APPROXIMATE HEDGING PROBLEM WITH TRANSACTION COSTS IN STOCHASTIC VOLATILITY MARKETS

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This paper studies the problem of option replication in general stochastic volatility markets with transaction costs, using a new specification for the volatility adjustment in Leland's algorithm. We prove several limit theorems for the normalized replication error of Leland's strategy, as well as that of the strategy suggested by Lépinette. The asymptotic results obtained not only generalize the existing results, but also enable us to fix the underhedging property pointed out by Kabanov and Safarian. We also discuss possible methods to improve the convergence rate and to reduce the option price inclusive of transaction costs.

KEY WORDS: Leland strategy, transaction costs, stochastic volatility, quantile hedging, approximate hedging, high-frequency markets.

1. INTRODUCTION

Leland (1985) suggests a simple method for pricing standard European options in markets with proportional transaction costs. He argues that transaction costs can be accounted for in the option price by increasing the volatility parameter in the classical Black–Scholes model (Black and Scholes 1973). Leland then claims, without giving a mathematically rigorous proof, that the replicating portfolio of the corresponding discrete delta strategy converges to the option payoff as the number of revisions n goes to infinity, if the transaction cost rate is a constant independent of n, or decreases to zero at the rate $n^{-1/2}$. The latter statement is proved by Lott (1993) in his PhD thesis. In fact, this property still holds if the transaction cost coefficient converges to zero at any power rate (Kabanov and Safarian 1997).

However, a careful analysis shows that the replicating portfolio does not converge to the option payoff when the cost rate is a constant independent of n. Kabanov and Safarian (1997) find an explicit limit for the hedging error, which is negative, showing that the

replication problem is not completely solved in Leland's framework. Pergamenshchikov (2003) obtains a weak convergence for the normalized hedging error and points out that for the case of constant transaction cost, the rate of convergence in Kabanov–Safarian's result is $n^{1/4}$. This limit theorem is of practical importance because it provides the asymptotic distribution of the hedging error. Note that the rate of convergence can be improved using nonuniform revisions (Darses and Lépinette 2014; Lépinette 2012). In these papers, the authors suggest a modification to Leland's strategy to solve the discrepancy identified by Kabanov and Safarian. A short review on recent results is given in Section 2. For detailed discussions, we refer the reader to Kabanov and Safarian (1997, 2009), Granditz and Schachinger (2001), Pergamenshchikov (2003), Lépinette (2008), Lépinette and Kabanov (2010), Gamys and Kabanov (2009), Darses and Lépinette (2014), and Lépinette (2012).

In this study, we examine the problem of approximate hedging of European style options in stochastic volatility (SV) markets with constant transaction costs (the reader is referred to, e.g., Fouque, Papanicolaou, and Sircar 2000 and the references therein for motivations and detailed discussions related to SV models). In particular, we establish a weak convergence for the normalized hedging error of Leland's strategy using a simple volatility adjustment in a general SV setting. The results obtained not only generalize the existing results, but also provide a method for improving the rate of convergence. Furthermore, it turns out that superhedging can be attained by controlling a model parameter.

Let us emphasize that the classic form for *adjusted volatility* proposed in Leland (1985) and applied in Kabanov and Safarian (1997, 2009) and Lépinette (2008, 2009, 2012) may not be applicable in SV models. The reason is that option pricing and hedging are intrinsically different in SV markets than in the classical Black–Scholes framework. In particular, the option price now depends on future realizations of the volatility process. In general, this information may not be statistically available for all investors. To treat this issue, we suggest a new specification for adjusted volatility in Leland's algorithm. Although we employ an artificially modified volatility, simpler than the well-known version used in the previous literature, the same asymptotic results are obtained for SV contexts. In addition, the rate of convergence can be improved by controlling a model parameter. Note that, in the above-mentioned papers, approximation procedures are mainly based on moment estimates. This essential technique no longer works in general SV models, unless some intrinsic conditions are imposed on the model parameters (Andersen and Piterbarge 2007; Lions and Musiela 2007). It is useful to remember that our goal is to establish a weak convergence for the normalized replicating error which only requires convergence in probability of the approximation terms. Thus, in the approximation procedure, the integrability issue can be avoided in order to keep our model setting as general as possible.

As discussed in Pergamenshchikov (2003), the option price (inclusive of transaction costs) in Leland's algorithm may be high (it, in fact, approaches the buy-and-hold price), even for small values of the revision number. Another practical advantage of our method is that the option price can be reduced as long as the option seller is willing to take a risk in option replication. This approach is inspired by the theory of quantile hedging (Föllmer and Leukart 1999).

The remainder of the paper is organized as follows. In Section 2, we give a brief review of Leland's approach. Section 3 is devoted to formulating the problem and presenting our main results. Section 4 presents some direct applications to pricing and hedging. Section 5 discusses common SV models that fulfill our condition on the volatility

function. A numerical result for Hull–White's model is also provided for illustration. Section 6 connects our results to high-frequency markets with proportional transaction costs. The proofs of our main results are reported in Section 7. Auxiliary lemmas can be found in the Appendix.

2. APPROXIMATE HEDGING WITH TRANSACTION COSTS: A REVIEW OF LELAND'S APPROACH

In a complete no-arbitrage model (i.e., there exists a unique equivalent martingale measure under which the stock price is a martingale), options can be completely replicated by a self-financing trading strategy. The option price, defined as the replication cost, is the initial capital that the investor must invest to obtain a complete hedge. In fact, the option price can be computed as the expectation of the discounted claim under the unique equivalent martingale measure. This principle plays a central role in the well-known Black–Scholes model. For simplicity, let us consider a continuous time model of a two-asset financial market on the time interval $[0, 1]$, where the bond price is equal to 1 at all times. The stock price dynamics follow the stochastic differential equation

$$(2.1) \qquad \mathrm{d}S_t = \sigma_0 S_t \mathrm{d}W_t, \quad S_0 \quad \text{given,}$$

where S_0 and σ_0 are positive constants and $(W_t)_{0 \le t \le 1}$ is a standard Wiener process. As usual, let $\mathcal{F}_t = \sigma\{W_u, \, 0 \le u \le t\}$. We recall that a financial strategy $(\beta_t, \gamma_t)_{0 \le t \le 1}$ is an *admissible self-financing strategy* if it is bounded from below, $(\mathcal{F}_t)$, adapted with $\int_0^t (|\beta_t| + \gamma_t^2)\, \mathrm{d}t < \infty$ a.s., and the portfolio value satisfies

$$V_t = \beta_t + \gamma_t S_t = V_0 + \int_0^t \gamma_u \mathrm{d}S_u, \quad t \in [0, 1].$$

The classic hedging problem is to find an admissible self-financing strategy (β_t, γ_t) whose terminal portfolio value exceeds the payoff $h(S_1) = \max(S_1 - K, 0)$, or

$$V_1 = V_0 + \int_0^1 \gamma_u \mathrm{d}S_u \ge h(S_1) \qquad \text{a.s.,}$$

where K is the strike price. The standard pricing principle shows that the option price $C(t, S_t)$ is given by the well-known formula (Black and Scholes 1973)

$$(2.2) \qquad C(t, x) = C(t, x, \sigma_0) = x\Phi(\widetilde{\mathbf{v}}(t, x)) - K\Phi(\widetilde{\mathbf{v}}(t, x) - \sigma_0\sqrt{1 - t}),$$

where

$$(2.3) \qquad \widetilde{\mathbf{v}}(t, x) = \mathbf{v}(\sigma_0^2(1 - t), x) \quad \text{and} \quad \mathbf{v}(\lambda, x) = \frac{\ln(x/K)}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{2}.$$

Here, Φ is the standard normal distribution function. In the following, we denote by φ the $\mathcal{N}(0, 1)$ density: $\varphi(z) = \Phi'(z)$. One can check directly that

$$(2.4) \qquad C_x(t, x) = \Phi(\widetilde{\mathbf{v}}(t, x)) \quad \text{and} \quad C_{xx}(t, x) = \frac{\varphi(\widetilde{\mathbf{v}}(t, x))}{x\sigma_0\sqrt{1 - t}}.$$

By assuming that continuous portfolio adjustments are possible with zero transaction costs, Black and Scholes (1973) argue that the option payoff can be dynamically replicated using the delta strategy (i.e., the partial derivative of the option price with respect to the stock price).

It is clear that the assumption of continuous portfolio revision is not realistic. More-over, continuous trading would be ruinously expensive in the case of nonzero constant proportional transaction costs because the delta strategy has infinite variation. This sim-ple intuition contradicts the argument of Black and Scholes that if trading takes places reasonably frequently, then hedging errors are relatively small. Therefore, option pricing and replication with nonzero trading costs are intrinsically different from those in the Black–Scholes setting. Note that it may be very costly to assure a given degree of accuracy in replication with transaction costs. In what follows, we show that Leland's increasing volatility principle (Leland 1985) is practically helpful in such contexts.

2.1. Constant Volatility Case

Leland's (1985) approach provides an efficient technique to deal with transaction costs. This method is simply based on the intuition that transaction costs can be accounted for in the option price as a reasonable extra fee, necessary for the option seller to cover the option return. It means that in the presence of transaction costs, the option becomes more expensive than in the classic Black–Scholes framework. This is intuitively equivalent to an increase in the volatility parameter in the Black–Scholes formula.

Let us briefly describe Leland's approach (Leland 1985; Kabanov and Safarian 1997). Suppose that for each trading activity, the investor has to pay a fee directly proportional to the trading volume, measured in dollar value. Assume that the transaction cost rate is given by the law $\kappa_* n^{-\alpha}$, where n is the number of revisions. Here, $0 \leq \alpha \leq 1/2$ and $\kappa_* > 0$ are two fixed parameters. The basic idea of Leland's method is to replace the true volatility parameter in the Black–Scholes model by $\widehat{\sigma}$, artificially modified as

$$(2.5) \qquad \widehat{\sigma}^2 = \sigma_0^2 + \varrho\, n^{1/2-\alpha} \quad \text{with} \quad \varrho = \kappa_* \sigma_0 \sqrt{8/\pi}\,.$$

In this case, the option price is given by $\widehat{C}(t, x) = C(t, x, \widehat{\sigma})$, the Black–Scholes formula. For the problem of option replication, Leland suggests the following discrete strategy, known as Leland's strategy,

$$(2.6) \qquad \gamma_t^n = \sum_{i=1}^{n} \widehat{C}_x(t_{i-1}, S_{t_{i-1}}) \mathbf{1}_{(t_{i-1}, t_i]}(t), \quad t_i = \frac{i}{n},\ i \in \{1, 2, .., n\}.$$

Here, the number of shares held in the interval $(t_{i-1}, t_i]$ is the delta strategy calculated at the left bound of this interval. Then, the replicating portfolio value takes the form

$$(2.7) \qquad V_1^n = V_0^n + \int_0^1 \gamma_u^n \mathrm{d}S_u - \kappa_* n^{-\alpha}\, J_n\,,$$

where the total trading volume is

$$J_n = \sum_{i=1}^{n} S_{t_i} |\gamma_{t_i}^n - \gamma_{t_{i-1}}^n|$$

(measured in dollar value). Recall that the option price $\widehat{C}(t, x)$ is the solution of the Black–Scholes PDE with the adjusted volatility $\widehat{\sigma}$

$$(2.8) \qquad \widehat{C}_t(t, x) + \frac{1}{2}\widehat{\sigma}^2 x^2 \widehat{C}_{xx}(t, x) = 0\,, \quad 0 \leq t < 1; \quad \widehat{C}(1, x) = h(x)\,.$$

Using Itô's formula, we can represent the tracking error, $V_1^n - h(S_1)$, as

$$(2.9) \qquad \int_0^1 \left(\gamma_t^n - \widehat{C}_x(t, S_t)\right) \mathrm{d}S_t + \frac{1}{2}(\widehat{\sigma}^2 - \sigma_0^2) \int_0^1 S_t^2 \widehat{C}_{xx}(t, S_t)\mathrm{d}t - \kappa_* n^{-\alpha} J_n \,.$$

REMARK 2.1 (Leland). The specific form (2.5) results from the following intuition: the Lebesgue integral in (2.9) is clearly well approximated by the Riemann sum of the terms $\sigma_0 S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})\Delta t$, while $S_{t_i}|\gamma_{t_i}^n - \gamma_{t_{i-1}}^n|$ can be replaced by

$$\approx \sigma_0 S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})|\Delta W_{t_i}| \approx \sigma_0 \sqrt{2/(n\pi)}\, S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}),$$

because $\mathbf{E}|\Delta W_{t_i}| = \sqrt{2/\pi}\sqrt{\Delta t} = \sqrt{2/(\pi n)}$. Hence, it is reasonable to expect that the modified volatility defined in (2.5) will give an appropriate approximation to compensate transaction costs.

Leland (1985) conjectures that the replication error converges in probability to zero as $n \to \infty$ for the case of constant proportional transaction cost (i.e., $\alpha = 0$). He also suggests, without giving a rigorous proof, that this property is also true for the case $\alpha = 1/2$. In fact, Leland's second conjecture for $\alpha = 1/2$ is correct and is proved by Lott (1993) in his PhD thesis.

THEOREM 2.2 (Leland 1985–Lott 1993). *For $\alpha = 1/2$, strategy (2.6) defines an approximately replicating portfolio as the number of revision intervals n tends to infinity*

$$\mathbf{P} - \lim_{n \to \infty} V_1^n = h(S_1).$$

This result is then extended by Ahn et al. (1998) to general diffusion models. Kabanov and Safarian (1997) observe that the Leland–Lott theorem remains true as long as the cost rate converges to zero as $n \to \infty$.

THEOREM 2.3 (Kabanov and Safarian 1997). *For any $0 < \alpha \le 1/2$, $\mathbf{P} - \lim_{n\to\infty} V_1^n = h(S_1)$.*

In Lépinette and Kabanov (2010) and Kabanov and Safarian (2009), the authors study the Leland–Lott approximation in the sense of L^2 convergence for the case $\alpha = 1/2$.[1]

THEOREM 2.4 (Lépinette and Kabanov 2010). *Let $\alpha = 1/2$. The mean-square approximation error of Leland's strategy, with ϱ defined in (2.5), satisfies the following asymptotic equality:*

$$\mathbf{E}\,(V_1^n - h(S_1))^2 = An^{-1} + o(n^{-1}) \quad as \quad n \to \infty,$$

where A is some positive function.

Theorem 2.4 suggests that the normalized replication error converges in law as $n \to \infty$.

THEOREM 2.5 (Lépinette and Kabanov 2010). *For $\alpha = 1/2$, the processes $Y^n = n^{1/2}(V^n - h(S_1))$ converge weakly in the Skorokhod space $\mathcal{D}[0, 1]$ to the distribution of the process $Y_\bullet = \int_0^\bullet B(S_t)\mathrm{d}Z_t$, where Z is an independent Wiener process.*

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[1]Seemingly, mean-square replication may not contain much useful information because gains and losses have different meaning in practice. Clearly, if $\alpha = 1/2$, the modified volatility is independent of n.

REMARK 2.6. An interesting connection between this case and the problem of hedging under proportional transaction costs in high-frequency markets is discussed in Section 6.

It is important to note that Leland's approximation in Remark 2.1 is not mathematically correct, and thus, his first conjecture is not valid for the case of constant transaction costs. In fact, as $n \to \infty$, the trading volume J_n can be approximated by the following sum (which converges in probability to $J(S_1, \varrho)$ defined in (2.11)):

$$- \sum_{i=1}^{n} \lambda_{i-1}^{-1/2} S_{t_{i-1}} \widetilde{\varphi}(\lambda_{i-1}, S_{t_{i-1}}) |\sigma_0 \varrho^{-1} Z_i + q(\lambda_{i-1}, S_{t_{i-1}})| \, \Delta \lambda_i ,$$

where $\lambda_i = \lambda_{t_i} = \widehat{\sigma}^2 (1 - t_i)$, $Z_i = \Delta W_{t_i} / \sqrt{\Delta t_i}$, and

$$(2.10) \qquad \widetilde{\varphi}(\lambda, x) = \varphi(\mathbf{v}(\lambda, x)), \quad q(\lambda, x) = \frac{\ln(x/K)}{2\lambda} - \frac{1}{4} .$$

A careful study confirms that there is a *nontrivial discrepancy* between the limit of the replicating portfolio and the payoff for the case $\alpha = 0$.

THEOREM 2.7 (Kabanov and Safarian 1997). *For* $\alpha = 0$, V_1^n *converges to* $h(S_1) + \min(S_1, K) - \kappa_* J(S_1, \varrho)$ *in probability, where*

$$(2.11) \qquad J(x, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) \, \mathbf{E} \, |\widetilde{\varrho} Z + q(\lambda, x)| \, d\lambda ,$$

with $\widetilde{\varrho} = \sigma_0 \varrho^{-1}$ *and* $Z \sim \mathcal{N}(0, 1)$ *independent of* S_1.

