Estimation of the Present Values of Life Annuities for the Different Actuarial Models

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Abstract—The paper deals with the problem of estimating the actuarial present value of the continuous whole life and n-year term life annuities. We synthesize nonparametric estimators of these statuses of life annuity. The main parts of their asymptotic mean square errors for these estimators and their limit distributions are found. By individuals' death moments, both parametric and nonparametric estimates are constructed for the models of the whole and n-year term life insurance. The asymptotic normality and mean square convergence of the proposed estimators are proved. The simulations show that the empirical mean square errors of life annuity estimates decrease when the sample size increases. Also, when the model distribution is changed, the nonparametric estimates are more adaptable in comparison with parametric estimates, oriented on the best results only for the given distributions.

Keywords—nonparametric estimation; life insurance; life annuity; n-year life annuities; asymptotic normality; bias; mean square error

I. Introduction

The essence of life annuity in accordance with [1, p. 170] is that, from the moment $t=0$ an individual once a year begins to get a certain money, which we take as the unit of money, and payments are made only for the lifetime of an individual. It is known that the calculation of the characteristics of life annuity is based on the characteristics of the respective type of insurance. Thus, the average total cost of the present continuous annuity is given as (see [1, p.184])

$$\bar{a}_x(\delta) = \frac{1-\bar{A}_x}{\delta},$$

where $\bar{A}_x$ is a net premium (the average value of the present value of a single sum of money in the insurance lifetime at the age x), δ is a force of interest. Let x be an individual age when the payments begin, X be the duration of his life, $T_x = X - x$ be the future lifetime. Introduce the random variable

$$z(x) = \frac{1-e^{-\delta T_x}}{\delta},\ T_x > 0. \tag{1}$$

Then, taking the expectation of the random variable $z(x)$ (1), the life annuity is determined by the formula (see [2-6])

$$\bar{a}_x(\delta) = M(z(x)) = \frac{1}{\delta}\left(1-\frac{\Phi(x,\delta)}{S(x)}\right), \tag{2}$$

where M is the symbol of the expectation, $S(x)=P(X>x)$ is a survival function,

$$\Phi(x,\delta) = -e^{\delta x}\int_x^{\infty} e^{-\delta t}\,dS(t).$$

In practice, often the n-year life annuity [1, p. 185] is used. Here, from the moment $t=0$ an individual yearly begins to get money, which we take as the unit sums of money, and payments are made only during n years when he is alive. It is known that the continuous n-year term annuity is defined by the net premium. Thus, the average current value of the continuous n-year annuity (see: [1, p.185]) is equal to

$$\bar{a}_{x:\overline{n}|}(\delta) = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta},$$

where $\bar{A}_{x:\overline{n}|}$ is the net premium (the average value of the present value of a unit sum of insured in the n-year lifetime insurance at the age x).

Define for this case the random variable

$$z(x,n) = \frac{1-e^{-\delta \cdot T_x}}{\delta},\ T_x < n. \tag{3}$$

Taking the expectation of the random variable $z(x,n)$ (3), get the actuarial present value for the n-year annuity:

$$\bar{a}_{x:\overline{n}|}(\delta) = M(z(x,n)) = \frac{1}{\delta}\left(1-\frac{\Phi(x,\delta,n)}{S(x)}\right), \tag{4}$$

$$\Phi(x,\delta,n) = -e^{\delta x}\int_x^{n} e^{-\delta t}\,dS(t).$$

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II. ESTIMATION OF ANNUITIES

Suppose we have a random sample $X_1 ... X_N$ of N lifetimes of individuals. Estimate separately the numerator and denominator in (2) and (4). Substituting the unknown function $S(x)$ by its non-parametric estimator

$$S_N(x) = \frac{1}{N}\sum_{i=1}^{N} \mathrm{I}(X_i > x),$$

where $I(A)$ is the indicator of an event A, we obtain the following estimators of the whole life and n-year term life annuities:

