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A constructive approach to bundles of geometric objects on a differentiable manifold[a)]

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A constructive approach to bundles of geometric objects of finite rank on a differentiable manifold is proposed, whereby the standard techniques of fiber bundle theory are extensively used. Both the point of view of transition functions (here directly constructed from the jets of local diffeomorphisms of the basis manifold) and that of principal fiber bundles are developed in detail. These, together with the absence of any reference to the current functorial approach, provide a natural clue from the point of view of physical applications. Several examples are discussed. In the last section the functorial approach is also presented in a constructive way, and the Lie derivative of a field of geometric objects is defined.

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1. INTRODUCTION

Classical tensor calculus, developed during the latter part of the last century by Ricci and Levi-Civita, was soon found to be the most appropriate formalism for studying local physical laws in an invariant way. After its application to special and general relativity,[1] tensor calculus became a common tool in mathematical physics and the main formal link between geometry and physics itself.

As early as 1918 it was, however, discovered that certain local structures which are relevant both to physics and geometry do not have tensorial character, the most well-known example being, of course, given by connections [Levi-Civita (1917),[1a] Weyl (1918),[2] and Cartan (1923)[3]]. Early attempts to give definitions of "geometric objects" general enough to also include such nontensorial entities date back to the thirties [Schouten and Haantjes (1936)[4]], but a fully satisfactory and intrinsic definition was found only after the work of Nijenhuis during the fifties [Nijenhuis (1952),[5] (1960),[6] Haantjes and Laman (1953 a,b),[7] and Kuiper and Yano (1955)[8]]. More recently, the matter was reconsidered by Salvioli (1972),[9] who gave a natural and beautiful description grounded on a "functorial approach".

Roughly speaking, an object defined on a differentiable manifold is a geometric object if we know its transformation law for any change of local coordinates. Tensors are obviously geometric objects, but of a very restricted type; their transformation laws are in fact "homogeneous" and involve only the Jacobian matrix of the coordinate transformation. To allow more general objects like, for example, connections, higher derivatives of the coordinate transformation must be taken into account.

In recent years, owing to their greater generality, geometric objects other than tensors began to enter physical applications, because in many cases using objects more general than tensors is essential [see, e.g., Anderson (1967),[10] Krupka (1979a,b),[11] Kijowski and Tulczyjew (1978),[12] Prastaro (1980),[13] (1981),[14] Modugno (1981),[15] Pommaret (1978),[16] Ferraris, and Francaviglia, and Reina (1981)[17]]. In fact, in spite of the widely known and systematic use of tensorial methods in mathematical physics, restricting ones attention to tensors may often turn out to be misleading.

Motivated by physical applications we have reconsidered the mathematical foundations of the theory of geometric objects, providing for them a new direct approach, which adapts the nice construction proposed by Haantjes and Laman (1953a,b) to the more flexible language of differential geometry of fiber bundles. Our approach is less general than that of Salvioli because it refers explicitly to geometric objects having finite rank. However, it has the advantage of being constructive and able to handle in a simple, intrinsic, and detailed way most of the bundles of geometric objects which are relevant to mathematical physics. It, in fact, provides explicit constructions for the "lifting functors" of Salvioli's method and allows much easier calculations.

2. FIBER BUNDLES ON MANIFOLDS

1. Fields of geometric objects naturally arise as sections of suitable bundles. In the following we shall restrict ourselves to the bundles of geometric objects having finite rank, because they have the property of being fiber bundles.

Therefore, let us begin by recalling the concepts of fiber bundle theory we shall need later. We adopt the following definition.

Definition 2.1: Let $\mathbb{M}$, $\mathbb{F}$ be C^∞-manifolds and G a Lie group. A fiber bundle over $\mathbb{M}$ (with structure group G and standard fiber $\mathbb{F}$) is a quintuple ($\mathbb{B}$, $\mathbb{M}$, π; G, $\mathbb{F}$), where π: $\mathbb{B} \rightarrow \mathbb{M}$ is a surjective map from a differentiable manifold $\mathbb{B}$ onto $\mathbb{M}$, if the following conditions are satisfied. (i) G acts effectively and differentiably on $\mathbb{F}$; (ii) there exist an open covering $\{U_a\}$ of $\mathbb{M}$ and homeomorphisms (called local tri-

