Published for SISSA by Springer

Received: *July 17, 2015*
Accepted: *August 19, 2015*
Published: *September 7, 2015*

Higher S-dualities and Shephard-Todd groups

Sergio Cecotti[a] **and Michele Del Zotto**[b]

[a] *Scuola Internazionale Superiore di Studi Avanzati, via Bonomea 265, 34100 Trieste, Italy*

[b] *Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, U.S.A.*

E-mail: cecotti@sissa.it, eledelz@gmail.com

Abstract: Seiberg and Witten have shown that in $\mathcal{N} = 2$ SQCD with $N_f = 2N_c = 4$ the S-duality group $\mathrm{PSL}(2,\mathbb{Z})$ acts on the flavor charges, which are weights of Spin(8), by triality. There are other $\mathcal{N} = 2$ SCFTs in which SU(2) SYM is coupled to strongly-interacting non-Lagrangian matter: their matter charges are weights of E_6, E_7 and E_8 instead of Spin(8). The S-duality group $\mathrm{PSL}(2,\mathbb{Z})$ acts on these weights: *what replaces Spin(8) triality for the E_6, E_7, E_8 root lattices?*

In this paper we answer the question. The action on the matter charges of (a finite central extension of) $\mathrm{PSL}(2,\mathbb{Z})$ factorizes trough the action of the exceptional Shephard-Todd groups G_4 and G_8 which should be seen as complex analogs of the usual triality group $\mathfrak{S}_3 \simeq \mathrm{Weyl}(A_2)$. Our analysis is based on the identification of S-duality for SU(2) gauge SCFTs with the group of automorphisms of the cluster category of weighted projective lines of tubular type.

Keywords: Supersymmetry and Duality, Extended Supersymmetry, Duality in Gauge Field Theories

ArXiv ePrint: 1507.01799

doi:10.1007/JHEP09(2015)035

JHEP09(2015)035

Contents

1 Introduction and summary

We have a complete classification of the 4d $\mathcal{N} = 2$ gauge theories where SU(2) SYM is coupled to vector-less matter (possibly non-Lagrangian) [1]. By *vector-less* matter we mean $\mathcal{N} = 2$ QFTs whose BPS spectra, in all chambers, consist only of hypermultiplets: the ones having a gaugeable SU(2) symmetry are precisely the Argyres-Douglas (AD) theories of type D_p ($p \geq 2$) [1, 2] quark doublets being the case $p = 2$. From the classification we learn that the SCFTs in this class are in one-to-one correspondence with the orbifolds of an elliptic curve $\mathcal{E}$ i.e.

$$\mathcal{E}/\mathbb{Z}_p \quad \text{where } p = 1, 2, 3, 4, 6. \tag{1.1}$$

JHEP09(2015)035

For $p > 2$ the curve $\mathcal{E}$ should have complex multiplication by the appropriate quadratic field namely[1] $\mathbb{Q}(\omega)$ for $p = 3, 6$ and $\mathbb{Q}(i)$ for $p = 4$. This observation plays a crucial role below.

We have five SCFTs in this class of SU(2) gauge theories. The SCFT is a Lagrangian model iff the modulus τ of $\mathcal{E}$ is a free parameter, i.e. for $p = 1, 2$ which correspond, respectively, to $\mathcal{N} = 2^*$ and SQCD with $N_f = 4$. $p = 3, 4, 6$ yield three additional non-Lagrangian ($\equiv$ intrinsically strongly coupled) SCFTs. From the viewpoint of [1] it is more natural to state this classification as a one-to-one correspondence between this class of SCFTs and the *star* graphs (possibly with multiple edges[2]) which are affine Dynkin diagrams. There are five such affine stars

$$\mathfrak{g}^{(1)} = A_1^{(1)},\ D_4^{(1)},\ E_6^{(1)},\ E_7^{(1)},\ E_8^{(1)}, \tag{1.2}$$

which correspond to the five orbifolds (1.1). In the first two models, the Lagrangian ones, $\mathfrak{g}^{(1)}$ is also the affinization of the flavor symmetry algebra $\mathfrak{g}$ which is, respectively, $\mathfrak{su}(2)$ and $\mathfrak{so}(8)$: thus in the Lagrangian models the flavor charges are weights of $\mathfrak{g}$. In the non-Lagrangian theories the matter consists of strongly interacting systems with their own conserved electric and magnetic charges in addition to the flavor ones. Although the Dirac pairing between the internal charges of the matter is no longer trivial, yet it remains true that the matter charges[3] take value in the weight lattice of the corresponding finite-dimensional Lie algebra $\mathfrak{g}$ which, for the non-Lagrangian models, is E_6, E_7, or E_8 (see section 2.1).

These theories, already described in [1, 3, 4], recently have been constructed also as toroidal compactifications of certain 6d $(1, 0)$ SCFTs [5]. From the internal torus, all five (mass deformed) 4d SCFTs inherit a $\mathrm{PSL}(2, \mathbb{Z})$ group of S-dualities. In the $p = 2$ case Seiberg and Witten [6] have shown that $\mathrm{PSL}(2, \mathbb{Z})$ acts on the flavor charges by SO(8) triality; the triality group $\mathfrak{S}_3$ being identified with the modular quotient $\mathrm{PSL}(2, \mathbb{Z})/\Gamma(2)$. By the same token, for $p = 3, 4, 6$ we have a non-trivial action of S-duality, hence of the modular group $\mathrm{PSL}(2, \mathbb{Z})$, on the lattice of the matter charges, i.e. on the weight lattices of E_6, E_7, E_8. This action should be thought of as a generalization of the triality action on the weights of SO(8) to the weights of the exceptional Lie groups E_6, E_7, E_8.

At first sight this statement seems rather odd: from Lie algebra theory we do not expect any higher rank analog of SO(8) triality. Yet physics *predicts* its existence.

The full duality group $\mathbb{S}$ is actually an extension of the modular group

$$1 \to \mathscr{W} \to \mathbb{S} \to \mathrm{PSL}(2, \mathbb{Z}) \to 1, \tag{1.3}$$

where $\mathscr{W}$ is the 'obvious' group of physical symmetries acting on the lattice of conserved charges. For the Lagrangian models $\mathscr{W}$ is simply the Weyl group of the flavor symmetry. In the general case $\mathscr{W}$ is a well-understood finite group of symmetries of the matter system (seen as decoupled from the Yang-Mills sector) which fixes the charge to be gauged. $\mathscr{W}$ is

[1] ω is a primitive third root of unity, i.e. a solution to the cyclotomic equation $\omega^2 + \omega + 1 = 0$.