$$\bar{a}_x^N(\delta) = \frac{1}{\delta}\left(1 - \frac{e^{\delta x}}{S_N(x)\cdot N}\sum_{i=1}^{N}\exp(-\delta X_i)\mathrm{I}(X_i > x)\right) = \frac{1}{\delta}\left(1 - \frac{\Phi_N(x,\delta)}{S_N(x)}\right), \quad (5)$$

$$\bar{a}_{x:\overline{n}|}^N(\delta) = \frac{1}{\delta}\left(1 - \frac{e^{\delta x}}{S_N(x)\cdot N}\sum_{i=1}^{N}\exp(-\delta X_i)\mathrm{I}(x < X_i < n)\right) = \frac{1}{\delta}\left(1 - \frac{\Phi_N(x,\delta,n)}{S_N(x)}\right). \quad (6)$$

Here,

$$\Phi_N(x,\delta) = \frac{e^{\delta x}}{N}\sum_{i=1}^{N}\exp(-\delta X_i)\mathrm{I}(X_i > x),$$

$$\Phi_N(x,\delta,n) = \frac{e^{\delta x}}{N}\sum_{i=1}^{N}\exp(-\delta X_i)\mathrm{I}(x < X_i < n).$$

III. MEAN SQUARE ERRORS

Prove results on the properties of the estimators (5) and (6). Give only the properties of the estimators (5) as the results for (6) can be got analogously.

First, we find the principal part of the asymptotic mean square error (MSE) and the convergence rates of the bias (5). We need Theorem 1 from [7], (here it is Lemma).

Introduce the notation according to [7]: $t_N = (t_{1N}, t_{2N}, ..., t_{sN})^{\mathrm{T}}$ is an s-dimensional vector with components $t_{jN} = t_{jN}(x) = t_{jN}(x; X_1, ..., X_N),\ j = \overline{1,s},\ x \in R^{\alpha}$, R^{α} is the α-dimensional Euclidean space; $H(t): R^s \to R^1$ is a function, where $t = t(x) = (t_1(x), ..., t_s(x))^{\mathrm{T}}$ is an s-dimensional bounded vector function; $N_s(\mu,\sigma)$ is the s-dimensional normally distributed random variable with a mean vector $\mu = \mu(x) = (\mu_1, ... \mu_s)^T$ and covariance matrix $\sigma = \sigma(x)$;

$$\nabla H(t) = (H_1(t), ..., H_s(t))^{\mathrm{T}},\ H_j(t) = \left.\frac{\partial H(z)}{\partial z_j}\right|_{z=t},\ j = \overline{1,s};$$

$\Rightarrow$ is the symbol of convergence in distribution (weak convergence); $\|x\|$ is the Euclidean norm of a vector x.

Lemma. Let

1) the function $H(t)$ be twice differentiable, and $\nabla H(t) \neq 0$;
2) $M\|t_N - t\|^i = \mathrm{O}(d_N^{-i/2}),\ i = 1,2,...$

Then $\forall k = 1,2,...$

$$\left| M\left[H(t_N) - H(t)\right]^k - M\left[\nabla H(t)\cdot(t_N - t)\right]^k \right| = \mathrm{o}\left(d_N^{-(k+1)/2}\right). \quad (7)$$

Note, if in the formula (7) k=1, we obtain the main part of the bias for $H(t_n)$, and at k=2, we have the main part of the MSE.

Theorem 1. If $S(x)>0$ and $S(t)$ is continuous at x, then

1) for the bias of (5), the following relation holds:

$$\left| b(\bar{a}_x^N(\delta)) \right| = \mathrm{o}(N^{-1});$$

2) the MSE of (5) is given by the formula

$$u^2(\bar{a}_x^N(\delta)) = M(\bar{a}_x^N(\delta) - \bar{a}_x(\delta))^2 = \frac{\sigma(\bar{a}_x(\delta))}{N} + \mathrm{o}\left(N^{-3/2}\right),$$

where $\sigma(\bar{a}_x(\delta))$ is defined below by the formula (8).