Underhedging: It is important to observe that the problem of option replication is not completely solved in the case of constant transaction costs. Indeed, considering that $\mathbf{E}|\widetilde{\varrho} Z| = 1/(2\kappa_*)$ and the identity

$$(2.12) \qquad x \int_0^{\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) \, d\lambda = 2 \min(x, K) ,$$

we obtain (for the parameter ϱ given in (2.5)) that $\min(x, K) - \kappa_* J(x, \varrho) = x\kappa_*$ equals $\int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) (\mathbf{E}|\widetilde{\varrho} Z| - \mathbf{E}|\widetilde{\varrho} Z + q(\lambda, x)|) d\lambda$. Now, Anderson's inequality (see, for example, Ibragimov and Hasminskii 1981, page 155) implies directly that for any $q \in \mathbb{R}$, $\mathbf{E}|\widetilde{\varrho} Z + q| \geq \mathbf{E}|\widetilde{\varrho} Z|$. Therefore, $\mathbf{P} - \lim_{n \to \infty}(V_1^n - h(S_1)) \leq 0$; thus, the option is asymptotically underhedged in this case, see Figure 2.1.

In approximation procedures, one should also pay attention to the fact that $\widehat{C}$ and its derivatives depend on the number of revisions when $0 \leq \alpha < 1/2$. In addition, the coefficient ϱ appearing in (2.5) can be chosen in an arbitrary way.

Pergamenshchikov (2003) shows that the rate of convergence in Kabanov–Safarian's theorem is $n^{1/4}$ and provides a weak convergence for the normalized replication error.

THEOREM 2.8 (Pergamenshchikov 2003). *Consider Leland's strategy* (2.6) *with* $\alpha = 0$, *and let* ϱ *in* (2.5) *be some fixed positive constant. Then, the sequence of random variables*

$$(2.13) \qquad n^{1/4}(V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J(S_1, \varrho))$$

weakly converges to a centered mixed Gaussian variable as $n \to \infty$.
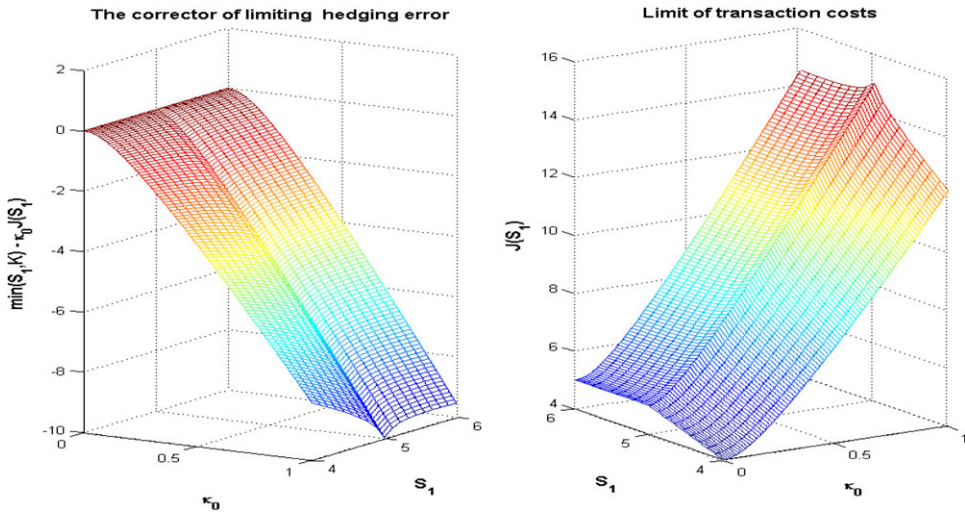
FIGURE 2.1. $\min(S_1, K) - \kappa_* J(S_1)$ on the left and $J(S_1)$ on the right with $K = 5$. [Color figure can be viewed at wileyonlinelibrary.com]

Theorem 2.8 is of practical importance because it not only gives the asymptotic information about the hedging error, but also provides a reasonable way to fix the underhedging issue (see Section 4).

Darses and Lépinette (2014) modify Leland's strategy in order to improve the convergence rate in Theorem 2.8 by applying a nonuniform revision policy $(t_i)_{1 \le i \le n}$, defined by

$$(2.14) \qquad t_i = g(i/n), \quad g(t) = 1 - (1 - t)^\mu \quad \text{for some} \quad \mu \ge 1.$$

The adjusted volatility is then taken as $\widehat{\sigma}_t^2 = \sigma_0^2 + \kappa_* \sigma_0 \sqrt{8/\pi} \sqrt{n f'(t)}$, where f is the inverse function of g. Furthermore, the discrepancy in Theorems 2.7 and 2.8 can be removed by employing the following modified strategy, known as Lépinette's strategy:

$$(2.15) \qquad \bar{\gamma}_t^n = \sum_{i=1}^n \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \int_0^{t_{i-1}} \widehat{C}_{xt}(u, S_u) du \right) \mathbf{1}_{(t_{i-1}, t_i]}(t).$$

THEOREM 2.9. *Let V_1^n be the terminal value of the strategy (2.15) with $\alpha = 0$. Then, for any $1 \le \mu < \mu_{\max}$, the sequence $n^\beta(V_1^n - h(S_1))$ weakly converges to a centered mixed Gaussian variable as $n \to \infty$, where*

$$(2.16) \qquad \beta = \frac{\mu}{2(\mu + 1)} \quad \text{and} \quad \mu_{\max} = \frac{3 + \sqrt{57}}{8}.$$

2.2. Time-Dependent Volatility Case

Assume now that σ is a positive nonrandom function and the payoff H is a continuous function with continuous derivatives, except at a finite number of points. Under the nonuniform rebalancing plan (2.14), the enlarged volatility should take the form

$$(2.17) \qquad \widehat{\sigma}_t^2 = \sigma^2(t) + \kappa_* \sigma(t) n^{1/2-\alpha} \sqrt{f'(t)8/\pi}.$$

THEOREM 2.10 (Lépinette 2008). *Let σ be a strictly positive Lipschitz and bounded function. Moreover, suppose that $H(\cdot)$ is a piecewise twice differentiable function and there exist $x_* \geq 0$ and $\delta \geq 3/2$, such that $\sup_{x \geq x_*} x^\delta |H''(x)| < \infty$. Then, for any $1/2 \geq \alpha > 0$, the replicating portfolio of Leland's strategy converges in probability to the payoff $H(S_1)$ as $n \to \infty$. Moreover, for $\alpha = 0$,*

$$\mathbf{P} - \lim_{n \to \infty} V_1^n = H(S_1) + H_1(S_1) - \kappa_* H_2(S_1),$$

where $H_1(\cdot)$ and $H_2(\cdot)$ are positive functions that depend on the payoff H.

REMARK 2.11. Theorem 2.9 still holds in the context of Theorem 2.10 (see Lépinette 2012).

2.3. Discussion

From Remark 2.1, the modified volatility defined by (2.5) would seem to give an appropriate approximation that accounts for transaction costs. However, this is not always the case because the option price inclusive of transaction costs now depends on the rebalancing number. In more general models, this specific choice may generate technical issues. For example, in local volatility models (Lépinette 2008), proving the existence of the solution to (2.8) requires patience and effort, because $\widehat{\sigma}$ depends on the stock price. On the other hand, it is interesting to observe that the true volatility plays no role in the approximation procedure from a mathematical point of view. In fact, all the results for the case $\alpha = 0$ can be obtained by using the form $\widehat{\sigma}_t^2 = \kappa_* \sigma(t) n^{1/2} \sqrt{f'(t)8/\pi}$, where the first term $\sigma^2(t)$ has been removed. More generally, we can take the following form:

$$(2.18) \qquad \widehat{\sigma}_t^2 = \varrho \sqrt{nf'(t)},$$

for some positive constant ϱ, which will be specified later. Of course, the limiting value of transaction costs will change slightly.

Let us emphasize that using the simple form (2.18) is important for two reasons. First, it allows us to carry out a far simpler approximation than is used in the existing literature. Second, Leland's strategy with $\widehat{\sigma}$ defined in (2.5) may no longer work in SV markets. Indeed, in those markets, option prices depend on future volatility realizations, which are not statistically available. We show in the remainder of the paper that the simple form (2.18) (a deterministic function of t) is helpful for approximate hedging in a very general SV setting. It should be noted that the approximation methodology developed here still works well for the classical form (2.5), if the volatility risk premium depends only on the current value of the volatility process (Pham and Touzi 1996; Renault and Touzi 1996).

We conclude the section by mentioning that Leland's algorithm is of practical importance due to its ease of implementation. The case of constant transaction costs $\alpha = 0$ should be investigated in more general situations, for instance, where volatility depends on external random factors, or jumps in stock prices are considered.

3. MODEL AND MAIN RESULTS

Let $(\Omega, \mathcal{F}_1, (\mathcal{F}_t)_{0 \le t \le 1}, \mathbf{P})$ be a standard filtered probability space with two standard independent $(\mathcal{F}_t)_{0 \le t \le 1}$ adapted Wiener processes $(W_t^{(1)})$ and $(W_t^{(2)})$, taking their values in $\mathbb{R}$. Our financial market consists of one risky asset governed by the following equations on the time interval $[0, 1]$

$$(3.1) \quad dS_t = \sigma(y_t) S_t dW_t^{(1)}; \quad dy_t = F_1(t, y_t)dt + F_2(t, y_t)(\mathbf{r}dW_t^{(1)} + \sqrt{1 - \mathbf{r}^2}dW_t^{(2)}),$$

where $-1 \le \mathbf{r} \le 1$ is the correlation coefficient. It is well known in the literature of SDEs that if $F_1(t, y)$ and $F_2(t, y)$ are measurable in $(t, y) \in [0, T] \times \mathbb{R}$, linearly bounded, and locally Lipschitz, there exists a unique solution y to the last equation of system (3.1). For this fundamental result, see Theorem 5.1 and Friedman (1975) and Liptser and Shiryaev (2001). For simplicity, assume that the interest rate equals zero. Thus, the nonrisky asset is chosen as the *numéraire*.

In this section, we consider the problem of approximate hedging with constant proportional costs using the principle of increasing volatility for model (3.1). As discussed in Subsection 2.3, the adjusted volatility is chosen as

$$(3.2) \qquad \widehat{\sigma}_t^2 = \varrho \sqrt{nf'(t)} = \mu^{-1/2} \varrho \sqrt{n}(1 - t)^{\frac{1-\mu}{2\mu}}, \quad 1 \le \mu < 2.$$

The replicating portfolio is revised at (t_i), as defined by (2.14). The parameter $\varrho > 0$ plays an important role in controlling the rate of convergence and is specified later. As shown below, the limiting value of the total trading volume is essentially related to the dependence of ϱ on the number of revisions.

REMARK 3.1. Intuitively, using an independent adjusted volatility seems unnatural because it fails to account for market information. However, the techniques developed in this note are well adapted to the case where the adjusted volatility depends on a volatility process driven by an independent Brownian motion. In such a context, if the volatility risk premium depends only on the current volatility process, then the no-arbitrage option price (without transaction costs) is the average of the Black–Scholes prices over the future paths of the volatility process (Pham and Touzi 1996; Renault and Touzi 1996).

Recall that $\widehat{C}(t, x)$ is the solution of the Cauchy problem (2.8) with two first derivatives, as described in (2.4): $\widehat{C}_x(t, x) = \Phi(\mathbf{v}(\lambda_t, x))$ and $\widehat{C}_{xx}(t, x) = x^{-1}\lambda_t^{-1/2} \widetilde{\varphi}(\lambda_t, x)$, where

$$(3.3) \qquad \lambda_t = \int_t^1 \widehat{\sigma}_s^2 \, ds = \widetilde{\mu} \varrho \sqrt{n}(1 - t)^{\frac{1}{4\beta}} \quad \text{and} \quad \widetilde{\mu} = 2\sqrt{\mu}/(\mu + 1).$$

REMARK 3.2. Section 4 will show that the underhedging situation pointed out in Kabanov and Safarian (1997) can be fixed by controlling the parameter ϱ.

We make use of the following condition on the volatility function.
$(\mathbf{C}_1)$ *Assume that σ is a C^2 function and there exists $\sigma_{\min}$ such that*

$$0 < \sigma_{\min} \le \sigma(y) \text{ for all } y \in \mathbb{R} \quad and \quad \sup_{0 \le t \le 1} \mathbf{E}[\sigma^2(y_t) + |\sigma'(y_t)|] < \infty.$$

Assumption $(\mathbf{C}_1)$ is not restrictive and is fulfilled in many popular SV models (see Section 5 and Pham 2002).

3.1. Asymptotic Results for Leland's Strategy

Let us study the replication error for Leland's strategy γ_t^n defined in (2.6).
The replicating portfolio V_1^n is defined by (2.7). Now, by Itô's formula,

$$(3.4) \qquad h(S_1) = \widehat{C}(1, S_1) = \widehat{C}(0, S_0) + \int_0^1 \widehat{C}_x(t, S_t)\,\mathrm{d}S_t - \frac{1}{2} I_{1,n},$$

where $I_{1,n} = \int_0^1 (\widehat{\sigma}_t^2 - \sigma^2(y_t)) S_t^2 \widehat{C}_{xx}(t, S_t)\,\mathrm{d}t$. Setting $V_0 = \widehat{C}(0, S_0)$, we can represent the replication error as

$$(3.5) \qquad V_1^n - h(S_1) = \frac{1}{2} I_{1,n} + I_{2,n} - \kappa_* J_n,$$

where $I_{2,n} = \int_0^1 (\gamma_t^n - \widehat{C}_x(t, S_t))\,\mathrm{d}S_t$ and J_n is defined as in (2.7).

Let us first emphasize that complete replication in SV models is far from obvious. In our setting, $I_{2,n}$ converges to zero faster than n^β, with β defined as in (2.16). The gamma error $I_{1,n}$ approaches $2\min(S_1, K)$ at the same rate. On the other hand, the total trading volume J_n converges in probability to the random variable $J(S_1, y_1, \varrho)$, defined by

$$(3.6) \qquad J(x, y, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x)\,\mathbf{E}\,\left|\sigma(y)\varrho^{-1} Z + q(\lambda, x)\right|\,\mathrm{d}\lambda,$$

where $Z \sim \mathcal{N}(0, 1)$ independent of S_1 and y_1. Our goal is to study the convergence of the normalized replication error corrected by these explicit limiting values, by applying the theory of limit theorems for martingales (Hall 1980). To do so, we search for the martingale part in the approximation of the above terms by developing a special discretization procedure in Section 7.

THEOREM 3.3. *Suppose that condition* $(\mathbf{C}_1)$ *holds and* $\varrho > 0$ *is a fixed positive constant. Then,*

$$n^\beta (V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J(S_1, y_1, \varrho))$$

weakly converges to a centered mixed Gaussian variable as $n \to \infty$.

REMARK 3.4. This theorem is a generalization that includes an improved convergence rate of the results in Kabanov and Safarian (1997) and Pergamenshchikov (2003), where the uniform revision is taken and the volatility is assumed to be a constant.

REMARK 3.5. Note that $h(x) + \min(x, K) = x$, where $h(x) = (x - K)^+$ is the payoff of a classical European call option. Then, from Theorem 3.3, the wealth process V_1^n approaches $S_1 - \kappa_* J(S_1, y_1, \varrho)$ as $n \to \infty$. This can be explained by the fact that the option is now sold at a higher price because $C(0, S_0, \hat{\sigma}) \to S_0$ as $\hat{\sigma} \to \infty$. In other words, Leland's strategy now converges to the well-known buy-and-hold strategy (Karatzas and Shreve 1998): buy a stock share at time $t = 0$ for price S_0 and keep it until expiry.

We now present a method for improving the rate of convergence in Theorem 3.3. To this end, by letting $\varrho \to \infty$, we observe that

$$(3.7) \qquad \lim_{\varrho \to \infty} J(x, y, \varrho) = x \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x)|q(\lambda, x)|\mathrm{d}\lambda := J^*(x),$$

which is independent of y. This suggests that the rate of convergence in Theorem 3.3 can be improved if ϱ is taken as a function of n. Our next result is established under the following condition on ϱ.

($\mathbf{C}_2$) *The parameter* $\varrho = \varrho(n)$ *is a function of* n *such that*

$$\lim_{n \to \infty} \varrho(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} \varrho \, n^{-\frac{\mu}{2(\mu+2)}} = 0.$$

THEOREM 3.6. *Under conditions* ($\mathbf{C}_1$) $-$ ($\mathbf{C}_2$),

$$\theta_n(V_1^n - h(S_1) - \min(S_1, K) + \kappa_* J^*(S_1)), \quad \text{with} \quad \theta_n = n^\beta \varrho^{2\beta},$$

weakly converges to a centered mixed Gaussian variable as $n \to \infty$.

REMARK 3.7. The asymptotic distributions in Theorems 3.3 and 3.6 are explicitly determined in the proofs in Section 7.

3.2. Asymptotic Result for Lépinette's Strategy

Let us study the replication error of Lépinette's strategy $\overline{\gamma}_t^n$, as defined in (2.15). As before, the replicating portfolio is $\overline{V}_1^n = \overline{V}_0^n + \int_0^1 \overline{\gamma}_t^n \, dS_t - \kappa_* \overline{J}_n$, where

$$(3.8) \qquad \overline{J}_n = \sum_{i=1}^n S_{t_i} |\overline{\gamma}_{t_i}^n - \overline{\gamma}_{t_{i-1}}^n|.$$

Now, by Itô's formula, the tracking error is

$$(3.9) \qquad \overline{V}_1^n - h(S_1) = \frac{1}{2} I_{1,n} + \overline{I}_{2,n} - \kappa_* \overline{J}_n,$$

where $\overline{I}_{2,n} = I_{2,n} + \sum_{i \geq 1}(S_{t_i} - S_{t_{i-1}}) \int_0^{t_{i-1}} \widehat{C}_{xt}(u, S_u) du$. Then, we have the following strengthening of Theorem 2.9.