[2] Non simply-laced star graphs corresponds to matter in SU(2) representations of isospin $> 1/2$.

[3] Throughout this paper, by *matter charges* we mean all conserved charges of the $\mathcal{N} = 2$ QFT but the SU(2) electric and magnetic ones.

JHEP09(2015)035

also the kernel of the action of $\mathbb{S}$ on the Yang-Mills electric/magnetic charges, on which only the quotient group $\mathrm{PSL}(2,\mathbb{Z})$ acts effectively. The quotient group of $\mathbb{S}$ which acts effectively on the matter charges, $\mathbb{S}_{\mathrm{matter}}$, is the *finite* group

$$1 \to \mathscr{W} \to \mathbb{S}_{\mathrm{matter}} \to \mathrm{PSL}(2,\mathbb{Z}/p\,\mathbb{Z}) \to 1, \qquad p = 1,2,3,4,6, \tag{1.4}$$

whose action on the root lattice of $\mathfrak{g}$ preserves the Cartan inner product as well as the Dirac skew-symmetric pairing. For $p = 1,2$ eq. (1.4) was obtained by Seiberg and Witten [6]. Its extension to the non-Lagrangian cases looks rather natural, a simple 'analytic continuation in p'.

Since the action of $\mathscr{W}$ on the matter charges is obvious, to understand the 'higher versions of triality' it is enough to understand the action of the quotient group

$$\mathrm{PSL}(2,\mathbb{Z}/p\,\mathbb{Z}) \simeq \mathrm{PSL}(2,\mathbb{Z})/\Gamma(p) \qquad \text{for } p = 2,3,4,6, \tag{1.5}$$

where $\Gamma(p) \subset \mathrm{PSL}(2,\mathbb{Z})$ is the principal congruence subgroup of level p [7]. $p = 6$ is special since[4]

$$\mathrm{PSL}(2,\mathbb{Z}/6\mathbb{Z}) = \mathrm{PSL}(2,\mathbb{Z}/2\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z}/3\mathbb{Z}). \tag{1.6}$$

However, it is preferable to study the action of a subgroup $G_{\mathrm{matter}} \subset \mathbb{S}_{\mathrm{matter}}$ which is a central extension of $\mathrm{PSL}(2,\mathbb{Z}/p\,\mathbb{Z})$ by a finite Abelian group of the form $(\mathbb{Z}/2\mathbb{Z})^k$. Studying the central extension G_{matter}, rather than $\mathrm{PSL}(2,\mathbb{Z}/p\,\mathbb{Z})$ itself, allows to discuss all five models in an unified way *via* the theory of reflection groups.

Let us explain. Weyl groups should be thought of as reflection groups defined over the rationals $\mathbb{Q}$, and Coxeter groups as real reflection groups. For $N_f = 4$ SQCD ($p = 2$) the action of $\mathrm{PSL}(2,\mathbb{Z}/2\mathbb{Z})$ is through SO(8) triality which is a *rational* reflection group: indeed, in concrete terms, the triality group is the quotient

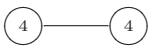
$$\mathbb{S}_{\mathrm{matter}}/\mathrm{Weyl}(\mathrm{SO}(8)) \equiv \mathrm{Weyl}(F_4)/\mathrm{Weyl}(\mathrm{SO}(8)). \tag{1.7}$$

In passing from $p = 2$ to $p = 3,4,6$ what we have to do is to replace reflection groups defined over $\mathbb{Q}$ with reflection groups defined over the appropriate complex multiplication fields $\mathbb{Q}(\omega)$ or $\mathbb{Q}(i)$. Roughly speaking, the abstract form of S-duality is the same for $p > 2$ as for the SQCD $p = 2$ model (where it is given by SO(8) triality) but structures that in the $p = 2$ case are defined over the ground field $\mathbb{Q}$ get replaced by structures defined over the complex quadratic fields $\mathbb{Q}(\omega)$ and $\mathbb{Q}(i)$.

Reflection groups defined over such quadratic fields are special instances of *complex reflection groups*. The *finite* complex reflection groups have been fully classified by Shephard and Todd [9]. From their classification we read the complete list of the "higher triality" groups G_{matter} which act on the matter charges of the $p = 3,4,6$ models, i.e. on the weight lattices $\Gamma^{\mathrm{w}}_{\mathfrak{g}}$ of E_6, E_7 and E_8. The classification also yields the decomposition of the vector space $\Gamma^{\mathrm{w}}_{\mathfrak{g}} \otimes \mathbb{C}$ in irreducible representations of G_{matter}, and thus completely specifies how $\mathrm{PSL}(2,\mathbb{Z}/p\,\mathbb{Z})$, and hence the full modular group $\mathrm{PSL}(2,\mathbb{Z})$, acts on the matter charges.

[4] There is a much stronger reason why $p = 6$ is different. As Klein proved in 1884 [8] the group $\mathrm{PSL}(2,\mathbb{Z})/N(6)$ has *infinite* order, while for $p = 1,2,3,4$, $\mathrm{PSL}(2,\mathbb{Z})/N(p) = \mathrm{PSL}(2,\mathbb{Z}/p\,\mathbb{Z})$ (see section 4.2.1).

JHEP09(2015)035

field $\mathbb{F}$	reflection group	abstract group	extension of a quotient of the modular group	reflection group graph[a]	McKay affine graph[b]
$\mathbb{Q}$	Weyl(A_2)	$\mathfrak{S}_3$	$\mathrm{PSL}(2,\mathbb{Z}/2\mathbb{Z})$		$D_5^{(1)}$
$\mathbb{Q}(\omega)$	G_4	binary tetrahedral	$\mathrm{SL}(2,\mathbb{Z}/3\mathbb{Z})$		$E_6^{(1)}$
$\mathbb{Q}(i)$	G_8	$\mathbb{Z}/2\mathbb{Z} \ltimes$ (binary octahedral)	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{SL}(2,\mathbb{Z}/4\mathbb{Z})$		$E_7^{(1)}$

[a] Usually the 2's in the nodes of the A_2 Dynkin graph are omitted.

