (-2, 0, 2) + \epsilon(1, -2, 5)$ and $\tilde{\theta} := (-1, -1, 4)$, for $\epsilon = 0.1$.

5.4 Application to $\mathbb{P}^1 \times \mathbb{P}^1$

Let Z be a 2-dimensional $\mathbb{K}$-vector space and set $X := \mathbb{P}_{\mathbb{K}}(Z) \times \mathbb{P}_{\mathbb{K}}(Z)$.

Take an ample divisor $A = aH + bF$ (for notation and formulae for invariants of sheaves, see 4.2.2.2) on X, meaning that $a \geq 1$ and $b \geq 1$.

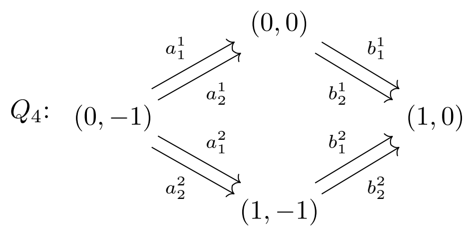
Consider the exceptional collections

$$\mathfrak{E} = (E_{(0,-1)}, E_{(0,0)}, E_{(1,-1)}, E_{(1,0)}) := (\mathcal{O}_X(0,-1)[-1], \mathcal{O}_X[-1], \mathcal{O}_X(1,-1), \mathcal{O}_X(1,0))\,,$$
$${}^{\vee}\mathfrak{E} = ({}^{\vee}E_{(1,0)}, {}^{\vee}E_{(1,-1)}, {}^{\vee}E_{(0,0)}, {}^{\vee}E_{(0,-1)}) = (\mathcal{O}_X(1,0), \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \tau_{\mathbb{P}^1}(-1), \tau_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}, \tau_{\mathbb{P}^1} \boxtimes \tau_{\mathbb{P}^1}(-1))$$

seen in Example 2.1.22.2 (note that the objects of ${}^{\vee}\mathfrak{E}$ are isomorphic to $\mathcal{O}_X(1,0)$, $\mathcal{O}_X(1,1)$, $\mathcal{O}_X(2,0)$, and $\mathcal{O}_X(2,1)$). We apply Theorem 2.2.5 to the full strong collection ${}^{\vee}\mathfrak{E}$. We have now the tilting bundle $T := \oplus_{i \in I} {}^{\vee}E_i$ (here $I = \{(0,-1),(0,0),(1,-1),(1,0)\}$) and its endomorphism algebra

$$\operatorname{End}_{\mathcal{O}_X}(T) = \begin{pmatrix} \mathbb{K} & & & \\ \mathbb{K} \otimes Z & \mathbb{K} & & \\ Z \otimes \mathbb{K} & 0 & \mathbb{K} & \\ Z \otimes Z & Z \otimes \mathbb{K} & \mathbb{K} \otimes Z & \mathbb{K} \end{pmatrix}.$$

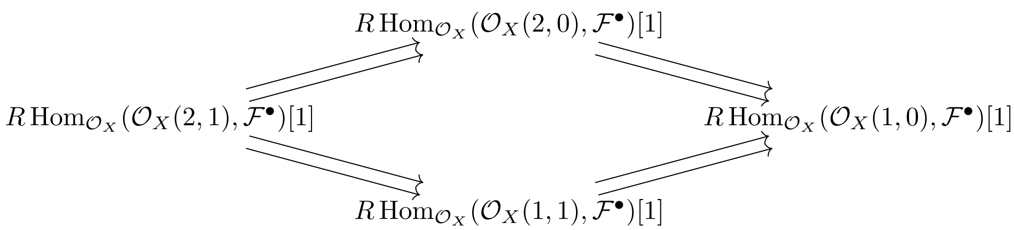
Choosing a basis $\{e_1, e_2\}$ of Z, $\operatorname{End}_{\mathcal{O}_X}(T)$ identifies with the opposite of the bound quiver algebra $\mathbb{K}Q_4/J$, where



and $J = (b_i^1 a_j^1 + b_j^2 a_i^2,\ i = 1,2)$. So we have again an equivalence

$$\Psi := \Phi_{{}^{\vee}\mathfrak{E}}[1] : D^b(X) \to D^b(Q_4; J)$$

which sends a complex $\mathcal{F}^\bullet \in D^b(X)$ to the complex of representations



