On additive bases in infinite abelian semigroups

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Abstract. In this paper, building on previous work by Lambert, Plagne and the third author, we study various aspects of the behavior of additive bases in a class of infinite abelian semigroups, which we term *translatable* semigroups. These include all numerical semigroups as well as all infinite abelian groups. We show that, for every such semigroup T, the number of essential subsets of any additive basis is finite, and also that the number $E_T(h,k)$ of essential subsets of cardinality k contained in an additive basis of order at most h can be bounded in terms of h and k alone. These results extend the reach of two theorems, one due to Deschamps and Farhi and the other to Hegarty, bearing upon $\mathbf{N}$. Also, using invariant means, we address a classical problem, initiated by Erdős and Graham and then generalized by Nash and Nathanson both in the case of $\mathbf{N}$, of estimating the maximal order $X_T(h,k)$ that a basis of cocardinality k contained in an additive basis of order at most h can have. Among other results, we prove that, whenever T is a translatable semigroup, $X_T(h,k)$ is $O(h^{2k+1})$ for every integer $k \geqslant 1$. This result is new even in the case where $k=1$ and T is an infinite abelian group. Besides the maximal order $X_T(h,k)$, the typical order $S_T(h,k)$ is also studied.

1. Introduction

Let $(T,+)$ be an abelian semigroup. If A,B are two subsets of T whose symmetric difference is finite, we write $A \sim B$. Also if $A \setminus B$ is finite, we write $A \subsetneq B$. Further, the Minkowski sum of A and B is defined as $\{a+b : (a,b) \in A \times B\}$ and denoted by $A+B$. For every integer $h \geqslant 1$, the Minkowski sum of h copies of A is denoted by hA. All semigroups in this paper are assumed to be *cancellative*, that is, if $a,b,c \in T$ and $a+c=b+c$, then $a=b$. It is well known that an abelian semigroup is cancellative if and only if it can be embedded in an abelian group (see Section 2.1).

A subset A of T is called an *additive basis* of T, or just a *basis* of T for brevity, whenever there exists an integer $h \geqslant 1$ for which all but finitely many elements of T can be represented as the sum of h not necessarily distinct elements of A. In other words, A is a basis of T if and only if $hA \sim T$ for some $h \geqslant 1$. The smallest possible integer $h \geqslant 1$ in the definition above is then denoted by $\mathrm{ord}^*_T(A)$ and is called the *order* of A over T. If A is not a basis of T, then we set $\mathrm{ord}^*_T(A) = \infty$.

The study of additive bases already has a rich history, especially in the special case where T is the semigroup $\mathbf{N}$ of nonnegative integers, but also for more general abelian semigroups. The interested reader is referred to [2, 14, 15, 17, 20, 22] for recent results in this field and [11, 16, 21] for instructive surveys on the subject. Some of the most natural and widely open

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problems in the area happen to deal with the "stability" of this notion. The present paper addresses classical questions of that nature, that we now proceed to describe in more detail.

1.1. **Essential subsets and the function $E_T(h,k)$.** Let A be an additive basis of T such that $\mathrm{ord}^*_T(A) \leqslant h$. What can be said about those subsets $F \subset A$ such that $A \setminus F$ is no longer an additive basis of T? Such an $F \subset A$ is called an *exceptional* subset of A and, from a set-theoretic point of view, it is readily seen that the set of exceptional subsets of A is an *upset* of $\mathcal{P}(A)$, in the sense that any subset of A containing an exceptional subset of A is exceptional itself. This last observation motivates the following definition. An exceptional subset which is minimal with respect to inclusion will be called an *essentiality* of A. A finite essentiality is called an *essential subset*. This notion was introduced by Deschamps and Farhi to tackle the aforementioned question and, in the special case where $T = \mathbf{N}$, they showed that the number of essential subsets in any given basis must be finite [7, Théorème 10].

Now observe that for general infinite abelian semigroups, this finiteness result fails dramatically. Indeed, let $T = (\mathbf{N}^*, \times)$. Let $A = \{2^k : k \in \mathbf{N}\} \cup \{2j+1 : j \in \mathbf{N}\}$. Then the decomposition of any positive integer as a product of a power of 2 and an odd integer shows that A is a basis of order 2. However, every prime is essential. Indeed, $h(A \setminus \{2\})$ does not meet $\{n \in \mathbf{N} : n \equiv 2 \mod 4\}$ for any $h \geqslant 1$. If p is an odd prime, the set $h(A \setminus \{p\})$ does not meet $\{2^k p : k \in \mathbf{N}\}$.

Therefore, if we want to keep the finiteness of the set of essential subsets, we need to work in a particular class of cancellative abelian semigroups. Note that a convenient property of a basis $A \subset \mathbf{N}$ is that for any $x \in \mathbf{N}$, the translated set $x + A \in \mathbf{N}$ is still a basis of the same order. We would like this property to hold for a basis A of a semigroup T. Taking $A = T$ a basis of order 1, we therefore want $x + T$ to be a basis of order 1, that is, cofinite in T, for any $x \in T$. This leads us to introduce the following class of semigroups.

A *translatable* semigroup is an infinite cancellative abelian semigroup $(T, +)$ such that for any $x \in T$, the set $T \setminus (x+T)$ is finite; in other words, $T \sim x + T$. Every infinite abelian group is a translatable semigroup. Other examples of translatable semigroups include $\mathbf{N}$, numerical semigroups (i.e. cofinite subsemigroups of $\mathbf{N}$) and also $C \times \mathbf{N}$ for any finite abelian group C. In contrast, neither $(\mathbf{N}^d, +)$ for $d \geqslant 2$ nor $(\mathbf{N}^*, \times)$ are translatable.

Given a translatable semigroup, let G_T be the Grothendieck group of T, i.e. $G_T = T - T$ (see Section 2.1 for a more precise definition). We introduce a mild generalization of additive bases. For $A \subset G_T$, we say that A is an *additive G_T-basis* (or simply a *G_T-basis*) of T if there exists $h \geqslant 1$ such that $T \subsetneq hA$. Again the order $\mathrm{ord}^*_T(A)$ of the basis A over T is then the minimal such h. Note that any basis of T is automatically also a G_T-basis of T of the same order. We naturally extend the definition of exceptional subsets, essentialities and essential subsets to subsets of a G_T-basis of T.

Deschamps-Farhi's method is applicable only to subsemigroups of $\mathbf{Z}$. Using an original argument, we extend the Deschamps-Farhi theorem to any translatable semigroup.

Theorem 1. *Every G_T-basis of a translatable semigroup T has finitely many essential subsets.*

To put the theorem above into perspective, we recall that, as proved by Lambert, Plagne and the third author, additive bases abound in infinite abelian groups indeed, since every such group admits at least one additive basis of every possible order $h \geqslant 1$ [17, Theorem 1]. We will show that the same is true of any translatable semigroup (Proposition 12). We also prove a structure result (see Corollary 20) for additive bases A of a translatable semigroup T. Then among all cosets K of subgroups of G such that $A \setminus K$ is finite, there is a minimal element K^* with respect to inclusion. Furthermore, all essential subsets of A are contained in $A \setminus K^*$. In the case of $\mathbf{N}$, the set $A \setminus K^*$ is called the *reservoir* of A by Deschamps-Farhi [7, p.172].

Going back to the special case where $T = \mathbf{N}$, Deschamps and Farhi observed that, for every integer $h \geqslant 2$, additive bases of order at most h can have an arbitrarily large number of essential subsets. However, and as we shall see, the situation changes drastically when we restrict our attention to the number of essential subsets of cardinality k that a basis of order at most h can have. Indeed, for any infinite abelian semigroup $(T, +)$ and any integers $h, k \geqslant 1$, let us define

$$E_T(h,k) = \max_{\substack{A \subset T \\ hA \sim T}} |\{F \subset A : F \text{ is essential and } |F| = k\}|,$$

and set $E_T(h) = E_T(h, 1)$. Further, we define an analogous function where bases are replaced by G_T-bases, thus

$$E'_T(h,k) = \max_{\substack{A \subset G \\ T \precsim hA}} |\{F \subset A : F \text{ is essential and } |F| = k\}|.$$

Note that $E_T(h,k) \leqslant E'_T(h,k)$ by definition. The function $E_{\mathbf{N}}(h)$ was introduced and first studied by Grekos [9] who proved that $E_{\mathbf{N}}(h) \leqslant h - 1$, which was later refined in [8]. Thanks to Plagne [22, Théorème], it is now known that

$$E_{\mathbf{N}}(h) \underset{h \to \infty}{\sim} 2\sqrt{\frac{h}{\log h}}. \tag{1}$$

For their part, Deschamps and Farhi asked if, for any integers $h, k \geqslant 1$, the function $E_{\mathbf{N}}(h,k)$ could be bounded in terms of h and k alone [7, Problème 1]. This was later confirmed by Hegarty [14, Theorem 2.2], who went on and obtained several asymptotic results such as

$$E_{\mathbf{N}}(h,k) \sim (h-1)\frac{\log k}{\log \log k} \tag{2}$$

for any fixed $h \geqslant 1$ as k tends to infinity, and

$$E_{\mathbf{N}}(h,k) \asymp_k \left(\frac{h^k}{\log h}\right)^{\frac{1}{k+1}} \tag{3}$$

for any fixed $k \geqslant 1$ as h tends to infinity [15, Theorems 1.1 & 1.2]. However, it is still an open problem to know whether, for all $k \geqslant 1$, there exists a constant $c_{\mathbf{N},k} > 0$ such that $E_{\mathbf{N}}(h,k) \sim c_{\mathbf{N},k}(h^k/\log h)^{1/(k+1)}$ as h tends to infinity, which would certainly be a nice extension of Plagne's estimate (1) to values of k other than 1.

In the framework of infinite abelian semigroups, far less is known concerning the function $E_T(h,k)$. In [17, Theorem 2], Lambert, Plagne and the third author proved that $E_G(h) \leqslant h-1$ for every infinite abelian group G and every integer $h \geqslant 1$, and also that, as far as infinite abelian groups are concerned, this inequality is best possible for all $h \geqslant 1$. The proof of [17, Theorem 2] carries over just as well with any translatable semigroup. However, beyond this result, even the finiteness of $E_T(h,k)$ when $h, k \geqslant 2$ was left to be established (note that it follows easily from the definition that $E_T(1,k) = 0$). We do so in this paper, showing further that for every translatable semigroup T, the number $E_T(h,k)$ can always be bounded in terms of h and k alone.

Theorem 2. *For any translatable semigroup T and any two integers $h, k \geqslant 2$,*

$$E'_T(h,k) \leqslant (50h \log k)^k. \tag{4}$$

From a quantitative point of view, this bound certainly seems modest when compared to the already achieved estimates that we just mentioned in the special case $T = \mathbf{N}$. However, Theorem 2 applies to such a wide variety of semigroups that such a comparison is not truly relevant, as illustrated by the fact that there are infinite abelian groups for which $E_G(h,k)$ is already at least as large as $(h-1)k$ (see Remark 1).

Finally, we show that to study E_T or E_{G_T} is equivalent when T is a translatable semigroup, where G_T is the Grothendieck group of T.

Theorem 3. *Let T be a translatable semigroup and G be its Grothendieck group. Then*

$$E_T(h,k) = E'_T(h,k) = E_G(h,k).$$

1.2. **Regular subsets and the function $X_T(h,k)$.** Let A be an additive basis of T such that $\mathrm{ord}^*_T(A) \leqslant h$. What can be said about $\mathrm{ord}^*_T(A \setminus F)$ for those subsets $F \subset A$ such that $A \setminus F$ remains an additive basis of T? Such an $F \subset A$ is called a *regular* subset of A and, from a set-theoretic point of view, it is readily seen that the set of regular subsets of A is a *downset* of $\mathcal{P}(A)$, in the sense that any subset of A contained in a regular subset of A is regular itself. Of course, the above is relevant for G_T-bases and will thus be investigated in this respect also.