[b] *More precisely*: the McKay graph of the SU(2) subgroup which has the same image in $P\mathrm{SU}(2)$.

Table 1. The relevant complex reflection groups.

The relevant reflection groups for the three complex multiplication fields $\mathbb{Q}$, $\mathbb{Q}(\omega)$ and $\mathbb{Q}(i)$ are listed in table 1. In the second column we write the group, seen as a concrete reflection group acting on a two-dimensional space, in the Shephard-Todd notation. In the third column we write the standard name of the corresponding abstract group. In the fourth column we describe the same group seen as a quotient of a central extension of the modular group $\mathrm{PSL}(2,\mathbb{Z})$; this column specifies how the action of S-duality group $\mathbb{S}$ on $\Gamma_{\mathfrak{g}}^{\mathrm{w}}$ factorizes through a representation of the reflection group G_{matter}. In the fifth column we draw the graph of the reflection group [10]: we stress that, in all three cases, it is the A_2 Dynkin graph but with order p at the nodes. The uniformity of the graph expresses our rough idea that the structure of S-duality is independent of p up to a change of the ground field. In particular, all three groups are realized as a concrete group of reflections by faithful *two*–dimensional unitary representations which we denote as W, R and F, respectively (see section 4.2.1 for full details). In the last column of the table we recall the affine Dynkin graph which is related to the given reflection group by the McKay correspondence [11]; more precisely, the affine graph shown in the table is the one associated to the finite SU(2) subgroup which has the same image in $P\mathrm{SU}(2)$ as the reflection group G_{matter} viewed as a subgroup of U(2) *via* its defining two-dimensional representation. Note that to the $E_6^{(1,1)}$ and $E_7^{(1,1)}$ SCFTs there correspond, respectively, the McKay graphs $E_6^{(1)}$ and $E_7^{(1)}$.

Summarizing, the reflection groups G_{matter} acting on the matter charges, which take values in the weight lattices $\Gamma_{\mathfrak{g}}^{\mathrm{w}}$, are (see section 4.2.1 for the definition of the groups G_4, G_8)

$$G_{\mathrm{matter}} = \begin{cases} \mathfrak{S}_3 & p = 2 \\ G_4 & p = 3 \\ G_8 & p = 4 \\ \mathfrak{S}_3 \times G_4 & p = 6. \end{cases} \tag{1.8}$$

For $p = 2$ the symmetric group $\mathfrak{S}_3$ acts on the weights of D_4 by triality. In this paper we describe the corresponding action of G_{matter} for $p = 3, 4, 6$. We do so in two ways. First we list the irreducible representations of the matter group in eq. (1.8) acting on the root

JHEP09(2015)035

vector space $\Gamma_{E_r} \otimes \mathbb{C}$

$$p = 3 \qquad \Gamma_{E_6} \otimes \mathbb{C} \simeq \mathbf{1} \oplus \mathbf{1} \oplus R \oplus \overline{R} \tag{1.9}$$

$$p = 4 \qquad \Gamma_{E_7} \otimes \mathbb{C} \simeq \mathbf{1} \oplus W \oplus F \oplus \overline{F} \tag{1.10}$$

$$p = 6 \qquad \Gamma_{E_8} \otimes \mathbb{C} \simeq \chi \oplus \overline{\chi} \oplus (W, \mathbf{1}) \oplus (\mathbf{1}, R) \oplus (\mathbf{1}, \overline{R}), \tag{1.11}$$

where: $\mathbf{1}$ is the trivial representation; W, R and F are the defining 2-dimensional representations of $\mathfrak{S}_3$, G_4 and G_8, respectively; $\overline{R}$, $\overline{F}$ are their conjugates (W is real); χ is a one-dimensional representation, namely a primitive character of the Abelian quotient[5]

$$\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \simeq \mathfrak{S}_3/\mathfrak{A}_3 \times \mathfrak{A}_4/V_4 \tag{1.12}$$

In eq. (1.10) by W we mean the 2-dimensional representation of $\mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})$ defined by the degree 2 representation W of $\mathfrak{S}_3$ *via* the canonical mod 2 surjection

$$\mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z}) \longrightarrow \mathrm{PSL}(2, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{W} GL(W). \tag{1.13}$$

The action of G_{matter} on $\Gamma_{E_r} \otimes \mathbb{C}$ preserves the lattice Γ_{E_r}, the inner product in root space given by the Cartan matrix, and the Dirac electro-magnetic pairing. For concreteness, in appendix A we also give a very explicit realization of the action of S-duality on the charges in terms of integral $(r(\mathfrak{g})+2) \times (r(\mathfrak{g})+2)$ matrices acting on the full charge lattice Γ (which includes the matter charges as well as the Yang-Mills electric and magnetic ones).

The rest of the paper is organized as follows. In section 2 we collect the basic tools of the homological approach to S-duality. In section 3 we reconsider $\mathcal{N} = 2$ SQCD with two colors and four flavors as a warm-up; here we recover the Seiberg-Witten result in two ways: first in a rather naive but very concrete approach, and then from a more intrinsic group-theoretical perspective. In section 4 we describe the $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ cases. In appendix A we write explicit $(r(\mathfrak{g})+2) \times (r(\mathfrak{g})+2)$ matrices which represent the action of S-duality on the conserved charges of the theory, and list some of the beautiful identities they satisfy. In appendix B we show how the present approach is related to the one in [1, 26] by *cluster-tilting*.

2 Homological approach to S-duality

In this section we review S-duality for the four (mass-deformed) SCFTs $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ following the mathematical literature [12–17] (see also [18]).

Remark. For definiteness, here we use the 4d definition of the relevant SCFTs, see refs. [1, 3]. Alternatively, one could have adopted the 6d viewpoint of ref. [5], and in particular their mirror Landau-Ginzburg description of the $\mathcal{E}/\mathbb{Z}_p$ orbifolds. By a theorem of Orlov ([19] **Theorem 2.5.***(iii)*, see also the discussion in [20]), the category of B-branes for the relevant Landau-Ginzburg models is *equivalent* to the derived category of coherent sheaves on the corresponding weighted projective lines of tubular type, which is the central

[5] From table 1 one has $G_4 \simeq \mathbb{Z}/2\mathbb{Z} \ltimes \mathfrak{A}_4$. $V_4 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the Klein *Vierergruppe* [8], the unique non-trivial normal subgroup of the alternating group $\mathfrak{A}_4$.