and the standard heart in $D^b(Q_4; J)$ corresponds to the heart

$$\mathcal{K} := \langle \mathcal{O}_X(0,-1)[1], \mathcal{O}_X, \mathcal{O}_X(1,-1), \mathcal{O}_X(1,0)[-1] \rangle_{\text{ext}}\,, \tag{5.4.1}$$

whose objects are *Kronecker complexes*

$$K_V : V_{0,-1} \otimes \mathcal{O}_X(0,-1) \to V_{0,0} \otimes \mathcal{O}_X \oplus V_{1,-1} \otimes \mathcal{O}_X(1,-1) \to V_{1,0} \otimes \mathcal{O}_X(1,0)$$

with the middle bundle in degree 0 (again, we have used Lemma 4.3.4). Also in this case we see immediately that $\mathfrak{E}$ is always monad-friendly with respect to A (Def. 5.2.1).

Let $\psi : K_0(X) \to K_0(Q_4; J)$ be the isomorphism induced by the equivalence Ψ; we have coordinates on these Grothendieck groups given by the isomorphisms

$$K_0(X) \overset{(\mathrm{rk}, \deg_H, \deg_F, \chi)}{\longrightarrow} \mathbb{Z}^4\,, \quad K_0(Q_4; J) \overset{\dim}{\longrightarrow} \mathbb{Z}^4\,,$$

and as usual we write

$$(d^v_{0,-1}, d^v_{0,0}, d^v_{1,-1}, d^v_{1,0}) = d^v := \underline{\dim}\,\psi(v)$$

for the coordinates of $\psi(v) \in K_0(Q_4; J)$ with respect to the basis of simple representations $S(i)$, where $i \in I = \{(0,-1),(0,0),(1,-1),(1,0)\}$; these are mapped to the objects $\mathcal{O}_X(0,-1)[1]$, $\mathcal{O}_X$, $\mathcal{O}_X(1,-1)$, and $\mathcal{O}_X(1,0)[-1]$, so we find the transformations

$$\begin{pmatrix} d^v_{0,-1} \\ d^v_{0,0} \\ d^v_{1,-1} \\ d^v_{1,0} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 2 & 0 & 2 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \operatorname{rk} v \\ \deg_H v \\ \deg_F v \\ \chi(v) \end{pmatrix}, \quad \begin{pmatrix} \operatorname{rk} v \\ \deg_H v \\ \deg_F v \\ \chi(v) \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} d^v_{0,-1} \\ d^v_{0,0} \\ d^v_{1,-1} \\ d^v_{1,0} \end{pmatrix}.$$

The arrays $\theta_{\mathrm{M},v}, \theta_{\chi,v} \in \mathbb{Z}^I$ of equation (5.2.3) are given by

$$\theta_{\mathrm{M},v} = \begin{pmatrix} -\deg_A v - b \operatorname{rk} v \\ \deg_A v \\ \deg_A v - (a-b)\operatorname{rk} v \\ -\deg_A v + a \operatorname{rk} v \end{pmatrix}, \quad \theta_{\chi,v} = \begin{pmatrix} -\chi(v) \\ -\operatorname{rk} v + \chi(v) \\ \chi(v) \\ 2\operatorname{rk} v - \chi(v) \end{pmatrix}.$$

Lemma 5.4.1. $\mathcal{R}_A = \mathcal{S}^\circ_A = \{v \in K_0(X) \mid \operatorname{rk} v > 0,\ -b \operatorname{rk} v < \deg_A(v) < a \operatorname{rk} v\}$.

Proof. We have

$$\min_i \mu_A({}^\vee E_i) = \mu_A(\mathcal{O}_X(1,0)) = a\,, \quad \max_i \mu_A({}^\vee E_i \otimes \omega_X) = \mu_A(\mathcal{O}_X(2,1) \otimes \mathcal{O}_X(-2,-2)) = -b\,,$$

so that

$$\mathcal{R}_A = \{\operatorname{rk} v > 0\,,\ -b < \mu_A(v) < a\} = \{\operatorname{rk} v > 0\,,\ -b \operatorname{rk} v < \deg_A(v) < a \operatorname{rk} v\}\,.$$