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vializations)

$$\tau_\alpha:\pi^{-1}(\mathsf{U}_\alpha)\to \mathsf{U}_\alpha\times \mathbb{F}$$

such that the diagram

$$\begin{array}{ccc} \pi^{-1}(\mathsf{U}_\alpha) & \xrightarrow{\tau_\alpha} & \mathsf{U}_\alpha\times\mathbb{F} \\ \pi\downarrow & \swarrow & \\ \mathsf{U}_\alpha & & \end{array}$$

is commutative; (iii) there exist maps $m_{\alpha\beta}:\mathsf{U}_{\alpha\beta}=\mathsf{U}_\alpha\cap\mathsf{U}_\beta\to\mathsf{G}$ (called transition functions) such that

$$\tau_\alpha\cdot\tau_\beta^{-1}(p,f)=(p,m_{\alpha\beta}(p)f)\quad \forall p\in\mathsf{U}_{\alpha\beta},\ f\in\mathbb{F}.$$

Remark 2.2: The transition functions above satisfy the compatibility relations $m_{\beta\alpha}(p)=[m_{\alpha\beta}(p)]^{-1}$ and $m_{\alpha\beta}(p)m_{\beta\gamma}(p)m_{\gamma\alpha}(p)=1\in\mathsf{G}$, $p\in\mathsf{U}_\alpha\cap\mathsf{U}_\beta\cap\mathsf{U}_\gamma$. Therefore, they form a 1-cocycle with values in the sheaf of germs of local differentiable functions from $\mathbb{M}$ to G [for more details see Hirzebruck (1978)[18]].

Remark 2.3: Note that given a covering $\{\mathsf{U}_\alpha\}$ of $\mathbb{M}$ and a set of G-valued transition functions $m_{\alpha\beta}$ satisfying the properties of Remark 2.2 one can construct a fiber bundle $\mathbb{B}$ over $\mathbb{M}$ with structure group G and standard fiber $\mathbb{F}$. We first form the disjoint union $\tilde{\mathbb{B}}$ of all the sets $\mathsf{U}_\alpha\times\mathbb{F}$. The bundle $\mathbb{B}$ is then obtained from $\tilde{\mathbb{B}}$ by identifying the points $(p,f)\in\mathsf{U}_\alpha\times\mathbb{F}$ and $(p,m_{\alpha\beta}(p)f)\in\mathsf{U}_\beta\times\mathbb{F}$ for any α,β and $p\in\mathsf{U}_{\alpha\beta}$. One can show that such a reconstruction does not depend on the choice of the covering $\{\mathsf{U}_\alpha\}$.

2. As examples of the preceding construction we may quote the following.

Example 2.4: A Lie group acts naturally on itself on the left (or on the right). Therefore, one can construct fiber bundles having the structure group G itself as standard fiber. These are called principal G–bundles and will be denoted by ($\mathbb{P}$, $\mathbb{M}$, π; G). Principal G bundles may be characterized as follows. A quadruplet ($\mathbb{P}$, $\mathbb{M}$, π; G) is a principal G bundle if and only if the following prescriptions are satisfied: (i) G is a Lie group, $\mathbb{P}$ and $\mathbb{M}$ are C^∞ manifolds, and π: $\mathbb{P}\to\mathbb{M}$ is a surjective map of maximal rank; (ii) there exists a right (or left) action R: $\mathbb{P}\times\mathsf{G}\to\mathbb{P}$ of G on $\mathbb{P}$ which is free [i.e., if $p\in\mathbb{P}$, $g\in\mathsf{G}$, and $R(p,g)=p$ then g is the identity of G], differentiable, and such that $\mathbb{M}=\mathbb{P}/\mathsf{G}$ [i.e., $\forall p\in\mathbb{P}$, $g\in\mathsf{G}$, $\pi[R(p,g)]=\pi(p)$].

Example 2.5: Let ($\mathbb{P}$, $\mathbb{M}$, π; G) be a principal G-bundle, $\mathbb{F}$ be a manifold, and $\rho:G\to\mathscr{D}(\mathbb{F})$ be a representation of G into the group $\mathscr{D}(\mathbb{F})$ of diffeomorphisms of $\mathbb{F}$. According to Remark 2.3 one can construct a fiber bundle ($\mathbb{B}$, $\mathbb{M}$, π'; ρ(G),$\mathbb{F}$) by using the composition of ρ with the transition functions of $\mathbb{P}$. An alternative well-known procedure consists in taking the quotient of the manifold $\mathbb{P}\times\mathbb{F}$ with respect to the equivalence relation defined by the following group action, $\hat{\rho}:\mathbb{P}\times\mathbb{F}\times\mathsf{G}\to\mathbb{P}\times\mathbb{F}$, induced by

$$\hat{\rho}:(p,f,g)\mapsto(pg,\rho(g)^{-1}f). \tag{1}$$

To this bundle we shall give the name of bundle of objects of type ρ associated with $\mathbb{P}$.

