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Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows

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Abstract

Aim of this paper is to discuss convergence of pointed metric measure spaces in absence of any compactness condition. We propose various definitions, show that all of them are equivalent and that for doubling spaces these are also equivalent to the well known measured-Gromov-Hausdorff convergence.

Then we show that the curvature conditions $CD(K, \infty)$ and $RCD(K, \infty)$ are stable under this notion of convergence and that the heat flow passes to the limit as well, both in the Wasserstein and in the L^2-framework. We also prove the variational convergence of Cheeger energies in the naturally adapted Γ-Mosco sense and the convergence of the spectra of the Laplacian in the case of spaces either uniformly bounded or satisfying the $RCD(K, \infty)$ condition with $K > 0$. When applied to Riemannian manifolds, our results allow for sequences with diverging dimensions.

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1 Introduction

The notion of convergence of metric measure spaces has been introduced by Fukaya in [23] as the natural variant of Gromov-Hausdorff convergence for metric structures endowed with a reference measure. Fukaya's interest was to study the behaviour of eigenvalues of the Laplacian on smooth Riemannian manifolds with some uniform curvature bound under a zeroth-order convergence: it turned out that the purely metric notion of Gromov-Hausdorff convergence was not enough to obtain reasonable results, and the additional requirement of convergence of volume measures was needed.

Since then the notion has been widely used, in particular in connection with lower Ricci curvature bounds, and proved to be useful to handle also non smooth limits of Riemannian manifolds (see for instance the series of papers by Cheeger-Colding [16, 17, 18] and the more recent developments by Colding-Minicozzi [20] and Colding-Naber [21]).

Almost ten years ago, Lott-Villani on one side [33] and Sturm on the other [42], [43] introduced an abstract notion of lower Ricci curvature bound on metric measure setting. The notion introduced is that of CD(K, N) space, meaning of a space with Ricci Curvature bounded from below by $K \in \mathbb{R}$ and Dimension bounded above by $N \in [1, \infty]$ (in [33], for $N < \infty$ only the case $K = 0$ was considered). A key feature of their definition is that it is stable w.r.t. convergence of the spaces. From the technical point of view, there are some

differences between the presentations of such stability result in the two approaches, although on compact and doubling metric measure spaces they produce the same convergence.

- Lott and Villani worked with proper (i.e. bounded closed sets are compact) pointed metric measure spaces, and showed that the $CD(K, N)$ condition is stable under pointed measured Gromov-Hausdorff convergence. This means, roughly said, that for any $R > 0$ there is measured Gromov-Hausdorff convergence of balls of radius R around the given points of the spaces (see Definition 3.24 for the precise statement).

- Sturm worked with normalized metric measure spaces with finite variance, i.e. spaces such that the reference measure is a probability measure with finite second moment. He then defined a distance $\mathbb{D}$ on this class of spaces putting

$$\mathbb{D}\Big((X_1, \mathsf{d}_1, \mathfrak{m}_1), (X_2, \mathsf{d}_2, \mathfrak{m}_2)\Big) := \inf W_2\big((\iota_1)_\sharp \mathfrak{m}_1, (\iota_2)_\sharp \mathfrak{m}_2\big),$$

the infimum being taken among all metric spaces (X, d) and all isometric embeddings $\iota_1 : (\mathrm{supp}(\mathfrak{m}_1), \mathsf{d}_1) \to (X, \mathsf{d})$ and $\iota_2 : (\mathrm{supp}(\mathfrak{m}_2), \mathsf{d}_2) \to (X, \mathsf{d})$. He then proved that Curvature-Dimension bounds are stable w.r.t. $\mathbb{D}$-convergence.

Given that $CD(K, N)$ spaces with $N < \infty$ are always proper, the approach of Lott-Villani fully covers this situation. On the other hand, the $CD(K, \infty)$ condition does not imply any sort of compactness, not even local, and hence to work with (pointed) measured Gromov-Hausdorff convergence is quite unnatural in this case. At least for spaces with finite variance, Sturm's $\mathbb{D}$ distance is suitable to discuss stability of these curvature bounds, as no compactness is required in order to work with it.

New problems arise when one considers sequences of spaces with non uniformly bounded measures and possibly diverging dimensions or simply deals with measures whose locally doubling constants are not uniformly bounded (as, e.g., in the infinite dimensional setting of [11, 22, 34]). This happens when one needs to consider σ-finite measures in the investigations of *blow ups* (i.e. tangent cones at a point) and *blow downs* (i.e. tangent cones at infinity), questions that have been addressed for instance in the aforementioned papers of Cheeger-Colding and in the article of Perelman [38] in the framework of measured Gromov-Hausdorff limits of Riemannian manifolds with uniform Ricci lower bound (let us also quote the work of Cheeger-Tian [19] and the aforementioned paper of Colding-Minicozzi regarding the asymptotic cone of Einstein manifolds).

Besides the issue of stability of the $CD(K, \infty)$ condition, other relevant questions have been recently considered in the framework of general metric measure spaces: they concern the stability of the heat flow, the variational convergence of the so-called Cheeger energy [7] and the stability of its quadratic behaviour, that is strictly linked with the infinitesimal Hilbertian structure of the ambient space [25]: all these aspects are intimately connected with the more restrictive notion of metric measure spaces with Riemannian Ricci curvature bounded from below, thus satisfying the so-called $RCD(K, \infty)$ condition (introduced in [9] and further analyzed in [10, 4, 27]). The class of $RCD(K, \infty)$ spaces has been proposed to somehow characterize the 'Riemannian-like' structures: it is the subclass of $CD(K, \infty)$ spaces where the Cheeger energy is a quadratic Dirichlet form so that the heat flow linearly depends on the initial data.

A quick overview

The present paper provides the tools to address all these questions in great generality.

Our aim is dual: on one side we propose a notion of convergence (that we call *pointed measured Gromov convergence*, pmG-convergence in short) of metric measure structures which works without any compactness assumptions on the metric structure and for possibly σ-finite measures, on the other we prove that lower Ricci bounds and the above mentioned constructions are stable w.r.t. this convergence.

The structures we shall work with are isomorphism classes of *pointed metric measure* (p.m.m.) spaces $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$, where

(X, d) is a complete and separable metric space, $\mathfrak{m}$ is a nonnegative and nonzero
Borel measure which is finite on bounded sets, and $\bar{x}$ is a point in $\mathrm{supp}(\mathfrak{m})$,

and $(X_1, \mathsf{d}_1, \mathfrak{m}_1, \bar{x}_1)$ is said isomorphic to $(X_2, \mathsf{d}_2, \mathfrak{m}_2, \bar{x}_2)$ provided there exists

an isometric embedding $\quad \iota : \mathrm{supp}(\mathfrak{m}_1) \to X_2$ such that $\iota_\sharp \mathfrak{m}_1 = \mathfrak{m}_2$ and $\iota(\bar{x}_1) = \bar{x}_2$.

The easiest (and most powerful) way to introduce the notion of pointed measured Gromov convergence (pmG in short) uses an extrinsic family of isometric imbeddings (see the more detailed discussions at the end of the Introduction and in Section 3): we say that a sequence of (isomorphism classes of) p.m.m. spaces $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ converges to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmG sense if there exist a complete and separable space (X, d) and isometric embeddings $\iota_n : X_n \to X$, $n \in \mathbb{N} \cup \{\infty\}$, in such a way that

$$\int \varphi \, \mathrm{d}(\iota_n)_\sharp \mathfrak{m}_n \longrightarrow \int \varphi \, \mathrm{d}(\iota_\infty)_\sharp \mathfrak{m}_\infty \quad \text{for every } \varphi \in \mathrm{C}_{bs}(X),$$

where $\mathrm{C}_{bs}(X)$ denotes the space of functions $\varphi : X \to \mathbb{R}$ which are continuous, bounded, and have bounded support.

Our main results are

i) to show that pmG convergence can be characterized in an intrinsic way, according to various equivalent approaches (Section 3.2 and Theorem 3.15),

ii) to prove that on doubling spaces, pmG-convergence is the same as pointed measured Gromov-Hausdorff convergence (Propositions 3.30 and 3.33), see also below for a brief description of these notions,

iii) to show that the curvature condition $\mathrm{CD}(K, \infty)$ is stable w.r.t. pmG-convergence (Theorem 4.9).

On top of this, we also prove that

iv) the heat flow, defined as the Wasserstein gradient flow of the relative entropy or as the L^2-gradient flow of the Cheeger energy, is stable w.r.t. pmG-convergence (Theorems 5.7 and 6.11),

v) the Cheeger energies of pmG-converging p.m.m. spaces are stable with respect to (a suitably adapted notion of) Mosco-convergence (Theorem 6.8).

The points iv) and v) above are particularly important when considering the class of RCD(K,∞) spaces, and it turns out that

vi) the RCD(K,∞) condition is also stable w.r.t. pmG-convergence (Theorem 7.2),

vii) in this case finer convergence properties of the heat flow can be proved (Theorem 7.7),

viii) whenever spaces satisfying a uniform weak Logarithmic-Sobolev-Talagrand inequality are considered (e.g. when $K > 0$ or the diameters are uniformly bounded), the spectrum of the (linear) Laplace operator is also stable (Theorem 7.8).

We devote the rest of this introduction to describe the main contributions in more detail and to state the most important theorems of the paper.

Stability of the Lott-Sturm-Villani curvature condition w.r.t. pmG-convergence.

Recall that a p.m.m. space $(X,\mathsf{d},\mathfrak{m},\bar{x})$ satisfies the Lott-Sturm-Villani CD(K,∞) curvature condition if the relative entropy functional $\mathrm{Ent}_\mathfrak{m} : \mathscr{P}_2(X) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\mathrm{Ent}_\mathfrak{m}(\mu) := \begin{cases} \displaystyle\int \rho \log \rho \, \mathrm{d}\mathfrak{m} & \text{if } \mu = \rho\mathfrak{m} \ll \mathfrak{m}, \quad \rho \log \rho \in L^1(X, \mathfrak{m}), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

is geodesically K-convex: here $\mathscr{P}_2(X)$ is the space of Borel probability measures with finite second moment endowed with the Kantorovich-Rubinstein-Wasserstein quadratic transportation distance W_2 (see Subsections 2.2 and 4.1.1 for more details). The set of μ's such that the entropy is finite is denoted by $D(\mathrm{Ent}_\mathfrak{m})$.

We will show that the curvature condition CD(K,∞) is stable w.r.t. pmG-convergence (**Theorem 4.9**):

Theorem I. *Let* $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, *be a sequence of* CD(K,∞) *p.m.m. spaces converging to* $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *in the* pmG-*convergence. Then* $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *is a* CD(K,∞) *space as well.* ∎

Stability of the Heat flows and Mosco-convergence of the Cheeger energies.

In a general p.m.m. space $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ there are (at least) two variational ways to introduce the heat flow. Adopting the point of view of probability measures and following the approach introduced in [30] and further developed by [5], one can consider the metric gradient flow à la De Giorgi of the relative entropy functional $\mathrm{Ent}_\mathfrak{m}$ in $(\mathscr{P}_2(X), W_2)$.

According to this notion, a curve $\mu : [0,\infty) \to D(\mathrm{Ent}_\mathfrak{m}) \subset \mathscr{P}_2(X)$ is the W_2-gradient flow of $\mathrm{Ent}_\mathfrak{m}$ starting from $\bar{\mu}$ provided it is locally absolutely continuous in $(\mathscr{P}_2(X), W_2)$, and

$$\mathrm{Ent}_\mathfrak{m}(\mu_t) = \mathrm{Ent}_\mathfrak{m}(\mu_s) + \frac{1}{2}\int_t^s |\dot{\mu}_r|^2 \, \mathrm{d}r + \frac{1}{2}\int_t^s |\mathrm{D}^-\mathrm{Ent}_\mathfrak{m}|^2(\mu_r) \, \mathrm{d}r \quad \forall 0 < t < s, \tag{1.2}$$

where $|\dot{\mu}_t|$ denotes the metric speed of the curve μ and $|\mathrm{D}^-\mathrm{Ent}_\mathfrak{m}| : \mathscr{P}_2(X) \to [0, +\infty]$ is the (descending) slope of the relative entropy $\mathrm{Ent}_\mathfrak{m}$. It is defined identically $+\infty$ outside $D(\mathrm{Ent}_\mathfrak{m})$, 0 at isolated measures in $D(\mathrm{Ent}_\mathfrak{m})$ and in all other cases as

$$|\mathrm{D}^-\mathrm{Ent}_\mathfrak{m}|(\mu) := \varlimsup_{W_2(\nu,\mu) \to 0} \frac{\left(\mathrm{Ent}_\mathfrak{m}(\mu) - \mathrm{Ent}_\mathfrak{m}(\nu)\right)^+}{W_2(\mu,\nu)}.$$

In a $\mathrm{CD}(K, \infty)$ space it is possibile to prove [7] that for every $\bar{\mu} \in D(\mathrm{Ent}_\mathfrak{m})$ there exists a unique solution $\mu_t = \mathscr{H}_t \bar{\mu}$ of (1.2) satisfying the initial condition $\mu_0 = \bar{\mu}$. Moreover, whenever $\bar{\mu} = f\mathfrak{m}$ with $f \in L^2(X, \mathfrak{m})$, one has

$$\mathscr{H}_t \mu = (\mathsf{H}_t f)\mathfrak{m}$$

where $(\mathsf{H}_t)_{t \geq 0}$ is the contraction semigroup in $L^2(X, \mathfrak{m})$ generated as the L^2-gradient flow (see Subsection 5.1.4) of the convex and lower semicontinuous Cheeger energy

$$\mathsf{Ch}(f) := \frac{1}{2} \int_X |\mathrm{D}f|_w^2 \, \mathrm{d}\mathfrak{m};$$

we refer to Section 4.1 for the precise definition of Sobolev spaces and of the weak upper gradient $|\mathrm{D}f|_w$ in a metric-measure framework.

It is then natural to investigate the asymptotic behaviour of the heat flows (both in the Wasserstein and L^2 sense) along a pmG-convergent sequence $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, of $\mathrm{CD}(K, \infty)$ p.m.m. spaces, where we set $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. This is possible by adopting the extrinsic point of view of pmG convergence, thus assuming that (X_n, d_n) are isometrically imbedded in a common metric space (X, d). In this way the convergence of the p.m.m. spaces is reduced to the weak convergence of the measures $\mathfrak{m}_n$ and it is also possible to define a weak/strong L^p convergence for real valued functions $f_n \in L^p(X, \mathfrak{m}_n)$, see Definition 6.1.

We will thus prove the following stability result (**Theorems 5.7 and 6.11**).

Theorem II. *Let $(X, \mathsf{d}, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, be a sequence of $\mathrm{CD}(K, \infty)$ spaces, let $\mathfrak{m}_n$ weakly converge to $\mathfrak{m}_\infty$*

in the duality with the bounded and continuous functions with bounded support of $\mathrm{C}_{bs}(X)$, and let $(\mathscr{H}_t^n)_{t \geq 0}$ (resp. $(\mathsf{H}_t^n)_{t \geq 0}$), $n \in \bar{\mathbb{N}}$, be the W_2 gradient flows of $\mathrm{Ent}_{\mathfrak{m}_n}$ in $\mathscr{P}_2(X)$ (resp. the $L^2(X, \mathfrak{m}_n)$-gradient flow of the Cheeger energy in $(X, \mathsf{d}, \mathfrak{m}_n)$).

If $(\bar{\mu}_n) \subset \mathscr{P}_2(X)$ is a sequence satisfying

$$\bar{\mu}_n \xrightarrow{W_2} \bar{\mu}_\infty, \qquad \mathrm{Ent}_{\mathfrak{m}_n}(\bar{\mu}_n) \to \mathrm{Ent}_{\mathfrak{m}_\infty}(\bar{\mu}_\infty) < \infty \qquad \text{as } n \to \infty, \tag{1.3}$$

then for every $t > 0$

$$\mathscr{H}_t^n \bar{\mu}_n \xrightarrow{W_2} \mathscr{H}_t^\infty \bar{\mu}_\infty, \qquad \mathrm{Ent}_{\mathfrak{m}_n}(\mathscr{H}_t^n \bar{\mu}_n) \longrightarrow \mathrm{Ent}_{\mathfrak{m}_\infty}(\mathscr{H}_t^n \mu_\infty) \qquad \text{as } n \to \infty. \tag{1.4}$$

If $\bar{f}_n \in L^2(X, \mathfrak{m}_n)$, $n \in \mathbb{N}$, L^2-strongly converges to $\bar{f}_\infty \in L^2(X, \mathfrak{m}_\infty)$, then

$$\mathsf{H}_t^n \bar{f}_n \xrightarrow{L^2} \mathsf{H}_t^\infty \bar{f}_\infty \quad \text{strongly for every } t \geq 0. \tag{1.5}$$

The L^2-convergence of the heat semigroups $(\mathsf{H}_t^n)_{t \geq 0}$ in the Theorem II above is strictly related to (a suitably adapted notion of) the Mosco-convergence for the corresponding Cheeger energies Ch_n (**Theorem 6.8**):

Theorem III. *Under the same assumptions of the previous Theorem, let us denote by Ch_n the Cheeger energy induced by $(X, \mathsf{d}, \mathfrak{m}_n)$. Then the following holds:*

- *Weak $\Gamma - \liminf$. For every sequence $n \mapsto f_n \in L^2(X_n, \mathfrak{m}_n)$ L^2-weakly converging to some $f_\infty \in L^2(X_\infty, \mathfrak{m}_\infty)$ we have*

$$\varliminf_{n \to \infty} \mathsf{Ch}_n(f_n) \geq \mathsf{Ch}_\infty(f_\infty).$$

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- *Strong* $\Gamma - \limsup$. *For every* $f_\infty \in L^2(X_\infty, \mathfrak{m}_\infty)$ *there exists a sequence* $n \mapsto f_n \in L^2(X_n, \mathfrak{m}_n)$ L^2-*strongly convergent to* f_∞ *such that*

$$\lim_{n \to \infty} \mathsf{Ch}_n(f_n) = \mathsf{Ch}_\infty(f_\infty).$$

The proof of some of these results would be easier under additional compactness assumptions (see e.g. [24] for the proof of the stability of the gradient flow of the entropy in the compact case), that also yields the finiteness of the quadratic moment (and thus of the mass) of the reference measure $\mathfrak{m}$. The crucial lower semicontinuity properties of the entropy functionals simplify a lot in this setting. Part of the added value of this paper is precisely to show that it is possible to work out the severe technical obstructions one encounters in working without any a priori compactness and to extend the stability issues from the Wasserstein to the L^2-setting.

Spaces with Riemannian Ricci curvature bounded from below: stability of the $\mathrm{RCD}(K, \infty)$ condition and spectral convergence.

The above results have particularly important and enhanced applications in the case of $\mathrm{RCD}(K, \infty)$ p.m.m. spaces, whose Riemannian Ricci curvature is bounded from below by K ([9, 4]).

As we already mentioned, the class of $\mathrm{RCD}(K, \infty)$ spaces is the subclass of $\mathrm{CD}(K, \infty)$ spaces where the Cheeger energy is a quadratic Dirichlet form,

i.e. there exists a symmetric bilinear form $\mathcal{E} : D(\mathsf{Ch}) \to \mathbb{R}$ such that $\mathsf{Ch}(f) = \frac{1}{2}\mathcal{E}(f, f)$ for every $f \in D(\mathsf{Ch})$.

It turns out that also this notion is stable w.r.t. pmG-convergence.

(**Theorem 7.2**):

Theorem IV. *Let* $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, *be a sequence of* $\mathrm{RCD}(K, \infty)$ *p.m.m. spaces converging to a limit space* $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *in the* pmG-*sense. Then* $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *is* $\mathrm{RCD}(K, \infty)$ *as well.*

In this case finer convergence properties of the Heat flow can be proved, obtaining a stability result analogous to Theorem II under weak convergence of the initial data (i.e. the assumption of the convergence of the Entropy in (1.3) can be avoided, see **Theorem 7.7**).

Whenever the spaces satisfy a uniform weak Logarithmic-Sobolev-Talagrand inequality (e.g. when $K > 0$ or the diameters are uniformly bounded; see Definition 6.4), the (linear) Laplace operator Δ_n, defined in terms of the (quadratic) Cheeger bilinear form $\mathcal{E}_n$:

$$- \int \Delta_n f \, g \, \mathrm{d}\mathfrak{m}_n = \mathcal{E}_n(f, g) \quad \forall f \in D(\Delta_n) \subset D(\mathsf{Ch}), g \in D(\mathsf{Ch}),$$

has a compact resolvent and its discrete spectrum can be ordered in nonnegative monotonic sequence $0 \leq \lambda_1(\Delta_n) \leq \lambda_2(\Delta_n) \leq \cdots \leq \lambda_k(\Delta_n) \leq \cdots$, with possibile repetitions according to the given multiplicities. It turns out that the spectra of Δ_n are also stable under pmG convergence (**Theorem 7.8**):

Theorem V. *Let* $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, *be a* pmG-*converging sequence of* $\mathrm{RCD}(K, \infty)$ *spaces. Assume that for some* $a, b \geq 0$, *the spaces satisfy the* wLSTI(A, B) *for every* $n \in \mathbb{N}$ *(in particular if* $K > 0$ *or* $\mathrm{diam}\big(\mathrm{supp}(\mathfrak{m}_n)\big)$ *is uniformly bounded). Then*

$$\lim_{n \to \infty} \lambda_k(\Delta_n) = \lambda_k(\Delta_\infty) \quad \text{for every } k \in \mathbb{N}.$$

As for the stability of the Heat flow, we stress that these results are new even in the case of non compact proper spaces endowed with σ-finite measures pointed measured Gromov Hausdorff converging to a limit space (in particular the reader may think of a sequence of non compact Riemannian n-manifolds satisfying a uniform lower bound on the Ricci curvature, more generally we allow also the sequence not to have dimension bounds).

We conclude the introduction by briefly describing the different approaches to the pmG convergence and the relations with the pointed measured Gromov Hausdorff convergence.

Pointed measured Gromov convergence: four possible approaches and their equivalence.

There are (at least) four different ways to introduce a convergence notion for p.m.m. spaces (for a different approach in the case of counting $\mathscr{H}^N$-rectifiable spaces, see [41]);

here we briefly recall the main ideas, for the details the reader is referred to Section 3.2 and in particular to the equivalence Theorem 3.15, showing that *all the approaches lead to equivalent definitions*, which thereby characterize the pointed measured Gromov convergence. **(A)** Intrinsic approach: use of Gromov reconstruction theorem. Given an integer $N \in \mathbb{N}$ and a continuous function with bounded support $\varphi : \mathbb{R}^{N \times N} \to \mathbb{R}$ we can consider the "cylindrical" function on the isomorphism classes of p.m.m. spaces $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ defined as

$$\varphi[X, \mathsf{d}, \mathfrak{m}, \bar{x}] := \int_{X^N} \varphi\big((\mathsf{d}(x_i, x_j))_{ij=1}^N\big) \, \mathrm{d}\delta_{\bar{x}}(x_1) \, \mathrm{d}\mathfrak{m}^{\otimes(N-1)}(x_2, \cdots, x_N).$$

It is clear from the above definition that $\varphi[X, \mathsf{d}, \mathfrak{m}, \bar{x}]$ only depends on the isomorphism class induced by $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$. Adapting a result due to Gromov for the case of spaces with finite mass, we shall see that the knowledge of all cylindrical functions in fact characterizes this class, since $(X_1, \mathsf{d}_1, \mathfrak{m}_1, \bar{x}_1)$ and $(X_2, \mathsf{d}_2, \mathfrak{m}_2, \bar{x}_2)$ are isomorphic if and only if

$$\varphi[X_1, \mathsf{d}_1, \mathfrak{m}_1, \bar{x}_1] = \varphi[X_2, \mathsf{d}_2, \mathfrak{m}_2, \bar{x}_2],$$

for any $N \in \mathbb{N}$ and φ as above, see Proposition 3.7.

This result suggests to declare that the sequence $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ intrinsically converges to $(X_\infty, \mathfrak{m}_\infty, \mathsf{d}_\infty, \bar{x}_\infty)$ provided

$$\varphi[X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n] \longrightarrow \varphi[X_\infty, \mathfrak{m}_\infty, \mathsf{d}_\infty, \bar{x}_\infty],$$

as $n \to \infty$ for every $N \in \mathbb{N}$ and φ as above (Definition 3.8).
(B) Extrinsic approach: embedding everything in a common space.
This is the point of view we have quickly recalled at the beginning of this discussion. We say that $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ converges to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ if there exists a complete and separable space (X, d) and isometric embeddings $\iota_n : X_n \to X$, $n \in \bar{\mathbb{N}}$, in such a way that

$$\int \varphi \, \mathrm{d}(\iota_n)_\sharp \mathfrak{m}_n \longrightarrow \int \varphi \, \mathrm{d}(\iota_\infty)_\sharp \mathfrak{m}_\infty \quad \text{for every } \varphi \in \mathrm{C}_{bs}(X).$$

We shall call this notion extrinsic convergence, as it relies on the additional data (X, d) and (ι_n) (Definition 3.9). Notice that a priori it is not clear that this definition is well-posed at the level of isomorphism classes, given that the ι_n's are required to be defined on the whole X_n's rather than on $\mathrm{supp}(\mathfrak{m}_n)$. Yet, in practice this is not an issue: in the simple Proposition

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3.10 we shall see that the domain of an isometry can always be extended provided we take the freedom of extending also the target space.

(C) <u>A variant of Sturm's distance $\mathbb{D}$.</u> As previously mentioned, in [42] Sturm worked with metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ such that $\mathfrak{m} \in \mathscr{P}_2(X)$ and introduced a distance $\mathbb{D}$ between their isomorphism classes whose relation to the quadratic transportation distance W_2 is essentially the same that the Gromov-Hausdorff distance has to the Hausdorff distance: he posed

$$\mathbb{D}\big((X_1, \mathsf{d}_1, \mathfrak{m}_1), (X_2, \mathsf{d}_2, \mathfrak{m}_2)\big) := \inf W_2((\iota_1)_\sharp \mathfrak{m}_1, (\iota_2)_\sharp \mathfrak{m}_2),$$

the infimum being taken among all spaces (X, d) and isometric embeddings $\iota_i : X_i \to X$, $i = 1, 2$.

If one is working with p.m.m. spaces with controlled volume growth, it is possible to adapt the definition of $\mathbb{D}$ in the following way. Say that $\psi : [0, \infty) \to (0, \infty)$ is a given continuous non-increasing map and denote by $\mathbb{X}^\psi$ the set of equivalence classes of spaces $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ such that

$$\int (1 + \mathsf{d}^2(\cdot, \bar{x})) \psi(\mathsf{d}(\cdot, \bar{x})) \, \mathrm{d}\mathfrak{m} < \infty.$$

Then we can define the distance $\mathbb{D}^\psi$ between $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathbb{X}^\psi$ as

$$\mathbb{D}^\psi\big(\mathfrak{X}_1, \mathfrak{X}_2\big) := \Big| \log \frac{\mathsf{z}_{1,\psi}}{\mathsf{z}_{2,\psi}} \Big| + \inf W_2((\iota_1)_\sharp \mathfrak{m}_{1,\psi}, (\iota_2)_\sharp \mathfrak{m}_{2,\psi}) + \mathsf{d}(\iota_1(\bar{x}_1), \iota_2(\bar{x}_2)),$$

where $\mathfrak{X}_i = [X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i]$ and

$$\mathsf{z}_{i,\psi} := \int \psi(\mathsf{d}_i(\cdot, \bar{x}_i)) \, \mathrm{d}\mathfrak{m}_i, \qquad \text{and} \qquad \mathfrak{m}_{i,\psi} := \frac{1}{\mathsf{z}_{i,\psi}} \psi(\mathsf{d}_i(\cdot, \bar{x}_i)) \, \mathrm{d}\mathfrak{m}_i,$$

for $i = 1, 2$ and the inf is taken among all spaces (X, d) and isometric embeddings $\iota_i : X_i \to X$, $i = 1, 2$, see Definition 3.11.

(D) <u>A metric independent of chosen weights: the pointed Gromov-Wasserstein distance</u>

In order to introduce a metric independent of a given weight we shall proceed as follows. First we fix a non-constant bounded concave function $c : [0, \infty) \to [0, \infty)$ with $c(0) = 0$ and notice that given a metric space (X, d), the map $(x, y) \mapsto c(\mathsf{d}(x, y))$ is a bounded distance inducing the same topology. Then we introduce the 1-transportation distance W_c built on top of it:

$$W_c(\mu, \nu) := \inf \int c(\mathsf{d}(x, y)) \, \mathrm{d}\boldsymbol{\gamma}(x, y), \qquad \text{the inf being taken among all transport plans } \boldsymbol{\gamma}.$$

Now for given (equivalence classes of) p.m.m. spaces $\mathfrak{X}_i = [X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i]$, $i = 1, 2$, with finite mass we can define the distance

$$\mathrm{p}\mathbb{G}_W^{fm}(\mathfrak{X}_1, \mathfrak{X}_2) := \Big| \log \frac{\mathfrak{m}_1(X_1)}{\mathfrak{m}_2(X_2)} \Big| + \inf \mathsf{d}(\iota_1(\bar{x}_1), \iota_2(\bar{x}_2)) + W_c((\iota_1)_\sharp \tilde{\mathfrak{m}}_1, (\iota_2)_\sharp \tilde{\mathfrak{m}}_2),$$

where $\tilde{\mathfrak{m}}_i := \mathfrak{m}_i(X_i)^{-1} \mathfrak{m}_i$, $i = 1, 2$, and the inf is taken as before among all spaces (X, d) and all isometric embeddings $\iota_i : X_i \to X$.

For the general case of p.m.m. spaces of possibly infinite mass we adopt a cut-off argument based on the fact that, by assumption, in our spaces bounded sets have finite mass. Hence

we fix a Lipschitz compactly supported function $\zeta : [0, \infty) \to [0, 1]$ with $\zeta \equiv 1$ on $[0, 1]$ and then for given $\mathcal{X} = (X, \mathsf{d}, \mathfrak{m}, \bar{x})$ and $k \in \mathbb{Z}$ we put

$$\mathrm{d}\mathfrak{m}_{[k]}(x) := \zeta(\mathsf{d}(x, \bar{x})2^{-k})\mathrm{d}\mathfrak{m}(x), \qquad \text{and} \qquad \mathcal{X}_{[k]} := (X, \mathsf{d}, \mathfrak{m}_{[k]}, \bar{x}),$$

and finally define the pointed-Gromov-weak distance as

$$\mathrm{p}\mathbb{G}_{\mathrm{W}}(\mathcal{X}_1, \mathcal{X}_2) := \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \min\left\{1, \mathrm{p}\mathbb{G}_{\mathrm{W}}^{fm}(\mathcal{X}_{1,[k]}, \mathcal{X}_{2,[k]})\right\},$$

see Definition 3.13. Notice that the choice to let k vary in $\mathbb{Z}$ rather than in $\mathbb{N}$ is made to get completeness of the distance $\mathrm{p}\mathbb{G}_{\mathrm{W}}$, see Theorem 3.17.

We shall see in **Theorem 3.15** that the intrinsic convergence is equivalent to the extrinsic one and they also coincide with convergence w.r.t. $\mathrm{p}\mathbb{G}_{\mathrm{W}}$ and $\mathbb{D}^\psi$ for ψ decreasing sufficiently fast (see also Remarks 3.19 and 3.20). As said, we shall call this convergence *pointed measured Gromov* convergence, pmG-convergence in short.

The link with pointed measured Gromov-Hausdorff convergence.

In Section 3.5, we shall analyze the relation between pmG-convergence and pointed measured Gromov Hausdorff one (pmGH for short). For compact spaces, the latter means that we can embed the given sequence of spaces in a common one to ensure pointed Gromov Hausdorff convergence at the level of metrics and weak convergence at the level of measures. The difference with pmG-convergence is thus in the fact that once the isometric embeddings are made, for pmGH-convergence both Hausdorff convergence of the metrics and weak convergence of measures are asked, while for pmG-convergence only weak convergence of measures is imposed. This is why we dropped the word 'Hausdorff' in denoting the notion of convergence we are proposing.

Perhaps the best way to see the differences between pmG and pmGH convergences is with an example. Thus consider the sequence $n \mapsto ([0, 1], \mathsf{d}_{\mathrm{Eucl}}, \mathfrak{m}_n, \frac{1}{2})$, where $\mathsf{d}_{\mathrm{Eucl}}$ is the standard metric on $[0, 1]$ and $\mathfrak{m}_n := \frac{1}{n}\mathcal{L}^1|_{[0,1]} + (1 - \frac{1}{n})\delta_{\frac{1}{2}}$. As $n \to \infty$ this sequence pmG-converges to the (isomorphism class of a) one-point space equipped with a Dirac delta. Yet this is not the case for pmGH-convergence as certainly $n \mapsto ([0, 1], \mathsf{d}_{\mathrm{Eucl}})$ does not converge to a one-point space in the GH sense. What one could say in this situation is that the given sequence converges to $([0, 1], \mathsf{d}_{\mathrm{Eucl}}, \delta_{\frac{1}{2}}, \frac{1}{2})$ (which is the approach we are going to choose in Section 3.5) but the drawback of this choice is that the notion of convergence would not be defined for isomorphism classes of spaces, but would rather depend on the portion of space which is outside the support of the measure.

We therefore see that while pmGH-convergence always implies pmG-convergence (see **Theorem 3.30**), the converse is not always true. Yet, at least in the class of doubling spaces a behaviour like that of the example proposed cannot occur and we shall see that in this case the two notions of convergence actually coincide (**Theorem 3.33**). In other words we will prove the following result.

Theorem VI. *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, be a sequence of p.m.m. spaces. If it is converging in the pmGH-sense then $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ w.r.t. pmG-convergence.*

Conversely, if $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ w.r.t. pmG-convergence, $\mathrm{supp}(\mathfrak{m}_\infty) = X_\infty$, and all the spaces $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ are c-doubling for some $\mathsf{c} > 0$ independent of n, then $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmGH sense.

2 Notation and preliminary results

2.1 Measures and weak/narrow convergence.

We assume that the reader is familiar with the basic notions of measure theory and optimal transport on abstract metric spaces (standard references are [3], [5] and [45]). Here we recall the basic definitions mainly in order to fix the notation.

Given a complete and separable metric space (X, d), we denote by $\mathscr{B}(X)$ the collection of its Borel sets, by $\mathscr{M}_{loc}(X)$ the set of Borel measures $\mu : \mathscr{B}(X) \to [0, \infty]$ which are finite on every bounded set, by $\mathscr{M}(X)$ the subset of finite measures μ, and by $\mathscr{P}(X)$ the collection of all Borel probability measures.