Given $x = ([z_1],[z_2]) \in X$, take $p_1, p_2 \in Z^\vee$ vanishing on z_1 and z_2 respectively. The complex

$$K_x : \mathcal{O}_X(0,-1) \overset{\binom{p_1}{p_2}}{\to} \mathcal{O}_X \oplus \mathcal{O}_X(1,-1) \overset{(\,p_2\ \ -p_1\,)}{\to} \mathcal{O}_X(1,0)$$

is quasi-isomorphic to $\mathcal{O}_x[-1]$. The classes v in $\mathcal{S}^\circ_A$ are those for which $\operatorname{rk} v > 0$ and K_x is $\nu_{\mathrm{M},v}$-stable. Any nontrivial subcomplex of K_x has dimension vector d' equal to one between $(0,0,0,1), (0,0,1,1), (0,1,0,1), (0,1,1,1)$, and the inequalities $\theta_{\mathrm{M},v} \cdot d' > 0$ read

$$-\deg_A v + a \operatorname{rk} v > 0\,, \quad b \operatorname{rk} v > 0\,, \quad a \operatorname{rk} v > 0\,, \quad b \operatorname{rk} v + \deg_A v > 0\,,$$

which are precisely the inequalities defining $\mathcal{R}_A$. □

Twisting by line bundles we can bring any sheaf of positive rank inside this region $\mathcal{R}_A$. Hence the following Theorem describes again all moduli spaces of semistable sheaves of positive rank:

Theorem 5.4.2. *Let $v \in \mathcal{R}_A$. We have isomorphisms*

$$\mathrm{M}^{\mathrm{ss}}_{X,A}(v) \simeq \mathrm{M}^{\mathrm{ss}}_{Q_4,J,\theta_{\mathrm{G},v}}(d^v) \quad \text{and} \quad \mathrm{M}^{\mathrm{st}}_{X,A}(v) \simeq \mathrm{M}^{\mathrm{st}}_{Q_4,J,\theta_{\mathrm{G},v}}(d^v)\,.$$

Remarks 5.4.3. Like after Theorem 5.3.3, we have some immediate remarks:

1 If for $v \in \mathcal{R}_A$ the dimension vector d^v is $\theta_{\mathrm{G},v}$-coprime, then $\gcd(\operatorname{rk} v, \deg_A v, \chi(v)) = 1$. In this case $\mathrm{M}^{\mathrm{ss}}_{X,A}(v) = \mathrm{M}^{\mathrm{st}}_{X,A}(v)$ and there is a universal family (by Remarks 4.2.10.2 or 4.1.15.2).

2 As we observed for $\mathbb{P}^2$, also in this case it follows from Cor. 4.3.6 that $\mathrm{M}^{\mathrm{st}}_{X,A}(v) \simeq \mathrm{M}^{\mathrm{st}}_{Q_4,J,\theta_{\mathrm{G},v}}(d^v)$ is smooth and its dimension is $\dim \mathrm{M}^{\mathrm{st}}_{Q_4,\theta_{\mathrm{G},v}}(d^v)$ minus the number $4d^v_{0,-1}d^v_{1,0}$ of relations imposed, which gives

$$\begin{aligned} \dim \mathrm{M}^{\mathrm{st}}_{X,A}(v) &= 2(d^v_{0,0} + d^v_{1,-1})(d^v_{0,-1} + d^v_{1,0}) - \textstyle\sum_{i\in I}(d^v_i)^2 + 1 - 4d^v_{0,-1}d^v_{1,0} \\ &= 1 - \operatorname{rk} v^2 + \Delta(v)\,, \end{aligned}$$

in agreement with Eq. (4.2.11).

3 For all $v \in \mathcal{R}_A$ we have $\theta^{(0,-1)}_{\mathrm{M},v} = -b \operatorname{rk} v - \deg_A v < 0$ and $\theta^{(1,0)}_{\mathrm{M},v} = a \operatorname{rk} v - \deg_A v > 0$.

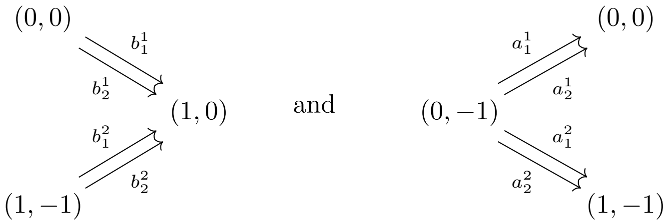
4 Notice that in this case there may be semistable Kronecker complexes in classes $w \in K_0(X)$ such that $P_{w,A} = 0$, so we do not have an analogue of Theorem 5.3.6.