THEOREM 3.8. *Suppose that* ($\mathbf{C}_1$) *is fulfilled. Then, for any* $\varrho > 0$, *the sequence*

$$n^\beta(\overline{V}_1^n - h(S_1) - \eta \min(S_1, K)), \quad \text{with} \quad \eta = 1 - \kappa_* \sigma(y_1)\varrho^{-1}\sqrt{8/\pi},$$

weakly converges to a centered mixed Gaussian variable as $n \to \infty$.

REMARK 3.9. Theorem 2.9 can be established from Theorem 3.8 with $\varrho = \kappa_* \sigma \sqrt{8/\pi}$ when the volatility is a constant. In addition, in our model, the parameter μ takes its values in the interval $[1, 2)$, which is slightly more general than the condition imposed in Theorem 2.9. Moreover, if the classical form of adjusted volatility is applied for Lépinette's strategy, then complete replication can be reached by taking $\varrho = \kappa_* \sqrt{8/\pi}$, and we again have the result established in Darses and Lépinette (2014).

COROLLARY 3.10. *Under conditions* ($\mathbf{C}_1$) $-$ ($\mathbf{C}_2$), *the wealth sequence* $\overline{V}_1^n$ *converges in probability to* $h(S_1) + \min(S_1, K) = S_1$.

Note that we do not obtain an improved convergence version of Theorem 3.8 because $\kappa_* \overline{J}_n$ converges to zero at the order of ϱ.

4. APPLICATION TO THE PRICING PROBLEM

In this section, we present an application to the problem of option pricing with transaction costs. We first emphasize that it is impossible to obtain a *nontrivial* perfect hedge in the presence of transaction costs, even in constant volatility models. In fact, the seller can take the *buy-and-hold* strategy, but this leads to a high option price. We show below that the price can be reduced in certain ways so that the payoff is covered with a given probability.

4.1. Superhedging with Transaction Costs

To be on the safe side, the investor searches for strategies with terminal values greater than the payoff. Such strategies are solutions to dynamic optimization problems. More precisely, let H be a general contingent claim and let $\mathcal{A}(x)$ and $V_T^{\pi,x}$ be the set of all *admissible strategies* π with initial capital x and the terminal value of strategy π, respectively. Then, the superreplication cost for H is determined as

$$(4.1) \qquad U_0 = \inf \left\{ x \in \mathbb{R} : \exists \pi \in A(x), \ V_T^{\pi,x} \geq H \quad \text{a.s.} \right\}$$

(see Karatzas and Shreve 1998 and the references therein for more details). In the presence of transaction costs, Cvitanić and Karatzas (1996) show that the *buy-and-hold strategy* is the unique choice for superreplication, and then S_0 is the superreplication price. In this section, we show that this property still holds for approximate superhedging. The following observation is a direct consequence of Theorem 3.6 when ϱ is a function of n.

PROPOSITION 4.1. *Under conditions* $(\mathbf{C}_1) - (\mathbf{C}_2)$, $\mathbf{P} - \lim_{n\to\infty} V_1^n \geq h(S_1)$. *The same property holds for Lépinette's strategy.*

Proof. First, note that $J^*(x) \leq \min(x, K)$, for all $x > 0$. Hence, by Theorem 3.6,

$$(4.2) \qquad \mathbf{P} - \lim_{n\to\infty} (V_1^n - h(S_1)) \geq (1 - \kappa_*) \min(S_1, K).$$

The left-hand side is obviously nonnegative as $\kappa_* < 1$. The conclusion follows from Theorem 3.8. $\qquad\qquad\square$

4.2. Asymptotic Quantile Pricing

As seen earlier, superhedging in the presence of transaction costs leads to a high option price. Practically, one can ask by how much the initial capital can be reduced in exchange for a shortfall probability at the terminal moment. More precisely, for a given significance level $0 \leq \varepsilon \leq 1$, the seller may look for hedges with a minimal initial cost

$$\inf \left\{ x \in \mathbb{R}, \exists \pi \in A(x) : \mathbf{P} \left(V_T^{\pi,x} \geq H \right) \geq 1 - \varepsilon \right\}.$$

This construction is motivated by quantile hedging theory, which goes back to Föllmer and Leukart (1999) and Novikov (1997). For related discussions, we refer to Föllmer and Leukart (1999), Novikov (1997), Pergamenshchikov (2003), Baran (2003), Bratyk and Mishura (2008), and Barski (2011). Here, we adapt this idea to the hedging problem. Recall that the superhedging price of Leland's algorithm is S_0. On the seller's side, we propose a price $\delta S_0 < S_0$ for the option, for a properly chosen $0 < \delta < 1$. We then follow

Leland's strategy for replication. To be safe at the terminal moment, we need to choose ϱ such that the probability of the terminal portfolio exceeding the sum of the real objective (i.e., the payoff) and the additional amount $(1 - \delta)S_0$ is greater than $1 - \varepsilon$. Here, ε is a significance level predetermined by the seller. By Proposition 4.1, this goal can be achieved for sufficiently large ϱ. To determine the option price, it now remains to choose the smallest value of δ. Motivated by (4.2), we define this by

(4.3) $\delta_\varepsilon = \inf \{a > 0 : \Upsilon(a) \geq 1 - \varepsilon\}, \quad \Upsilon(a) = \mathbf{P}\left((1 - \kappa_*)\min(S_1, K) > (1 - a)S_0\right).$

Thus, the reduction in the option price is given by $(1 - \delta_\varepsilon)S_0$. Clearly, smaller values of δ_ε yield cheaper options.

Next, we show that the option price is significantly reduced, compared with powers of the parameter ε.

PROPOSITION 4.2. *Assume that $\sigma_{\max} = \sup_{y \in \mathbb{R}} \sigma(y) < \infty$. Then, for any $r > 0$ and δ_ε defined by (4.3),*

(4.4)
$$\lim_{\varepsilon \to 0}(1 - \delta_\varepsilon)\varepsilon^{-r} = +\infty.$$

Proof. We first observe that $0 < \delta_\varepsilon \leq 1$ and δ_ε tends to 1 as $\varepsilon \to 0$. Set $b = 1 - \kappa_*$. Then, for sufficiently small ε such that $\delta_\varepsilon > a > 1 - bK/S_0$, one has

$$1 - \varepsilon > \mathbf{P}(b\min(S_1, K) > (1 - a)S_0) = 1 - \mathbf{P}(S_1/S_0 \leq (1 - a)/b).$$

Therefore,

(4.5)
$$\varepsilon < \mathbf{P}(S_1/S_0 \leq (1 - a)/b) \leq \mathbf{P}(X_1 \leq -z_a),$$

where $X_t = \int_0^t \sigma(y_t)dW_t^{(1)}$ and $z_a = \ln(b/(1 - a)) - \sigma_{\max}^2/2$. To estimate this probability, we note that for any integer $m \geq 1$, $\mathbf{E}(X_1)^{2m} \leq \sigma_{\max}^{2m}(2m - 1)!!$ (see Liptser and Shiryaev 2001, lemma 4.11, p. 130). Set $R(v) = 2v\sigma_{\max}^2$. For any $0 < v < 1/2\sigma_{\max}^2$,

$$\mathbf{E}\,e^{vX_1^2} = \sum_{m=0}^{\infty} \frac{v^m}{m!}\mathbf{E}(X_1)^{2m} \leq \sum_{m=0}^{\infty} \frac{v^m}{m!}\sigma_{\max}^{2m}(2m - 1)!! \leq \frac{1}{1 - R(v)}.$$

Therefore, for sufficiently small $\varepsilon > 0$, we have

$$\varepsilon \leq \mathbf{P}(X_1 \leq -z_a) = \mathbf{P}(-X_1 \geq z_a) \leq e^{-vz_a^2}\,\mathbf{E}\,e^{vX_1^2} \leq \frac{e^{-vz_a^2}}{1 - R(v)}.$$

Then, $1 - a \geq be^{-\iota_\varepsilon(v)}$, where $\iota_\varepsilon(v) = \sqrt{|\ln \varepsilon(1 - R(v))|/v} + \sigma_{\max}^2/2$. Letting $a \to \delta_\varepsilon$, we get $1 - \delta_\varepsilon \geq be^{-\iota_\varepsilon(v)}$, which implies (4.4). $\square$

The boundedness of the volatility function is essential for the above comparison result. If we wish to relax this assumption, the price reduction will be smaller than that in Proposition 4.2.

PROPOSITION 4.3. *Suppose that $\mathbf{E}\exp\{\alpha\int_0^1 \sigma^2(y_s)ds\} < \infty$, for some constant $\alpha > 1/2$. Then, for $r_\alpha = (2\sqrt{2\alpha} + 1)/2\alpha$,*

(4.6)
$$\liminf_{\varepsilon \to 0} \varepsilon^{-r_\alpha}(1 - \delta_\varepsilon) > 0.$$

Proof. For any positive constant L, we set

$$(4.7) \qquad \tau = \tau_L = \inf \left\{ t > 0 : \int_0^t \sigma^2(y_s) \mathrm{d}s \geq L \right\} \wedge 1,$$

which is understood to be the first time that the log-price's variance passes level L. Then, from (4.5),

$$(4.8) \qquad \varepsilon \leq \mathbf{P}\left(\mathcal{E}_1^{-1}(\sigma) \geq u_a, \int_0^1 \sigma^2(y_s)\mathrm{d}s \leq L \right) + \mathbf{P}\left(\int_0^1 \sigma^2(y_s)\mathrm{d}s \geq L \right),$$

where $\mathcal{E}_t(\sigma) = e^{\int_0^t \sigma(y_s)\mathrm{d}W_s^{(1)} - \frac{1}{2}\int_0^t \sigma^2(y_s)\mathrm{d}s}$, $u_a = (1 - \kappa_*)/(1 - a)$, and $\delta_\varepsilon > a > 1 - bK/S_0$. Note that for any $p > 0$, the stopped process $\chi_t = \mathcal{E}_{\tau \wedge t}(-p\sigma)$ is a martingale and $\mathbf{E}\chi_t = 1$. Therefore, the first probability on the right side of (4.8) can be estimated as

$$(u_a)^{-p}\, \mathbf{E}\, \mathcal{E}_\tau^{-p}(\sigma) = (u_a)^{-p}\, \mathbf{E}\, \chi_1\, e^{\check{p}\int_0^\tau \sigma^2(y_s)\mathrm{d}s} \leq (u_a)^{-p}\, e^{\check{p}L},$$

where $\check{p} = (p^2 + p)/2$.

By the hypothesis and Chebysev's inequality, we have

$$\mathbf{P}\left(\int_0^1 \sigma^2(y_s)\mathrm{d}s \geq L \right) \leq C_\alpha e^{-\alpha L}, \quad \text{with} \quad C_\alpha = \mathbf{E}\exp\left\{ \alpha \int_0^1 \sigma^2(y_s)\mathrm{d}s \right\}.$$

Hence, $\varepsilon \leq (u_a)^{-p} e^{\check{p}L} + C_\alpha e^{-\alpha L}$. By choosing $L = \alpha^{-1}\ln(2C/\varepsilon)$ and letting $a \to \delta_\varepsilon$, one deduces that for any $p > 0$ and for some positive constant $\tilde{C}_\alpha$,

$$1 - \delta_\varepsilon \geq \tilde{C}_\alpha\, \varepsilon^{\gamma^*(p)}, \quad \text{where} \quad \gamma^*(p) = (p + 1)/(2\alpha) + p^{-1}.$$

Note that $r_\alpha = \min_{p>0} \gamma^*(p) = \gamma^*(\sqrt{2\alpha})$. Then, including in the last inequality $p = \sqrt{2\alpha}$, we obtain property (4.6). $\qquad\square$

REMARK 4.4. It is clear that $r_\alpha < 1$ for $\alpha > 3/2 + \sqrt{2}$. The condition used in Proposition 4.3 holds for such α, when σ is linearly bounded and y_t follows an Orstein–Uhlenbeck process (see Appendix C). The same quantile pricing result can be established for the Lépinette strategy.

5. EXAMPLES AND NUMERICAL RESULTS

In this section, we list some well-known SV models for which condition $(\mathbf{C}_1)$ is fulfilled. To this end, we need some moment estimates for solutions to general SDEs:

$$(5.1) \qquad \mathrm{d}y_t = F_1(t, y_t)\mathrm{d}t + F_2(t, y_t)\mathrm{d}Z_t, \quad y(0) = y_0,$$

where Z is a standard Wiener process and F_1, F_2 are two smooth functions. We first recall the well-known result in SDE theory (see, for example, Friedman 1975, theorem 2.4, p. 107).

THEOREM 5.1. *Suppose that $F_1(t, y)$ and $F_2(t, y)$ are measurable in $(t, y) \in [0, T] \times \mathbb{R}$, linearly bounded and locally Lipschitz. If $\mathbf{E}|y_0|^{2m} < \infty$ for some integer $m \geq 1$, then there exists a unique solution (y_t) to (5.1) and*

$$\mathbf{E}|y_t|^{2m} < (1 + \mathbf{E}|y_0|^{2m})e^{\alpha t}, \quad \mathbf{E}\sup_{0 \leq s \leq t}|y_s|^{2m} < M(1 + \mathbf{E}|y_0|^{2m}),$$

where α, M are positive constants dependent on t, m.

In the context of Theorem 5.1, condition $(\mathbf{C}_1)$ holds if the volatility function and its derivative have polynomial growth $|\sigma(y)| \leq C(1 + |y|^m)$, for some positive constant C and $m \geq 1$.

Hull–White models: Assume that y_t follows a geometric Brownian motion

$$(5.2) \qquad \mathrm{d}S_t = (y_t + \sigma_{\min})S_t\mathrm{d}W_t \quad \text{and} \quad \mathrm{d}y_t = y_t(a\mathrm{d}t + b\mathrm{d}Z_t),$$

where $\sigma_{\min} > 0$, a and b are some constants, and Z is a standard Brownian motion correlated with W_t. Put $y^* = \sup_{0 \leq t \leq 1} |y_t|$. Then, by Theorem 5.1, we have

$$\mathbf{E}\,(y^*)^{2m} \leq C(1 + \mathbf{E}|y_0|^{2m}) < \infty, \quad m \geq 1,$$

as long as $\mathbf{E}|y_0|^{2m} < \infty$. Therefore, condition $(\mathbf{C}_1)$ is fulfilled in (5.2).

Uniform elliptic volatility models: Suppose that volatility is driven by a mean-reverting Orstein–Uhlenbeck process

$$(5.3) \qquad \mathrm{d}S_t = (y_t^2 + \sigma_{\min})S_t\mathrm{d}W_t \quad \text{and} \quad \mathrm{d}y_t = (a - by_t)\mathrm{d}t + \mathrm{d}Z.$$

In this case, $\sigma(y) = y^2 + \sigma_{\min}$. Thus, condition $(\mathbf{C}_1)$ is verified throughout Theorem 5.1.

Stein–Stein models: Assume that

$$(5.4) \qquad \mathrm{d}S_t = \sqrt{y_t^2 + \sigma_{\min}}\,S_t\mathrm{d}W_t \quad \text{and} \quad \mathrm{d}y_t = (a - by_t)\mathrm{d}t + \mathrm{d}Z_t.$$

We have $\sigma(y) = \sqrt{y^2 + \sigma_{\min}}$ and condition $(\mathbf{C}_1)$ is also verified by Theorem 5.1.

Heston models: Heston (1993) proposes an SV model where volatility is driven by a Cox-Ingersoll-Ross process, which is also known as a square root process. This model can be used in our context. Indeed, assume now that the price dynamics are given by the following:

$$(5.5) \quad \mathrm{d}S_t = \sqrt{y_t + \sigma_{\min}}\,S_t\mathrm{d}W_t \quad \text{and} \quad \mathrm{d}y_t = (a - by_t)\mathrm{d}t + \sqrt{y_t}\,\mathrm{d}Z_t, \quad y_0 \geq 0.$$

For any a and $b > 0$, the last equation admits a unique strong solution $y_t > 0$. Note that the Lipschitz condition in Theorem 5.1 is violated, but by using the stopping times technique, we can directly show that $\mathbf{E}y^* < \infty$. Hence, this implies that condition $(\mathbf{C}_1)$ is satisfied for model (5.5).

Similarly, we can check that $(\mathbf{C}_1)$ also holds for Ball–Roma models (Ball and Roma 1994) or, more generally, for a class of processes with bounded diffusion satisfying the following condition.