To tackle this problem, we define the function[1]

$$X_T(h,k) = \max_{\substack{A \subset T \\ T \underset{\sim}{\subsetneq} hA}} \{\mathrm{ord}^*_T(A \setminus F) : F \subset A, F \text{ is regular and } |F| = k\},$$

and set $X_T(h) = X_T(h,1)$. Again, we also define

$$X'_T(h,k) = \max_{\substack{A \subset G \\ T \underset{\sim}{\subsetneq} hA}} \{\mathrm{ord}^*_T(A \setminus F) : F \subset A, F \text{ is regular and } |F| = k\},$$

and $X'_T(h) = X'_T(h,1)$. Note that $X_T \leqslant X'_T$ always holds, with equality if $T = G_T$ is a group.

In other words, $X_T(h,k)$ (resp. $X'_T(h,k)$) is the maximum order of a basis (resp. G_T-basis) obtained by removing a regular subset of cardinality k from a basis (resp. G_T-basis) of order at most h.

The function $X_{\mathbf{N}}(h)$ was introduced by Erdős and Graham in [5], and the best bounds currently known are due to Plagne [20], namely

$$\left\lfloor \frac{h(h+4)}{3} \right\rfloor \leqslant X_{\mathbf{N}}(h) \leqslant \frac{h(h+1)}{2} + \left\lceil \frac{h-1}{3} \right\rceil. \tag{5}$$

It follows that $X_{\mathbf{N}}(1) = 1$, $X_{\mathbf{N}}(2) = 4$ and $X_{\mathbf{N}}(3) = 7$. However, the exact value of $X_{\mathbf{N}}(4)$ is still unknown, and a conjecture of Erdős and Graham [6] asserting that $X_{\mathbf{N}}(h) \sim d_{\mathbf{N}} h^2$ for some absolute constant $d_{\mathbf{N}} > 0$ as h tends to infinity still stands to this day.

[1] In $\mathbf{N}$, this function is also denoted by $G_k(h)$ in the literature. Our notation accommodates the fact that we will be working with an infinite abelian semigroup denoted by T, and also unifies different notations for the cases $k = 1$ and $k > 1$.

In the context of infinite abelian groups, Lambert, Plagne and the third author [17, Theorem 3] proved that, for a rather large class of infinite abelian groups G (including $\mathbf{Z}^d$, any divisible group and the group $\mathbf{Z}_p$ of p-adic integers), one has

$$X_G(h) = O_G(h^2). \tag{6}$$

However, the techniques do not carry over from these particular groups to arbitrary infinite abelian groups and, until now, it was not even known if $X_G(h)$ is finite for all infinite abelian groups G and integers $h \geqslant 1$. We now confirm that this is indeed the case.

Theorem 4. *For any translatable semigroup T and any integer $h \geqslant 1$,*

$$X_T'(h) \leqslant \frac{2h^3}{3} + O(h^2).$$

We do not know whether this bound is optimal. However, even the fact that $X_T(h)$ is finite and can be bounded independently of T is already new.

In the semigroup $\mathbf{N}$, the function $X_{\mathbf{N}}(h,k)$ was first introduced by Nathanson [19]. For fixed $k \geqslant 1$ and $h \to \infty$, Nash and Nathanson [18, Theorem 4] proved that

$$X_{\mathbf{N}}(h,k) \asymp_k h^{k+1}. \tag{7}$$

It is an open problem to know whether, for every $k \geqslant 1$, one has $X_{\mathbf{N}}(h,k) \sim d_{\mathbf{N},k} h^{k+1}$ for some constant $d_{\mathbf{N},k} > 0$ as h tends to infinity. As mentioned above, this is unknown even for $k = 1$. For a more detailed account and more precise estimates of $X_{\mathbf{N}}(h,k)$, we refer the reader to the survey [16]. We will prove the following analogue, which holds for all translatable semigroups of the theorem of Nash and Nathanson.

Theorem 5. *For any translatable semigroup T and integer $k \geqslant 1$,*

$$X_T'(h,k) \leqslant \frac{h^{2k+1}}{k!^2}\,(1 + o_k(1)) \quad \textit{as } h \to \infty.$$

In particular, $X_T'(h,k) = O_T(h^{2k+1})$ for every $k \geqslant 1$. Nash-Nathanson's proof of (7) uses Kneser's theorem[2] on the lower asymptotic density of sumsets in $\mathbf{N}$. Now such a theorem is not available in every translatable semigroup. Our main tool in proving Theorems 4 and 5 will be *invariant means*, that is, finitely-additive translation-invariant probability measures on G. Invariant means are similar in many ways to the asymptotic density, but they are defined abstractly and it is less straightforward to infer properties of a set from its probability measure. In [17, Theorem 7], invariant means were already used, but their use in the study of X_T is new. We believe that invariant means will become part of the standard toolbox to study additive problems in abelian semigroups.

Imposing specific conditions on the semigroup T allows one to control the function $X_T'(h,k)$ more finely. We found a class of abelian groups for which a bound of the shape (7) may be achieved. We say that a group G is *σ-finite* if there exists a nondecreasing sequence $(G_n)_{n\in\mathbf{N}}$ of subgroups such that $G = \bigcup_{n\geqslant 0} G_n$. Examples include $(C[x], +)$ for any finite abelian group C or $\bigcup_{n\geqslant 1} U_{d_n}$ where U_k is the group of k-th roots of unity and $(d_n)_{n\geqslant 1}$ is a sequence of integers satisfying $d_n \mid d_{n+1}$ for any $n \geqslant 1$; the latter example includes the so-called Prüfer

[2] This is not the same as Kneser's better known theorem on the cardinality of the sumset of two finite sets in an abelian group.

p-groups U_{p^∞}. Combining a result of Hamidoune and Rødseth [13] on this class of groups with the argument of Nash and Nathanson, we will prove the following bound.

Theorem 6. *Let G be an infinite σ-finite abelian group. Then $X_G(h,k) \leqslant 2\frac{h^{k+1}}{k!} + O(h^k)$.*

In [17, Theorem 5], it was shown that for infinite abelian groups G having a fixed exponent p, where p is prime, $X_G(h)$ is in fact linear in h: $2h + O_p(1) \leqslant X_G(h) \leqslant ph + O_p(1)$. We now extend this to all infinite abelian groups having a prime power as an exponent, and show the same phenomenon for $X_G(h,k)$.

Theorem 7. *Let G be an infinite abelian group of finite exponent ℓ. Then, the following two statements hold.*

(1) $X_G(h,k) \leqslant \ell^{2k}(h+1) - \ell^k + h$.
(2) If ℓ is a prime power, then $X_G(h) \leqslant \ell h + \ell^2 - \ell$.

1.3. **The "typical order" and the function $S_T(h,k)$.** Define $S_T(h)$ to be the minimum s such that for any basis A with $\mathrm{ord}^*_T(A) \leqslant h$, there are only finitely many elements $a \in A$ such that $\mathrm{ord}^*_T(A \setminus \{a\}) > s$. In particular $S_T(h) \leqslant X_T(h)$. Again we define S' analogously but with G_T-bases. Grekos [10] introduced the function $S = S_{\mathbf{N}}$ and conjectured that $S_{\mathbf{N}}(h) < X_{\mathbf{N}}(h)$. In [2], Cassaigne and Plagne proved that

$$h + 1 \leqslant S_{\mathbf{N}}(h) \leqslant 2h, \tag{8}$$

for all $h \geqslant 2$ and $S_{\mathbf{N}}(2) = 3$, settling Grekos' conjecture in view of equation (5).

In [17, Theorem 7], using invariant means, the authors showed that we also have $h + 1 \leqslant S_G(h) \leqslant 2h$ for every infinite abelian group G. It is an open problem to find the exact asymptotic of $S_{\mathbf{N}}(h)$, or $S_T(h)$ for *any* fixed translatable semigroup T. The proof of [17, Theorem 7] also gives a bound for the number of "bad" elements, that is, elements g of a basis A of order at most h such that $S_G(h) < \mathrm{ord}^*_G(A \setminus \{g\})$. The proof of [17, Theorem 7] implies that the number of such elements is at most h^2. We now give a slight generalization of this fact to translatable semigroups, while showing that in the case of groups we do have a sharper bound.

Theorem 8. *Let T be a translatable semigroup, and let $h \geqslant 2$ be an integer. If A is a G_T-basis of T of order at most h, then there are at most $h(h-1)$ elements a of A such that $\mathrm{ord}^*_T(A \setminus \{a\}) > 2h$. If T is a group then the number of such elements is at most $2(h-1)$.*

While we do not know if $2(h-1)$ is best possible, it is nearly so because certainly $E_G(h)$ is a lower bound for the maximal number of bad elements, and it was observed in [17, Theorem 2] that for the group $G = \mathbf{F}_2[t]$, one has $E_G(h) = h - 1$ for any $h \geqslant 1$.

As a generalization, define $S_T(h,k)$ to be the minimum value of s such that for any basis A with $\mathrm{ord}^*_T(A) \leqslant h$, there are only finitely many regular subsets $F \subset A, |F| = k$ with the property that $s < \mathrm{ord}^*_T(A \setminus F)$. Thus $S_T(h,1) = S_T(h)$. We have the trivial bound $S_T(h,k) \leqslant X_T(h,k)$, and it is interesting to know if this inequality is strict. We have a partial positive answer.

Theorem 9. *Let T be a translatable semigroup, and let $h \geqslant 1$ be an integer. Then*

$$S_T(h,2) \leqslant 2X_T(h). \tag{9}$$

Furthermore, if A is a G_T-basis of T of order at most h, there are at most $O\left(h^2 X_T(h)^2\right)$ regular pairs $F \subset A$ such that $\mathrm{ord}^*_T(A \setminus F) > 2X_T(h)$. If T is a group, then the number of such pairs is at most $4h(X_T(h) - 1)$.

Naturally, letting $S'_T(h, 2)$ being the corresponding function regarding G_T-bases, we could also prove using the same method that $S'_T(h, 2) \leqslant 2X'_T(h)$. We underline that at least for the semigroups $T = \mathbf{N}$ or $\mathbf{Z}$, the bound (9) is nontrivial because $X_T(h)$ is much smaller than $X_T(h, 2)$. Indeed, a construction of Nathanson and Nash [18, Theorem 4], devised for $\mathbf{N}$ but straightforwardly adaptable to $\mathbf{Z}$, shows that $X_T(h, k) \gg_k h^{k+1}$; in particular $X_T(h, 2) \gg h^3$ while $X_T(h) = O(h^2)$ by (6).

The organization of the paper is as follows. In Section 2 we introduce some tools used in our proofs, including a generalization of the Erdős-Graham criterion for finite exceptional subsets. In Sections 3, 4 and 5, we prove results on the functions $E_T(h, k)$, $X_T(h, k)$, and $S_T(h, k)$ respectively.

2. Preliminary results

2.1. **Translatable semigroups and their Grothendieck groups.** Let T be a cancellative abelian semigroup. We let G_T be the quotient of the product semigroup $T \times T$ (with coordinate-wise addition) by the equivalence relation $(a_1, a_2) \sim (b_1, b_2)$ if $a_1 + b_2 = a_2 + b_1$. It is clear that the equivalence relation is compatible with the addition, so that the quotient is again an abelian semigroup. Further the class of (x, x) is a neutral element which we denote by 0 and $(a_1, a_2) + (a_2, a_1) = 0$, so that G_T is an abelian group. Also T is embedded in G_T via the map $x \mapsto (x + t, t)$ (for any $t \in T$). This group is an explicit realization of the *Grothendieck group* of T, which may also be defined by a universal property.