Proof. For the estimator $\bar{a}_x^N(\delta)$ (5) in the notation of Lemma, we have:

$$t_N = (\Phi_N(x,\delta), S_N(x))^{\mathrm{T}};\ d_N = N;\ t = (\Phi(x,\delta), S(x))^{\mathrm{T}};$$

$$H(t) = \frac{1}{\delta}\left(1 - \frac{\Phi(x,\delta)}{S(x)}\right) = \bar{a}_x;\ H(t_N) = \frac{1}{\delta}\left(1 - \frac{\Phi_N(x,\delta)}{S_N(x)}\right) = \bar{a}_x^N;$$

$$\nabla H(t) = \left(H_1(t), H_2(t)\right)^T = \left(\frac{1}{\delta}\frac{1}{S(x)}, -\frac{1}{\delta}\frac{\Phi(x,\delta)}{S^2(x)}\right)^T \neq 0.$$

We know that $S_N(x)$ is an unbiased and consistent estimator of $S(x)$. Show that $\Phi_N(x,\delta)$ is an unbiased estimator of the functional $\Phi(x,\delta)$:

$$M\Phi_N(x,\delta) = \frac{e^{\delta x}}{N} M\left\{\sum_{i=1}^{N}\exp(-\delta X_i)\mathrm{I}(X_i > x)\right\} = \Phi(x,\delta).$$

Now, calculate the variance of $\Phi_N(x,\delta)$:

$$D\Phi_N(x,\delta) = D\left\{\frac{e^{\delta x}}{N}\sum_{i=1}^{N} \mathrm{I}(X_i > x)e^{-\delta X_i}\right\} = \frac{e^{2\delta x}}{N^2}\sum_{i=1}^{N} D\left\{\mathrm{I}(X_i > x)e^{-\delta X_i}\right\}$$
$$= \frac{1}{N}\left(\Phi(x,2\delta) - \Phi^2(x,\delta)\right)$$

The ratio of two unbiased estimators can have a bias. We find the order of the bias with making use of the results from [7]. In view of $M(t_N - t) = 0$, we obtain

$$\left|M(\bar{a}_x^N(\delta) - \bar{a}_x(\delta)) - M[\nabla H(t)(t_N - t)]\right| = \left|M(\bar{a}_x^N(\delta) - \bar{a}_x(\delta))\right| = \mathrm{o}(N^{-1}).$$

Find the components of the covariance matrix

$$\sigma(\bar{a}_x(\delta)) = \begin{bmatrix} \sigma_{11}\sigma_{12} \\ \sigma_{21}\sigma_{22} \end{bmatrix}$$

for the statistics t_N :

$$\sigma_{11} = ND\{\Phi_N(x,\delta)\} = \Phi(x,2\delta) - \Phi^2(x,\delta);$$
$$\sigma_{22} = ND\{S_N(x)\} = S(X)(1 - S(x));$$
$$\sigma_{12} = \sigma_{21} = N\operatorname{cov}(S_N(x), \Phi_N(x,\delta))$$
$$= N(M\{S_N(x)\Phi_N(x,\delta)\})$$
$$-M\{S_N(x)\}M\{\Phi_N(x,\delta)\} = (1 - S(x))\Phi(x,\delta).$$

Using the previous result on the bias and the covariance matrix, we obtain

$$u^2(\bar{a}_x^N(\delta)) = M[\nabla H(t)(t_N - t)]^2 + \mathrm{O}(N^{-3/2}) = \frac{\sigma(\bar{a}_x(\delta))}{N} + \mathrm{O}(N^{-3/2}),$$

where

$$\sigma(\bar{a}_x(\delta)) = \sum_{p=1}^{2}\sum_{j=1}^{2} H_j(t)\sigma_{jp}H_p(t) = H_1^2(t)\sigma_{11} + H_2^2(t)\sigma_{22} +$$
$$+2H_1(t)H_2(t)\sigma_{12} = \frac{1}{\delta^2}\left(\frac{\Phi(2\delta)}{S^2(x)} - \frac{3\Phi^2(x,\delta)}{S^3(x)} + \frac{2\Phi^2(x,\delta)}{S^2(x)}\right). \quad (8)$$

Theorem 1 is proved.

