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Hedging of the Barrier Put Option in a Diffusion (B, S) – Market in case of Dividends Payment on a Risk Active

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Abstract: Barrier European put option formed by additional clause putting in option contract with payment limitation for issuer and guaranteed income for holder of the security are researched when dividends on base risk active are paid. The equitable price, the optimal portfolio and a size of the capital answered the hedging strategy are founded for the options under consideration on diffusion (*B*, *S*)-financial market. Comparative price analysis for two option classes is carried out and specific properties of decision and decision under limiting are explored.

Keywords: Financial market, stochastic financial mathematics, option price, hedging strategy, barrier European put option, put option with guaranteed income for holder of the security, dividends.

1. INTRODUCTION

As of today the financial instruments of trading and risks hedging (Hull, 2013) on the derivatives market are presented by futures, forwards and options, particularly the exotic options (Rubinstein, 2013, Burenin, 2011ab, Shiryaev et al, 2006) The lasts are of interest for investor due to variety of the option's payment liabilities (Shiryaev, 1999) and are the stochastic financial mathematics object (Melnikov et al., 2002). An European put option is a derivative (secondary) security, it is the contract giving option's buyer (the holder) the right to sell stipulated underlying asset by a certain date for a certain price, and option's seller must satisfy an agreement when exercising for an option premium (Hull, 2013).

The research is devoted to barrier European put option on stocks when dividends on base risk active are paid. The payoff function determined the payment size when the option under consideration exercising is

$$f_T(S_T) = \min\{(K_1 - S_T)^+, K_2\} I[S_T > H] + (H - S_T) I[S_T \le H], \quad (1)$$

where S_T is risk asset's spot price at expiration date T, K_1 is exercise price or strike price, K_2 – contracted constant restricted payment of the option writer, on the one hand, and guaranteed income for option, H is barrier for price S_T ($0 \le H < K_1 - K_2$). In accordance to (1) the exotic European put option payoff liability assumed as (2) is base for barrier option under study

$$f_T^{base}(S_T) = \min\{(K_1 - S_T)^+, K_2\}, \quad (2)$$

and it goes on when intersection of barrier H by the spot price S_T top-down (in drop in prices phase). If at the moment T the market state such as $S_T > H$ then the option holder gets the size $f_T^{base}(S_T)$; in other cases (if $S_T \le H$) the option buyer earns rebate $(H - S_T)$.

We denote the mathematical expectation by $E\{\cdot\}$, the normal (Gaussian) density with the parameters a and b by $N\{a; b\}$; $\Phi(x) = \int_{-\infty}^{x} \varphi(y)dy$, $\varphi(y) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}$ are Laplace distribution function and probability density function respectively.

2. STATEMENT OF THE PROBLEM

Let us consider complete, without arbitrage and risk-neutral financial market of two assets, notably: risk (stocks) and risk free (bank deposit) active. The stocks price evolution is given on stochastic basis $(\Omega, F, \mathbf{F} = (F_t)_{t>0}, \mathbf{P})$ (Shiryaev, 1999, Shiryaev et al., 2006). The current prices of the securities S_t и B_t, $t \in [0,T]$, are specified by (3) and (4) respectively

$$dS_t = S_t(\mu dt + \sigma dW_t),\ S_t = S_0 \exp\{(\mu - (\sigma^2/2))t + \sigma W_t\}. \quad (3)$$

$$dB_t = rB_t dt,\ B_t = B_0 \exp\{rt\}, \quad (4)$$

where W_t is a standard Wiener process, $S_0 > 0$ is the stock initial cost, $\mu \in R = (-\infty, +\infty)$ is the percentage drift, $\sigma > 0$ is the percentage volatility in a geometric Brownian motion, $B_0 > 0$ is the risk free asset initial price, $r > 0$ is interest rate.