By identifying $x \in T$ with $(x, 0) \in G_T$, we have $T \subset G_T$, and we observe that $G_T = T - T$. We will often omit the index and let $G = G_T$.

Recall that a translatable semigroup is an infinite cancellative abelian semigroup with the property that for any $x \in T$, the set $T \setminus (x + T)$ is finite, or equivalently $T \sim x + T$. We now list some immediate consequences of this property that we will use frequently.

Lemma 10. *Let T be a translatable semigroup, $G = G_T$, and H be a subgroup of finite index of G. Then*

1. *For any $x \in G$, we have $T \sim x + T$.*
2. *If A is a subset of G, then for any $x \in G$, we have $T \cap (x + A) \sim x + T \cap A$.*
3. *If F is a finite subset of G, then there is $t \in T$ such that $t + F \subset T$.*
4. *For any $x \in G$, $T \cap (x + H)$ is infinite.*
5. *$T \cap H$ is also a translatable semigroup. Furthermore, $H = T \cap H - T \cap H$.*
6. *If R contains a system of representatives of G/H and $S \subset G$ satisfies $T \cap H \precsim S$, then $T \precsim R + S$.*

Proof. Since $G = T - T$, we may write $x = a - b$ where $(a, b) \in T^2$. Then

$$x + T = (T + a) - b \sim T - b \sim (T + b) - b = T$$

because the relation $\sim$ is transitive.

If $A \subset G$, then

$$T \cap (x + A) = x + (T - x) \cap A \sim x + T \cap A$$

since $T \sim T - x$.

For part (3), for each $x \in F$ we write $x = a_x - b_x$ where $a_x, b_x \in T$. Thus $t = \sum_{x \in F} b_x$ satisfies $t \in T$ and $t + x \in T$ for all $x \in F$.

If H has finite index, there exists a finite set F such that $G = \bigcup_{x \in F}(x + H)$. By the pigeonhole principle, one of the sets $(x + H) \cap T$ for $x \in F$ must be infinite. Hence all of them are infinite by part (2).

For part (5), the translatability of $T \cap H$ follows from part (2), since for any $x \in T \cap H$, we have $x + T \cap H \sim T \cap (x + H) = T \cap H$. Now let x be any element of H. Then there exist $a, b \in T$ such that $x = a - b$. By part (4), there exists $c \in T$ such that $a + c \in H$. We also have $b + c \in H$. Therefore, $x = (a + c) - (b + c) \in T \cap H - T \cap H$.

For part (6), notice that we may assume that R is finite, and in this case,

$$T = \bigcup_{r \in R} (r + H) \cap T \sim \bigcup_{r \in R} (r + H \cap T) \subsetneq \bigcup_{r \in R} (r + S) = R + S.$$

□

Although we will not need it, we give here a structure theorem for translatable semigroups.

Proposition 11. *Let T be a translatable semigroup. Then either T is a group (i.e. T equals its Grothendieck group G_T), or $T \sim C \oplus x\mathbf{N}$, where C is a finite subgroup of G_T.*

Proof. Suppose that T is not a group. Let $G = G_T$ be its Grothendieck group. Since $T \neq G$, we have $T \not\subset -T$. Let $x \in T \setminus (-T)$. Then the order of x in T is necessarily infinite, since if $kx = mx$ for some $k > m$ then $-x = (k - m - 1)x \in T$, a contradiction. Therefore x generates an infinite subgroup $x\mathbf{Z}$ of G and also a subsemigroup $x\mathbf{N}^*$ (isomorphic to $\mathbf{N}^*$) of T.

Let $R = T \setminus (x + T)$, a finite set. Let $u \in T$ be arbitrary. If $u - kx \in T$ for infinitely many positive integers k, then since $T \sim u + T$, we have $u - kx \in u + T$ and $-kx \in T$ for some positive integer k. Therefore $-x = -kx + (k - 1)x \in T$, which contradicts our hypothesis on x. So we let $u' = u - kx$ where k is the maximum nonnegative integer so that $u - kx \in T$; then $u' \notin x + T$. As a result, every element of T may be uniquely decomposed as a sum of an element of R and an element of $x\mathbf{N}$, so $T = R + x\mathbf{N}$ and $G = T - T = R - R + x\mathbf{Z}$. Consequently, $x\mathbf{Z}$ has finite index in G.

By the classification theorem of finitely generated abelian groups, there exists a finite subgroup C of G such that $G = C \oplus x\mathbf{Z}$. By Lemma 10 part (4), $T \cap (c + x\mathbf{Z}) \neq \emptyset$ for any $c \in C$. On the other hand, we have $T \cap (c + x\mathbf{Z}^-) = \emptyset$. If not, then since c has finite order, we have $-\ell x \in T$ for some $\ell \in \mathbf{Z}^+$, so $-x = (-\ell x) + (\ell - 1)x \in T$, a contradiction. Thus for every $c \in C$, there exists a minimal $k \in \mathbf{N}$ such that $c + kx \in T$. We conclude that $T \sim C \oplus x\mathbf{N}$. □

We point out that this structure theorem implies that any translatable semigroup T admits a basis of any order $h \geqslant 2$.

Proposition 12. *For every translatable semigroup T and every integer $h \geqslant 2$, there exists a basis of T of order h.*

Proof. We may assume that T has a neutral element 0. Indeed, supposing that T does not have a neutral element, there exists $x \in T \setminus (-T)$; then A is a basis of $T \cup \{0\}$ if and only if $A + x \subset T$ is a basis of T, and $\mathrm{ord}^*_T(A + x) = \mathrm{ord}^*_{T\cup\{0\}}(A)$. We shall construct an infinite sequence $\Lambda = (\Lambda_i)_{i\geqslant 0}$ of subsets of T such that $\{0\} \subsetneq \Lambda_i$ for every $i \geqslant 0$ and for any $x \in T$, there exists a unique sequence $\lambda(x) = (\lambda_i(x))_{i\geqslant 0}$ of finite support such that $x = \sum_{i=0}^{\infty} \lambda_i(x)$ where $\lambda_i(x) \in \Lambda_i$. The support $\mathrm{supp}(s)$ of a sequence $s = (s_i)_{i\geqslant 0} \in T^{\mathbf{N}}$ is the set $\{j \in \mathbf{N} : s_j \neq 0\}$. As shown in [17, Theorem 1] (the arguments there do not use the group structure, only the semigroup structure), such a sequence Λ gives rise to a basis of order exactly h.

Either T is a group, in which case we can use [17, Proposition 1]; or there is a finite subset $\{0\} \subset R \subset T$ and $x \in T$ such that any $t \in T$ may be uniquely written as $t = r + kx$ for some $(r, k) \in R \times \mathbf{N}$. Let $n = \sum_{i=0}^{\infty} a_i(n)2^i$ be the unique binary decomposition of any integer $n \in \mathbf{N}$, where $a_i(n) \in \{0, 1\}$; then we set $\Lambda_i = \{0, 2^{i-1}x\}$ for any $i \geqslant 1$, and $\Lambda_0 = R$ if $R \neq \{0\}$, and $\Lambda_i = \{0, 2^i x\}$ for any $i \geqslant 0$ otherwise. The sequence Λ has then the desired property. □

2.2. **A generalization of the Erdős-Graham criterion.** In the early eighties, Erdős and Graham proved [5, Theorem 1] that if A is a basis of $\mathbf{N}$ and $a \in A$, then $A \setminus \{a\}$ is a basis of $\mathbf{N}$ if and only if $\gcd(A \setminus \{a\} - A \setminus \{a\}) = 1$. This criterion was generalized to groups in [17, Lemma 7], as we now recall. Let T be a translatable semigroup and G_T be its Grothendieck group. For $B \subset G_T$ (in particular for $B \subset T$), let $\langle B\rangle$ be the subgroup of $G = G_T$ generated by B. The criterion states that if A is a basis of G and $a \in A$, then $A \setminus \{a\}$ is a basis of G if and only if $\langle A \setminus \{a\} - A \setminus \{a\}\rangle = G$. We now generalize further this criterion to translatable semigroups and exceptional subsets instead of exceptional elements.

We first prove the following more general form of [17, Lemma 7].

Lemma 13. *Let T be a translatable semigroup and G be its Grothendieck group. Let $s, t, h \geqslant 1$. Suppose $B \subset G$ and $a \in G$ satisfy*

$$T \precsim \bigcup_{i=h-t+1}^{h} (iB + (h-i)a).$$

Suppose $(sB + a) \cap (s+1)B \neq \emptyset$ (in particular, this is the case if $sB - sB = G$). Then $T \precsim h'B$ where $h' = (t-1)s + h$.

Proof. Suppose $c \in (sB + a) \cap (s + 1)B$. Then $2c \in (2sB + 2a) \cap ((2s + 1)B + a) \cap (2s + 2)B$. Continuing in this way yields

$$(t-1)c \in \bigcap_{i=0}^{t-1} \big(((t-1)(s+1) - i)B + ia\big).$$

For all but finitely many $x \in T$, we have $x \in (t-1)c + T$, and the hypothesis implies that for all but finitely many of them,

$$x \in (t-1)c + \bigcup_{i=0}^{t-1}((i + h - t + 1)B + (t - 1 - i)a).$$

It follows that for all but finitely many $x \in T$, we have

$$x = (x - (t-1)c) + (t-1)c \in ((t-1)(s+1) + h - t + 1)B + (t-1)a = h'B + (t-1)a.$$

Since $T \sim T + (t-1)a$, this implies that $T \precsim h'B$, as desired. □

We can now state our generalization of the Erdős-Graham criterion.

Lemma 14. *Let T be a translatable semigroup of Grothendieck group G. Let A be a G-basis of T. Let F be a finite subset of A. Then $A \setminus F$ is a G-basis of T if and only if $\langle A \setminus F - A \setminus F \rangle = G$.*

In the case of $\mathbf{N}$, this was proved by Nash and Nathanson in [18, Theorem 3]. Their proof uses the fact that, in $\mathbf{N}$, any set of positive Schnirelmann density is a basis. Our argument is different from theirs.

Proof. Let $B = A \setminus F$. To prove the "only if" direction, let us suppose that $H = \langle B - B \rangle \subsetneq G$. Then, for any $\ell \geqslant 1$, the sumset ℓB lies in a coset $x + H$ for some $x \in T$. In particular, $(x + H) \cap T$ is infinite. Let $y \in G \setminus (x + H)$; by Lemma 10 part (2), we have $(y + H) \cap T \sim y - x + (x + H) \cap T$ so $(y + H) \cap T$ is an infinite subset of T that does not meet ℓB. In other words, $T \not\precsim \ell B$.