On $\mathscr{M}_{loc}(X)$ we consider the topology induced by the duality with $C_{bs}(X)$, the space of bounded continuous functions with *bounded support*: a sequence $(\mu_n) \subset \mathscr{M}_{loc}(X)$ converges weakly to $\mu_\infty \in \mathscr{M}_{loc}(X)$ provided

$$\lim_{n \to \infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu_\infty \qquad \text{for every } f \in C_{bs}(X). \tag{2.1}$$

When $\mu_n, \mu_\infty \in \mathscr{P}(X)$, (2.1) is equivalent to narrow convergence, i.e. convergence in $\mathscr{M}(X)$ with respect to the duality with the space $C_b(X)$ of bounded continuous functions:

$$\lim_{n \to \infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu_\infty \qquad \text{for every } f \in C_b(X).$$

The following lemma provides a useful lower semicontinuity result. We set $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Lemma 2.1. *Let (X, d) be a complete and separable metric space and $F_n : X \to [0, +\infty]$ (resp. $G_n : X \to \mathbb{R}$), $n \in \bar{\mathbb{N}}$, lower semicontinuous (resp. continuous and uniformly bounded) functionals satisfying*

$$\mathsf{d}(x_n, x_\infty) \to 0 \qquad \Rightarrow \qquad F_\infty(x_\infty) \le \varliminf_{n \to \infty} F_n(x_n), \quad G_\infty(x_\infty) = \lim_{n \to \infty} G_n(x_n). \tag{2.2}$$

Then for every sequence $(\nu_n) \subset \mathscr{M}(X)$ weakly converging to $\nu_\infty \in \mathscr{M}(X)$ it holds

$$\int F_\infty \, \mathrm{d}\nu_\infty \le \varliminf_{n \to \infty} \int F_n \, \mathrm{d}\nu_n, \qquad \int G_\infty \, \mathrm{d}\nu_\infty = \lim_{n \to \infty} \int G_n \, \mathrm{d}\nu_n. \tag{2.3}$$

Proof. Providing $\bar{\mathbb{N}}$ by the usual polish topology, it is easy to check that (2.2) yields the lower semicontinuity of the functional $\bar{F} : X \times \bar{\mathbb{N}} \to [0, +\infty]$ defined by $\bar{F}(x, n) := F_n(x)$, $(x, n) \in X \times \bar{\mathbb{N}}$. We then consider the product measures $\bar{\nu}_n : \nu_n \times \delta_n$, $n \in \bar{\mathbb{N}}$: since $\bar{\nu}_n \to \bar{\nu}_\infty$ weakly in $\mathscr{P}(X \times \bar{\mathbb{N}})$ as $n \to \infty$ and $\bar{F}_n$ are nonnegative, we obtain

$$\int F_\infty \, \mathrm{d}\nu_\infty = \int \bar{F} \, \mathrm{d}\bar{\nu}_\infty \le \varliminf_{n \to \infty} \int \bar{F} \, \mathrm{d}\bar{\nu}_n = \varliminf_{n \to \infty} \int F_n \, \mathrm{d}\nu_n.$$

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The second property of (2.3) then follows by applying the previous lower semicontinuity result to $F_n(x) := G_n(x) - C$ and to $F_n(x) := C - G_n(x)$, where $C \geq 0$ is a sufficiently big constant such that $-C \leq G_n(x) \leq C$ for every $x \in X$ and $n \in \mathbb{N}$. $\qquad\square$

A simple application of Lemma 2.1 yields the lower semicontinuity of second moments w.r.t. weak convergence:

$$\mu_n \to \mu_\infty \text{ weakly in } \mathscr{P}(X), \ \bar{x}_n \to \bar{x}_\infty \ \Rightarrow \ \int \mathsf{d}^2(\cdot, \bar{x}_\infty) \, \mathrm{d}\mu_\infty \leq \varliminf_{n\to\infty} \int \mathsf{d}^2(\cdot, \bar{x}_n) \, \mathrm{d}\mu_n. \quad (2.4)$$

Convergence in (2.4) holds if (μ_n) is 2-*uniformly integrable*; recall that a subset $K \subset \mathscr{P}_2(X)$ satisfies this property if

$$\lim_{R \to +\infty} \sup_{\mu \in K} \int_{B_R^c(\bar{x})} \mathsf{d}^2(\cdot, \bar{x}) \, \mathrm{d}\mu = 0 \quad \text{for some (and thus every) } \bar{x} \in X. \qquad (2.5)$$

In this case

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu \quad \text{for every } f \in \mathrm{C}(X) \text{ with quadratic growth}, \qquad (2.6)$$

i.e. satisfying

$$|f(x)| \leq A(1 + \mathsf{d}^2(x, \bar{x})) \text{ for some } A \geq 0, \ \bar{x} \in X \text{ and every } x \in X.$$

Relative weak compactness in $\mathscr{P}(X)$ can be characterized by Prokhorov's Theorem. Let us first recall that set $\mathcal{K} \subset \mathscr{P}(X)$ is said to be tight provided for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that

$$\mu(X \setminus K_\varepsilon) \leq \varepsilon \quad \text{for every } \mu \in \mathcal{K}.$$

The we have the following classical result:

Theorem 2.2 (Prokhorov)**.** *Let* (X, d) *be complete and separable and let* $\mathcal{K} \subset \mathscr{P}(X)$*. Then the following are equivalent.*

- $\mathcal{K}$ *is precompact in the weak topology.*

- $\mathcal{K}$ *is tight.*

- *There exists a function* $\psi : X \to [0, +\infty]$ *with compact sublevels such that*

$$\sup_{\mu \in \mathcal{K}} \int \psi \, \mathrm{d}\mu < \infty.$$

2.2 Transport and L^2-Wasserstein distances.

If (Y, d) is a separable metric space and $\boldsymbol{t} : X \to Y$ is a Borel map, every probability measure $\mu \in \mathscr{P}(X)$ can be pushed to a measure $\boldsymbol{t}_\sharp \mu \in \mathscr{P}(Y)$ by the formula $(\boldsymbol{t}_\sharp \mu)(B) := \mu(\boldsymbol{t}^{-1}(B))$ for every $B \in \mathscr{B}(Y)$. If $\boldsymbol{t}^{-1}(B)$ is also bounded whenever B is bounded (e.g. when $\boldsymbol{t}$ is an isometry), then the same formula applies to $\mu \in \mathscr{M}_{loc}(X)$ and defines a measure $\boldsymbol{t}_\sharp \mu \in \mathscr{M}_{loc}(Y)$.

Given $\mu_i \in \mathscr{P}(X_i)$, the set $\mathrm{Adm}(\mu_1, \mu_2) \subset \mathscr{P}(X_1 \times X_2)$ is the set of admissible transport plans, i.e. the set of those γ such that $\pi^i_\sharp \gamma = \mu_i$, $i = 1, 2$, where π^1, π^2 are the projections on the first and second coordinate respectively.

By $\mathscr{P}_2(X)$ we denote the space of probability measures with finite second moment, i.e. the set of those measures $\mu \in \mathscr{P}(X)$ such that

$$\int \mathsf{d}^2(x, \bar{x}) \, \mathrm{d}\mu(x) < \infty, \qquad \text{for some (and thus any) } \bar{x} \in X.$$

We endow such space with the quadratic transportation distance W_2 defined as

$$W_2^2(\mu, \nu) := \inf_{\gamma \in \mathrm{Adm}(\mu,\nu)} \int \mathsf{d}^2(x, y) \, \mathrm{d}\gamma(x, y).$$

It is well known that $(\mathscr{P}_2(X), W_2)$ is a complete and separable metric space and that W_2-convergence can be characterized as in the following proposition (for a proof, see e.g. Proposition 7.1.5 in [5]).

Proposition 2.3. *Let (X, d) be a complete separable metric space and $(\mu_n) \subset \mathscr{P}_2(X)$ a sequence weakly converging to some $\mu_\infty \in \mathscr{P}(X)$. Then $\mu_\infty \in \mathscr{P}_2(X)$ and $W_2(\mu_n, \mu_\infty) \to 0$ if one of the following holds.*

- *(μ_n) is 2-uniformly integrable, i.e. (2.5) holds.*

- *(μ_n) satisfies (2.6).*

- *for some sequence $(\bar{x}_n) \subset X$ converging to $\bar{x}_\infty \in X$ it holds*

$$\lim_{n \to \infty} \int \mathsf{d}^2(\cdot, \bar{x}_n) \, \mathrm{d}\mu_n = \int \mathsf{d}^2(\cdot, \bar{x}_\infty) \, \mathrm{d}\mu_\infty < \infty.$$

- *for some sequence $(\bar{x}_n) \subset X$ converging to $\bar{x}_\infty \in X$ it holds*

$$\varlimsup_{n \to \infty} \int \mathsf{d}^2(\cdot, \bar{x}_n) \, \mathrm{d}\mu_n \leq \int \mathsf{d}^2(\cdot, \bar{x}_\infty) \, \mathrm{d}\mu_\infty < \infty.$$

Conversely, if $(\mu_n) \subset \mathscr{P}_2(X)$ is a sequence W_2-converging to some $\mu_\infty \in \mathscr{P}_2(X)$, then the sequence also weakly converges and all the four properties above are true.

In order to freely deal with all measures in $\mathscr{P}(X)$, rather than in the restricted set $\mathscr{P}_2(X)$, it will be convenient to introduce the distance W_{c} as follows. We fix once and for all a function $c \in \mathrm{C}([0, \infty))$ such that

$$c \text{ is concave and non-constant}, \quad c(0) = 0, \quad \lim_{d \to \infty} c(d) < \infty, \tag{2.7}$$

typical examples being $c(d) := \tanh(d)$ and $c(d) := d \wedge 1$. Then if (X, d) is a metric space, the cost function

$$\mathsf{c}(x, y) := c(\mathsf{d}(x, y)) \quad x, y \in X,$$

is a bounded and complete distance on X, inducing the same topology of d. The Kantorovich-Rubinstein-Wasserstein distance W_{c} is then introduced as

$$W_{\mathsf{c}}(\mu, \nu) := \inf_{\gamma \in \mathrm{Adm}(\mu_1,\mu_2)} \int \mathsf{c}(x, y) \, \mathrm{d}\gamma(x, y), \quad \mathsf{c} \text{ defined as in (2.7)}. \tag{2.8}$$

Recall that regardless of the choice of c as in (2.7)

$$\text{the space } (\mathscr{P}(X), W_{\mathsf{c}}) \text{ is a complete and separable metric space}$$
$$\text{whose topology induces the weak convergence of probability measures ,} \qquad (2.9)$$

see for instance Chapter 6 of [45] for a proof.

3 Pointed metric measure spaces and their convergence

3.1 Pointed metric measure spaces and their equivalence classes

In this paper we will deal with *pointed metric measure (p.m.m.) spaces*. A metric measure space is a structure $(X, \mathsf{d}, \mathfrak{m})$ where

$$(X, \mathsf{d}) \text{ is a complete separable metric space,} \quad \mathfrak{m} \in \mathscr{M}_{loc}(X), \ \mathfrak{m} \neq 0 \qquad (3.1a)$$

and a pointed metric measure space $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ is a metric measure space together with

$$\text{a given point } \bar{x} \in \text{supp}(\mathfrak{m}) \subset X. \qquad (3.1b)$$

Two metric measure spaces $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$, $i = 1, 2$ are called *isomorphic* if there exists

$$\text{an isometric imbedding } \iota : \text{supp}(\mathfrak{m}_1) \to X_2 \quad \text{such that} \quad \iota_\sharp \mathfrak{m}_1 = \mathfrak{m}_2, \qquad (3.2a)$$

in case of pointed metric measure spaces $(X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i)$, $i = 1, 2$, we further require that

$$\iota(\bar{x}_1) = \bar{x}_2. \qquad (3.2b)$$

Any such ι is called isomorphism from X_1 to X_2.

We shall typically denote by $\mathfrak{X}^* := [X, \mathsf{d}, \mathfrak{m}]$ the equivalence class of the given metric measure space $(X, \mathsf{d}, \mathfrak{m})$ and by $\mathfrak{X} := [X, \mathsf{d}, \mathfrak{m}, \bar{x}]$ the one of the pointed metric measure space $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$. Also, we shall denote by $\mathbb{X}$ the collection of all equivalence classes of p.m.m. spaces and by $\mathbb{X}_{fm}$ the subclass of (equivalence classes of) p.m.m. spaces with finite mass, i.e. such that $\mathfrak{m}(X) < \infty$.

As usual, the foundational issue concerning the possibility of considering the equivalence classes of 'all' p.m.m.s. can be easily avoided via the use of Kuratowski embedding which allows to isometrically embed any separable metric space in ℓ^∞. We won't insist on this point any further.

Remark 3.1 (Only the support of the measure matters)**.** We stress that the portion of spaces outside the support of the measures is irrelevant, so that $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ (resp. $(X, \mathsf{d}, \mathfrak{m})$) is always isomorphic to $(\text{supp}(\mathfrak{m}), \mathsf{d}, \mathfrak{m}, \bar{x})$ (resp. $(\text{supp}(\mathfrak{m}), \mathsf{d}, \mathfrak{m})$). ∎

In what comes next we shall need the following compactness result, which is of its own interest:

Proposition 3.2 (Compactness of the class of isomorphisms)**.** *Let* $\mathfrak{X}_i = [X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i]$, $i = 1, 2$, *be p.m.m. spaces. Then the collection of isomorphisms from* X_1 *to* X_2, *i.e. of all maps* $\iota : \text{supp}(\mathfrak{m}_1) \to X_2$ *as in (3.2), is sequentially compact w.r.t. pointwise convergence.*

Proof. Using the fact that $\mathfrak{m}_1$ is finite on bounded sets, it is easy to produce a continuous non-increasing function $\psi : [0, \infty) \to (0, \infty)$ such that the measure $\tilde{\mathfrak{m}}_1 = \psi(\mathsf{d}_1(\cdot, \bar{x}_i))\mathfrak{m}_1$ has finite mass (see for instance the explicit construction given in Proposition 3.12 below). Fix such ψ, notice that the image measure $\tilde{\mathfrak{m}}_2 := \iota_\sharp \tilde{\mathfrak{m}}_1$ does not depend on the particular isomorphism ι chosen, let (ι_n) be a sequence of isomorphisms and define $\boldsymbol{\gamma}_n := (\boldsymbol{i}, \iota_n)_\sharp \tilde{\mathfrak{m}}_1 \in \mathscr{M}(X_1 \times X_2)$, $n \in \mathbb{N}$, $\boldsymbol{i}$ being the identity map. Then we have $\pi^i_\sharp \boldsymbol{\gamma}_n = \tilde{\mathfrak{m}}_i$ for $i = 1, 2$ and $n \in \mathbb{N}$ and given that $\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2$ are tight (because they have finite mass), we easily deduce that the family $\{\boldsymbol{\gamma}_n\}_{n \in \mathbb{N}}$ is tight as well.

Thus by Theorem 2.2 we know that up to pass to a subsequence, not relabelled, we may assume that $(\boldsymbol{\gamma}_n)$ weakly converges to a limit plan $\boldsymbol{\gamma} \in \mathscr{M}(X_1 \times X_2)$. Pick $(x'_1, x'_2), (x''_1, x''_2) \in \text{supp}(\boldsymbol{\gamma})$ and notice that the weak convergence of $(\boldsymbol{\gamma}_n)$ to $\boldsymbol{\gamma}$ ensures that there are $(x'_{1,n}, x'_{2,n}), (x''_{1,n}, x''_{2,n}) \in \text{supp}(\boldsymbol{\gamma}_n)$ such that $(x'_{1,n}, x'_{2,n}) \to (x'_1, x'_2)$ and $(x''_{1,n}, x''_{2,n}) \to (x''_1, x''_2)$ in $X_1 \times X_2$ as $n \to \infty$. Given that by construction it holds $\mathsf{d}_1(x'_{1,n}, x''_{1,n}) = \mathsf{d}_2(x'_{2,n}, x''_{2,n})$ for every $n \in \mathbb{N}$, we deduce that $\mathsf{d}_1(x'_1, x''_1) = \mathsf{d}_2(x'_2, x''_2)$, i.e. that $\boldsymbol{\gamma}$ is concentrated on the graph of an isometry ι defined $\mathfrak{m}_1$-a.e. with values in X_2. By continuity, we can extend ι to an isometry defined on the whole $\text{supp}(\mathfrak{m}_1)$, so that it remains to show that $\iota_n(x) \to \iota(x)$ as $n \to \infty$ for every $n \in \mathbb{N}$ and $x \in \text{supp}(\mathfrak{m}_1)$. But this is obvious, indeed for given $x \in \text{supp}(\mathfrak{m}_1)$ we can find, using as before the weak convergence of the $\boldsymbol{\gamma}_n$'s, points $(x_n, \iota_n(x_n)) \in \text{supp}(\boldsymbol{\gamma}_n)$ converging to $(x, \iota(x)) \in \text{supp}(\boldsymbol{\gamma})$ in $X_1 \times X_2$ as $n \to \infty$, so that we can conclude with

$$\mathsf{d}_2(\iota(x), \iota_n(x)) \le \mathsf{d}_2(\iota(x), \iota_n(x_n)) + \mathsf{d}_2(\iota_n(x_n), \iota_n(x)) = \mathsf{d}_2(\iota(x), \iota_n(x_n)) + \mathsf{d}_1(x_n, x) \to 0,$$

as $n \to \infty$. $\qquad\square$

The following simple lemma will be useful to adapt some results already available for metric measure spaces (as Gromov reconstruction Theorem, see [29, $3\frac{1}{2}$.7] and the next Theorem 3.7) to the case of p.m.m. spaces.

Lemma 3.3. *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i)$, $i = 1, 2$, be two p.m.m. spaces with equal finite mass $m = \mathfrak{m}_1(X_1) = \mathfrak{m}_2(X_2) < \infty$.*

Then they are isomorphic as p.m.m. spaces if and only if there exists $\lambda > m$ such that $(X_1, \mathsf{d}_1, \mathfrak{m}_1 + \lambda \delta_{\bar{x}_1})$ is isomorphic to $(X_2, \mathsf{d}_2, \mathfrak{m}_2 + \lambda \delta_{\bar{x}_2})$ as m.m. spaces.

Proof. One implication is obvious; in order to prove the converse one, let $\iota : \text{supp}(\mathfrak{m}_1) \to X_2$ (by (3.1b) $\bar{x}_1 \in \text{supp}(\mathfrak{m}_1)$) be an isometry such that $\iota_\sharp(\mathfrak{m}_1 + \lambda \delta_{\bar{x}_1}) = \mathfrak{m}_2 + \lambda \delta_{\bar{x}_2}$ and let $y = \iota(\bar{x}_1) \in \text{supp}(\mathfrak{m}_2)$, $\mathfrak{n} = \iota_\sharp \mathfrak{m}_1$. Since $\iota_\sharp(\lambda \delta_{\bar{x}_1}) = \lambda \delta_y$ we have

$$\mathfrak{n} + \lambda \delta_y = \mathfrak{m}_2 + \lambda \delta_{\bar{x}_2}.$$

Since $\mathfrak{m}_2(X_2) < \lambda$ the only possibility is that $y = \bar{x}_2$ and therefore $\mathfrak{m}_2 = \mathfrak{n}$. $\qquad\square$

3.2 Equivalent notions of convergence of pointed metric measure spaces

3.2.1 A) Intrinsic approach: use of Gromov reconstruction theorem

It is a remarkable result due to Gromov [29] that equivalence classes of normalized metric-measure spaces can be characterized in terms of integral functionals depending on the distances between points. In order to first state such result and then adapt it to the case of general p.m.m. spaces we need to introduce some notation.

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First of all we introduce the finite dimensional spaces of (pseudo)metrics in $\mathbb{R}^{N \times N}$

$$\mathbb{G}^N := \Big\{ \boldsymbol{g} \in \mathbb{R}^{N \times N} : \boldsymbol{g} = (g_{ij})_{1 \leq i,j < N} :$$

$$0 = g_{ii} \leq g_{ji} = g_{ij} \leq g_{ih} + g_{hj} \quad \text{for every } i,j,h \in \{1, \cdots, N\} \Big\},$$

endowed with the canonical product distance and topology inherited as convex subset of $\mathbb{R}^{N \times N}$; we also consider the continuous map

$$j_X^N : X^N \to \mathbb{G}^N, \quad j_X^N : \boldsymbol{x} = (x_i)_{1 \leq i \leq N} \mapsto \boldsymbol{g} = (g_{ij})_{1 \leq i,j \leq N} : \quad g_{ij} := \mathsf{d}(x_i, x_j),$$

and the image measures

$$\mathfrak{m}_{\mathcal{X}}^N := (j_X^N)_\sharp (\delta_{\bar{x}} \otimes \mathfrak{m}^{\otimes(N-1)}) \in \mathscr{M}_{loc}(\mathbb{G}^N). \tag{3.3}$$

It is immediate to check that $\mathfrak{m}_{\mathcal{X}}^N$ only depend on the equivalence class $\mathcal{X} = [X, \mathsf{d}, \mathfrak{m}, \bar{x}]$. Those measures naturally arise when one considers "cylindrical" integral functionals of the type

$$\varphi[\mathcal{X}] := \int_{X^N} \varphi\big((\mathsf{d}(x_i, x_j))_{ij=1}^N\big) \, \mathrm{d}\delta_{\bar{x}}(x_1) \, \mathrm{d}\mathfrak{m}^{\otimes(N-1)}(x_2, \cdots, x_N) = \int_{\mathbb{G}^N} \varphi(\boldsymbol{g}) \, \mathrm{d}\mathfrak{m}_{\mathcal{X}}^N(\boldsymbol{g}), \tag{3.4}$$

for every $\varphi \in \mathrm{C}_{bs}(\mathbb{G}^N)$. In fact, for $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{X}$ by definition one has

$$\varphi[\mathcal{X}_1] = \varphi[\mathcal{X}_2] \text{ for every } \varphi \in \mathrm{C}_{bs}(\mathbb{G}^N), \, N \in \mathbb{N} \quad \Leftrightarrow \quad \mathfrak{m}_{\mathcal{X}_1}^N = \mathfrak{m}_{\mathcal{X}_2}^N \text{ for every } N \in \mathbb{N}.$$

Remark 3.4. Notice that we could equivalently consider the map

$$\tilde{j}_{\mathcal{X}}^N : X^{N-1} \to \mathbb{G}^N, \quad \boldsymbol{x} = (x_i)_{2 \leq i \leq N} \mapsto (g_{ij})_{1 \leq i,j \leq N} : \quad g_{ij} := \begin{cases} \mathsf{d}(x_i, x_j) & \text{if } i,j \geq 2, \\ \mathsf{d}(\bar{x}, x_j) & \text{if } i = 1, \\ \mathsf{d}(x_i, \bar{x}) & \text{if } j = 1, \end{cases}$$

and $\mathfrak{m}_{\mathcal{X}}^N = (\tilde{j}_{\mathcal{X}}^N)_\sharp \mathfrak{m}^{\otimes(N-1)}$. ∎

Remark 3.5. When $\mathfrak{m}(X) < \infty$, (3.4) makes sense for every $\varphi \in \mathrm{C}_b(\mathbb{G}^N)$ and it is not difficult to check by 'expanding' the product $(\lambda \delta_{\bar{x}} + \mathfrak{m})^{\otimes N}$ that the knowledge of all the measures $(\mathfrak{m}_{\mathcal{X}}^N)_{N \in \mathbb{N}}$ also determines all the measures of the type

$$(j_{\mathcal{X}}^N)_\sharp \Big(\big(\lambda \delta_{\bar{x}} + \mathfrak{m}\big)^{\otimes N} \Big) \in \mathscr{M}(\mathbb{G}^N) \quad N \in \mathbb{N},$$

for every $\lambda > 0$. ∎

A similar construction is possible for non-pointed metric measure spaces: denoting by $\mathcal{X}^*$ the equivalence class of $(X, \mathsf{d}, \mathfrak{m})$ and given $N \in \mathbb{N}$ and $\varphi \in \mathrm{C}_{bs}(\mathbb{G}^N)$, we define

$$\varphi^*[\mathcal{X}^*] := \int_{X^N} \varphi\big((\mathsf{d}(x_i, x_j))_{ij=1}^N\big) \, \mathrm{d}\mathfrak{m}^{\otimes N}(x_1, \cdots, x_N) = \int_{\mathbb{G}^N} \varphi(\boldsymbol{g}) \, \mathrm{d}(j_X^N)_\sharp \mathfrak{m}^{\otimes N}(\boldsymbol{g}).$$

Gromov reconstruction theorem for metric measure spaces can then be formulated as follows, see [29, $3\frac{1}{2}$.7] for the proof:

Theorem 3.6 (Gromov reconstruction for m.m. spaces of finite mass)**.** *Let* $\mathcal{X}_i^* := [X_i, \mathsf{d}_i, \mathfrak{m}_i]$, *$i = 1, 2$, be two metric measure spaces with finite mass. Assume that*

$$\varphi^*[\mathcal{X}_1^*] = \varphi^*[\mathcal{X}_2^*] \qquad \text{for every } N \in \mathbb{N} \text{ and } \varphi \in C_{bs}(\mathbb{G}^N).$$

Then $\mathcal{X}_1^* = \mathcal{X}_2^*$.

The compactness given by Proposition 3.2 allows to extend the reconstruction theorem for general p.m.m. spaces:

Theorem 3.7 (Gromov reconstruction for p.m.m. spaces)**.** *Let* $\mathcal{X}_i = [X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i]$, *$i = 1, 2$, be p.m.m. spaces such that*

$$\varphi[\mathcal{X}_1] = \varphi[\mathcal{X}_2] \qquad \text{for every } N \in \mathbb{N} \text{ and } \varphi \in C_{bs}(\mathbb{G}^N).$$

Then $\mathcal{X}_1 = \mathcal{X}_2$.

Proof. Let $\zeta : [0, \infty) \to [0, 1]$ be a Lipschitz cut-off function identically 1 on $[0, 1]$, positive on $[1, 2)$ and identically 0 on $[2, \infty)$. Fix $R > 0$ and let $\bar{\zeta}_{R,i} : X_i \to [0, \infty)$, $i = 1, 2$, be given by $\bar{\zeta}_{R,i}(x) := \zeta(\mathsf{d}_i(\cdot, \bar{x}_i)/R)$, and $Z_R : \mathbb{G}^N \to \mathbb{R}$ defined by

$$Z_R(\boldsymbol{g}) := \zeta\left(\frac{g_{12}}{R}\right) \zeta\left(\frac{g_{13}}{R}\right) \cdots \zeta\left(\frac{g_{1N}}{R}\right), \quad \boldsymbol{g} = (g_{ij})_{1 \le i, j \le N} \in \mathbb{G}^N. \tag{3.5}$$

Now observe that

$$(\bar{\zeta}_{R,i} \mathfrak{m}_i)_{\mathcal{X}_i}^N := (j_X^N)_\sharp \left(\delta_{\bar{x}_i} \otimes (\bar{\zeta}_{R,i} \mathfrak{m}_i)^{\otimes N} \right) = Z_R \cdot (\mathfrak{m}_i)_{\mathcal{X}_i}^N, \qquad \forall N \in \mathbb{N}, \tag{3.6}$$

indeed, for every $\varphi \in C_{bs}(\mathbb{G}^N)$ we have

$$\int_{\mathbb{G}^N} \varphi(\boldsymbol{g}) \, \mathrm{d}(\bar{\zeta}_{R,i}\mathfrak{m}_i)_{\mathcal{X}_i}^N = \int_{X^{N+1}} \varphi\left((\mathsf{d}_i(x_j, x_k))_{j,k}\right) \prod_{\ell=1}^N \zeta\left(\frac{\mathsf{d}_i(\bar{x}_i, x_\ell)}{R}\right) \mathrm{d}(\delta_{\bar{x}_i} \otimes \mathfrak{m}_i^{\otimes N})$$

$$= \int_{X^{N+1}} \varphi\left((\mathsf{d}_i(x_j, x_k))_{j,k}\right) Z_R\left((\mathsf{d}_i(x_j, x_k))_{j,k}\right) \mathrm{d}(\delta_{\bar{x}_i} \otimes \mathfrak{m}_i^{\otimes N})$$

$$= \int_{\mathbb{G}^N} \varphi(\boldsymbol{g}) Z_R(\boldsymbol{g}) \, \mathrm{d}(\mathfrak{m}_i)_{\mathcal{X}_i}^N.$$

From (3.6), the fact that $\bar{\zeta}_{R,i} \mathfrak{m}_i(X_i) < \infty$, $i = 1, 2$, and taking into account Remark 3.5 we deduce that

$$(j_{\mathcal{X}_1}^N)_\sharp \left(\left(\lambda \delta_{\bar{x}_1} + \bar{\zeta}_{R,1}\mathfrak{m}_1\right)^{\otimes N} \right) = (j_{\mathcal{X}_2}^N)_\sharp \left(\left(\lambda \delta_{\bar{x}_2} + \bar{\zeta}_{R,2}\mathfrak{m}_2\right)^{\otimes N} \right),$$

for every $\lambda > 0$, $N \in \mathbb{N}$ and $\varphi \in C_{bs}(\mathbb{G}^N)$. Thus we can apply Gromov reconstruction Theorem 3.6 for metric measure space of finite mass to obtain $[X_1, \mathsf{d}_1, \bar{\zeta}_{R,1}\mathfrak{m}_1 + \lambda\delta_{\bar{x}_1}] = [X_2, \mathsf{d}_2, \bar{\zeta}_{R,2}\mathfrak{m}_2 + \lambda\delta_{\bar{x}_2}]$ for every $\lambda > 0$.

Using Lemma 3.3 we deduce that there exists an isometry

$$\iota_R : B_{2R}(\bar{x}_1) \cap \mathrm{supp}(\mathfrak{m}_1) \to B_{2R}(\bar{x}_2) \cap \mathrm{supp}(\mathfrak{m}_2)$$

such that

$$\iota_R(\bar{x}_1) = \bar{x}_2, \qquad \text{and} \qquad (\iota_R)_\#(\bar{\zeta}_{R,1}\mathfrak{m}_1) = \bar{\zeta}_{R,2}\mathfrak{m}_2. \tag{3.7}$$

From the definition of $\bar{\zeta}_{R,i}$, $i = 1, 2$, and the fact that ι_R is an isometry we also have the consistency property

$$(\iota_R)_\#(\bar{\zeta}_{R',1}\mathfrak{m}_1) = \bar{\zeta}_{R',2}\mathfrak{m}_2, \qquad \forall R' < R. \tag{3.8}$$

Now let $R_n \uparrow \infty$ be a chosen sequence and ι_{R_n} the corresponding sequence of isomorphisms. For any given $k \in \mathbb{N}$, the validity of (3.7) and (3.8) ensures that the maps $\{\iota_{R_n}\}_{n\geq k}$ are isomorphisms of $(X_1, \mathsf{d}_1, \bar{\zeta}_{R_k,1}\mathfrak{m}_1, \bar{x}_1)$ and $(X_2, \mathsf{d}_2, \bar{\zeta}_{R_k,2}\mathfrak{m}_2, \bar{x}_2)$, hence applying Proposition 3.2 we deduce that there exists a subsequence of (R_n) whose corresponding isomorphisms pointwise converge to a limit isomorphism, call it $\tilde{\iota}_k$, of $(X_1, \mathsf{d}_1, \bar{\zeta}_{R_k,1}\mathfrak{m}_1, \bar{x}_1)$ and $(X_2, \mathsf{d}_2, \bar{\zeta}_{R_k,2}\mathfrak{m}_2, \bar{x}_2)$.

By a standard diagonalization argument we can ensure that the limit map $\tilde{\iota}_k$ does not depend on k. Denote it as $\tilde{\iota}$ and notice that the construction and (3.7) and (3.8) grant that

$$\tilde{\iota}(\bar{x}_1) = \bar{x}_2, \qquad \text{and} \qquad \tilde{\iota}_\#(\bar{\zeta}_{R,1}\mathfrak{m}_1) = \bar{\zeta}_{R,2}\mathfrak{m}_2, \qquad \forall R > 0,$$

which, by definition of $\bar{\zeta}_{R,i}$, $i = 1, 2$, ensures that $\tilde{\iota}$ is an isomorphism of $(X_1, \mathsf{d}_1, \mathfrak{m}_1, \bar{x}_1)$ and $(X_2, \mathsf{d}_2, \mathfrak{m}_2, \bar{x}_2)$. $\qquad\square$

Theorem 3.7 suggests the following notion of convergence:

Definition 3.8 (Intrinsic notion of convergence of equivalence classes of p.m.m.s.)**.** *Let* $\mathcal{X}_n = [X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n] \in \mathbb{X}$, $n \in \mathbb{N}$, *and* $\mathcal{X}_\infty = [X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty] \in \mathbb{X}$.
We say that the sequence $n \mapsto \mathcal{X}_n$ *intrinsically converges to* $\mathcal{X}_\infty$ *provided:*

$$\text{for any } N \in \mathbb{N} \text{ and } \varphi \in \mathrm{C}_{bs}(\mathbb{G}^N) \text{ we have} \qquad \lim_{n\to\infty} \varphi[\mathcal{X}_n] = \varphi[\mathcal{X}_\infty].$$

3.2.2 B) Extrinsic approach: embedding everything in a common space

Recalling the definition of weak convergence of measures in $\mathscr{M}_{loc}(X)$ given in (2.1), where (X, d) is a complete and separable metric space, we can give the following definition of convergence:

Definition 3.9 (Extrinsic notion of convergence of equivalence classes of p.m.m.s.)**.** *Let* $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, *be a sequence of p.m.m. spaces. We say that the corresponding sequence of classes* $n \mapsto \mathcal{X}_n$ *extrinsically converges to* $\mathcal{X}_\infty$ *provided there exists a complete and separable metric space* (X, d) *and isometric embeddings* $\iota_n : X_n \to X$, $n \in \bar{\mathbb{N}}$, *such that*

$$(\iota_n)_\sharp\mathfrak{m}_n \to (\iota_\infty)_\sharp\mathfrak{m}_\infty \text{ weakly in } \mathscr{M}_{loc}(X), \quad \iota_n(\bar{x}_n) \to \iota_\infty(\bar{x}_\infty) \in \mathrm{supp}((\iota_\infty)_\sharp\mathfrak{m}_\infty) \quad \text{in } X. \tag{3.9}$$

The system $\big((X, \mathsf{d}, \iota_n)\big)_{n\in\bar{\mathbb{N}}}$ *is called an* effective realization *of the extrinsic convergence.*

Notice that a priori it is not clear that this definition is well-posed, because the isometries ι_n are required to be defined on the whole spaces X_n and not just on the supports of the measures $\mathfrak{m}_n$, so that a priori the convergence might depend on the chosen elements in the equivalence classes. In fact, this is not an issue thanks to the following simple result:

Proposition 3.10 (Extension of isometric immersions)**.** *Let* (X_n, d_{X_n}), $n \in \mathbb{N}$, *and* (Y, d_Y) *be separable metric spaces,* $F_n \subset X_n$ *given subsets and* $\iota_{F_n} : F_n \to Y$ *isometric embeddings. Then there exists a complete separable metric space* (Z, d_Z) *and isometric embeddings* $\iota_Y : Y \to Z$, $\iota_{X_n} : X_n \to Z$, $n \in \mathbb{N}$, *such that*

$$\iota_{X_n}\big|_{F_n} = \iota_Y \circ \iota_{F_n}, \qquad \forall n \in \mathbb{N}.$$

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Proof. Put $(Z_0, \mathsf{d}_{Z_0}) := (Y, \mathsf{d}_Y)$. Then define the set $Z_1 := Z_0 \sqcup (X_0 \setminus F_0)$ and the distance d_{Z_1} on it by requiring d_{Z_1} to coincide with d_{Z_0} on $(Z_0)^2$, with d_{X_0} on $(X_0 \setminus F_0)^2$, defined as

$$\mathsf{d}_{Z_1}(z', z'') := \inf_{x \in F_0} \mathsf{d}_{Z_0}(z', \iota_{F_0}(x)) + \mathsf{d}_{X_0}(x, z'')$$

on $Z_0 \times (X_0 \setminus F_0)$ and extended by symmetry on $(X_0 \setminus F_0) \times Z_0$. It is immediate to check that (Z_1, d_{Z_1}) is a separable metric space, that the natural inclusion $\iota_{Z_0}^{Z_1}$ of Z_0 in Z_1 is an isometry and that the map $\iota_{X_0}^{Z_1} : X_0 \to Z_1$ defined as

$$\iota_{X_0}^{Z_1}(x) := \iota_{Z_0}^{Z_1}(\iota_{F_0}(x)), \quad \text{if } x \in F_0, \qquad \iota_{X_0}^{Z_1}(x) := x \quad \text{if } x \in X_0 \setminus F_0,$$

is an isometry as well for which the identity $\iota_{X_0}^{Z_1}|_{F_0} = \iota_{Z_0}^{Z_1} \circ \iota_{F_0}$ holds.

Iterating the construction we produce a sequence of separable metric spaces (Z_n, d_{Z_n}), $n \in \mathbb{N}$, and isometric embeddings $\iota_{Z_{n-1}}^{Z_n}$, $\iota_{X_{n-1}}^{Z_n}$ of Z_{n-1}, X_{n-1} respectively in Z_n satisfying

$$\iota_{X_{n-1}}^{Z_n}\big|_{F_{n-1}} = \iota_{Z_{n-1}}^{Z_n} \circ \cdots \circ \iota_{Z_0}^{Z_1} \circ \iota_{F_0}.$$

Eventually defining $Z := \cup_{n \in \mathbb{N}} Z_n$, equipping it with the natural distance induced by the inclusions of the Z_n's and taking, if necessary, its completion, we conclude the proof. $\qquad \square$

3.2.3 C) A variant of Sturm's distance $\mathbb{D}$

In [42] Sturm worked in the class of m.m. spaces $(X, \mathsf{d}, \mathfrak{m})$ *normalized* and with *finite variance*, these two meaning that $\mathfrak{m} \in \mathscr{P}_2(X)$. For this sort of spaces he defined the distance $\mathbb{D}$ as

$$\mathbb{D}\Big((X_1, \mathsf{d}_1, \mathfrak{m}_1), (X_2, \mathsf{d}_2, \mathfrak{m}_2)\Big) := \inf W_2\big((\iota_1)_\sharp \mathfrak{m}_1, (\iota_2)_\sharp \mathfrak{m}_2\big),$$

the infimum being taken among all metric spaces (X, d) and all isometric immersions $\iota_i : X_i \to X$, $i = 1, 2$. In fact, in the aforementioned reference the isometries ι_1, ι_2 were required to be defined only on $\operatorname{supp} \mathfrak{m}_i$, $i = 1, 2$, but as for the case of the extrinsic notion of convergence, Proposition 3.10 shows that it is not restrictive to assume them to be defined on the whole X_1, X_2.

