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Complete extension of the symmetry axis of the Tomimatsu-Sato solution of the Einstein equations

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The symmetry axis of the simplest Tomimatsu-Sato field is considered. Since this manifold is not geodesically complete for every value of the parameters occurring in the metric, a complete extension is given, and it is shown that its causal structure is very similar to that of the symmetry axis of the Kerr field.

1. INTRODUCTION

Ten years after the discovery by Kerr[1] of the first axisymmetric rotating solution of Einstein's equations, new rotating fields were found by Tomimatsu and Sato.[2,3] The main difference between these two classes of solutions is that the quadrupole moment of the T–S fields is larger than that of the Kerr field, and that the former solutions exhibit a number of ring singularities among which the outermost is unshielded by an event horizon.

The structure of these manifolds has been investigated by several authors, who have considered the geodesic problem[3,4] and, in the case of the simplest T–S metric, the behavior of the metric near the poles ($x=1$, $y=\pm 1$ in prolate spheroidal coordinates).[5,6] In particular, Ernst[5] introduced a new representation of this T–S metric, showing that the full four-dimensional geodesic problem can be completely solved in the neighborhood of the poles.

In this paper the bidimensional metric on the axis of the simplest T–S field is studied, using extensively the method adopted by Carter[7] in the case of the axis of the Kerr solution. Although this problem is rather more restricted than the maximal extension of the full four-dimensional metric, it is nevertheless significant to have found a complete extension of the bidimensional metric which is exact and C_0 on its domain.

In spite of the differences between the Kerr and T–S solutions, it is found that they have a very similar causal structure when restricted to the axis.

2. T-S FIELD IN QUASISPHEROIDAL COORDINATES

The axisymmetric line element in canonical coordinates reads

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2\, d\varphi^2] - f(dt - \omega\, d\varphi)^2, \tag{1}$$

where f, ω, γ are functions of ρ and z only. In prolate spheroidal coordinates (x, y) defined by the mapping

$$\begin{cases} \rho = k(x^2-1)^{1/2}(1-y^2)^{1/2}, \\ z = kxy, \end{cases} \tag{2}$$

the line element (1) takes the form

$$ds^2 = k^2 f^{-1}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)\,d\varphi^2\right] - f(dt-\omega d\varphi)^2. \tag{3}$$

For the T–S fields the metric functions f, γ, ω are expressed in terms of three polynomials A, B, C of x, y in the following way[3]:

$$f = \frac{A}{B}, \quad \omega = \frac{2mq}{A}(1-y^2)C, \quad e^{2\gamma} = \frac{A}{p^{2\delta}(x^2-y^2)^{\delta^2}} \tag{4}$$

where m is a parameter describing the mass of the source of the field, p and q are real constants subjected to the condition $p^2+q^2=1$, and δ is an integer parameter taking the values 2, 3, 4. The explicit expressions for A, B, C depend on the chosen value of δ. The simplest T–S metric was obtained[2] for $\delta=2$, yielding

$$\begin{aligned} A &= p^4(x^2-1)^4 + q^4(1-y^2)^4 - 2p^2q^2(x^2-1)(1-y^2) \\ &\quad \times [2(x^2-1)^2 + 2(1-y^2)^2 + 3(x^2-1)(1-y^2)], \\ B &= [p^2(x^2+1)(x^2-1) - q^2(y^2+1)(1-y^2) + 2px(x^2-1)]^2 \\ &\quad + 4q^2y^2[px(x^2-1) + (px+1)(1-y^2)]^2, \\ C &= p^3x(x^2-1)[2(x^2+1)(x^2-1) + (x^2+3)(1-y^2)] \\ &\quad - p^2(x^2-1)[4x^2(x^2-1) + (3x^2-1)(1-y^2)] \\ &\quad + q^2(px+1)(1-y^2)^3. \end{aligned} \tag{5}$$

For the explicit form of the polynomials A, B, C for $\delta=3$ the work by T–S is cited.[3]