We now prove the "if" direction. First, note that there exists $s \geqslant 1$ such that $sB \cap (s+1)B \neq \emptyset$. Indeed, let $b \in B$. Since $b \in G = \langle B - B \rangle$, there exists $s \geqslant 1$ such that $b \in s(B - B)$. Therefore, there exists $(x, y) \in (sB)^2$ such that $b = y - x$. Now $y = x + b \in sB \cap (s+1)B$ yields the desired nonempty intersection. According to Lemma 13 (with $a = 0$), it now suffices to show that $T \precsim \bigcup_{i=1}^{\ell} iB$ for some $\ell \geqslant 1$. Since $\langle B - B \rangle = G$, each element $x \in F$ has a representation of the form

$$x = \sum_{i=1}^{s_x} (a_i(x) - b_i(x)), \tag{10}$$

where $s_x \in \mathbf{N}$ and $a_i(x), b_i(x) \in B$. Since A is a G-basis of T, let $h \geqslant 1$ satisfy $T \precsim hA$. All but finitely many elements $g \in T$ can be written as

$$g = \sum_{x \in F} m_x(g) x + y,$$

where $m_x(g) \geqslant 0$ and $\sum_{x \in F} m_x(g) \leqslant h$ whereas $y \in (h - \sum_x m_x(g))B$. Replacing each occurrence of $x \in F$ with (10) and translating by $g_0 = h \sum_{x \in F} \sum_{i=1}^{s_x} b_i(x) \in T$, we find that

$$g + g_0 = \sum_{x \in F} \sum_{i=1}^{s_x} (m_x(g) a_i(x) + (h - m_x(g)) b_i(x)) + y,$$

where the right-hand side is a sum of

$$h \sum_{x \in F} s_x + h - \sum_{x \in F} m_x(g)$$

elements in B. Let $\ell = h \sum_{x \in F} s_x + h$. This shows that $g_0 + T \precsim \bigcup_{i=1}^{\ell} iB$ and by translatability, we conclude. □

As pointed out by Nash and Nathanson [18], the conclusion of Lemma 14 is no longer true for the semigroups $T = \mathbf{N}$ or $T = \mathbf{Z}$ if $F \subset A$ is allowed to be infinite. For example, consider $A = \{1\} \cup \{2n : n \in T\}$, a basis of order 2 of T, and $F = \{n \in T : \forall k \geqslant 1, n \neq 6^k\}$.

More generally, let T be a translatable semigroup and $h \geqslant 2$. We invoke the construction of a basis A of order h in Proposition 12. With the notation of that construction, let $B = \bigcup_{i=0}^{\infty} \Lambda_i \subset A$ and $F = A \setminus B$. Then $\langle B - B \rangle = G_T$. However, for any $\ell \geqslant 1$, the sumset ℓB misses all elements whose support has cardinality strictly larger than ℓ, so B is not a basis. This means that in any translatable semigroup, the finiteness of F is crucial for Lemma 14.

2.3. **Characterizations of exceptional and essential subsets.** As demonstrated by Lemma 14, the subgroups $\langle A \setminus F - A \setminus F \rangle$, where F is a finite subset of a given basis A, play an important role. We now prove some preliminary results on these subgroups. The next lemma states that whenever F is a finite subset of A, the subgroup $\langle A \setminus F - A \setminus F \rangle$ cannot be too small.

Lemma 15. *Let T be a translatable semigroup of Grothendieck group G. Let A be a subset of G such that $T \subsetneq hA$ for some $h \geqslant 2$ and let F be a finite subset of A. Let $H = \langle A \setminus F - A \setminus F \rangle$. Then for any $x \in A \setminus F$, we have $(h-1)(F \cup \{x\}) + H = G$. Consequently,*

$$[G : H] \leqslant \binom{h + |F| - 1}{h - 1}.$$

Proof. By the definition of H, we have $A \setminus F \subset x + H$, so that $A \subset (x + H) \cup F$ and A meets a finite number of cosets of H. This fact and the finiteness of $T \setminus hA$ imply that the projection of T in G/H is finite. However, $T - T = G$, so G/H is finite.

Let $g \in G/H$. We may write $g = t + H$ for some $t \in T$. Now $hA \subset hF \cup \bigcup_{i=0}^{h-1}(iF + (h-i)x + H)$. Note that $hF \cup (T \setminus hA)$ is finite and $(t + H) \cap T$ is infinite by Lemma 10. This implies that $g = t' + H$ for some $t' \in \bigcup_{i=0}^{h-1}(iF + (h-i)x)$. Finally,

$$G \subset H + \bigcup_{i=0}^{h-1}(iF + (h-i)x) = H + x + (h-1)(F \cup \{x\})$$

as desired. This implies that $[G : H] \leqslant |(h-1)F'|$ where $F' = F \cup \{x\}$ has cardinality $|F| + 1$. The bound follows from counting the number of $(h-1)$-combinations of elements from F' with repetition allowed. $\square$

Lemma 15 implies that if G does not have proper subgroups of index at most $\binom{h+k-1}{h-1}$, then a basis A of order at most h cannot contain an exceptional (and in particular essential) subset F of cardinality at most k. This implies that, as far as infinite abelian semigroups are concerned, we cannot expect lower bounds for $E_T(h,k)$ other than the trivial one, that is $E_T(h,k) \geqslant 0$.

Lemma 14 gives the following characterization of essential subsets of a basis.

Corollary 16. *Let T be a translatable semigroup of Grothendieck group G and A be a G-basis of T and $E \subset A$ be a finite subset. Then E is an essential subset of A if and only if the following two statements hold.*

(1) *$H = \langle A \setminus E - A \setminus E \rangle$ is a proper subgroup of G.*
(2) *G/H is generated by $\overline{x - a}$, where x is any element of E and a is any element of $A \setminus E$.*

In particular, if E is essential then G/H is a finite cyclic group.

Proof. Lemma 14 implies that E is essential precisely when $G \neq H$, but $G = \langle (A \setminus E) \cup \{x\} - (A \setminus E) \cup \{x\} \rangle$ for any $x \in E$. The claimed characterization follows by noting that $\langle (A \setminus E) \cup \{x\} - (A \setminus E) \cup \{x\} \rangle$ is generated by $H \cup \{x - a\}$ for any $a \in A \setminus E$.

The second claim follows from the fact that G/H is finite, by Lemma 15. □

The next lemma gives a correspondence between essential subsets and proper subgroups.

Lemma 17. *Let T be a translatable semigroup of Grothendieck group G and A be a G-basis of T. Let E be an essential subset of the basis A and F be any subset of A such that $E \not\subset F$. Then $\langle A \setminus (E \cup F) - A \setminus (E \cup F) \rangle \subsetneq \langle A \setminus F - A \setminus F \rangle$.*

Proof. We have $\langle A \setminus (E \cup F) - A \setminus (E \cup F) \rangle \subset \langle A \setminus E - A \setminus E \rangle \cap \langle A \setminus F - A \setminus F \rangle$. Further, since $A \setminus (E \cap F) = (A \setminus E) \cup (A \setminus F)$, we have

$$\langle A \setminus E - A \setminus E \rangle + \langle A \setminus F - A \setminus F \rangle = \langle A \setminus (E \cap F) - A \setminus (E \cap F) \rangle.$$

Since $E \cap F \subsetneq E$, it follows from the essentiality of E and Lemma 14 that the right-hand side is $G \neq \langle A \setminus E - A \setminus E \rangle$. So $\langle A \setminus F - A \setminus F \rangle \not\subset \langle A \setminus E - A \setminus E \rangle$, which finally yields the desired result. □

2.4. **Invariant means.** Let $(T, +)$ be an abelian semigroup. Let $\ell^\infty(T)$ denote the set of all bounded functions from T to $\mathbf{R}$. An *invariant mean* on T is a linear functional $\Lambda : \ell^\infty(T) \to \mathbf{R}$ satisfying:

(M1) Λ is *nonnegative*: if $f \geqslant 0$ on T, then $\Lambda(f) \geqslant 0$,
(M2) Λ has *norm* 1: $\Lambda(1_T) = 1$ where 1_T is the characteristic function of T,
(M3) Λ is *translation-invariant*: $\Lambda(\tau_x f) = \Lambda(f)$ for any $f \in \ell^\infty(T)$ and $x \in T$, where τ_x is the translation by x: $\tau_x f(t) = f(x + t)$.

Note that by restricting Λ to indicator functions of subsets of T, we induce a function $d : \mathcal{P}(T) \to [0, 1]$, that we will usually call *density* satisfying the following three properties.

(D1) d is *finitely additive*, i.e. if $A_1, \ldots, A_n \subset T$ are disjoint, then
$$d\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n d(A_i).$$
(D2) d is *translation-invariant*, i.e. for all $A \subset T$ and $x \in T$, we have $d(x + A) = d(A)$.
(D3) d is a *probability measure*, i.e. $d(T) = 1$.

Note that the axiom (D1) implies that for any $A_1, \ldots, A_n \subset T$, we have $d(\bigcup_{i=1}^n A_i) \leqslant \sum_{i=1}^n d(A_i)$. Also, if A is finite, then $d(A) = 0$.

If there exists an invariant mean on T, then T is said to be *amenable*. It is known that all abelian semigroups are amenable (for a proof, see [3, Theorem 6.2.12]). However, even in $\mathbf{N}$, all known proofs of the existence of invariant means are nonconstructive[3], and require the axiom of choice in one way or another (e.g. the Hahn-Banach theorem or ultrafilters).

[3]Observe that popular densities such as the lower asymptotic one do not satisfy the first axiom: only an inequality is true in general.

In Sections 4 and 5, we will use the existence of invariant means as a blackbox and make crucial use of their properties to prove our results. For now, we record the following simple fact, which is an immediate extension of the so-called prehistorical lemma to invariant means.

Lemma 18. *Let T be a cancellative abelian semigroup, G be its Grothendieck group and d be a density on T. If $A, B \subset T$ and $d(A) + d(B) > 1$ then $T \subset A - B \subset G$. In particular, if T is a group then $T = A - B$.*

Proof. Let $t \in T$. By (D2), $d(A) + d(t + B) = d(A) + d(B) > 1$. By axioms (D1) and (D3), we infer that $A \cap (t + B) \neq \emptyset$. Let $a = t + b \in A \cap (t + B)$, then $t = a - b \in A - B$. ☐

We will also make use of the following observation, which says that if T is translatable, then any invariant mean on T can be extended to all of G in a trivial way.

Lemma 19. *Let T be a translatable semigroup, G be its Grothendieck group and Λ be an invariant mean on T. For $f \in \ell^\infty(G)$, define $\Lambda'(f) = \Lambda(f|_T)$, where $f|_T$ is the restriction of f on T. Then Λ' is an invariant mean on G.*

Proof. Since $G = T - T$, it suffices to verify (M3) for any $f \in \ell^\infty(G)$ and $x \in T$. We have

$$\begin{aligned} \Lambda'(\tau_x f) &= \Lambda((\tau_x f)|_T) = \Lambda(\tau_x(f|_{T-x})) \\ &= \Lambda(\tau_x(f|_T)) + \Lambda(\tau_x(f|_{(T-x)\setminus T})) \\ &= \Lambda(f|_T) + \Lambda(f|_{T\setminus(T+x)}) \\ &= \Lambda'(f) \end{aligned}$$

since $T \setminus (T + x)$ is finite and f is bounded. ☐

When T is a group, in proving Theorem 8, we will require the following additional property of d.

(D4) d is invariant with respect to inversion, i.e. $d(A) = d(-A)$ for all $A \subset T$.

This property may not be satisfied by all invariant means, but invariant means having this property abound (see for instance [4, Theorem 1]).

3. Essential subsets of an additive basis

3.1. **Finiteness of the set of essential subsets.** We first prove Theorem 1.

Proof of Theorem 1. Let T be a translatable semigroup and G be its Grothendieck group. Let also A be an additive G-basis of order $h \geqslant 1$ over T. We assume for a contradiction that the set $\mathcal{F}_A$ of all essential subsets of A is infinite. It follows that $h \geqslant 2$ and there exists an infinite sequence $(F_i)_{i\geqslant 1}$ of pairwise distinct elements of $\mathcal{F}_A$. In addition, extracting an appropriate infinite subsequence of $(F_i)_{i\geqslant 1}$ if need be, we may assume that $F_{i+1} \not\subset \bigcup_{j=1}^{i} F_j$ for all $i \geqslant 1$.