During time interval $t \in [0,T]$ the investor forms self-financing portfolio $\pi_t = (\beta_t, \gamma_t)$, where F_t-measurable processes β_t and γ_t are parts of the risk free and risk assets

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at investment portfolio respectively, and this portfolio secures investor capital $X_t = \beta_t B_t + \gamma_t S_t$. As in (Shiryaev, 1999, Shiryaev et al., 2006), for holding asset dividends are paid in accordance to process D_t at the rate of $\delta\gamma_t S_t$, $\delta > 0$, that is $dD_t = \delta\gamma_t S_t dt$. Then capital change trajectory is described by equation $dX_t = \beta_t dB_t + \gamma_t dS_t + dD_t$. And as $dX_t = \beta_t dB_t + +\gamma_t dS_t + +B_t d\beta_t + S_t d\gamma_t$, then $B_t d\beta_t + S_t d\gamma_t = dD_t$ is a balance correlation replacing term $B_t d\beta_t + S_t d\gamma_t = 0$ for self financing portfolio in the standard problem (Burenin, 2011ab).

The problem involves the fact that to form the portfolio (hedging strategy) $\pi_t = (\beta_t, \gamma_t)$, the evolution of the capital X_t has option price $P_T = X_0$ in accordance to the payoff function (1), as well as, the hedging strategy and corresponding capital, ensuring the fulfillment of payment liability $X_T = f_T(S_T)$.

3. PRELIMINARY RESULTS

All results below are obtained on the assumption of the sole risk-neutral measure existence. Relative to this measure the process of the risk asset capitalized price S_t/B_t is martingale, and that condition guarantees the assigned problem solvability (Burenin, 2011ab, Shiryaev, 1999, Shiryaev et al, 2006, Melnikov et al., 2002).

Theorems 1, 2 are proved with a glance of the base financial relations (5)–(7) (Burenin, 2011ab, Shiryaev, 1999, Shiryaev et al 2006, Melnikov et al., 2002)

$$P_T = e^{-rT} E^*\{f_T(S_T)\}, \quad (5)$$

$$X_t = E^*\{e^{-r(T-t)} f_T(S_T) | S_t\}, \quad (6)$$

$$\gamma_t = \left.\frac{X_t(s)}{\partial s}\right|_{s=S_t}, \quad \beta_t = \frac{(X_t - \gamma_t S_t)}{B_t}, \quad (7)$$

where $E^*\{\cdot\}$ – risk-neutral measure averaging.

Statement 1 (Shiryaev, 1999, Shiryaev et al., 2006). Let us that risk-neutral (martingale) measure $\mathbf{P}^* = \mathbf{P}^{\mu-r+\delta}$ is associated with source measure $\mathbf{P}$ by transformation which looks like

$$d\mathbf{P}_t^{\mu-r+\delta} = Z_t^{\mu-r+\delta} d\mathbf{P}_t, \quad (8)$$

where

$$Z_t^{\mu-r+\delta} = \exp\left\{-\frac{\mu-r+\delta}{\sigma} W_t - \frac{1}{2}\left(\frac{\mu-r+\delta}{\sigma}\right)^2 t\right\}. \quad (9)$$

Then stochastic properties of the process defined by equation

$$dS_t = S_0(\mu dt + \sigma dW_t), \quad (10)$$

with regard to measure $\mathbf{P}^{\mu-r+\delta}$ are coinciding with properties of the process $S(r,\delta)$ defined by equation

$$dS_t(r,\delta) = S_0((r-\delta)dt + \sigma dW_t), \quad (11)$$

with respect to the measure $\mathbf{P}$, where

$$W_t^{\mu-r+\delta} = W_t^* = W_t + \frac{(\mu - r + \delta)}{\sigma} t \quad (12)$$

is Wiener process with respect to measure $\mathbf{P}^{\mu-r+\delta} = \mathbf{P}^*$.

4. MAIN RESULTS

Theorem 1. Let us consider the function below (13)–(16)

$$y_1(T,S_0) = \left[\ln(K_1/S_0) - \left(r - \delta - (\sigma^2/2)\right)T\right]/\sigma\sqrt{T}, \quad (13)$$

$$y_2(T,S_0) = \left[\ln((K_1 - K_2)/S_0) - \left(r - \delta - (\sigma^2/2)\right)T\right]/\sigma\sqrt{T}, \quad (14)$$

$$y_3(T,S_0) = \left[\ln(H/S_0) - \left(r - \delta - (\sigma^2/2)\right)T\right]/\sigma\sqrt{T}, \quad (15)$$