5.4.1 Examples

Examples 5.4.4. We use the notation $\mathrm{M}^{\mathrm{ss/st}}_{X,A}(\operatorname{rk} v, \deg_H v, \deg_F v, \chi(v)) := \mathrm{M}^{\mathrm{ss/st}}_{X,A}(v)$.

1. Let r be a positive integer. Taking $d^v = (0, r, 0, 0)$ we get $\mathrm{M}^{\mathrm{ss}}_{X,A}(r, 0, 0, r) = \{\mathcal{O}_X^{\oplus r}\}$, while for $d^v = (0, 0, r, 0)$ we find $\mathrm{M}^{\mathrm{ss}}_{X,A}(r, r, -r, 0) = \{\mathcal{O}_X(1, -1)^{\oplus r}\}$.

2. If we choose $v \in K_0(X)$ with at least one between $d^v_{0,-1}$ and $d^v_{1,0}$ vanishing, then the representations we are considering reduce to representations of the quivers



respectively, and the relations J are trivially satisfied. These are the cases considered in [Kul97].

3. Let ℓ be a positive integer. The choice $(\operatorname{rk} v, \deg_H v, \deg_F v, \chi(v)) = (1, 0, 0, 1 - \ell)$ gives the Hilbert scheme of points and corresponds to the dimension vector $d^v = (\ell, \ell + 1, \ell, \ell)$, so

$$\operatorname{Hilb}^\ell(X) = \mathrm{M}^{\mathrm{ss/st}}_{X,A}(v) \simeq \mathrm{M}^{\mathrm{ss/st}}_{Q_4,J,\theta_{\mathrm{G},v}}(\ell, \ell + 1, \ell, \ell)\,,$$

where $\theta_{\mathrm{G},v} = (-tb + (\ell - 1), -\ell, t(b - a) + (1 - \ell), ta + (\ell + 1))$.
In particular, taking $\ell = 1$, we find that

$$\mathrm{M}^{\mathrm{ss}}_{Q_4,J,\theta_{\mathrm{G},v}}(1, 2, 1, 1) = \mathrm{M}^{\mathrm{st}}_{Q_4,J,\theta_{\mathrm{G},v}}(1, 2, 1, 1) \simeq \mathbb{P}^1 \times \mathbb{P}^1\,.$$

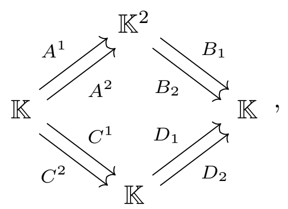
It is instructive to prove this isomorphism directly, for example with the choice $a = b = 1$: first, the arrays

$$\theta_{\mathrm{G},v} = (-t, -1, 0, t + 2) \quad \text{and} \quad \tilde{\theta} = (-1, -1, 0, 3)$$

are equivalent by looking at the walls in $(1, 2, 1, 1)^\perp$ (see Figure 5.8).[7] Then by the symmetry $Q_4 \simeq Q_4^{\mathrm{op}}$ we also see that

$$\mathrm{M}^{\mathrm{ss}}_{Q_4,J,(-1,-1,0,3)}(1, 2, 1, 1) \simeq \mathrm{M}^{\mathrm{ss}}_{Q_4,J,(-3,1,0,1)}(1, 2, 1, 1)\,.$$

So we are interested in understanding $(-3, 1, 0, 1)$-stability for representations



where

$$A = (\, A^1 \mid A^2 \,) \in \mathrm{M}_{2\times 2}(\mathbb{K})\,, \quad B = \left(\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right) \in \mathrm{M}_{2\times 2}(\mathbb{K})\,,$$
$$C = (\, C^1 \mid C^2 \,) \in \mathrm{M}_{1\times 2}(\mathbb{K})\,, \quad D = \left(\begin{smallmatrix} D_1 \\ D_2 \end{smallmatrix}\right) \in \mathrm{M}_{2\times 1}(\mathbb{K})\,.$$

Note that the relations J read $BA + DC = 0$. By looking at all the possible subrepresentations, one can easily check that a representation (A, B, C, D) of Q_4 is $(-3, 1, 0, 1)$-semistable if and only if $\det A \neq 0$ and $(B, C) \neq (0, 0)$ and $(B, D) \neq (0, 0)$, and it is stable if and only

[7] Observe that $\theta_{\mathrm{G},v} = (-t, -1, 0, t + 2)$ and $\tilde{\theta}$ are not in a chamber, but they lie on the wall $W(0, 0, 1, 0) = W(1, 2, 0, 1)$. However, we will see that the relations rule out strictly semistable objects. Moreover, we could have equivalently chosen the polarization $(a, b) = (2, 1)$, and in this case $\theta_{\mathrm{G},v}$ would have been in a chamber.