IV. Asymptotic Normality

To find the limit distribution of (3), we need the following two theorems.

Theorem 2 (Central Limit Theorem in the multidimensional case) [3, p. 178-202]. If $t_1, t_2, \ldots, t_N, \ldots$ is a sequence of independent and identically distributed s-dimensional vectors,

$$M\{t_s\} = 0, \sigma(x) = M\{t_s^{\mathrm{T}} t_s\}, \quad S_N = \sum_{s=1}^{N} t_s,$$

then as $N \to \infty$

$$\frac{S_N}{\sqrt{N}} \Rightarrow N_s(0, \sigma(x)).$$

Theorem 3 (asymptotic normality of $H(t_N)$) [7-13]. Let

1. $\sqrt{N} \cdot t_N \Rightarrow N_s\{\mu, \sigma(x)\}$;
2. function *H(z)* be differentiable in the point μ, $\nabla H(\mu) \neq 0$.

Then

$$\sqrt{N}(H(t_N) - H(\mu)) \Rightarrow N_1\left\{\sum_{j=1}^{s} H_j(\mu)\mu_j, \sum_{p=1}^{s}\sum_{j=1}^{s} H_j(\mu)\sigma_{jp}H_p(\mu)\right\}.$$

Theorem 4. Under the conditions of Theorem 1

$$\sqrt{N}(\bar{a}_x^N(\delta) - \bar{a}_x(\delta)) \Rightarrow N_1(0, \sigma(\bar{a}_x(\delta))). \quad (9)$$

Proof. In the notation of Theorem 2, we have $s = 2, \sigma(x) = \sigma(\bar{a}_x(\delta))$, and

$$\sqrt{N}\{(\Phi_N(x,\delta), S_N(x)) - (\Phi(x,\delta), S(x))\} \Rightarrow N_2\left(0, \sigma(\bar{a}_x(\delta))\right).$$

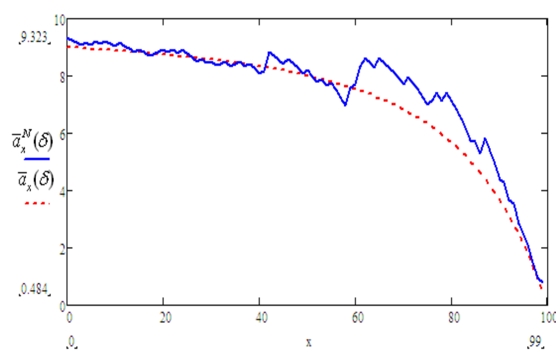
The function *H(z)* is differentiable at the point $t = (\Phi(x,\delta), S(x))$ and $\nabla H(t) \neq 0$. Consequently, all the conditions of Theorem 3 hold. Theorem 4 is proved.

IV. Simulations

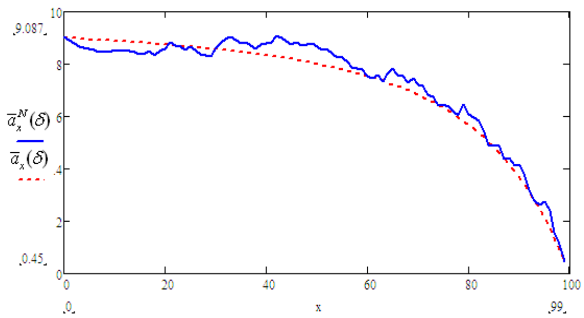
Consider the model of de Moivre, for which the lifetime of the individual X is uniformly distributed in the interval (0,100), where ω is the limit age. The whole life annuity, in accordance with (1), takes the form:

$$\bar{a}_x(\delta) = \frac{\delta(\omega - x) - 1 + e^{-\delta(\omega - x)}}{\delta^2(\omega - x)}.$$