JHEP09(2015)035

object of our analysis, see section 2.2. Then all our considerations apply directly to the Landau-Ginzburg set up. Needless to say, the equivalence of the two categories is an instance of the $2d/4d$ correspondence advocated in [21].

2.1 Overview

The relation between eqs. (1.1), (1.2) and the mass-deformed SCFTs is as follows.[6] The BPS particles of the QFT are given by the (quantization of) continuous families of *stable* objects in a certain orbifold category $\mathcal{C}(\mathcal{E}/\mathbb{Z}_p)$ of the derived category of coherent sheaves on the orbifold $\mathcal{E}/\mathbb{Z}_p$ or, equivalently [22], on the orbifold $\mathbb{P}^1/\Gamma_p$ where $\Gamma_p \subset \mathrm{SU}(2)$ is the finite subgroup associated to the affine Lie algebra $\mathfrak{g}^{(1)}$ by the McKay correspondence[7] [11]

$$\mathcal{C}(\mathcal{E}/\mathbb{Z}_p) = \mathsf{D}^b(\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p)/\mathscr{C} \simeq \mathsf{D}^b(\mathsf{coh}\,\mathbb{P}^1/\Gamma_p)/\mathscr{C}. \tag{2.1}$$

Here $\mathscr{C}$ is an infinite cyclic subgroup of $\mathrm{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p)$ to be described later. $\mathcal{C}(\mathcal{E}/\mathbb{Z}_p)$ is the *cluster category* [17, 36] of the orbifold $\mathcal{E}/\mathbb{Z}_p$ whose relevance for the physics of the BPS sector stems from the Kontsevich-Soibelman wall-crossing formula [23], see refs. [21, 24]. In the present context, *stable* means that the lift of the object in $\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$ is stable in the sense of ref. [25]; the stability condition on $\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$ is defined by the mass deformation we consider and the chosen point in the Coulomb branch. Since all five QFTs are complete in the sense of [1], we have only zero– and one-dimensional families of stable objects which yield, respectively, hypermultiplets and vector multiplets of $\mathcal{N} = 2$ SUSY.

Charge lattices. The lattice of conserved QFT charges, Γ, may then be identified with the Grothendieck group of the associated additive categories[8]

$$\Gamma = K_0(\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p) \simeq K_0(\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p). \tag{2.2}$$

Given an object $X \in \mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$, we write $[X] \in \Gamma$ for its Grothendieck class; if X is stable, $[X]$ is the charge vector of a corresponding BPS state. In all chambers the charges of BPS particles generate Γ. On Γ we have a bilinear form, the Euler pairing

$$\langle [X], [Y] \rangle_E = \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathrm{Hom}^\bullet(X, Y[k]), \tag{2.3}$$

where $Y \to Y[1]$ denotes the shift equivalence in the triangle category $\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$. The Dirac pairing in Γ is just the anti-symmetric part of the Euler one

$$\langle [X], [Y] \rangle_{\mathrm{Dirac}} = \langle [X], [Y] \rangle_E - \langle [Y], [X] \rangle_E. \tag{2.4}$$

The Tits form is the integral quadratic form on Γ

$$q([X]) = \langle [X], [X] \rangle_E. \tag{2.5}$$

The symmetric bilinear form associated to the Tits quadratic form will be written $\langle \cdot, \cdot \rangle_{\mathrm{sym}}$.

[6] For the justification of these assertions, see footnote 15. More technical details in appendix B.

[7] We write $\mathsf{coh}\,X/G$ as a shorthand for the category of G-equivariant coherent sheaves on X.

[8] The fact that the Grothendieck group is a finite-rank lattice follows from the fact that the Abelian category $\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$ admits a tilting object [12–14].

JHEP09(2015)035

For X stable one has [26, 27]

$$q([X]) = 2\big(1 - \mathrm{MaxSpin}([X])\big), \tag{2.6}$$

where $\mathrm{MaxSpin}([X])$ is the largest possible spin for a BPS particle of charge $[X]$. Since for a complete $\mathcal{N} = 2$ theory [1, 27], the spin of the BPS particles is bounded by 1, the quadratic form $q([X])$ is positive semi-definite; its radical

$$\operatorname{rad} q \equiv \Big\{ [X] \in \Gamma \;\Big|\; q([X]) = 0 \Big\} \subset \Gamma, \tag{2.7}$$

is a sublattice of rank 2 which may be identified with the lattice of electric/magnetic Yang-Mills SU(2) charges; in particular, the restriction to $\operatorname{rad} q$ of the Dirac pairing has the form[9]

$$p \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.8}$$

We define the *matter charges lattice* as Γ modulo the sublattice of Yang-Mills charges

$$\Gamma_{\text{matter}} \simeq \Gamma \big/ \operatorname{rad} q. \tag{2.9}$$

The above categorical identification of the charges is not the conventional one in physics (but it coincides for Lagrangian QFTs). For the three non-Lagrangian models, our SU(2) electric charge is a linear combination of the physical electric charge and matter ones; the present conventions make the group actions more transparent (in facts, they are the obvious generalization of the factor 2 difference of normalizations for the $p = 1$ and $p = 2$ models pointed out in section 16 of the original Seiberg-Witten paper [6]).

By general theory of integral quadratic forms [28] (reviewed in the present context in [27]) the Tits form q induces an integral quadratic form $\overline{q}$ on $\Gamma \big/ \operatorname{rad} q$ which is positive-definite, hence $\mathbb{Z}$-equivalent to the Tits form $q_{\mathfrak{g}}$ on the *root* lattice $\Gamma_{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ of ADE type

$$q_{\mathfrak{g}}(x_i) = \frac{1}{2} C_{ij} x_i x_j, \qquad C_{ij} \text{ the Cartan matrix of } \mathfrak{g}. \tag{2.10}$$

For our five categories $\mathsf{D}^b(\mathsf{coh}\, \mathcal{E}/\mathbb{Z}_p)$ with $p = 1, 2, 3, 4, 6$ one finds (see section 2.2)

$$\overline{q} \simeq q_{\mathfrak{g}} \text{ hence } \Gamma_{\text{matter}} \simeq \Gamma_{\mathfrak{g}} \quad \text{where } \mathfrak{g} = A_1,\ D_4,\ E_6,\ E_7,\ E_8 \text{ respectively.} \tag{2.11}$$

In the language of refs. [1, 27] the two statements $\operatorname{rank} \operatorname{rad} q = 2$ and $\overline{q} \simeq q_{\mathfrak{g}}$ are summarized in the fact that the mutation class of quivers with superpotentials $(Q, \mathcal{W})$ which describe the BPS sector of our five theories is given by the elliptic Dynkin graphs $\mathfrak{g}^{(1,1)}$ of respective type $\mathfrak{g} = A_1, D_4, E_6, E_7, E_8$ (with all triangles oriented). The *elliptic* (or *toroidal* [29]) Lie algebra $\mathfrak{g}^{(1,1)}$ of type $\mathfrak{g}$ is obtained by affinization of the affine Lie algebra $\mathfrak{g}^{(1)}$ of the same type; see [27] for more details.