$$\bar{y}_k(T,S_0) = y_k(T,S_0) - \sigma\sqrt{T}, \quad k = 1;2;3. \quad (16)$$

Then the value of the barrier European put option with payoff function (1) when dividends on risk asset are paid is defined as (17)

$$\begin{aligned} P_T &= K_1 e^{-rT}\left[\Phi(y_1(T,S_0)) - \Phi(y_2(T,S_0))\right] - S_0 e^{-\delta T} \\ &\cdot\left[\Phi(\bar{y}_1(T,S_0)) - \Phi(\bar{y}_2(T,S_0)) + \Phi(\bar{y}_3(T,S_0))\right] + K_2 e^{-rT} \quad (17) \\ &\cdot\left[\Phi(y_2(T,S_0)) - \Phi(y_3(T,S_0))\right] + He^{-rT}\,\Phi(y_3(T,S_0)). \end{aligned}$$

Proof. Accordance to (1), (5) and using changes of variables $z = x/\sqrt{T}$, $y = z + ((\mu - r + \delta)/\sigma)\sqrt{T}$ we obtain

$$\begin{aligned} P_T &= e^{-rT} E^*\left\{\min\{(K_1 - S_T)^+, K_2\} I[S_T > H] + (H - S_T)\right. \\ &\times I[S_T \le H]\Big\} = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-(y^2/2)\} \min\Big\{\big(K_1 - S_0 \exp\{(r - \delta \\ &- (\sigma^2/2))T + y\sigma\sqrt{T}\}\big)^+, K_2\Big\} I\big[S_0 \exp\{(r - \delta - (\sigma^2/2))T \\ &+ y\sigma\sqrt{T}\} > H\big] + \big(H - S_0 \exp\{(r - \delta - (\sigma^2/2))T + y\sigma\sqrt{T}\}\big) \\ &\times I\big[S_0 \exp\{(r - \delta - (\sigma^2/2))T + y\sigma\sqrt{T}\} \le H\big] dy. \end{aligned}$$

Obviously that (13)–(15) are roots of equations below

$$S_0 \exp\{(r - \delta - (\sigma^2/2))T + \sigma\sqrt{T}\,y\} = K_1,$$

$$S_0 \exp\{(r - \delta - (\sigma^2/2))T + \sigma\sqrt{T}\,y\} = K_1 - K_2,$$

$$S_0 \exp\{(r - \delta - (\sigma^2/2))T + \sigma\sqrt{T}\,y\} = H.$$

So we get

$$\begin{aligned} P_T &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{y_3(T,S_0)}^{+\infty} \exp\left\{-\frac{y^2}{2}\right\} \min\left\{\left(K_1 - S_0 \exp\left\{\left(r - \delta\right.\right.\right.\right. \\ &\left.\left.\left.\left.- \frac{\sigma^2}{2}\right)T + y\sigma\sqrt{T}\right\}\right)^+, K_2\right\} dy + \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{y_3(T,S_0)} \exp\left\{-\frac{y^2}{2}\right\} \quad (18) \\ &\times \left(H - S_0 \exp\{(r - \delta - (\sigma^2/2))T + y\sigma\sqrt{T}\}\right) dy = P_T^1 + P_T^2. \end{aligned}$$

Summands P_T^1 and P_T^2 are defined by the formulas

$$\begin{aligned} P_T^1 = K_1 e^{-rT}\left[\Phi(y_1(T,S_0)) - \Phi(y_2(T,S_0))\right] \\ - S_0 e^{-\delta T}\left[\Phi(\bar{y}_1(T,S_0)) - \Phi(\bar{y}_2(T,S_0))\right] + \\ + K_2 e^{-rT}\left[\Phi(y_2(T,S_0)) - \Phi(y_3(T,S_0))\right], \end{aligned} \tag{19}$$

$$P_T^2 = He^{-rT}\Phi(y_3(T,S_0)) - S_0 e^{-\delta T}\Phi(\bar{y}_3(T,S_0)). \tag{20}$$

Then, (17) holds if we substitute (19), (20) into (18).

