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NUT-like generalization of axisymmetric gravitational fields*

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The complex potential formulation of the axisymmetric problem discussed by Ernst enables us to construct new solutions from a given one, by multiplying the corresponding potential by a unit complex number. This rotation introduces naturally the NUT parameter in the metric. The generalized Kerr, Weyl, and Tomimatsu–Sato solutions are explicity constructed.

I. INTRODUCTION

In 1963 Newman, Tamburino, and Unti[1] found a family of solutions of the Einstein equations, which contains as a special case the Schwarzschild solution. The interest in the NUT fields is mainly mathematical, since the only member of the family which is flat at infinity is the Schwarzschild solution itself.

A generalization of the Kerr field, analogous to that proposed by NUT, was obtained by Demianski and Newman[2] by means of a mathematical trick, involving a complex coordinate transformation.

In this paper it is shown that the complex potential formalism introduced by Ernst[3] leads naturally to the NUT and to the Demianski and Newman solutions, the NUT parameter being related to an arbitrary phase constant in the Ernst potential ξ_0. The generalization can be extended to any axisymmetric solution, and in particular it is given here for the Tomimatsu–Sato field.

II. NUT AND DEMIANSKI-NEWMAN FIELDS

In canonical cylindric coordinates the most general axisymmetric electrovac line element reads[4]

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega\, d\varphi)^2, \tag{1}$$

where the potentials f, γ, ω are functions of ρ, z. It was shown by Ernst[3] that the potentials can be derived from a complex function ξ_0, satisfying the equation,

$$(\xi_0\xi_0^* - 1)\nabla^2\xi_0 = 2\xi_0^*\nabla\xi_0\nabla\xi_0, \tag{2}$$

where ∇^2 is the flat space three-dimensional operator. The equations relating f, ω, γ to ξ_0 are

$$f = \mathrm{Re}\frac{\xi_0 - 1}{\xi_0 + 1}, \tag{3}$$

$$\nabla\omega = \frac{2\rho}{(\xi_0\xi_0^* - 1)^2}\mathrm{Im}[(\xi_0^* - 1)^2\hat{\mathbf{n}}\times\nabla\xi_0], \tag{4}$$

$$\frac{\partial\gamma}{\partial\rho} = \frac{\rho}{(\xi_0\xi_0^* - 1)^2}\left[\frac{\partial\xi_0}{\partial\rho}\frac{\partial\xi_0^*}{\partial\rho} - \frac{\partial\xi_0}{\partial z}\frac{\partial\xi_0^*}{\partial z}\right], \tag{5}$$

$$\frac{\partial\gamma}{\partial z} = \frac{\rho}{(\xi_0\xi_0^* - 1)^2}\left[\frac{\partial\xi_0}{\partial\rho}\frac{\partial\xi_0^*}{\partial z} + \frac{\partial\xi_0^*}{\partial\rho}\frac{\partial\xi_0}{\partial z}\right], \tag{6}$$

where $\hat{n}$ is the azimuth direction.

It was noted by Ernst[5] that from a given solution ξ_0 of Eq. (2), one can generate in a number of ways new solutions, which, however, in general are not physically meaningful. In particular we show that the transformation

$$\xi = \exp(i\alpha)\xi_0 \tag{7}$$

yields the NUT and Demianski–Newman fields for ξ_0 corresponding to the Schwarzschild and Kerr solutions respectively.

In prolate spheroidal coordinates (x,y) $[\rho = k(x^2-1)^{1/2} \times (1-y^2)^{1/2}$; $z = kxy$, k being a scale factor$]$ the Kerr solution corresponds to

$$\xi_0 = px + iqy,$$

with $p^2 + q^2 = 1$. The transformation (7) together with Eqs. (3) and (4) gives

$$f = 1 - 2\frac{p\cos\alpha x - q\sin\alpha y + 1}{(px+1)^2 + q^2y^2 + 2p(\cos\alpha - 1)x - 2q\sin\alpha y}, \tag{8}$$

$$\omega = -\frac{2k}{p}q\frac{(1-y^2)(q\sin\alpha y - p\cos\alpha x - 1)}{p^2x^2 + q^2y^2 - 1} - 2\frac{k}{p}\sin\alpha y. \tag{9}$$

Since Eqs. (5) and (6) are independent of α, the potential γ is unchanged by the transformation (7), and therefore,

$$\exp(2\gamma) = (p^2x^2 + q^2y^2 - 1)/p^2(x^2 - y^2). \tag{10}$$

By the coordinate transformation

$$x = (r - m)/k, \quad y = \cos\vartheta$$

the metric is mapped into the form

$$ds^2 = \frac{r^2 + (a\cos\vartheta - l)^2}{r^2 - 2mr + a^2 - l^2}dr^2 + [r^2 + (a\cos\vartheta - l)^2] \times\left(d\vartheta^2 + \frac{r^2 - 2mr + a^2 - l^2}{r^2 - 2mr + a^2\cos^2\vartheta - l^2}d\varphi^2\right) - \left(1 - 2\frac{mr + l(l - a\cos\vartheta)}{r^2 + (a\cos\vartheta - l)^2}\right) \times\left[dt - \left(\frac{2a\sin^2\vartheta[mr + l(l - a\cos\vartheta)]}{r^2 - 2mr - l^2 + a^2\cos^2\vartheta} - 2l\cos\vartheta\right)d\varphi\right]^2, \tag{11}$$

where m, l, and a are related to p, q, α, and k by

$$k^2 = m^2 + l^2 - a^2,$$

$$p = k/(m^2 + l^2)^{1/2}, \quad q = a/(m^2 + l^2)^{1/2},$$

$$\cos\alpha = m/(m^2 + l^2)^{1/2}, \quad \sin\alpha = l/(m^2 + l^2)^{1/2}.$$

The line element (11) coincides with the Demianski–Newman uncharged metric, which reduces to the usual Boyer and Lindquist form of the Kerr metric for $l = 0$ and to the NUT generalization of the Schwarzschild metric for $a = 0$.