The idea to define a distance on pointed non-normalized spaces is the following. We fix a

non-increasing and absolutely continuous function $\psi : [0, \infty) \to (0, \infty)$ with $\lim_{r \to \infty} \psi(r) = 0$ (3.10)

and introduce the class $\mathbb{X}^\psi \subset \mathbb{X}$ of those (equivalence classes of) p.m.m. spaces $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ such that

$$\int (1 + \mathsf{d}^2(\cdot, \bar{x})) \psi(\mathsf{d}(\cdot, \bar{x})) \, \mathrm{d}\mathfrak{m} < \infty.$$

For $[X, \mathsf{d}, \mathfrak{m}, \bar{x}] \in \mathbb{X}^\psi$ we define the normalization constant $\mathsf{z}_\psi \in (0, \infty)$ and the renormalized measure $\mathfrak{m}_\psi \in \mathscr{P}_2(X)$ as

$$\mathsf{z}_\psi := \int \psi(\mathsf{d}(\cdot, \bar{x})) \, \mathrm{d}\mathfrak{m}, \qquad \mathrm{d}\mathfrak{m}_\psi := \frac{1}{\mathsf{z}_\psi} \psi(\mathsf{d}(\cdot, \bar{x})) \, \mathrm{d}\mathfrak{m}. \tag{3.11}$$

Then we introduce the distance:

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Definition 3.11 (The distance $\mathbb{D}^\psi$ on $\mathbb{X}^\psi$). *For $\mathfrak{X}_i := [X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i]$, $i = 1, 2$, with $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathbb{X}^\psi$ we define*

$$\mathbb{D}^\psi(\mathfrak{X}_1, \mathfrak{X}_2) := \left| \log \frac{\mathsf{z}_{1,\psi}}{\mathsf{z}_{2,\psi}} \right| + \inf \left(W_2 \big((\iota_1)_\sharp \mathfrak{m}_{1,\psi}, (\iota_2)_\sharp \mathfrak{m}_{2,\psi} \big) + \mathsf{d} \big(\iota_1(\bar{x}_1), \iota_2(\bar{x}_2) \big) \right),$$

the infimum being taken among all separable metric spaces (X, d) and isometric embeddings $\iota_1 : X_1 \to X$ and $\iota_2 : X_2 \to X$.

It is clear that the definition is well posed, in the sense that $\mathbb{D}^\psi(\mathfrak{X}_1, \mathfrak{X}_2)$ only depends on $\mathfrak{X}_1, \mathfrak{X}_2$ and not on the chosen elements in the equivalence class. Also, using the fact that $\mathbb{D}$ is a distance on the class of normalized spaces with finite variance (see [42]), it is immediate to verify that $\mathbb{D}^\psi$ is a distance on $\mathbb{X}^\psi$. We omit the simple proof.

It is obvious that spaces in $\mathbb{X}^\psi$ have a controlled volume growth. In the next statement we show that also the converse implication holds, a fact which will turn useful later on. Notice that in the bound (3.13) below the third power of the distance appears: this will help in the proof of Theorem 3.15 to gain 2-uniform integrability (see also Remark 3.20).

Proposition 3.12. *Let $\phi : [0, \infty) \to [0, \infty)$ be a given continuous non-decreasing function and $[X, \mathsf{d}, \mathfrak{m}, \bar{x}] \in X$ be such that*

$$\mathfrak{m}(B_R(\bar{x})) \le \phi(R), \qquad \forall R \ge 0.$$

Define $\psi : [0, \infty) \to (0, \infty)$ as

$$\psi(r) := \int_r^\infty \frac{1}{(1 + s^3)^2 \phi(s)} \, \mathrm{d}s. \tag{3.12}$$

Then ψ is as in (3.10), $[X, \mathsf{d}, \mathfrak{m}, \bar{x}] \in X^\psi$ and

$$\int (1 + \mathsf{d}^3(\cdot, \bar{x})) \psi(\mathsf{d}(\cdot, \bar{x})) \, \mathrm{d}\mathfrak{m} \le \int_0^\infty \frac{1}{1 + r^3} \, \mathrm{d}r < \infty. \tag{3.13}$$

Proof. It is obvious that formula (3.12) defines a non-increasing and absolutely continuous map such that $\lim_{r \to \infty} \psi(r) = 0$. Conclude using the identity

$$\int \psi(\mathsf{d}(x, \bar{x})) f(x) \, \mathrm{d}\mathfrak{m}(x) = - \int_0^\infty \psi'(r) \left(\int_{B_r(\bar{x})} f(x) \, \mathrm{d}\mathfrak{m}(x) \right) \mathrm{d}r,$$

valid for any such ψ and every continuous $f : X \to [0, \infty)$. $\qquad \square$

3.2.4   D) A metric independent of chosen weights: the $\mathrm{p}\mathbb{G}_\mathrm{W}$ distance

One drawback of the distances $\mathbb{D}^\psi$ is that they depend on the chosen weight function ψ. In order to build a 'universal' distance we can use the distance W_c (recall the definition (2.8)) in place of W_2 and proceed as follows.

For two normalized metric measure spaces $\mathfrak{X}_i^* := [X_i, \mathsf{d}_i, \mathfrak{m}_i]$, $i = 1, 2$, we define

$$\mathbb{G}_\mathrm{W}(\mathfrak{X}_1^*, \mathfrak{X}_2^*) := \inf W_c((\iota_1)_\sharp \mathfrak{m}_1, (\iota_2)_\sharp \mathfrak{m}_2),$$

where here and below the inf being taken, as usual, among all complete and separable spaces (X, d) and isometric immersions $\iota_i : X_i \to X$, $i = 1, 2$.

In case $\mathcal{X}_1^*, \mathcal{X}_2^*$ have finite mass, the natural variant of $\mathbb{G}_W$ is given by

$$\mathbb{G}_W^{fm}(\mathcal{X}_1^*, \mathcal{X}_2^*) := \left| \log \frac{\mathfrak{m}_1(X_1)}{\mathfrak{m}_2(X_2)} \right| + \inf W_c((\iota_1)_\sharp \tilde{\mathfrak{m}}_1, (\iota_2)_\sharp \tilde{\mathfrak{m}}_2),$$

where the measures $\tilde{\mathfrak{m}}_i := \frac{\mathfrak{m}_i}{\mathfrak{m}_i(X_i)}$ are the normalizations of $\mathfrak{m}_i$, $i = 1, 2$.

For pointed metric measure spaces of finite mass $\mathcal{X}_i := [X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i]$, $i = 1, 2$, we can then consider the distance

$$\mathrm{p}\mathbb{G}_W^{fm}(\mathcal{X}_1, \mathcal{X}_2) := \left| \log \frac{\mathfrak{m}_1(X_1)}{\mathfrak{m}_2(X_2)} \right| + \inf \mathsf{d}(\iota_1(\bar{x}_1), \iota_2(\bar{x}_2)) + W_c((\iota_1)_\sharp \tilde{\mathfrak{m}}_1, (\iota_2)_\sharp \tilde{\mathfrak{m}}_2).$$

The fact that all these are separable distances on the corresponding classes can be seen arguing exactly as in [42, 28], we omit the details.

For the general case of p.m.m. spaces of possibly infinite mass we adopt a cut-off argument: fix once and for all a function

$$\zeta : [0, \infty) \to [0, 1] \text{ Lipschitz}, \quad \text{with } \zeta \equiv 1 \text{ on } [0, 1] \text{ and } \zeta \equiv 0 \text{ on } [2, \infty). \tag{3.14}$$

Then for given $(X, \mathsf{d}, \mathfrak{m}, \bar{x}) \in \mathbb{X}$ and $k \in \mathbb{Z}$ we consider the rescaled measures $\mathfrak{m}_{[k]}$ on X defined as

$$\mathrm{d}\mathfrak{m}_{[k]}(x) := \zeta(\mathsf{d}(x, \bar{x})2^{-k})\mathrm{d}\mathfrak{m}(x), \tag{3.15}$$

and the corresponding isomorphism class

$$\mathcal{X}_{[k]} := [X, \mathsf{d}, \mathfrak{m}_{[k]}, \bar{x}]. \tag{3.16}$$

Then we define:

Definition 3.13 (The distance $\mathrm{p}\mathbb{G}_W$)**.**

$$\mathrm{p}\mathbb{G}_W(\mathcal{X}', \mathcal{X}'') := \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \min \left\{ 1, \mathrm{p}\mathbb{G}_W^{fm}(\mathcal{X}'_{[k]}, \mathcal{X}''_{[k]}) \right\}.$$

Knowing that $\mathrm{p}\mathbb{G}_W^{fm}$ is a distance, it is obvious that $\mathrm{p}\mathbb{G}_W$ is a distance on $\mathbb{X}$ and that $\mathrm{p}\mathbb{G}_W(\mathcal{X}_1, \mathcal{X}_2) \to 0$ if and only if $\mathrm{p}\mathbb{G}_W^{fm}(\mathcal{X}_{1,[k]}, \mathcal{X}_{2,[k]}) \to 0$ for every $k \in \mathbb{Z}$.

Notice that we are letting k vary on $\mathbb{Z}$ rather than on $\mathbb{N}$: we did so to get completeness of $\mathrm{p}\mathbb{G}_W$, see Theorem 3.17.

3.3 Proof of the equivalence of the various notions of convergence

Aim of this section is to prove that the various notions of convergence previously introduced coincide. As already said in the introduction, such equivalence is already known under particular circumstances, like compactness or normalization of the spaces and our scope is to generalize such previously known facts. In doing so, we shall use the fact that, shortly said, 'intrinsic convergence implies convergence w.r.t. the distance $\mathbb{G}_W$ for normalized spaces' as proved in Proposition 10.5, Proposition 10.1, and Theorem 5 of [28]. The rigorous statement is the following:

Theorem 3.14. *Let $\mathcal{X}_n^* := (X_n, \mathsf{d}_n, \mathfrak{m}_n)$, $n \in \bar{\mathbb{N}}$, be normalized metric measure spaces. Assume that*

$$\lim_{n \to \infty} \varphi^*[\mathcal{X}_n^*] = \varphi^*[\mathcal{X}_\infty^*],$$

for every $N \in \mathbb{N}$ and $\varphi \in C_{bs}(\mathbb{G}^N)$. Then $\lim_{n \to \infty} \mathbb{G}_W(\mathcal{X}_n^*, \mathcal{X}_\infty^*) = 0$.

We can now turn to the main result of this section:

Theorem 3.15. *Let $\mathcal{X}_n = [X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n]$, $n \in \bar{\mathbb{N}}$. Then the following are equivalent:*

(A) *The sequence $n \mapsto \mathcal{X}_n$ intrinsically converges to $\mathcal{X}_\infty$ in the sense of Definition 3.8.*

(B) *The sequence $n \mapsto \mathcal{X}_n$ extrinsically converges to $\mathcal{X}_\infty$ in the sense of Definition 3.9.*

(C) *There exists $\psi : [0, \infty) \to (0, \infty)$ nonincreasing and absolutely continuous such that $\mathcal{X}_n \in \mathbb{X}^\psi$ for every $n \in \bar{\mathbb{N}}$ and $\lim_{n \to \infty} \mathbb{D}^\psi(\mathcal{X}_n, \mathcal{X}_\infty) = 0$.*

(D) $\lim_{n \to \infty} \mathrm{p}\mathbb{G}_W(\mathcal{X}_n, \mathcal{X}_\infty) = 0$.

Proof.
(C) $\Rightarrow$ (B) According to (3.11) for every $n \in \bar{\mathbb{N}}$ we define

$$\mathsf{z}_{n,\psi} := \int \psi(\mathsf{d}_n(\cdot, \bar{x}_n)) \, \mathrm{d}\mathfrak{m}_n, \qquad \mathfrak{m}_{n,\psi} := \frac{1}{\mathsf{z}_{n,\psi}} \psi(\mathsf{d}_n(\cdot, \bar{x}_n)) \mathfrak{m}_n.$$

By definition of $\mathbb{D}^\psi$ we know that for every $n \in \mathbb{N}$ there exists a complete separable metric space (Y_n, d_{Y_n}) and isometric embeddings ι_n, ι_n^∞ of (X_n, d_n), $(X_\infty, \mathsf{d}_\infty)$ respectively in Y_n such that

$$\left| \log \frac{\mathsf{z}_{n,\psi}}{\mathsf{z}_{\infty,\psi}} \right| + \mathsf{d}_{Y_n}(\iota_n(\bar{x}_n), \iota_n^\infty(\bar{x}_\infty)) + W_2((\iota_n)_\sharp \mathfrak{m}_{n,\psi}, (\iota_n^\infty)_\sharp \mathfrak{m}_{\infty,\psi}) \leq \mathbb{D}^\psi(\mathcal{X}_n, \mathcal{X}_\infty) + \frac{1}{n}. \quad (3.17)$$

Now define $Y := \sqcup_{n \in \bar{\mathbb{N}}} X_n$ and a pseudodistance d_Y on Y by declaring that

$$\mathsf{d}_Y(y, y') :=$$
$$\begin{cases} \mathsf{d}_n(y, y'), & \text{if } y, y' \in X_n \text{ for some } n \in \bar{\mathbb{N}}, \\ \mathsf{d}_{Y_n}(\iota_n(y), \iota_n^\infty(y')), & \text{if } y \in X_n \text{ for some } n \in \mathbb{N}, y' \in X_\infty, \\ \mathsf{d}_{Y_n}(\iota_n^\infty(y), \iota_n(y')), & \text{if } y \in X_\infty, y' \in X_n \text{ for some } n \in \mathbb{N}, \\ \inf_{x \in X_\infty} \mathsf{d}_{Y_n}(\iota_n^\infty(y), \iota_n^\infty(x)) + \mathsf{d}_{Y_m}(\iota_m^\infty(y), \iota_m^\infty(x)), & \text{if } y \in X_n, y' \in X_m \text{ for some } n, m \in \mathbb{N}. \end{cases}$$

Next we identify points $x, y \in Y$ such that $\mathsf{d}_Y(x, y) = 0$: it is readily verified that the resulting space, which we shall still denote by (Y, d_Y) is separable and up to passing to the completion we can also assume that Y is complete. By construction, the set $\iota_n(X_n) \cup \iota_n^\infty(X_\infty) \subset Y_n$ endowed with the distance d_{Y_n} is canonically isometrically embedded in (Y, d_Y) so that there are canonical isometric embeddings ι_n' of X_n into Y, $n \in \bar{\mathbb{N}}$. We claim that (Y, d_Y) and the ι_n''s provide an effective realization of the extrinsic convergence. The fact that $\mathsf{d}_Y(\iota_n'(\bar{x}_n), \iota_\infty'(\bar{x}_\infty)) \to 0$ is obvious by (3.17) and similarly we obtain that $\mathsf{z}_{n,\psi} \to \mathsf{z}_{\infty,\psi}$. For the weak convergence of $(\iota_n')_\sharp \mathfrak{m}_n$ to $(\iota_\infty')_\sharp \mathfrak{m}_\infty$ fix $\varphi \in C_{bs}(Y)$ and notice that the boundedness of $\mathrm{supp}(\varphi)$ and the continuity of ψ ensure that

$$\text{the sequence} \quad \frac{\mathsf{z}_{n,\psi} \varphi(\cdot)}{\psi(\mathsf{d}_Y(\cdot, \iota_n'(\bar{x}_n)))} \quad \text{uniformly converges to} \quad \frac{\mathsf{z}_{\infty,\psi} \varphi(\cdot)}{\psi(\mathsf{d}_Y(\cdot, \iota_\infty'(\bar{x}_\infty)))} \quad (3.18)$$

on $\operatorname{supp}(\varphi)$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \int \varphi \, \mathrm{d}(\iota'_n)_\sharp \mathfrak{m}_n = \lim_{n\to\infty} \int \frac{z_{n,\psi}\varphi(\cdot)}{\psi(\mathsf{d}_Y(\cdot,\iota'_n(\bar{x}_n)))} \, \mathrm{d}(\iota'_n)_\sharp \mathfrak{m}_{n,\psi}$$

$$= \int \frac{z_{\infty,\psi}\varphi(\cdot)}{\psi(\mathsf{d}_Y(\cdot,\iota'_\infty(\bar{x}_\infty)))} \, \mathrm{d}(\iota'_\infty)_\sharp \mathfrak{m}_{\infty,\psi} = \lim_{n\to\infty} \int \varphi \, \mathrm{d}(\iota'_\infty)_\sharp \mathfrak{m}_\infty,$$

having used (3.18) and the weak convergence of $(\iota'_n)_\sharp \mathfrak{m}_{n,\psi}$ to $(\iota'_\infty)_\sharp \mathfrak{m}_{\infty,\psi}$ granted by the fact that $W_2((\iota'_n)_\sharp \mathfrak{m}_{n,\psi}, (\iota'_\infty)_\sharp \mathfrak{m}_{\infty,\psi}) \to 0$ (from (3.17)) and Proposition 2.3.

(B) $\Rightarrow$ (A) Keeping the same notation of (3.3), we observe that (A) is equivalent to

for every $N \in \mathbb{N}$ the sequence $\mathfrak{m}_n^N := (j_{X_n}^N)_\sharp (\delta_{\bar{x}_n} \otimes (\mathfrak{m}_n)^{\otimes(N-1)})$, $n \in \mathbb{N}$,

weakly converges to $\mathfrak{m}_\infty^N := (j_{X_\infty}^N)_\sharp (\delta_{\bar{x}_\infty} \otimes (\mathfrak{m}_\infty)^{\otimes(N-1)})$ in $\mathscr{M}_{loc}(\mathbb{G}^N)$ as $n \to \infty$. (3.19)

Now notice that by (B) and Definition (3.9), there exists a separable metric space (Y, d_Y) and isometries $\iota_n : X_n \to Y$ such that $\mathfrak{n}_n = (\iota_n)_\sharp \mathfrak{m}_n$ and $\bar{y}_n = \iota_n(\bar{x}_n)$ satisfy (3.9). In particular the measures $\mathfrak{m}_n^N$ of (3.19) admit the representations $\mathfrak{m}_n^N = (j_Y^N)_\sharp (\delta_{\bar{y}_n} \otimes (\mathfrak{n}_n)^N)$ for every $n \in \bar{\mathbb{N}}$, so that (3.19) follows immediately from (3.9).

(A) $\Rightarrow$ (D) It is sufficient to prove that for every $k \in \mathbb{Z}$ we have $\mathrm{p}\mathbb{G}_W^{fm}(\mathcal{X}_{n,[k]}, \mathcal{X}_{\infty,[k]}) \to 0$ as $n \to \infty$. Thus fix $k \in \mathbb{Z}$, recalling the definition of Z_R given by (3.5) and set $\varphi_k := Z_{2^k}\varphi$ to deduce that $\varphi[\mathcal{X}_{n,[k]}] = \varphi_k[\mathcal{X}_n]$ for every $n \in \bar{\mathbb{N}}$. Hence by assumption for every function $\varphi \in C_b(\mathbb{G}^N)$ we have $\lim_{n\to\infty} \varphi[\mathcal{X}_{n,[k]}] = \varphi[\mathcal{X}_{\infty,[k]}]$.

Now pick $\lambda > \sup_{n\in\bar{\mathbb{N}}} \mathfrak{m}_{n,[k]}(X_n)$ and consider the (non-pointed) metric measure spaces $\mathcal{X}_{n,[k],\lambda}^* := [X_n, \mathsf{d}_n, \mathfrak{m}_{n,[k]} + \lambda\delta_{\bar{x}_n}]$: by Remark 3.5, Theorem 3.14 and taking into account that $\mathfrak{m}_{n,[k]}(X_n) \to \mathfrak{m}_{\infty,[k]}(X_\infty)$ we deduce that $\mathbb{G}_W(\mathcal{X}_{n,[k],\lambda}^*, \mathcal{X}_{\infty,[k],\lambda}^*) \to 0$ as $n \to \infty$.

With the very same construction used to prove the implication (C)$\Rightarrow$(B) above, we can produce a complete separable space (Y, d_Y) and isometric immersions $\iota_n : X_n \to Y$, $n \in \bar{\mathbb{N}}$ such that $(\iota_n)_\sharp(\mathfrak{m}_{n,[k]} + \lambda\delta_{\bar{x}_n})$ weakly converges to $(\iota_\infty)_\sharp(\mathfrak{m}_{\infty,[k]} + \lambda\delta_{\bar{x}_\infty})$.

To conclude is therefore sufficient to show that $\iota_n(\bar{x}_n) \to \iota_\infty(\bar{x}_\infty)$, as this would also give that $(\iota_n)_\sharp \mathfrak{m}_{n,[k]}$ weakly converges to $(\iota_\infty)_\sharp \mathfrak{m}_{\infty,[k]}$. But this is obvious by the choice of λ. Indeed, for $r > 0$ we can consider the map $\chi_r \in C_{bs}(Y)$ given by $\chi_r(y) := 0 \vee (1 - \frac{\mathsf{d}(y,\iota_\infty(x_\infty))}{r})$ so that

$$\lambda + \varliminf_{n\to\infty} \chi_r(\iota_n(\bar{x}_n)) > \varliminf_{n\to\infty} \int \chi_r \, \mathrm{d}(\iota_n)_\sharp(\mathfrak{m}_{n,[k]} + \lambda\delta_{\bar{x}_n}) = \int \chi_r \, \mathrm{d}(\iota_\infty)_\sharp(\mathfrak{m}_{\infty,[k]} + \lambda\delta_{\bar{x}_\infty}) \geq \lambda,$$

which gives $\varliminf_{n\to\infty} \chi_r(\iota_n(\bar{x}_n)) > 0$ and thus $\varlimsup_{n\to\infty} \mathsf{d}(\iota_n(\bar{x}_n), \iota_\infty(\bar{x}_\infty)) < r$. Being $r > 0$ arbitrary, the claim follows.

(D) $\Rightarrow$ (C) By the assumption we have $\mathrm{p}\mathbb{G}_W^{fm}(\mathcal{X}_{n,[k]}, \mathcal{X}_{\infty,[k]}) \to 0$ as $n \to \infty$ for every $k \in \mathbb{Z}$. Given $R > 0$ we can find $k \in \mathbb{Z}$ such that $R < 2^k$ so that recalling the definition of the measures $\mathfrak{m}_{[k]}$ given in (3.15) we deduce

$$\lim_{n\to\infty} \int 0 \vee \big(1 - \mathsf{d}_n(\cdot, B_R(\bar{x}_n))\big) \, \mathrm{d}\mathfrak{m}_n = \int 0 \vee \big(1 - \mathsf{d}_\infty(\cdot, B_R(\bar{x}_\infty))\big) \, \mathrm{d}\mathfrak{m}_\infty < \infty.$$

Using the bound $\mathfrak{m}_n(B_R(\bar{x}_n)) \leq \int 0 \vee (1 - \mathsf{d}_n(\cdot, B_R(\bar{x}_n))) \, \mathrm{d}\mathfrak{m}_n$ valid for every $n \in \mathbb{N}$, we deduce that there exists a continuous non-decreasing $\phi : [0,\infty) \to [0,\infty)$ such that

$$\sup_{n\in\bar{\mathbb{N}}} \mathfrak{m}_n(B_R(\bar{x}_n)) \leq \phi(R), \qquad \forall R > 0.$$

Now define $\psi : [0, \infty) \to (0, \infty)$ as in (3.12) and use Proposition 3.12 to deduce that $\mathcal{X}_n \in \mathbb{X}^\psi$ for every $n \in \bar{\mathbb{N}}$ and

$$\sup_{n \in \bar{\mathbb{N}}} \int (1 + \mathsf{d}_n^3(\cdot, \bar{x}_n)) \psi(\mathsf{d}_n(\cdot, \bar{x}_n)) \, \mathrm{d}\mathfrak{m}_n \le \int_0^\infty \frac{1}{1 + r^3} \, \mathrm{d}r =: \mathsf{S} < \infty,$$

from which it follows that

$$\int_{X_n \setminus B_R(\bar{x}_n)} \psi(\mathsf{d}_n(\cdot, \bar{x}_n)) \, \mathrm{d}\mathfrak{m}_n \le \frac{\mathsf{S}}{R^3}, \qquad \int_{X_n \setminus B_R(\bar{x}_n)} \mathsf{d}_n^2(\cdot, \bar{x}_n) \psi(\mathsf{d}_n(\cdot, \bar{x}_n)) \, \mathrm{d}\mathfrak{m}_n \le \frac{\mathsf{S}}{R}, \quad (3.20)$$

for every $n \in \bar{\mathbb{N}}$. Fix $k \in \mathbb{Z}$, recall that by definition of $\mathrm{p}\mathbb{G}_W$ we have $\mathrm{p}\mathbb{G}_W^{fm}(\mathcal{X}_{n,[k]}, \mathcal{X}_{\infty,[k]}) \to 0$ as $n \to \infty$ and then use the very same construction used in the proof of (C)$\Rightarrow$(B) above to deduce the existence of a complete and separable metric space (Y, d_Y) and isometric embeddings ι_n of X_n in Y, $n \in \bar{\mathbb{N}}$, such that

$$\lim_{n \to \infty} \left| \log \frac{\mathfrak{m}_{n,[k]}(X_n)}{\mathfrak{m}_{\infty,[k]}(X_\infty)} \right| + \mathsf{d}_Y(\iota_n(\bar{x}_n), \iota_\infty(\bar{x}_\infty)) + W_c((\iota_n)_\sharp \tilde{\mathfrak{m}}_{n,[k]}, (\iota_\infty)_\sharp \tilde{\mathfrak{m}}_{\infty,[k]}) = 0, \quad (3.21)$$

where $\mathrm{d}\mathfrak{m}_{n,[k]} := \zeta(\mathsf{d}_n(\cdot, \bar{x}_n) 2^{-k}) \mathrm{d}\mathfrak{m}_n$ (ζ being given in (3.14)) and $\tilde{\mathfrak{m}}_{n,[k]} = \mathfrak{m}_{n,[k]}(X_n)^{-1} \mathfrak{m}_{n,[k]}$, for every $n \in \bar{\mathbb{N}}$. In particular, the sequence of measures $n \mapsto (\iota_n)_\sharp \mathfrak{m}_{n,[k]} = \mathfrak{m}_{n,[k]}(X_n) \cdot (\iota_n)_\sharp \tilde{\mathfrak{m}}_{n,[k]}$ weakly converges to $(\iota_\infty)_\sharp \mathfrak{m}_{\infty,[k]} = \mathfrak{m}_{\infty,[k]}(X_\infty) \cdot (\iota_\infty)_\sharp \tilde{\mathfrak{m}}_{\infty,[k]}$ and therefore

$$\lim_{n \to \infty} \int \psi(\mathsf{d}_n(\cdot, \bar{x}_n)) \zeta(\mathsf{d}_n(\cdot, \bar{x}_n) 2^{-k}) \, \mathrm{d}\mathfrak{m}_n = \int \psi(\mathsf{d}_\infty(\cdot, \bar{x}_\infty)) \zeta(\mathsf{d}_\infty(\cdot, \bar{x}_\infty) 2^{-k}) \, \mathrm{d}\mathfrak{m}_\infty, \quad (3.22)$$

from which it follows, taking into account that $\mathrm{supp}(1 - \zeta(\mathsf{d}_n(\cdot, \bar{x}_n) 2^{-k})) \subset X_n \setminus B_{2^k}(\bar{x}_n)$ for every $n \in \bar{\mathbb{N}}$ and the first in (3.20), that

$$\varlimsup_{n \to \infty} \left| \int \psi(\mathsf{d}_n(\cdot, \bar{x}_n)) \, \mathrm{d}\mathfrak{m}_n - \int \psi(\mathsf{d}_\infty(\cdot, \bar{x}_\infty)) \, \mathrm{d}\mathfrak{m}_\infty \right| \le \frac{\mathsf{S}}{2^{3k-2}}.$$

We now introduce the constants $\mathrm{z}_{n,[k],\psi}$ (resp. $\mathrm{z}_{n,\psi}$) and the probability measures $\mathfrak{m}_{n,[k],\psi}$ (resp. $\mathfrak{m}_{n,\psi}$) as in (3.11), starting from $\mathfrak{m}_{n,[k]}$ (resp. from $\mathfrak{m}_n$). By (3.22) we know that $\lim_{n \to \infty} \mathrm{z}_{n,[k],\psi} = \mathrm{z}_{\infty,[k],\psi}$. On the other hand, the last inequality can be rewritten as $\varlimsup_{n \to \infty} |\mathrm{z}_{n,\psi} - \mathrm{z}_{\infty,\psi}| \le \frac{\mathsf{S}}{2^{3k-2}}$, and given that this holds for every $k \in \mathbb{Z}$ we deduce

$$\lim_{n \to \infty} \mathrm{z}_{n,\psi} = \mathrm{z}_{\infty,\psi}. \quad (3.23)$$

Now define $\rho_n := \frac{\mathrm{d}\mathfrak{m}_{n,[k],\psi}}{\mathrm{d}\mathfrak{m}_{n,\psi}}$, $n \in \bar{\mathbb{N}}$, and build a transport plan $\boldsymbol{\gamma}_{n,k} \in \mathrm{Adm}(\mathfrak{m}_{n,\psi}, \mathfrak{m}_{n,[k],\psi})$ by 'letting the mass in common stand still and uniformly distributing the rest' or, more rigorously, define

$$\boldsymbol{\gamma}_{n,k} := (\boldsymbol{i}, \boldsymbol{i})_\sharp (\mathfrak{m}_{n,\psi}|_{\{\rho_n \ge 1\}}) + \frac{\mathfrak{m}_{n,\psi}|_{\{\rho_n < 1\}} \otimes (\mathfrak{m}_{n,[k],\psi} - \mathfrak{m}_{n,\psi}|_{\{\rho_n \ge 1\}})}{\mathfrak{m}_{n,\psi}|_{\{\rho_n < 1\}}(X_n)}.$$

It is readily checked that indeed $\boldsymbol{\gamma}_{n,k} \in \mathrm{Adm}(\mathfrak{m}_{n,\psi}, \mathfrak{m}_{n,[k],\psi})$. Since $\mathfrak{m}_{n,[k]} \le \mathfrak{m}_n$, we have $\mathrm{z}_{n,[k],\psi} \le \mathrm{z}_{n,\psi}$; moreover, $\rho_n \equiv \mathrm{z}_{n,\psi}/\mathrm{z}_{n,[k],\psi} \ge 1$ on $B_{2^k}(\bar{x}_n)$. Therefore, by construction, for

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$(x, y) \in \mathrm{supp}(\boldsymbol{\gamma}_{n,k})$ and $x \neq y$ we have $x \notin B_{2^k}(\bar{x}_n)$ and $y \in B_{2^{k+1}}(\bar{x}_n)$ and thus $\mathsf{d}_n(x, y) \leq 3\mathsf{d}_n(x, \bar{x}_n)$. Taking into account the second in (3.20) we deduce

$$
\begin{aligned}
W_2^2(\mathfrak{m}_{n,\psi}, \mathfrak{m}_{n,[k],\psi}) &\leq \int_{\{x \neq y\}} \mathsf{d}_n^2(x, y) \, \mathrm{d}\boldsymbol{\gamma}_{n,k}(x, y) \\
&\leq \frac{9}{\mathsf{z}_{n,\psi}} \int_{X \setminus B_{2^k}(\bar{x}_n)} \mathsf{d}_n^2(\cdot, \bar{x}_n) \psi(\mathsf{d}_n(\cdot, \bar{x}_n)) \, \mathrm{d}\mathfrak{m}_n \leq \frac{\mathsf{c}}{2^k}, \qquad \forall n \in \bar{\mathbb{N}},
\end{aligned}
\tag{3.24}
$$

where $\mathsf{c} := \sup_{n \in \bar{\mathbb{N}}} 9(\mathsf{z}_{n,\psi})^{-1} \mathsf{S}$ (notice that $\mathsf{c} < \infty$ by (3.23) and the fact that $\mathsf{z}_{\infty,\psi} \neq 0$). To conclude, recall that by the same argument we used in the proof of the implication $(\mathrm{C}) \Rightarrow (\mathrm{B})$ $(\iota_n)_\sharp \mathfrak{m}_{n,[k],\psi}$ weakly converges to $(\iota_\infty)_\sharp \mathfrak{m}_{\infty,[k],\psi}$ as $n \to \infty$ in $\mathscr{P}(Y)$ and thus, given that they have uniformly bounded support, by Proposition 2.3 we get that $W_2((\iota_n)_\sharp \mathfrak{m}_{n,[k],\psi}, (\iota_\infty)_\sharp \mathfrak{m}_{\infty,[k],\psi}) \to 0$ as $n \to \infty$. Hence

$$
\begin{aligned}
\varlimsup_{n \to \infty} W_2((\iota_n)_\sharp \mathfrak{m}_{n,\psi}, (\iota_\infty)_\sharp \mathfrak{m}_{\infty,\psi}) &\leq \varlimsup_{n \to \infty} W_2((\iota_n)_\sharp \mathfrak{m}_{n,\psi}, (\iota_n)_\sharp \mathfrak{m}_{n,[k],\psi}) \\
&\quad + \varlimsup_{n \to \infty} W_2((\iota_\infty)_\sharp \mathfrak{m}_{\infty,[k],\psi}, (\iota_\infty)_\sharp \mathfrak{m}_{\infty,\psi}) \overset{(3.24)}{\leq} 2\sqrt{\frac{\mathsf{c}}{2^k}}.
\end{aligned}
$$

In summary, choosing (Y, d_Y) and the embeddings $\{\iota_n\}_{n \in \bar{\mathbb{N}}}$ in the definition of $\mathbb{D}^\psi$, by (3.21), (3.23) and the last bound we get $\varlimsup_{n \to \infty} \mathbb{D}^\psi(\mathfrak{X}_n, \mathfrak{X}_\infty) \leq 2\sqrt{\frac{\mathsf{c}}{2^k}}$. Eventually letting $k \to +\infty$ we conclude. $\qquad \square$

By virtue of Theorem 3.15 we can propose the following definition:

Definition 3.16 (Pointed measured Gromov (pmG) convergence). *Let $\mathfrak{X}_n = [X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n]$, $n \in \bar{\mathbb{N}}$. We say that the sequence $n \mapsto \mathfrak{X}_n$ converges to $\mathfrak{X}_\infty$ in the pointed measured Gromov sense (pmG-sense, for short) provided any of the 4 equivalent statements in Theorem 3.15 holds.*

3.4 Basic topological properties of the pmG-convergence and some comments

We collect here some basic properties of the space of p.m.m. spaces equipped with pmG-convergence. We start with the following result:

Theorem 3.17. *The space $\mathbb{X}$ endowed with the distance $\mathrm{p}\mathbb{G}_\mathrm{W}$ is a complete and separable metric space.*

Proof.
Completeness. Let $n \mapsto \mathfrak{X}_n = [X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n] \in \mathbb{X}$ be a $\mathrm{p}\mathbb{G}_\mathrm{W}$-Cauchy sequence. Then for $k \in \mathbb{Z}$ the sequence $(\mathfrak{X}_{n,[k]}) \subset \mathbb{X}_{fm}$ is $\mathrm{p}\mathbb{G}_\mathrm{W}^{fm}$-Cauchy. With the same gluing procedure used in the proof of $(\mathrm{C}) \Rightarrow (\mathrm{B})$ in Theorem 3.15 we can find a complete separable space (Y_k, d_{Y_k}) and isometric embeddings $\iota_{n,k} : X_n \to Y_k$, $n \in \bar{\mathbb{N}}$, such that

$$
\lim_{n,m \to \infty} \left| \log \frac{\mathfrak{m}_{n,[k]}(X_n)}{\mathfrak{m}_{m,[k]}(X_m)} \right| + \mathsf{d}_{Y_k}(\iota_{n,k}(\bar{x}_n), \iota_{m,k}(\bar{x}_m)) + W_c((\iota_{n,k})_\sharp \tilde{\mathfrak{m}}_{n,[k]}, (\iota_{m,k})_\sharp \tilde{\mathfrak{m}}_{m,[k]}) = 0,
\tag{3.25}
$$

the measures $\mathfrak{m}_{n,[k]}$ being defined as in (3.15) and $\tilde{\mathfrak{m}}_{n,[k]}$ being their normalization. In particular, we see that

$$\text{the sequence } n \mapsto \mathfrak{m}_{n,[k]}(X_n) \text{ has limit } \mathsf{z}_k \in (0,\infty) \text{ for every } k \in \mathbb{Z},$$

and that

$$\text{the sequence } n \mapsto \iota_{n,k}(\bar{x}_n) \text{ has limit } \bar{y}_k \in Y_k \text{ for every } k \in \mathbb{Z},$$

and from the completeness of $(\mathscr{P}(Y_k), W_c)$ (recall (2.9)) we also deduce that there exists $\mathfrak{n}_k \in \mathscr{P}(Y_k)$ such that $W_c(\mathfrak{n}_k, (\iota_{n,k})_\sharp \tilde{\mathfrak{m}}_{n,[k]}) \to 0$ as $n \to \infty$. It is then clear that the sequence of non-pointed metric measure spaces $n \mapsto [X_n, \mathsf{d}_n, \mathfrak{m}_{n,[k]}]$ converges to $[Y_k, \mathsf{d}_{Y_k}, \mathsf{z}_k \mathfrak{n}_k]$ w.r.t. $\mathbb{G}_{\mathrm{W}}^{fm}$.