(**A**) *There exist positive constants a, b, and M such that*

$$yF_1(t, y) \leq a - by^2 \quad \text{and} \quad |F_2(t, y)| \leq M, \quad \text{for all} \quad t > 0, y \in \mathbb{R}.$$

PROPOSITION 5.2. *Under condition (**A**), there exists $\alpha > 0$ such that $\mathbf{E}e^{\alpha y^{*2}} < \infty$.*

Proof. The proof uses the same method as in proposition 1.1.2 in Kabanov and Pergamenshchikov (2003). □

Scott models: Suppose that volatility follows an Orstein–Uhlenbeck, as in Stein–Stein models, and the function σ takes the exponential form

$$(5.6) \qquad \mathrm{d}S_t = (e^{\delta y_t} + \sigma_{\min})S_t\mathrm{d}W_t^{(1)} \quad \text{and} \quad \mathrm{d}y_t = (a - by_t)\mathrm{d}t + \mathrm{d}Z_t,$$

TABLE 5.1
Convergence for Lépinette's Strategy with $\kappa_* = 0.01$, $\varrho = 2$

n	Gain/loss	Corrected error	Lower bound	Upper bound	Price	Strategy
10	0.1523845	−0.2225988	−0.2363122	−0.2088854	0.7914033	0.9013901
50	0.2966983	−0.0596194	−0.0670452	−0.0521936	0.9399330	0.9706068
100	0.3086120	−0.0288526	−0.0350141	−0.0226911	0.9746527	0.9875094
500	0.2955755	0.0032387	−0.0005821	0.0070594	0.9991733	0.9995891
1,000	0.2851002	0.0012409	−0.0021596	0.0046415	0.9999300	0.9999652

TABLE 5.2
Convergence for Lépinette's Strategy with $\kappa_* = 0.001$, $\varrho = 4$

n	Gain/loss	Corrected error	Lower bound	Upper bound	Price	Strategy
10	0.2859197	−0.0744180	−0.0813544	−0.0674816	0.9246420	0.9659700
50	0.3172523	−0.0069238	−0.0115426	−0.0023049	0.9921661	0.9962377
100	0.3033519	0.0007474	−0.0030916	0.0045864	0.9984346	0.9992385
500	0.3618707	0.0001296	−0.0024741	0.0027333	0.9999977	0.9999989
1,000	0.3334375	0.0003996	−0.0020559	0.0028550	1	1

where a, b, and $\sigma_{\min} > 0$ are constants. Here, $\delta > 0$ is chosen such that $2\delta \leq \alpha$, defined as in Proposition 5.2. Clearly, $\sigma(y) = e^{\delta y} + \sigma_{\min}$ and condition ($\mathbf{C}_1$) is fulfilled because

$$\mathbf{E} \sup_{0 \leq t \leq 1} |\sigma(y)|^2 \leq 2\sigma_{\min}^2 + 2\mathbf{E}\left(e^{2\delta}\mathbf{1}_{\{|y_t| \leq 1\}} + e^{2\delta|y|_1^2}\mathbf{1}_{\{|y_t| > 1\}}\right) < \infty.$$

Numerical result for the Hull–White model: We provide a numerical example for the approximation result of Lépinette's strategy in the Hull–White model (5.2). By Theorem 3.8, the corrected replication error is given by $V_1^n - \max(S_1 - K, 0) - \eta \min(S_1, K)$, where $\eta = 1 - \kappa_* \sigma(y_1)\varrho^{-1}\sqrt{8/\pi}$.

The difference $V_1^n - \max(S_1 - K, 0)$ can be seen as the gain/loss of strategy $\bar{\gamma}^n$. For a numerical evaluation, we simulate $N = 500$ trajectories in a crude Monte-Carlo method, where the correlation coefficient of the two Brownian motions is 0.05 and the other initial values are $S_0 = K = 1$, $y_0 = 2$, $\sigma_{\min} = 2$, $a = -2$, and $b = 1$. For each value of n, we estimate the average value of the corrected error and give the corresponding 95% intervals defined by lower and upper bounds. Initial numbers of shares held are given in the last column of Tables 5.1 and 5.2. It turns out that strategy $\bar{\gamma}_t^n$ converges to the buy-and-hold strategy and the option prices approach the superhedging price S_0. We also see that the convergence of the corrected replication error to zero is somehow slow. In fact, increasing values of ϱ can provide a faster convergence, but this unexpectedly leads to superreplication more rapidly.
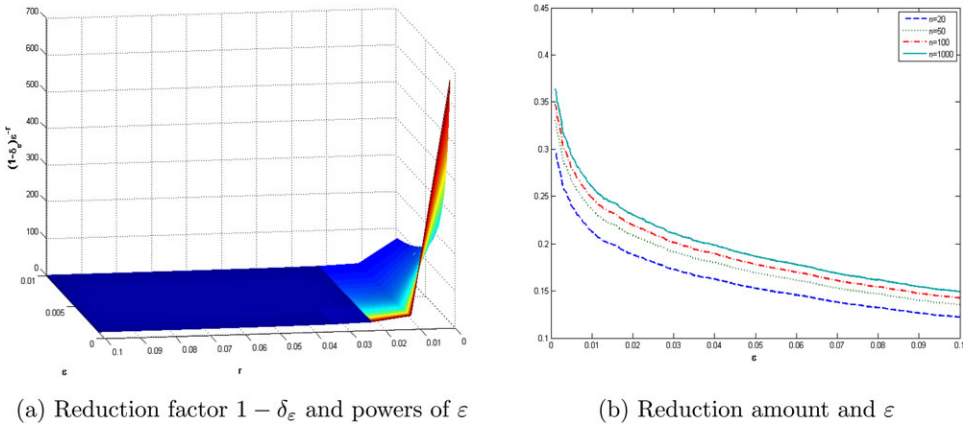
(a) Reduction factor $1 - \delta_\varepsilon$ and powers of ε (b) Reduction amount and ε

FIGURE 5.1. (a) Reduction factor $1 - \delta_\varepsilon$ and powers of ε and (b) reduction amount and ε. [Color figure can be viewed at wileyonlinelibrary.com]

We now provide a numerical illustration for the quantile hedging result of Proposition 4.2. For simplicity, suppose that $\sigma(y) = \sin^2(y) + 0.1$ and that y follows a geometric Brownian motion as above. To compare the reduction factor $1 - \delta_\varepsilon$ with powers of significance level ε, we compute $(1 - \delta_\varepsilon)\varepsilon^{-r}$ for $0.001 \leq \varepsilon \leq 0.1$ and $0 \leq r \leq 0.1$, with $\kappa_* = 0.001$. Then, (4.4) is confirmed by the simulation result (see Figure 5.1a). The simulation also shows that the option price inclusive of transaction costs is $1 - 0.385 = 0.615$, which is cheaper than the superhedging price $S_0 = 1$, for a shortfall probability less than 0.1%. Of course, it is reasonable to replace S_0 by the option price inclusive of transaction costs $\widehat{C}(0, S_0)$. The simulated reduction in the option price $(1 - \delta_\varepsilon)\widehat{C}(0, S_0)$ is then given in Figure 5.1(b).

6. HIGH-FREQUENCY MARKETS

We now assume that purchases of the risky asset are carried out at a higher ask price $S_t + \varepsilon_t$, whereas sales earn a lower bid price $S_t - \varepsilon_t$. Here, the midprice S_t is given as in model (3.1) and ε_t is the halfwidth of the bid-ask spread. Then, for any trading strategy of finite variation ψ_t, the wealth process can be determined by

$$(6.1) \qquad V_t = V_0 + \int_0^t \psi_s \, dS_s - \int_0^t \varepsilon_s \, d|\psi|_s,$$

where $|\psi|$ is the total variation of ψ_t. Observe that the first two terms are the classic components in frictionless frameworks, and, respectively, describe the initial capital and gains from trading. The last integral in (6.1) accounts for transaction costs incurred from the trading activities by weighting the total variation[2] of the strategy with the halfwidth of the spread.

For optimal investment and consumption with small transaction costs Kallsen and Muhle-Karbe (2017), the additional terms should be added in the formulation of V_t. In such cases, approximate solutions are usually determined through an asymptotic

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[2]It is important to know that the classical Black–Scholes strategy is not finite variation.

expansion around zero of the halfwidth spread ε, where the leading corrections are obtained by collecting the inputs from the frictionless problem.

In this section, we are only interested in replication using discrete strategies in Leland's spirit. Assume that for replication, the option seller applies a discrete hedging strategy $\psi_t^{n,\varepsilon}$, revised at n dates defined by $t_i = g(i/n)$ as in Section 3. The corresponding wealth process is now given by

$$(6.2) \qquad V_t^{n,\varepsilon} = V_0^{n,\varepsilon} + \int_0^t \psi_s^{n,\varepsilon} \, \mathrm{d}S_s - \sum_{i=1}^{n} \varepsilon_{t_i} |\psi_{t_i}^{n,\varepsilon} - \psi_{t_{i-1}}^{n,\varepsilon}|.$$

To treat the risk of transaction costs, we again apply the increasing volatility principle. Note that in high-frequency markets, the bid-ask spread is, in general, of the same order of magnitude as price jumps.[3] Hence, ε_t should be of the form $\kappa_* n^{-1/2} S_t$, for some positive constant κ_*. Then, this case corresponds to the Leland–Lott framework with $\alpha = 1/2$.

In our context, the method in Section 3 is still helpful when $\psi_t^{n,\varepsilon}$ is replaced by Leland's or Lépinette's strategy.

PROPOSITION 6.1. *Let* $\varepsilon_t = \kappa_* n^{-1/2} S_t$, *and assume that the adjusted volatility is of the form* $\widehat{\sigma}^2 = \varrho \sqrt{n f'(t)}$ *as in* (3.2). *For both Leland's and Lépinette's strategies, the sequence of replicating portfolio values* $V_1^{n,\varepsilon}$ *converges in probability to* $h(S_1) + \min(S_1, K) = S_1$. *In particular,* $n^\beta(V_1^{n,\varepsilon} - S_1)$ *converges to a mixed Gaussian variable as* $n \to \infty$.

Proof. The proof is a direct consequence of Theorem 3.3, because the total transaction cost now converges to zero. □

Note that the case $\alpha = 0$ studied in Section 3 corresponds to the assumption $\varepsilon_t = \kappa_* S_t$, for some constant κ_*. This specific form means that the market is more illiquid and the bid-ask spread is now proportional to the spot price in every trade. Therefore, approximate hedging results for this case are the same as those in Section 3.

We conclude the section by supposing that the stock spreads remain constant at all times, regardless of the current stock price. In other words, $\varepsilon_t = \kappa_*$ for some positive constant κ_*. Intuitively, transaction costs are now based on the volume of traded shares, instead of the traded amount of money as in the literature and Section 3. It is interesting to see that our methodology still works in this case. The following result is just an analog of Theorem 3.3, with a small modification to the limiting value of transaction costs, defined by

$$(6.3) \qquad J_0(x, y, \varrho) = \int_0^{+\infty} \lambda^{-1/2} \widetilde{\varphi}(\lambda, x) \, \mathbf{E} \left| \sigma(y) \varrho^{-1} Z + \frac{\ln(x/K)}{2\lambda} - \frac{1}{4} \right| \mathrm{d}\lambda,$$

where $Z \sim \mathcal{N}(0, 1)$ independent of S_1, y_1

PROPOSITION 6.2. *Suppose that* $\varepsilon_t = \kappa_* > 0$ *and* $\widehat{\sigma}^2 = \varrho \sqrt{n f'(t)}$. *For Leland's strategy under condition* $(\mathbf{C}_1)$, *the sequence* $n^\beta (V_1^{n,\varepsilon} - h(S_1) - \min(S_1, K) + \kappa_* J_0(S_1, y_1 \varrho))$ *weakly converges to a centered mixed Gaussian variable as* $n \to \infty$. *Furthermore, for Lépinette's strategy,* $n^\beta (\overline{V}_1^{n,\varepsilon} - h(S_1) - (1 - \eta_0) \min(S_1, K))$ *weakly converges to a centered mixed Gaussian variable, where* $\eta_0 = \sigma(y_1) \varrho^{-1} S_1^{-1} \sqrt{8/\pi}$.

Proof. The proof is similar to that of Theorem 3.3 (see Section 7). □

REMARK 6.3. When $\varrho \to \infty$ under condition $(\mathbf{C}_2)$, one obtains an improved-rate version of Proposition 6.2, as in Theorem 3.6.

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[3] We thank an anonymous referee for pointing out the correspondence of the case $\alpha = 1/2$ to this setting.

7. PROOFS

Our main results are proved in the following generic procedure.

Step 1: Determine the principal term of the hedging error. In particular, we will show that the gamma term $I_{1,n}$ converges to $2 \min(S_1, K)$, while the cumulative transaction cost approaches J defined in (3.6). Both convergences are at rate $\theta_n = n^\beta \varrho^{2\beta}$.

Step 2: Represent the residual terms as discrete martingales. To this end, stochastic integrals will be discretized by following a special procedure setup in Subsection 7.2.

Step 3: Determine the limit distribution of the normalized replication error by applying Theorem 7.3. This result is the key tool, but we need in fact special versions adapted to our context. These will be explicitly constructed in Subsection 7.3.

7.1. Preliminary

Note first that $\widehat{C}(t, x)$ and its derivatives can be represented as functions of λ_t and x, where

$$(7.1) \qquad \lambda_t = \lambda_0 (1 - t)^{\frac{1}{4\beta}} := \lambda_0 \nu(t) \quad \text{and} \quad \lambda_0 = \widetilde{\mu} \varrho \sqrt{n}.$$

Moreover, the function $\widetilde{\varphi}(\lambda, x)$, which appears in all kth ($k \geq 2$) degree derivatives of $\widehat{C}$ with respect to x and derivatives in time via the relation (2.8), is exponentially decreasing to zero when λ tends to zero or infinity. This property motives our analysis in terms of variable λ. In particular, let us fix two functions l_*, l^* and let $1 \leq m_1 < m_2 \leq n$ be two integers such that $l_* = \lambda_0 \nu(g(m_2/n))$ and $l^* = \lambda_0 \nu(g(m_1/n))$. Then, all terms corresponding to index $j \notin [m_1, m_2]$ can be ignored at a certain order which depends on the choice of l_* and l^*.

For our purpose, the desired order is $\theta_n \sim \lambda_0^{2\beta}$. Therefore, we take, for example, $l_* = 1/\ln^3 n$, $l^* = \ln^3 n$ and define

$$(7.2) \qquad m_1 = n - \left[n \, (l^*/\lambda_0)^{2/(\mu+1)} \right] + 1 \quad \text{and} \quad m_2 = n - \left[n \, (l_*/\lambda_0)^{2/(\mu+1)} \right],$$

where the notation $[x]$ stands for the integer part of a real number x. Below, we focus on the subsequence (t_j) of trading times and the corresponding sequence (λ_j), defined as

$$(7.3) \qquad t_j = 1 - (1 - j/n)^\mu \quad \text{and} \quad \lambda_j = \lambda_0 (1 - t_j)^{\frac{1}{4\beta}}, \quad m_1 \leq j \leq m_2.$$

Note that (t_j) is an increasing sequence taking values in $[t^*, t_*]$, where $t_* = 1 - (l_*/\lambda_0)^{4\beta}$ and $t^* = 1 - (l^*/\lambda_0)^{4\beta}$, whereas (λ_j) is decreasing in $[l_*, l^*]$, see Figure 7.1. Therefore, we use the notations $\Delta t_j = t_j - t_{j-1}$ and $\Delta \lambda_j = \lambda_{j-1} - \lambda_j$, for $m_1 \leq j \leq m_2$, to avoid recopying the negative sign in discrete sums.

Below, Itô stochastic integrals will be discretized through the following sequences of independent normal random variables

$$(7.4) \qquad Z_{1,j} = \frac{W_{t_j}^{(1)} - W_{t_{j-1}}^{(1)}}{\sqrt{t_j - t_{j-1}}} \quad \text{and} \quad Z_{2,j} = \frac{W_{t_j}^{(2)} - W_{t_{j-1}}^{(2)}}{\sqrt{t_j - t_{j-1}}}.$$
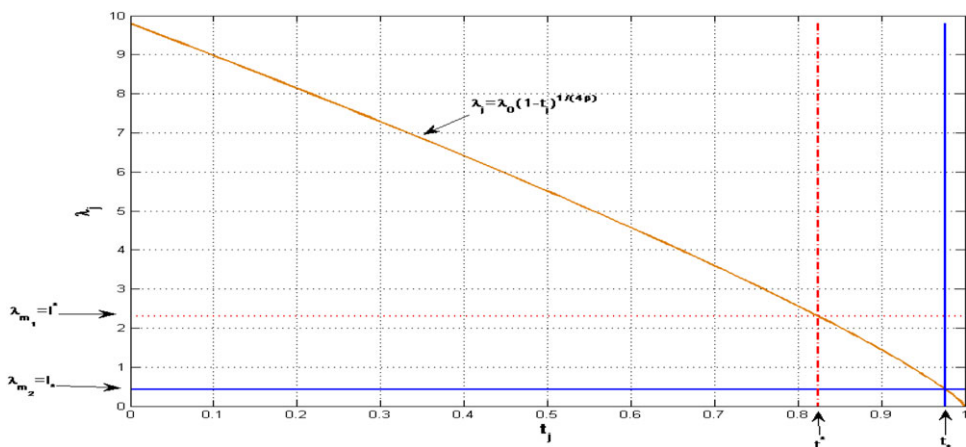
FIGURE 7.1. The sequences (λ_j) and (t_j) defined by (7.3). [Color figure can be viewed at wileyonlinelibrary.com]

We set

(7.5)
$$p(\lambda, x, y) = \frac{\varrho}{\sigma(y)} \left(\frac{\ln(x/K)}{2\lambda} - \frac{1}{4} \right)$$

and write for short $p_{j-1} = p(\lambda_{j-1}, S_{t_{j-1}}, y_{t_{j-1}})$. This style of reduced notation is abusively applied for functions appearing in the approximation procedure. Define

(7.6)
$$\begin{cases} Z_{3,j} = |Z_{1,j} + p_{j-1}| - \mathbf{E}\left(|Z_{1,j} + p_{j-1}| \,|\, \mathcal{F}_{j-1} \right), \\ Z_{4,j} = |Z_{1,j}| - \mathbf{E}\left(|Z_{1,j}| \,|\, \mathcal{F}_{j-1} \right) = |Z_{1,j}| - \sqrt{2/\pi}. \end{cases}$$

The sequences $(Z_{3,j})$ and $(Z_{4,j})$ will help to find the Dood decomposition of our approximation terms. In order to represent the limit of transaction costs, we introduce

(7.7)
$$\begin{cases} G(a) = \mathbf{E}\left(|Z + a| \right) = 2\varphi(a) + a\left(2\Phi(a) - 1 \right), \\ \Lambda(a) = \mathbf{E}\left(|Z + a| - \mathbf{E}|Z + a| \right)^2 = 1 + a^2 - G^2(a), \end{cases}$$

for $a \in \mathbb{R}$ and $Z \sim \mathcal{N}(0, 1)$. We also write $o(a_n^{-r})$ for generic sequences of random variables (X_n) satisfying $\mathbf{P} - \lim_{n \to \infty} a_n^r X_n = 0$.