III. GENERALIZED WEYL AND TOMIMATSU-SATO FIELDS

The transformation (7) can be applied to algebraically general fields as well. We consider the special family of Weyl solutions

$$\xi_0 = [(x+1)^\delta + (x-1)^\delta]/[(x+1)^\delta - (x-1)^\delta], \qquad (12)$$

which for $\delta = 1$ is the Schwarzschild solution and for $\delta = 2, 3, 4$ are the static counterparts of the Tomimatsu–Sato solutions.

Applying the transformation (7) and solving for the potentials f, ω, γ, we have

$$f = 2(x^2-1)^\delta/[(\cos\alpha+1)(x+1)^{2\delta} + (\cos\alpha-1)(x-1)^{2\delta}], \qquad (13)$$

$$\omega = 2k\delta \sin\alpha\, y, \qquad (14)$$

$$\exp(2\gamma) = (x^2-1)^{\delta^2}/(x^2-y^2)^{\delta^2}. \qquad (15)$$

For $\delta = 1$ this reduces to the NUT field.

The Tomimatsu–Sato complex potential for $\delta = 2$ reads[6]

$$\xi_0 = (u+iv)/(m+in),$$

where

$$u = p^2x^4 + q^2y^4 - 1, \quad v = -2pq\times y(x^2-y^2),$$

$$m = 2px(x^2-1), \quad n = -2qy(1-y^2).$$

The rotation (7) yields

$$\xi = [\cos\alpha\, u - \sin\alpha\, v + i(\sin\alpha\, u + \cos\alpha\, v)]/(m+in)$$

and therefore

$$f = A_0/B,$$

where

$$A_0 = u^2 + v^2 - m^2 - n^2,$$

$$B = B_0 + 2(\cos\alpha - 1)\eta - 2\sin\alpha\, \epsilon,$$

$$B_0 = (u+m)^2 + (v+n)^2,$$

$$\eta = mu + nv, \quad \epsilon = mv - nu.$$

(Hereinafter a subscript 0 indicates the quantities which are unchanged with respect to the Tomimatsu–Sato case.) The potential γ is that given by Tomimatsu–Sato,

$$\exp(2\gamma_0) = A_0/p^4(x^2-y^2)^4.$$

Equations (4) in prolate spheroidal coordinates yield

$$\frac{\partial}{\partial x}(\omega - \cos\alpha\, \omega_0) = -k\frac{(1-y^2)}{A_0^2}\left[2(1-\cos\alpha)\left(\eta\frac{\partial\epsilon}{\partial y} - \epsilon\frac{\partial\eta}{\partial y}\right) + \sin\alpha\left(B_0\frac{\partial\eta}{\partial y} - \eta\frac{\partial B_0}{\partial y}\right)\right], \qquad (16)$$

$$\frac{\partial}{\partial y}(\omega - \cos\alpha\, \omega_0) = k\frac{(x^2-1)}{A_0^2}\left[2(1-\cos\alpha)\left(\eta\frac{\partial\epsilon}{\partial x} - \epsilon\frac{\partial\eta}{\partial x}\right) + \sin\alpha\left(B_0\frac{\partial\eta}{\partial x} - \eta\frac{\partial B_0}{\partial x}\right)\right], \qquad (17)$$

where ω_0 reads

$$\omega_0 = -2mq\frac{(1-y^2)}{A_0}\{p^3x(x^2-1)[2(x^4-1) + (x^3+3)(1-y^2)] + p^2(x^2-1)[4x^2(x^2-1) + (3x^2+1)(1-y^2)] - q^2(px+1)(1-y^2)^3]\}. \qquad (18)$$

From Eqs. (16) and (17) it can be easily shown that ω must be of the form,

$$\omega = \cos\alpha\, \omega_0 + kq\frac{(1-y^2)}{A_0}[2(\cos\alpha-1)C + \sin\alpha\, D] + h\sin\alpha\, y, \qquad (19)$$

where C, D are polynomials of x, y and h is a constant independent of α. The presence of the last term in Eq. (19) and the condition that it must reduce to the form (14) for $q=0$ is sufficient to show that also this metric is not asymptotically flat. It does not seem therefore very interesting to work out the explicit form of ω.

IV. CONCLUSIONS

We have shown that the Ernst formulation of the axisymmetric problem leads directly to the generalizations of Schwarzschild and Kerr fields given originally by NUT and Demianski and Newman. An advantage of this derivation is that it can be extended to algebraically general fields as the Weyl and Tomimatsu–Sato fields.

It is obvious that the method can be applied also to electrovac solutions. In fact, by using the results of Ernst[7] it is clear that, multiplying ξ_0 by a complex number with modulus different from 1, one obtains the charged NUT-like generalization of any given solution.

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