Now fix $k' < k-2$ and let $f_n : Y_k \to \mathbb{R}$ be given by $f_n(x) := \zeta\big(\mathsf{d}_{Y_k}(x, \iota_{n,k}(\bar{x}_n))2^{-k'-1}\big)$. Let $\bar{y}_k \in Y_k$ be the limit of $n \mapsto \iota_{n,k}(\bar{x}_n)$ (whose existence is granted by (3.25)) and notice that (f_n) uniformly converges to $f(x) := \zeta\big(\mathsf{d}_{Y_k}(x, \bar{y}_k)2^{-k'-1}\big)$ from which it follows that $f_n \iota_{n,k}\mathfrak{m}_{n,[k]}$ weakly converges to $\mathsf{z}_k f\mathfrak{n}_k$. In other words, for $k' < k$ the space (Y_k, d_{Y_k}) and the embeddings $\{\iota_{n,k}\}_{n\in\mathbb{N}}$ can be used as competitor in the definition of $\mathbb{G}_{\mathrm{W}}^{fm}([X_n, \mathsf{d}_n, \mathfrak{m}_{n,[k']}], [X_m, \mathsf{d}_m, \mathfrak{m}_{m,[k']}])$ to check that such sequence is $\mathbb{G}_{\mathrm{W}}^{fm}$-Cauchy. It is then clear that it converges to $[Y_k, \mathsf{d}_{Y_k}, \mathsf{z}_k f\mathfrak{n}_k]$.

The same argument also gives

$$(\mathsf{z}_k \mathfrak{n}_k)(B_{2^{k'+2}}(\bar{y}_k)) \geq \int f \, \mathsf{d}\mathsf{z}_k\mathfrak{n}_k = \lim_{n\to\infty} \int f_n \, \mathsf{d}\mathfrak{m}_{n,[k]}$$
$$\geq \varlimsup_{n\to\infty} \mathfrak{m}_{n,[k']}(B_{2^{k'+1}}(\bar{x}_n)) = \varlimsup_{n\to\infty} \mathfrak{m}_{n,[k']}(X_n) = \mathsf{z}_{k'} > 0,$$

and being this true for every $k' < k-2$, letting $k' \downarrow -\infty$ we obtain $\bar{y}_k \in \mathrm{supp}(\mathfrak{n}_k)$. This shows that $n \mapsto \mathfrak{X}_{n,[k]}$ converges to $(Y_k, \mathsf{d}_k, \mathfrak{n}_k, \bar{y}_k)$ w.r.t. $\mathrm{p}\mathbb{G}_{\mathrm{W}}^{fm}$.

To conclude, we need to show that there exists a p.m.m. space $\mathcal{Y} := [Y, \mathsf{d}, \mathfrak{n}, \bar{y}]$ such that $\mathcal{Y}_{[k]} = [Y_k, \mathsf{d}_k, \mathfrak{n}_k, \bar{y}_k]$, but this also follows by the above compatibility argument. Indeed, for $k' < k$ we know that there exists an isometry $\iota_{k'}^k : (\mathrm{supp}\,\mathfrak{n}_{k'}, \mathsf{d}_{Y_{k'}}) \to (\mathrm{supp}\,\mathfrak{n}_k, \mathsf{d}_{Y_k})$ such that $\iota_{k'}^k(\bar{y}_{k'}) = \bar{y}_k$ and $(\iota_{k'}^k)_\sharp \mathfrak{n}_{k'} = \zeta(\mathsf{d}_{Y_k}(\cdot, \bar{y}_k)2^{-k'})\mathfrak{n}_k$, hence the conclusion follows with the same gluing argument we already used in the proof of (C)$\Rightarrow$(B) of Theorem 3.15.

Separability. We simply observe that the map

$$B : \mathbb{X} \quad \to \quad \prod_{N=1}^{\infty} \mathscr{M}_{loc}(\mathbb{G}^N)$$

$$\mathcal{X} \quad \mapsto \quad (\mathfrak{m}_{\mathcal{X}}^N)_{N\in\mathbb{N}}$$

is a homeomorphism of $\mathbb{X}$ with its image $B(\mathbb{X})$ endowed with the product topology inherited by $\prod_{N=1}^{\infty} \mathscr{M}_{loc}(\mathbb{G}^N)$. Since this topology is separable and metrizable, we conclude that $\mathbb{X}$ is separable as well. $\qquad\square$

Remark 3.18. It is worth to underline that for any given ψ as in (3.10), the space $(\mathbb{X}^\psi, \mathbb{D}^\psi)$ is *not* complete. The problem is that the condition $\bar{x} \in \mathrm{supp}(\mathfrak{m})$ can be not satisfied in the limit. $\qquad\blacksquare$

Remark 3.19 (Different weight functions). In the proof of the implication (D)$\Rightarrow$(C) of Theorem 3.15 we used the explicit formula (3.12) for the weight function ψ only to deduce

the bounds (3.20). The very same arguments used there actually show that the following slightly stronger statement holds: if $n \mapsto \mathcal{X}_n = [X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n]$ is a sequence converging to $\mathcal{X}_\infty = [X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty]$ in the pmG-sense and ψ as in (3.10) is such that

$$\mathcal{X}_n \in \mathbb{X}^\psi, \ \forall n \in \mathbb{N}, \qquad \text{and} \qquad \lim_{R \to \infty} \sup_{n \in \mathbb{N}} \int_{X_n \setminus B_R(\bar{x}_n)} \mathsf{d}^2(\cdot, \bar{x}_n) \psi(\mathsf{d}(\cdot, \bar{x}_n)) \, \mathrm{d}\mathfrak{m}_n = 0,$$
(3.26)

then

$$\lim_{n \to \infty} \mathbb{D}^\psi(\mathcal{X}_n, \mathcal{X}_\infty) = 0.$$

In particular, we see that if $\psi_1. \psi_2$ are as in (3.10) and $\psi_1(r) \le \psi_2(r)$ for every r sufficiently big, then $\mathbb{X}^{\psi_2} \subset \mathbb{X}^{\psi_1}$ and

$$(\mathcal{X}_n)_{n \in \bar{\mathbb{N}}} \subset \mathbb{X}^{\psi_2}, \qquad \lim_{n \to \infty} \mathbb{D}^{\psi_2}(\mathcal{X}_n, \mathcal{X}_\infty) = 0 \qquad \Rightarrow \qquad \lim_{n \to \infty} \mathbb{D}^{\psi_1}(\mathcal{X}_n, \mathcal{X}_\infty) = 0.$$

∎

Remark 3.20 (A limit case concerning weights). In connection with Remark 3.19 above it is worth to underline that it may happen that $(\mathcal{X}_n)$ pmG-converges to $\mathcal{X}_\infty$, that for some ψ as in (3.10) we have $\mathcal{X}_n \in \mathbb{X}^\psi$ for every $n \in \bar{\mathbb{N}}$, yet that $\mathbb{D}^\psi(\mathcal{X}_n, \mathcal{X}_\infty)$ does not go to 0.

An explicit example in this direction is given by $\mathcal{X}_n := [\mathbb{R}, \mathsf{d}_{\mathrm{Eucl}}, \mu_n, 0]$, $n \in \bar{\mathbb{N}}$, where $\mathsf{d}_{\mathrm{Eucl}}$ is the standard Euclidean distance and $n \mapsto \mu_n \in \mathscr{P}_2(\mathbb{R})$ is a sequence weakly converging to $\mu_\infty \in \mathscr{P}_2(\mathbb{R})$ which is not W_2-converging. Then the choice $\psi \equiv 1$ produces the desired behaviour.

It is then easy to see that if $\mathcal{X}_n \in \mathbb{X}^\psi$ for every $n \in \bar{\mathbb{N}}$ and $\mathcal{X}_n \overset{\mathrm{pmG}}{\to} \mathcal{X}_\infty$ but $\mathbb{D}^\psi(\mathcal{X}_n, \mathcal{X}_\infty)$ does not go to 0, then for every ψ' such that $\lim_{r \to \infty} \frac{\psi'(r)}{\psi(r)} = 0$ we have $\mathbb{D}^{\psi'}(\mathcal{X}_n, \mathcal{X}_\infty) \to 0$, as the faster decrease of ψ' ensures the 2-uniform integrability of the rescaled measures. ∎

Remark 3.21 (Distortion distances). In [44] Sturm introduced the distortion distance between normalized m.m. spaces. The induced notion of convergence of sequences of m.m. spaces can easily be adapted to the case of p.m.m. spaces along the same lines used here. Then Corollary 2.10 of [44] shows that for a given sequence $(\mathcal{X}_n)$ of normalized p.m.m. spaces with uniformly bounded diameter, convergence in the distortion distance to a limit space $\mathcal{X}_\infty$ is equivalent to pmG-convergence. ∎

By a diagonal argument, the characterization of Gromov-weak convergence given by Theorem 3.15(C), and Proposition 7.1 in [28] we obtain the following compactness result.

Corollary 3.22 (Compactness). *Let* $\mathcal{X}_n = [X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n]$, $n \in \mathbb{N}$ *be a sequence of p.m.m. spaces and let* $\mathcal{X}_{n,[k]}$ *be defined as in* (3.16). $(\mathcal{X}_n)_{n \in \mathbb{N}}$ *is precompact in* $\mathbb{X}$ *if and only if for every* $k \in \mathbb{Z}$ *the sequence* $(\mathcal{X}_{n,[k]})_{n \in \mathbb{N}}$ *is precompact in* $\mathbb{X}_{fm}$.

In particular $(\mathcal{X}_n)_{n \in \mathbb{N}}$ *is precompact in* $\mathbb{X}$ *if and only if*

1. *for every* $k \in \mathbb{Z}$ *the sequence* $n \mapsto \log\left(\mathfrak{m}_n(B_{2^k}(\bar{x}_k))\right)$ *is bounded;*

2. *for every* $k \in \mathbb{Z}$ *and* $\varepsilon > 0$ *there exists* $N_{\varepsilon,k} \in \mathbb{N}$ *and subsets* $X_{\varepsilon,k,n} \subset B_{2^k}(\bar{x}_n) \subset X_n$, $n \in \mathbb{N}$, *such that* $\mathfrak{m}_n(B_{2^k}(\bar{x}_n) \setminus X_{\varepsilon,k,n}) \le \varepsilon$ *and* $X_{\varepsilon,k,n}$ *can be covered by at most* $N_{\varepsilon,k}$ *balls of radius* ε.

We conclude the section pointing out that the extrinsic approach to pmG-convergence is the only one, out of the 4 that we analyzed, which relies on some additional structure: it needs not only the equivalence classes $\{\mathfrak{X}_n\}_{n \in \bar{\mathbb{N}}}$, but also the space (X, d) and the isometric embeddings $\{\iota_n\}_{n \in \bar{\mathbb{N}}}$.

As said, we refer to such (X, d) and ι_n's as an *effective realization* of the pmG-convergence. To choose an effective realization amounts, in a sense, to choose 'the way the $\mathfrak{X}_n$'s are converging to $\mathfrak{X}_\infty$'. To see this, consider the following definition:

Definition 3.23 (Convergence of points belonging to converging spaces)**.** *Let* $\mathfrak{X}_n = [X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n]$ *be converging to* $\mathfrak{X}_\infty = [X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty]$ *in the pmG-sense and* (X, d), $\{\iota_n\}_{n \in \bar{\mathbb{N}}}$ *an effective realization of the convergence.*

Then we say that $n \mapsto x_n \in \operatorname{supp}(\mathfrak{m}_n)$ *is converging to* $x_\infty \in \operatorname{supp}(\mathfrak{m}_\infty)$ *via such realization provided*

$$\mathsf{d}\big(\iota_n(x_n), \iota_\infty(x_\infty)\big) \to 0,$$

as $n \to \infty$.

In other words, we are identifying $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ with the isomorphic space $(X, \mathsf{d}, (\iota_n)_\sharp \mathfrak{m}_n, \iota_n(\bar{x}_n))$, $n \in \bar{\mathbb{N}}$, and then reading the convergence at the level of points in X.

It is possible to see that if the limit space $\mathfrak{X}_\infty$ admits no non-trivial automorphisms, then such notion of convergence is in fact independent on the effective realization (this is in fact an if-and-only-if, the argument makes use of the compactness given by Proposition 3.2, we omit the details). Yet, in general, to fix an effective realization truly affects this convergence; to see this just consider the case where all the $\mathfrak{X}_n$'s are $\mathbb{R}^2$ equipped with the Euclidean distance, the Lebesgue measure and pointed at the origin. It is then clear that in embedding all of them in a common, say, $\mathbb{R}^2$, we are free to 'rotate' each one of any desired angle.

In this sense, to fix an effective realization is a non-intrinsic choice. Yet, from the technical point of view to have at disposal the above notion of convergence of points is very useful in order both to state and to prove results about converging sequences of spaces. Hence we shall adopt this point of view when speaking of pmG-convergence of spaces with a lower bound on the Ricci, see Remark 4.6. The fact that the results proven this way are intrinsic can then be recovered noticing that they do not rely on the particular effective realization chosen.

3.5 Relation with pointed measured Gromov-Hausdorff convergence

We now analyze the relation between pmG-convergence and pointed measured Gromov-Hausdorff convergence. The definition below is adapted from Definition 8.1.1 in [15] adding the requirement of weak convergence at the level of measures.

Definition 3.24 (Pointed measured Gromov Hausdorff convergence)**.** *Let* $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, *be pointed metric measure spaces as in* (3.1a,b)*. We say that* $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *in the pointed measured Gromov Hausdorff (pmGH) sense, provided for any* $\varepsilon, R > 0$ *there exists* $N(\varepsilon, R) \in \mathbb{N}$ *such that for all* $n \geq N(\varepsilon, R)$ *there exists a Borel map* $f_n^{R,\varepsilon} : B_R(\bar{x}_n) \to X_\infty$ *such that*

- $f_n^{R,\varepsilon}(\bar{x}_n) = \bar{x}_\infty$,

- $\sup_{x,y \in B_R(\bar{x}_n)} |\mathsf{d}_n(x, y) - \mathsf{d}_\infty(f_n^{R,\varepsilon}(x), f_n^{R,\varepsilon}(y))| \leq \varepsilon$,

- *the ε-neighbourhood of $f_n^{R,\varepsilon}(B_R(\bar{x}_n))$ contains $B_{R-\varepsilon}(\bar{x}_\infty)$,*

- *$(f_n^{R,\varepsilon})_\sharp(\mathfrak{m}_n|_{B_R(\bar{x}_n)})$ weakly converges to $\mathfrak{m}_\infty|_{B_R(x_\infty)}$ as $n \to \infty$, for a.e. $R > 0$.*

Remark 3.25 (All the space matters)**.** Notice the technical difference between this definition and the one of pmG-convergence. Here the entire sets X_n, $n \in \bar{\mathbb{N}}$, matter, and not only the portion $\mathrm{supp}(\mathfrak{m}_n)$. This means in particular that this notion of convergence is only defined for p.m.m. spaces and not for their equivalence classes.

It is certainly possible to force the choice of the representative in the class by requiring to deal with spaces $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ such that $\mathrm{supp}(\mathfrak{m}) = X$, but this would produce examples of non-convergence like the one discussed at the end of the introduction. $\blacksquare$

Remark 3.26 (Uniform vs weak convergence)**.** Another, more important, difference between pmGH convergence and pmG-convergence is that in the pmGH case distances are asked to converge uniformly on bounded sets, whereas for pmG the convergence of distances is only required implicitly via the weak convergence of measures. This means that in the pmG-case distances are only required to converge in an appropriate weighted, or weak, sense. $\blacksquare$

Remark 3.27. Due to the aforementioned uniform convergence on bounded sets, typically when speaking about pmGH-convergence one assumes the spaces to be proper (i.e. bounded closed set are compact). We didn't do so just to keep a slightly higher level of generality, but this is not the main point in the discussion. $\blacksquare$

It is worth to point out that the maps $f_n^{R,\varepsilon}$ in the definition of pmGH-convergence can be chosen to be independent on R, ε:

Proposition 3.28 (Equivalent definition of pointed mGH convergence)**.** *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, be pointed metric measure spaces as in (3.1a,b). Then the following are equivalent.*

A) *$(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pointed measured Gromov-Hausdorff sense.*

B) *There are sequences $R_n \uparrow +\infty$, $\varepsilon_n \downarrow 0$ and Borel maps $f_n : X_n \to X_\infty$ such that*

     1) $f_n(\bar{x}_n) = \bar{x}_\infty$,
     2) $\sup_{x,y \in B_{R_n}(\bar{x}_n)} |\mathsf{d}_n(x,y) - \mathsf{d}_\infty(f_n(x), f_n(y))| \le \varepsilon_n$,
     3) the ε_n-neighbourhood of $f_n(B_{R_n}(\bar{x}_n))$ contains $B_{R_n - \varepsilon_n}(\bar{x}_\infty)$,
     4) for any $\varphi \in \mathrm{C}_b(X_\infty)$ with bounded support it holds $\lim_{n \to \infty} \int \varphi \circ f_n \, \mathrm{d}\mathfrak{m}_n = \int \varphi \, \mathrm{d}\mathfrak{m}_\infty$.

Proof. The implication $(B) \Rightarrow (A)$ is obvious, so we prove $(A) \Rightarrow (B)$. Let $\bar{R}_k \uparrow +\infty$, $\bar{\varepsilon}_k \downarrow 0$ be two arbitrary sequences and put $N_k := N(\bar{\varepsilon}_k, \bar{R}_k)$. It is not restrictive to assume that $N_{k+1} > N_k$ for any $k \in \mathbb{N}$ and that $N_k \uparrow +\infty$. Define $R_n \uparrow +\infty$ and $\varepsilon_n \downarrow 0$ as: R_n and ε_n are arbitrary for $n < N_1$, while for $N_k \le n < N_{k+1}$ we put $R_n := \bar{R}_k$ and $\varepsilon_n := \bar{\varepsilon}_k$.

The functions $f_n : X_n \to X_\infty$ are then defined as follows: f_n is arbitrary for $n < N_1$, for $N_k \le n < N_{k+1}$, f_n is arbitrary on $B_{R_n}^c(\bar{x}_n)$ as well, while on $B_{R_n}(\bar{x}_n)$ we put $f_n := f_n^{R_n, \varepsilon_n}$.

Possibly redefining (R_n), (ε_n) and (f_n) for $n < N_1$ in order to achieve $(B2)$ and $(B3)$ even for small n's, it is readily checked that the construction gives the thesis. $\square$

Remark 3.29. With a little bit of work one can see that if the spaces (X_n, d_n), $n \in \mathbb{N}$ are length spaces (i.e. the distance is realized as infimum of length of curves), then (B) above can be replaced by

B') There are sequences $R_n \uparrow +\infty$, $\varepsilon_n \downarrow 0$ and Borel maps $f_n : X_n \to X_\infty$ such that

1') $f_n(\bar{x}_n) = \bar{x}_\infty$,

2') $\sup_{x,y \in B_{R_n}(\bar{x}_n)} |d_n(x,y) - d_\infty(f_n(x), f_n(y))| \leq \varepsilon_n$ and $f(B_{R_n}(\bar{x}_n)) \subset B_{R_n}(\bar{x}_\infty)$

3') the ε_n-neighbourhood of $f_n(B_{R_n}(\bar{x}_n))$ contains $B_{R_n}(\bar{x}_\infty)$

4') for any $\varphi \in C_b(X_\infty)$ with bounded support it holds $\lim_{n \to \infty} \int \varphi \circ f_n \, d\mathfrak{m}_n = \int \varphi \, d\mathfrak{m}_\infty$.

Indeed, starting from (B) we double ε_n to achieve $(3')$, then we modify a bit f_n so that the image of $f_n(B_{R_n}(\bar{x}_n))$ of $B_{R_n}(\bar{x}_n)$ is contained in $B_{R_n}(\bar{x}_\infty)$, in this way we achieve $(2)'$: this is done by shrinking a bit the original image along almost minimizing curves connecting to $\bar{x}_\infty$. With this procedure we alter distances of at most $3\varepsilon_n$, thus we keep the desired convergence.

The approach (B') is the one chosen by Villani in Definition 27.30 (third part) of [45], so that all the stability statements proved in Chapter 29 of [45] under pointed mGH convergence are still true in our framework provided the involved spaces are length spaces (which will be always the case in the paper).

Let us finally mention that (B') gives the same convergence as the Gromov-Hausdorff-Prokhorov metric studied in [39] in case of locally compact length spaces endowed with locally finite measures; here the authors prove that such a metric induces a structure of Polish space on the equivalence classes of p.m.m.s. as above. ∎

Due to Remark 3.26, the following result is not surprising.

Theorem 3.30 (From pmGH to pmG-convergence). *Let* $(X_n, d_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, *be p.m.m. spaces converging to* $(X_\infty, d_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *in the pmGH-sense. Then* $[X_n, d_n, \mathfrak{m}_n, \bar{x}_n] \to [X_\infty, d_\infty, \mathfrak{m}_\infty, \bar{x}_\infty]$ *w.r.t.* pmG-*convergence.*

Proof. Let (R_n), (ε_n) and (f_n) as in part (B) of Proposition 3.28. Define $Y := \sqcup_{n \in \bar{\mathbb{N}}} X_n$ and the separable pseudodistance d on Y as

$$d(y_1, y_2) := \begin{cases} d_n(y_1, y_2), & \text{if } y_1, y_2 \in X_n, \\ d_\infty(y_1, y_2), & \text{if } y_1, y_2 \in X_\infty, \\ \displaystyle\inf_{y_1' \in B_{R_n}(x_n)} d_n(y_1', y_1) + d_\infty(f_n(y_1'), y_2), & \text{if } y_1 \in X_n, \ y_2 \in X_\infty, \\ d(y_2, y_1), & \text{if } y_1 \in X_\infty, \ y_2 \in X_n. \end{cases}$$

Identifying points $x, y \in Y$ such that $d(x, y) = 0$ and taking if necessary the completion, it is easy to see that (Y, d) is complete and separable. Also, it holds

$$d(y, f_n(y)) \leq \varepsilon_n \qquad \forall y \in X_n \subset Y, \ \text{such that } d(y, \bar{x}_n) \leq R_n.$$

By definition we have $d(\bar{x}_n, \bar{x}_\infty) = \varepsilon_n \downarrow 0$.

Let us now prove that

$$\phi(r) := \sup_n \mathfrak{m}_n(B_r(\bar{x}_n)) < \infty \quad \text{for every } r \geq 0;$$

Let us set $\varepsilon := \sup_n \varepsilon_n$ and observe that choosing $\varphi(y) := 0 \vee (r + 1 + \varepsilon - d(y, \bar{x}_\infty)) \wedge 1$ we have

$$\varphi(f_n(y)) = 1 \quad \text{if } y \in X_n \cap B_r(\bar{x}_n) \text{ and } R_n \geq r,$$

so that B4) yields

$$\limsup_{n\to\infty} \mathfrak{m}_n(B_r(\bar{x}_n)) \leq \limsup_{n\to\infty} \int_{X_n} \varphi \circ f_n \, \mathrm{d}\mathfrak{m}_n = \int_X \varphi \, \mathrm{d}\mathfrak{m}_\infty \leq \mathfrak{m}_\infty(B_{r+1+\varepsilon}(\bar{x}_\infty)).$$

In order to conclude our proof it is sufficient to check that

$$\lim_{n\to\infty} \int \zeta \, \mathrm{d}\mathfrak{m}_n = \int \zeta \, \mathrm{d}\mathfrak{m}_\infty \tag{3.27}$$

for every $\zeta \in C_{bs}(Y)$. A standard approximation argument (see e.g. [5, § 5.1]) shows that it is not restrictive to assume ζ 1-Lipschitz. If $\mathrm{supp}(\zeta) \subset B_R(\bar{x}_\infty)$ and $R + \varepsilon \leq R_n$ we have

$$\left| \int \zeta \, \mathrm{d}\mathfrak{m}_n - \int \zeta \circ f_n \, \mathrm{d}\mathfrak{m}_n \right| \leq \int_{B_{R+\varepsilon}(\bar{x}_n)} |\zeta - \zeta \circ f_n| \, \mathrm{d}\mathfrak{m}_n \leq \phi(R+\varepsilon)\varepsilon_n \to 0 \quad \text{as } n \uparrow \infty.$$

(3.27) then follows by B4). $\qquad\square$

It is worth underlying the difference between 'uniform' and 'weak' convergence of distances alluded to in Remark 3.26 with an explicit example (see also the end of the introduction).

Example 3.31. Let $X_n := [0, 1-1/n] \cup [2, 2+1/n] \subset \mathbb{R}$ and $X_\infty := [0, 1]$, all of them endowed with the (restriction of) Euclidean distance and Lebesgue measure. The fact that $\mathrm{diam} X_n \geq 2$ for every $n \in \mathbb{N}$ and $\mathrm{diam} X_\infty = 1$ shows that there is no (pointed) mGH convergence of X_n to X_∞. On the other hand, pmG-convergence is obvious. $\qquad\blacksquare$

Hence in general pmG-convergence does not imply pointed mGH convergence. A natural assumption which allows to have such reverse implication is to assume the spaces to be uniformly doubling. Recall that $(X, \mathsf{d}, \mathfrak{m})$ is called c-doubling provided it holds

$$\mathfrak{m}(B_{2R}(x)) \leq \mathsf{c}\,\mathfrak{m}(B_R(x)), \qquad \forall x \in X, \ R > 0. \tag{3.28}$$

Notice that the doubling condition imposed as in (3.28) forces $\mathrm{supp}(\mathfrak{m}) = X$ and in particular it does not pass to the quotient to a condition on equivalence classes of p.m.m. spaces (to get such property one could impose (3.28) only for $x \in \mathrm{supp}(\mathfrak{m})$).

It is easy to see that a family of bounded and c-doubling spaces is uniformly totally bounded (i.e. for any ε there exists n_ε such that each of the spaces can be covered by at most n_ε balls of radius ε), whence the following compactness criterion holds (see e.g. Theorem 8.1.10 of [15] for a proof).

Lemma 3.32. *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, be a sequence of p.m.m. spaces as in (3.1a), (3.1b). Assume that for some $\mathsf{c} > 0$, all of them are c-doubling and there exists $r > 0$ such that $\sup_n \mathfrak{m}_n(B_r(\bar{x}_n)) < \infty$.*
Then the sequence is precompact in the pmGH topology and any limit space $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ is c-doubling as well.

Under the doubling assumption, we can show that pmG-convergence implies pmGH convergence (following the lines used by Sturm in [42]).

Theorem 3.33 (From pmG to pmGH). *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, be p.m.m. spaces as in (3.1a,b). Assume that*

i) $[X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n] \to [X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty]$ *w.r.t.* pmG-*convergence,*

ii) $\operatorname{supp}(\mathfrak{m}_\infty) = X_\infty$,

iii) for some $\mathsf{c} > 0$ *the spaces* $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ *are all* c-*doubling.*

Then $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *in the pmGH sense.*

Proof. Assumption *(iii)* and Lemma 3.32 give that some subsequence $k \mapsto (X_{n_k}, \mathsf{d}_{n_k}, \mathfrak{m}_{n_k}, \bar{x}_{n_k})$ converges to some $(X'_\infty, \mathsf{d}'_\infty, \mathfrak{m}'_\infty, \bar{x}'_\infty)$ in the pointed measured Gromov-Hausdorff sense.

Theorem 3.30 shows that $k \mapsto [X_{n_k}, \mathsf{d}_{n_k}, \mathfrak{m}_{n_k}, \bar{x}_{n_k}]$ converges to $[X'_\infty, \mathsf{d}'_\infty, \mathfrak{m}'_\infty, \bar{x}'_\infty]$ w.r.t. pmG-convergence as well. Hence $[X'_\infty, \mathsf{d}'_\infty, \mathfrak{m}'_\infty, \bar{x}'_\infty] = [X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty]$ and recalling the approach to pmG-convergence given by the extrinsic approach (Definition 3.9), Proposition 3.28 and using assumption (ii) we easily deduce that $k \mapsto (X_{n_k}, \mathsf{d}_{n_k}, \mathfrak{m}_{n_k}, \bar{x}_{n_k})$ pmGH-converges to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$.

Being this result independent on the converging subsequence chosen at the beginning, the thesis is proved. $\qquad\square$

4 Stability of lower Ricci curvature bounds under pmG-convergence

4.1 Preliminaries

4.1.1 Relative entropy

From now on, we shall only work with p.m.m. spaces as in (3.1) such that the bound

$$\mathfrak{m}(B_r(\bar{x})) \leq \mathsf{c}_1 \mathrm{e}^{\mathsf{c}_2 r^2}, \qquad \forall r > 0, \tag{4.1}$$

holds for some $\mathsf{c}_1, \mathsf{c}_2$. As we shall see in Remark 4.4 below, such restriction is fully justified when working on $\mathrm{CD}(K, \infty)$ spaces.

Given a p.m.m. space $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ as in (3.1a,b) satisfying (4.1), the relative entropy functional $\operatorname{Ent}_{\mathfrak{m}} : \mathscr{P}_2(X) \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\operatorname{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \displaystyle\int \rho \log(\rho) \, \mathrm{d}\mathfrak{m} & \text{if } \mu = \rho\mathfrak{m} \ll \mathfrak{m}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2}$$

To check that this is a good definition, recall that if the reference measure $\mathfrak{m}$ is a probability measure, then it is well known that $\operatorname{Ent}_{\mathfrak{m}}$ is lower semicontinuous w.r.t. weak convergence and non negative. For spaces satisfying (4.1), we pick $\mathsf{C} > \mathsf{c}_2$ and define

$$\mathsf{z} := \int \mathrm{e}^{-\mathsf{C}\mathsf{d}^2(x, \bar{x})} \, \mathrm{d}\mathfrak{m}(x) \in (0, \infty), \quad \text{so that} \quad \tilde{\mathfrak{m}} := \frac{1}{\mathsf{z}} \mathrm{e}^{-\mathsf{C}\mathsf{d}^2(\cdot, \bar{x})} \mathfrak{m} \in \mathscr{P}_2(X). \tag{4.3}$$

Then we observe that for $\mu = \rho\mathfrak{m}$ it holds $\mu = \mathsf{z}\rho\mathrm{e}^{\mathsf{C}\mathsf{d}^2(\cdot, \bar{x})}\tilde{\mathfrak{m}}$ and that since $\tilde{\mathfrak{m}} \in \mathscr{P}(X)$, the negative part of $\rho \log(\rho)$ is in $L^1(\mathfrak{m})$, so that we get

$$\operatorname{Ent}_{\mathfrak{m}}(\mu) = \operatorname{Ent}_{\tilde{\mathfrak{m}}}(\mu) - \mathsf{C} \int \mathsf{d}^2(\cdot, \bar{x}) \, \mathrm{d}\mu - \log \mathsf{z}. \tag{4.4}$$

In particular, the definition in (4.2) is well-posed and $\text{Ent}_{\mathfrak{m}}$ is lower semicontinuous w.r.t. W_2.

We will denote by $D(\text{Ent}_{\mathfrak{m}}) \subset \mathscr{P}_2(X)$ the set of μ's such that $\text{Ent}_{\mathfrak{m}}(\mu) < \infty$.

A crucial property of the relative entropy w.r.t. probability measures is that it is jointly lower semicontinuous w.r.t. weak convergence on the two variables, i.e.

$$\left.\begin{array}{l}
(\tilde{\mathfrak{m}}_n), (\mu_n) \subset \mathscr{P}(X), \\
\tilde{\mathfrak{m}}_n \to \tilde{\mathfrak{m}}_\infty \in \mathscr{P}(X) \text{ weakly} \\
\mu_n \to \mu_\infty \in \mathscr{P}(X) \text{ weakly}
\end{array}\right\} \quad \Rightarrow \quad \text{Ent}_{\tilde{\mathfrak{m}}_\infty}(\mu_\infty) \leq \varliminf_{n\to\infty} \text{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_n). \qquad (4.5)$$

See for instance [45, Theorem 29.20], [11, Lemma 6.2] for a proof.

4.1.2 Compactness criterions

Let (Z, d_Z) be a complete and separable metric space.

A useful fact that we will often use in the following is that sublevels of the relative entropy w.r.t. probability measures are automatically tight:

Proposition 4.1. *Let (Z, d_Z) be a complete separable metric space and let $(\tilde{\mathfrak{m}}_n), (\mu_n) \subset \mathscr{P}(Z)$ be two sequences of measures. Assume that $(\tilde{\mathfrak{m}}_n)$ is tight and that $\sup_n \text{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_n) < \infty$. Then (μ_n) is tight as well.*

Proof. Notice that $r \mapsto r\log(r)$ is convex and bounded from below by $-\frac{1}{e}$. Hence by Jensen's inequality, for any Borel set $E \subset Z$ and for $\mu_n = \rho_n \tilde{\mathfrak{m}}_n$ it holds

$$\mu_n(E) \log\left(\frac{\mu_n(E)}{\tilde{\mathfrak{m}}_n(E)}\right) \leq \int_E \rho_n \log(\rho_n) \, \mathrm{d}\tilde{\mathfrak{m}}_n \leq \text{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_n) + \frac{\tilde{\mathfrak{m}}_n(Z \setminus E)}{e} \leq \sup_n \text{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_n) + \frac{1}{e}.$$

Hence if $\sup_n \tilde{\mathfrak{m}}_n(E_k) \to 0$, then also $\sup_n \mu_n(E_k) \to 0$. Conclude using the tightness of $(\tilde{\mathfrak{m}}_n)$. $\qquad\square$

4.1.3 Geodesics

Let (Z, d_Z) be a complete and separable metric space and $z^0, z^1 \in Z$. A curve $[0,1] \ni t \mapsto z_t \in Z$ is said geodesic connecting z_0 to z_1 provided $z_0 = z^0$, $z_1 = z^1$ and

$$\mathsf{d}_Z(z_t, z_s) = |s - t|\mathsf{d}_Z(z_0, z_1), \qquad \forall t, s \in [0,1].$$

The set of all geodesics is denoted by $\text{Geo}(Z)$ and it is a closed subset of the complete and separable metric space $C^0([0,1]; Z)$ endowed with the sup distance (possibly reduced to trivial constant curves, if Z has no geodesics connecting two distinct points). We will denote by $e_t : C^0([0,1]; Z) \to Z$, $t \in [0,1]$, the evaluation map $e_t(z) := z_t$.