Example 2.6: In particular, whenever G admits a representation λ:G$\to$GL(V) in the linear group of some vector space V, by the same procedure we can construct bundles having V as standard fiber and λ (G) as structure group. These are called vector bundles over $\mathbb{M}$ (associated with $\mathbb{P}$).

Example 2.7: Whenever G admits a representation α:G$\to$IGL (A) in the affine group of some affine space A we obtain affine bundles (associated with $\mathbb{P}$), having A as standard fiber. Note that any vector bundle can be improperly considered as an affine bundle by identifying GL (V) with its isomorphic copy contained in IGL(V), where V is considered as an affine space. This procedure can be inverted, in the sense that given an affine bundle ($\mathbb{B}$, $\mathbb{M}$, π; G, A) we may define an associated vector bundle $\mathbb{B}'$ having as fiber the vector space V underlying the affine fibers A of $\mathbb{B}$.

3. BUNDLES OF GEOMETRIC OBJECTS

1. Among the fiber bundles over $\mathbb{M}$ with given fiber $\mathbb{F}$ and structure group G, we shall describe here an important subclass, whose transition functions $m_{\alpha\beta}(p)$ are constructed starting only from the differentiable structure of $\mathbb{M}$. This is the original viewpoint of Haantjes and Laman, which will here be briefly recalled and set up in slightly different language, in order to prepare us for the alternative description which will be given later.

Let $\{(\mathsf{U}_\alpha,\varphi_\alpha)\}$ be an atlas of $\mathbb{M}$. For any pair of overlapping charts $((\mathsf{U}_\alpha,\varphi_\alpha),(\mathsf{U}_\beta,\varphi_\beta))$, there exists a (local) C^∞ diffeomorphism

$$\Phi_{\alpha\beta}=\varphi_\alpha\cdot\varphi_\beta^{-1}:\varphi_\beta(\mathsf{U}_{\alpha\beta})\to\varphi_\alpha(\mathsf{U}_{\alpha\beta}) \tag{2}$$

between open subsets of $\mathbb{R}^n$. The local diffeomorphisms $\Phi_{\alpha\beta}$ are usually called coordinate transformations. Our next task will then be to construct transition functions out of these local diffeomorphisms of $\mathbb{R}^n$.

First of all, we note that for any point $p\in\mathsf{U}_{\alpha\beta}$ and for any $\Phi_{\alpha\beta}$ one can construct a local diffeomorphism $\tilde{\Phi}_{\alpha\beta}(p)$ of $\mathbb{R}^n$ into itself such that $\tilde{\Phi}_{\alpha\beta}(p)(0)=0$, by defining

$$\tilde{\Phi}_{\alpha\beta}(p):x\mapsto\Phi_{\alpha\beta}[x+\varphi_\beta(p)]-\varphi_\alpha(p) \tag{3}$$

for any $x\in\mathbb{R}^n$ such that $x+\varphi_\beta(p)\in\varphi_\beta(\mathsf{U}_{\alpha\beta})$. The local diffeomorphisms $\tilde{\Phi}_{\alpha\beta}(p)$ satisfy the following conditions: $[\tilde{\Phi}_{\alpha\beta}(p)]^{-1}=\tilde{\Phi}_{\beta\alpha}(p)$ and $\tilde{\Phi}_{\alpha\beta}(p)\cdot\tilde{\Phi}_{\beta\sigma}(p)\cdot\tilde{\Phi}_{\sigma\alpha}(p)=\mathrm{id}_{\mathbb{M}}$ for any $p\in\mathsf{U}_\alpha\cap\mathsf{U}_\beta\cap\mathsf{U}_\sigma$.