Let us set $H_i = \langle A \setminus \bigcup_{j=1}^{i} F_j - A \setminus \bigcup_{j=1}^{i} F_j \rangle$ for all $i \geqslant 1$. On the one hand, it follows from Lemma 17 that $(H_i)_{i\geqslant 1}$ is a decreasing sequence of proper subgroups of G, and from Lemma 15 that, for every $i \geqslant 1$, the quotient group $G_i = G/H_i$ is finite (in particular, H_i is infinite).

On the other hand, for every $i \geqslant 1$, there is a unique coset K_i^* of H_i such that $A \setminus K_i^*$ is finite. In particular, one has $K_j^* \subset K_i^*$ for any $j \geqslant i$.

Now, for each $i \geqslant 1$, let us define $d_i = \min\{\ell \geqslant 1 : |(\ell A) \cap (x + H_i)| = \infty,\ \forall x \in G\}$. In other words, d_i is the smallest integer $\ell \geqslant 1$ such that every coset of H_i has an infinite intersection with ℓA. Alternatively, one also has $d_i = \min\{\ell \geqslant 1 : G_i \subset K_i^* + (\ell - 1)\pi_i(A)\}$ where, for every $i \geqslant 1$, π_i denotes the canonical epimorphism from G to G_i.

It is easily noticed that by definition, the sequence $(d_i)_{i \geqslant 1}$ is nondecreasing. Also, since H_i is a proper subgroup of G and K_i^* is the only coset of H_i having an infinite intersection with A, one has $2 \leqslant d_i$ for all i. Finally, since $T \setminus hA$ is finite by assumption and each coset of H_i has an infinite intersection with T, one has $d_i \leqslant h$ for all i.

At this stage, observe that by translatability, any translation of the original additive G-basis A by an element $a \in G$ results in a new additive G-basis $A' = a + A$ of order h itself over T. The sequence $(F'_i)_{i \geqslant 1}$ obtained by translating each F_i by a then is an infinite sequence of essential subsets of A' satisfying $F'_{i+1} \not\subset \bigcup_{j=1}^{i} F'_j$ for all $i \geqslant 1$, and starting from which the previous definitions yield the very same sequences $(H_i)_{i \geqslant 1}$ and $(d_i)_{i \geqslant 1}$ as for A itself.

Our aim is to prove that, starting from any given $i \geqslant 1$, the nondecreasing sequence $(d_j)_{j \geqslant i}$ cannot be constant, which will give the desired contradiction. To do so, let us fix some $i \geqslant 1$ and let $x_i \in G$ such that $K_i^* = x_i + H_i$. Now, using the just described translation-invariance of $(H_i)_{i \geqslant 1}$ and $(d_i)_{i \geqslant 1}$, we can assume from now on that $x_i = 0$ and $K_i^* = H_i$. In particular, $d_i = \min\{\ell \geqslant 1 : G_i \subset (\ell - 1)\pi_i(A)\}$.

It follows from the minimality of $d_i \geqslant 2$ that there exists at least one coset $\overline{K_i}$ of H_i belonging to $(d_i - 1)\pi_i(A) \setminus (d_i - 2)\pi_i(A)$. Now, pick an integer $j \geqslant i$. Since we are done if $d_j > d_i$, assume that $d_j = d_i$ and let K_j be any coset of H_j such that $K_j \subset \overline{K_i}$.

Since $d_j = d_i$, one has $K_j \in K_j^* + (d_j - 1)\pi_j(A) = K_j^* + (d_i - 1)\pi_j(A)$. Let $K_{\ell_2}, \dots, K_{\ell_{d_i}}$ be any $d_i - 1$ elements of $\pi_j(A)$ such that in G_j, one has

$$K_j = K_j^* + K_{\ell_2} + \cdots + K_{\ell_{d_i}}.$$

For every $j \geqslant i$, let $f_j^i : G_j \to G_i$ be the group homomorphism sending every coset K_j of H_j to the unique coset K_i of H_i such that $K_j \subset K_i$. Note also that by definition, one has $f_j^i \circ \pi_j = \pi_i$. Since $f_j^i(K_j^*) = K_i^* = H_i$, applying f_j^i to both sides of the equality above in G_j results in the following relation in G_i,

$$\overline{K_i} = f_j^i(K_{\ell_2}) + \cdots + f_j^i(K_{\ell_{d_i}}).$$

For every $2 \leqslant k \leqslant d_i$, there exists by definition an element $a_k \in A$ such that $K_{\ell_k} = \pi_j(a_k)$. However, $a_k \in K_i^*$ would imply $f_j^i(K_{\ell_k}) = (f_j^i \circ \pi_j)(a_k) = \pi_i(a_k) = K_i^* = H_i$ and readily give $\overline{K_i} \in (d_i - 2)\pi_i(A)$, which is a contradiction. As a result, $K_{\ell_k} \in \pi_j(A \setminus K_i^*)$, for every $2 \leqslant k \leqslant d_i$. We now have all we need to complete our proof.

On the one hand, each K_{ℓ_k} can take at most $|\pi_j(A \setminus K_i^*)| \leqslant |A \setminus K_i^*|$ values, so that the number of possible sums of the form $K_{\ell_2} + \cdots + K_{\ell_{d_i}}$ in G_j is at most

$$\binom{|A \setminus K_i^*| + d_i - 1}{d_i},$$

which is independent of j.

On the other hand, there are $[H_i : H_j]$ cosets of H_j that are contained in $\overline{K_i}$, and in order for each of them to be an element of $K_j^* + (d_i - 1)\pi_j(A)$, we must have

$$[H_i : H_j] \leqslant \binom{|A \setminus K_i^*| + d_i - 1}{d_i}.$$

Since $[H_i : H_j]$ tends to infinity when j does so, the previous inequality holds only for finitely many integers $j \geqslant i$, so that at least one of them satisfies $d_j > d_i$. Since $i \geqslant 1$ was chosen arbitrarily, we obtain that $(d_i)_{i\geqslant 1}$ tends to infinity when i does so, which contradicts the fact that $d_i \leqslant h$ for all i, and proves the desired result. □

The proof of Theorem 1 gives the following structure theorem for additive bases in translatable semigroups. Corresponding to any G_T-basis A of T, we associate the family

$$\mathcal{M}(A) = \{K \subset G : K \text{ is a coset of a subgroup of } G \text{ and } A \setminus K \text{ is finite}\}.$$

Then we have

Corollary 20. *Let A be a G_T-basis of the translatable semigroup T. Then $\mathcal{M}(A)$ is finite and admits a minimal element K^* with respect to inclusion. Furthermore, all essential subsets of A are contained in $A \setminus K^*$.*

Proof. We first claim that if $K_1, K_2 \in \mathcal{M}(A)$, then $K_1 \cap K_2 \in \mathcal{M}(A)$ as well. Suppose K_1 and K_2 are cosets of subgroups $H_1 = K_1 - K_1$ and $H_2 = K_2 - K_2$ of G. Since $A \setminus K_1$ and $A \setminus K_2$ are finite, it follows that $A \setminus (K_1 \cap K_2)$ is finite. In particular, $K_1 \cap K_2 \neq \emptyset$. Let $g \in K_1 \cap K_2$ be an arbitrary element. Then $K_1 = g + H_1$ and $K_2 = g + H_2$. Therefore, $K_1 \cap K_2 = g + H_1 \cap H_2$ is a coset of the subgroup $H_1 \cap H_2$.

If $K \in \mathcal{M}(A)$, then by applying Lemma 15 to $F = A \setminus K$, we see that the subgroup $\langle A \cap K - A \cap K\rangle$ has finite index in G. This in turn implies that the subgroup $K - K$ has finite index. The proof of Theorem 1 shows that $\mathcal{M}(A)$ does not contain an infinite decreasing sequence with respect to inclusion. Since $\mathcal{M}(A)$ is closed under intersection, it has a minimal element K^*, and $K^* \subset K$ for all $K \in \mathcal{M}(A)$. Since $K^* - K^*$ has finite index, there are only finitely many subgroups of G containing $K^* - K^*$, and for any such subgroup, there is only one coset containing K^*. Thus $\mathcal{M}(A)$ is finite.

Let E be any essential subset of A, then $\langle A \setminus E - A \setminus E\rangle$ is a proper subgroup of G. Let a be an arbitrary element of $(A \setminus E) \cap K^*$. Then the coset $K_E = \langle A \setminus E - A \setminus E\rangle + a$ is in $\mathcal{M}(A)$ since $A \setminus K_E \subset E$ is finite. Therefore, $K^* \subset K_E$. Suppose for a contradiction that $E \not\subset A \setminus K^*$. Let x be any element of $E \setminus (A \setminus K^*) = E \cap K^*$. By Corollary 16, G is generated by $\langle A \setminus E - A \setminus E\rangle \cup \{x - a\}$. But $x \in K^*$ and $a \in K^*$, so $x - a \in K^* - K^* \subset K_E - K_E$. This implies that $\langle A \setminus E - A \setminus E\rangle = G$, a contradiction. □

3.2. **Bounding the number of essential subsets.** We now prove Theorem 2, that is, a bound for the number of essential subsets of cardinality k.

Proof of Theorem 2. Let A be a G_T-basis of order at most h over T, and let $k \geqslant 1$ be an integer. It readily follows from Theorem 1 that the set $\mathcal{F}$ of essential subsets of cardinality k of A is finite. Our aim is to bound $N = |\mathcal{F}|$ in terms of h and k alone.

We take a minimal sequence $F_1, \dots, F_n$ of elements of $\mathcal{F}$ with the property that $\bigcup_{i\leqslant n} F_i = \bigcup_{F\in\mathcal{F}} F$. By minimality, note that $F_i \not\subset \bigcup_{j<i} F_j$ for any i. Also, $|\bigcup_{F\in\mathcal{F}} F| \leqslant nk$. Using the elementary bound $\binom{a}{b} \leqslant (\frac{ea}{b})^b$, we have

$$N \leqslant \binom{nk}{k} \leqslant (en)^k \tag{11}$$

so it suffices to bound n in terms of h and k.

Let $H_i = \langle A \setminus \bigcup_{j=1}^{i} F_j - A \setminus \bigcup_{j=1}^{i} F_j \rangle$. By Lemma 17, one has

$$H_n \subsetneq H_{n-1} \subsetneq \cdots \subsetneq H_1 \subsetneq G.$$

Therefore,

$$[G : H_n] \geqslant 2^n. \tag{12}$$

On the other hand, it follows from Lemma 15 that

$$[G : H_n] \leqslant \binom{h+nk-1}{h-1} \tag{13}$$

since $|\bigcup_{j=1}^{n} F_j| \leqslant kn$. Combining (12) and (13), one has $2^n \leqslant \binom{h+nk-1}{h-1}$. Using the same bound as above for binomial coefficients, we have

$$2^n \leqslant \left(\frac{e(h+nk)}{h}\right)^h \leqslant \left(\frac{2enk}{h}\right)^h$$

(we may assume that $n \geqslant h$; otherwise the bound (11) is already stronger than the desired bound). This implies that $\frac{n}{h} \leqslant 2\log(6\frac{nk}{h})$. A quick analysis of the real function $x \mapsto x - 2\log(6xk)$ reveals that this inequality may only be satisfied if $n \leqslant 15h\log k$. The bound (11) implies that $N \leqslant (50h\log k)^k$, as desired. $\square$

Remark 1. Although our bound above may not be optimal, Hegarty's asymptotic (8) is too much to hope for general translatable semigroups. Let $G = \mathbf{F}_p[t]$ (where p is prime) and $G_r = \{f \in G : \deg f < r\}$. Fix $h \geqslant 2$ and consider

$$A = G_r \cup t^r G_r \cup \cdots \cup t^{r(h-2)} G_r \cup t^{r(h-1)} G.$$

This is a basis of order h of G. To see that, consider for any element $f = \sum_{i=0}^{\infty} f_i t^i \in G$ the decomposition

$$f = \sum_{j=0}^{h-2} \sum_{i=jr}^{(j+1)r-1} f_i t^i + \sum_{i\geqslant(h-1)r} f_i t^i.$$

The complement in A of a hyperplane of the subspace $t^i G_r$ is essential as may be seen by Lemma 14. Such sets have cardinality $k = p^r - p^{r-1}$, and there are $\frac{p^r-1}{p-1}p \geqslant k$ of them. This implies that for infinitely many k, there exists a group G such that $E_G(h,k) \geqslant (h-1)k$ for any h.