Theorem 2. For the barrier put option with payoff function (1) the current values of the minimal hedging portfolio $\pi_t = (\beta_t, \gamma_t)$ and the accordance investment portfolio X_t are described by (21)–(23)

$$\begin{aligned} \gamma_t = -e^{-\delta(T-t)}\left[\Phi(\bar{y}_1(T-t,S_t)) - \Phi(\bar{y}_2(T-t,S_t))\right. \\ \left. + \Phi(\bar{y}_3(T-t,S_t))\right] + \frac{K_2 e^{-r(T-t)}}{S_t\sigma\sqrt{T-t}}\varphi(y_3(T-t,S_t)), \end{aligned} \tag{21}$$

$$\begin{aligned} \beta_t = (K_1/B_T)\left[\Phi(\bar{y}_1(T-t,S_t)) - \Phi(\bar{y}_2(T-t,S_t))\right] \\ + (K_2/B_T)\left[\Phi(y_2(T-t,S_t)) - \Phi(y_3(T-t,S_t))\right] \\ + \frac{H}{B_T}\Phi(y_3(T-t,S_t)) - \frac{K_2}{B_T\sigma\sqrt{T-t}}\varphi(y_3(T-t,S_t)), \end{aligned} \tag{22}$$

$$\begin{aligned} X_t = K_1 e^{-r(T-t)}\left[\Phi(y_1(T-t,S_t)) - \Phi(y_2(T-t,S_t))\right] \\ - S_t e^{-\delta(T-t)}\left[\Phi(\bar{y}_1(T-t,S_t)) - \Phi(\bar{y}_2(T-t,S_t))\right. \\ \left. + \Phi(\bar{y}_3(T-t,S_t))\right] + K_2 e^{-r(T-t)}\left[\Phi(y_2(T-t,S_t))\right. \\ \left. - \Phi(y_3(T-t,S_t))\right] + He^{-r(T-t)}\Phi(y_3(T-t,S_t)), \end{aligned} \tag{23}$$

where $y_k(T-t,S_t)$, $\bar{y}_k(T-t,S_t)$, $k = 1;2;3$, are defined by formulas (13)–(16) with substitutions $T \to (T-t)$, $S_0 \to S_t$.

Proof. In accordance to (5), (6) formula (23) arises from (17) with replacements $T \to (T-t)$, $S_0 \to S_t$.

According to (7), (23), we obtain

$$\begin{aligned} \gamma_t = -e^{-\delta(T-t)}\left[\Phi(\bar{y}_1(T-t,S_t)) - \Phi(\bar{y}_2(T-t,S_t))\right. \\ \left. + \Phi(\bar{y}_3(T-t,S_t))\right] + K_1 e^{-r(T-t)}\left[\left.\frac{\partial\Phi(y_1(T-t,S_t))}{\partial s}\right|_{s=S_t}\right. \\ \left. - \left.\frac{\partial\Phi(y_2(T-t,S_t))}{\partial s}\right|_{s=S_t}\right] - S_t e^{-\delta(T-t)}\left[\left.\frac{\partial\Phi(\bar{y}_1(T-t,S_t))}{\partial s}\right|_{s=S_t}\right. \\ \left. - \left.\frac{\partial\Phi(\bar{y}_2(T-t,S_t))}{\partial s}\right|_{s=S_t} + \left.\frac{\partial\Phi(\bar{y}_3(T-t,S_t))}{\partial s}\right|_{s=S_t}\right] \\ + K_2 e^{-r(T-t)}\left[\left.\frac{\partial\Phi(y_2(T-t,S_t))}{\partial s}\right|_{s=S_t} - \left.\frac{\partial\Phi(\bar{y}_3(T-t,S_t))}{\partial s}\right|_{s=S_t}\right] \\ + He^{-r(T-t)}\partial\Phi(y_3(T-t,S_t))/\partial s\big|_{s=S_t}. \end{aligned} \tag{24}$$