In the following it will be useful to work in quasispheroidal coordinates (r, θ) defined by

$$\frac{1}{\delta}px + 1 = \frac{r}{m}, \quad y = \cos\theta. \tag{6}$$

The line element (3) becomes

$$\begin{aligned} ds^2 &= \frac{\tilde{B}}{[(r-m)^2 - H^2\cos^2\theta]^{\delta^2-1}}\left(\frac{dr^2}{r^2-2mr+\alpha^2}+d\theta^2\right) \\ &\quad + \frac{B}{A}(r^2-2mr+\alpha^2)\sin^2\theta\, d\varphi^2 \\ &\quad - \frac{A}{B}\left(dt - \frac{2mq}{A}\sin^2\theta\cdot C\,d\varphi\right)^2, \end{aligned} \tag{7}$$

where the arbitrariness of the scale k is used in order to put $k = H = pm/\delta$, and where

$$\alpha = m^2 - H^2, \quad \tilde{B} = (H^{2\delta^2}/p^{2\delta})B. \tag{8}$$

On the symmetry axis $\sin\theta = 0$ (i.e., $y^2=1$) this metric reduces to the form

$$ds^2_{\rm ax} = \frac{\rho^4}{\Delta^4}dr^2 - \frac{\Delta^4}{\rho^4}dt^2, \tag{9}$$

where

$$\rho^4 = (r-m)^2(r^2+m^2) + (m^2-\alpha^2)(r^2-\alpha^2), \tag{10}$$

$$\Delta^4 = (r^2 - 2mr + \alpha^2)^2. \tag{11}$$

Note that for $m \neq \alpha$, $\rho^4 > 0$ for all r, while for $m = \alpha$, $\rho^4 = 0$ for $r = m$. Since Δ^4 has no real zeros for $m^2 < \alpha^2$, in this case the manifold $-\infty < r < +\infty$, $-\infty < t < +\infty$, with the metric (9) is complete.

3. GEODESIC COMPLETENESS

In order to see the necessity of an extension for $m^2 \geq \alpha^2$, we introduce new coordinates t', r' defined by

$$dt' = dt' + \frac{\Delta^4 - \rho^4}{\Delta^4}\, dr', \quad dr = dr'. \tag{12}$$

The metric becomes

$$ds^2_{ax} = (1+\tau)\, dr'^2 + 2\tau\, dr'\, dt' - (1-\tau)\, dt'^2, \tag{13}$$

where

$$\tau = \frac{2mr(r-m)^2 + (m^2-\alpha^2)[(r-m)^2 - 2(mr-\alpha^2)]}{(r^2+m^2)(r-m)^2 + (m^2-\alpha^2)(r^2-\alpha^2)}. \tag{13a}$$

Upon introducing a null coordinate u such that

$$t' = u - r', \quad r = r', \tag{14}$$

as was done by Finkelstein for the Schwarzschild manifold, the metric (13) becomes

$$ds^2_{ax} = -(1-\tau)\, du^2 + 2\, du\, dr. \tag{15}$$

Geodesic trajectories can be derived from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}[2\dot{u}\dot{r} - (1-\tau)\dot{u}^2], \tag{16}$$

where the dot indicates the derivative with respect to an affine parameter, λ say. The Euler–Lagrange equation obtained varying the action with respect to u is immediately integrated, giving

$$-(1-\tau)\dot{u} + \dot{r} = -E,$$

where E is a constant. This equation together with the normalization condition $\mathcal{L} = \epsilon$ ($\epsilon = 0, \pm 1$ for null, spacelike and timelike geodesics, respectively) yields the two equations

$$\dot{u} = \frac{E \pm \sqrt{E^2 + \epsilon(1-\tau)}}{(1-\tau)},$$

$$\dot{r} = \pm\sqrt{E^2 + \epsilon(1-\tau)}.$$

For $m^2 < \alpha^2$, the expression $1 - \tau = \rho^4/\Delta^4$ has no real zeros and $\dot{u}$, $\dot{r}$ are bounded functions of r. This implies that each geodesic $u(\lambda)$, $r(\lambda)$ can be continued to arbitrary values of the affine parameter λ. Therefore, the manifold $-\infty < r < +\infty$, $-\infty < t < +\infty$, with the metric (15) is geodesically complete in this case.

