Heisenberg-Clifford superalgebra and instantons

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Instanton moduli spaces. Let $\mathcal{M}_0^{reg}(r,n)$ be the moduli space of framed instantons on $\mathbb{R}^4$ of rank r and instanton charge n (taken modulo gauge transformations fixing the framing). It is a smooth complex affine variety of complex dimension $2rn$. It is not compact. A "partial compactification" is obtained by adding the so-called *ideal instantons*. An ideal instanton may regarded as a collection of m points x_i in $\mathbb{R}^4$ (with $0 < m \leq k$) and a framed instanton (∇, ϕ) on $\mathbb{R}^4 - \{x_1, \dots, x_m\}$ of instanton charge $k - m$, such that the measure associated with the curvature of the ASD connection ∇ approaches the Dirac delta concentrated at x_i when $x \to x_i$. In this way one gets a moduli space $\mathcal{M}_0(r,n)$ which is singular at the points corresponding to the ideal instantons. Resolving the singularities one obtains a space $\mathcal{M}(r,n)$ which is a quasi-projective smooth variety, and may be regarded as a moduli space parametrizing geometric objects, namely, torsion-free coherent sheaves $\mathcal{E}$ on the complex projective plane $\mathbb{P}^2$, which are locally free in a neighbourhood of a fixed line $\ell_\infty \subset \mathbb{P}^2$, and are equipped with an isomorphism $\Phi \colon \mathcal{E}_{|\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$.

Heisenberg-Clifford superalgebra. We want to define operators $q_i[u]$, where i is an integer, and u is a homology class (with compact support) in $\mathbb{R}^4$. This will act as a linear map

$$q_i[u] : H_\bullet(\mathcal{M}(r,n), \mathbb{Q}) \to H_\bullet(\mathcal{M}(r,n+i), \mathbb{Q})$$

(the integer r will be kept fixed during the whole treatment). When $i \geq 0$ we consider the cartesian product $\mathcal{M}(r,n) \times \mathcal{M}(r,n+i) \times X$ with projections

$$X \xleftarrow{p_1} \mathcal{M}(r,n) \times \mathcal{M}(r,n+1) \times X \xrightarrow{p_2} \mathcal{M}(r,n) \times \mathcal{M}(r,n+i)\,.$$

We define the closed subscheme $\mathcal{M}_r^{[n,i]}$ of $\mathcal{M}(r,n) \times \mathcal{M}(r,n+i) \times X$ whose elements are the triples $(\mathcal{E}, \mathcal{E}', x)$ such that the sheaves $\mathcal{E}$, $\mathcal{E}'$ fit into an exact sequence $0 \to \mathcal{E} \to \mathcal{E}' \to A_x \to 0$, where A_x is a skyscraper sheaf concentrated at the point x. We define $\omega(u) \in H_\bullet(\mathcal{M}(r,n) \times M(r,n+i), \mathbb{Q})$ by letting $\omega(u) = p_{2*}\big(p_1^* u \cap [\mathcal{M}_r^{[n,i]}]\big)$, and define the linear map $q_i[u]$ by letting

$$q_i[u](\alpha) = \pi_{2*}\big(\pi_1^*\alpha \cap \omega(u)\big)$$

where π_1, π_2 are the projections of $\mathcal{M}(r,n) \times \mathcal{M}(r,n+i)$ onto the factors. When i is negative, the operator $q_i[u]$ is defined by replacing the product $\mathcal{M}(r,n) \times \mathcal{M}(r,n+i)$ with the product $\mathcal{M}(r,n+i) \times \mathcal{M}(r,n)$ and proceeding as above.

We define among these operators the graded commutator

$$\big[q_i[u], q_j[v]\big] = q_i[u] \circ q_j[v] - (-1)^{\deg(u)\cdot\deg(v)} q_j[v] \circ q_i[u]\,.$$

Theorem. *The operators $q_i[u]$ verify the commutation relations*

$$\big[q_i[u], q_j[v]\big] = (-1)^{ri-1} ri\ \langle u, v\rangle\ \delta_{i+j,0} \cdot Id\,, \tag{1}$$

where $\langle u, v\rangle$ is the intersection product of the homology classes u, v.

This result has been proved in the case $r = 1$ by Nakajima [6] and Grojnowski [5].

The case $r = 1$. In this case the moduli space $\mathcal{M}(1,n)$ reduces to the Hilbert scheme $\mathcal{H}_n = (\mathbb{C}^2)^{[n]}$ parametrizing 0-dimensional subschemes of length n of the space $\mathbb{C}^2$. The commutation relations (1) may be proved as follows. Define the closed subschemes $\mathcal{M}_n$ and $\mathcal{M}_n(p)$ of $\mathcal{H}_n$ (where p is a point in $\mathbb{C}^2$) as follows:

$$\mathcal{M}_n = \{Z \in \mathcal{H}_n /\ Z \text{ is topologically supported at one point }\},$$

$$\mathcal{M}_n(p) = \{Z \in \mathcal{H}_n /\ Z \text{ is topologically supported at } p\}.$$

Briançon has shown [2] that $\mathcal{M}_n$ and $\mathcal{M}_n(p)$ are irreducible projective varieties, with $\dim(\mathcal{M}_n) = n+1$ and $\dim \mathcal{M}_n(p) = n-1$. Moreover, Ellingsrud and Strømme [4] have computed the intersection product of these subschemes of $\mathcal{H}_n$, obtaining

$$[\mathcal{M}_n] \cap [\mathcal{M}_n(p)] = (-1)^{n-1} n\,.$$

This computes the constants in the commutation relations (1), since $q_n[\mathrm{pt}]\mathbb{I} = [\mathcal{M}_n(p)]$ and $q_n[X]\mathbb{I} = [\mathcal{M}_n]$, where $\mathbb{I}$ is the generator of $H_\bullet(\emptyset)$, and X is the fundamental class in the homology of $\mathbb{R}^4$ with compact support.

The constants may also be computed by noting that

$$[\mathcal{M}_n] \cap [\mathcal{M}_n(p)] = s_{n-1}(\mathcal{M}_n(p))$$

where $s_{n-1}(\mathcal{M}_n(p))$ is the top Segre class of the scheme $\mathcal{M}_n(p)$ [7].

The instanton case. For $r > 1$ one introduces the subschemes of $\mathcal{M}(r,n)$

$$\mathrm{Quot}(r,n) = \{\mathcal{O}_X^{\oplus r} \to A \to 0\}, \qquad \mathrm{Quot}_p(r,n) = \{\mathcal{O}_X^{\oplus r} \to A_p \to 0\},$$

where A is a rank zero sheaf whose topological support is a point, and A_p is a rank zero sheaf whose topological support is a fixed point p. The sets $\mathrm{Quot}(r,n)$ and $\mathrm{Quot}_p(r,n)$ are irreducible projective varieties of dimension $rn+1$ and $rn-1$, respectively [1, 3]. Again the intersection product $[\mathrm{Quot}(r,n)] \cdot \mathrm{Quot}_p[(r,n)]$ computes the constants in the commutation relations (1). Moreover, also in this case the one has the identification

$$[\mathrm{Quot}(r,n)] \cdot \mathrm{Quot}_p[(r,n)] = s_{2n-1}(\mathrm{Quot}_p(r,n))$$

The idea is to compute this Segre classe using a Bott formula for the equivariant cohomology of the moduli space $\mathcal{M}(r,n)$ with respect to a naturally defined action of $\mathbb{C}^*$ [8].

References

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