A geodesic $(\nu_t) \subset \mathscr{P}_2(Z)$ w.r.t. the W_2 distance can always be lifted to a 'geodesic plan' on $\mathscr{P}(\text{Geo}(Z))$ in the following sense (see [32] for a proof):

Proposition 4.2. *Let $(\nu_t) \subset \mathscr{P}_2(Z)$ be a geodesic in $(\mathscr{P}_2(Z), W_2)$. Then there exists a plan $\boldsymbol{\pi} \in \mathscr{P}(\text{Geo}(Z))$ such that*

$$(e_t)_\sharp \boldsymbol{\pi} = \nu_t, \qquad \forall t \in [0,1],$$

$$W_2^2(\nu_0, \nu_1) = \int \mathsf{d}_Z^2(\gamma_0, \gamma_1) \, \mathrm{d}\boldsymbol{\pi}(\gamma).$$

4.1.4 CD(K, ∞) spaces

Definition 4.3 (CD(K, ∞) spaces). *Let $K \in \mathbb{R}$. We say that a (pointed) metric measure space $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ as in (3.1a,b) with (4.3) has Ricci curvature bounded from below by $K \in \mathbb{R}$ provided for any $\mu^0, \mu^1 \in D(\mathrm{Ent}_\mathfrak{m})$ there exists a W_2-geodesic (μ_t) such that $\mu_0 = \mu^0$, $\mu_1 = \mu^1$ and*

$$\mathrm{Ent}_\mathfrak{m}(\mu_t) \leq (1-t)\mathrm{Ent}_\mathfrak{m}(\mu_0) + t \, \mathrm{Ent}_\mathfrak{m}(\mu_1) - \frac{K}{2} t(1-t) W_2^2(\mu_0, \mu_1).$$

Notice that in this definition, only the portion of the space in supp$(\mathfrak{m})$ plays a role, because if $\mu(X \setminus \mathrm{supp}(\mathfrak{m})) > 0$ then certainly $\mathrm{Ent}_\mathfrak{m}(\mu) = \infty$. It follows that the property of being a CD(K, ∞) space passes to the quotient and is well defined for equivalence classes $\mathcal{X}^*, \mathcal{X}$ of m.m. spaces and p.m.m. spaces respectively.

Remark 4.4. In the present approach, we defined the relative entropy only on spaces satisfying the exponential growth condition (4.3), which grants, as already noticed, that $\mathrm{Ent}_\mathfrak{m}$ is well defined and W_2-lower semicontinuous on $(\mathscr{P}_2(X), W_2)$.

It would be possible to define the relative entropy functional also on general p.m.m. spaces not satisfying (4.3), e.g. by restricting its domain of definition: if $\mathfrak{m}$ is finite on bounded sets, then certainly $\mathrm{Ent}_\mathfrak{m}$ is well defined on probability measures with bounded support (see also, e.g., [42] for a different approach). It is however important to underline that whatever - meaningful - definition of $\mathrm{Ent}_\mathfrak{m}$ one chooses, on a CD(K, ∞) space $(X, \mathsf{d}, \mathfrak{m})$, for every $\bar{x} \in \mathrm{supp}(\mathfrak{m})$ the volume function $v(r) := \mathfrak{m}(B_r(\bar{x}))$ satisfies the exponential growth rate

$$v(R) \leq \mathsf{c}(\varepsilon, v(2\varepsilon), v(2\varepsilon)/v(\varepsilon)) \exp\left((1 + K_-)R^2\right) \quad \text{for every } R \geq 2\varepsilon > 0, \qquad (4.6)$$

where $\mathsf{c} : [0, \infty) \times [0, \infty) \times (0, \infty) \to (0, \infty)$ is a constant continuously depending on its parameters, see the argument in Theorem 4.24 in [42]. In particular, (4.3) always holds on CD(K, ∞) spaces. ∎

4.2 Setting up the problem

We now discuss the stability of lower Ricci curvature bounds under pmG-convergence. To this aim, notice that Theorem 3.33 and Remark 3.29 imply that all the curvature dimension conditions CD(K, N), CD$^*(K, N)$, MCP(K, N) with finite N are stable w.r.t. pmG-convergence (see [43], [33], [13] for the various definitions). Indeed if $(X, \mathsf{d}, \mathfrak{m})$ fulfils any of these, it is doubling and $(\mathrm{supp}(\mathfrak{m}), \mathsf{d})$ is geodesic. Hence we can apply first Theorem 3.33 to get that the considered sequence is converging in the pointed mGH sense, then use Remark 3.29 to reduce to the definition of pointed mGH convergence used by Villani and finally use his stability statements Theorem 29.25 in [45] (he deals only with CD(K, N), but the proofs directly generalize to CD$^*(K, N)$ and MCP(K, N) - also, it should be pointed out that there are slight variants in the definition of CD(K, N), but all of these are stable w.r.t. pointed mGH convergence and hence w.r.t. pmG-convergence).

Therefore the only case left out of the analysis is the one of CD(K, ∞) spaces, as this condition does not enforce doubling nor any sort of local compactness. In the rest of the work we thus concentrate on convergence of this sort of spaces.

In order to fix the ideas and the notation that will be used later on, we start with the following simple proposition:

Proposition 4.5. *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ be a sequence of $\mathrm{CD}(K, \infty)$ spaces converging to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ under pmG-convergence.*

Then there exists a constant $\mathsf{c}_1 > 0$ such that

$$\mathfrak{m}_n(B_R(\bar{x}_n)) \leq \mathsf{c}_1 \exp\left((1 + K_-)\, R^2\right) \quad \text{for every } n \in \mathbb{N}, \ R \geq 0, \tag{4.7}$$

and choosing

$$\psi(r) := \mathrm{e}^{-\mathsf{C}r^2} \ \text{ with } \mathsf{C} > 1 + K_-,$$

we have

$$\mathcal{X}_n := [X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n] \in \mathbb{X}^\psi \text{ for every } n \in \bar{\mathbb{N}} \qquad and \qquad \lim_{n \to \infty} \mathbb{D}^\psi(\mathcal{X}_n, \mathcal{X}_\infty) = 0.$$

Proof. Choosing e.g. $\varepsilon = 1$ in (4.6) and observing that

$$\liminf_{n \to \infty} \mathfrak{m}_n(B_1(\bar{x}_n)) \geq \mathfrak{m}_\infty(B_1(\bar{x}_\infty)) > 0, \quad \limsup_{n \to \infty} \mathfrak{m}_n(B_2(\bar{x}_n)) \leq \mathfrak{m}_\infty(B_3(\bar{x}_\infty)) < \infty,$$

(4.6) yields the uniform bound (4.7). The other assertions follow from Theorem 3.15 and Remark 3.19, noticing that the property (3.26) holds for the chosen ψ. $\qquad\square$

We collect in the next remark the main assumptions and notation that we are going to use in the rest of the paper.

Remark 4.6 (Main assumptions and notation)**.**

- $\mathcal{X}_n := [X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n]$, $n \in \mathbb{N}$, is a given family of p.m.m. spaces pmG-converging to a limit space $\mathcal{X}_\infty := [X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty]$.

- There is a complete and separable space (X, d) containing isometrically all the (X_n, d_n), $n \in \bar{\mathbb{N}}$ and which realizes the extrinsic convergence as in Definition 3.9. In particular

$$\lim_{n \to \infty} \mathsf{d}(\bar{x}_n, \bar{x}_\infty) = 0.$$

- For some $\mathsf{C} > 0$ we have $\mathcal{X}_n \in \mathbb{X}^\psi$ for every $n \in \bar{\mathbb{N}}$, where $\psi(r) := \mathrm{e}^{-\mathsf{C}r^2}$ and $\mathbb{D}^\psi(\mathcal{X}_n, \mathcal{X}_\infty) \to 0$ as $n \to \infty$. In particular, for some $\mathsf{c}_1, \mathsf{c}_2 > 0$ all the spaces $\mathcal{X}_n$, $n \in \bar{\mathbb{N}}$, satisfy the bound (4.1).

- According to (4.3), for $n \in \bar{\mathbb{N}}$ we define

$$\mathsf{z}_n := \int \mathrm{e}^{-\mathsf{C}\mathsf{d}_n^2(x, \bar{x}_n)} \, \mathrm{d}\mathfrak{m}_n(x), \quad \text{and} \quad \tilde{\mathfrak{m}}_n := \frac{1}{\mathsf{z}_n} \mathrm{e}^{-\mathsf{C}\mathsf{d}_n^2(\cdot, \bar{x}_n)} \mathfrak{m}_n \in \mathscr{P}_2(X_n) \subset \mathscr{P}_2(X),$$

so that the convergence in $(\mathbb{X}^\psi, \mathbb{D}^\psi)$ gives

$$\lim_{n \to \infty} \mathsf{z}_n = \mathsf{z}_\infty, \qquad \lim_{n \to \infty} W_2(\tilde{\mathfrak{m}}_n, \tilde{\mathfrak{m}}_\infty) = 0.$$

- We shall say that $n \mapsto x_n \in \mathrm{supp}(\mathfrak{m}_n)$ converges to $x_\infty \in \mathrm{supp}(\mathfrak{m}_\infty)$ provided $\lim_{n \to \infty} \mathsf{d}(x_n, x_\infty) = 0$.

- We shall say that $n \mapsto \mu_n \in \mathscr{P}(\mathrm{supp}(\mathfrak{m}_n))$ weakly converges to $\mu \in \mathscr{P}(\mathrm{supp}(\mathfrak{m}_\infty))$ provided (μ_n) weakly converges to μ_∞ as sequence in $\mathscr{P}(X)$. Similarly for measures in $\mathscr{P}_2(\mathrm{supp}(\mathfrak{m}_n))$ and W_2-convergence.

$\blacksquare$

Notice that we are directly assuming that (X, d) contains isometrically the X_n's rather than asking for the existence of isometric embeddings. Up to identifying (X_n, d_n) with its isometric image this is always possible, but this particular choice allows to avoid to introduce the isometric embeddings ι_n, thus simplifying the notation.

In line with the discussion made at the end of Section 3.4, we also point out that although the choice of the space (X, d) is non-intrinsic and might affect the notion of convergence of points and measures as given above (in line with Definition 3.23), in fact all the statements that we are going to prove are intrinsic in nature, being independent of the particular choice made. We won't insist on this point any further.

4.3 Stability of $\mathrm{CD}(K, \infty)$

We start our discussion on the stability of Ricci curvature bounds. The first result is about so-called Γ-convergence of the relative entropies w.r.t. W_2-convergence.

Proposition 4.7 (Γ-convergence of the Entropies)**.** *With the same assumptions and notation as in Remark 4.6, the sequence of relative entropies $(\mathrm{Ent}_{\mathfrak{m}_n})$ Γ-converges to $\mathrm{Ent}_{\mathfrak{m}_\infty}$ w.r.t. the W_2-convergence, i.e.:*

- $\underline{\Gamma - \liminf}$ *for any $\mu_\infty \in \mathscr{P}_2(X_\infty)$ and any sequence (μ_n) such that $W_2(\mu_n, \mu_\infty) \to 0$ we have*

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) \leq \varliminf_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n),$$

- $\underline{\Gamma - \limsup}$ *for any $\mu_\infty \in \mathscr{P}_2(X_\infty)$ there exists a sequence (μ_n) such that $W_2(\mu_n, \mu_\infty) \to 0$ and*

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) \geq \varlimsup_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n). \tag{4.8}$$

Proof.
Γ-$\liminf$. Let $(\mu_n) \subset \mathscr{P}_2(X)$ be a sequence W_2-converging to some $\mu_\infty \in \mathscr{P}_2(X)$. By Proposition 2.3 we have $\int \mathsf{d}^2(\cdot, \bar{x}_n) \, \mathrm{d}\mu_n \to \int \mathsf{d}^2(\cdot, \bar{x}_\infty) \, \mathrm{d}\mu_\infty$. Thus from (4.4) and the fact that $\mathsf{z}_n \to \mathsf{z}_\infty \in (0, \infty)$, to conclude it is enough to check that

$$\varliminf_{n \to \infty} \mathrm{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_n) \geq \mathrm{Ent}_{\tilde{\mathfrak{m}}_\infty}(\mu_\infty),$$

which follows by (4.5).
Γ-$\limsup$. Using (4.4) and $\mathsf{z}_n \to \mathsf{z}_\infty$, to conclude it is enough to prove that for any $\mu_\infty \in \mathscr{P}_2(X)$ there exists a sequence $(\mu_n) \subset \mathscr{P}_2(X)$ W_2-converging to μ_∞ and such that

$$\varlimsup_{n \to \infty} \mathrm{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_n) \leq \mathrm{Ent}_{\tilde{\mathfrak{m}}_\infty}(\mu_\infty).$$

The proof of this fact is not new (see for instance [11, Lemma 6.2]), but for completeness let us recall the arguments.

If $\mathrm{Ent}_{\tilde{\mathfrak{m}}_\infty}(\mu_\infty) = \infty$ there is nothing to prove. Thus we can assume that $\mu_\infty \ll \tilde{\mathfrak{m}}_\infty$. Let $\mu_\infty = \rho\tilde{\mathfrak{m}}_\infty$ and assume for the moment that ρ is bounded.

For any $n \in \mathbb{N}$, let $\boldsymbol{\gamma}_n \in \mathrm{Opt}(\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n)$ be an optimal plan. Define $\boldsymbol{\gamma}'_n \in \mathscr{P}_2(X^2)$ as $\mathrm{d}\boldsymbol{\gamma}'_n(x,y) := \rho(x)\mathrm{d}\boldsymbol{\gamma}_n(x,y)$ and $\mu_n := \pi^2_\sharp \boldsymbol{\gamma}'_n \in \mathscr{P}_2(X)$. By construction, $\boldsymbol{\gamma}'_n \ll \boldsymbol{\gamma}_n$, hence $\mu_n \ll \pi^2_\sharp \boldsymbol{\gamma}_n = \tilde{\mathfrak{m}}_n$. Let $\mu_n = \eta_n\tilde{\mathfrak{m}}_n$. It is readily checked from the definition that it holds $\eta_n(y) = \int \rho(x)\,\mathrm{d}(\boldsymbol{\gamma}_n)_y(x)$, where $\{(\boldsymbol{\gamma}_n)_y\}$ is the disintegration of $\boldsymbol{\gamma}_n$ w.r.t. the projection on the second marginal. By Jensen's inequality applied to the convex function $u(z) = z\log(z)$ we have

$$
\begin{aligned}
\mathrm{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_n) = \int u(\eta_n)\,\mathrm{d}\tilde{\mathfrak{m}}_n &= \int u\left(\int \rho(x)\,\mathrm{d}(\boldsymbol{\gamma}_n)_y(x)\right)\mathrm{d}\tilde{\mathfrak{m}}_n(y) \\
&\leq \iint u(\rho(x))\,\mathrm{d}(\boldsymbol{\gamma}_n)_y(x)\,\mathrm{d}\tilde{\mathfrak{m}}_n(y) = \iint u(\rho(x))\,\mathrm{d}\boldsymbol{\gamma}_n(x,y) \\
&= \int u(\rho)\,\mathrm{d}\pi^1_\sharp \boldsymbol{\gamma}_n = \int u(\rho)\,\mathrm{d}\tilde{\mathfrak{m}}_\infty = \mathrm{Ent}_{\tilde{\mathfrak{m}}_\infty}(\mu_\infty).
\end{aligned}
$$

Since by construction we have $\boldsymbol{\gamma}'_n \in \mathrm{Adm}(\mu_\infty, \mu_n)$, it holds

$$
W_2^2(\mu_\infty, \mu_n) \leq \int \mathsf{d}^2(x,y)\,\mathrm{d}\boldsymbol{\gamma}'_n(x,y) = \int \rho(x)\mathsf{d}^2(x,y)\,\mathrm{d}\boldsymbol{\gamma}_n(x,y) \leq \big(\sup\rho\big)W_2^2(\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n),
$$

and therefore $W_2(\mu_\infty, \mu_n) \to 0$. Thus in this case the thesis is proved.

If ρ is not bounded, for $k \in \mathbb{N}$ define $\rho^k := c_k \min\{\rho, k\}$, c_k being such that $\mu^k := \rho^k\tilde{\mathfrak{m}}_\infty \in \mathscr{P}_2(X)$. Clearly, it holds

$$
\varlimsup_{k\to\infty} \mathrm{Ent}_{\tilde{\mathfrak{m}}_\infty}(\mu^k) \leq \mathrm{Ent}_{\tilde{\mathfrak{m}}_\infty}(\mu_\infty), \qquad \lim_{k\to\infty} W_2(\mu^k, \mu_\infty) = 0.
$$

Then apply the previous argument to μ^k and conclude with a diagonalization argument. $\qquad\square$

We shall also make use of the following general 2-uniform integrability criterion:

Proposition 4.8. *Let (Y, d) be a complete and separable metric space and $(\mu_n), (\nu_n) \subset \mathscr{P}_2(Y)$ two 2-uniformly integrable sequences. Assume that for every $n \in \mathbb{N}$ there exists a W_2-geodesic $t \mapsto \mu_{n,t}$ connecting μ_n to ν_n.*

Then the family $\{\mu_{n,t}\}_{n\in\mathbb{N}, t\in[0,1]}$ is 2-uniformly integrable.

Proof. For every $n \in \mathbb{N}$, let $\boldsymbol{\pi}_n \in \mathscr{P}(\mathrm{Geo}(Y))$ be a plan representing the geodesic $t \mapsto \mu_{n,t}$ as in Proposition 4.2. Clearly it holds

$$
\int_{B_R^c(\bar{x})} \mathsf{d}^2(\bar{x}, \cdot)\,\mathrm{d}\mu_{n,t} = \int_{\{\gamma : \mathsf{d}^2(\bar{x}, \gamma_t) > R\}} \mathsf{d}^2(\bar{x}, \gamma_t)\,\mathrm{d}\boldsymbol{\pi}_n(\gamma), \qquad \forall R > 0, \ t \in [0,1], \ n \in \mathbb{N}. \quad (4.9)
$$

The trivial inequality

$$
\mathsf{d}(\gamma_t, \bar{x}) \leq 2\big(\mathsf{d}(\gamma_0, \bar{x}) + \mathsf{d}(\gamma_1, \bar{x})\big),
$$

ensures that $\{\gamma : \mathsf{d}(\gamma_t, \bar{x}) > R\} \subset A_R^0 \cup A_R^1$, where $A_R^0, A_R^1 \subset \mathrm{Geo}(Y)$ are given by

$$
\begin{aligned}
A_R^0 &:= \big\{\gamma \ : \ \mathsf{d}(\gamma_0, \bar{x}) \geq R/4, \ \mathsf{d}(\gamma_0, \bar{x}) \geq \mathsf{d}(\gamma_1, \bar{x})\big\}, \\
A_R^1 &:= \big\{\gamma \ : \ \mathsf{d}(\gamma_1, \bar{x}) \geq R/4, \ \mathsf{d}(\gamma_1, \bar{x}) \geq \mathsf{d}(\gamma_0, \bar{x})\big\}.
\end{aligned}
$$

From the bound

$$\mathsf{d}(\gamma_t, \bar{x}) \leq 2\big(\mathsf{d}(\gamma_0, \bar{x}) + \mathsf{d}(\gamma_1, \bar{x})\big) \leq 4\mathsf{d}(\gamma_0, \bar{x}), \qquad \forall \gamma \in A_R^0,$$

we get

$$\int_{A_R^0} \mathsf{d}^2(\gamma_t, \bar{x})\, \mathrm{d}\boldsymbol{\pi}_n(\gamma) \leq 16 \int_{A_R^0} \mathsf{d}^2(\gamma_0, \bar{x})\, \mathrm{d}\boldsymbol{\pi}_n(\gamma) \leq 16 \int_{B_{R/4}^c(\bar{x})} \mathsf{d}^2(\cdot, \bar{x})\, \mathrm{d}\mu_n.$$

Arguing symmetrically for A_R^1 and taking (4.9) into account we obtain

$$\int_{B_R^c(\bar{x})} \mathsf{d}^2(\bar{x}, \cdot)\mathrm{d}\mu_{n,t} \leq 16 \int_{B_{R/4}^c(\bar{x})} \mathsf{d}^2(\cdot, \bar{x})\, \mathrm{d}\mu_n + 16 \int_{B_{R/4}^c(\bar{x})} \mathsf{d}^2(\cdot, \bar{x})\, \mathrm{d}\nu_n,$$

for any $R > 0$, $t \in [0,1]$, $n \in \mathbb{N}$, and the conclusion follows from the 2-uniform integrability of (μ_n), (ν_n). $\qquad\square$

With these tools at disposal, we can now prove the general stability of the $\mathrm{CD}(K, \infty)$ condition along the same lines used by Lott-Villani [33] and Sturm [42]:

Theorem 4.9 (Stability of $\mathrm{CD}(K, \infty)$)**.** *Let $\mathfrak{X}_n$, $n \in \mathbb{N}$, be a sequence of $\mathrm{CD}(K, \infty)$ p.m.m. spaces converging to $\mathfrak{X}_\infty$ in the pmG-convergence. Then $\mathfrak{X}_\infty$ is a $\mathrm{CD}(K, \infty)$ space as well.*

Proof. By Proposition 4.5, without loss of generality we may use the assumptions and notation of Remark 4.6.

Let $\mu_\infty, \nu_\infty \in D(\mathrm{Ent}_{\mathfrak{m}_\infty})$. Use the $\Gamma - \overline{\lim}$ inequality in Proposition 4.7 to find sequences $(\mu_n), (\nu_n) \subset \mathscr{P}_2(X)$ which W_2-converge to μ_∞, ν_∞ respectively and satisfy

$$\varlimsup_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) \leq \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty), \qquad\qquad \varlimsup_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\nu_n) \leq \mathrm{Ent}_{\mathfrak{m}_\infty}(\nu_\infty). \qquad (4.10)$$

In particular, for n large enough we have $\mu_n, \nu_n \in D(\mathrm{Ent}_{\mathfrak{m}_n})$, thus by assumption there are W_2-geodesics $t \mapsto \mu_{n,t}$ connecting μ_n to ν_n such that

$$\mathrm{Ent}_{\mathfrak{m}_n}(\mu_{n,t}) \leq (1-t)\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) + t\, \mathrm{Ent}_{\mathfrak{m}_n}(\nu_n) - \frac{K}{2}t(1-t)W_2^2(\mu_n, \nu_n). \qquad (4.11)$$

Therefore, for n large enough the values of $\mathrm{Ent}_{\mathfrak{m}_n}(\mu_{n,t})$ are uniformly bounded in n, t. Since the second moments of $\mu_{n,t}$ are also - clearly - uniformly bounded in n, t, from (4.4) and $\mathsf{z}_n \to \mathsf{z}_\infty \in (0, \infty)$ we deduce that the values of $\mathrm{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_{n,t})$ are uniformly bounded in n, t, as well. Hence by Proposition 4.1, for every $t \in [0,1]$ the sequence $n \mapsto \mu_{n,t}$ is tight. From Proposition 4.8 we also know that it is 2-uniformly integrable, thus it is relatively compact in $(\mathscr{P}_2(X), W_2)$ (Theorem 2.2 and Proposition 2.3).

Since geodesics are equi-Lipschitz, applying the metric version of Arzelà-Ascoli Theorem we find a subsequence $n_k \uparrow +\infty$ and a limit geodesic $(\mu_{\infty,t}) \subset \mathscr{P}_2(X)$ such that $W_2(\mu_{n_k,t}, \mu_{\infty,t}) \to 0$ as $k \to \infty$ for every $t \in [0,1]$. Hence the $\Gamma - \underline{\lim}$ inequality in Proposition 4.7 yields

$$\varliminf_{k \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_{n,t}) \geq \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,t}) \quad \text{for every } t \in [0,1]. \qquad (4.12)$$

The inequalities (4.10), (4.11) and (4.12) give

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,t}) \leq (1-t)\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) + t\, \mathrm{Ent}_{\mathfrak{m}_\infty}(\nu_\infty) - \frac{K}{2}t(1-t)W_2^2(\mu_\infty, \nu_\infty),$$

for any $t \in [0,1]$. $\qquad\square$

5 Compactness and stability of the heat flows

Aim of this section is to show that the gradient flow of the relative entropy is stable w.r.t. pmG-convergence under a uniform $\mathrm{CD}(K, \infty)$ condition. We call such gradient flow 'heat flow' by analogy with the common Euclidean setting.

5.1 Preliminaries

5.1.1 Absolutely continuous curves

Let (Z, d_Z) be a complete separable metric space and $I \subset \mathbb{R}$ a non trivial interval. A curve $I \ni t \mapsto z_t \in Z$ is absolutely continuous (resp. locally absolutely continuous) provided there exists a function $f \in L^1(I)$ (resp. in $L^1_{loc}(I)$) such that

$$\mathsf{d}_Z(z_t, z_s) \leq \int_t^s f(r)\, \mathrm{d}r, \qquad \forall t, s \in I,\ t < s. \tag{5.1}$$

For absolutely continuous (resp. locally a.c.) curves, the limit $\lim_{h \to 0} \frac{\mathsf{d}_Z(z_{t+h}, z_t)}{|h|}$ exists for a.e. $t \in I$, defines an $L^1(I)$ (resp. $L^1_{loc}(I)$) function denoted by $|\dot{z}_t|$ called metric speed, and it is the minimal function - in the a.e. sense - that can be put in the right hand side of (5.1) (see Theorem 1.1.2 in [5] for a proof). We denote by $\mathrm{AC}(I; Z)$ (respectively $\mathrm{AC}^p(I; Z)$) the space of all absolutely continuous curves (resp. with metric derivative in $L^p(I)$).

The map

$$\mathrm{C}(I, Z) \ni \gamma \mapsto \mathcal{E}_2[\gamma] := \begin{cases} \int_I |\dot{\gamma}_t|^2 \, \mathrm{d}t, & \text{if } \gamma \in \mathrm{AC}^2(I; Z), \\ +\infty, & \text{otherwise,} \end{cases}$$

is lower semicontinuous. In the following we will often write the expression $\int_I |\dot{\gamma}_t|^2 \, \mathrm{d}t$ even for curves γ not absolutely continuous, in this case the value will be understood as $+\infty$ implicitly.

Given an absolutely continuous curve $\mu \in \mathrm{AC}(I; (\mathscr{P}_2(Z), W_2))$, we will denote by $|\dot{\mu}_t|$ its metric speed in the space $(\mathscr{P}_2(Z), W_2)$. If $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{C}(I, Z))$ is a plan satisfying $(\mathrm{e}_t)_\sharp \boldsymbol{\pi} = \mu_t$ for any $t \in I$, it is easy to see that it holds

$$\int_I |\dot{\mu}_t|^2 \, \mathrm{d}t \leq \int \mathcal{E}_2[\gamma] \, \mathrm{d}\boldsymbol{\pi}(\gamma). \tag{5.2}$$

In [32] it has been showed that with an appropriate selection of $\boldsymbol{\pi}$ equality can hold in (5.2):

Proposition 5.1. *Let (Z, d_Z) be complete and separable and $\mu \in \mathrm{AC}^2(I; (\mathscr{P}_2(Z), W_2))$. Then there exists $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{C}(I, Z))$ such that*

$$(\mathrm{e}_t)_\sharp \boldsymbol{\pi} = \mu_t \text{ for any } t \in I, \qquad \int_I |\dot{\mu}_t|^2 \, \mathrm{d}t = \int \mathcal{E}_2[\gamma] \, \mathrm{d}\boldsymbol{\pi}(\gamma). \tag{5.3}$$

5.1.2 Bits of Sobolev calculus

In what follows (specifically, Proposition 5.12), we will need to deal with a little bit of Sobolev calculus. Here we recall the basic definitions and facts used later.

There are several, equivalent, definitions of Sobolev functions from a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ to $\mathbb{R}$: we consider here only the case $p = 2$ following the approach proposed in [7];

we refer to [8, 1] for the case of a general summability exponent $p \in [1, \infty)$ and to [25] for further developments and a deeper analysis of the duality relations between weak differentials and gradients in metric measure spaces.

Definition 5.2 (Test plans). *Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be a p.m.m. space as in (3.1a,b). We say that $\boldsymbol{\pi} \in \mathscr{P}(C([0,1], X))$ is a test plan provided there exists a constant $c > 0$ such that*

$$(\mathrm{e}_t)_\sharp \boldsymbol{\pi} \leq c\mathfrak{m} \quad \text{for every } t \in [0, 1], \qquad \int \mathcal{E}_2[\gamma] \mathrm{d}\boldsymbol{\pi}(\gamma) < \infty.$$

Definition 5.3 (Sobolev class and weak upper gradients). *Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be a p.m.m. space as in (3.1a,b) and let $f : X \to \mathbb{R}$ be a Borel function. We say that f belongs to the Sobolev class $\mathrm{S}^2(X, \mathsf{d}, \mathfrak{m})$ provided there exists $G \in L^2(X, \mathfrak{m})$ such that*

$$\int |f(\gamma_1) - f(\gamma_0)| \, \mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t) |\dot{\gamma}_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\boldsymbol{\gamma}), \qquad \text{for every test plan } \boldsymbol{\pi}.$$

Any such G is called weak upper gradient of f.

It turns out that this notion is invariant with respect to modification of f in $\mathfrak{m}$-negligible sets and for $f \in \mathrm{S}^2(X, \mathsf{d}, \mathfrak{m})$ there exists a minimal - in the $\mathfrak{m}$-a.e. sense - weak upper gradient G, which we will denote by $|\mathrm{D}f|_w$. Notice that if $f : X \to \mathbb{R}$ is Lipschitz, then certainly the function identically equal to the Lipschitz constant $\mathrm{Lip}(f)$ of f is a weak upper gradient, so we get the inequality

$$|\mathrm{D}f|_w \leq \mathrm{Lip}(f), \qquad \mathfrak{m} - a.e.. \tag{5.4}$$

Basic calculus rules are

$$\begin{aligned} |\mathrm{D}(\alpha f + \beta g)|_w &\leq |\alpha| |\mathrm{D}f|_w + |\beta| |\mathrm{D}g|_w, &\forall f, g \in \mathrm{S}^2(X, \mathsf{d}, \mathfrak{m}), \ \alpha, \beta \in \mathbb{R}, \\ |\mathrm{D}(fg)|_w &\leq |f| |\mathrm{D}g|_w + |g| |\mathrm{D}f|_w, &\forall f \in \mathrm{S}^2(X, \mathsf{d}, \mathfrak{m}), \ g \text{ Lipschitz}, \end{aligned} \tag{5.5}$$

these inequalities being valid $\mathfrak{m}$-a.e..

The Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is then defined as $L^2(X, \mathfrak{m}) \cap \mathrm{S}^2(X, \mathsf{d}, \mathfrak{m})$ endowed with the norm

$$\|f\|_{W^{1,2}}^2 := \|f\|_{L^2}^2 + \||\mathrm{D}f|_w\|_{L^2}^2.$$

Notice that in general $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is not a Hilbert space (consider for instance the case of finite dimensional Banach spaces). In particular, in general there is no natural Dirichlet form on $L^2(X, \mathfrak{m})$. The potentially non quadratic object which replaces the Dirichlet energy is called - following [7] - Cheeger energy: it is the lower semicontinuous and convex functional $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ defined by

$$\mathsf{Ch}(f) := \begin{cases} \dfrac{1}{2} \displaystyle\int |\mathrm{D}f|_w^2 \, \mathrm{d}\mathfrak{m}, & \text{if } f \in \mathrm{S}^2(X, \mathsf{d}, \mathfrak{m}), \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.6}$$

Notice that the notions of (minimal) weak upper gradient and of Cheeger energy are invariant under isomorphisms of p.m.m. spaces, in the sense that if $(X_i, \mathsf{d}_i, \mathfrak{m}_i, \bar{x}_i)$, $i = 1, 2$ are isomorphic according to (3.2) then

$$\mathsf{Ch}_{X_1}(f \circ \iota) = \mathsf{Ch}_{X_2}(f) \quad \text{for every } f \in L^2(X_2, \mathfrak{m}_2).$$

As usual, in the following we will follow the notation of Remark 4.6 and deal with spaces $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, which are all subspaces of a common space (X, d). In particular, the spaces $(X, \mathsf{d}, \mathfrak{m}_n, \bar{x}_n)$, $n \in \bar{\mathbb{N}}$, are isomorphic to $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ and we will consider the Cheeger energies $\mathsf{Ch}_n : L^2(X, \mathfrak{m}_n) \to [0, \infty]$, $n \in \bar{\mathbb{N}}$.

5.1.3 W_2-Gradient flow of the entropy

Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be a p.m.m. space as in (3.1a,b) satisfying the growth condition (4.7). The (descending) slope $|\mathrm{D}^- \mathrm{Ent}_\mathfrak{m}| : \mathscr{P}_2(X) \to [0, +\infty]$ of the relative entropy $\mathrm{Ent}_\mathfrak{m}$ is defined as identically $+\infty$ outside $D(\mathrm{Ent}_\mathfrak{m})$, as 0 at isolated measures in $D(\mathrm{Ent}_\mathfrak{m})$ and in all other cases as

$$|\mathrm{D}^- \mathrm{Ent}_\mathfrak{m}|(\mu) := \varlimsup_{W_2(\nu,\mu) \to 0} \frac{\left(\mathrm{Ent}_\mathfrak{m}(\mu) - \mathrm{Ent}_\mathfrak{m}(\nu)\right)^+}{W_2(\mu,\nu)},$$

where by $(\cdot)^+$ we intend the positive part.

If $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ is a $\mathrm{CD}(K, \infty)$ space for some $K \in \mathbb{R}$, the slope admits the representation

$$|\mathrm{D}^- \mathrm{Ent}_\mathfrak{m}|(\mu) := \sup_{\nu \neq \mu} \left(\frac{\mathrm{Ent}_\mathfrak{m}(\mu) - \mathrm{Ent}_\mathfrak{m}(\nu)}{W_2(\mu,\nu)} + \frac{K}{2} W_2(\mu,\nu) \right)^+ \quad \text{for every } \mu \in \mathscr{P}_2(X), \quad (5.7)$$

see [5, Chap. 2] which shows in particular that

If $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ is a $\mathrm{CD}(K, \infty)$ space, then $|\mathrm{D}^- \mathrm{Ent}_\mathfrak{m}|$ is W_2-lower semicontinuous. $\quad (5.8)$

A deep result obtained in [7] is that on $\mathrm{CD}(K, \infty)$ spaces the slope admits the representation formula

$$|\mathrm{D}^- \mathrm{Ent}_\mathfrak{m}|^2(\rho \mathfrak{m}) = 8 \mathsf{Ch}(\sqrt{\rho}) = \int_{\{\rho > 0\}} \frac{|\mathrm{D}\rho|_w^2}{\rho} \, \mathrm{d}\mathfrak{m} \quad (5.9)$$

which is the standard representation formula for the slope of the entropy in terms of the Fisher information, well known in a smooth setting.

Another non trivial consequence (see Corollary 2.4.10 in [5] for a proof) of the K-geodesic convexity of $\mathrm{Ent}_\mathfrak{m}$ is that the slope is an upper gradient for $\mathrm{Ent}_\mathfrak{m}$, i.e. it holds

$$|\mathrm{Ent}_\mathfrak{m}(\mu_0) - \mathrm{Ent}_\mathfrak{m}(\mu_1)| \leq \int_0^1 |\dot{\mu}_t| |\mathrm{D}^- \mathrm{Ent}_\mathfrak{m}|(\mu_t) \, \mathrm{d}t,$$

whenever $(\mu_t) \subset D(\mathrm{Ent}_\mathfrak{m})$, $t \in [0,1]$, is an absolutely continuous curve. In particular, for any locally absolutely continuous curve $(\mu_t) \subset D(\mathrm{Ent}_\mathfrak{m})$ defined on some interval $I \subset \mathbb{R}$ it holds

$$\mathrm{Ent}_\mathfrak{m}(\mu_t) \leq \mathrm{Ent}_\mathfrak{m}(\mu_s) + \frac{1}{2} \int_t^s |\dot{\mu}_r|^2 \, \mathrm{d}r + \frac{1}{2} \int_t^s |\mathrm{D}^- \mathrm{Ent}_\mathfrak{m}|^2(\mu_r) \, \mathrm{d}r, \qquad \forall t, s \in I, \ t < s. \ (5.10)$$

Gradient flows are defined as those curves for which equality holds:

Definition 5.4 (Gradient flow of the relative entropy). *Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be a $\mathrm{CD}(K, \infty)$ space and $\bar{\mu} \in D(\mathrm{Ent}_\mathfrak{m})$. A curve $\mu : [0, \infty) \to D(\mathrm{Ent}_\mathfrak{m}) \subset \mathscr{P}_2(X)$ is the W_2-gradient flow of $\mathrm{Ent}_\mathfrak{m}$ starting from $\bar{\mu}$ provided it is locally absolutely continuous in $(\mathscr{P}_2(X), W_2)$, $\mu_0 = \bar{\mu}$ and*

$$\mathrm{Ent}_\mathfrak{m}(\mu_t) = \mathrm{Ent}_\mathfrak{m}(\mu_s) + \frac{1}{2} \int_t^s |\dot{\mu}_r|^2 \, \mathrm{d}r + \frac{1}{2} \int_t^s |\mathrm{D}^- \mathrm{Ent}_\mathfrak{m}|^2(\mu_r) \, \mathrm{d}r \quad \forall 0 < t < s. \quad (5.11)$$

Thanks to (5.10), (5.11) is equivalent to

$$\mathrm{Ent}_\mathfrak{m}(\bar{\mu}) \geq \mathrm{Ent}_\mathfrak{m}(\mu_T) + \frac{1}{2} \int_0^T |\dot{\mu}_t|^2 \, \mathrm{d}t + \frac{1}{2} \int_0^T |\mathrm{D}^- \mathrm{Ent}_\mathfrak{m}|^2(\mu_t) \, \mathrm{d}t \quad \text{for every } T > 0. \quad (5.12)$$

We remark that this definition is invariant under isomorphisms of p.m.m. spaces, since it just involves the distance and measures in the domain of the entropy: notice that, with the notation of (3.2),

$$\mathrm{Ent}_{\mathfrak{m}_2}(\iota_\sharp\mu) = \mathrm{Ent}_{\mathfrak{m}_1}(\mu) \quad \text{for every } \mu \in \mathscr{P}_2(X_1),\ \mu \ll \mathfrak{m}_1.$$

Therefore the property of being a gradient flow of the entropy is not affected if our space $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ is isometrically embedded into some bigger space X as in Remark 4.6.