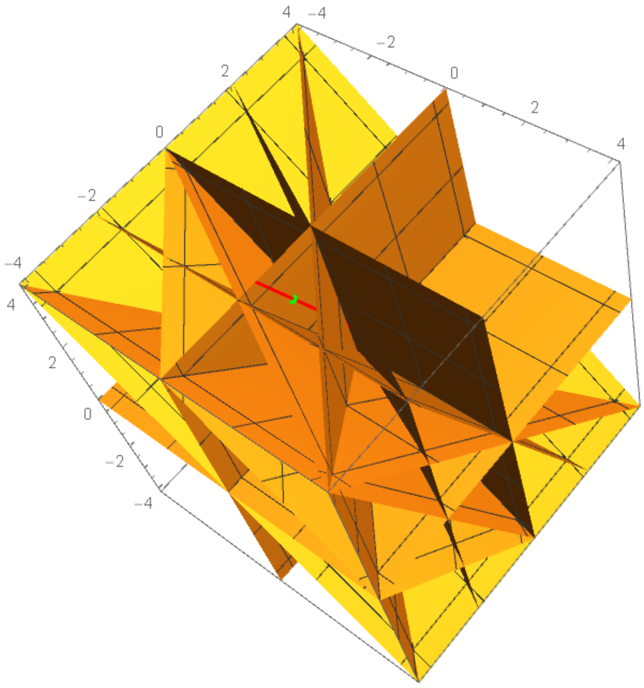


Figure 5.8: The hyperplane $(1,2,1,1)^\perp$ in $K_0(Q_4)\otimes_{\mathbb{Z}}\mathbb{R}\simeq\mathbb{R}^4$, represented with respect to the basis $\{(-1,0,0,1),(-1,0,1,0),(-2,1,0,0)\}$. The planes are the numerical walls, the ray $\theta_{\mathrm{M},(1,2,1,1)}+\epsilon\theta_{\chi,(1,2,1,1)}$ is displayed in red, while the green dot is the point $\tilde{\theta}:=(-1,-1,0,3)$.

if, in addition, we have $C\neq 0$ and $D\neq 0$. But if the representation is subject to the relation $BA=-DC$ we see that $C=0$ or $D=0$ imply $B=0$, so there are no strictly semistable representations of (Q_4,J) and

$$\begin{aligned} R^{\mathrm{st}}_{(1,2,1,1),(-3,1,0,1)}\cap X_{(1,2,1,1),J} &= R^{\mathrm{ss}}_{(1,2,1,1),(-3,1,0,1)}\cap X_{(1,2,1,1),J} \\ &= \{(A,B,C,D)\mid \det A\neq 0,\ C\neq 0,\ D\neq 0,\ BA=-DC\}\,. \end{aligned}$$

So we have a well-defined map

$$R^{\mathrm{st}}_{(1,2,1,1),(-3,1,0,1)}\cap X_{(1,2,1,1),J}\to\mathbb{P}^1\times\mathbb{P}^1$$

sending (A,B,C,D) to $([C],[D])$. This map is clearly $G_{(1,2,1,1)}$-invariant; it is surjective because any $([C],[D])$ is the image of $(\mathbb{I}_2,-DC,C,D)$, and it separates orbits since if (A,B,C,D) and (A',B',C',D') have the same image, then $C=\lambda_1 C'$, $\lambda_2 D=D'$ for some $\lambda_1,\lambda_2\in\mathbb{K}^\times$, and we see that

$$(\lambda_1,g,1,\lambda_2)\cdot(A,B,C,D)=(gA\lambda_1^{-1},\lambda_2 Bg^{-1},C\lambda_1^{-1},\lambda_2 D)=(A',B',C',D')\,,$$

where $g:=\lambda_1 A'A^{-1}$. So we conclude that the map descends to an isomorphism

$$\mathrm{M}^{\mathrm{ss}}_{Q_4,J,(-3,1,0,1)}(1,2,1,1)\to\mathbb{P}^1\times\mathbb{P}^1\,.$$

5.5 Other applications and related projects

We conclude by briefly describing some research projects based on the content of this chapter.