2. Now let $\mathscr{P}_0(\mathbb{R}^n)$ be the pseudogroup of all local diffeomorphisms Ψ of $\mathbb{R}^n$ into itself such that $\Psi(0)=0$. We remind the reader that $[D\Psi(0)]^{-1}$ exists, where the linear map $D\Psi(0):\mathbb{R}^n\to\mathbb{R}^n$ denotes the derivative of Ψ at 0. For any $\Psi\in\mathscr{P}_0(\mathbb{R}^n)$ we define $t^k(\Psi)$ to be the Taylor expansion of Ψ at 0 up to and including the order $k\geqslant0$. Two local diffeomorphisms Ψ, $\Psi'\in\mathscr{P}_0(\mathbb{R}^n)$ are said to agree to the order k (at 0) if $t^k(\Psi)=t^k(\Psi')$. This is obviously an equivalence relation. The equivalence class $j^k(\Psi)$ may be represented as follows:

$$j^k(\Psi)=(0,D\Psi(0),D^2\Psi(0),\dots D^k\Psi(0)),$$

where the symmetric r-linear operators $D^r\Psi(0):(\mathbb{R}^n)^r\to\mathbb{R}^n$ denote the rth order derivatives of Ψ at 0.

Let $G^k(n;\mathbb{R})=\{j^k(\Psi)\,|\,\Psi\in\mathscr{P}_0(\mathbb{R}^n)\}$ denote the quotient set of $\mathscr{P}_0(\mathbb{R}^n)$ under the above equivalence relation. It is easy to show that $G^k(n;\mathbb{R})$ is a (real) Lie group with respect to the natural composition law:

$$j^k(\Psi)\cdot j^k(\Psi')=j^k(\Psi\cdot\Psi')$$

In particular, when $k = 1$ we recover the general linear group $GL(n;\mathbb{R})$.

3. Since all the local diffeomorphisms $\widetilde{\Phi}_{\alpha\beta}(p)$ defined in Sec. 2.1 belong to $\mathscr{P}_0(\mathbb{R}^n)$ we may consider their k th order jets $j^k(\widetilde{\Phi}_{\alpha\beta}(p)) \in G^k(n;\mathbb{R})$. Then, for any $U_{\alpha\beta}$ and any integer $k \geqslant 0$ we may define functions $\Phi^k_{\alpha\beta}: U_{\alpha\beta} \to G^k(n;\mathbb{R})$ as follows:

$$\Phi^k_{\alpha\beta}: p \mapsto j^k[\widetilde{\Phi}_{\alpha\beta}(p)]. \tag{4}$$

One may easily check that the functions $\Phi^k_{\alpha\beta}$ defined in this way are C^∞ functions from $U_{\alpha\beta}$ into $G^k(n;\mathbb{R})$. Moreover, they satisfy the following conditions:

(i) $[\Phi^k_{\alpha\beta}(p)]^{-1} = \Phi^k_{\beta\alpha}(p)$

(ii) $\Phi^k_{\alpha\beta}(p) \cdot \Phi^k_{\beta\sigma}(p) \cdot \Phi^k_{\sigma\alpha}(p) = \mathbb{1}$,

where $\mathbb{1} = j^k(\mathrm{id}_n)$ denotes the identity of the group $G^k(n;\mathbb{R})$.

Remark 3.1: By this last result we see that the functions $\Phi^k_{\alpha\beta}$ may be considered as transition functions for a fiber bundle having structure group $G^k(n;\mathbb{R})$, because they form a 1-cocycle with values in the group $G^k(n;\mathbb{R})$.

4. Now let $\mathbb{F}$ be a manifold on which a Lie group G acts effectively and differentiably and ρ: $G^k(n;\mathbb{R}) \to$ G be a group homomorphism onto G. Then we have maps

$$m_{\alpha\beta}: U_{\alpha\beta} \xrightarrow{\Phi^k_{\alpha\beta}} G^k(n;\mathbb{R}) \xrightarrow{\rho} G$$

given by $m_{\alpha\beta}(p) = \rho(\Phi^k_{\alpha\beta}(p))$, which "lift" the differentiable structure of $\mathbb{M}$ into G-valued transition functions. From these data we can construct a fiber bundle $\mathbb{B}$ over $\mathbb{M}$ with standard fiber $\mathbb{F}$ and structure group G, which will be denoted by $(\mathbb{B}, \mathbb{M}, \pi; \mathbb{F}, G, \rho)$. We then give the following definition:

Definition 3.2: $(\mathbb{B}, \mathbb{M}, \pi; \mathbb{F}, G, \rho)$, where $\rho: G^k(n;\mathbb{R}) \to$ G, is called a bundle of geometric objects of type ρ of finite rank $(\leqslant k)$.