3.3. Comparing E_T and E_{G_T}.

We conclude this section with a comparison of the functions E_T and E_{G_T}. We first need the following generalization of Lemma 14.

Lemma 21. *Let T be a translatable semigroup of Grothendieck group G. Let A be a G-basis of T and $F \subset A$ be any finite subset. Put $B = A \setminus F$ and $H = \langle B - B \rangle$. Let b be an arbitrary element of B. Then $T \cap H$ is a translatable semigroup of Grothendieck group H and $B - b$ is an H-basis of $T \cap H$.*

Clearly, Lemma 14 is a special case of Lemma 21 when $H = G$. In $\mathbf{N}$, Lemma 21 was proved by Nash-Nathanson [18, Theorem 1]. Again, Nathanson-Nash's proof is very specific to $\mathbf{N}$ (it uses Schnirelmann density and Schnirelmann's theorem). Our proof is different from theirs and works for any translatable semigroup. In fact, we use Lemma 14 to prove Lemma 21, while Nathanson and Nash proceeded the other way round.

Proof. The fact that $T \cap H$ is a translatable semigroup of Grothendieck group H is Lemma 10 part (5). For h large enough, and by translatability, we have

$$\begin{aligned} T \sim T - hb \subsetneq h(A-b) &\sim \bigcup_{i=0}^{h} (i(F-b) + (h-i)(A \setminus F - b)) \\ &\subset \bigcup_{i=0}^{h} (i(F-b) + h(A \setminus F - b)) \qquad \text{since } 0 \in B - b. \end{aligned}$$

In particular,

$$T \cap H \subsetneq \bigcup_{i=0}^{h} (i(F-b) + h(A \setminus F - b)).$$

Since F is finite, this means that there are finitely many translates $a_1 + h(B-b), \ldots, a_k + h(B-b)$ of $h(B-b)$ such that

$$T \cap H \subsetneq \bigcup_{i=1}^{k} (a_i + h(A \setminus F - b)).$$

A priori $a_1, \ldots, a_k \in G$. But a translate $a_i + h(B-b)$ can have nonempty intersection with H only if $a_i \in H$. Thus we may assume that $a_1, \ldots, a_k \in H$. Let $A' = h(A \setminus F - b) \cup \{a_1, \ldots, a_k\} \subset H$, then the equation above shows that $T \cap H \subsetneq 2A'$. Clearly $\langle hB - hB \rangle = H$. We now invoke Lemma 14 with the set A' and the translatable semigroup $T \cap H$ (whose Grothendieck group is H, by Lemma 10 part (5)), and conclude that for some $k \geqslant 1$, $T \cap H \subsetneq kh(A \setminus F - b)$, as desired. □

Next we need the following lemma of independent interest, which is reminiscent of Hegarty's reduction [15] of the study of $E_{\mathbf{N}}(h,k)$ to the postage stamp problem.

Lemma 22. *Let T be a translatable semigroup of Grothendieck group G. Let H be a subgroup of G of finite index. Let B be a subset of G satisfying $\langle B - B \rangle = H$ and b be an arbitrary element of B. Let F be a finite subset of G disjoint from B and $A = F \cup B$. Then the following are equivalent:*

(1) A is a G-basis of T.
(2) (a) $B - b$ is an H-basis of $T \cap H$, and
(b) $\langle F - b + H \rangle = G$ (i.e. $\overline{F - b}$ generates G/H).

Further, if h_1 is minimal such that $h_1((F-b) \cup \{0\}) + H = G$, $h_2 = \operatorname{ord}^*_{T \cap H}(B - b)$, and $h = \operatorname{ord}^*_T(A)$, then we have $h_1 + 1 \leqslant h \leqslant h_1 + h_2$.

Proof. If (1) holds, then (2a) follows from Lemma 21 and (2b) follows from Lemma 15.

Now suppose (2) holds. Let h_1 be minimal such that $h_1((F-b)\cup\{0\})+H=G$ and $h_2=\mathrm{ord}^*_{T\cap H}(B-b)$. If $T \subsetneq hA$, then by Lemma 15 we have $(h-1)((F-b)\cup\{0\})+H=G$ and therefore $h \geqslant h_1+1$. We will now prove that $T \subsetneq (h_1+h_2)A$. We have

$$\begin{aligned}(h_1+h_2)(A-b) &= \bigcup_{i=0}^{h_1+h_2} (i(F-b)+(h_1+h_2-i)(B-b)) \\ &\supset \bigcup_{i=0}^{h_1} (i(F-b)+h_2(B-b)) \qquad \text{since } 0\in B-b \\ &= h_2(B-b)+h_1((F-b)\cup\{0\}).\end{aligned}$$

Since $h_2(B-b)$ misses only finitely many elements of $T\cap H$ and $h_1((F-b)\cup\{0\})$ meets every coset of H, by Lemma 10 part (6) we know that $T \subsetneq (h_1+h_2)(A-b)$, and $T \subsetneq (h_1+h_2)A$ by translatability. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let us first prove that $E'_T(h,k) \leqslant E_G(h,k)$. Let A be a G-basis of T of order at most h. Our aim is to prove that A has at most $E_G(h,k)$ essential subsets of cardinality k.

By Theorem 1, we already know that A has finitely many essential subsets. Let F be the union of all essential subsets of A. From now on, and since the desired inequality readily holds true otherwise, we assume that F is nonempty. Let $B=A\setminus F$ and $H=\langle B-B\rangle$. By definition, $A=F\cup B$ and taking an arbitrary element $b\in B$, we have $B\subset H+b$.

By Lemma 15, H is a subgroup of finite index of G so that Lemma 22 applies to the partition $A=F\cup B$. It follows that, since A is a G-basis of T, the condition (2a) of Lemma 15 is satisfied, that is to say $B-b$ is a H-basis of $T\cap H$.

Also, let us prove that $F\cap(H+b)=\emptyset$. Assume to the contrary that $x\in F\cap(H+b)$. Then, there exists an essential subset E' of A such that $x\in E'$. Since $E'\subset F$, we obtain $b\in A\setminus F\subset A\setminus E'$. Letting $H_{E'}=\langle A\setminus E'-A\setminus E'\rangle$, we have $H\subset H_{E'}$, that is $H+b\subset H_{E'}+b$. By Corollary 16, G is generated by $H_{E'}\cup\{x-b\}$. Yet $x-b\in H_{E'}$ which yields $G=H_{E'}$, a contradiction.

By Lemma 15, $(h-1)(F\cup\{b\})+H=G$. Let $A'=F\cup(H+b)\subset G$. Then A' is a basis of G of order at most h. Also, $\langle A'\setminus F-A'\setminus F\rangle=H$ is a subgroup of finite index of G so that Lemma 22 applies to the partition $A'=F\cup(H+b)$. Finally, the condition (2a) of Lemma 15 is trivially satisfied in this case.

Now let $E\subset F$ be any subset. We know that $B\subset H+b$ and $E\cap(H+b)=\emptyset$. Since H is a subgroup of finite index of G, it follows that $A\setminus E=(F\setminus E)\cup B$ and $A'\setminus E=(F\setminus E)\cup(H+b)$ are two partitions to which Lemma 22 applies. Note also that the condition (2a) of that lemma has already been proved to hold in both cases. This gives

$$A\setminus E \text{ is a } G\text{-basis of } T \iff \langle F\setminus E-b+H\rangle=G \iff A'\setminus E \text{ is a basis of } G.$$

Consequently, each essential subset of A (all of which are subsets of F) is an essential subset of A'. Now A' has at most $E_G(h,k)$ essential subsets of cardinality k by definition, whence $E'_T(h,k)\leqslant E_G(h,k)$.

To prove that $E_G \leqslant E_T$, we argue similarly; thus let A be a basis of G of order at most h and let F be the union of its essential subsets. From now on, and since the desired inequality readily holds true otherwise, we assume that F is nonempty. Using Lemma 10 part (3), by translating A by some $t \in T$, and since translations preserve bases and the number of essential subsets, we may assume that $F \subset T$. By Lemma 14, the subgroup $H = \langle A \setminus F - A \setminus F \rangle$ of G is proper and of finite index, and $A = F \cup B$ where $B = A \setminus F \subset x + H$ for some $x \in G$. We may assume $x \in T$ by Lemma 10 part (4). We have again $(h-1)(F \cup \{x\}) + H = G$ by Lemma 15. Let $A' = F \cup (x + T \cap H) \subset T$. Then $hA' \supset (h-1)(F \cup \{x\}) + T \cap H \sim T$ by Lemma 10 part (6). Using Lemma 22 in the same way as before, we see that if $E \subset F$ then

$$A \setminus E \text{ is a basis of } G \iff \langle F \setminus E - x + H \rangle = G \iff A' \setminus E \text{ is a basis of } T.$$

This shows that all essential subsets of A are essential subsets of A', so A has at most $E_T(h,k)$ essential subsets of size k, and finally $E_G(h,k) \leqslant E_T(h,k)$. Together with the trivial inequality $E_T \leqslant E'_T$, this concludes the proof. □

4. The function $X_T(h,k)$

We fix a translatable semigroup T of Grothendieck group $G = G_T$ and an invariant mean Λ on T. By Lemma 19, we extend it to an invariant mean on G by letting $\Lambda(f) = \Lambda(f|_T)$ for any $f \in \ell^\infty(G)$, where $f|_T$ is the restriction of f to T. For a set $A \subset G$, we refer to $d(A) = \Lambda(1_A)$ as the "density" of A. Note that $d(T) = 1$. We first prove some lemmas on the densities of certain sumsets.

Lemma 23. *Let $B, C \subset G$. Then either $d(B+C) \geqslant 2d(C)$ or $B - B \subset C - C$.*

Proof. Suppose there are two distinct elements b, b' of B such that $b+C$ and $b'+C$ are disjoint. Then $d(B+C) \geqslant d((b+C) \cup (b'+C)) = 2d(C)$. Otherwise, for any $b \neq b'$ of B we have $(b+C) \cap (b'+C) \neq \emptyset$, which implies that $b - b' \in C - C$, so that $B - B \subset C - C$. □

We shall deduce by iteration the following corollary.

Corollary 24. *Let $A \subset G$. Let $r \geqslant 1$ be an integer. For any $i \geqslant 0$, let $s_i = 2^i r + 2^i - 1$. Then either $d(s_i A) \geqslant 2^i d(rA)$ or $i \geqslant 1$ and $s_{i-1}(A - A) = \langle A - A \rangle$.*

Proof. We argue by induction. For $i = 0$ the claim is trivial.