The following theorem collects some of the main properties of the gradient flow of the entropy and a useful a priori estimate.

Theorem 5.5. *Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be a $\mathrm{CD}(K, \infty)$ p.m.m. space. Then for every $\bar{\mu} \in D(\mathrm{Ent}_\mathfrak{m})$ there exists a unique W_2-gradient flow $\mu_t := \mathscr{H}_t\bar{\mu}$, $t \geq 0$, of the entropy starting from $\bar{\mu}$. The curve (μ_t) has also the following properties:*

$$|\dot{\mu}_t| = |\mathrm{D}^-\mathrm{Ent}_\mathfrak{m}|(\mu_t) \qquad \text{for a.e. } t > 0, \tag{5.13}$$

$$t \mapsto \mathrm{e}^{Kt}|\mathrm{D}^-\mathrm{Ent}_\mathfrak{m}|(\mu_t) \qquad \text{is lower semicontinuous and non increasing.} \tag{5.14}$$

Moreover, if $\mathsf{C} > 1 + K_-$ so that (4.3) holds, and $T := \frac{1}{8\mathsf{C}}$, then

$$\frac{1}{2}\int_0^T |\dot{\mu}_t|^2\,\mathrm{d}t \leq 2\mathrm{Ent}_\mathfrak{m}(\bar{\mu}) + 4\mathsf{C}\int \mathsf{d}^2(\cdot, \bar{x})\,\mathrm{d}\bar{\mu} + 2\log(\mathsf{z}). \tag{5.15}$$

Proof. Here we limit to check the estimate (5.15), referring to [7] for a proof of the main part of the statement (see also [6] for a survey in the case of compact spaces).

By definition of gradient flows and (4.4) we know that

$$\mathrm{Ent}_\mathfrak{m}(\mu_0) \geq \mathrm{Ent}_\mathfrak{m}(\mu_T) + \frac{1}{2}\int_0^T |\dot{\mu}_t|^2\,\mathrm{d}t = \mathrm{Ent}_{\tilde{\mathfrak{m}}}(\mu_T) - \mathsf{C}\int \mathsf{d}^2(\cdot, \bar{x})\,\mathrm{d}\mu_T - \log(\mathsf{z}) + \frac{1}{2}\int_0^T |\dot{\mu}_t|^2\,\mathrm{d}t.$$

We also have the simple bound

$$\int \mathsf{d}^2(\cdot, \bar{x})\,\mathrm{d}\mu_T = W_2^2(\mu_T, \delta_{\bar{x}}) \leq \left(W_2(\mu_0, \delta_{\bar{x}}) + \int_0^T |\dot{\mu}_t|\,\mathrm{d}t\right)^2 \leq 2W_2^2(\mu_0, \delta_{\bar{x}}) + 2T\int_0^T |\dot{\mu}_t|^2\,\mathrm{d}t$$

$$= 2\int \mathsf{d}^2(\cdot, \bar{x})\,\mathrm{d}\bar{\mu} + \frac{1}{4\mathsf{C}}\int_0^T |\dot{\mu}_t|^2\,\mathrm{d}t.$$

(5.15) comes from these two inequalities taking into account that $\mathrm{Ent}_{\tilde{\mathfrak{m}}} \geq 0$, since $\tilde{\mathfrak{m}}$ is a probability measure. $\qquad\square$

5.1.4 L^2-gradient flow of the Cheeger energy

The main identification result of [7] shows that in $\mathrm{CD}(K, \infty)$ spaces the Wasserstein gradient flow of the entropy coincides with the L^2-gradient flow $(\mathsf{H}_t)_{t\geq 0}$ of the Cheeger energy: the latter is the semigroup of contractions $\mathsf{H}_t : L^2(X, \mathfrak{m}) \to L^2(X, \mathfrak{m})$ whose trajectories $f_t = \mathsf{H}_t\bar{f}$, $t \geq 0$, belong to $\mathrm{Lip}_{\mathrm{loc}}((0, \infty); L^2(X, \mathfrak{m}))$ and are the unique solution of the differential inclusion (see e.g. [14] and [7, §4])

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t + \partial\mathsf{Ch}(f_t) \ni 0 \quad \text{a.e. in } (0, \infty), \quad \lim_{t\downarrow 0} f_t = \bar{f} \quad \text{in } L^2(X, \mathfrak{m}).$$

$\partial\mathsf{Ch}(f)$ denotes the (possibly multivalued) subdifferential of Ch in $L^2(X, \mathfrak{m})$, it is a convex subset of $L^2(X, \mathfrak{m})$; when $\partial\mathsf{Ch}(f)$ is not empty, its element of minimal norm defines the Laplacian $\Delta_{\mathsf{d},\mathfrak{m}}$ of f.

Theorem 5.6 ([7], Thm 9.3). *Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be a $\mathrm{CD}(K, \infty)$ p.m.m. space. If $\bar{\mu} = \bar{f}\mathfrak{m} \in \mathscr{P}_2(X)$ with $f \in L^2(X, \mathfrak{m})$ then*

$$\mathscr{H}_t \bar{\mu} = (\mathsf{H}_t \bar{f})\mathfrak{m} \quad \text{for every } t \geq 0.$$

5.2 Convergence of Heat flows in the general non linear case

This section is devoted to the proof of the following general convergence result:

Theorem 5.7 (Convergence of heat flows). *Let $\mathfrak{X}_n$, $n \in \mathbb{N}$, be a sequence of $\mathrm{CD}(K, \infty)$ spaces converging to a limit space $\mathfrak{X}_\infty$ in the pmG-sense. Then with the notation of Remark 4.6 the following holds.*

Let $(\bar{\mu}_n) \subset \mathscr{P}_2(X)$ be such that

$$\mathrm{Ent}_{\mathfrak{m}_n}(\bar{\mu}_n) \to \mathrm{Ent}_{\mathfrak{m}_\infty}(\bar{\mu}_\infty) < \infty, \qquad \bar{\mu}_n \xrightarrow{W_2} \bar{\mu}_\infty, \tag{5.16}$$

as $n \to \infty$. Then the solutions $\mu_{n,t} = \mathscr{H}_{n,t}(\bar{\mu}_n)$, $t \geq 0$, of the W_2 gradient flow of $\mathrm{Ent}_{\mathfrak{m}_n}$ satisfy

$$\mu_{n,t} \xrightarrow{W_2} \mu_{\infty,t} \qquad \qquad \text{for every } t > 0, \tag{5.17a}$$
$$\mathrm{Ent}_{\mathfrak{m}_n}(\mu_{n,t}) \longrightarrow \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,t}) \qquad \qquad \text{for every } t > 0, \tag{5.17b}$$
$$|\mathrm{D}^- \mathrm{Ent}_{\mathfrak{m}_n}|(\mu_{n,t}) \longrightarrow |\mathrm{D}^- \mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_{\infty,t}) \qquad \text{for every } t \in (0, \infty) \setminus \mathcal{S}, \tag{5.17c}$$
$$|\dot{\mu}_{n,t}| \longrightarrow |\dot{\mu}_{\infty,t}| \qquad \qquad \text{for a.e. } t > 0, \tag{5.17d}$$

where $\mathcal{S}$ is the set (at most countable) of discontinuity points of $t \mapsto |\mathrm{D}^- \mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_{\infty,t})$.

We split the *proof of Theorem 5.7* in various steps.

Tightness and compactness results

Lemma 5.8. *Let $\psi : X \to [0, +\infty]$ be a function with compact sublevels. Then for every non trivial interval $I = [a, b] \subset \mathbb{R}$ the functional $\Psi : \mathrm{C}(I, X) \to [0, +\infty]$ defined by*

$$\Psi(\gamma) := \int_I \psi(\gamma_t) \, dt + \mathcal{E}_2[\gamma] = \begin{cases} \int_I \left(\psi(\gamma_t) + |\dot{\gamma}_t|^2 \right) dt, & \text{if } \gamma \in \mathrm{AC}^2(I; X), \\ +\infty & \text{otherwise,} \end{cases}$$

has compact sublevels on $\mathrm{C}(I, X)$ (recall that the latter is endowed with the sup distance).

Proof. Since $\mathcal{E}_2[\cdot]$ is a l.s.c. functional, Fatou's Lemma and the lower semicontinuity of ψ on X ensure that Ψ is lower semicontinuous on $\mathrm{C}(I, X)$.

Let $(\gamma^n) \in \mathrm{AC}^2(I; X)$ be a sequence satisfying $\Psi(\gamma^n) \leq C < \infty$. The bound $\mathcal{E}_2[\gamma^n] \leq C$ easily yields the Hölder equicontinuity estimate

$$\mathsf{d}(\gamma^n_{t+h}, \gamma^n_t) \leq \sqrt{C\,h}, \quad \text{whenever} \quad a \leq t \leq t+h \leq b. \tag{5.18}$$

By [40, Theorem 2, (1.24)] (γ^n) is relatively compact with respect to the convergence in measure: in particular, we can find a subsequence (γ^{n_k}), a $\mathscr{L}^1$-negligible set $N \subset I$, and a limit curve $\gamma^\infty : I \setminus N \to X$ such that $\lim_{k \to \infty} \mathsf{d}(\gamma^{n_k}_t, \gamma^\infty_t) = 0$ for every $t \in I \setminus N$. The uniform bound (5.18) shows that γ^∞ is also $1/2$ Hölder continuous, it can therefore be extended to I by the density of $I \setminus N$ and the completeness of X, and the resulting convergence of γ^{n_k} to γ^∞ is uniform. □

Proposition 5.9 (Compactness in $\mathscr{P}(C([0,T),X)))$. *Let* $(\bar{\mu}_n) \subset \mathscr{P}_2(X)$ *be such that*

$$\sup_{n \in \mathbb{N}} \mathrm{Ent}_{\mathfrak{m}_n}(\bar{\mu}_n) < \infty, \qquad \sup_{n \in \mathbb{N}} \int \mathsf{d}^2(\cdot, \bar{x}_n) \, \mathrm{d}\bar{\mu}_n < \infty, \qquad (5.19)$$

and let $\mu_{n,t} = \mathscr{H}_{n,t}(\bar{\mu}_n)$, $t \geq 0$. *For every* $0 < T \leq \frac{1}{8\mathsf{C}}$ *the sequence of plans* $\boldsymbol{\pi}_n \in \mathscr{P}(C([0,T],X))$ *associated to the absolutely continuous curve* $[0,T] \ni t \mapsto \mu_{n,t}$ *via Proposition 5.1, is tight in* $\mathscr{P}(C([0,T),X))$.

Proof.
Step 1: tightness of $\{\mu_{n,t}\}_{n,t}$. Taking into account that $t \mapsto \mathrm{Ent}_{\mathfrak{m}_n}(\mu_{n,t})$ is non increasing and (4.4), for any $t \in [0,T]$ we get

$$\begin{aligned}
\mathrm{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_{n,t}) &= \mathrm{Ent}_{\mathfrak{m}_n}(\mu_{n,t}) + \mathsf{C}W_2^2(\mu_{n,t}, \delta_{\bar{x}_n}) + \log(\mathsf{z}_n) \\
&\leq \mathrm{Ent}_{\mathfrak{m}_n}(\bar{\mu}_n) + \mathsf{C}\left(W_2(\bar{\mu}_n, \delta_{\bar{x}_n}) + \int_0^t |\dot{\mu}_{n,s}| \, \mathrm{d}s\right)^2 + \log(\mathsf{z}_n) \\
&\leq \mathrm{Ent}_{\mathfrak{m}_n}(\bar{\mu}_n) + 2\mathsf{C}W_2^2(\bar{\mu}_n, \delta_{\bar{x}_n}) + 2\mathsf{C}T \int_0^T |\dot{\mu}_{n,s}|^2 \, \mathrm{d}s + \log(\mathsf{z}_n).
\end{aligned}$$

Hence from the bound (5.15), assumption (5.19) and the fact that $\sup_n \log(\mathsf{z}_n) < \infty$ we conclude that

$$\sup_{\substack{n \in \mathbb{N} \\ t \in [0,T]}} \mathrm{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_{n,t}) < \infty,$$

and the tightness of $\{\mu_{n,t}\}_{n \in \mathbb{N}, t \in [0,T]}$ follows from Proposition 4.1.
Step 2: tightness of $(\boldsymbol{\pi}_n)$. Since the set $\{\mu_{n,t}\}_{t \in [0,T], n \in \mathbb{N}}$ is tight, by Theorem 2.2 there exists a function $\psi : X \to [0, +\infty]$ with compact sublevels such that

$$\sup_{t \in [0,T], n \in \mathbb{N}} \int \psi \, \mathrm{d}\mu_{n,t} < \infty. \qquad (5.20)$$

Define $\Psi : C([0,T], X) \to [0, +\infty]$ by $\Psi(\gamma) := \mathcal{E}_2[\gamma] + \int_0^T \psi(\gamma_t) \, \mathrm{d}t$. Lemma 5.8 ensures that the sublevels of Ψ are compact in $C([0,T], X)$. Using (5.3) we get

$$\int \Psi \, \mathrm{d}\boldsymbol{\pi}_n = \iint_0^T \psi(\gamma_t) \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) + \int \mathcal{E}_2[\gamma] \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) = \int_0^T \left(\int \psi \, \mathrm{d}\mu_{n,t} + |\dot{\mu}_{n,t}|^2\right) \mathrm{d}t.$$

The right hand side of this expression is uniformly bounded in n thanks to (5.20) and (5.15), thus the conclusion follows by Prokhorov's Theorem 2.2. $\qquad \square$

Γ-$\liminf$ **estimates**

The next two propositions are valid without any Ricci curvature assumption (but still assuming the growth condition (4.7)) and are of independent interest.

Proposition 5.10 (Γ-$\underline{\lim}$ for Entropy + second moment). *With the same assumptions and notation of Remark 4.6, let* $(\mu_n) \subset \mathscr{P}_2(X)$ *be a sequence weakly converging to some* $\mu_\infty \in \mathscr{P}_2(X)$. *Then*

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) + \mathsf{C} \int \mathsf{d}^2(\cdot, \bar{x}_\infty) \, \mathrm{d}\mu_\infty \leq \varliminf_{n \to \infty} \left(\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) + \mathsf{C} \int \mathsf{d}^2(\cdot, \bar{x}_n) \, \mathrm{d}\mu_n\right).$$

Proof. Direct consequence of (4.4), $z_n \to z_\infty$ and (4.5). $\qquad\square$

Proposition 5.11 (Γ-$\underline{\lim}$ for kinetic energy minus second moment)**.** *With the same assumptions and notation of Remark 4.6, let $T \leq \frac{1}{4\mathsf{C}}$ and $(\boldsymbol{\pi}_n) \subset \mathscr{P}(C([0,T],X))$ a sequence weakly converging to some $\boldsymbol{\pi}_\infty \in \mathscr{P}(C([0,T],X))$. Assume also that*

$$(\mathrm{e}_0)_\sharp \boldsymbol{\pi}_n \in \mathscr{P}_2(X), \ \forall n \in \bar{\mathbb{N}}, \quad \text{with} \quad W_2\big((\mathrm{e}_0)_\sharp \boldsymbol{\pi}_n, (\mathrm{e}_0)_\sharp \boldsymbol{\pi}_\infty\big) \to 0.$$

Then

$$\int \left(\frac{1}{2}\mathcal{E}_2[\gamma] - \mathsf{Cd}^2(\gamma_T, \bar{x}_\infty) \right) \mathrm{d}\boldsymbol{\pi}_\infty(\gamma) \leq \varliminf_{n\to\infty} \int \left(\frac{1}{2}\mathcal{E}_2[\gamma] - \mathsf{Cd}^2(\gamma_T, \bar{x}_n) \right) \mathrm{d}\boldsymbol{\pi}_n(\gamma). \qquad (5.21)$$

If we further assume that

$$\lim_{n\to\infty} \iint_0^T |\dot{\gamma}_t|^2 \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) = \iint_0^T |\dot{\gamma}_t|^2 \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}_\infty(\gamma) < \infty, \qquad (5.22)$$

then $(\mathrm{e}_t)_\sharp \boldsymbol{\pi}_n$ converges to $(\mathrm{e}_t)_\sharp \boldsymbol{\pi}_\infty$ in $(\mathscr{P}_2(X), W_2)$ for any $t \in [0,T]$.

Proof. Let $F_n : C([0,T],X) \to [0, +\infty]$, $n \in \bar{\mathbb{N}}$ be the l.s.c. functionals given by

$$F_n(\gamma) := \frac{1}{2}\mathcal{E}_2[\gamma] - \mathsf{Cd}^2(\gamma_T, \bar{x}_n) + 2\mathsf{Cd}^2(\gamma_0, \bar{x}_n).$$

Since $\bar{x}_n \to \bar{x}_\infty$, we easily get

$$\sup_{t\in[0,T]} \mathsf{d}(\gamma_{n,t}, \gamma_{\infty,t}) \to 0, \qquad \Rightarrow \qquad F_\infty(\gamma_\infty) \leq \varliminf_n F_n(\gamma_n).$$

We claim that the F_n's are non negative. Indeed, from $T \leq \frac{1}{4\mathsf{C}}$ we have

$$\mathsf{Cd}^2(\gamma_T, \bar{x}_n) \leq \mathsf{C} \left(\int_0^T |\dot{\gamma}_t| \, \mathrm{d}t + \mathsf{d}(\gamma_0, \bar{x}_n) \right)^2 \leq \frac{1}{2} \int_0^T |\dot{\gamma}_t|^2 \, \mathrm{d}t + 2\mathsf{Cd}^2(\gamma_0, \bar{x}_n).$$

Therefore applying Lemma 2.1 with $C([0,T],X)$ in place of X, $\nu_n := \boldsymbol{\pi}_n$, and recalling that $(\mathrm{e}_0)_\sharp \boldsymbol{\pi}_n \to (\mathrm{e}_0)_\sharp \boldsymbol{\pi}_\infty$ in $(\mathscr{P}_2(X), W_2)$ by assumption, we get (5.21).

For the second part of the statement, notice that the lower semicontinuity of the kinetic energy and assumption (5.22) give that $\lim_{n\to\infty} \iint_0^t |\dot{\gamma}_s|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) = \iint_0^t |\dot{\gamma}_s|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}_\infty(\gamma)$ for any $t \in [0,T]$. Hence, up to diminishing T, we can assume without loss of generality that $t = T$. Now observe that the positivity of F_n, (5.21) and (5.22) give

$$-\infty < -\mathsf{C} \int \mathsf{d}^2(\cdot, \bar{x}_\infty) \, \mathrm{d}(\mathrm{e}_T)_\sharp \boldsymbol{\pi}_\infty \leq \varliminf_{n\to\infty} -\mathsf{C} \int \mathsf{d}^2(\cdot, \bar{x}_n) \, \mathrm{d}(\mathrm{e}_T)_\sharp \boldsymbol{\pi}_n.$$

The weak convergence of $\boldsymbol{\pi}_n$ to $\boldsymbol{\pi}$ gives the weak convergence of $(\mathrm{e}_T)_\sharp \boldsymbol{\pi}_n$ to $(\mathrm{e}_T)_\sharp \boldsymbol{\pi}_\infty$, thus the conclusion comes from Proposition 2.3. $\qquad\square$

In the next proposition, to get the Γ-$\underline{\lim}$ inequality for the slope, the uniform lower Ricci curvature bound plays a crucial role.

Proposition 5.12 (Γ-$\underline{\lim}$ for the slope). *With the same notation and assumption as in Remark 4.6, assume furthermore that for some $K \in \mathbb{R}$ the spaces $\mathcal{X}_n$ are all $\mathrm{CD}(K, \infty)$ spaces (so that by Theorem 4.9 also the limit space $\mathcal{X}_\infty$ is a $\mathrm{CD}(K, \infty)$ space).*

Then for any sequence $n \mapsto \mu_n \in \mathscr{P}_2(X_n)$ weakly converging to some $\mu_\infty \in \mathscr{P}_2(X)$ it holds

$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_\infty) \leq \varliminf_{n \to \infty} |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|(\mu_n).$$

Proof.
Step 1: From weak to W_2 convergence. For $R > 0$, let $h_R : \mathbb{R}^+ \to [0, 1]$ be given by $h_R(z) := \max\{\{\min\{2 - z/R, 1\}, 0\}$ and define the $1/R$-Lipschitz functions $\chi_{n,R} : X \to [0, 1]$, $n \in \bar{\mathbb{N}}$, as $\chi_{n,R}(x) := h_R(\mathsf{d}_n(x, \bar{x}_n))$.

Clearly, for some R_0, n_0, the probability measures $\mu_{\infty,R} := c_{\infty,R}\chi_{\infty,R}^2\mu_\infty$ and $\mu_{n,R} := c_{n,R}\chi_{n,R}^2\mu_n$ are well defined for any $R > R_0$ and $n \geq n_0$, where $c_{\infty,R}, c_{n,R}$ are the normalization constants, and it holds $c_{n,R} \to 1$ as $R \to \infty$ and $c_{n,R} \to c_{\infty,R}$ as $n \to \infty$.

From the convergence of $\bar{x}_n$ to $\bar{x}_\infty$, we get the weak convergence of $(\mu_{n,R})$ to $\mu_{\infty,R}$ as $n \to \infty$. Therefore, since their supports are uniformly bounded, the convergence is in the W_2-topology by Proposition 2.3.

Step 2: control of the slope under cutoff. Given that the slope is $+\infty$ outside the domain of the entropy, up to pass to subsequences, with no loss of generality we can assume that $\mu_n \ll \mathfrak{m}_n$ for every $n \in \mathbb{N}$. Let $\mu_n = f_n^2\mathfrak{m}_n$, so that $\mu_{n,R} = c_{n,R}(f_n\chi_{n,R})^2\mathfrak{m}_n$. From (5.4) and (5.5) we get

$$|\mathrm{D}(f_n\chi_{n,r})|_w \leq |\mathrm{D}f_n|_w + |f_n|/R.$$

Squaring, integrating, recalling that $\int f_n^2 \, \mathrm{d}\mathfrak{m}_n = 1$ and the definition of Ch_n given in (5.6) and the discussion thereafter, we get

$$\mathsf{Ch}_n(\sqrt{c_{n,R}}f_n\chi_{n,R}) \leq c_{n,R}\left(\mathsf{Ch}_n(f_n)\left(1 + \frac{1}{2R}\right) + \frac{1}{2R} + \frac{1}{2R^2}\right), \qquad \forall R \geq R_0, n \geq n_0.$$

Therefore using (5.9) twice, we get that for $R > R_0$ it holds

$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|^2(\mu_{n,R}) \leq c_{n,R}\left(|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|^2(\mu_n)\left(1 + \frac{1}{2R}\right) + \frac{4}{R} + \frac{4}{R^2}\right). \tag{5.23}$$

Step 3: Γ-$\underline{\lim}$ under W_2-convergence. Fix $\nu \in D(\mathrm{Ent}_{\mathfrak{m}_\infty})$ and use the Γ-$\overline{\lim}$ inequality in Proposition 4.7 to find a sequence $(\nu_n) \subset \mathscr{P}_2(X)$ such that $\mathrm{Ent}_{\mathfrak{m}_n}(\nu_n) \to \mathrm{Ent}_{\mathfrak{m}_\infty}(\nu)$ and $W_2(\nu_n, \nu) \to 0$ as $n \to \infty$. Using the W_2-convergence of $\mu_{n,R}$ to $\mu_{\infty,R}$ as $n \to \infty$ and the Γ-$\underline{\lim}$ inequality in Proposition 4.7 we have

$$\frac{\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,R}) - \mathrm{Ent}_{\mathfrak{m}_\infty}(\nu)}{W_2(\mu_{\infty,R}, \nu)} + \frac{K}{2}W_2(\mu_{\infty,R}, \nu) \leq \varliminf_{n \to \infty} \frac{\mathrm{Ent}_{\mathfrak{m}_n}(\mu_{n,R}) - \mathrm{Ent}_{\mathfrak{m}_n}(\nu_n)}{W_2(\mu_{n,R}, \nu_n)} + \frac{K}{2}W_2(\mu_{n,R}, \nu_n)$$

$$\leq \varliminf_{n \to \infty} |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|(\mu_{n,R}),$$

having used (5.7) in the last step. Taking the positive parts, the supremum over $\nu \in D(\mathrm{Ent}_{\mathfrak{m}_\infty})$ and using again (5.7) we conclude

$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_{\infty,R}) \leq \varliminf_{n \to \infty} |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|(\mu_{n,R}), \qquad \forall R \geq R_0. \tag{5.24}$$

Step 4: back to μ and conclusion. From the definition it is clear that $W_2(\mu_{\infty,R}, \mu_\infty) \to 0$ as $R \to \infty$ (because there is weak convergence and 2-uniform integrability), hence recalling the W_2-lower semicontinuity of $|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|$ (see (5.8)) we get

$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_\infty) \leq \varliminf_{R \to \infty} |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_{\infty,R}). \tag{5.25}$$

The inequalities (5.23), (5.24) and (5.25) give the conclusion. $\qquad\qquad\square$

Remark 5.13. It is worth underlying that although the structure of the proof of Proposition 5.12 is very similar to the one presented in [24] for the compact case, actually it is conceptually more involved. Indeed, to gain W_2-convergence from weak one we used a cut-off argument, and to control the variation of the slope under this procedure (inequality (5.23)) we used as crucial tool the non-trivial identification of the squared slope with the Fisher information (identity (5.9)), which in turn is a consequence of the fine analysis carried out in [7]. $\qquad\blacksquare$

Conclusion of the proof of Theorem 5.7

Proof. Let $T := \frac{1}{8C}$ and, for every $n \in \mathbb{N}$, $\boldsymbol{\pi}_n \in \mathscr{P}(C([0,T], X))$ a plan associated to $[0,T] \ni t \mapsto \mu_{n,t}$ via Proposition 5.1, so that in particular

$$\int_0^T |\dot\mu_{n,t}|^2 \, \mathrm{d}t = \int \mathcal{E}_2[\gamma] \, \mathrm{d}\boldsymbol{\pi}_n(\gamma). \tag{5.26}$$

Thanks to (5.16), Proposition 5.9 is applicable, thus $(\boldsymbol{\pi}_n)$ is tight in $\mathscr{P}(C([0,T], X))$ and up to pass to a subsequence, not relabelled, we can assume that it weakly converges to some $\boldsymbol{\pi}_\infty$. The convergences in (5.16), the bound (5.15), the identity (5.26), and the lower semicontinuity of the kinetic energy combined with Lemma 2.1, give

$$\int \mathcal{E}_2[\gamma] \, \mathrm{d}\boldsymbol{\pi}_\infty(\gamma) \leq \varliminf_{n \to \infty} \int \mathcal{E}_2[\gamma] \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) < \infty.$$

Put $\mu_t := (\mathrm{e}_t)_\sharp \boldsymbol{\pi}_\infty$ and recalling (5.2), we get that $\mu \in \mathrm{AC}^2([0,T]; (\mathscr{P}_2(X), W_2))$ with

$$\int_0^T |\dot\mu_t|^2 \, \mathrm{d}t \leq \int \mathcal{E}_2[\gamma] \, \mathrm{d}\boldsymbol{\pi}_\infty(\gamma). \tag{5.27}$$

The weak convergence of $(\boldsymbol{\pi}_n)$ to $\boldsymbol{\pi}_\infty$ ensures that $n \mapsto \mu_{n,t}$ weakly converges to μ_t for any $t \in [0,T]$. Proposition 5.11 yields

$$\underbrace{\int \left(\frac{1}{2} \mathcal{E}_2[\gamma] - \mathsf{C}\mathsf{d}^2(\gamma_T, \bar{x}_\infty) \right) \mathrm{d}\boldsymbol{\pi}_\infty(\gamma)}_{=:\mathrm{a}} \leq \varliminf_{n \to \infty} \underbrace{\int \left(\frac{1}{2} \mathcal{E}_2[\gamma] - \mathsf{C}\mathsf{d}^2(\gamma_T, \bar{x}_n) \right) \mathrm{d}\boldsymbol{\pi}_n(\gamma)}_{=:\mathrm{a}_n}. \tag{5.28}$$

Applying Proposition 5.10 to $\mu_n := (\mathrm{e}_T)_\sharp \boldsymbol{\pi}_n$, $\mu_\infty := (\mathrm{e}_T)_\sharp \boldsymbol{\pi}_\infty$ we obtain

$$\underbrace{\mathrm{Ent}_{\mathfrak{m}_\infty}\big((\mathrm{e}_T)_\sharp \boldsymbol{\pi}_\infty\big) + \mathsf{C}\int \mathsf{d}^2(\gamma_T, \bar{x}_\infty) \, \mathrm{d}\boldsymbol{\pi}_\infty(\gamma)}_{=:\mathrm{b}} \leq \varliminf_{n \to \infty} \underbrace{\left(\mathrm{Ent}_{\mathfrak{m}_n}\big((\mathrm{e}_T)_\sharp \boldsymbol{\pi}_n\big) + \mathsf{C}\int \mathsf{d}^2(\gamma_T, \bar{x}_n) \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) \right)}_{=:\mathrm{b}_n}.$$

$$\tag{5.29}$$

Also, from Proposition 5.12 and Fatou's lemma we get

$$\underbrace{\frac{1}{2}\int_0^T |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|^2(\mu_t)\,\mathrm{d}t}_{=:\mathrm{c}} \le \varliminf_{n\to\infty} \underbrace{\frac{1}{2}\int_0^T |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|^2(\mu_{n,t})\,\mathrm{d}t}_{=:\mathrm{c}_n}. \tag{5.30}$$

Adding up (5.28), (5.29) and (5.30) and taking (5.27) into account we get

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_T) + \frac{1}{2}\int_0^T |\dot\mu_t|^2\,\mathrm{d}t + \frac{1}{2}\int_0^T |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|^2(\mu_t)\,\mathrm{d}t \le \varliminf_{n\to\infty}(\mathrm{a}_n + \mathrm{b}_n + \mathrm{c}_n). \tag{5.31}$$

From the fact that $t \mapsto \mu_{n,t}$ is a gradient flow of $\mathrm{Ent}_{\mathfrak{m}_n}$, (5.11), (5.16) and (5.26), we obtain

$$\mathrm{a}_n + \mathrm{b}_n + \mathrm{c}_n = \mathrm{Ent}_{\mathfrak{m}_n}(\bar\mu_n) \to \mathrm{Ent}_{\mathfrak{m}_\infty}(\bar\mu_\infty). \tag{5.32}$$

Therefore (5.31) yields

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_T) + \frac{1}{2}\int_0^T |\dot\mu_t|^2\,\mathrm{d}t + \frac{1}{2}\int_0^T |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|^2(\mu_t)\,\mathrm{d}t \le \mathrm{Ent}_{\mathfrak{m}_\infty}(\bar\mu_\infty), \tag{5.33}$$

which means, according to (5.12), that $[0,T] \ni t \mapsto \mu_t$ is the (restriction to $[0,T]$ of the) gradient flow of $\mathrm{Ent}_{\mathfrak{m}_\infty}$ starting from $\bar\mu_\infty$. Thus $\mu_t = \mu_{\infty,t}$ for any $t \in [0,T]$.

By (5.10), the opposite inequality in (5.33) also holds, and since in proving (5.33) we used (5.31), which in turn was proved using (5.27), we deduce that equality holds in (5.27), i.e.

$$\int_0^T |\dot\mu_t|^2\,\mathrm{d}t = \iint_0^T |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}_\infty(\gamma). \tag{5.34}$$

Notice also that equality in (5.33) reads as $\mathrm{a} + \mathrm{b} + \mathrm{c} = \mathrm{Ent}_{\mathfrak{m}_\infty}(\bar\mu_\infty)$, therefore from (5.28), (5.29), (5.30), (5.31) and (5.32) we get

$$\mathrm{Ent}_{\mathfrak{m}_\infty}(\bar\mu_\infty) = \mathrm{a} + \mathrm{b} + \mathrm{c} \le \varliminf_{n\to\infty}\mathrm{a}_n + \varliminf_{n\to\infty}\mathrm{b}_n + \varliminf_{n\to\infty}\mathrm{c}_n \le \varliminf_{n\to\infty}(\mathrm{a}_n + \mathrm{b}_n + \mathrm{c}_n)$$
$$\le \varlimsup_{n\to\infty}(\mathrm{a}_n + \mathrm{b}_n + \mathrm{c}_n) = \varlimsup_{n\to\infty}\mathrm{Ent}_{\mathfrak{m}_n}(\bar\mu_n) = \mathrm{Ent}_{\mathfrak{m}_\infty}(\bar\mu_\infty),$$

which forces $\mathrm{a}_n \to \mathrm{a}$, $\mathrm{b}_n \to \mathrm{b}$ and $\mathrm{c}_n \to \mathrm{c}$. Proposition 5.12 and the convergence $\mathrm{c}_n \to \mathrm{c}$ give

$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|^2(\mu_t) \le \varliminf_{n\to\infty} |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|^2(\mu_{n,t}), \qquad a.e.\ t \in [0,T],$$

and

$$\int_0^T |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|^2(\mu_t)\,\mathrm{d}t = \lim_{n\to\infty}\int_0^T |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|^2(\mu_{n,t})\,\mathrm{d}t, \tag{5.35}$$

which implies

$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|^2(\mu_t) = \lim_{n\to\infty} |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|^2(\mu_{n,t}), \qquad a.e.\ t \in [0,T].$$

Taking the regularity property (5.14) into account we get the convergence for all $t \in [0,T]$ with the possible exception of the countable discontinuity set of $t \mapsto |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_{\infty,t})$. From (5.13) we also get $|\dot\mu_{\infty,t}| = \lim_n |\dot\mu_{n,t}|$ for a.e. $t \in [0,T]$ which together with (5.35) and the equalities (5.34), (5.26) give that (5.22) holds. Therefore from the second part of Proposition

5.11 we get that $(\mu_{n,t})$ converges to $\mu_{\infty,t}$ w.r.t. W_2 for any $t \in [0,T]$. Finally, the convergence $\mathrm{Ent}_{\mathfrak{m}_n}(\mu_{n,t}) \to \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_t)$ follows from $b_n \to b$ and the convergence of the second moments (replace T by $t \in [0,T]$ in the whole argument to get this latter convergence).