The bundles of geometric objects defined in this way fit into the scheme of Salvioli. It can be proved, in fact, that they satisfy all the properties listed in Salvioli (Ref. 9, p. 259).

We remark that the definitions given by Salvioli extend to also cover geometric objects of infinite rank. The direct approach we outlined above may also be extended to this case by relying on suitable Fréchet manifolds, i.e., by allowing the use of infinite jets of mappings.

5. We remark that, in differential geometry and in its recent applications to physics, a central role is played by principal fiber bundles and that, moreover, all fiber bundles can be considered as associated with some suitable principal fiber bundle.

Our next task is then to show that the construction we outlined above is, in fact, in agreement with this spirit, in the sense that all the bundles of geometric objects covered by Definition 3.2 are associated with certain principal bundles. This will provide us an alternative approach to the class of bundles considered, which, as we shall see later, is more manageable for applications.

Let us then proceed as follows. Given a C^∞-manifold $\mathbb{M}$ and an integer k $(1 \leqslant k < \infty)$, for any C^∞-function $h \in C^\infty(\mathbb{R}^n, \mathbb{M})$, the k th order jet $j^k(h)$ of h at $0 \in \mathbb{R}^n$ is naturally defined by reverting to any local parametrization of $\mathbb{M}$. We denote by $\mathbb{L}^k(\mathbb{M})$ the set of all the jets $j^k(h)$ such that h^{-1} exists. Let us now consider the quadruple $(\mathbb{L}^k(\mathbb{M}), \mathbb{M}, \pi^k; G^k(n,\mathbb{R}))$, where π^k: $\mathbb{L}^k(\mathbb{M}) \to \mathbb{M}$ is the canonical projection defined by $\pi^k[j^k(h)] = h(0)$. From the construction above, we see that there exists a canonical right-action of $G^k(n;\mathbb{R})$ on $\mathbb{L}^k(\mathbb{M})$, which is given by

$$(j^k(h), j^k(\Psi)) \mapsto j^k(h \cdot \Psi).$$

It is easy to check that this defines a principal $G^k(n;\mathbb{R})$-bundle over $\mathbb{M}$ [see Example 2.5].

Now let $\rho: G^k(n;\mathbb{R}) \to$ G be a group homomorphism and $\mathbb{F}$ a manifold on which the Lie group G acts effectively and differentiably. We see immediately that the bundles of geometric objects of Definition 2.3 are the bundles of type ρ associated with $\mathbb{L}^k(\mathbb{M})$ in the sense of Example 2.6. Our claim is thence proved.

The principal bundles $\mathbb{L}^k(\mathbb{M})$ will be called bundles of k th order frames on $\mathbb{M}$. This terminology is motivated by the fact that $\mathbb{L}^1(\mathbb{M})$ is isomorphic with the bundle of linear frames of $\mathbb{M}$.

4. EXAMPLES

1. According to our previous remarks, all the bundles of geometric objects of types ρ are associated with some of the principal bundles $\mathbb{L}^k(\mathbb{M})$, which therefore are, in a sense, the prototype of such bundles.

Note that $\mathbb{L}^k(\mathbb{M})$ is associated with $\mathbb{L}^{k'}(\mathbb{M})$ whenever $k' \geqslant k$, thanks to the existence of a canonical epimorphism from $G^{k'}(n;\mathbb{R})$ onto $G^k(n;\mathbb{R})$. As a consequence, if a bundle $\mathbb{B}$ of geometric objects of type ρ is associated with $\mathbb{L}^k(\mathbb{M})$ it is also associated with all principal bundles $\mathbb{L}^{k'}(\mathbb{M})$ with $k' \geqslant k$. The smallest integer k such that $\mathbb{B}$ is associated with $\mathbb{L}^k(\mathbb{M})$ is called the rank of $\mathbb{B}$.

Example 4.1: All the bundles of tensors over $\mathbb{M}$ may be obtained as vector bundles associated with the bundle of geometric objects $\mathbb{L}^1(\mathbb{M})$, by means of suitable linear representations of $G^1(n;\mathbb{R})$.