As a basis of the matter charges we take the simple roots α_a of $\mathfrak{g}$. Then the matter charges of the BPS particle corresponding to the stable object $X \in \mathsf{D}^b\, \mathsf{coh}\, \mathcal{E}/\mathbb{Z}_p$ are

$$f_a(X) = \langle [X], \alpha_a \rangle_{\text{sym}} \in \mathbb{Z}, \qquad a = 1, 2, \ldots, r(\mathfrak{g}). \tag{2.12}$$

[9] In the Lagrangian case, $p = 2$, the overall coefficient 2 is interpreted as the Cartan matrix of SU(2).

JHEP09(2015)035

In particular, for $p = 1, 2$ we get back that the flavor charges take values in the weight lattice of Sp(1) and SO(8), respectively. The same statement holds, in the present sense, for $p = 3, 4, 6$.

Serre duality and Coxeter transformation. The Abelian category $\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$ satisfies Serre duality in the form[10]

$$\mathrm{Ext}^1(X, Y) = D\,\mathrm{Hom}(Y, \tau X), \qquad \tau X \equiv \omega \otimes X, \tag{2.13}$$

where ω is the dualizing sheaf. τ is an auto-equivalence of $\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$, and hence of the derived category $\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$; τ also plays the role of Auslander-Reiten translation [30]. Given that τ is an auto-equivalence, $\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$ has no non-zero injectives or projectives and the Abelian category $\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$ is *hereditary* (global dimension 1).

Since $\omega^p \simeq \mathcal{O}$, one has $\tau^p = \mathrm{Id}$. The Coxeter transformation $\boldsymbol{\Phi}\colon \Gamma \to \Gamma$ is defined by

$$[\tau X] = \boldsymbol{\Phi} \cdot [X]. \tag{2.14}$$

In particular, for all $[X], [Y] \in \Gamma$ we have

$$\langle \boldsymbol{\Phi} \cdot [X], \boldsymbol{\Phi} \cdot [Y] \rangle_E = \langle [X], [Y] \rangle_E, \tag{2.15}$$

$$\langle [X], \boldsymbol{\Phi} \cdot [Y] \rangle_E = -\langle [Y], [X] \rangle_E, \tag{2.16}$$

which implies

$$[X] \in \mathrm{rad}\, q \quad \Longleftrightarrow \quad \boldsymbol{\Phi} \cdot [X] = [X], \tag{2.17}$$

i.e. the Yang-Mills magnetic/electric charges are the $(+1)$-eigenvectors of $\boldsymbol{\Phi}$. Likewise, the flavor charges are the (-1)-eigenvectors of $\boldsymbol{\Phi}$. The eigenvectors associated to eigenvalues $\lambda \neq \pm 1$ correspond to internal electric/magnetic charges of the matter AD systems. Note that $\boldsymbol{\Phi}^p = 1$, in fact (for $p > 1$)

$$\det[z - \boldsymbol{\Phi}] = \frac{\prod_{i=1}^{s}(z^{p_i} - 1)}{(z-1)^{s-2}}, \tag{2.18}$$

where s is the number of branches of the associated star graph and p_i are the number of nodes in the i-th branch (counting the vertex node), while $p \equiv \mathrm{l.c.m.}\{p_i\}$.

Auto-equivalences of the derived category. Suppose

$$K\colon \mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p \to \mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p \tag{2.19}$$

is an auto-equivalence (of triangulated categories). K induces an automorphism $\boldsymbol{K}$ of the Grothendieck group Γ

$$\boldsymbol{K}\colon \Gamma \to \Gamma, \quad \text{given by} \quad [KX] = \boldsymbol{K} \cdot [X]. \tag{2.20}$$

Choosing a $\mathbb{Z}$-basis in Γ, $\boldsymbol{\Phi}$ and $\boldsymbol{K}$ may be seen as $\mathrm{rank}\,\Gamma \times \mathrm{rank}\,\Gamma$ matrices with integral entries. For all auto-equivalences K we have $\tau K = K\tau$ (since the Auslander-Reiten translation is unique). Then $\boldsymbol{\Phi}$ and $\boldsymbol{K}$, as matrices, commute

$$\boldsymbol{K}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{K}, \tag{2.21}$$

[10] Here D stands for the usual duality over the ground field $\mathbb{C}$, i.e. $D(-) = \mathrm{Hom}(-, \mathbb{C})$.

JHEP09(2015)035

and, in particular, $\boldsymbol{K}$ preserves the radical sublattice $\operatorname{rad} q \subset \Gamma$; this already follows from the fact that all auto-equivalences are isometries of the Euler form

$$\langle \boldsymbol{K}\cdot[X], \boldsymbol{K}\cdot[Y]\rangle_E = \langle [X],[Y]\rangle_E. \tag{2.22}$$

Comparing with (2.9), we see that all auto-equivalence K of the derived category $\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p$ induces a reduced additive map

$$\overline{\boldsymbol{K}}\colon \Gamma_{\text{matter}} \to \Gamma_{\text{matter}} \tag{2.23}$$

which commutes with the reduced Coxeter element $\overline{\boldsymbol{\Phi}}$ and is an isometry of the reduced Tits form $\overline{q}(\cdot)$

$$\overline{q}\big(\overline{\boldsymbol{K}}\cdot f\big) = \overline{q}\big(f\big), \qquad \forall\, f\in\Gamma_{\text{matter}}. \tag{2.24}$$

It follows that the image of the automorphism group of the derived category,

$$\operatorname{Aut}\big(\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p\big),$$

under the homomorphism $\varrho\colon K\mapsto \overline{\boldsymbol{K}}$ is a subgroup of the *finite* group $O(\Gamma_{\mathfrak{g}})$ of the $\mathbb{Z}$-isometries of the positive-definite Tits form $q_{\mathfrak{g}}$. More precisely, the image is a subgroup of the centralizer of the reduced Coxeter element $\overline{\boldsymbol{\Phi}}$

$$\varrho\big(\operatorname{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p)\big) \subset Z(\overline{\boldsymbol{\Phi}}) \subset O(\Gamma_{\mathfrak{g}}) = \operatorname{Weyl}(\mathfrak{g}) \ltimes \operatorname{Aut}(D_{\mathfrak{g}}), \tag{2.25}$$

where $D_{\mathfrak{g}}$ is the Dynkin graph of $\mathfrak{g}$.