For $m^2 > \alpha^2$, u diverges at $r = r_\pm$, and the manifold is incomplete. This can be shown explicitly for null geodesics ($\epsilon = 0$). Redefining λ so that $E = 1$, one has the following equations for "ingoing" and "outgoing" goedesics:

$$u = C_1, \quad r = -\lambda, \quad u = C_2 + F(r), \quad r = \lambda,$$

where C_1, C_2 are constants, and

$$F(r) = \int \frac{2\, dr}{1-\tau} = 2r + D_1 \ln|r - r_+| - D_2 \ln|r - r_-| - D_3 \frac{r}{(r-r_+)(r-r_-)}, \tag{17}$$

with

$$D_1 = 2\left(\frac{2m^2 - \alpha^2}{(m^2-\alpha^2)^{1/2}} + m\right),$$

$$D_2 = 2\left(\frac{2m^2 - \alpha^2}{(m^2-\alpha^2)^{1/2}} - m\right),$$

$$D_3 = 2(m^2 - \alpha^2).$$

Since $F(r)$ diverges for $r = r_\pm$, outgoing geodesics cannot penetrate the surface $r = r_+$.

For $m^2 = \alpha^2$, it has been shown[8] that all T–S spaces are equivalent to extreme Kerr ($m^2 = a^2$), so the complete extension has been given already[7] for this case.

4. COMPLETE EXTENSION

Introduce now a second null coordinate w, defined by

$$F(r) = u + w. \tag{18}$$

Since $F(r)$ is monotonic in the regions,

I: $r > r_+$,

II: $r_+ > r > r_-$,

III: $r_- > r$,

one must specify to which region one is referring, in order that the mapping (18) be well defined. With the coordinates u, w the metric assumes the canonical double null form

$$ds^2_{ax} = (1-\tau)\, dw\, du, \tag{19}$$

where again the factor $1 - \tau$ is degenerate at $r = r_\pm$.

Following Carter, one can introduce the manifold $\mathcal{M}^*$ spanned by coordinates ψ, ξ, ranging from $-\infty$ to $+\infty$. Let r_n, a_m be the lines

$$\begin{aligned} r_n)\quad & \psi = -\xi + \pi/2 + n\pi, \\ a_m)\quad & \psi = \xi + \pi/2 + m\pi \\ & (m, n = 0, \pm 1, \pm 2, \cdots), \end{aligned} \tag{20}$$

and let Q_{nm} be the intersections of the two strips bounded by the lines a_m, a_{m-1} and r_n, r_{n-1}, respectively. The ψ, ξ coordinates are defined by the relations

$$\begin{aligned} u &= \tan\tfrac{1}{2}(\psi + \xi), \\ w &= \cot\tfrac{1}{2}(\psi - \xi). \end{aligned} \tag{21}$$

The squares Q_{kk} are images of the region II, $Q_{h,h-1}$ are images of the region III if h is odd and are images of the region I if h is even, and finally the squares $Q_{j-1,j}$ are images of the region I if j is odd and of the region III if j is even.

The metric becomes

$$ds^2_{ax} = \Omega^2\, d\psi\, d\xi \tag{22}$$

where

$$\Omega^2 = (1-\tau)\, \tfrac{1}{4}\sec^2\left(\frac{\psi+\xi}{2}\right)\, \csc^2\left(\frac{\psi-\xi}{2}\right).$$

It can be easily shown that this conformal factor is continuous and positive definite on the manifold $\mathcal{M}^*$.

The conformal diagram for the axis of the T—S solution considered here is identical to that for the Kerr axis.[4] Therefore, the two axis have the same causal structure, i. e., they are conformally related.

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