For example, the tangent bundle $\mathbb{T}\mathbb{M}$ is obtained from the canonical isomorphism $i: G^1(n;\mathbb{R}) \to GL(n;\mathbb{R})$ while the contangent bundle $\mathbb{T}^*\mathbb{M}$ is obtained from the inverse transpose isomorphism $i^*: G^1(n;\mathbb{R}) \to GL(n;\mathbb{R})$ defined by

$$i^*: j^1(\Psi) \mapsto {}^t[i(j^1(\Psi^{-1}))]. \tag{5}$$

The tensor bundles $\mathbb{T}^p_q(\mathbb{M})$ are then obtained by tensorizing the above constructions; analogously for the bundle $\Lambda^p(\mathbb{M})$ of differential p-forms.

Example 4.2: Let det: GL $(n;\mathbb{R}) \to \mathbb{R}^*$ be the determinant homomorphism. We denote by Δ the composition $\Delta = (\mathrm{det}) \cdot i: G^1(n;\mathbb{R}) \to \mathbb{R}^*$. From the linear representation Δ we can construct a line bundle det($\mathbb{M}$), called the determinant bundle of $\mathbb{M}$, whose sections are the fields of n-vectors on $\mathbb{M}$. Analogously, we can construct the dual bundle det*($\mathbb{M}$) by using the linear representation $\Delta^* = (\mathrm{det}) \cdot i^*$. Its sections are the fields of n-covectors on $\mathbb{M}$ and, therefore, there is a natural isomorphism between the bundles det*($\mathbb{M}$) and $\Lambda^n(\mathbb{M})$.

Example 4.3: Let U(1) be the unitary group. By relying on det($\mathbb{M}$) one can construct a principal U(1)-bundle of geometric objects U($\mathbb{M}$). This can be done by considering the epimorphism $\alpha: G^1(n;\mathbb{R}) \to U(1)$ defined by

$$u: j^1(\Psi) \mapsto \exp[i \ln|\Delta(j^1(\Psi))|], \tag{6}$$

or shortly $u = \exp(i \ln|\Delta|)$. The conjugate bundle $U^*(\mathbb{M})$ is obtained in a completely analogous way, by relying instead on the epimorphism $u^* = \exp(-i \ln|\Delta|)$. To the bundle $U(\mathbb{M})$, which enters some recent unified theory of gravitation and electromagnetism [Ferraris and Kijowski (1981)[19]], we shall give the name of unitary bundle of $\mathbb{M}$.

Example 4.4: Let us now consider the bundle $\mathbb{L}^2(\mathbb{M})$. We define a natural left action of $G^2(n;\mathbb{R})$ on the vector space $T^1_2(\mathbb{R}^n) = \mathbb{R}^n \otimes (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$ by the following explicit relation:

$$(g^i_j, g^i_{jk})(\Gamma^a_{bc}) = (g^i_a \Gamma^a_{bc} \bar{g}^b_j \bar{g}^c_k + g^i_a \bar{g}^a_{jk}), \tag{7}$$

where (g^i_j, g^i_{jk}) and Γ^a_{bc} are canonical coordinates in $G^2(n;\mathbb{R})$ and $T^1_2(\mathbb{R}^n)$, respectively, and $(\bar{g}^i_j, \bar{g}^i_{jk})$, denotes the inverse of (g^i_j, g^i_{jk}). The fiber bundle $C(\mathbb{M})$ associated with $\mathbb{L}^2(\mathbb{M})$ via the affine representation (7) above is an affine bundle of geometric objects, whose sections are easily recognized to be the linear connections over $\mathbb{M}$. For this reason the bundle $C(\mathbb{M})$ will be called the connection bundle of $\mathbb{M}$. It is easy to check that the vector bundle canonically associated with $C(\mathbb{M})$ is the tensor bundle $T^1_2(\mathbb{M})$.