Remark. The triality of $\mathfrak{so}(8)$ is a group of outer automorphisms; from the point of view of eq. (2.25) this means that its image is not contained in $\operatorname{Weyl}(\mathfrak{so}(8))$. For, say, $\mathfrak{g}=E_8$, $\operatorname{Aut}(D_{\mathfrak{g}})$ is trivial and the image of $\operatorname{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathcal{E}/\mathbb{Z}_p)$ is a subgroup of the Weyl group. The reader then may wonder in which sense the S-duality action is a generalization of triality which is an *outer* action. The point is that the duality action is outer with respect to the natural group of 'inner' automorphisms which is $\mathscr{W}$ (see eq. (2.101)). For SQCD $N_f=4$ $\mathscr{W}$ is the full Weyl group but it is a small subgroup for E_6, E_7, E_8.

2.2 Coherent sheaves on weighted projective lines

With the exclusion[11] of $\mathcal{N}=2^*$, the complete $\mathcal{N}=2$ gauge theories with gauge group SU(2) are in one-to-one correspondence with the weighted projective lines having non-negative Euler characteristic $\chi\geq 0$ [1, 3]. SU(2) SYM coupled to a set of AD matter systems of types D_{p_i} $(i=1,..,s)$ corresponds to the weighted projective line with weights $(\boldsymbol{p}) = (p_1,p_2,\ldots,p_s)$. The superconformal theories in this class are precisely the ones associated with the $\chi=0$ weighted projective lines;[12] there are four such lines with weights

$$(\boldsymbol{p}) = (2,2,2,2), \quad (3,3,3), \quad (2,4,4), \quad (2,3,6), \tag{2.26}$$

[11] The case of $\mathcal{N}=2^*$, i.e. $p=1$, is rather similar. Indeed, the theory of coherent sheaves on the weighted projective lines of tubular type was constructed by Geigle and Lenzing in [12] using as a model the Atiyah description of $\mathsf{coh}\,\mathcal{E}$ [31]. The main technical difference is that for $p=1$ the canonical sheaf is trivial, while for $p>1$ is a p-torsion sheaf. Then for $p=1$ there is no tilting object.

[12] Indeed, the coefficient of the β-function of the Yang-Mills coupling, g_{YM}, is -2χ, see [1, 3, 4].

JHEP09(2015)035

the i-th weight p_i being equal to the number of nodes in the i-th branch of the corresponding affine star graph (counting the vertex). The $\chi = 0$ weighted projective lines will be written $\mathbb{X}_p$ (where $p \equiv \text{l.c.m.}(p_i) = 2, 3, 4, 6$) or simply $\mathbb{X}$. We have [12, 22, 32]

$$\mathbb{X}_p = \mathcal{E}/\mathbb{Z}_p, \tag{2.27}$$

and passing from elliptic orbifolds $\mathcal{E}/\mathbb{Z}_p$ to weighted projective lines $\mathbb{X}_p$ is just a convenient shift in language.

Weighted projective lines [12–16, 33, 34]. Given a set of positive integral weights[13] $\boldsymbol{p} = (p_1, p_2, \ldots, p_s)$ we define $L(\boldsymbol{p})$ to be the Abelian group over the generators $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_s$ subjected to the relations

$$\vec{c} = p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_s\vec{x}_s. \tag{2.28}$$

$\vec{c}$ is called the *canonical* element of $L(\boldsymbol{p})$, while the *dual* element is

$$\vec{\omega} = (s-2)\vec{c} - \sum_{i=1}^{s} \vec{x}_i \in L(\boldsymbol{p}). \tag{2.29}$$

Given the weights $\boldsymbol{p}$ and s distinct points $(\lambda_i : \mu_i) \in \mathbb{P}^1$ we define a ring graded by $L(\boldsymbol{p})$

$$S(\boldsymbol{p}) = \bigoplus_{\vec{a} \in L(\boldsymbol{p})} S_{\vec{a}} = \mathbb{C}[X_1, X_2, \cdots, X_s, u, v] \Big/ \left(X_1^{p_1} - \lambda_1 u - \mu_1 v,\ \cdots,\ X_s^{p_s} - \lambda_s u - \mu_s v\right) \tag{2.30}$$

where the degree of X_i is $\vec{x}_i$ and the degree of u, v is $\vec{c}$. The weighted projective line $\mathbb{X}(\boldsymbol{p})$ is defined to be the projective scheme $\mathsf{Proj}\, S(\boldsymbol{p})$. Its Euler characteristic is

$$\chi(\boldsymbol{p}) = 2 - \sum_{i=1}^{s} (1 - 1/p_i). \tag{2.31}$$

The Picard group of $\mathbb{X}(\boldsymbol{p})$ (i.e. the group of its invertible coherent sheaves $\equiv$ line bundles) is isomorphic to the group $L(\boldsymbol{p})$

$$\mathsf{Pic}\, \mathbb{X}(\boldsymbol{p}) = \left\{ \mathcal{O}(\vec{a}) \;\middle|\; \vec{a} \in L(\boldsymbol{p}) \right\}, \tag{2.32}$$

i.e. all line bundles are obtained from the structure sheaf $\mathcal{O} \equiv \mathcal{O}(0)$ by shifting its degree in $L(\boldsymbol{p})$. The dualizing sheaf is $\mathcal{O}(\vec{\omega})$. Hence

$$\tau\, \mathcal{O}(\vec{a}) = \mathcal{O}(\vec{a} + \vec{\omega}). \tag{2.33}$$