Observe now that all these results do not really depend on the particular weakly converging sequence $(\boldsymbol{\pi}_n)$ chosen at the beginning, so that they are valid for the full original sequence. Thus we proved all the stated convergence properties for $t \in [0, \frac{1}{8C}]$. In particular, we have that

$$W_2(\mu_{n,1/(8C)}, \mu_{\infty,1/(8C)}) \to 0, \qquad \mathrm{Ent}_{\mathfrak{m}_n}(\mu_{n,1/(8C)}) \to \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,1/(8C)}) < \infty.$$

Hence the argument can be repeated with $\mu_{n,1/(8C)}, \mu_{\infty,1/(8C)}$ in place of $\bar{\mu}_n, \bar{\mu}_\infty$. Iterating, we get the result on the whole $[0,\infty)$. $\qquad\square$

5.3   A consequence: Mosco-convergence of the slopes

Now that we have at disposal the convergence of the heat flows, we can improve the result of Proposition 5.12 and obtain the full Mosco-convergence of the slopes of the relative entropies:

Corollary 5.14 (Mosco-convergence of the slopes)**.** *With the same notation and assumption as in Remark 4.6, assume furthermore that for some $K \in \mathbb{R}$ the spaces $\mathfrak{X}_n$ are all $\mathrm{CD}(K,\infty)$ spaces (so that by Theorem 4.9 also the limit space $\mathfrak{X}_\infty$ is a $\mathrm{CD}(K,\infty)$ space). Then the following holds:*

- *<u>weak $\Gamma - \liminf$</u> For any sequence $n \mapsto \mu_n \in \mathscr{P}_2(X_n)$ weakly converging to some $\mu_\infty \in \mathscr{P}_2(X_\infty)$ it holds*
$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_\infty) \leq \varliminf_{n\to\infty} |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|(\mu_n).$$

- *<u>strong $\Gamma - \limsup$</u> For any $\mu_\infty \in \mathscr{P}_2(X_\infty)$ there exists a sequence $n \mapsto \mu_n \in \mathscr{P}_2(X_n)$ W_2-converging to μ_∞ such that*
$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_\infty) \geq \varlimsup_{n\to\infty} |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|(\mu_n).$$

Proof. The first part is precisely the content of Proposition 5.12, so we pass to the second. Without loss of generality we assume $|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_\infty) < \infty$ (otherwise there is nothing to prove) so that in particular $\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) < \infty$. Invoking the second part of Proposition 4.7 we can find a sequence μ_n W_2-converging to μ_∞ such that $\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) \to \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty)$. Setting $\mu_{n,t} := \mathscr{H}_{n,t}(\mu_n)$, (5.15) yields the uniform Hölder estimate

$$W_2(\mu_{n,t}, \mu_n) \leq C\sqrt{t} \quad t \in [0,T],$$

for some $C,T > 0$ independent of n. Let us choose a vanishing sequence of times $t_k \in (0,T)\backslash \mathcal{S}$, $\mathcal{S}$ being as in Theorem 5.7. Applying (5.17c), we can find an increasing sequence $k \mapsto n_k \in \mathbb{N}$ such that for every $n \geq n_k$

$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|(\mu_{n,t_k}) \leq |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_{\infty,t_k}) + 2^{-k} \leq \mathrm{e}^{Kt_k}|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_\infty) + 2^{-k},$$

where in the last inequality we applied (5.14). The sequence $\nu_n := \mu_{n,t_k}$ whenever $n_k \leq n < n_{k+1}$ thus satisfies the desired inequality. $\qquad\square$

6 Mosco-convergence of the Cheeger energies.

In this section we will prove a Γ-convergence result for Cheeger energies in the same spirit of Proposition 4.7.

As usual, we adopt the assumptions and notation of Remark 4.6. We denote by Ch_n the Cheeger energy associated to the measure $\mathfrak{m}_n$ and by $L^p_{\mathrm{loc}}(X, \mathfrak{m}_n)$ the Lebesgue space of $\mathfrak{m}_n$-measurable functions whose restriction to each bounded set $B \subset X$ belongs to $L^p(B, \mathfrak{m}_n)$.

6.1 Weak and strong L^2-convergence w.r.t. varying measures.

Let us first introduce a notion of weak/strong convergence for real valued functions defined in varying L^p-spaces and strongly inspired by the theory of Young measures, see e.g. [2]. We refer to [5, §5.4] for similar definitions and the relevant proofs, that can be easily adapted to cover the present case of measures in $\mathscr{M}_{loc}(X)$ by the rescaling approach we adopted in Section 3. The basic idea is to study the convergence of a sequence of functions $f_n \in L^1_{\mathrm{loc}}(X, \mathfrak{m}_n)$ by the corresponding measures $\boldsymbol{\mu}_n = (\boldsymbol{i} \times f_n)_\sharp \mathfrak{m}_n$ in $\mathscr{M}_{loc}(X \times \mathbb{R})$, $\boldsymbol{i} : X \to X$ being the identity map. Here we state everything in the equivalent terms of weak convergence and test functions.

Definition 6.1. Let $f_n \in L^1_{\mathrm{loc}}(X, \mathfrak{m}_n)$, $n \in \bar{\mathbb{N}}$. We say that (f_n) weakly converges to f_∞ if

$$\int \varphi f_n \, \mathrm{d}\mathfrak{m}_n \longrightarrow \int \varphi f_\infty \, \mathrm{d}\mathfrak{m}_\infty \quad \text{for every } \varphi \in \mathrm{C}_{bs}(X). \tag{6.1}$$

For $p \in (1, \infty)$ we say that (f_n) L^p-weakly converges to f_∞ provided (6.1) holds and moreover

$$\sup_{n \in \mathbb{N}} \int |f_n|^p \, \mathrm{d}\mathfrak{m}_n < \infty. \tag{6.2}$$

If (f_n) is L^p-weakly converging to f_∞ and furthermore

$$\overline{\lim_{n \to \infty}} \int |f_n|^p \, \mathrm{d}\mathfrak{m}_n \le \int |f_\infty|^p \, \mathrm{d}\mathfrak{m}_\infty < \infty, \tag{6.3}$$

then we say that it is L^p-strongly converging to f_∞.

If $f_n \in L^1_+(X, \mathfrak{m}_n)$, we say that (f_n) converge to f_∞ in the W_2-Entropy sense if (6.1) holds and

$$
\begin{aligned}
\int \mathsf{d}^2(x, \bar{x}_n) f_n \, \mathrm{d}\mathfrak{m}_n &\longrightarrow \int \mathsf{d}^2(x, \bar{x}) f_\infty \, \mathrm{d}\mathfrak{m}_\infty < \infty \\
\int f_n \log f_n \, \mathrm{d}\mathfrak{m}_n &\longrightarrow \int f_\infty \log f_\infty \, \mathrm{d}\mathfrak{m}_\infty < \infty.
\end{aligned}
\tag{6.4}
$$

Remark 6.2. In the Hilbertian case $p = 2$ a different approach, still leading to similar notions and results, can be found in [31, Section 2.2]. It is based on the construction of a family of linear maps $\Phi_n : \mathcal{C} \to L^2(X, \mathfrak{m}_n)$ defined in a dense subspace $\mathcal{C}$ of $L^2(X, \mathfrak{m}_\infty)$ and satisfying $\lim_{n \to \infty} \|\Phi_n f\|_{L^2(X, \mathfrak{m}_n)} = \|f\|_{L^2(X, \mathfrak{m}_\infty)}$. In our situation the maps Φ_n could be defined by a disintegration technique, as in the proof of Proposition 4.7; however we prefer to adopt the more direct approach of Definition 6.1 above, that takes advantage of the particular structure of the spaces $L^2(X, \mathfrak{m}_n)$ associated to the weakly convergent measures $\mathfrak{m}_n$. ∎

It would not be difficult to show, arguing as for Proposition 4.7 (see also [5, § 5.2]) that along any weakly convergent sequence (f_n) we always have

$$\varliminf_{n \to \infty} \int |f_n|^p \, d\mathfrak{m}_n \geq \int |f_\infty|^p \, d\mathfrak{m}_\infty, \tag{6.5}$$

so that (6.3) is in fact a limit. Moreover the same argument of [5, Theorem 5.4.4] shows that strong convergence in the L^p sense (6.1)–(6.3) implies

$$\int \zeta(y, f_n(y)) \, d\mathfrak{m}_n(y) \longrightarrow \int \zeta(y, f_\infty(y)) \, d\mathfrak{m}_\infty(y) \tag{6.6}$$

for every $\zeta \in C(X \times \mathbb{R})$ with $|\zeta(y, r)| \leq \varphi(y) + C|r|^p, \quad \varphi \in C_{bs}(X), \ C \geq 0.$

A similar argument, based on the strict convexity of the function $r \mapsto r \log r$ and on the decomposition (4.4) shows that convergence in the W_2-Entropy sense yields (6.6) with $p = 1$.

We also remark that if $n \mapsto f_n \in L^p(X, \mathfrak{m}_n)$, $p \in (1, \infty)$, fulfils (6.2), then there is a subsequence L^p-weakly converging to some limit $f_\infty \in L^p(X, \mathfrak{m}_\infty)$.

In the following we shall need the convergence result

$$\left.\begin{array}{l} f_n \xrightarrow{L^2} f_\infty, \text{ strongly} \\ g_n \xrightarrow{L^2} g_\infty, \text{ weakly} \end{array}\right\} \quad \Rightarrow \quad \int f_n \, g_n \, d\mathfrak{m}_n \longrightarrow \int f_\infty \, g_\infty \, d\mathfrak{m}_\infty, \tag{6.7}$$

which can be proved by picking $\varepsilon \in \mathbb{R}$, applying (6.2) and (6.5) to the L^2-weakly converging sequence $(f_n + \varepsilon g_n)$ and (6.3) to (f_n) to pass to the limit in

$$2\varepsilon \int f_n g_n \, d\mathfrak{m}_n = \int |f_n + \varepsilon g_n|^2 \, d\mathfrak{m}_n - \int |f_n|^2 \, d\mathfrak{m}_n - \varepsilon^2 \int |g_n|^2 \, d\mathfrak{m}_n$$

and obtain

$$\varliminf_{n \to \infty} 2\varepsilon \int f_n g_n \, d\mathfrak{m}_n \geq \int |f_\infty + \varepsilon g_\infty|^2 \, d\mathfrak{m}_\infty - \int |f_\infty|^2 \, d\mathfrak{m}_\infty - \varepsilon^2 S$$

$$\geq 2\varepsilon \int f_\infty g_\infty \, d\mathfrak{m}_\infty - \varepsilon^2 S,$$

where $S := \sup_{n \in \mathbb{N}} \int |g_n|^2 \, d\mathfrak{m}_n < \infty$. Then the claim follows by dividing by $\varepsilon > 0$ (resp. $\varepsilon < 0$) and letting $\varepsilon \downarrow 0$ (resp. $\varepsilon \uparrow 0$).

Finally, as for Proposition 4.7, every $f_\infty \in L^p(X, \mathfrak{m}_\infty)$ can be approximated in the strong L^p-sense by a sequence $f_n \in L^p(X, \mathfrak{m}_n)$.

The next result provides a useful criterium to improve weak convergence, whenever one knows a bound on the Fisher information or on the Cheeger energy.

Theorem 6.3. *Let $\mathbb{X}_n$, $n \in \bar{\mathbb{N}}$, and (X, d) as in Remark 4.6, assume furthermore that for some $K \in \mathbb{R}$ the space $\mathbb{X}_n$ is $\mathrm{CD}(K, \infty)$ for every $n \in \bar{\mathbb{N}}$ and let $f_n \in L^1_{\mathrm{loc}}(X, \mathfrak{m}_n)$, $n \in \bar{\mathbb{N}}$. Then the following are true.*

i) Let $\mu_n = f_n \mathfrak{m}_n \in \mathscr{P}_2(X)$, $n \in \mathbb{N}$, be such that

$$\sup_{n \in \mathbb{N}} W_2(\mu_n, \delta_{\bar{x}_n}) + |\mathrm{D}^- \mathrm{Ent}_{\mathfrak{m}_n}|(\mu_n) < \infty. \tag{6.8}$$

Then there exists a subsequence weakly converging to some $\mu_\infty = f_\infty \mathfrak{m}_\infty \in \mathscr{P}_2(X)$.

If furthermore we assume that

$$W_2(\mu_n, \mu_\infty) \longrightarrow 0, \tag{6.9}$$

as $n \to \infty$, then (f_n) converges to f_∞ in the W_2-Entropy sense (thus (6.4) and (6.6) hold with $p = 1$).

ii) *Let (f_n) be converging to f_∞ in the L^2-weak sense. Then*

$$\mathsf{Ch}_\infty(f_\infty) \le \varliminf_{n \to \infty} \mathsf{Ch}_n(f_n). \tag{6.10}$$

If furthermore we assume that

$$\sup_{n \in \mathbb{N}} \mathsf{Ch}_n(f_n) < \infty \qquad \text{and} \qquad \lim_{R \to \infty} \sup_n \int_{X \setminus B_R(\bar{x}_n)} f_n^2 \, d\mathfrak{m}_n = 0, \tag{6.11}$$

then (f_n) converge to f_∞ in the L^2-strong sense.

Proof.
i) Let us first recall that (5.7) yields for every $\nu_n \in D(\mathrm{Ent}_{\mathfrak{m}_n})$

$$\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) \le \mathrm{Ent}_{\mathfrak{m}_n}(\nu_n) + |\mathrm{D}^- \mathrm{Ent}_{\mathfrak{m}_n}|(\mu_n) W_2(\mu_n, \nu_n) + \frac{K_-}{2} W_2^2(\mu_n, \nu_n). \tag{6.12}$$

Choose now ν_n such that

$$\sup_{n \in \mathbb{N}} \mathrm{Ent}_{\mathfrak{m}_n}(\nu_n) + W_2(\nu_n, \delta_{\bar{x}_n}) < \infty$$

(it is easy to see that this choice is possible) and recall (6.8) to deduce that $\sup_n \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) < \infty$. Recalling (6.8) again and using the identity

$$\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) = \mathrm{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_n) - \mathsf{C} \int \mathsf{d}^2(\cdot, \bar{x}_n) \, d\mu_n - \log \mathsf{z}_n, \tag{6.13}$$

we obtain that $\sup_n \mathrm{Ent}_{\tilde{\mathfrak{m}}_n}(\mu_n) < \infty$. Proposition 4.1 then ensures that (μ_n) is tight and hence by Theorem 2.2 that it has a subsequence weakly converging to some μ_∞ with (by (4.5)) $\mathrm{Ent}_{\tilde{\mathfrak{m}}_\infty}(\mu_\infty) < \infty$. In particular $\mu_\infty \ll \tilde{\mathfrak{m}}_\infty \ll \mathfrak{m}_\infty$. Moreover, the uniform bound on the second moments of the μ_n's granted by assumption (6.8) ensures that $\mu_\infty \in \mathscr{P}_2(X_\infty)$ and therefore by (6.13) with $n = \infty$ we deduce $\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) < \infty$.

If we further assume (6.9), so that in particular the full sequence (μ_n) W_2-converges to μ_∞, then passing to the limit in (6.13) gives $\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty) \le \varliminf_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n)$. Choosing now in (6.12) ν_n W_2-converging to μ_∞ with $\varlimsup_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\nu_n) \le \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty)$ as in (4.8), since the triangle inequality yields $W_2(\nu_n, \mu_n) \longrightarrow 0$, we obtain from (6.12) that

$$\varlimsup_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n) \le \varlimsup_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\nu_n) \le \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty).$$

ii) In order to prove (6.10), without loss of generality we can assume that $\sup_n \mathsf{Ch}_n(f_n) < \infty$. Initially we will also suppose that

$$f_n(x) \equiv 0 \text{ for } \mathfrak{m}_n\text{-a.e. } x \in X_n \text{ with } \mathsf{d}(x, \bar{x}_n) > R, \tag{6.14}$$

for some $R > 0$ and that $f_n \geq 0$ $\mathfrak{m}_n$-a.e. for every $n \in \bar{\mathbb{N}}$.

Now observe that up to extracting a subsequence we can assume that $Z_n := \int f_n^2 \, \mathrm{d}\mathfrak{m}_n \to Z_\infty$. If $Z_\infty = 0$ then $f_\infty = 0$ and there is nothing to prove. Thus we can assume $Z_\infty > 0$. Put $\mu_n := Z_n^{-1} f_n^2 \mathfrak{m}_n$, recall that the identity (5.9) gives $|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|^2(\mu_n) = 8Z_n^{-1}\mathsf{Ch}_n(f_n)$, the uniform bound on $\mathsf{Ch}_n(f_n)$ and notice that (6.14) grants 2-uniform integrability of (μ_n). Hence by point (i) we deduce that there is a subsequence of (μ_n), not relabelled, converging to some $\mu_\infty = Z_\infty^{-1} g_\infty \mathfrak{m}_\infty$ in the W_2-Entropy sense. Choosing $\zeta(y, r) = \varphi(y)\sqrt{r^+}$ in (6.6) we obtain that $g_\infty = f_\infty^2$ and the convergence of Z_n to Z_∞ yields (6.3) with $p = 2$, i.e. the L^2-strong convergence of (f_n) to f_∞.

Inequality (6.10) then follows by Proposition 5.12 and identity (5.9) again:

$$8Z_\infty^{-1}\mathsf{Ch}_\infty(f_\infty) = |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|^2(\mu_\infty) \leq \varliminf_{n\to\infty} |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|^2(\mu_n) = \varliminf_{n\to\infty} 8Z_n^{-1}\mathsf{Ch}_n(f_n).$$

To drop the assumption on the positivity of the f_n's, apply what we just proved to (f_n^+) and (f_n^-) to deduce that up to subsequences they L^2-strongly converge to some f^+, f^- respectively and that

$$\mathsf{Ch}_\infty(f^\pm) \leq \varliminf_{n\to\infty} \mathsf{Ch}_n(f_n^\pm). \tag{6.15}$$

Since we know that (f_n) is L^2-weakly converging to f_∞, we deduce that $f_\infty^\pm = f^\pm$ and the claim follows adding up the two inequalities in (6.15) and observing that

$$\mathsf{Ch}_n(f_n) = \mathsf{Ch}_n(f_n^+) + \mathsf{Ch}_n(f_n^-), \qquad \forall n \in \bar{\mathbb{N}}.$$

We consider now the general case, by showing how to remove the auxiliary assumption (6.14). Let $\chi : [0, \infty) \to [0, 1]$ be a 1-Lipschitz function with compact support identically 1 on $[0, 1]$ and for $R > 0$ define the truncated functions

$$f_{R,n} := \chi(\mathsf{d}_n(\cdot, \bar{x}_n)/R)f_n, \qquad \forall n \in \bar{\mathbb{N}}.$$

The locality property of the minimal weak upper gradient and the Leibniz rule (5.5) yield

$$|\mathrm{D}f_{R,n}|_w \leq |\mathrm{D}f_n|_w + \frac{1}{R}|f_n| \qquad \mathfrak{m}_n\text{-a.e..}$$

Squaring and integrating we deduce

$$\mathsf{Ch}_n(f_{R,n}) \leq \mathsf{Ch}_n(f_n)\Big(1 + \frac{1}{R}\Big) + \frac{S}{2}\Big(\frac{1}{R} + \frac{1}{R^2}\Big), \qquad \forall n \in \mathbb{N},$$

where $S := \sup_{n\in\mathbb{N}} \int f_n^2 \, \mathrm{d}\mathfrak{m}_n < \infty$. Taking first the $\varliminf$ as $n \to \infty$ and then as $R \uparrow \infty$ we deduce

$$\varliminf_{n\to\infty} \mathsf{Ch}_n(f_n) \geq \varliminf_{R\uparrow\infty} \varliminf_{n\to\infty} \mathsf{Ch}_n(f_{R,n}).$$

Now observe that the $f_{R,n}$'s are L^2-weakly converging to $f_{R,\infty}$ as $n \to \infty$ and clearly satisfy (6.14), so that by what we previously proved we get $\varliminf_{n\to\infty} \mathsf{Ch}_n(f_{R,n}) \geq \mathsf{Ch}_\infty(f_{R,\infty})$ for every $R > 0$. Eventually noticing that $f_{R,\infty} \to f_\infty$ in $L^2(X, \mathfrak{m}_\infty)$ and taking into account the $L^2(X, \mathfrak{m}_\infty)$-lower semicontinuity of Ch_∞ we obtain (6.10).

It remains to prove L^2-strong convergence under the additional assumptions (6.11). To this aim notice that for given $R > 0$ the functions $(f_{R,n})$ satisfy the assumption (6.14), so

by what we previously proved they L^2-strongly converge to $f_{R,\infty}$ as $n \to \infty$. Moreover, as already noticed, $f_{R,\infty} \to f_\infty$ as $R \to \infty$ in $L^2(X, \mathfrak{m}_\infty)$ while the second in (6.11) grants that

$$\limsup_{R \uparrow \infty} \int |f_n - f_{R,n}|^2 \, \mathrm{d}\mathfrak{m}_n = 0.$$

The conclusion follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the particular case of a single measure $\mathfrak{m}_n \equiv \mathfrak{m}$, $n \in \bar{\mathbb{N}}$, and a $\mathrm{CD}(K, \infty)$ space (X, d), the previous Theorem provides a simple criterium for the compact imbedding of the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ in $L^2(X, \mathfrak{m})$.

In order to make the connection more evident we recall the useful notion of Logarithmic-Sobolev-Talagrand inequality:

Definition 6.4. *We say that a pointed metric measure space $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ satisfies a weak Logarithmic-Sobolev-Talagrand inequality* wLSTI(A, B) *with constants $A, B \geq 0$ if we have*

$$\left(\int_X \mathsf{d}^2(x, \bar{x}) f^2(x) \, \mathrm{d}\mathfrak{m} \right)^{1/2} \leq A \|f\|_{L^2(X, \mathfrak{m})} + B \sqrt{\mathsf{Ch}(f)}, \qquad \forall f \in L^2(X, \mathfrak{m}). \qquad (6.16)$$

We are calling (6.16) a weak Logarithmic-Sobolev-Talagrand inequality because it can be seen as a consequence of a combination of the log-Sobolev inequality and of the Talagrand inequality, as made precise by the following simple proposition:

Proposition 6.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathrm{CD}(K, \infty)$ space with finite mass, put $\mathfrak{m}^* := \mathfrak{m}(X)^{-1}\mathfrak{m}$ and assume that for some $a, b \geq 0$ we have*

$$W_2(\mu, \mathfrak{m}^*) \leq a + b|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}^*}|(\mu) \quad \text{for every } \mu \in \mathscr{P}_2(X). \qquad (6.17)$$

Then for every $\bar{x} \in \mathrm{supp}(\mathfrak{m})$ the space $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ satisfies a wLSTI(A, B) *with*

$$A := 2a + 2b\left(\frac{\mathfrak{m}(B_2(\bar{x}))}{\mathfrak{m}(B_1(\bar{x}))}\right)^{1/2} + 2, \quad B := \sqrt{8}\, b.$$

Proof. We start proving that $\mathfrak{m}^* \in \mathscr{P}_2(X)$. Indeed, choosing $\mu = Z_R^{-1}h_R^2(x)\mathfrak{m}^*$ with $h_R(x) := \left((2 - \mathsf{d}(x, \bar{x})/R) \wedge 1\right) \vee 0$ and $Z_R = \int_X h_R^2 \, \mathrm{d}\mathfrak{m}^*$ we get

$$|\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}^*}|(\mu) = \frac{2}{\sqrt{Z_R}}\left(\int_X |\mathrm{D}h_R|_w^2 \, \mathrm{d}\mathfrak{m}^* \right)^{1/2} \leq \frac{2}{R\sqrt{Z_R}}\left(\mathfrak{m}^*(B_{2R}(\bar{x})) \right)^{1/2},$$

and therefore

$$\left(\int_X \mathsf{d}^2(x, \bar{x}) \, \mathrm{d}\mathfrak{m}^* \right)^{1/2} \leq W_2(\mathfrak{m}^*, \mu) + W_2(\mu, \delta_{\bar{x}}) \leq a + \frac{2b}{R\sqrt{Z_R}}\left(\mathfrak{m}^*(B_{2R}(\bar{x})) \right)^{1/2} + 2R.$$

Since $Z_R \geq \mathfrak{m}^*(B_R(\bar{x}))$ we eventually get

$$\left(\int_X \mathsf{d}^2(x, \bar{x}) \, \mathrm{d}\mathfrak{m}^* \right)^{1/2} \leq a + \frac{2b}{R}\left(\frac{\mathfrak{m}(B_{2R}(\bar{x}))}{\mathfrak{m}(B_R(\bar{x}))} \right)^{1/2} + 2R, \quad \text{for every } R > 0. \qquad (6.18)$$

Now let $M := \mathfrak{m}(X)$ and $\mathfrak{m}^* = M^{-1}\mathfrak{m}$, pick $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and let $\int f^2 \, \mathrm{d}\mathfrak{m} = M F^2$. Then (5.9) and (6.17) yield

$$W_2((f/F)^2 \mathfrak{m}^*, \mathfrak{m}^*) \leq a + \frac{b}{\sqrt{M}}\sqrt{8\mathsf{Ch}(f/F)}$$

so that the triangle inequality for W_2 yields

$$\left(\int_X \mathsf{d}^2(x, \bar{x})(f/F)^2(x) \, \mathrm{d}\mathfrak{m}^* \right)^{1/2} \le a + \frac{b}{\sqrt{M}} \sqrt{8\mathsf{Ch}(f/F)} + \left(\int_X \mathsf{d}^2(x, \bar{x}) \, \mathrm{d}\mathfrak{m}^* \right)^{1/2}.$$

Taking into account (6.18) with $R = 1$, (6.16) follows. $\qquad\square$

Remark 6.6. Otto and Villani showed in [36] that in the smooth setting a $CD(K, \infty)$ bound yields a Logarithmic-Sobolev inequality of constant $\rho > 0$ which in turn yields (6.17) with $a = 0$ and $b = \rho^{-1}$ and hence the wLSTI$(0, \sqrt{8}\rho^{-1})$. These implications have been later generalized to the non-smooth setting: see for instance the calculus tools developed in [7] to see that $CD(K, \infty)$ yields the log-Sobolev and [26] and references therein for the implication from log-Sobolev to the (6.17).

In particular if $(X, \mathsf{d}, \mathfrak{m})$ is a $CD(K, \infty)$ space of finite mass with $K > 0$ then it satisfies wLSTI$(0, K^{-1})$. It is also obvious that whenever $\mathfrak{m}$ has bounded support of diameter D, then wLSTI$(D, 0)$ holds. $\qquad\blacksquare$

The relevance of the wLSTI(A, B) in our discussion is due to the following fact:

Proposition 6.7 (Compact embedding of $W^{1,2}$ in L^2)**.** *Let* $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ *be a* $CD(K, \infty)$ *p.m.m. space satisfying* wLSTI(A, B) *for some* $A, B \ge 0$ *(e.g. when* $K > 0$ *and* $\mathfrak{m}$ *is finite or* $\mathfrak{m}$ *has bounded support).*

Then the imbedding of $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ *in* $L^2(X, \mathfrak{m})$ *is compact.*

Proof. The trivial inequality $\int_{X \setminus B_R(\bar{x})} f^2 \, \mathrm{d}\mathfrak{m} \le \frac{1}{R^2} \int_X \mathsf{d}^2(\cdot, \bar{x}) f^2 \, \mathrm{d}\mathfrak{m}$ and (6.16) ensure that whenever a sequence f_n is bounded in $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ it holds

$$\lim_{R \to \infty} \sup_n \int_{X \setminus B_R(\bar{x})} f_n^2 \, \mathrm{d}\mathfrak{m} = 0.$$

Hence the conclusion comes from point (ii) of Theorem 6.3. $\qquad\square$

6.2 Mosco-convergence of Cheeger energies

We apply the previous results to study the variational convergence of the Cheeger energies.

Even if the functionals are not imbedded in a common Hilbert space, it is natural to call the property below "Mosco"-convergence (see [12, § 3.3]). It would also be possible to imbed all the domains of the functionals in a common topological space, as e.g. [37], and state the results in terms of Γ-convergence, or to adopt the general approach of [31, Section 2.5]. Nevertheless, we think that the more direct formulation in terms of functions in varying L^p-spaces would be simpler and still sufficient for the applications.

Theorem 6.8 (Mosco-convergence of Cheeger energies)**.** *Let* $\mathfrak{X}_n$, $n \in \mathbb{N}$, *be a sequence of* $CD(K, \infty)$ *spaces converging to a limit space* $\mathfrak{X}_\infty$ *in the pmG-sense. Then with the notation of Remark 4.6 the following holds:*

- *Weak* $\Gamma - \liminf$ *For every sequence* $n \mapsto f_n \in L^2(X, \mathfrak{m}_n)$ L^2-*weakly converging to some* $f_\infty \in L^2(X, \mathfrak{m}_\infty)$ *we have*
$$\varliminf_{n \to \infty} \mathsf{Ch}_n(f_n) \ge \mathsf{Ch}_\infty(f_\infty). \tag{6.19}$$

- *Strong* $\Gamma - \limsup$ *For every* $f_\infty \in L^2(X, \mathfrak{m}_\infty)$ *there exists a sequence* $n \mapsto f_n \in L^2(X, \mathfrak{m}_n)$ L^2-*strongly convergent to* f_∞ *such that*

$$\lim_{n\to\infty} \mathsf{Ch}_n(f_n) = \mathsf{Ch}_\infty(f_\infty). \tag{6.20}$$

Proof. Inequality (6.19) has already beed proved in point (ii) of Theorem 6.3, so we consider (6.20).

By a simple diagonalization argument we see that it is sufficient to approximate functions f_∞ in a set dense in energy. Furthermore, given that we are going to build an L^2-strongly converging sequence, by using the identity $\mathsf{Ch}(f) = \mathsf{Ch}(f^+) + \mathsf{Ch}(f^-)$ we can also reduce to the case of non-negative f_∞.

Hence after a truncation, localization and normalization arguments we can then assume that f_∞ is essentially bounded with bounded support and, setting $g_\infty := f_\infty^2$, that $\mu_\infty := g_\infty \mathfrak{m}_\infty$ is in $\mathscr{P}_2(X)$. By (5.9) and the same argument of the proof or Theorem 6.3 it is then sufficient to find a sequence

$$n \mapsto \nu_n = g_n \mathfrak{m}_n \ W_2\text{-converging to } \mu_\infty, \text{ with } |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_n}|(\nu_n) \longrightarrow |\mathrm{D}^-\mathrm{Ent}_{\mathfrak{m}_\infty}|(\mu_\infty).$$

This is precisely the content of the second part of Corollary 5.14, so the thesis is achieved. $\square$

Remark 6.9. By the stability of the $\mathrm{CD}(K, \infty)$ condition granted by Theorem 4.9 and the Mosco-convergence of the Cheeger energies just proved it is easy to deduce that the class of $\mathrm{CD}(K, \infty)$ spaces satisfying wLSTI(A, B) is closed w.r.t. pmG-convergence. $\blacksquare$

We provide two useful corollaries to Theorem 6.8, concerning the convergence of the resolvents and of the L^2 gradient flows associated to the Cheeger energies. Both the results are well known in the case of functionals defined in the same Hilbert space (see e.g. [12]), so we only give a brief sketch of the proofs.

For every $\tau > 0, n \in \bar{\mathbb{N}}$ we define the resolvent map $J_{n,\tau} : L^2(X, \mathfrak{m}_n) \to D(\mathsf{Ch}_n)$ as

$$J_{n,\tau}(f) := (I + \tau \partial \mathsf{Ch}_n)^{-1}(f) = \operatorname*{argmin}_{L^2(X,\mathfrak{m}_n)} \Phi_{n,\tau}(\cdot; f) \tag{6.21}$$

where

$$\Phi_{n,\tau}(g; f) := \frac{1}{2\tau} \int |g - f|^2 \, \mathrm{d}\mathfrak{m}_n + \mathsf{Ch}_n(g) \quad \text{for every } g \in L^2(X, \mathfrak{m}_n).$$

Corollary 6.10. *Under the same assumptions of Theorem 6.8, for every sequence* (f_n) L^2-*strongly converging to* $f_\infty \in L^2(X, \mathfrak{m}_\infty)$ *we have*

$$J_{n,\tau}(f_n) \xrightarrow{L^2} J_{\infty,\tau}(f_\infty), \qquad and \qquad \mathsf{Ch}_n(J_{n,\tau}(f_n)) \longrightarrow \mathsf{Ch}_\infty(J_{\infty,\tau}(f_\infty)), \tag{6.22}$$

for every $\tau > 0$.

Proof. Let $g_n = J_{n,\tau}(f_n)$, for every $n \in \bar{\mathbb{N}}$. Choosing 0 as competitor in the definition of g_n we easily see that $\int g_n^2 \, \mathrm{d}\mathfrak{m}_n \leq 4 \int f_n^2 \, \mathrm{d}\mathfrak{m}_n$. Hence we can extract a subsequence (still denoted by g_n) L^2-weakly converging to some $g \in L^2(X, \mathfrak{m}_\infty)$. Applying the second part of Theorem 6.8 we can also find a sequence $\tilde{g}_n \in D(\mathsf{Ch}_n)$ such that $\Phi_{n,\tau}(\tilde{g}_n; f_n) \longrightarrow \Phi_{\infty,\tau}(g_\infty; f_\infty)$. Passing to the limit in the inequalities $\Phi_{n,\tau}(g_n, f_n) \leq \Phi_{n,\tau}(\tilde{g}_n, f_n)$ thanks to the first part of Theorem 6.8 we find that g is a minimizer of $\Phi_{\infty,\tau}(\cdot; f_\infty)$, so that $g = g_\infty$. Since the

limit is unique, we conclude that the whole sequence g_n is L^2-weakly converging to g_∞ and $\Phi_{n,\tau}(g_n, f_n) \to \Phi_{\infty,\tau}(g_\infty, f_\infty)$. A further application of the lower semicontinuity results (6.5) and (6.19) provides the second in (6.22) and

$$\int |g_n - f_n|^2 \, d\mathfrak{m}_n \to \int |g_\infty - f_\infty|^2 \, d\mathfrak{m}_\infty.$$

Expanding the squares, using the L^2-strong convergence of the f_n's and recalling (6.7) we deduce that $\int |g_n|^2 \, d\mathfrak{m}_n \to \int |g_\infty|^2 \, d\mathfrak{m}_\infty$, i.e. the first in (6.22). $\qquad \square$

Theorem 6.11 (L^2-convergence of the Heat flows). *Let $\mathfrak{X}_n$, $n \in \mathbb{N}$, be a sequence of* $\mathrm{CD}(K, \infty)$ *spaces converging to a limit space $\mathfrak{X}_\infty$ in the pmG-sense and adopt the notation of Remark 4.6. Furthermore, let $(\mathsf{H}^n)_{t\geq0}$ be the L^2-gradient flows of the corresponding Cheeger energies as in § 5.1.4.*

Then for every sequence $n \mapsto \bar{f}_n \in L^2(X, \mathfrak{m}_n)$ L^2-strongly converging to $\bar{f}_\infty \in L^2(X, \mathfrak{m}_\infty)$ we have

$$\mathsf{H}_t^n \bar{f}_n \xrightarrow{L^2} \mathsf{H}_t^\infty \bar{f}_\infty \quad \text{strongly for every } t \geq 0. \tag{6.23}$$

Proof. Let us first suppose that $\mathsf{Ch}_n(\bar{f}_n) \leq C < \infty$ for every $n \in \bar{\mathbb{N}}$. We denote by $J_{n,\tau}^k$ the iterated resolvent $(J_{n,\tau})^k$; uniform convergence estimates (see e.g. [5, Thm. 4.0.4]) show that

$$\int \left| \mathsf{H}_t^n(\bar{f}_n) - J_{n,t/k}^k(\bar{f}_n) \right|^2 d\mathfrak{m}_n \leq C \frac{t}{k} \quad \text{for every } n \in \bar{\mathbb{N}}, \; k \in \mathbb{N}.$$

Since the estimate is uniform w.r.t. n and $J_{n,t/k}^k \bar{f}_n \xrightarrow{L^2} J_{\infty,t/k}^k \bar{f}_\infty$ as $n \to \infty$ thanks to the previous Corollary, we easily get (6.23).

We then use the L^2-contraction property of $(\mathsf{H}_t^n)_{t\geq0}$ and the Γ-$\limsup$ estimate (6.20) to extend the result to the general case (see a similar argument in the proof of Theorem 7.7 below). $\qquad \square$

Remark 6.12. In the case of $\mathrm{RCD}(K, \infty)$ spaces, when $(\mathsf{H}_t)_{t\geq0}$ are linear operators, Theorem 6.11 can be directly deduced by the corresponding Wasserstein result, Theorem 5.7, see [10, §5.2].