Example 4.5: We can now define a further bundle by "taking the trace" of $C(\mathbb{M})$, namely by considering the following left action of $G^2(n;\mathbb{R})$ on $(\mathbb{R}^n)^*$:

$$(g^i_j, g^i_{jk})(A_a) = (A_a \bar{g}^a_j + g^k_a \bar{g}^a_{jk}), \tag{8}$$

where A_a are coordinates in $(\mathbb{R})^*$. The mapping (8) is obtained by taking a suitable trace in (7). It is easily seen that (8) is truly an action of $G^2(n;\mathbb{R}) \times (\mathbb{R}^n)^*$ into $(\mathbb{R}^n)^*$ and that it defines an affine bundle $D^*(\mathbb{M})$ over $\mathbb{M}$, which will be called the dilatation bundle of $\mathbb{M}$. This terminology is suggested by the fact that the sections of $D^*(\mathbb{M})$ are linear connections on the vector bundle $\det^*(\mathbb{M}) \simeq \Lambda^n(\mathbb{M})$, whose structure group is the group of dilatations in $\mathbb{R}^n$. We can easily realize that the vector bundle associated with $D^*(\mathbb{M})$ is the cotangent bundle $T^*\mathbb{M}$. There exists, of course, a dual construction, which gives a bundle $D(\mathbb{M})$ whose sections are connections on $\det(\mathbb{M})$.

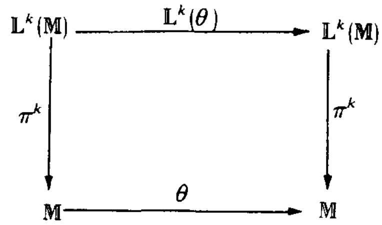
2. Other constructions involving the bundles of affine frames, projective frames, and spinor frames are currently under investigation and they will be the subject of further publication.

5. LIFT OF DIFFEOMORPHISMS AND LIE DERIVATIVES

In this last section we shall prove our main concern, i.e., we shall show that the construction presented above enables one to define in an intrinsic and canonical way the functorial lift of (local) diffeomorphisms of $\mathbb{M}$ to any bundle of geometric objects of type ρ and finite rank. This canonical lifting will provide more explicit formulas for the Lie derivative of a field of geometric objects.

Note added in proof: A more extended version, containing a detailed discussion of $U(\mathbb{M})$ bundles and their role in providing a possible characterization of the electric charge, will appear in J. Math. Pures Appl. Phys.

1. Let $k \geqslant 1$ be an integer. Let θ: $\mathbb{M} \to \mathbb{M}$ be a local diffeomorphism of $\mathbb{M}$. There exists a canonical lift $\mathbb{L}^k(\theta): \mathbb{L}^k(\mathbb{M}) \to \mathbb{L}^k(\mathbb{M})$ such that the following diagram is commutative:



and $\mathbb{L}^k(\theta)$ is a local diffeomorphism which commutes with the natural right action of $G^k(n;\mathbb{R})$ on $\mathbb{L}^k(\mathbb{M})$. In fact, the local diffeomorphism $\mathbb{L}^k(\theta)$ is defined by the following relation:

$$\mathbb{L}^k(\theta): j^k(h) \mapsto j^k(\theta \cdot h), \tag{9}$$

where $h: U(0) \subset \mathbb{R}^n \to \mathbb{M}$ is a local diffeomorphism.

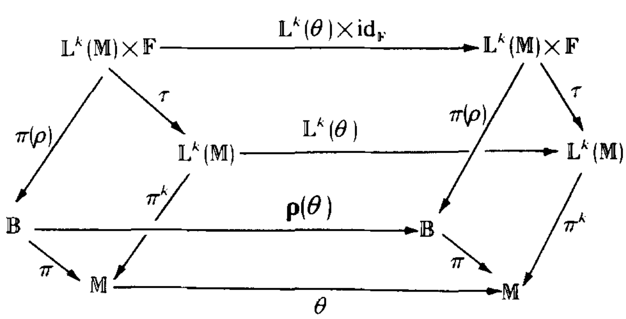
It is easy to prove that the lifting $\mathbb{L}^k: \theta \to \mathbb{L}^k(\theta)$ so defined satisfies the following properties:

$$\mathbb{L}^k(id_{\mathbb{M}}) = id_{\mathbb{L}^k(\mathbb{M})}, \tag{10}$$

$$\mathbb{L}^k(\theta_1 \cdot \theta_2) = \mathbb{L}^k(\theta_1) \cdot \mathbb{L}^k(\theta_2). \tag{11}$$

Therefore, $\mathbb{L}^k$ defines a (covariant) functor from the category of manifolds with local diffeomorphisms to the category of principle fiber bundles with principal fiber bundle morphisms.