One has

$$\mathrm{Hom}(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{b})) \simeq S_{\vec{b}-\vec{a}}, \qquad \mathrm{Ext}^1(\mathcal{O}(\vec{a}), \mathcal{O}(\vec{b})) \simeq D\, S_{\vec{a}+\vec{\omega}-\vec{b}}. \tag{2.34}$$

Any non-zero morphism between line bundles is a monomorphism [13, 14]. In particular, for all line bundles L, $\mathrm{End}\, L = \mathbb{C}$. Hence, if $(\lambda : \mu) \in \mathbb{P}^1$ is not one of the special s points $(\lambda_i : \mu_i)$, we have the exact sequence

$$0 \to \mathcal{O} \xrightarrow{\lambda u + \mu v} \mathcal{O}(\vec{c}) \to \mathcal{S}_{(\lambda:\mu)} \to 0 \tag{2.35}$$

[13] For definiteness we write the p_i's in a non-decreasing order. Without loss we may assume $p_i \geq 2$.

which defines a coherent sheaf $\mathcal{S}_{(\lambda:\mu)}$ concentrated at $(\lambda:\mu)\in\mathbb{P}^1$. It is a simple object in the category $\mathsf{coh}\,\mathbb{X}(\boldsymbol{p})$ (the 'skyscraper'). At the special points $(\lambda_i:\mu_i)\in\mathbb{P}^1$ the skyscraper is not a simple object but rather it is an indecomposable of length p_i. The simple sheaves localized at the i-th special point $(\lambda_i:\mu_i)$ are the $\mathcal{S}_{i,j}$ (where $j\in\mathbb{Z}/p_i\mathbb{Z}$) defined by the exact sequences

$$0\to\mathcal{O}(j\vec{x}_i)\to\mathcal{O}((j+1)\vec{x}_i)\to\mathcal{S}_{i,j}\to 0. \tag{2.36}$$

Applying τ to these sequences we get

$$\tau\mathcal{S}_{(\lambda;\mu)}=\mathcal{S}_{(\lambda;\mu)},\qquad \tau\mathcal{S}_{i,j}=\mathcal{S}_{i,j-1}. \tag{2.37}$$

In conclusion we have[14] [13, 14]

$$\mathsf{coh}\,\mathbb{X}(\boldsymbol{p})=\mathcal{H}_+\vee\mathcal{H}_0, \tag{2.38}$$

where $\mathcal{H}_0$ is the full Abelian subcategory of finite length objects (which is a uniserial category) and $\mathcal{H}_+$ is the subcategory of *bundles*. Any non-zero morphism from a line bundle L to a bundle E is a monomorphism. For all bundles E we have a filtration [13, 14]

$$0=E_0\subset E_1\subset E_2\subset\cdots\subset E_\ell=E, \tag{2.39}$$

with E_{i+1}/E_i line bundles. Then we have an additive function $\mathsf{rank}\colon K_0(\mathsf{coh}\,\mathbb{X}(\boldsymbol{p}))\to\mathbb{Z}$, the *rank*, which is τ-invariant, zero on $\mathcal{H}_0$ and positive on $\mathcal{H}_+$. $\mathsf{rank}\,E$ is the length ℓ of the filtration (2.39); line bundles have rank 1.

In physical terms [3] $\mathcal{H}_0$ is the 'light category' which encodes the zero Yang-Mills coupling limit $g_{\mathrm{YM}}\to 0$; hence $\mathcal{H}_0$ is well understood in terms of 'perturbative' physics [3].

We define the additive function *degree*, $\mathsf{deg}\colon K_0(\mathsf{coh}\,\mathbb{X}(\boldsymbol{p}))\to\frac{1}{p}\mathbb{Z}$, by

$$\mathsf{deg}\,\mathcal{O}\Big(\textstyle\sum_i n_i\vec{x}_i\Big)=\displaystyle\sum_i\frac{n_i}{p_i}. \tag{2.40}$$

deg satisfies the four properties: *(i)* the degree is τ stable; *(ii)* $\mathsf{deg}\,\mathcal{O}=0$; *(iii)* if $\mathcal{S}$ is a simple of τ-period q one has $\mathsf{deg}\,\mathcal{S}=1/q$; *(iv)* $\mathsf{deg}\,X>0$ for all non-zero objects in $\mathcal{H}_0$.

Physically, rank is the Yang-Mills magnetic charge while deg is (a linear combination of) the Yang-Mills electric charge (and matter charges) normalized so that the W boson has charge +1. For the four weighted projective lines $\mathbb{X}_p$ with $\chi(\boldsymbol{p})=0$, eq. (2.26), the Riemann-Roch theorem reduces to the equality [12–14]

$$\frac{1}{p}\sum_{j=0}^{p-1}\big\langle[\tau^jX],[Y]\big\rangle_E=\mathsf{rank}\,X\,\mathsf{deg}\,Y-\mathsf{deg}\,X\,\mathsf{rank}\,Y. \tag{2.41}$$

[14] The notation in the RHS [13, 14, 28] stands for two properties: *(i)* all object X of $\mathsf{coh}\,\mathbb{X}(\boldsymbol{p})$ has the form $X_+\oplus X_0$ with $X_+\in\mathcal{H}_+$, $X_0\in\mathcal{H}_0$, and *(ii)* $\mathrm{Hom}(\mathcal{H}_0,\mathcal{H}_+)=0$.

JHEP09(2015)035

Explicit formulae in the canonical basis. To write explicit expressions, it is convenient to choose a set of homological generators of $\mathsf{coh}\,\mathbb{X}(\boldsymbol{p})$; their classes then give a $\mathbb{Z}$-basis of the Grothendieck group $K_0(\mathsf{coh}\,\mathbb{X}(\boldsymbol{p}))$. It is convenient to choose the generators to be the direct summands of a tilting object of $\mathsf{coh}\,\mathbb{X}(\boldsymbol{p})$ [12–14]. We choose the *canonical*[15] such tilting object whose endomorphism algebra is the Ringel canonical algebra $\Lambda(\boldsymbol{p})$ of type $(\boldsymbol{p})$ [12–14]. The canonical generating set consists of the following $n \equiv \sum_i (p_i - 1) + 2$ line bundles

$$\mathcal{O}, \qquad \mathcal{O}(\ell \vec{x}_i) \text{ (with } i = 1, \ldots, s,\ \ell = 1, \ldots, p_i - 1), \qquad \mathcal{O}(\vec{c}). \tag{2.42}$$

By definition of tilting object, Ext^1 vanishes between any pair of sheaves in eq. (2.42), while the only non-zero Hom spaces are

$$\dim \mathrm{Hom}(\mathcal{O}, \mathcal{O}(\vec{c})) = 2, \qquad \dim \mathrm{Hom}(\mathcal{O}(k_i \vec{x}_i), \mathcal{O}(\ell_i \vec{x}_i)) = 1, \quad 0 \leq k_i \leq \ell_i \leq p_i, \tag{2.43}$$

where, for all i, $\mathcal{O}(0\,\vec{x}_i) \equiv \mathcal{O}$ and $\mathcal{O}(p_i \vec{x}_i) \equiv \mathcal{O}(\vec{c})$.