Fix some $i \geqslant 0$ and let us show that either $d(s_{i+1}A) \geqslant 2^{i+1} d(rA)$ or $s_i(A - A) = \langle A - A \rangle$. We apply Lemma 23, to $C = s_i A$ and $B = (s_i + 1)A$. Then $B + C = s_{i+1}A$. If $B - B \subset C - C$, we have for any $s \geqslant s_i$ the inclusion $s(A - A) \subset s_i(A - A)$. Since $\langle A - A \rangle = \bigcup_{j=1}^\infty j(A - A)$, this implies that $s_i(A - A) = \langle A - A \rangle$.

Otherwise, we must have $d(s_{i+1}A) = d(B + C) \geqslant 2d(s_i A)$. Further, note that $s_i(A - A) \neq \langle A - A \rangle$, and therefore for any $s \leqslant s_i$ we know that $s(A - A) \neq \langle A - A \rangle$. If $i = 0$ we are done. Otherwise, applying the induction hypothesis, we see that $d(s_j) \geqslant 2d(s_{j-1}A)$ for any $j \leqslant i$. By a straightforward induction, we conclude that $d(s_{i+1}A) \geqslant 2^i d(rA)$. □

We now show that if $d(hA) > 0$, then $A - A$ must be a basis of bounded order of the group it generates.

Lemma 25. *Suppose* $A \subset G$, $h \geqslant 1$ *and* $d(hA) = \alpha > 0$. *Then there exists* $s \leqslant \frac{1}{\alpha}(h+1) - 1$ *such that* $sA - sA = \langle A - A \rangle$.

Proof. We apply Corollary 24 to the set hA, the integer $r = h$ and $i = i_0$ the smallest integer such that $2^{i_0}\alpha > 1$. Since the density cannot exceed 1, we have $s_i(A - A) = \langle A - A \rangle$ where $s_i = 2^{i_0-1}h + 2^{i_0-1} - 1 \leqslant \frac{1}{\alpha}(h+1) - 1$ (since $\frac{1}{\alpha} \geqslant 2^{i_0-1}$). This yields the desired conclusion. □

We are now equipped for the proof of Theorem 4.

Proof of Theorem 4. Let $A \subset G$. Suppose $T \lesssim hA$ and $a \in A$ is a regular element. Write $B = A \setminus \{a\}$, then $\langle B - B \rangle = G$ by Lemma 14. We have

$$T \lesssim hB \cup ((h-1)B + a) \cup \cdots \cup (B + (h-1)a)$$

and consequently

$$1 \leqslant \sum_{i=1}^{h} d(iB + (h-i)a) = \sum_{i=1}^{h} d(iB).$$

Suppose $h(B - B) \neq G$. Let $r \in \mathbf{N}$ be defined by $2^r - 1 < h \leqslant 2^{r+1} - 1$. Let I_j be the interval $\left[\lfloor \frac{h-2^{j+1}+1}{2^j} \rfloor + 1, \lfloor \frac{h-2^j+1}{2^j} \rfloor\right]$. So $[1, h] \subset \bigcup_{j=0}^{r} I_j$. For $k \in I_j$, note that $h \geqslant 2^j k + 2^j - 1$, so by Corollary 24, we have $d(hB) \geqslant 2^j d(kB)$. Further, $|I_j| \leqslant \frac{h-2^j+1}{2^j} - \frac{h-2^{j+1}+1}{2^{j+1}} + 1 \leqslant \frac{h+1}{2^{j+1}} + 1$. Therefore we have

$$1 \leqslant \sum_{i=0}^{r} \left(\frac{h+1}{2^i} + 1 \right) 2^{-i} d(hB).$$

Summing the series, we infer $1 \leqslant \left(2 + \frac{2(h+1)}{3}\right) d(hB)$.

By Lemma 25, there exists $s \leqslant s_0 = \left(2 + \frac{2(h+1)}{3}\right)(h+1) - 1$ such that $sB - sB = \langle B - B \rangle = G$. Note that this is also true if $h(B - B) = G$ since $h \leqslant s_0$. By Lemma 13 (with $t = h$) we have $T \lesssim h'B$ and B is a basis of T of order at most h', where $h' = (h-1)s + h \leqslant \frac{2(h+1)^2(h-1)}{3} + 2(h^2 - 1) + 1 = \frac{1}{3}(2h^3 + 8h^2 - 2h - 5)$. □

The following lemma can be regarded as an analogue of [18, Lemma 3].

Lemma 26. *Let* $B \subset G$ *satisfy* $\langle B - B \rangle = G$. *Suppose there exist* $h, m \geqslant 1$ *and* $x_1, \ldots, x_m$ *in* T *such that* $T \lesssim \bigcup_{i=1}^{m}(x_i + hB)$. *Then* B *is a* G*-basis of* T *of order at most* $h + m^2(h+1) - m$.

Proof. The hypothesis and axioms of a density imply $d(hB) \geqslant 1/m$. By Lemma 25, we infer that there exists $s \leqslant m(h+1) - 1$ such that $sB - sB = \langle B - B \rangle = G$. Thus, for each $1 \leqslant i \leqslant m$ we may write $x_i = a_i - b_i$ where $a_i \in sB$ and $b_i \in sB$. Hence

$$T \lesssim \bigcup_{i=1}^{m}(hB + a_i - b_i).$$

By adding $\sum_{i=1}^{m} b_i$ to both sides and using translatability, we have

$$T \lesssim \bigcup_{i=1}^{m}(hB + a_i + \sum_{j \neq i} b_j)$$

which shows that all except finitely many elements of T can be expressed as a sum of $h+ms$ elements of B. Since $h+ms \leqslant h+m^2(h+1)-m$, we are done. $\square$

We may now deal with the effect of removing a regular subset from a basis.

Proof of Theorem 5. Let A be a G-basis of order at most h and $F \subset A$ be a regular subset of cardinality k. Let $B = A \setminus F$. Since F is regular, by Lemma 14, we have $\langle B - B\rangle = G$. We observe that

$$T \precsim hB \cup ((h-1)B + F) \cup \cdots \cup (B + (h-1)F). \tag{14}$$

Let $b \in B$. Since $iB \subset hB - (h-i)b$, we have $iB + hb \subset hB + ib$ and by translatability

$$T \precsim (hB + hb) \cup (hB + F + (h-1)b) \cup \cdots \cup (hB + (h-1)F + b).$$

Therefore we may apply Lemma 26 with

$$m = \left|\bigcup_{j=1}^{h-1}(jF + (h-j)b)\right| \leqslant \left|\left\{(t_1,\ldots,t_k) \in \mathbf{N}^k : \sum_{i=1}^{k} t_i \leqslant h-1\right\}\right| = \binom{h+k-1}{k}. \tag{15}$$

We infer that B is a G-basis of order at most

$$(h+1)\binom{h+k-1}{k}^2 - \binom{h+k-1}{k} + h = \frac{h^{2k+1}}{k!^2}(1+o_k(1)),$$

which is the desired result. $\square$

Remark 2. In the case $k=1$, the previous proof gives the bound $X_G(h) \leqslant (h+1)h^2$, which is slightly weaker than the bound we already obtained in Theorem 4.

In the case of σ-finite groups, we can do better.

Proof of Theorem 6. Let T be a σ-finite infinite abelian group. Let $(G_n)_{n\geqslant 0}$ be a nondecreasing sequence of subgroups such that $T = \bigcup_{n\geqslant 0} G_n$. For $C \subset T$, let $\overline{\mathrm{d}}(C) = \limsup_{n\to\infty} \frac{|C\cap G_n|}{|G_n|}$ be its upper asymptotic density. Let A be a basis of G of order at most $h \geqslant 2$. Let F be a regular subset of A of cardinality k and $B = A \setminus F$. Note that $\langle B\rangle \supset \langle B - B\rangle = T$ by Lemma 14. By equation (14), we have $\overline{\mathrm{d}}(hB + \bigcup_{j=0}^{h-1} jF) = \overline{\mathrm{d}}(T) = 1$. Note that for any two subsets X, Y of T, for any $\epsilon > 0$, we have

$$\frac{|(X\cup Y)\cap G_n|}{|G_n|} \leqslant \frac{|X\cap G_n| + |Y \cap G_n|}{|G_n|} \leqslant \overline{\mathrm{d}}(X) + \overline{\mathrm{d}}(Y) + \epsilon.$$

Taking the upper limit, we find that $\overline{\mathrm{d}}(X\cup Y) \leqslant \overline{\mathrm{d}}(X) + \overline{\mathrm{d}}(Y) + \epsilon$. Finally, letting ϵ tend to 0, we see that $\overline{\mathrm{d}}(X\cup Y) \leqslant \overline{\mathrm{d}}(X) + \overline{\mathrm{d}}(Y)$.

Because of the translation-invariance of the density, the just obtained inequality and equation (15), we infer that $\overline{\mathrm{d}}(hB)\binom{h+k-1}{k} \geqslant 1$. We are now in position to apply [13, Theorem 1], which yields that hB is a basis of $\langle hB\rangle = G$ of order at most $1 + 2/\overline{\mathrm{d}}(hB) \leqslant 1 + 2\binom{h+k-1}{k} = \frac{h^k}{k!} + O(h^{k-1})$. Therefore, B itself is a basis of order at most $h\,\mathrm{ord}^*_G(hB) \leqslant 2\frac{h^{k+1}}{k!} + O(h^k)$. $\square$

Remark 3. Instead of appealing to [13, Theorem 1], we could have used Kneser's theorem for the lower asymptotic density [1] and the fact that any set A of lower asymptotic density larger than $1/2$ satisfies $A+A \sim G$ and argued like Nash and Nathanson in the integers.

Note that a Kneser-type theorem is available in any countable abelian group G for the upper *Banach* density [12]. However, that density has the drawback that a set $A \subset G$ satisfying $\mathrm{d}^*(A) > 1/2$, even $\mathrm{d}^*(A) = 1$, may not be a basis of any order of the group it generates. For instance, take $B = \bigcup_{i \geqslant 1} [2^i, 2^i + i) \subset \mathbf{Z}$ and $A = A \cup (-A)$; it generates $\mathbf{Z}$ but is far too sparse to be a basis of $\mathbf{Z}$, of any order. Yet its upper Banach density is 1.

We conclude the section with the case of infinite abelian groups of finite exponent.

Proof of Theorem 7. Let G be an infinite abelian group of exponent ℓ. For part (1), we proceed identically to the proof of Theorem 5 with the group G in the place of T. The difference is that, since G has exponent ℓ,

$$m = \left| \bigcup_{j=1}^{h-1} jF \right| \leqslant \left| \left\{ (t_1, \ldots, t_k) \in \mathbf{N}^k : t_i \leqslant \ell - 1, \sum_{i=1}^{k} t_i \leqslant h-1 \right\} \right| \leqslant \ell^k.$$

Thus by Lemma 26, B is a basis of order at most $(h+1)\ell^{2k} - \ell^k + h$ as desired.