5.5.1 Extension to other rational surfaces

It is natural to ask whether the results of §5.2 can be applied to other surfaces than $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$. As mentioned in 2.1.3, all Del Pezzo and Hirzebruch surfaces admit full strong exceptional collections of vector bundles.

On Del Pezzo surfaces all the exceptional objects are shifts of torsion sheaves or slope-stable (at least with respect to the anticanonical polarization) vector bundles, and one should be able to construct some monad-friendly collection (Def. 5.2.1) by using e.g. the results of [KN98] on three-blocks collections. On Hirzebruch surfaces, on the other hand, this might be problematic as it is possible to have exceptional objects which are complexes with nontrivial cohomology in more than one degree, and these likely arise when one tries to compute the dual collection of a given one.

In any case, it should be possible to extend all the arguments of §5.2, using a slightly different approach, under the only hypothesis of having a full strong exceptional sequence made of (shifts of) semistable vector bundles: dropping assumption (A2) in Def. 5.2.1 we lose the concept of Kronecker complex, and we cannot use most of the theory of monads anymore, but the proof of the main theorem could be adapted to this situation by working directly with quiver representations instead of Kronecker complexes.

As a further generalization, it would be interesting to understand if it is possible to adapt these methods to show existence and projectivity of some moduli spaces of Bridgeland-semistable objects. It has been shown in [Ohk10, AM17] that, for $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown-up in one point, semistable objects with respect to any geometric stability condition (see §1.2.3) are parameterized by projective moduli spaces. However, the techniques used do not seem to work on other rational surfaces, for which the existence of moduli spaces of Bridgeland-semistable objects is still unknown.

5.5.2 Invariants of moduli spaces

Having observed that moduli spaces of sheaves on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ have a simpler description as quiver moduli spaces, we could try to use this fact to investigate further their geometric properties. For example, Ellingsrud and Strømme described in [ES93] the Chow rings of the moduli spaces on $\mathbb{P}^2$ using the construction of [DLP85] (i.e. our Theorem 5.3.9), and one expects that the same can be done starting from the other isomorphisms obtained in the last two sections. Indeed, we can apply to our cases the results of [KW95] extending the technique of [ES93] to certain moduli spaces of representations of algebras, to get some information on the Chow rings: doing this for $\mathbb{P}^2$ gives back the result of [ES93], while for $\mathbb{P}^1 \times \mathbb{P}^1$ we can now use Theorem 5.4.2 to get:

Theorem 5.5.1. *Take an ample divisor A on $\mathbb{P}^1 \times \mathbb{P}^1$ and a class $v \in K_0(\mathbb{P}^1 \times \mathbb{P}^1)$ such that* $\operatorname{rk} v > 0$ *and* $\gcd(\operatorname{rk} v, \deg_A v, \chi(v)) = 1$. *Then* $\mathrm{M} := \mathrm{M}^{\mathrm{st}}_{\mathbb{P}^1\times\mathbb{P}^1,A}(v)$ *is smooth and projective, and its Chow ring* $A^*(\mathrm{M})$ *has the following properties:*

1. *as an abelian group, $A^*(\mathrm{M})$ is free, and numerical and algebraic equivalence coincide;*
2. *for $\mathbb{K} = \mathbb{C}$, the cycle map $A^*(\mathrm{M}) \to H^{2*}(\mathrm{M}, \mathbb{Z})$ is an isomorphism; in particular, M has no odd cohomology and no torsion in even cohomology;*
3. *suppose that $v \in \mathcal{R}_A$ (see Lemma (5.4.1)); then, as a $\mathbb{Z}$-algebra, $A^*(\mathrm{M})$ is generated by the Chern classes of the bundles*
$$R^1(\mathrm{pr}_\mathrm{M})_*\mathcal{U}(-2,-1)\,,\ R^1(\mathrm{pr}_\mathrm{M})_*\mathcal{U}(-2,0)\,,\ R^1(\mathrm{pr}_\mathrm{M})_*\mathcal{U}(-1,-1)\,,\ R^1(\mathrm{pr}_\mathrm{M})_*\mathcal{U}(-1,0)\,,$$
where $\mathcal{U}$ is the universal sheaf on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{M}$.