7.2. Approximation for Stochastic Integrals

For any $L > 0$, we consider the stopping time

(7.8)
$$\tau^* = \tau_L^* = \inf \left\{ t \geq 0 : \sigma(y_t) + |\sigma'(y_t)| > L \right\},$$

and denote by $S_t^* = S_{\tau^* \wedge t}$ and $y_t^* = y_{\tau^* \wedge t}$ the corresponding stopped processes. We provide an approximation procedure for Itô stochastic integrals through the sequences $(Z_{1,j})$ and $(Z_{2,j})$. The discrete approximation concerns the class of functions satisfying the technical condition,

(H) $A : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is a continuously differentiable function satisfying the following: there exist $\gamma > 0$ and a positive function U such that for any $x \ge 0$, $y \in \mathbb{R}$,*

$$\sup_{\lambda > 0} \min(\lambda^\gamma, 1) |A(\lambda, x, y)| \le U(x, y) \quad \text{and} \quad \sup_{0 \le t \le 1} \mathbf{E}\, (S_t^*)^m U^{2r}(S_t^*, y_t^*) < \infty,$$

for any $-\infty < m < +\infty$, $r \ge 0$ and $L > 0$.

REMARK 7.1. We can check directly that for $k \ge 2$, $\partial_x^k \widehat{C}(\lambda, x) = x^{k-1} \lambda^{-k/2} \widetilde{\varphi}(\lambda, x) P(\ln(x/K))$, where P is some polynomial. Therefore, all functions A_0 appearing in the approximation below are of the form $\lambda^{-k/2} x^m \bar{\sigma}(y) P(\ln(x/K))$, where $\bar{\sigma}$ can be a power of σ or of its two first derivatives σ', σ''. In bounded volatility settings, it can be shown with some computational effort (see, e.g., Darses and Lépinette 2014; Lépinette 2008, 2012) that

$$(7.9) \qquad \sup_{0 \le t \le 1} \mathbf{E} S_t^m \ln^{2r} S_t < \infty, \quad \text{for any} \quad m \in \mathbb{R}, r \ge 0.$$

The latter property is, however, not always fulfilled for SV models with unbounded volatility. In fact, it has been demonstrated in Andersen and Piterbarge (2007) and Lions and Musiela (2007) that the stock price does not admit integrable moments in general SV markets, unless some natural conditions are imposed on the correlation and the volatility dynamics coefficients. Thus, asymptotic analysis using L^2 estimates as in the existing works may be impossible in general SV frameworks. Nevertheless, note that (7.9) is true for processes stopped by τ^*. Below, the approximation analysis will be established in the sense of convergence in probability, in order to avoid this integrability issue.

For simplicity, we use the notation $\check{S} = (S, y)$. The following technique is frequently applied in our asymptotic analysis.

PROPOSITION 7.2. *Let $A(\lambda, x, y) = A_0(\lambda, x, y) \widetilde{\varphi}(\lambda, x)$, where $A_0 = A_0(\lambda, x, y)$ is a function satisfying* **(H)**. *Then, for $i = 1, 2$,*

$$(7.10) \qquad \int_0^1 \widehat{\sigma}_t^2 \left(\int_t^1 A(\lambda_t, \check{S}_u) \mathrm{d} W_u^{(i)} \right) \mathrm{d}t = \varrho^{-1} \sum_{j=m_1}^{m_2} \overline{A}_{j-1} Z_{i,j} \Delta \lambda_j + o(\theta_n^{-1}),$$

where $\theta_n = n^\beta \varrho^{2\beta}$, $\overline{A}_j = \overline{A}(\lambda_j, \check{S}_{t_j})$ and $\overline{A}(\lambda, x, y) = \int_\lambda^\infty A(z, x, y)\, \mathrm{d}z$.

Proof. By making use of the stochastic Fubini theorem, we get

$$\widehat{I}_n = \int_0^1 \widehat{\sigma}_t^2 \left(\int_t^1 A(\lambda_t, \check{S}_u) \mathrm{d} W_u^{(i)} \right) \mathrm{d}t = \int_0^1 \left(\int_0^u \widehat{\sigma}_t^2 A(\lambda_t, \check{S}_u) \mathrm{d}t \right) \mathrm{d} W_u^{(i)}.$$

Then, changing the variables $v = \lambda_t$ for the inner integral yields

$$\int_0^u \widehat{\sigma}_t^2 A(\lambda_t, \check{S}_u) \mathrm{d}t = \int_{\lambda_u}^{\lambda_0} A(v, \check{S}_u) \mathrm{d}v = \overline{A}(\lambda_u, \check{S}_u) - \overline{A}(\lambda_0, \check{S}_u).$$

In other words, $\widehat{I}_n = \widehat{I}_{1,n} - \widehat{I}_{2,n}$, where $\widehat{I}_{1,n} = \int_0^1 \check{A}_u \mathrm{d} W_u^{(i)}$, $\check{A}_u = \overline{A}(\lambda_u, \check{S}_u)$, and $\widehat{I}_{2,n} = \int_0^1 \overline{A}(\lambda_0, \check{S}_u) \mathrm{d} W_u^{(i)}$. Moreover, we have

$$(7.11) \quad \widehat{I}_{1,n} = \int_0^{t^*} \check{A}_u \mathrm{d} W_u^{(i)} + \int_{t^*}^{t_*} \check{A}_u \mathrm{d} W_u^{(i)} + \int_{t_*}^1 \check{A}_u \mathrm{d} W_u^{(i)} := R_{1,n} + R_{2,n} + R_{3,n}.$$

Let $\varepsilon > 0$ and $b > 0$. One observes that $\mathbf{P}(\theta_n|\widehat{I}_{2,n}| > \varepsilon)$ is bounded by $\mathbf{P}(\tau_L^* < 1) + \mathbf{P}(\theta_n|\widehat{I}_{2,n}| > \varepsilon, \ \tau_L^* = 1)$. By condition $(\mathbf{C}_1)$, we have

$$\limsup_{L\to\infty} \mathbf{P}(\tau_L^* < 1) = 0. \tag{7.12}$$

In view of $(\mathbf{H})$, one deduces $\overline{A}(\lambda_0, x, y)| \leq C\sqrt{K}\widetilde{U}(x, y)e^{-\lambda_0/8}$, where $\widetilde{U}(x, y) = x^{-1/2}U(x, y)$. Now, putting $\check{A}_u^* = \check{A}_{u\wedge\tau^*}$ and $\widehat{\overline{I}}_{2,n}^* = \int_0^1 \check{A}_u^* \, dW_u^{(i)}$, one has $\mathbf{P}(\theta_n|\widehat{I}_{2,n}| > \varepsilon, \ \tau_L^* = 1) = \mathbf{P}(\theta_n|\widehat{\overline{I}}_{2,n}^*| > \varepsilon)$. Using the Chebychev inequality, we obtain

$$\mathbf{P}(\theta_n|\widehat{\overline{I}}_{2,n}^*| > \varepsilon) \leq \varepsilon^{-2}\theta_n^2\mathbf{E}(\widehat{\overline{I}}_{2,n}^*)^2 \leq C\varepsilon^{-2}\theta_n^2 e^{-\lambda_0/8} \sup_{0\leq t\leq 1} \mathbf{E}\,\widetilde{U}^2(\check{S}_t^*).$$

Hence, due to condition $(\mathbf{H})$, $\widehat{I}_{2,n} = o(\theta_n^{-1})$ as $n \to\infty$. Similarly, taking into account that $l^* \leq \lambda_u \leq \lambda_0$ for $0 \leq u \leq t^*$, we get $R_{1,n} = o(\theta_n^{-1})$.

Next, let us show the same behavior for the last term in (7.11). Indeed, for some fixed $\eta > 0$ and $L > 0$, one has

$$\mathbf{P}\left(\theta_n|R_{3,n}| > \varepsilon\right) \leq \mathbf{P}\left(\theta_n|R_{3,n}| > \varepsilon, \ \Gamma_{1,\eta,L}\right) + \mathbf{P}\left(\Gamma_{1,\eta,L}^c\right), \tag{7.13}$$

where $\Gamma_{1,\eta,L} = \{\inf_{t_*\leq u\leq 1} |\ln(S_u/K)| > \eta, \ \tau_L^* = 1\}$. Then, by taking into account Lemma A.3 and the integrability condition $(\mathbf{C}_1)$, one gets

$$\lim_{\eta\to 0}\overline{\lim}_{n\to\infty}\overline{\lim}_{L\to\infty}\mathbf{P}(\Gamma_{1,\eta,L}^c) = 0.$$

On $\Gamma_{1,\eta,L}$, we have $\check{A} = \check{A}^*$ and

$$|\check{A}_u^*| \leq U(\check{S}_u^*)\int_{\lambda_u}^{\infty}(1 + z^{-\gamma})\widetilde{\varphi}(z, S_u^*)dz \leq \widetilde{U}(\check{S}_u^*)\check{f}_u^*,$$

where $\check{f}_u^* = \sqrt{K/(2\pi)}\int_{\lambda_u}^{\infty}(1 + z^{-\gamma})e^{-\eta^2/(2z)-z/8}dz$. Set $\Gamma_{3,j} = \{|\check{A}_u| \leq \widetilde{U}(\check{S}_u^*)\check{f}_u^*\}, \widehat{A}_u^* = \check{A}_u^*\mathbf{1}_{\Gamma_{3,j}}$, and $\widehat{R}_{3,n} = \int_{t_*}^1 \widehat{A}_u^* dW_u^{(i)}$. By Chebychev's inequality, we obtain

$$\mathbf{P}\left(\theta_n|R_{3,n}| > \varepsilon, \Gamma_{1,\eta,L}\right) \leq \theta_n^2\varepsilon^{-2}\int_{t_*}^1 \mathbf{E}(\widehat{A}_u^*)^2 du \leq \theta_n^2\varepsilon^{-2}\sup_{0\leq u\leq 1}\mathbf{E}\,\widetilde{U}^2(\check{S}_u^*)\int_{t_*}^1 (\check{f}_u^*)^2 du,$$

which converges to zero as $\int_{t_*}^1(\check{f}_u^*)^2 du \leq C\lambda_0^{-4\beta}l_*$. Hence, $R_{3,n} = o(\theta_n^{-1})$. It remains to discretize the integral term $R_{2,n}$ via the sequence $(Z_{i,j})$. The key steps for this aim are the following. First, we represent $R_{2,n} = \int_{t^*}^{t_*} \check{A}_u dW_u^{(i)} = \sum_{j=m_1}^{m_2}\int_{t_{j-1}}^{t_j} \check{A}_u dW_u^{(i)}$ and replace the Itô integral in the last sum with $\overline{A}_{j-1}Z_{i,j}\sqrt{\Delta t_j}$.

Next, Lemma A.1 enables us to substitute $\sqrt{\Delta t_j} = \varrho^{-1}\Delta\lambda_j$ into the last sum to obtain the martingale $\mathcal{M}_{m_2}$ defined by $\mathcal{M}_k = \varrho^{-1}\sum_{j=m_1}^k \overline{A}_{j-1}Z_{i,j}\Delta\lambda_j$. We need to show that $|R_{2,n} - \mathcal{M}_{m_2}| = o(\theta_n^{-1})$ or equivalently, $\sum_{j=m_1}^{m_2} B_{j,n} = o(\theta_n^{-1})$, where $B_{j,n} = \int_{t_{j-1}}^{t_j} \widetilde{A}_{u,j}dW_u^{(i)}$ and $\widetilde{A}_{u,j} = \overline{A}(\lambda_u, \check{S}_u) - \overline{A}(\lambda_{j-1}, \check{S}_{t_{j-1}})$. For this aim, we introduce the set

$$\Gamma_{2,b} = \left\{\sup_{t^*\leq u\leq 1}\sup_{z\in\mathbb{R}}\left(|A(z, \check{S}_u)| + \left|\partial_x\overline{A}(z, \check{S}_u)\right| + \left|\partial_y\overline{A}(z, \check{S}_u)\right|\right) \leq b\right\}.$$

Then, for any $\varepsilon > 0$, $\mathbf{P}(\theta_n | \sum_{j=m_1}^{m_2} B_{j,n} | > \varepsilon)$ is bounded by $\mathbf{P}(\Gamma_{2,b}^c) + \mathbf{P}(\tau^* < 1) + \varpi_n$, where $\varpi_n = \mathbf{P}(\theta_n | \sum_{j=m_1}^{m_2} B_{j,n} | > \varepsilon, \Gamma_{2,b}, \tau^* = 1)$. Let $\widehat{B}_{j,n} = \int_{t_{j-1}}^{t_j} \widehat{A}_{u,j} \mathrm{d} W_u^{(i)}$, where

$$\widehat{A}_{u,j} = \widetilde{A}_{u,j} \mathbf{1}_{\{|\widetilde{A}_{u,j}| \le b(|\lambda_u - \lambda_{j-1}| + |S_u^* - S_{t_{j-1}}^*| + |y_u^* - y_{t_{j-1}}^*|)\}}.$$

Then, $\varpi_n = \mathbf{P}(\theta_n | \sum_{j=m_1}^{m_2} \widehat{B}_{j,n} | > \varepsilon)$, which is smaller than $\varepsilon^{-2} \theta_n^2 \sum_{j=m_1}^{m_2} \mathbf{E}\, \widehat{B}_{j,n}^2$ by Chebychev's inequality. Clearly, $\mathbf{E}\, \widehat{B}_{j,n}^2$ is bounded by

$$3b^2 \left(\int_{t_{j-1}}^{t_j} ((\lambda_u - \lambda_{j-1})^2 + \mathbf{E}(S_u^* - S_{t_{j-1}}^*)^2 + \mathbf{E}(y_u^* - y_{t_{j-1}}^*)^2) \mathrm{d}u \right) \le (\Delta \lambda_j)^3 + (\Delta t_j)^2$$

up to a multiple constant. Consequently, $\theta_n^2 \sum_{j=m_1}^{m_2} \mathbf{E}\, \widehat{B}_{j,n}^2 \le C \theta_n^2 \sum_{j=m_1}^{m_2} (\Delta \lambda_j)^3 + (\Delta t_j)^2$, which converges to 0 by Lemma A.1 and condition $(\mathbf{C}_2)$. On the other hand, by Lemma A.4, we get $\lim_{\to \infty} \overline{\lim}_{n \to \infty} \mathbf{P}(\Gamma_{2,b}^c) = 0$. The proof is complete.

7.3. Limit Theorem for Approximations

We first recall the following result in Hall (1980), which is useful for studying asymptotic distributions of discrete martingales.

THEOREM 7.3. *[Theorem 3.2 and Corollary 3.1, p. 58 in Hall (1980)] Let $\mathcal{M}_n = \sum_{i=1}^{n} X_i$ be a zero-mean, square integrable martingale, and ς be an a.s. finite random variable. Assume that the following convergences are satisfied in probability:*

$$\sum_{i=1}^{n} \mathbf{E}\left(X_i^2 \mathbf{1}_{\{|X_i| > \delta\}} | \mathcal{F}_{i-1} \right) \longrightarrow 0 \quad \text{for any} \quad \delta > 0 \quad \text{and} \quad \sum_{i=1}^{n} \mathbf{E}\left(X_i^2 | \mathcal{F}_{i-1} \right) \longrightarrow \varsigma^2.$$

Then, $(\mathcal{M}_n)$ converges in law to X whose characteristic function is $\mathbf{E} \exp(-\frac{1}{2} \varsigma^2 t^2)$, i.e., X has a Gaussian mixture distribution.