As for part (2), we will generalize the argument in [17, Theorem 5]. Suppose $F = \{a\}$. By translating A by $-a$ if necessary, we may assume that $a = 0$. Since G has exponent ℓ, we have $sB \subset (s+\ell)B$ for any s. Therefore,

$$G \sim \bigcup_{i=1}^{h} iB \sim \bigcup_{i=h-\ell+1}^{h} iB. \tag{16}$$

For any $x \in G$, since B is infinite, $(x - B) \cap \bigcup_{i=h-\ell+1}^{h} iB$ is nonempty and therefore

$$G = \bigcup_{i=h-\ell+2}^{h+1} iB.$$

We now claim that there are u, v such that $h+2 \leqslant u < u+v \leqslant h+\ell+1$, $uB \cap (u+v)B \neq \emptyset$ and $\gcd(v, \ell) = 1$. Suppose for a contradiction that this is not true. Then we have disjoint unions

$$G = \bigcup_{i \in I_1 + \ell} iB \sqcup \bigcup_{i \in I_2 + \ell} iB$$

and

$$G = \bigcup_{i \in I_1} iB \sqcup \bigcup_{i \in I_2} iB,$$

where

$$I_1 := \{ j \in [h-\ell+2, h+1] : p \mid j \}$$

and

$$I_2 := \{ j \in [h-\ell+2, h+1] : p \nmid j \},$$

where p is the unique prime divisor of ℓ. It follows that $\bigcup_{i \in I_1} iB = \bigcup_{i \in I_1 + \ell} iB$. By repeatedly adding ℓB to both sides, we have $\bigcup_{i \in I_1} iB = \bigcup_{i \in I_1 + s\ell} iB$ for any $s \geqslant 1$. For s sufficiently large, this implies that $\bigcup_{i\in I_1} iB = G$ (since we already know that B is a basis). This is a contradiction and the claim is proved.

We now proceed similarly to the proof of Lemma 13. If $c \in uB \cap (u+v)B$, then

$$(\ell-1)c \in (\ell-1)uB \cap ((\ell-1)u+v)B \cap \cdots \cap (\ell-1)(u+v)B.$$

Let $y_i = (\ell-1)u+iv$. For each $i \in [0, \ell-1]$, there exists $x_i \in [(\ell-1)(u+v-1), (\ell-1)(u+v)]$ satisfying $x_i \equiv y_i \mod \ell$ and $x_i > y_i$. Further, since $\gcd(v,\ell)=1$, we have $\{x_0, \ldots, x_{\ell-1}\} = [(\ell-1)(u+v-1), (\ell-1)(u+v)]$. Therefore,

$$(\ell-1)c \in \bigcap_{i=(\ell-1)(u+v-1)}^{(\ell-1)(u+v)} iB. \tag{17}$$

For all but finitely many $x \in G$, from (16) and (17), we have

$$x = (x-(\ell-1)c) + (\ell-1)c \in ((\ell-1)(u+v)+h-\ell+1)\, B.$$

Therefore, B is a basis of order at most $(\ell-1)(u+v)+h-\ell+1 \leqslant (\ell-1)(h+\ell+1)+h-\ell+1 = h\ell+\ell^2-\ell$. ☐

Remark 4. What we need about ℓ in the proof is that whenever $\gcd(a,\ell)=1$ and $\gcd(b,\ell)\neq 1$, then $\gcd(a-b,\ell)=1$. Obviously, prime powers are the only integers having this property.

Knowing the exact asymptotic of $X_G(h)$ or more generally $X_G(h,k)$ for any specific group G is already interesting. For instance, if G has exponent 2, then $X_G(h) \sim 2h$ as $h \to \infty$ ([17, Theorem 5]) and these are the only groups for which the exact asymptotic of $X_G(h)$ is known so far.

5. The function $S_T(h,k)$

Again, in this section we fix a translatable semigroup T and an invariant mean Λ on T. Recall that Λ extends to an invariant mean on $G = G_T$ by setting $\Lambda(f) = \Lambda(f|_T)$ for all $f \in \ell^\infty(G)$, where $f|_T$ is the restriction of f to T. For $A \subset G$, we write $d(A) = \Lambda(1_A)$.

We first prove the following observation already used in [17, Section 6].

Lemma 27. *Suppose $A \subset G$, $a \in A$ satisfy $T \subsetneq hA$ and $d(T \setminus h(A\setminus\{a\})) < \frac{1}{h}$. Then $T \subsetneq 2h(A\setminus\{a\})$.*

Proof. Let a_0 be an element in $A\setminus\{a\}$. Let $B = T\setminus h(A\setminus\{a\})$, then $d(B) < 1/h$. Since d is translation-invariant, we have

$$\sum_{i=0}^{h-1} d(B+(h-i)a+ia_0) < 1$$

and consequently there are infinitely many $x \in T$ such that $x+h(a+a_0) \notin B+(h-i)a+ia_0$ for all $i = 0,1,\ldots,h-1$. In other words, $x+ia+(h-i)a_0 \in h(A\setminus\{a\})$ and $x+i(a-a_0) \in h(A\setminus\{a\}-a_0)$ for all $i=0,1,\ldots,h-1$.

Now for all but finitely many $t \in T$, we have $t - x \in h(A - a_0)$ and $t - x \neq h(a - a_0)$. If i is the number of occurrences of $a - a_0$ in some representation of $t - x$ as a sum of h elements of $A - a_0$, then $0 \leqslant i \leqslant h - 1$ and $t - x - i(a - a_0) \in (h - i)(A \setminus \{a\} - a_0)$. Thus

$$t = (t - x - i(a - a_0)) + (x + i(a - a_0)) \in (2h - i)(A \setminus \{a\} - a_0) \subset 2h(A \setminus \{a\} - a_0),$$

and the lemma is proved. □

Proof of Theorem 8. We first strengthen slightly an observation already used in [17, Section 6].

Claim 1. For any finite subset $I \subset A$, for all but finitely many $x \in T$, there are at most $h-1$ elements $a \in I$ such that $x \in T \setminus h(A \setminus \{a\})$.

Since $T \setminus hA$ is finite, we may assume $x \in hA$. Fix a representation

$$x = a_1 + \cdots + a_h,$$

where $a_i \in A$ for $i = 1, \ldots, h$. If $x \in T \setminus h(A \setminus \{a\})$, then a must be one of $a_1, \ldots, a_h$. This already implies that there are at most h elements $a \in I$ such that $x \in T \setminus h(A \setminus \{a\})$. Furthermore, if $x \in T \setminus h(A \setminus \{a\})$ for h elements $a \in I$, then necessarily $x \in hI$. Since hI is finite, this proves the claim.

Let I be an arbitrary finite subset of A. Let $f(x) = \sum_{a \in I} 1_{T \setminus h(A \setminus \{a\})}(x)$. Then for all but finitely many x, we have $f(x) \leqslant h - 1$. By evaluating $\Lambda(f)$ and the fact that finite sets have density 0, we have the following

Claim 2. For any finite set $I \subset A$, we have $\sum_{a \in I} d\big(T \setminus h(A \setminus \{a\})\big) \leqslant h - 1$.

Suppose now T is a group. We may assume that Λ satisfies the property (A4) in Section 2.4. We have

Claim 3. If $B \subset T$ and $d(B) > 1/2$, then $2B = T$.

This immediately follows from Lemma 18.

Let J be the set of all $a \in A$ such that $\mathrm{ord}^*_T(A \setminus \{a\}) > 2h$. For all $a \in J$, we have $d(h(A \setminus \{a\})) \leqslant 1/2$ (if not, we will have $2h(A \setminus \{a\}) = T$) and therefore $d\big(T \setminus h(A \setminus \{a\})\big) \geqslant 1/2$. Since $\sum_{a \in I} d\big(T \setminus h(A \setminus \{a\})\big) \leqslant h - 1$ for any finite subset I of J, this shows that J is finite and $|J| \leqslant 2(h-1)$, and the second part of Theorem 8 is proved.

For general translatable semigroups, we use Lemma 27 instead of Claim 3. For all $a \in J$, we have $d(h(A \setminus \{a\})) \leqslant 1/h$ and therefore $|J| \leqslant h(h-1)$. □

We now generalize these ideas to prove Theorem 9.

Proof of Theorem 9. Suppose $A \subset T$ and $hA \sim T$ (the proof for G_T-bases is almost identical, with only notational differences). Let R be the set of all regular pairs $\{a, b\} \subset A$ such that $\mathrm{ord}^*_T(A \setminus \{a, b\}) > 2X_T(h)$. Also, let U be the set of all regular elements $a \in A$ such that $\mathrm{ord}^*_T(A \setminus \{a\}) > S_T(h)$. By Theorem 8 we know that $|U| = O(h^2)$.

Claim 1. For all but finitely many $x \in T$, there are at most $hX_T(h)(X_T(h)-1)$ pairs $F \in R$ such that $x \in T \setminus h(A \setminus F)$. If T is a group then the number of such pairs is at most $2h(X_T(h)-1)$.

Since $T \setminus hA$ is finite, we may assume $x \in hA$. Fix a representation

$$x = a_1 + \cdots + a_h,$$

where $a_i \in A$ for $i = 1, \ldots, h$. If $x \in T \setminus h(A \setminus F)$, then $a_i \in F$ for some i. Let $F = \{a_i, b\}$, then b is a regular element of the basis $A \setminus \{a_i\}$ (note that a_i has to be regular in the first place). By the definition of $X_T(h)$, we have

$$\mathrm{ord}_T^*(A \setminus \{a_i, b\}) > 2X_T(h) \geqslant 2\mathrm{ord}_T^*(A \setminus \{a_i\}).$$

By Theorem 8, there are at most $X_T(h)(X_T(h)-1)$ choices for b, and this number can be replaced by $2(X_T(h)-1)$ if T is a group. Thus Claim 1 is proved.

Let I be a finite subset of R. Let $f(x) = \sum_{F\in I} 1_{T\setminus h(A\setminus F)}(x)$. Again evaluating $\Lambda(f)$ yields the following bound.

Claim 2. For any finite subset $I \subset R$, we have

$$\sum_{F\in I} d\big(T \setminus h(A \setminus F)\big) \leqslant \begin{cases} hX_T(h)(X_T(h)-1) & \text{for any } T, \\ 2h(X_T(h)-1) & \text{when } T \text{ is a group.} \end{cases}$$

We are now able to conclude the proof when T is a group. For all $F \in R$, we have $d\big(T \setminus h(A \setminus F)\big) \geqslant 1/2$. If not, we will have $\mathrm{ord}_T^*(h(A \setminus F)) \leqslant 2$ and $\mathrm{ord}_T^*(A \setminus F) \leqslant 2h \leqslant 2X_T(h)$, which contradicts the definition of R. This implies that R is finite and furthermore, $|R| \leqslant 4h(X_G(h)-1)$.

If T is an arbitrary translatable semigroup, then we apply Lemma 27 to the basis $A \setminus \{a\}$ and get

Claim 3. If $F = \{a, b\}$ is regular, $\mathrm{ord}_T^*(A \setminus \{a\}) = k$, $d(T \setminus k(A \setminus F)) < \frac{1}{k}$, then $\mathrm{ord}_T^*(A \setminus F) \leqslant 2k$. Consequently, if $F \in R$ and $a \notin U$, then $d(T \setminus h(A \setminus F)) \geqslant d(T \setminus 2h(A \setminus F)) \geqslant \frac{1}{2h}$.

From Claims 2 and 3, the number of pairs $F \in R$, at least one of whose elements is not in U, is at most $hX_T(h)(X_T(h)-1) \cdot 2h = O(h^2 X_T(h)^2)$. Clearly the number of pairs $F \in R$, both of whose elements are in U, is $O(h^4)$. This concludes the proof of Theorem 9. $\square$

We point out that the argument used in the proof of Theorem 9 may be applied to bound $S_T(h,k)$ for $k \geqslant 3$, but it seems to yield bounds which are worse than trivial.

Theorem 8 prompts the following question.

Question. If $T \subsetneq hA$, then we know that there are at most $h-1$ elements $a \in A$ such that $\mathrm{ord}_T^*(A \setminus \{a\}) = \infty$, and these are characterized by the Erdős-Graham criterion, i.e. $\langle A \setminus \{a\} - A \setminus \{a\} \rangle \neq G$. Can one find a nice algebraic characterization for elements a for which $\mathrm{ord}_T^*(A \setminus \{a\}) > 2h$?

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