Another interesting question is if we can determine more precisely the homology of these moduli spaces, that is compute the Betti numbers. For example, for $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2}(v)$ they have been computed so far for low ranks in [ES87, Yos94, Man11, Man17] (always in the coprime case, i.e. when this is a smooth projective variety) via toric localization and wall-crossing. The two descriptions $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2}(v)$ of §5.3 suggest other approaches to this problem: one is again via toric localization, but using the natural torus actions on quiver moduli spaces, as done e.g. in [Wei13] for studying Euler characteristics of Kronecker quivers. Another approach is via a Harder-Narasimhan recursion applied to the moduli spaces of representations of the Beilinson quiver B_3 with relations J or J': as explained in Ex. 4.1.15.3, every (stacky) motivic invariant (such as Betti numbers and Hodge numbers) can be computed on $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2}(v)$ if we know how to compute it for (the projectivizations of) the subvarieties $X_{d,J}$ and $X_{d,J}$ in $R_d(B_3)$ cut by the above relations, for certain d. Unfortunately, a systematic computation of these numbers via topological or Hodge-theoretic methods seems to be out of reach.[8] Probably it is more feasible to count their points over finite fields to get from the motivic formula of Ex. 4.1.15.3 the counting polynomials of $\mathrm{M}^{\mathrm{st}}_{\mathbb{P}^2}(v)$.

5.5.3 Moduli spaces of framed sheaves

Another approach to the construction of moduli spaces of coherent sheaves (or vector bundles) consists, instead of restricting to Gieseker-semistable sheaves, of putting additional structures to rigidify them (i.e. remove their automorphism, which are the main obstruction to the existence of moduli spaces) and studying the classification problem for the pairs (sheaf, extra structure). A relevant situation in which this idea is applied is the study of sheaves $\mathcal{E}$ on a variety X together with a fixed trivialization φ of the restriction $\mathcal{E}\restriction_D$ to a subvariety $D \subset X$ of codimension 1; the pair $(\mathcal{E}, \varphi)$ is called a *framed sheaf*. Framed sheaves are indeed shown in [HL95, BM11] to admit well-behaved moduli spaces. The importance of framed sheaves lies also in their relations to gauge theory: their moduli spaces provide nonsingular partial compactifications of moduli spaces of instantons (i.e. anti-self dual connections), as discussed notably in [Don84]. This is why moduli spaces of framed sheaves have been used by theoretical physicists for the computation of partition functions of supersymmetric field theories via localization techniques [Nek03, BPT11].

Moduli spaces of framed sheaves on a surface X can also be constructed in many cases via linear data, that is as quotients of some space of linear maps. This kind of approach goes back to the work [Don84] of Donaldson, who worked out the case of sheaves on $X = \mathbb{P}^2$ framed on a line (this also generalizes the well-known description of the Hilbert schemes of points on $\mathbb{C}^2$ via linear data). Similar descriptions have been carried out e.g. for sheaves on the blown-up $\mathbb{P}^2$ framed on a general line [Kin89] and for sheaves on Hirzebruch surfaces Σ_e with $e \neq 0$, framed on the relative hyperplane H [BBR15]. The case $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is not included, the reason being that, unlike for $e \neq 0$, the results of [BM11] do not apply to sheaves framed on H, and indeed a simple argument shows that a moduli space for such sheaves cannot exist. However, this problem does not arise if one considers instead sheaves framed on the cross $F \cup H$ (as this case reduces to that of [Don84]), or on the diagonal line $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$, for which some of the techniques of §5.4 can be used. The upshot is that sheaves framed on Δ can be shown to belong to the heart $\mathcal{K} \subset D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ of Eq. (5.4.1), which means that they can be written as middle cohomologies of monads

$$V_{0,-1} \otimes \mathcal{O}_X(0,-1) \to V_{0,0} \otimes \mathcal{O}_X \oplus V_{1,-1} \otimes \mathcal{O}_X(1,-1) \to V_{1,0} \otimes \mathcal{O}_X(1,0)\,,$$

and this can be used to build the moduli spaces via linear data based on representations of the quiver Q_4.

[8] While we have seen (based on Cor. 4.3.6) that the intersections of these varieties with the semistable loci $R^{\mathrm{ss}}_{d,\theta} \subset R_d$ are smooth complete intersections, the computations require taking into account the unstable loci as well, which may be very far from being this regular.

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