In this subsection, we establish special versions of Theorem 7.3. In fact, our aim is to study the asymptotic distribution of discrete martingales resulting from approximation (7.10) in Proposition 7.2. First, we define

$$(7.14) \qquad \qquad \mathcal{M}_k = \sum_{j=m_1}^{k} \upsilon_j, \quad m_1 \le k \le m_2,$$

where $\upsilon_j = \sum_{i=1}^{3} A_{i,j-1} Z_{i,j} \Delta \lambda_j$, $A_{i,j} = A_i(\lambda_j, \check{S}_{t_j})$, and $Z_{i,j}$ defined as in (7.4) and (7.6). To describe the asymptotic variance of $(\mathcal{M})$, we introduce the following function:

$$\mathbf{L}(\lambda, x, y) = A_1^2(\lambda, x, y) + 2 A_1(\lambda, x, y) A_3(\lambda, x, y)(2\Phi(p) - 1)$$
$$(7.15) \qquad \qquad + A_3^2(\lambda, x, y) \Lambda(p) + A_2^2(\lambda, x, y),$$

where p is defined in (7.5). Set

$$(7.16) \qquad \qquad \check{\mu} = \frac{1}{2}(\mu + 1)\widetilde{\mu}^{\frac{2}{\mu+1}} \quad \text{and} \quad \widehat{\mu} = (\mu - 1)/(\mu + 1).$$

PROPOSITION 7.4. *Let $A_i^0 = A_i^0(\lambda, x, y)$, $i = 1, 2, 3$ be functions having property* (**H**) *and $A_i(\lambda, x, y) = A_i^0(\lambda, x, y)\widetilde{\varphi}(\lambda, x)$. Then, for any fixed $\varrho > 0$, the sequence $(n^\beta \mathcal{M}_{m_2})_{n \geq 1}$ weakly converges to a mixed Gaussian variable with mean zero and variance ς^2 defined as $\varsigma^2 = \varsigma^2(\check{S}_1) = \breve{\mu}\varrho^{\frac{2}{\mu+1}} \int_0^{+\infty} \lambda^{\widehat{\mu}} \mathbf{L}(\lambda, \check{S}_1)\mathrm{d}\lambda$. The same property still holds if some (or all) of the functions A_i are of the form $\int_\lambda^\infty A_i^0(z, x, y)\widetilde{\varphi}(z, x)\mathrm{d}z$.*

Proof. Note that the square integrability property is not guaranteed for (υ_j). To overcome this issue, we consider the "stopped version" (υ_j^*), which is obtained by substituting $\check{S}_{t_{j-1}}$ by $\check{S}_{t_{j-1}}^*$ in A_i, i.e., $\upsilon_j^* = \sum_{i=1}^3 A_i(\lambda_j, \check{S}_{t_j}^*) Z_{i,j}\Delta\lambda_j$. Let $\mathcal{M}_k^* = \sum_{j=m_1}^k \upsilon_j^*$, the corresponding stopped martingale. First, we show, throughout Theorem 7.3, that for any $L > 0$, this martingale weakly converges to a mixed Gaussian variable with mean zero and variance $\varsigma^{*2}(L) = \varsigma^2(\check{S}_1^*)$. To this end, setting $\Gamma_{1,\eta} = \{\inf_{t^* \leq u \leq 1} |\ln(S_u^*/K)| > \eta\}$ and $\mathbf{a}_j^* = \mathbf{E}(\upsilon_j^{*2} \mathbf{1}_{\{|\upsilon_j^*|>\delta\}}|\mathcal{F}_{j-1})$, we obtain

$$(7.17) \qquad \mathbf{P}\left(n^{2\beta}| \sum_{j=m_1}^{m_2} \mathbf{a}_j^*| > \varepsilon\right) \leq \mathbf{P}\left(n^{2\beta}| \sum_{j=m_1}^{m_2} \mathbf{a}_j^*| > \varepsilon, \Gamma_{1,\eta}\right) + \mathbf{P}(\Gamma_{1,\eta}^c).$$

It suffices to show that the first probability on the right side of (7.17) converges to zero. Indeed, from the proof of Proposition 7.2, one observes that on the set $\Gamma_{1,\eta}$,

$$(7.18) \qquad \max_{i=1,2,3} \left|A_i(\lambda_u, \check{S}_u^*)\right| \leq \widetilde{U}(\check{S}_u^*)(1 + \lambda_u^{-\gamma}), \quad t^* \leq u \leq t_*,$$

for some $\gamma > 0$ and $\widetilde{U}(\check{S}) = S^{-1/2}U(\check{S})$. Set $\widehat{\upsilon}_j^* = \upsilon_j^* \mathbf{1}_{\Gamma_{3,j}}$ and $\widehat{\mathbf{a}}_j^* = \mathbf{E}(\widehat{\upsilon}_j^{*2}\mathbf{1}_{\{|\widehat{\upsilon}_j^*|>\delta\}}|\mathcal{F}_{j-1})$, where

$$\Gamma_{3,j} = \left\{\max_{1 \leq i \leq 3} \left|A_i(\lambda_u, \check{S}_u^*)\right| \leq \widetilde{U}(\check{S}_u^*)(1 + \lambda_u^{-\gamma})\right\}.$$

We have

$$\mathbf{P}\left(n^{2\beta}| \sum_{j=m_1}^{m_2} \mathbf{a}_j^*| > \varepsilon, \Gamma_{1,\eta,L}\right) = \mathbf{P}\left(n^{2\beta}| \sum_{j=m_1}^{m_2} \widehat{\mathbf{a}}_j^*| > \varepsilon\right) \leq \varepsilon^{-1}n^{2\beta} \sum_{j=m_1}^{m_2} \mathbf{E}\,\widehat{\mathbf{a}}_j^*,$$

by Markov's inequality. By using Chebychev's inequality and then again Markov's inequality, we observe that

$$\mathbf{E}\,\widehat{\mathbf{a}}_j^* \leq \sqrt{\mathbf{E}\,\widehat{\upsilon}_j^{*4}}\sqrt{\mathbf{P}(|\widehat{\upsilon}_j^*| > \delta)} \leq \delta^{-2}\mathbf{E}\,\widehat{\upsilon}_j^{*4} \leq 9\delta^{-2}(1 + \lambda_u^{-\gamma})^4(\Delta\lambda_j)^4 \mathbf{E}\,\widetilde{U}^4(\check{S}_u^*)\sum_{i=1}^3 Z_{i,j}^4.$$

Note that $Z_{i,j}$ has bounded moments. Then, by using (7.18), we obtain $\varepsilon^{-1}n^{2\beta} \sum_{j=m_1}^{m_2} \mathbf{E}\,\widehat{\mathbf{a}}_j^*$ is bounded by $9\varepsilon^{-1}\delta^{-2}n^{2\beta} \sum_{j=m_1}^{m_2}(1 + \lambda_u^{-\gamma})^4(\Delta\lambda_j)^4$, which converges to zero by Lemma A.1.

We now verify the limit of the sum of conditional variances $\mathbf{E}(\upsilon_j^{*2}|\mathcal{F}_{j-1})$. Set $\upsilon_{i,j}^* = A_{i,j-1}^* Z_{i,j} \Delta\lambda_j$. As $Z_{1,j}$ and $Z_{2,j}$ are independent, $\mathbf{E}(\upsilon_{1,j}^* \upsilon_{3,j}^*|\mathcal{F}_{j-1}) = \mathbf{E}(\upsilon_{2,j}^* \upsilon_{3,j}^*|\mathcal{F}_{j-1}) = 0$. It follows that

$$\mathbf{E}(\upsilon_j^{*2}|\mathcal{F}_{j-1}) = \mathbf{E}(\upsilon_{1,j}^{*2}|\mathcal{F}_{j-1}) + \mathbf{E}(\upsilon_{2,j}^{*2}|\mathcal{F}_{j-1}) + \mathbf{E}(\upsilon_{3,j}^{*2}|\mathcal{F}_{j-1}) + 2\mathbf{E}(\upsilon_{1,j}^* \upsilon_{2,j}^*|\mathcal{F}_{j-1}).$$

Now, observe that for $Z \sim N(0, 1)$ and some constant a, $\mathbf{E}(Z|Z + a|) = 2\Phi(a) - 1$ and $\mathbf{E}(Z + a)^2 - (\mathbf{E}|Z + a|)^2 = \Lambda(a)$. On the other hand, $\Delta\lambda_j = n^{-2\beta}(1 + o(1))\breve{\mu}\varrho^{\frac{2}{\mu+1}}\lambda_{j-1}^{\widehat{\mu}}$

by Lemma A.1. Therefore,

$$n^{2\beta} \mathbf{E}(\upsilon_j^{*2}|\mathcal{F}_{j-1}) = (1 + o(1))\check{\mu}\, \varrho^{\frac{2}{\mu+1}}\, \lambda_{j-1}^{\widehat{\mu}}\, \mathbf{L}(\lambda_{j-1}, \check{S}_{t_{j-1}}^*)\Delta\lambda_j.$$

By Lemma A.5, $n^{2\beta} \sum_{j=m_1}^{m_2} \mathbf{E}(\upsilon_j^{*2}|\mathcal{F}_{j-1})$ converges in probability to $\varsigma^{*2}(L)$.

Thus, $n^{\beta}\mathcal{M}_{m_2}^*$ weakly converges to $\mathcal{N}(0, \varsigma^{*2}(L))$ by Theorem 7.3. Moreover, property (7.12) implies that for any $\delta > 0$,

$$\lim_{L\to\infty} \overline{\lim_{n\to\infty}} \mathbf{P}(n^{\beta}|\mathcal{M}_{m_2} - \mathcal{M}_{m_2}^*| > \delta) = 0.$$

Therefore, by taking into account that $\varsigma^{*2}(L)$ converges a.s. to ς^2 as $L \to \infty$, we conclude that $n^{\beta}\mathcal{M}_{m_2}$ converges in law to $\mathcal{N}(0, \varsigma^2)$. This completes the proof.

Next, we study the asymptotic property of the following martingale

$$(7.19) \qquad \overline{\mathcal{M}}_k = \sum_{j=m_1}^{k} \left(A_{1,j-1}\, Z_{1,j} + A_{2,j-1}\, Z_{2,j} + A_{4,j-1}\, Z_{4,j} \right) \Delta\lambda_j.$$

The limiting variance will be defined throughout the function

$$(7.20) \qquad \overline{\mathbf{L}}(\lambda, x, y) = A_1^2(\lambda, x, y) + A_2^2(\lambda, x, y) + (1 - 2/\pi) A_4^2(\lambda, x, y).$$

The following result is similar to Proposition 7.4.

PROPOSITION 7.5. *Let $A_i^0 = A_i^0(\lambda, x, y)$, $i = 1, 2, 4$ be functions having property* (**H**) *and $A_i(\lambda, x, y) = A_i^0(\lambda, x, y)\widetilde{\varphi}(\lambda, x)$. Then, for any fixed $\varrho > 0$, the sequence $(n^{\beta}\, \overline{\mathcal{M}}_{m_2})_{n\geq 1}$ weakly converges to a mixed Gaussian variable with mean zero and variance $\overline{\varsigma}^2$ given by $\overline{\varsigma}^2 = \check{\mu}\, \varrho^{\frac{2}{\mu+1}} \int_0^{+\infty} \lambda^{\widehat{\mu}}\, \overline{\mathbf{L}}(\lambda, \check{S}_1)\mathrm{d}\lambda$. The same property still holds if some (or all) A_i are of the form $\int_{\lambda}^{\infty} A_i^0(z, x, y)\widetilde{\varphi}(z, x)\mathrm{d}z$.*

Proof. The conclusion follows directly from the proof of Proposition 7.4 and the observation that $\mathbf{E}\, Z_{4,j}^2 = \mathbf{E}(|Z_{1,j}| - \sqrt{2/\pi})^2 = 1 - 2/\pi$, and $\mathbf{E}(Z_{i,j} Z_{4,j}) = 0$, for $i = 1, 2$ and $m_1 \leq j \leq m_2$. □

In the rest of the subsection, we establish a limit theorem for a martingale of the following form:

$$\check{\mathcal{M}}_k = \sum_{j=m_1}^{k} \left(A_{1,j-1}\, Z_{1,j} + A_{3,j-1}\, Z_{3,j} \right) \Delta\lambda_j := \sum_{j=m_1}^{k} \check{\upsilon}_j, \quad m_1 \leq k \leq m_2,$$

where $A_i(\lambda, x, y) = A_i^0(\lambda, x, y)\widetilde{\varphi}(\lambda, x)$ and A_i^0, $i = 1, 3$ are functions having property (**H**). The following result is helpful for the case when ϱ diverges to infinity as in Theorem 3.6.

PROPOSITION 7.6. *Under condition* (**C$_2$**), *the sequence $(n^{\beta} \varrho^{\frac{-1}{\mu+1}}\, \check{\mathcal{M}}_{m_2})$ weakly converges to a mixed Gaussian variable with mean zero and variance $\check{\varsigma}^2 = \check{\mu} \int_0^{+\infty} \lambda^{\widehat{\mu}}\, \check{\mathbf{L}}(\lambda, S_1)\mathrm{d}\lambda$, where $\check{\mathbf{L}}(\lambda, x, y) = A_1^2(\lambda, x, y) + 2 A_1(\lambda, x, y) A_3(\lambda, x, y) + A_3^2(\lambda, x, y)$. The same property still holds if some (or all) A_i are of the form $\int_{\lambda}^{\infty} A_i^0(z, x, y)\widetilde{\varphi}(z, x)\mathrm{d}z$.*

Proof. We determine the limit of conditional variances of $n^{\beta} \varrho^{\frac{-1}{\mu+1}}\, \check{\mathcal{M}}_{m_2}$. We first observe that

$$(7.21) \qquad n^{2\beta} \varrho^{\frac{-2}{\mu+1}}\, \mathbf{E}(\check{\upsilon}_j^2|\mathcal{F}_{j-1}) = \check{\mu}(1 + o(1)) \lambda_{j-1}^{\widehat{\mu}}\, \check{\mathbf{Q}}(\lambda_{j-1}, \check{S}_{t_{j-1}})\Delta\lambda_j,$$

where $\check{\mathbf{Q}}(\lambda, x, y) = A_1^2(\lambda, x, y) + A_3^2(\lambda, x, y)\, \Lambda(p) + 2\, A_1(\lambda, x, y)\, A_3(\lambda, x, y)(2\Phi(|p|) - 1)$. Moreover, it can be checked directly that the function $G(\cdot)$ defined in (7.7) satisfies the following inequalities: $|a| \le G(a) \le |a| + 2\varphi(a)$, for any $a \in \mathbb{R}$. This implies that $|\Lambda(a) - 1| \le 4|a|\varphi(a) + \varphi^2(a)$, hence, $\sup_{a \in \mathbb{R}} |\Lambda(a)| < \infty$. Also, note that $\check{\mathbf{Q}} \to \check{\mathbf{L}}$ a.s. as $n \to \infty$ because $p(\lambda, x, y) \to \infty$ as $\varrho = \varrho(n) \to \infty$, for any $x > 0$ and $\lambda \ne 2\ln(x/K)$. Using Lemma A.5, we claim that the sum of the terms on the right-hand side of (7.21) converges in probability to ξ^2. The proof is completed by running again the argument in the proof of Proposition 7.4. □

7.4. Proof of Theorem 3.3

We first observe that $I_{1,n}$ approaches $2\min(S_1, K)$ at order θ_n. In particular, setting $\bar{I}_{1,n} = \int_0^1 \lambda_t^{-1/2} \hat{\sigma}_t^2 (S_t \tilde{\varphi}(\lambda_t, S_t) - S_1 \tilde{\varphi}(\lambda_t, S_1))\, \mathrm{d}t$ and changing variables $v = \int_t^1 \hat{\sigma}_s^2 \mathrm{d}s$, we can represent $I_{1,n} = S_1 \int_0^{\lambda_0} v^{-1/2} \tilde{\varphi}(v, S_1)\, \mathrm{d}v + \bar{I}_{1,n} + o(\theta_n^{-1})$. The first integral in the right side converges a.s. to $2\min(S_1, K)$ by (2.12), while $\bar{I}_{1,n}$ is approximated by $\int_0^1 \hat{\sigma}_t^2 (\int_t^1 \sigma(y_u) S_u H(\lambda_t, S_u) \mathrm{d}W_u^{(1)}) \mathrm{d}t$, where $H = (2^{-1}\lambda^{-1/2} - \lambda^{-3/2} \ln(x/K))\tilde{\varphi}(\lambda, x)$. The discretization technique of Proposition 7.2 can be applied to replace the latter double integral by $\mathcal{U}_{1,m_2}$, defined by

$$(7.22) \qquad \mathcal{U}_{1,k} = \varrho^{-1} \sum_{j=m_1}^{k} \sigma(y_{t_{j-1}}) S_{t_{j-1}} \check{H}_{j-1}\, Z_{1,j}\, \Delta\lambda_j, \quad m_1 \le k \le m_2,$$

where $\check{H}(\lambda, x) = \int_\lambda^\infty (z^{-1/2}/2 - z^{-3/2} \ln(x/K))\tilde{\varphi}(z, x)\mathrm{d}z$. We summarize the asymptotic form of $I_{1,n}$ in the following.