2. The functorial construction above can be extended to any bundle of geometric objects of type ρ and finite rank k $(\mathbb{B}, \mathbb{M}, \pi; \mathbb{F}, \mathbb{G}, \rho)$ by the following procedure. First we remind the reader that, according to Sec. 3.5, the bundle $\mathbb{B}$ is associated with the principal bundle $\mathbb{L}^k(\mathbb{M})$ via the canonical projection $\pi(\rho)$: $\mathbb{L}^k(\mathbb{M}) \times \mathbb{F} \to \mathbb{B}$ defined by the group action ρ [in the sense of Example 2.5]. Let us denote by τ the projection of $\mathbb{L}^k(\mathbb{M}) \times \mathbb{F}$ onto the first factor $\mathbb{L}^k(\mathbb{M})$. Then there exists a local diffeomorphism $\rho(\theta): \mathbb{B} \to \mathbb{B}$ such that the following (three-dimensional) diagram is commutative:



In fact, $\rho(\theta)$ is defined by the following prescription:

$$\rho(\theta): \pi(\rho)\,[j^k h, f] \to \pi(\rho)[j^k(\theta \cdot h), f], \tag{12}$$

for any $(j^k h, f) \in \mathbb{L}^k(\mathbb{M}) \times \mathbb{F}$. The relation (12) is well defined, because $\mathbb{L}^k$ commutes with the group action of $G^k(n;\mathbb{R})$.

It is easy to show that (12) defines a local isomorphism of bundles $\rho(\theta): \mathbb{B} \to \mathbb{B}$ which, moreover, satisfies the required functorial properties:

$$\rho(id_{\mathbb{M}}) = id_{\mathbb{B}}, \tag{13}$$

$$\rho(\theta_1 \cdot \theta_2) = \rho(\theta_1) \cdot \rho(\theta_2). \tag{14}$$

Therefore, setting $\mathbb{B} = \rho(\mathbb{M})$ we have a covariant functor ρ from the category of manifolds with local diffeomorphisms to the category of bundles of geometric objects of finite rank with local bundle-isomorphisms. It is obvious that in the particular case $\mathbb{F} = G^k(n;\mathbb{R})$ and $\rho = id_{G^k(n;\mathbb{R})}$ the functor ρ

reduces to the functor $\mathbb{L}^k$.

It is straightforward to prove that the covariant functor $\boldsymbol{\rho}$ defined above satisfies all the required properties in order to make $(\mathbb{B}, \mathbb{M}, \boldsymbol{\rho})$ a bundle of geometric objects in the sense of Salvioli.

3. In order to define the Lie derivative of a field of geometric objects along a vector field X on $\mathbb{M}$, we may now apply the standard procedure described in Salvioli (1972), making explicit the functor $\boldsymbol{\rho}$.

Then let θ_t be the local 1-parameter group of diffeomorphisms generated by a vector field X on $\mathbb{M}$ and let $\beta:\mathbb{M}\to\mathbb{B}$ be a (local) section of a bundle of geometric objects $(\mathbb{B},\mathbb{M},\pi;\mathbb{F},\mathbb{G},\rho)$ of finite rank $k\geqslant 1$. The following relation,

$$\beta_t: x\in\mathbb{M}\to\boldsymbol{\rho}(\theta_t)^{-1}[\beta\cdot\theta_t(x)]\in\pi^{-1}(x), \tag{15}$$

defines a one-parameter family of local sections of $\mathbb{B}$. Accordingly, we may define the Lie derivative of the (local) field of geometric objects β as follows:

$$L_X\beta: x\in\mathbb{M} \to \frac{d}{dt}[\beta_t(x)]\Big|_{t=0}. \tag{16}$$

It is easy to check that $L_X\beta$ defines a (local) field of vertical vectors over β, i.e., the following conditions hold;

(i) $\pi_{\mathbf{B}}\cdot(L_X\beta) = \beta$,

where $\pi_{\mathbf{B}}:\mathbb{TB}\to\mathbb{B}$ is the canonical projection;

(ii) $[\mathbb{T}\pi\cdot(L_X\beta)](x) = x, \quad \forall x\in\mathbb{M}$,

where $\mathbb{T}\pi:\mathbb{TB}\to\mathbb{TM}$ is the tangent map of the bundle projection π.

For further properties of Lie derivatives of geometric objects we refer the reader to Salvioli (1972) or Yano (1955).[20]

Note added in proof. A more extended version, containing a detailed discussion of U(M) bundles and their role in providing a possible characterization of the electric charge, will appear in Annales Inst. H. Poincaré.

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