In consideration of form of functions (13)–(16), we have the expressions

$$\begin{aligned} \left.\frac{\partial\Phi(y_k(T-t,S_t))}{\partial s}\right|_{s=S_t} = -\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(y_k(T-t,S_t))^2}{2}\right\} \\ \times\left(1/S_t\sigma\sqrt{T-t}\right) = -\left(\varphi(y_k(T-t,S_t))/S_t\sigma\sqrt{T-t}\right),\ k = 1;2;3, \end{aligned}$$

$$\left.\frac{\partial\Phi(\bar{y}_1(T-t,S_t))}{\partial s}\right|_{s=S_t} = -\frac{\varphi(y_1(T-t,S_t))}{S_t\sigma\sqrt{T-t}}\cdot\frac{K_1}{S_t}\cdot e^{-(r-\delta)(T-t)},$$

$$\left.\frac{\partial\Phi(\bar{y}_2(T-t,S_t))}{\partial s}\right|_{s=S_t} = -\frac{\varphi(y_2(T-t,S_t))}{S_t\sigma\sqrt{T-t}}\cdot\frac{K_1-K_2}{S_t}\cdot e^{-(r-\delta)(T-t)},$$

$$\left.\frac{\partial\Phi(\bar{y}_3(T-t,S_t))}{\partial s}\right|_{s=S_t} = -\frac{\varphi(y_3(T-t,S_t))}{S_t\sigma\sqrt{T-t}}\cdot\frac{H}{S_t}\cdot e^{-(r-\delta)(T-t)},$$

using of which in (24) brings us to (21). Formula (22) arises from (7), (21) and (23).

5. DECISION PROPERTIES

Statement 2. Sensivity coefficients that determine the dependences of the barrier put option value with payoff function (1) on the stock initial cost $P_T^{S_0} = \partial P_T/\partial S_0$; on the strike price $P_T^{K_1} = \partial P_T/\partial K_1$; on the contracted constant restricted payment of the option writer, on the one hand, and guaranteed income for option $P_T^{K_2} = \partial P_T/\partial K_2$; on the barrier $P_T^H = \partial P_T/\partial H$ are defined like this

$$\begin{aligned} P_T^{S_0} = -e^{-\delta T}\left[\Phi(\bar{y}_1(T,S_0)) - \Phi(\bar{y}_2(T,S_0))\right. \\ \left. + \Phi(\bar{y}_3(T,S_0))\right] + \left(K_2 e^{-rT}/S_0\sigma\sqrt{T}\right)\varphi(y_3(T,S_0)), \end{aligned} \tag{25}$$

$$P_T^{K_1} = e^{-rT}\left[\Phi(y_1(T,S_0)) - \Phi(y_2(T,S_0))\right], \tag{26}$$

$$P_T^{K_2} = e^{-rT}\left[\Phi(y_2(T,S_0)) - \Phi(y_3(T,S_0))\right], \tag{27}$$

$$P_T^H = e^{-rT}\Phi(y_3(T,S_0)) - \frac{e^{-rT}\varphi(y_3(T,S_0))}{\sigma\sqrt{T}}\cdot\frac{K_2}{H}. \tag{28}$$

The format of (25)–(28) follows from the definition of $P_T^{S_0}$, $P_T^{K_1}$, $P_T^{K_2}$, P_T^H with (17).

Statement 3. The sensivity coefficients (25)–(28) of barrier put option value with the option payoff function (1) satisfy the inequalities (29)

$$P_T^{S_0} \wedge 0, \quad P_T^{K_1} > 0, \quad P_T^{K_2} > 0, \quad P_T^H \wedge 0. \tag{29}$$

Remark 1. According to (1), (2)

$$\lim_{H\to 0} f_T(S_T) = f_T^{base}(S_T) = \min\left\{(K_1 - S_T)^+, K_2\right\}$$

Statement 4. When $H \to 0$, we have $P_T \to P_T^{base}$, $X_t \to X_t^{base}$, $\gamma_t \to \gamma_t^{base}$, $\beta_t \to \beta_t^{base}$, where P_T^{base}, X_t^{base}, $\pi_t^{base} = \left(\beta_t^{base}, \gamma_t^{base}\right)$, are value, capital and investment portfolio of the exotic put option with payoff function (2) that defined in (Andreeva, 2010).

Remark 2. Statement 4 and (17), expressions for the exotic put option value from (Andreeva, 2010) make possible to compare options prices and obtain that $P_T \le P_T^{base}$.