PROPOSITION 7.7. *If ϱ either is constant or satisfies condition (C$_2$), then*

$$\mathbf{P} - \lim_{n \to \infty} \theta_n \left| I_{1,n} - 2\min(S_1, K) - \mathcal{U}_{1,m_2} \right| = 0.$$

Next, we claim that $I_{2,n} = o(\theta_n^{-1})$.

PROPOSITION 7.8. *If ϱ either is a positive constant or satisfies condition (C$_2$), then $\theta_n I_{2,n}$ converges to zero in probability as $n \to \infty$.*

Proof. See Appendix A.3.

Let us study the trading volume J_n. First, it is easy to check that for any $v > 0$, $1 - \Phi(v) \le v^{-1}\varphi(v)$. Now, observe that $|\gamma_{t_i}^n - \gamma_{t_{i-1}}^n| \le |1 - \gamma_{t_i}^n| + |1 - \gamma_{t_{i-1}}^n|$, which almost surely converges to zero more rapidly than any power of n when $\inf_{1 \le i \le n} \lambda_i \ge l^* \iff i \le m_1$. The same property can be deduced for the case $\sup_i \lambda \le l_* \iff i \ge m_2$. To see this, we note that for $\lambda_u \le l_*$, $S_u(\omega) = S_{1-(\lambda_u/\lambda_0)^{4\beta}}(\omega)$ converges to $S_1(\omega)$ as $n \to \infty$ uniformly in $\lambda_u \in [0, l_*]$, for any ω outside the zero probability set $\{S_1 = K\}$. Therefore, one can truncate the sum and keep only the part corresponding to index $m_1 \le j \le m_2$. In other words, J_n is approximated by $J_{1,n} = \sum_{j=m_1}^{m_2} S_{t_j} |\Delta\Phi_j|$. Putting $b_j = |\Delta\Phi_j| - \tilde{\varphi}_{j-1}|\Delta\mathbf{v}_j|$, we can represent $J_{1,n} = J'_{1,n} + \varepsilon_{1,n} + \varepsilon_{2,n}$, where $J'_{1,n} = \sum_{j=m_1}^{m_2} S_{t_{j-1}} \tilde{\varphi}_{j-1}|\Delta\mathbf{v}_j|$, $\varepsilon_{1,n} = \sum_{j=m_1}^{m_2} \Delta S_{t_{j-1}} |\Delta_j \Phi|$, and $\varepsilon_{2,n} = \sum_{j=m_1}^{m_2} S_{t_{j-1}} b_j$.

In view of (A.1) and condition (C$_2$), $\varepsilon_{1,n} = o(\theta_n^{-1})$ as $n \to \infty$. Furthermore, by using the Taylor expansion, we obtain $|\varepsilon_{2,n}| \le S_{\sup} \sum_{j=m_1}^{m_2} |\Delta\mathbf{v}_j|^2$ up to a multiple constant,

where $S_{\sup} = \sup_{0 \le t \le 1} S_t$. Now, by taking into account that

$$\mathbf{E}\left|\mathbf{v}_{j-1} - \mathbf{v}_j\right|^2 \le \frac{1}{n\lambda_{j-1}} + \left(\lambda_{j-1}^{1/2} - \lambda_j^{1/2}\right)^2 + \left(\lambda_{j-1}^{-1/2} - \lambda_j^{-1/2}\right)^2$$

up to a multiple constant and using condition $(\mathbf{C}_2)$ together with (A.1), we get $|\varepsilon_{2,n}| = o(\theta_n^{-1})$. Next, by using Itô's Lemma and the substitution $\lambda_j = \lambda_0 (1 - t_j)^{4\beta}$, we replace $J'_{1,n}$ by

$$(7.23) \quad J_{2,n} = \sum_{j=m_1}^{m_2} \lambda_{j-1}^{-1/2} S_{t_{j-1}} \widetilde{\varphi}_{j-1} |\varkappa_j| \Delta\lambda_j := \sum_{j=m_1}^{m_2} \zeta_j, \quad \varkappa_j = \varrho^{-1}\sigma(y_{t_{j-1}})Z_{1,j} + q_{j-1},$$

where q is defined in (2.10). We need to determine the limit of J_n throughout the Doob's decomposition of $J_{2,n}$ w.r.t. the filtration $(\mathcal{F}_j)_{m_1 \le j \le m_2}$. To this end, note that

$$\mathbf{E}(\zeta_j|\mathcal{F}_{j-1}) = \lambda_{j-1}^{-1/2} S_{t_{j-1}} \widetilde{\varphi}_{j-1} \Delta\lambda_j \, \mathbf{E}(|\varkappa_j||\mathcal{F}_{j-1}),$$

where $\mathbf{E}(|\varkappa_j||\mathcal{F}_{j-1}) = \varrho^{-1}\sigma(y_{t_{j-1}})G(p_{j-1}) := D_{j-1}$ and $G(p)$ defined in (7.7). Let

$$(7.24) \qquad B(\lambda, x, y) = \lambda^{-1/2} x\widetilde{\varphi}(\lambda, x)D(\lambda, x, y) \quad \text{and} \quad J_{3,n} = \sum_{j=m_1}^{m_2} B_{j-1}\Delta\lambda_j.$$

We observe that $J_{2,n} = J_{3,n} + \mathcal{U}_{2,m_2}$, where

$$(7.25) \qquad \mathcal{U}_{2,k} = \sum_{j=m_1}^{k} \lambda_{j-1}^{-1/2} S_{t_{j-1}} \widetilde{\varphi}_{j-1} \overline{\varkappa}_j \Delta\lambda_j \quad \text{and} \quad \overline{\varkappa}_j := |\varkappa_j| - D_{j-1}.$$

By substituting $\check{S}_{t_{j-1}}$ by $\check{S}_1$ everywhere in $J_{3,n}$, we write $J_{3,n} = J_{4,n} + J_{5,n}$, where $J_{4,n} = \sum_{j=m_1}^{m_2} B(\lambda_{j-1}, \check{S}_1)\Delta\lambda_j$, $J_{5,n} = \sum_{j=m_1}^{m_2} B_{j-1}^* \Delta\lambda_j$, and $B_{j-1}^* = B(\lambda_{j-1}, \check{S}_{t_{j-1}}) - B(\lambda_{j-1}, \check{S}_1)$. Then, by Lemma A.2, we can check that $J_{4,n}$ converges a.s. to $J(S_1, y_1, \varrho)$ at rate θ_n. Now, an application of Itô's Lemma for B_{j-1}^* yields stochastic integrals with respect to the Wiener processes. Owing to Proposition 7.2, the sum of these integrals can be approximated by $\mathcal{U}_{3,m_2}$, defined by

$$\mathcal{U}_{3,k} = \varrho^{-1} \sum_{i=1}^{2} \sum_{j=m_1}^{k} Q_{i,j-1} Z_{i,j} \Delta\lambda_j, \quad m_1 \le k \le m_2,$$

where $Q_1 = \int_\lambda^\infty (x\sigma(y)\partial_x B + \mathbf{r}F_2(t(\lambda), y)\partial_y B)\mathrm{d}z$ and $Q_2 = \sqrt{1 - \mathbf{r}^2}F_2(t(\lambda), y)\int_\lambda^\infty \partial_y B\mathrm{d}z$ and $t(\lambda) = 1 - (\lambda/\lambda_0)^{4\beta}$. The asymptotic form of J_n is summarized in the following.

PROPOSITION 7.9. *For any fixed* $\varrho > 0$,

$$\mathbf{P} - \lim_{n \to \infty} \theta_n \left|J_n - J(S_1, y_1, \varrho) - (\mathcal{U}_{2,m_2} + \mathcal{U}_{3,m_2})\right| = 0.$$

Now, the martingale part of the hedging error is given by $\mathcal{M}_{m_2}$, defined by

$$\mathcal{M}_k = \frac{1}{2}\mathcal{U}_{1,k} - \kappa_*(\mathcal{U}_{2,k} + \mathcal{U}_{3,k}) = \varrho^{-1}\sum_{j=m_1}^{k}\sum_{i=1}^{3} A_{i,j-1}Z_{i,j}\Delta\lambda_j, \quad m_1 \le k \le m_2,$$

where $A_1 = -\sigma(y)\,x\check{H}/2$, $A_2 = \kappa_* Q_2$ and $A_3 = -\kappa_*\sigma(y)\lambda^{-1/2}x\widetilde{\varphi}(\lambda, x)$. It is easy to see that the assumption of Proposition 7.4 is fulfilled for A_i, $i = 1, 2, 3$. Hence, the sequence $(n^\beta \mathcal{M}_{m_2})_{n\geq 1}$ converges in law to a mixed Gaussian variable by Proposition 7.4, which proves Theorem 3.3. $\qquad\square$

7.5. Proof of Theorem 3.6

When $\varrho \to \infty$ under condition $(\mathbf{C}_2)$, the approximation for J_n is slightly different. In particular, observe first that for any $b \in \mathbb{R}$, $\mathbf{E}|aZ + b|$ can be approximated by $b(2\Phi(b/a) - 1)$ as $a \to 0$.

Therefore, we can replace $J_{3,n}$ in (7.24) by the sum $\widehat{J}_{3,n} = \sum_{j=m_1}^{m_2} \widehat{B}_{j-1}\Delta\lambda_j$, where $\widehat{B}(\lambda, x) = \lambda^{-1/2}\,x\widetilde{\varphi}(\lambda, x)q(\lambda, x)\widetilde{\Phi}(\varrho q(\lambda, x))$, with $\widetilde{\Phi}(q) = 2\Phi(\varrho\, q) - 1$ and $q(\lambda, x)$ defined in (2.10). Putting $\widehat{J}_{4,n} = \sum_{j=m_1}^{m_2} \widehat{B}(\lambda_{j-1}, S_1)\Delta\lambda_j$ and $\widehat{B}^*_{j-1} = \widehat{B}(\lambda_{j-1}, S_{t_{j-1}}) - \widehat{B}(\lambda_{j-1}, S_1)$, we represent $\widehat{J}_{5,n} := \widehat{J}_{3,n} - \widehat{J}_{4,n} = \sum_{j=m_1}^{m_2} \widehat{B}^*_{j-1}\Delta\lambda_j$. Now, using Lemma A.2, we can directly show that $|\widehat{J}_{4,n} - J^*(S_1)| = o(\theta_n^{-1})$. Furthermore, owing to Itô's formula, we replace $\widehat{B}^*_{j-1}$ by $\int_{t_{j-1}}^1 \partial_x \widehat{B}(\lambda_{j-1}, S_u)\mathrm{d}S_u$. Direct calculations yield that

$$\partial_x \widehat{B} = \lambda^{-1/2}\widetilde{\varphi}(\lambda, x)[-2q^2(\lambda, x)\widetilde{\Phi}(\lambda, x) + \frac{1}{2\lambda}\widetilde{\Phi}(\lambda, x) + \frac{\varrho}{\lambda}\varphi(\varrho\, q(\lambda, x))].$$

Clearly, $\widetilde{\Phi}(\varrho q) \to \mathrm{sign}(q)$ and $\varphi(\varrho\, q) \to 0$ as $\varrho \to \infty$. Now, using Proposition 7.2, we can approximate $\widehat{J}_{5,n}$ by $\widehat{\mathcal{U}}_{3,m_2}$, defined by $\widehat{\mathcal{U}}_{3,k} = \varrho^{-1}\sum_{j=m_1}^k \sigma(y_{t_{j-1}})S_{t_{j-1}}N_{j-1}\,Z_{1,j}\Delta\lambda_j$, where $N(\lambda, x) = \int_\lambda^{+\infty} z^{-1/2}\widetilde{\varphi}(z, x)(-2q^2(z, x) + 1/(2z))\mathrm{sign}(q(z, x))\mathrm{d}z$. The asymptotic representation of the trading volume is summarized in the following.

PROPOSITION 7.10. *Under conditions* $(\mathbf{C}_1) - (\mathbf{C}_2)$,

$$\mathbf{P} - \lim_{n\longrightarrow\infty} \theta_n|J_n - J^*(S_1) - (\mathcal{U}_{2,m_2} + \widehat{\mathcal{U}}_{3,m_2})| = 0.$$

Now, the martingale part $\varrho^{-1}\check{\mathcal{M}}_{m_2}$ of the hedging error is determined by

$$\check{\mathcal{M}}_k = \frac{\varrho}{2}\mathcal{U}_{1,k} - \kappa_*\varrho(\mathcal{U}_{2,k} + \widehat{\mathcal{U}}_{3,k}) = \sum_{j=m_1}^k (\check{A}_{1,j-1}Z_{1,j} + \check{A}_{3,j-1}Z_{3,j})\Delta\lambda_j,$$

where $\check{A}_i$, $i = 1, 2$, are explicitly determined and satisfy the assumption of Proposition 7.6. Then, in view of $\theta_n\varrho^{-1}\check{\mathcal{M}}_{m_2} = n^\beta\varrho^{-\frac{1}{\mu+1}}\check{\mathcal{M}}_{m_2}$, Theorem 3.6 is proved throughout Proposition 7.6.

7.6. Proof of Theorem 3.8

The key technique in Proposition 7.2 can be used to obtain a smart martingale approximation for the sum $\sum_{i\geq 1}\Delta S_{t_i}\int_0^{t_{i-1}}\widehat{C}_{xt}(u, S_u)\mathrm{d}u$.

PROPOSITION 7.11. *If ϱ either is a positive constant or satisfies condition* $(\mathbf{C}_2)$, *then* $|\overline{I}_{2,n} - \overline{\mathcal{U}}_{1,m_2}| = o(\theta_n^{-1})$, *where* $Y(\lambda, x) = \int_\lambda^\infty z^{-3/2}\ln(x/K)\widetilde{\varphi}(z, x)\mathrm{d}z$ *and*

$$\overline{\mathcal{U}}_{1,k} = \varrho^{-1}\sum_{j=m_1}^k \sigma(y_{t_{j-1}})S_{t_{j-1}}Y_{j-1}\,Z_{1,j}\,\Delta\lambda_j, \quad m_1 \leq k \leq m_2.$$

Proof. The proof follows from the substitution ΔS_{t_j} by $\varrho^{-1}\sigma(y_{t_{j-1}})S_{t_{j-1}}\Delta\lambda_{t_j}$ as in Proposition 7.2.

$\square$

Let us now study the trading volume $\overline{J}_n$ by following the procedure in the approximation of J_n. First, by Itô's lemma,

$$\overline{\gamma}_{t_i} - \overline{\gamma}_{t_{i-1}} = \int_{t_{i-1}}^{t_i} \widehat{C}_{xx}(u, S_u)\mathrm{d}S_u + \frac{1}{2}\int_{t_{i-1}}^{t_i} \widehat{C}_{xxx}(u, S_u)\sigma^2(y_u)S_u^2\mathrm{d}u,$$

where the time correction, which involves the term q_{j-1} in the formula of $\varkappa_j$ defined by (7.23), has been removed. We now approximate $\overline{J}_n$ by

$$\overline{J}_{1,n} = \varrho^{-1}\sum_{j=m_1}^{m_2}\overline{B}_{j-1}\left|Z_{1,j}\right|\Delta\lambda_j \quad\text{and}\quad \overline{B}(\lambda, x, y) = \sigma(y)\,x\,\lambda^{-1/2}\widetilde{\varphi}(\lambda, x).$$

As $\mathbf{E}|Z| = \sqrt{2/\pi}$, for $Z \sim \mathcal{N}(0, 1)$, the Dood' decomposition of $\overline{J}_{1,n}$ is given by $\overline{J}_{2,n} + \bar{\mathcal{U}}_{2,m_2}$, where $\overline{J}_{2,n} = \varrho^{-1}\sqrt{2/\pi}\sum_{j=m_1}^{m_2}\overline{B}_{j-1}\Delta\lambda_j$ and $\bar{\mathcal{U}}_{2,m_2} = \varrho^{-1}\sum_{j=m_1}^{m_2}\overline{B}_{j-1}Z_{4,j}\Delta\lambda_j$. Now, putting $\overline{B}_{j-1}^* = \overline{B}(\lambda_{j-1}, \check{S}_{t_{j-1}}) - \overline{B}(\lambda_{j-1}, \check{S}_1)$, we write $\overline{J}_{2,n} = \overline{J}_{4,n} + \overline{J}_{3,n}$, where

$$\overline{J}_{4,n} = \varrho^{-1}\sqrt{2/\pi}\sum_{j=m_1}^{m_2}\overline{B}_{j-1}\Delta\lambda_j, \quad \overline{J}_{3,n} = \varrho^{-1}\sqrt{2/\pi}\sum_{j=m_1}^{m_2}\overline{B}_{j-1}^*\Delta\lambda_j.$$

Observe that $\overline{J}_{4,n}$ converges a.s. to $\eta\min(S_1, K)$ by Lemma A.2 and (2.12). We now find the suitable martingale approximation for $\overline{J}_{3,n}$. By Itô's formula, $\overline{B}_{j-1}^*$ can be replaced by $\sum_{i=1}^{2}\int_t^1 \overline{Q}_i(\lambda_{j-1}, \check{S}_u)\mathrm{d}W_u^{(i)}$, where $\overline{Q}_1 = \sigma(y)x\partial_x\overline{B} + \mathbf{r}F_2(t(\lambda), y)\partial_y\overline{B}$ and $\overline{Q}_2 = \sqrt{1-\mathbf{r}^2}F_2(t(\lambda), y)\partial_y\overline{B}$. Direct calculations show that $\partial_x\overline{B} = \sigma(y)(2^{-1}\lambda^{-1/2} - \lambda^{-3/2}\ln(X/K))\widetilde{\varphi}(\lambda, x)$ and $\partial_y\overline{B} = \sigma'(y)\lambda^{-1/2}x\widetilde{\varphi}(\lambda, x)$. Now, Proposition 7.2 can be applied to approximate $\overline{J}_{3,n}$ by the martingale $\bar{\mathcal{U}}_{3,m_2}$, defined by

$$\bar{\mathcal{U}}_{3,k} = \varrho^{-1}\sum_{j=m_1}^{k}(\overline{A}_{1,j-1}Z_{1,j} + \overline{A}_{2,j-1}Z_{2,j})\Delta\lambda_j, \quad m_1 \le k \le m_2,$$

for explicit functions $\overline{A}_i$, $i = 1, 2$. The final asymptotic form of $\overline{J}_n$ is given below.