7 Stability, convergence and spectral properties for $\mathrm{RCD}(K, \infty)$ spaces.

7.1 The $\mathrm{RCD}(K, \infty)$ condition and its stability.

In [9] a study of $\mathrm{CD}(K, \infty)$ with linear heat flow has been initiated, the definition being the following:

Definition 7.1 ($\mathrm{RCD}(K, \infty)$ spaces). *$(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ is a $\mathrm{RCD}(K, \infty)$ p.m.m. space if it satisfies the $\mathrm{CD}(K, \infty)$ condition and the W_2-heat flow $(\mathscr{H}_t)$ (see Theorem 5.5) is linear, i.e.*

$$\mathscr{H}_t(\alpha\mu + \beta\nu) = \alpha\mathscr{H}_t\mu + \beta\mathscr{H}_t\nu \quad \text{for every } t \geq 0, \; \mu, \nu \in D(\mathrm{Ent}_\mathfrak{m}), \; \alpha, \beta \in [0, 1], \; \alpha + \beta = 1, \tag{7.1}$$

or, equivalently, if the Cheeger energy Ch is a quadratic form in $L^2(X, \mathfrak{m})$, i.e.

$$\mathsf{Ch}(f + g) + \mathsf{Ch}(f - g) = 2\mathsf{Ch}(f) + 2\mathsf{Ch}(g) \quad \text{for every } f, g \in L^2(X, \mathfrak{m}). \tag{7.2}$$

The acronym RCD stands for Riemannian Curvature Dimension, indeed it is well known that Finsler geometries are included in the class of $\mathrm{CD}(K, \infty)$ spaces, and that the heat flow on a smooth Finsler manifold is linear if and only if the manifold is Riemannian. Hence the idea behind the definition of the subclass $\mathrm{RCD}(K, \infty)$ of $\mathrm{CD}(K, \infty)$ spaces is somehow to isolate those spaces which have a 'Riemannian' behaviour, see [9] and [4] for results in this direction. We remark that in [9] the $\mathrm{RCD}(K, \infty)$ was required both by asking the linearity of the heat flow as above, and enforcing a bit the $\mathrm{CD}(K, \infty)$ with the requirement that the entropy was K-geodesically convex along a family of geodesics larger than the one appearing in Definition 4.3. It has been later understood in [4] that this enforcement of the $\mathrm{CD}(K, \infty)$ assumption was actually unnecessary.

As for the $\mathrm{CD}(K, \infty)$ condition, the $\mathrm{RCD}(K, \infty)$ one is invariant with respect to isomorphism of p.m.m. spaces. It is also stable w.r.t. pmG convergence, as we show now. Such stability property can be achieved both passing to the limit in (7.2) thanks to the Mosco-convergence of the Cheeger energies that we just proved, or passing to the limit in (7.1) with an argument based on the properties of the relative entropy. We think that both approaches are interesting and thus we are going to propose both proofs.

Theorem 7.2 (Stability of $\mathrm{RCD}(K, \infty)$ under pmG-convergence). *Let $\mathfrak{X}_n$, $n \in \mathbb{N}$, be a sequence of p.m.m. spaces converging to a limit space $\mathfrak{X}_\infty$ in the pmG-sense. Assume that $\mathfrak{X}_n$ is an $\mathrm{RCD}(K, \infty)$ space for every $n \in \mathbb{N}$. Then $\mathfrak{X}_\infty$ is $\mathrm{RCD}(K, \infty)$ as well.*

Proof. We shall adopt the notation of Remark 4.6.
Proof via the use of the Cheeger energy. By assumption we know that for every $n \in \mathbb{N}$ it holds

$$\mathsf{Ch}_n(f_n + g_n) + \mathsf{Ch}_n(f_n - g_n) = 2\mathsf{Ch}_n(f_n) + 2\mathsf{Ch}_n(g_n) \qquad \text{for every } f_n, g_n \in L^2(X, \mathfrak{m}_n). \quad (7.3)$$

Now pick $f_\infty, g_\infty \in L^2(X, \mathfrak{m}_\infty)$ and use the second part of Theorem 6.8 to find sequences (f_n), (g_n) L^2-strongly converging to f_∞, g_∞ respectively such that

$$\lim_{n \to \infty} \mathsf{Ch}_n(f_n) = \mathsf{Ch}_\infty(f_\infty), \qquad\qquad \lim_{n \to \infty} \mathsf{Ch}_n(g_n) = \mathsf{Ch}_\infty(g_\infty).$$

Notice that $n \mapsto f_n \pm g_n$ L^2-strongly converges to $f_\infty \pm g_\infty$, use the first part of Theorem 6.8 and pass to the limit in (7.3) to get

$$\mathsf{Ch}_\infty(f_\infty + g_\infty) + \mathsf{Ch}_\infty(f_\infty - g_\infty) \leq 2\mathsf{Ch}_\infty(f_\infty) + 2\mathsf{Ch}_\infty(g_\infty).$$

Repeat the argument with $f'_\infty := f_\infty + g_\infty$ and $g'_\infty := f_\infty - g_\infty$ in place of f_∞, g_∞ and recall that the Cheeger energy is 2-homogeneous to get the other inequality and the conclusion.
Proof via the use of the relative entropy. We denote by $\mathscr{H}_n$, $n \in \bar{\mathbb{N}}$, the W_2-gradient flow of $\mathrm{Ent}_{\mathfrak{m}_n}$ in $\mathscr{P}_2(X)$. Let $\mu_\infty^i \in D(\mathrm{Ent}_{\mathfrak{m}_\infty})$, $i = 0, 1$, and $\mu_\infty^\alpha := (1 - \alpha)\mu_\infty^0 + \alpha\mu_\infty^1$, $\alpha \in (0, 1)$.
The $\Gamma\text{-}\overline{\lim}$ inequality in Proposition 4.7 provides sequences $n \mapsto \mu_n^i \in D(\mathrm{Ent}_{\mathfrak{m}_n})$ such that

$$\lim_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^i) = \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty^i), \qquad \mu_n^i \xrightarrow{W_2} \mu_\infty^i \qquad i = 0, 1.$$

By the general convergence result Theorem 5.7 we know that

$$\mathscr{H}_{n,t}(\mu_n^\alpha) = (1 - \alpha)\mathscr{H}_{n,t}(\mu_n^0) + \alpha\mathscr{H}_{n,t}(\mu_n^1) \xrightarrow{W_2} (1 - \alpha)\mathscr{H}_{\infty,t}(\mu_\infty^0) + \alpha\mathscr{H}_{\infty,t}(\mu_\infty^1).$$

Hence our thesis follows if we show that

$$\lim_{n\to\infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^\alpha) = \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty^\alpha), \qquad \mu_n^\alpha \xrightarrow{W_2} \mu_\infty^\alpha, \tag{7.4}$$

so that Theorem 5.7 yields $\mathscr{H}_{n,t}(\mu_t^\alpha) \xrightarrow{W_2} \mathscr{H}_{\infty,t}(\mu_\infty^\alpha)$. In order to prove (7.4) let us first observe that the W_2-convergence of μ_n^α to μ_∞^α is a direct consequence of the convexity of the squared Wasserstein distance W_2^2.

Since $\mu_n^\alpha \in D(\mathrm{Ent}_{\mathfrak{m}_n})$ and $\mu_n^0, \mu_n^1 \ll \mu_n^\alpha$ with bounded density if $\alpha \in (0,1)$, for every $n \in \bar{\mathbb{N}}$ we have

$$\mathrm{Ent}_{\mu_n^\alpha}(\mu_n^i) = \int \log\left(\frac{\mathrm{d}\mu_n^i}{\mathrm{d}\mu_n^\alpha}\right) \mathrm{d}\mu_n^i = \int \log\left(\frac{\mathrm{d}\mu_n^i}{\mathrm{d}\mathfrak{m}_n}\right) - \log\left(\frac{\mathrm{d}\mu_n^\alpha}{\mathrm{d}\mathfrak{m}_n}\right) \mathrm{d}\mu_n^i$$
$$= \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^i) - \int \log\left(\frac{\mathrm{d}\mu_n^\alpha}{\mathrm{d}\mathfrak{m}_n}\right) \mathrm{d}\mu_n^i.$$

Taking appropriate convex combinations we get

$$(1-\alpha)\mathrm{Ent}_{\mu_n^\alpha}(\mu_n^0) + \alpha\mathrm{Ent}_{\mu_n^\alpha}(\mu_n^1) = (1-\alpha)\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^0) + \alpha\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^1) - \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^\alpha).$$

Therefore from the W_2-convergence of $(\mu_n^i), (\mu_n^\alpha)$ to $\mu_\infty^i, \mu_\infty^\alpha$ respectively and the $\Gamma\text{-}\underline{\lim}$ inequality in Proposition 4.7 we get

$$\varliminf_{n\to\infty} (1-\alpha)\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^0) + \alpha\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^1) - \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^\alpha) = \varliminf_{n\to\infty} (1-\alpha)\mathrm{Ent}_{\mu_n^\alpha}(\mu_n^0) + \alpha\mathrm{Ent}_{\mu_n^\alpha}(\mu_n^1)$$
$$\geq (1-\alpha)\mathrm{Ent}_{\mu_\infty^\alpha}(\mu_\infty^0) + \alpha\mathrm{Ent}_{\mu_\infty^\alpha}(\mu_\infty^1)$$
$$= (1-\alpha)\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty^0) + \alpha\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty^1) - \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty^\alpha).$$

This fact together with the assumption $\lim_n \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^i) = \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty^i)$, $i = 0,1$, give

$$\varliminf_{n\to\infty} -\mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^\alpha) \geq -\mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty^\alpha).$$

Since the other inequality in ensured by the $\Gamma\text{-}\underline{\lim}$ part in Proposition 4.7, the thesis is achieved. $\qquad\square$

7.2 Refined estimates on the convergence of the Heat flow.

One of the main contributions of [9] (see also [4]) is the proof that the linearity condition on the heat flow grants additional regularity properties for the flow itself. A crucial one is the following contractivity statement.

Theorem 7.3 (W_2-contraction). *Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be a $\mathrm{RCD}(K, \infty)$ space. For every $\bar{\mu}, \bar{\nu} \in D(\mathrm{Ent}_{\mathfrak{m}})$ it holds*

$$W_2(\mathscr{H}_t\bar{\mu}, \mathscr{H}_t\bar{\nu}) \leq \mathrm{e}^{-Kt} W_2(\bar{\mu}, \bar{\nu}) \qquad \text{for every } t \geq 0. \tag{7.5}$$

Notice that in a smooth setting, (7.5) is specific of Riemannian geometry, because Ohta-Sturm proved in [35] that no exponential contraction holds in $(\mathbb{R}^d, \|\cdot\|, \mathcal{L}^d)$ if the norm $\|\cdot\|$ does not come from a scalar product.

A direct consequence of (7.5) is that the W_2-gradient flow $(\mathscr{H}_t)_{t\geq 0}$ of the entropy can be extended from $D(\mathrm{Ent}_{\mathfrak{m}})$ to its W_2-closure, which is $\mathscr{P}_2(\mathrm{supp}(\mathfrak{m}))$ (i.e. the subset of $\mathscr{P}_2(X)$

made of measures μ such that $\text{supp}(\mu) \subset \text{supp}(\mathfrak{m})$). In other words, a unique one parameter family of maps $\mathscr{H}_t : \mathscr{P}_2(\text{supp}(\mathfrak{m})) \to \mathscr{P}_2(\text{supp}(\mathfrak{m}))$ is defined by the following two properties:

$$W_2(\mathscr{H}_t(\mu), \mathscr{H}_t(\nu)) \le e^{-Kt} W_2(\mu, \nu) \quad \text{for every } t \ge 0, \ \mu, \nu \in \mathscr{P}_2(\text{supp}(\mathfrak{m})), \qquad (7.6)$$

and

$\forall \mu \in D(\text{Ent}_\mathfrak{m})$ the curve $t \mapsto \mathscr{H}_t(\mu)$ is the gradient flow of the entropy starting from μ in the sense of Definition 5.4.

As for the heat flow and the $\text{CD}(K, \infty)$, $\text{RCD}(K, \infty)$ conditions, these maps are invariant w.r.t. isomorphisms of p.m.m. spaces.

A more explicit characterization is provided by the following theorem, showing in parti- cular that $\mathscr{H}_t(\mu) \in D(\text{Ent}_\mathfrak{m})$ for $t > 0$ and every $\mu \in \mathscr{P}_2(\text{supp}(\mathfrak{m}))$. The proof can be found, for instance, in the preliminary section of [9].

In the statement of the result and thereafter, the function $I_K : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$I_K(t) := \int_0^t e^{Ks} \, ds = \begin{cases} \frac{e^{Kt} - 1}{K} & \text{if } K \ne 0 \\ t & \text{if } K = 0. \end{cases}$$

Theorem 7.4 (A priori estimates). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\text{RCD}(K, \infty)$ space. Then for every $\mu, \nu \in \mathscr{P}_2(\text{supp}(\mathfrak{m}))$ and any $t > 0$ it holds*

$$I_K(t)\text{Ent}_\mathfrak{m}(\mathscr{H}_t(\mu)) + \frac{I_K(t)^2}{2}|\text{D}^-\text{Ent}_\mathfrak{m}|^2(\mathscr{H}_t(\mu)) \le I_K(t)\text{Ent}_\mathfrak{m}(\nu) + \frac{1}{2}W_2^2(\nu, \mu). \qquad (7.7)$$

As a direct consequence of (7.7), we have the following a priori control on the entropy and its slope along the flow, which we state and prove only for t close to 0, which is the regime we will need later.

Corollary 7.5. *Let $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ be a $\text{RCD}(K, \infty)$ p.m.m. space, $\mathsf{C} > 1 + K_-$ and $\mathsf{z}, \tilde{\mathfrak{m}}$ as in (4.3). For every $\mu \in \mathscr{P}_2(\text{supp}(\mathfrak{m}))$ and $t \in [0, \frac{1}{8\mathsf{C}}]$ it holds*

$$I_K(t)\text{Ent}_\mathfrak{m}(\mathscr{H}_t(\mu)) \le -I_K(t) \log \mathsf{z} + (1 - \mathsf{C} \, I_K(t)) \int \mathsf{d}^2(\cdot, \bar{x}) \, d\tilde{\mathfrak{m}} + \int \mathsf{d}^2(\cdot, \bar{x}) \, d\mu \quad (7.8)$$

$$\frac{I_K(t)^2}{2}|\text{D}^-\text{Ent}_\mathfrak{m}|^2(\mathscr{H}_t(\mu)) \le (1 + 3\mathsf{C} \, I_K(t)e^{-2Kt}) \int \mathsf{d}^2(\cdot, \bar{x}) \, d\mu + \mathsf{c}(K, t) \int \mathsf{d}^2(\cdot, \bar{x}) \, d\tilde{\mathfrak{m}}, \quad (7.9)$$

where $\mathsf{c}(K, t)$ is given by $\mathsf{c}(K, t) := 1 - \mathsf{C} \, I_K(t) + 3\mathsf{C}I_K(t)(4\mathsf{C}t + (1 + e^{-Kt})^2)$.

Proof. The bound (7.8) follows plugging $\nu := \tilde{\mathfrak{m}}$ in (7.7), neglecting the (non negative) term in the squared slope and using the identity

$$\text{Ent}_\mathfrak{m}(\tilde{\mathfrak{m}}) = \int \log(\mathsf{z}^{-1}e^{-\mathsf{C}\mathsf{d}^2(\cdot, \bar{x})}) \, d\tilde{\mathfrak{m}} = -\log \mathsf{z} - \mathsf{C}W_2^2(\tilde{\mathfrak{m}}, \delta_{\bar{x}}). \qquad (7.10)$$

For (7.9) we argue as follows. Let (ν_t) be the gradient flow of $\text{Ent}_\mathfrak{m}$ starting from $\tilde{\mathfrak{m}}$ and notice that

$$W_2(\mathscr{H}_t(\mu), \delta_{\bar{x}}) \le W_2(\mathscr{H}_t(\mu), \nu_t) + W_2(\nu_t, \tilde{\mathfrak{m}}) + W_2(\tilde{\mathfrak{m}}, \delta_{\bar{x}})$$

$$\le e^{-Kt} W_2(\mu, \tilde{\mathfrak{m}}) + \int_0^t |\dot{\nu}_s| \, ds + W_2(\tilde{\mathfrak{m}}, \delta_{\bar{x}})$$

$$\le e^{-Kt} W_2(\mu, \delta_{\bar{x}}) + \sqrt{t \int_0^t |\dot{\nu}_s|^2 \, ds} + W_2(\tilde{\mathfrak{m}}, \delta_{\bar{x}})(1 + e^{-Kt}).$$

Squaring, using the trivial inequality $(a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2$ and the a priori estimate (5.15) (here we use that $t \in [0, \frac{1}{8C}]$) we get

$$\frac{1}{3} W_2^2(\mathscr{H}_t(\mu), \delta_{\bar{x}}) \leq e^{-2Kt} W_2^2(\mu, \delta_{\bar{x}}) + 4t \mathrm{Ent}_{\mathfrak{m}}(\tilde{\mathfrak{m}}) + W_2^2(\tilde{\mathfrak{m}}, \delta_{\bar{x}})\big(8Ct + (1 + e^{-Kt})^2\big) + 4t \log z.$$

Taking into account the identity (7.10), this bound reduces to

$$\frac{1}{3} W_2^2(\mathscr{H}_t(\mu), \delta_{\bar{x}}) \leq e^{-2Kt} W_2^2(\mu, \delta_{\bar{x}}) + W_2^2(\tilde{\mathfrak{m}}, \delta_{\bar{x}})\big(4Ct + (1 + e^{-Kt})^2\big).$$

Plugging this bound into (4.4) and using the fact that $\mathrm{Ent}_{\tilde{\mathfrak{m}}} \geq 0$ we obtain

$$\mathrm{Ent}_{\mathfrak{m}}(\mathscr{H}_t(\mu)) \geq -\log z - 3Ce^{-2Kt} W_2^2(\mu, \delta_{\bar{x}}) - 3CW_2^2(\tilde{\mathfrak{m}}, \delta_{\bar{x}})\big(4Ct + (1 + e^{-Kt})^2\big).$$

The conclusion follows using this inequality in (7.7), choosing $\nu := \tilde{\mathfrak{m}}$ and using again (7.10). $\qquad \square$

Remark 7.6. Corollary 7.5 grants that for every $\mu \in \mathscr{P}_2(\mathrm{supp}(\mathfrak{m}))$, the curve $t \mapsto \mathscr{H}_t(\mu)$ is continuous and satisfies $\mathscr{H}_t(\mu) \in D(\mathrm{Ent}_{\mathfrak{m}})$ for any $t > 0$. In particular, for any $t_0 > 0$ the curve $[0, \infty) \ni t \mapsto \mathscr{H}_{t+t_0}(\mu)$ is the gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ starting from $\mathscr{H}_{t_0}(\mu)$ in the sense of Definition 5.4. For these reasons, we still call the curve $[0, \infty) \ni t \mapsto \mathscr{H}_t(\mu)$ the gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ starting from μ even for $\mu \in \mathscr{P}_2(\mathrm{supp}(\mathfrak{m})) \setminus D(\mathrm{Ent}_{\mathfrak{m}})$. $\qquad \blacksquare$

The a priori estimates in Corollary 7.5 and the contraction property (7.6) allow to refine, in the case of $\mathrm{RCD}(K, \infty)$ spaces, the convergence result on the heat flow given in Theorem 5.7. Notice indeed that in the statement below only W_2-convergence of the initial data is required, without any assumption on the behaviour of the entropies.

Theorem 7.7 (Improved stability of the heat flow). *Let $\mathfrak{X}_n$, $n \in \mathbb{N}$, be a sequence of $\mathrm{RCD}(K, \infty)$ spaces converging to a limit space $\mathfrak{X}_\infty$ in the pmG-sense and adopt the notation of Remark 4.6.*

If $(\bar{\mu}_n) \subset \mathscr{P}_2(X)$ satisfies

$$\bar{\mu}_n \xrightarrow{W_2} \bar{\mu}_\infty \in \mathscr{P}_2(\mathrm{supp}(\mathfrak{m}_\infty)),$$

then the solutions $\mu_{n,t} = \mathscr{H}_{n,t}(\bar{\mu}_n)$, $t \geq 0$, of the W_2-gradient flows of $\mathrm{Ent}_{\mathfrak{m}_n}$ satisfy all the properties (5.17a,b,c,d).

Proof. Let $k \mapsto \mu_\infty^k \subset D(\mathrm{Ent}_{\mathfrak{m}_\infty})$ be a sequence such that

$$\lim_{k \to \infty} \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty^k) = \mathrm{Ent}_{\mathfrak{m}_\infty}(\bar{\mu}_\infty), \qquad \lim_{k \to \infty} W_2(\mu_\infty^k, \bar{\mu}_\infty) = 0.$$

For every $k \in \mathbb{N}$, use the $\Gamma\text{-}\overline{\lim}$ part of Proposition 4.7 to find a sequence $n \mapsto \mu_n^k \in \mathscr{P}_2(X)$ such that

$$\lim_{n \to \infty} \mathrm{Ent}_{\mathfrak{m}_n}(\mu_n^k) = \mathrm{Ent}_{\mathfrak{m}_\infty}(\mu_\infty^k), \qquad \lim_{n \to \infty} W_2(\mu_n^k, \mu_\infty^k) = 0. \qquad (7.11)$$

Taking (7.6) into account we get

$$W_2\big(\mu_{\infty,t}, \mu_{n,t}\big)$$
$$\leq W_2\big(\mu_{\infty,t}, \mathscr{H}_{\infty,t}(\mu_\infty^k)\big) + W_2\big(\mathscr{H}_{\infty,t}(\mu_\infty^k), \mathscr{H}_{n,t}(\mu_n^k)\big) + W_2\big(\mathscr{H}_{n,t}(\mu_n^k), \mu_{n,t}\big)$$
$$\leq e^{-Kt} W_2(\bar{\mu}_\infty, \mu_\infty^k) + W_2\big(\mathscr{H}_{\infty,t}(\mu_\infty^k), \mathscr{H}_{n,t}(\mu_n^k)\big) + e^{-Kt} W_2(\mu_n^k, \bar{\mu}_n)$$
$$\leq e^{-Kt}\big(2W_2(\bar{\mu}_\infty, \mu_\infty^k) + W_2(\bar{\mu}_\infty, \bar{\mu}_n) + W_2(\mu_\infty^k, \mu_n^k)\big) + W_2\big(\mathscr{H}_{\infty,t}(\mu_\infty^k), \mathscr{H}_{n,t}(\mu_n^k)\big).$$
$$(7.12)$$

The choices (7.11) and the assumption $\mu_\infty^k \in D(\text{Ent}_{\mathfrak{m}_\infty})$ ensure that the hypotheses of Theorem 5.7 are fulfilled with μ_n^k in place of $\bar{\mu}_n$ and μ_∞^k in place of $\bar{\mu}_\infty$. Hence it holds $\lim_{n\to\infty} W_2\big(\mathscr{H}_{n,t}(\mu_n^k), \mathscr{H}_{\infty,t}(\mu_\infty^k)\big) = 0$, so that passing to the limit first as $n \to \infty$ then as $k \to \infty$ in (7.12) we get

$$\lim_{n\to\infty} W_2\big(\mu_{\infty,t}, \mu_{n,t}\big) = 0 \qquad \text{for every } t \geq 0.$$

Now fix $\varepsilon \in (0, \frac{1}{8\mathsf{C}}]$ and notice that from $W_2\big(\mu_{\infty,\varepsilon}, \mu_{n,\varepsilon}\big) \to 0$ and the $\Gamma\text{-}\underline{\lim}$ part of Proposition 4.7 we get $\underline{\lim}_{n\to\infty} \text{Ent}_{\mathfrak{m}_n}(\mu_{n,\varepsilon}) \geq \text{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,\varepsilon})$. We claim that $\overline{\lim}_{n\to\infty} \text{Ent}_{\mathfrak{m}_n}(\mu_{n,\varepsilon}) \leq \text{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,\varepsilon}) < \infty$ as well. The fact that $\mu_{\infty,\varepsilon} \in D(\text{Ent}_{\mathfrak{m}_\infty})$ follows from (7.8). Use the $\Gamma\text{-}\overline{\lim}$ part of Proposition 4.7 to find a sequence $(\nu_n) \subset \mathscr{P}_2(X)$ such that

$$\lim_{n\to\infty} \text{Ent}_{\mathfrak{m}_n}(\nu_n) = \text{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,\varepsilon}), \qquad \lim_{n\to\infty} W_2(\nu_n, \mu_{\infty,\varepsilon}) = 0.$$

From (7.9) we get $\sup_n |\text{D}^-\text{Ent}_{\mathfrak{m}_n}|(\mu_{n,\varepsilon}) =: S < \infty$, so that formula (5.7) gives

$$\text{Ent}_{\mathfrak{m}_n}(\mu_{n,\varepsilon}) \leq \text{Ent}_{\mathfrak{m}_n}(\nu_n) + S W_2(\nu_n, \mu_{n,\varepsilon}) - \frac{K}{2} W_2^2(\nu_n, \mu_{n,\varepsilon}).$$

Letting $n \to \infty$ and using the fact that $W_2(\nu_n, \mu_{n,\varepsilon}) \to 0$ we get $\overline{\lim}_{n\to\infty} \text{Ent}_{\mathfrak{m}_n}(\mu_{n,\varepsilon}) \leq \text{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,\varepsilon}) < \infty$, as claimed.

In summary, we proved that it holds

$$\lim_{n\to\infty} \text{Ent}_{\mathfrak{m}_n}(\mu_{n,\varepsilon}) = \text{Ent}_{\mathfrak{m}_\infty}(\mu_{\infty,\varepsilon}), \qquad \lim_{n\to\infty} W_2(\mu_{n,\varepsilon}, \mu_{\infty,\varepsilon}) = 0,$$

therefore Theorem 5.7 is applicable with $\mu_{n,\varepsilon}$ in place of $\bar{\mu}_n$ and $\mu_{\infty,\varepsilon}$ in place of $\bar{\mu}_\infty$. The conclusion comes from the arbitrariness of $\varepsilon \in [0, \frac{1}{8\mathsf{C}}]$. $\qquad\square$

7.3 Spectral convergence

Since in a RCD(K, ∞) space $(X, \mathsf{d}, \mathfrak{m})$ the Laplacian $\Delta_{\mathsf{d},\mathfrak{m}}$ (see § 5.1.4) is a selfadjoint linear closed operator in $L^2(X, \mathfrak{m})$, it would be interesting to study the behaviour of its spectrum w.r.t. pmG-convergence. We consider here the case of a sequence $[X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n] \in \mathbb{X}$, $n \in \bar{\mathbb{N}}$, of RCD(K, ∞) spaces satisfying a uniform weak Logarithmic-Sobolev-Talagrand inequality wLSTI(A, B), so that in particular

$$W^{1,2}(X_n, \mathsf{d}_n, \mathfrak{m}_n) \quad \text{is compactly imbedded in} \quad L^2(X_n, \mathfrak{m}_n) \quad \text{for every } n \in \bar{\mathbb{N}},$$

by Proposition 6.7. Let $\Delta_n := \Delta_{\mathsf{d}_n,\mathfrak{m}_n}$ be the corresponding Laplace operators; since the resolvent $(I + \tau\Delta_n)^{-1}$ as in (6.21) are compact operators, we can find an enumeration

$$0 \leq \lambda_1(\Delta_n) \leq \lambda_2(\Delta_n) \leq \cdots \leq \lambda_k(\Delta_n) \leq \cdots, \qquad k \in \mathbb{N},$$

of the eigenvalues of Δ_n, each repeated according to its own multiplicity. The eigenvalues are invariant under isomorphisms of p.m.m. spaces, so $\lambda_k(\Delta_n)$ only depends on the equivalence class $\mathfrak{X}_n$ of the space $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$.

Theorem 7.8 (Convergence of the eigenvalues of the Laplace operator)**.** *Let $\mathfrak{X}_n$ be a sequence of RCD(K, ∞) spaces converging to a limit $\mathfrak{X}_\infty$ space in the pmG-sense. Assume that, for some $A, B \geq 0$, $\mathfrak{X}_n$ satisfies the wLSTI(A, B) according to Definition 6.4 for every $n \in \mathbb{N}$ (in*

particular if $K > 0$ and the masses are finite or $\mathrm{diam}\big(\mathrm{supp}(\mathfrak{m}_n)\big)$ is uniformly bounded as in Remark 6.6).

Then

$$\lim_{n\to\infty} \lambda_k(\Delta_n) = \lambda_k(\Delta_\infty) \quad \text{for every } k \in \mathbb{N}.$$

Proof. The proof is a typical application of the Mosco-convergence result proven in Theorem 6.8: the only difference with respect to the canonical setting concerns the fact that the domains of the energies are not imbedded in a common Hilbert space. We shall adopt the notation of Remark 4.6.

If V is a subspace of $L^2(X, \mathfrak{m}_n)$, we denote by $B(V)$ the closed unit ball in V w.r.t. the $L^2(X, \mathfrak{m}_n)$-norm. We call $\mathcal{V}_{k,n}$ the collection of all the k-dimensional subspaces contained in $D(\mathsf{Ch}_n)$, and we recall the minimum-maximum principle characterizing the eigenvalues

$$\lambda_k(\Delta_n) = \min_{V \subset \mathcal{V}_{k,n}} \max_{f \in B(V)} \mathcal{E}_n(f, f), \tag{7.13}$$

where $\mathcal{E}_n : D(\mathsf{Ch}_n) \times D(\mathsf{Ch}_n) \to \mathbb{R}$ is the symmetric bilinear form associated to $2\mathsf{Ch}_n$. Let us fix $k \in \mathbb{N}$ and first prove that

$$\varlimsup_{n\to\infty} \lambda_k(\Delta_n) \le \lambda_k(\Delta_\infty). \tag{7.14}$$

Let $V_\infty \in \mathcal{V}_{k,\infty}$ be a vector space realizing the minimum in (7.13) and let $(f_{i,\infty}) \in \mathcal{V}_{k,\infty}$, $i = 1, \ldots, k$, be an $L^2(X, \mathfrak{m}_n)$-orthonormal system of V_∞, so that

$$\lambda_k(\Delta_\infty) \ge \mathcal{E}_\infty(f_{i,\infty}, f_{i,\infty}) \quad \forall i = 1, \ldots, k \quad \& \quad \mathcal{E}_\infty(f_{i,\infty}, f_{j,\infty}) = 0 \quad \forall i \ne j = 1, \ldots, k.$$

By Theorem 6.8 we find k sequences $n \mapsto f_{i,n} \in D(\mathsf{Ch}_n)$, $i = 1, \ldots, k$, strongly converging in the L^2-sense and in Cheeger energy to $f_{i,\infty}$. It is obviously not restrictive to assume that $\int f_{i,n}^2 \, d\mathfrak{m}_n = 1$ for every $n \in \mathbb{N}$ and $i = 1, \ldots, k$, furthermore by (7.13) we thus know that for every constant $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ sufficiently big so that for every $n > n_\varepsilon$ we have

$$\left| \int f_{i,n} f_{j,n} \, d\mathfrak{m}_n \right| + \left| \mathcal{E}_n(f_{i,n}, f_{j,n}) \right| \le \varepsilon, \quad \forall i \ne j, \qquad \mathcal{E}_n(f_{i,n}, f_{i,n}) \le \lambda_k(\Delta_\infty) + \varepsilon, \quad \forall i.$$

Setting $V_{k,n} = \mathrm{span}(f_{1,n}, \ldots f_{k,n})$ we easily find that for every $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$

$$\int \Big(\sum_{i=1}^k \alpha_i f_{i,n} \Big)^2 \, d\mathfrak{m}_n = |\boldsymbol{\alpha}|^2 + \sum_{i \ne j} \alpha_i \alpha_j \int f_{i,n} f_{j,n} \, d\mathfrak{m}_n \ge (1 - \varepsilon)|\boldsymbol{\alpha}|^2$$

so that $\dim(V_{k,n}) = k$ and every $f \in B(V_{k,n})$ can be represented as $f = \sum_{i=1}^k \alpha_i f_{i,n}$ with $|\boldsymbol{\alpha}|^2 \le (1 - \varepsilon)^{-1}$. Using this linear decomposition, we get

$$\mathcal{E}_n(f, f) = \sum_{i,j=1}^k \alpha_i \alpha_j \mathcal{E}_n(f_{i,n}, f_{j,n}) \le (\lambda_k(\Delta_\infty) + 2\varepsilon)|\boldsymbol{\alpha}|^2 \le (1 - \varepsilon)^{-1}(\lambda_k(\Delta_\infty) + 2\varepsilon)$$

for every $f \in B(V_{k,n})$: thus $\lambda_k(\Delta_n) \le (1 - \varepsilon)^{-1}(\lambda_k(\Delta_\infty) + 2\varepsilon)$ if $n \ge n_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves (7.14).

Let us now prove the $\liminf$ inequality

$$\varliminf_{n\to\infty} \lambda_k(\Delta_n) \ge \lambda_k(\Delta_\infty).$$

By (7.13) for every $n \in \mathbb{N}$ we can find a space $V_{k,n} \subset D(\mathsf{Ch}_n)$ generated by the $L^2(X, \mathfrak{m}_n)$-orthonormal system $(f_{i,n})$, $i = 1, \ldots, k$, such that for every $n \in \mathbb{N}$ and for every linear combination

$$f_{\boldsymbol{\alpha},n} := \sum_{i=1}^{k} \alpha_i f_{i,n} \quad \text{with } \boldsymbol{\alpha} = (\alpha_i)_i \in \mathbb{R}^k, \quad |\boldsymbol{\alpha}|^2 = \sum_{i=1}^{k} \alpha_i^2 = 1, \tag{7.15}$$

we have

$$\int f_{i,n} f_{j,n} \, d\mathfrak{m}_n = \delta_{ij}, \quad \lambda_k(\Delta_n) \geq \mathcal{E}_n(f_{\boldsymbol{\alpha},n}, f_{\boldsymbol{\alpha},n}). \tag{7.16}$$

The previous estimate shows in particular that $\sup_{\boldsymbol{\alpha},n} \mathcal{E}_n(f_{\boldsymbol{\alpha},n}, f_{\boldsymbol{\alpha},n}) < \infty$: we can then extract an increasing subsequence n' such that for every $\boldsymbol{\alpha} \in \mathbb{R}^k$ with $|\boldsymbol{\alpha}| = 1$

$$\lambda_k(\Delta_n) \longrightarrow \lambda, \quad f_{\boldsymbol{\alpha},n'} \longrightarrow f_{\boldsymbol{\alpha},\infty} = \sum_{i=1}^{k} \alpha_i f_{i,\infty} \quad \text{weakly}, \quad \mathcal{E}_\infty(f_{\boldsymbol{\alpha},\infty}, f_{\boldsymbol{\alpha},\infty}) \leq \lambda, \tag{7.17}$$

where for the last inequality we applied (6.19).

Applying ii) of Theorem 6.3 thanks to the uniform estimate provided by the wLSTI, we can reinforce the weak convergence of $n \mapsto f_{i,n}$ to a strong convergence in L^2-sense, so that we can pass to the limit in the identities of (7.16) and prove that $(f_{i,\infty})$ is an orthonormal system generating a k-dimensional subspace $V_{k,\infty}$. Every $f \in B(V_{k,\infty})$ can then be represented as $f_{\boldsymbol{\alpha},\infty}$ as in (7.15) so that (7.17) yields $\mathcal{E}_\infty(f, f) \leq \lambda$ for every $f \in B(V_{k,\infty})$. Using (7.13) with $n = \infty$ we conclude $\lambda_k(\Delta_\infty) \leq \lambda$, as desired. $\qquad\square$

Remark 7.9. When the pmG-convergent sequence of $\text{RCD}(K, \infty)$ spaces $\mathfrak{X}_n$ does not satisfy a uniform wLSTI condition, the Mosco-convergence result of Theorem 6.8 is still sufficient to prove that every point λ in the spectrum $\sigma(\Delta_\infty)$ is the limit of points $\lambda_n \in \sigma(\Delta_n)$: one can argue as in the proof of [31, Proposition 2.5] (see also Remark 6.2), where it is shown that (a suitably adapted) Mosco-convergence of Dirichlet forms always implies lower semicontinuity of the spectra.

The uniform wLSTI condition provides a further asymptotical compactness analogous to [31, Definition 2.12]: in this case Theorem 7.8 could also be proven as for Theorem 2.6 in [31]. $\blacksquare$

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