6. CONCLUSIONS

According to (29) analytically obtained properties $P_T^{K_1} > 0$, $P_T^{K_2} > 0$ are corroborated graphically (Fig. 1, 2) and can be interpreted with (1) as follows: strike price K_1 increment leads to probability that K_1 ranks over S_T increase. Thus, payment size under exercising increases (if $S_T > H$) and derivative cost increases too. When $S_T > H$ the more size of the K_2 the more payment size for option emitter respectively. Option buyer risk decreases, and for less risk should pay more.

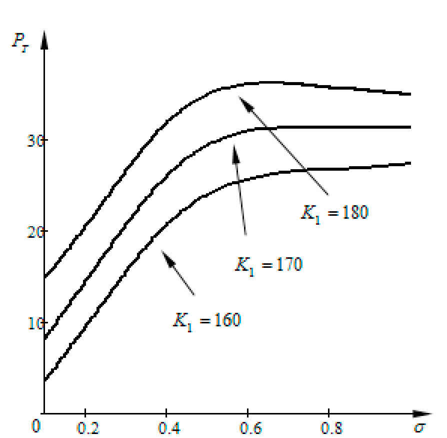

Fig. 1. Value of option with payoff function (1) with various K_1 and fixed K_2, S_0, H.

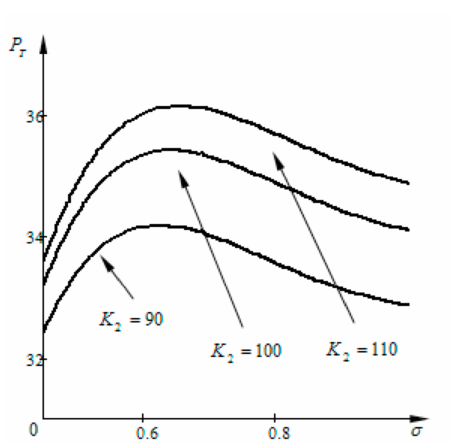

Fig. 2. Value of option with payoff function (1) with various K_2 and fixed K_1, S_0, H.

It is not succeed to establish analytically derivative value dependence on stock initial cost S_0 and on barrier H. However graphical solution (Fig. 3, 4) shows that $P_T^{S_0} < 0$, $P_T^H < 0$ and these properties can be explained as follows: at the average spot price increment is expected when value S_0 is more. Probability that S_T ranks over exercise price K_1 increases. In this case, option buyer risk increases, and for this risk should pay less. The more size of the barrier H the less probability that S_T ranks over H. Consequently probability that $(H - S_T)$ will be paid increases. As $(H - S_T)$ is less than $f_T^{base}(S_T)$, barrier option price is decreasing function of barrier H.

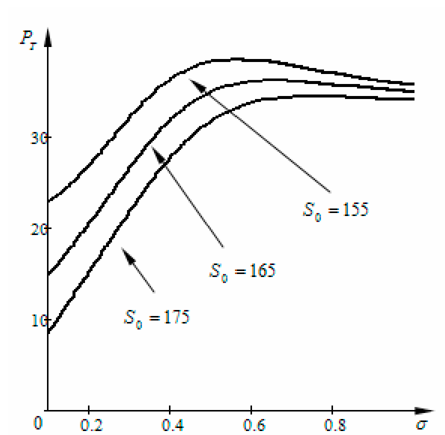

Fig. 3. Value of option with payoff function (1) with various S_0 and fixed K_1, K_2, H.

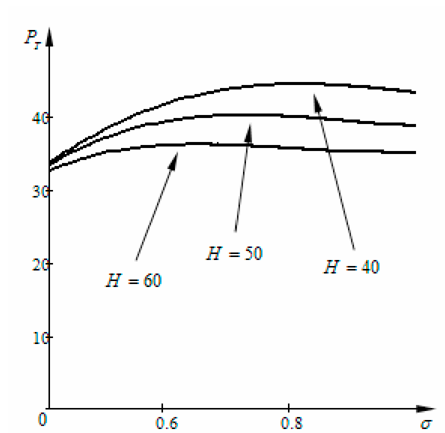

Fig. 4. Value of option with payoff function (1) with various H and fixed K_1, K_2, S_0.

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