PROPOSITION 7.10. *If ϱ is a positive constant independent of n then,*

$$\mathbf{P} - \lim_{n\to\infty}\theta_n|\overline{J}_n - \eta\min(S_1, K) - (\overline{\mathcal{U}}_{2,m_2} + \overline{\mathcal{U}}_{3,m_2})| = 0.$$

Hence, the martingale part of the hedging error for Lépinette's strategy is determined by $\overline{\mathcal{M}}_{m_2} = \mathcal{U}_{1,m_2} + \overline{\mathcal{U}}_{1,m_2} - \kappa_*(\overline{\mathcal{U}}_{2,m_2} + \overline{\mathcal{U}}_{3,m_2})$. The latter martingale sum can be represented in the form

$$\overline{\mathcal{M}}_k = \varrho^{-1}\sum_{j=m_1}^{k}(A_{1,j-1}Z_{1,j} + A_{4,j}Z_{4,j-1} + A_{2,j-1}Z_{2,j})\Delta\lambda_j, \quad m_1 \le k \le m_2,$$

for explicit functions A_i holding the assumption of Proposition 7.5. Then, $(n^\beta\overline{\mathcal{M}}_{m_2})_{n\ge1}$ converges in law to a mixed Gaussian variable, which completes the proof. $\square$

8. CONCLUSION

We studied the problem of approximate option replication in SV settings using a new specification for adjusted volatility. Although our model employed a simpler adjusted volatility than in the previous literature, we obtain the same asymptotic results for both Leland' and Lépinette's strategies in general SV markets. A possible connection to high-frequency markets with proportional transaction costs was also discussed. As an application, we showed that the option price inclusive of transaction costs can be reduced by adapting the theory of quantile hedging. Note that our approach is still helpful for more general settings, for example, when the friction rule admits a separate variable representation (Nguyen 2013). This generalization includes the case where trading costs are based on the physical number of traded shares. Lastly, in a companion paper (Nguyen 2014), we extended the method to multidimensional frameworks for European options with general payoffs written on several assets.

APPENDIX A: AUXILIARY LEMMAS

LEMMA A.1. *There exist two positive constants C_1, C_2 such that*

$$(A.1) \quad C_1 \, n^{-2\beta} \varrho^{\frac{2}{\mu+1}} v_0(l_*) \leq \inf_{m_1 \leq j \leq m_2} |\Delta\lambda_j| \leq \sup_{m_1 \leq j \leq m_2} |\Delta\lambda_j| \leq C_2 n^{-2\beta} \varrho^{\frac{2}{\mu+1}} v_0(l^*),$$

where $v_0(x) = x^{(\mu-1)/(\mu+1)}$. Moreover, for any $m_1 \leq j \leq m_2$,

$$(A.2) \quad \Delta\lambda_j = n^{-2\beta} \varrho^{\frac{2}{\mu+1}} v_0(\lambda_{j-1})(1+o(1)) \quad and \quad \Delta\lambda_j \, (\Delta t_j)^{-1/2} = \varrho(1+o(1)).$$

Proof. It follows directly from the relation (7.3). □

A technical condition ($\mathbf{H}_0$): $A : \mathbb{R}_+ \to \mathbb{R}$ *is a continuously differentiable function having absolutely integrable derivative A' and*

$$\lim_{n\to\infty} \theta_n \left(\int_0^{l_*} |A(\lambda)| d\lambda + \int_{l^*}^{+\infty} |A(\lambda)| d\lambda \right) = 0, \quad where \quad \theta_n = n^\beta \varrho^{2\beta}.$$

The following result is straightforward to check.

LEMMA A.2. *Let ϱ either be a positive constant or satisfy condition ($\mathbf{C}_2$). Then, for any function A satisfying condition ($\mathbf{H}_0$),*

$$(A.3) \quad \lim_{n\to\infty} \theta_n \left| \sum_{j=m_1}^{m_2} \mathbf{1}_{\{\lambda_{j-1} \geq a\}} A(\lambda_{j-1}) \Delta\lambda_j - \int_a^\infty A(\lambda) d\lambda \right| = 0.$$

In particular, $\lim_{n\to\infty} \theta_n | \sum_{j=m_1}^{m_2} A(\lambda_{j-1})\Delta\lambda_j - \int_0^\infty A(\lambda)d\lambda| = 0$.

LEMMA A.3. *For any $K > 0$, $\lim_{\varepsilon\to 0} \limsup_{v\to 1} \mathbf{P}(\inf_{v \leq u \leq 1} |\ln(S_u/K)| \leq \varepsilon) = 0$.*

Proof. It follows from the fact that conditioning on the σ-field generated by the volatility process, the log-price process $\ln S_t$ has Gaussian distribution. □

LEMMA A.4. *Suppose that A_0 and its derivatives $\partial_x A_0$, $\partial_y A_0$ verify condition* (**H**). *Set $A(\lambda, x, y) = A_0(\lambda, x, y)\widetilde{\varphi}(\lambda, x)$, $\bar{A}(\lambda, x, y) = \int_\lambda^\infty A(z, x, y)\mathrm{d}z$ and define*

$$r_n = \sup_{(z,r,d)\in[l_*,l^*]\times\mathcal{B}} \left(|\partial_\lambda \bar{A}(z, r, d)| + |\partial_x \bar{A}(z, r, d)| + |\partial_y A(z, r, d)|\right),$$

where $\mathcal{B} = [S_{\min}, S_{\max}] \times [y_{\min}, y_{\max}]$ with $S_{\min} = \inf_{t^*\leq u\leq t_*} S_u$, $S_{\max} = \sup_{t^*\leq u\leq t_*} S_u$ and $y_{\min} = \inf_{t^*\leq u\leq t_*} y_u$, $y_{\max} = \sup_{t^*\leq u\leq t_*} y_u$. Then, $\lim_{b\to\infty} \overline{\lim}_{n\to\infty} \mathbf{P}(r_n > b) = 0$.

Proof. Let $\varepsilon > 0$. On the set $\Gamma_{1,\varepsilon} = \{\inf_{t^*\leq u\leq 1} |\ln(S_u/K)| \geq \varepsilon\}$,

$$\sup_{S_{\min}\leq r\leq S_{\max}} \widetilde{\varphi}(q, r) \leq (2\pi)^{-1/2}\sqrt{Kr^{-1}} \exp\{-\varepsilon^2/(2q) - q/8\}.$$

By condition (**H**), there exists $\gamma > 0$ such that

$$|\bar{A}_x(z, r, d)| \leq C|\widetilde{U}(r, d)| \int_z^\infty (q^{-1/2} + q^\gamma)e^{-\varepsilon^2/(2q)-q/8}\mathrm{d}q \leq C_\epsilon\widetilde{U}(r, d),$$

where $\widetilde{U}$ is some function verifying $\sup_{0\leq t\leq 1} \mathbf{E}\,\widetilde{U}(\check{S}_t^*) < \infty$. For any $\eta > 0$ and $N > 0$, let

$$\Gamma_{2,\eta} = \{ \sup_{(r,d)\in\mathcal{B}} |\widetilde{U}(r, d) - \widetilde{U}(\check{S}_1)| < \eta \}\bigcap\{|\widetilde{U}(\check{S}_1)| < N\}.$$

It is clear that $|\widetilde{U}(r, d)| < N + \eta$ on the set $\Gamma_{2,\eta}$. Similarly, taking into account $\partial_\lambda \bar{A}(z, r, d) = -A(z, r, d)$, $\partial_y \bar{A}(z, r, d) = \int_\lambda^\infty \partial_y A_0(z, x, y)\widetilde{\varphi}(z, x)\mathrm{d}z$, we deduce that both $|\partial_\lambda \bar{A}(z, r, d)|$ and $|\partial_y \bar{A}(z, r, d)|$ are bounded on $\Gamma_{2,\eta}$ by a constant $C_{N,\eta}$ independent of b. Now, for $b > N + \eta + 2C_{N,\eta}$, $\mathbf{P}(r_n > b)$ is bounded by

$$\mathbf{P}(\Gamma_{1,\varepsilon}^c) + \mathbf{P}(\sup_{(r,d)\in\mathcal{B}} |\widetilde{U}(r, d) - \widetilde{U}(\check{S}_1^*)| \geq \eta) + \mathbf{P}(|\widetilde{U}(\check{S}_1^*)| > N) + \mathbf{P}(\tau^* < 1).$$

By Lemma A.3, $\lim_{\varepsilon\to 0} \lim_{n\to\infty} \mathbf{P}(\Gamma_{1,\varepsilon}^c) \to 0$. Due to the continuity of the functions S_t and y_t, one gets $\lim_{n\to\infty} \mathbf{P}(\sup_{(r,d)\in\mathcal{B}} |\widetilde{U}(r, d) - \widetilde{U}(\check{S}_1^*)| \geq \eta) = 0$. Moreover, the integrability of $\widetilde{U}(\check{S}_1^*)$ implies that $\mathbf{P}(|\widetilde{U}(\check{S}_1^*)| > N)$ converges to zero as $N \Rightarrow \infty$. By (7.12), $\mathbf{P}(\tau^* < 1)$ converges to 0 as $L \to \infty$, which completes the proof. $\square$

LEMMA A.5. *Let $\overline{A}(\lambda, x, y) = \int_\lambda^\infty A^0(z, x, y)\widetilde{\varphi}(z, x)\mathrm{d}z$, $\widetilde{A} = \overline{A}^2$, where A^0 is a function having property* (**H**). *Then, for any $\gamma > 0$,*

$$\mathbf{P} - \lim_{n\to\infty} \left|\sum_{j=m_1}^{m_2} \lambda_{j-1}^\gamma \widetilde{A}(\lambda_{j-1}, \check{S}_{t_{j-1}})\Delta\lambda_j - \int_0^\infty \lambda^\gamma \widetilde{A}(\lambda, \check{S}_1)\mathrm{d}\lambda\right| = 0,$$

where $\check{S}_t = (S_t, y_t)$. The same property still holds if $\overline{A}(\lambda, x, y) = A^0(\lambda, x, y)\widetilde{\varphi}(x, y)$ or is a product of these above kinds.

Proof. We prove for the first case $\overline{A}(\lambda, x, y) = \int_\lambda^\infty A^0(z, x, y)\widetilde{\varphi}(z, x)\mathrm{d}z$, as the same argument can be made for the other cases. First, we split the expression under the absolute sign as $\sum_{j=m_1}^{m_2} \lambda_{j-1}^\gamma \widetilde{A}(\lambda_{j-1}, \check{S}_1)\Delta\lambda_j + \sum_{j=m_1}^{m_2} \Delta_{j,n}\Delta\lambda_j$, where $\Delta_{j,n} = \widehat{A}(\lambda_{j-1}, \check{S}_{t_{j-1}}) - \widehat{A}(\lambda_{j-1}, \check{S}_1)$ and $\widehat{A}(\lambda, x, y) = \lambda^\gamma \widetilde{A}(\lambda, x, y)$. It is clear that for any (x, y), the function $\widehat{A}(\cdot, x, y)$ satisfies condition (**H$_0$**). Hence, $\sum_{j=m_1}^{m_2} \widehat{A}(\lambda_{j-1}, \check{S}_1)\Delta\lambda_j$ converges a.s. to zero

by Lemma A.2. It remains to show that $\mathbf{P}(|\Delta_n| > \varepsilon) \to 0$ for any given $\varepsilon > 0$, but it can be done by the same way as in Lemma A.3. $\qquad\square$

APPENDIX B: PROOF OF PROPOSITION 7.8

The singularity of $\widehat{C}$ at the maturity $T = 1$ requires a separate treatment. Let $\varepsilon_n = n^{-2\beta}\varrho^{-4\beta}l_*$. We then represent $I_{2,n} = \int_0^{1-\varepsilon_n} \varpi_n(t)\mathrm{d}W_t^{(1)} + \int_{1-\varepsilon_n}^1 \varpi_n(t)\mathrm{d}W_t^{(1)}$, where $\varpi_n(t) = (\gamma_t^n - \widehat{C}_x(t, S_t))\sigma(y_t)S_t$. Taking into account that $|\gamma_t^n - \widehat{C}_x(t, S_t)| \leq 1$, we obtain $\lim_{n\to\infty} \theta_n^2 \mathbf{E} \int_{1-\varepsilon_n}^1 \varpi_n^2(t)\mathrm{d}t = 0$. Now put $\widehat{t}_j = \min(t_j, 1 - \varepsilon_n)$. It then remains to prove that $\sum_{j=1}^n \int_{\widehat{t}_{j-1}}^{\widehat{t}_j} \mathbf{E}(\gamma_t^n - \widehat{C}_x(t, S_t))^2\mathrm{d}t = o(\theta_n^{-2})$.

Let us introduce the discrete sums $w_1(t) = \sum_{j=1}^n \lambda_t^{-1}(x_t - x_{\widehat{t}_{j-1}})^2\xi_j(t)$, $w_2(t) = \sum_{j=1}^n x_t^2(\lambda_t^{-1/2} - \lambda_{\widehat{t}_{j-1}}^{-1/2})^2\xi_j(t)$, and $w_3(t) = \sum_{j=1}^n (\lambda_t^{1/2} - \lambda_{\widehat{t}_{j-1}}^{1/2})^2\xi_j(t)$, where $\xi_j(t) = \mathbf{1}_{(\widehat{t}_{j-1}, \widehat{t}_j]}(t)$ and $x_t = \ln(S_t/K)$. Clearly, $|\gamma_t^n - \widehat{C}_x(t, S_t)|^2 \leq w_1(t) + w_2(t) + w_3(t)$. By taking into account that

$$\sup_{n,\, 1 \leq j \leq n} n \sup_{0 \leq t \leq 1} \mathbf{E}(x_t - x_{\widehat{t}_{j-1}})^2\xi_j(t) < \infty \quad \text{and} \quad \sup_{0 \leq t \leq 1} \mathbf{E}\, x_t^2 < \infty,$$

we have $\theta_n^2\mathbf{E} \int_0^{1-\varepsilon_n} w_1(t)\mathrm{d}t \leq Cn^{2\beta-3/2}\varrho^{4\beta-1}$, which converges to zero by $(\mathbf{C}_2)$. Now, the particular choice of ε ensures that $\theta_n^2\mathbf{E} \int_0^{1-\varepsilon_n} w_2(t)\mathrm{d}t \leq C\theta_n^2 n^{-2}(\varepsilon_n)^{-(4\beta+1)/4\beta}\lambda_0^{-1}$, which tends to zero. The convergence for $w_3(t)$ can be shown in the same way. $\qquad\square$

APPENDIX C: MOMENTS OF ORSTEIN–UHLENBECK'S PROCESSES

LEMMA C.1. *Suppose that $\sigma(z) \leq \gamma(1 + |z|)$ for all $z \in \mathbb{R}$, for some constant $\gamma > 0$. Let y_t be an Orstein–Uhlenbeck process defined by $\mathrm{d}y_t = (a - by_t)\mathrm{d}t + \mathrm{d}Z_t$ with some constants a and $b > 0$. Put $X_\alpha = \exp\{2\alpha\gamma^2 \int_0^1 y_s^2\mathrm{d}s\}$ and $\alpha_* = b^2(2\gamma^2(2b + a^2))^{-1}$. Then, $\mathbf{E}X_\alpha < \infty$ for any $0 < \alpha < \alpha_*$.*

Proof. Remark that $(a - by)y \leq a^2/(2b) - by^2/2$. Then, by adapting proposition 1.1.5 in Kabanov and Pergamenshchikov (2003, p. 24), we can show that $\mathbf{E}\,|y_t|^{2m} \leq m!(2/b + a^2/b^2)^m$ for any integer $m \geq 1$. It follows that for any $0 < \alpha < \alpha_*$,

$$\mathbf{E}X_\alpha \leq \sum_{m=0} (\alpha 2\gamma^2)^m(m!)^{-1}\mathbf{E}\,|y_t|^{2m} \leq \sum_{m=0} \left(2/b + a^2/b^2\right)^m (\alpha 2\gamma^2)^m < \infty.$$

If y_t is mean-reverting, then b takes very large values. Hence, it is possible to choose $\alpha > 3/2 + \sqrt{2}$ as discussed in Remark 4.4. $\qquad\square$

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