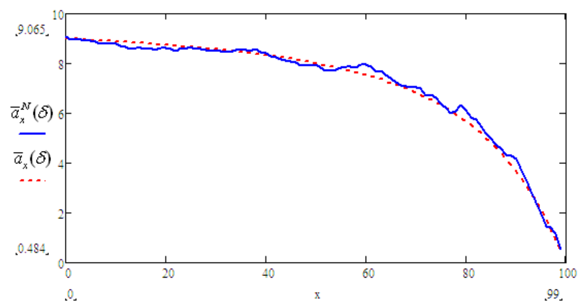
The annuities and their estimates are presented in Fig. 1 for the random samples $X_1, \ldots, X_N$ of the sizes N = *50,100,500*, uniformly distributed on the interval (0, 100). Let the force of interest δ =0,09531(9,531%). Note that for this δ the effective annual interest rate $i = e^{\delta} - 1 = 0{,}1(10\%)$.



a)



b)



c)

Figure 1. Dependence of the whole annuity and its estimate on the age by the sample sizes N: a) 50; b) 100; c) 500.

We will characterize the quality of the estimates by the empirical MSE:

$$G(N,\delta)=\frac{\sum_{x=1}^{100}(\bar{a}_x(\delta)-\bar{a}_x^N(\delta))^2}{N}.$$

The calculation results are given in Table I.

TABLE I. EMPIRICAL MSEs FOR DIFFERENT SAMPLE SIZES

N	25	50	100	250	500
$G(N,\delta)$	1.048	0.524	0.262	0.105	0.052

The present values of the whole life annuities for persons of the different ages x, δ=0,09531(9,531%), and the monthly payment of 1000 rubles, are presented in Table II.

TABLE II. PRESENT VALUES OF ANNUITIES FOR DIFFERENT AGES

x	35	40	45	50	55
$12000\cdot\bar{a}_x(\delta)$	105623	103960	102014	99710	96952

Similarly, for the n-year term annuities we have

$$G(N,n,\delta)=\frac{\sum_{x=1}^{\omega-n}(\bar{a}_{x:\overline{n}|}^N(\delta)-\bar{a}_{x:\overline{n}|}(\delta))^2}{N},\ N=50,100,500.$$

The calculation results are given in Table III.

TABLE III. EMPIRICAL MSEs FOR DIFFERENT SAMPLE SIZES

N	50	100	250	500
$G(N,n,\delta)$	0.790871	0.632723	0.2319	0.051219

It is seen from Tables I and III, the quality of estimation is improving with increasing sample size.

The present values of the 5-year term life annuities for persons of the different ages x, δ=0,09531(9,531%), and the monthly payment of 1000 rubles, are presented in Table IV.

TABLE IV. PRESENT VALUES OF 5-YEAR ANNUITIES FOR DIFFERENT AGES

X	35	40	45	50	55
$12000\cdot\bar{a}_{x:\overline{5}|}(\delta$	105718	104036	102048	99662	96747

If n=10, the present value of the 10-year annuity for a person at age x = 45 years, δ=0.09531(9.531%), and the monthly payment of 1000 rubles, is equal to

$$12000\cdot\bar{a}_{45:\overline{10}|}(0.09531)=12000\cdot 8.079335=96952.$$

V. CONCLUSION

The paper deals with the problem of estimating the present values of the continuous whole life and n-year term life annuities. We prove asymptotic properties of the proposed estimators: unbiasedness, consistency and normality. Also, we found the main parts of the asymptotic MSEs of these estimators. The simulation results for the model of de Moivre show that the quality of evaluation by the criteria $G(N,n,\delta)$ is improving with increasing sample size.

Note that one can obtain the improved estimators of life annuities (5) and (6) changing the empirical survival functions in the denominators of (5) and (6) by the smooth empirical survival functions (cf. [14-20]).

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