We write $(\phi_1, \ldots, \phi_n)$ for the elements of the basis of $K_0(\mathsf{coh}\,\mathbb{X}(\boldsymbol{p}))$ given by the Grothendieck classes of the n line bundles in eq. (2.42) ordered so that the $\phi_1 = [\mathcal{O}]$, $\phi_n = [\mathcal{O}(\vec{c})]$ while the $\{\phi_a\}_{a=2}^{n-1}$ are the $[\mathcal{O}(j\vec{x}_i)]$ listed in the (i, j) lexicographic order. The YM magnetic and electric charges of the generating sheaves (2.42) are

$$(\mathsf{rank}\,\phi_1, \ldots, \mathsf{rank}\,\phi_n) = (1, 1, \ldots, 1, 1) \equiv M^t \tag{2.44}$$

$$p(\mathsf{deg}\,\phi_1, \ldots, \mathsf{deg}\,\phi_n) = (0, q_{1,1}, q_{1,2}, \ldots, q_{1,p_1-1}, \ldots\ldots, q_{s,1}, q_{s,2}, \ldots, q_{s,p_s-1}, p) \equiv Q^t. \tag{2.45}$$

Specializing to the $\chi(\boldsymbol{p}) = 0$ case. For the $\chi(\boldsymbol{p}) = 0$ weighted projective lines, eq. (2.26), the electric charges

$$q_{i,1}, \cdots, q_{i,p_i-1}, p \tag{2.46}$$

of the sheaves (2.42) are just the Coxeter labels on the i-th branch of the associated affine star Dynkin graph $\mathfrak{g}^{(1)}$ numbered in increasing order from the most peripheral node to the vertex of the star (which has label p). Explicitly,

$$q_{i,j} = \frac{p}{p_i}\, j. \tag{2.47}$$

The Euler form between the elements of the basis $E_{ab} = \langle \phi_a, \phi_b \rangle_E$ is given by the unipotent (upper triangular) block matrix

$$E = \left(\begin{array}{c|c|c|c|c} 1 & 1^t_{p_1} & \cdots & 1^t_{p_s} & 2 \\ \hline 0 & T_{p_1} & \cdots & 0 & 1_{p_1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & T_{p_s} & 1_{p_s} \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right) \tag{2.48}$$

[15] In view of [3] the existence of this tilting object justifies our claim that the BPS particles of the relevant QFT correspond to stable objects of the derived category $\mathsf{D}^b\mathsf{coh}\,\mathbb{X}(\boldsymbol{p}) \equiv \mathsf{D}^b\mathsf{mod}\,\Lambda(\boldsymbol{p})$. A more detailed analysis is presented in appendix B.

JHEP09(2015)035

where 1_p stands for the column $(p-1)$-vector with 1's in all entries, and T_p is the $(p-1)\times(p-1)$ triangular matrix with 1's along the main diagonal and everywhere above it.

In the canonical basis the Coxeter element is represented by the matrix $\mathbf{\Phi}_{ab}$ such that $\tau\phi_a = \mathbf{\Phi}_{ab}\,\phi_b$; comparing with eqs. (2.15), (2.16) we get

$$\mathbf{\Phi} = -E(E^t)^{-1}. \tag{2.49}$$

The radical of the Tits form q is generated by the two vectors

$$R_1 = E^{-1}M \equiv (-1, 0, \cdots, 0, 1)^t, \tag{2.50}$$

$$R_2 = E^{-1}Q. \tag{2.51}$$

In particular, we note that the last two entries of the radical vector $E^{-1}(pM - Q)$ are $(\cdots, 1, 0)^t$. It follows that in this basis we may identify the matter charge lattice $\Gamma_{\text{matter}} \equiv \Gamma/\operatorname{rad} q$ with the sublattice $\widehat{\Gamma}_{\text{matter}} \subset \Gamma$ of vectors of the form

$$\boldsymbol{x} \equiv (x_\star, x_{1,1}, \cdots, x_{1,p_1-1}, \cdots\cdots, x_{s,1}, \cdots, x_{s,p-2}, 0, 0) \subset \mathbb{Z}^n \simeq \Gamma. \tag{2.52}$$

Since E is upper triangular and unimodular, it maps a basis of the sublattice $\widehat{\Gamma}_{\text{matter}}$ into a $\mathbb{Z}$-equivalent basis. Then we write the vectors (2.52) in the form

$$\boldsymbol{x} = E^{-1}\,\boldsymbol{y}, \qquad \boldsymbol{y} \in \widehat{\Gamma}_{\text{matter}}. \tag{2.53}$$

Let $\dot{E}$ (resp. $\ddot{E}$) be the principal submatrix of E obtained by omitting the last (resp. the last two) row(s) and column(s). Essentially by definition,

$$\dot{E}^{-1} + (\dot{E}^{-1})^t = C_{\mathfrak{g}^{(1)}} = \left[\begin{array}{l}\text{the Cartan matrix of the}\\ \text{affine Lie algebra } \mathfrak{g}^{(1)}\end{array}\right. \tag{2.54}$$

$$\ddot{E}^{-1} + (\ddot{E}^{-1})^t = C_{\mathfrak{g}} = \left[\begin{array}{l}\text{the Cartan matrix of the}\\ \text{finite-type Lie algebra } \mathfrak{g}.\end{array}\right. \tag{2.55}$$

Then the restricted Tits form $\overline{q}$ on $\Gamma_{\text{matter}} \simeq \widehat{\Gamma}_{\text{matter}}$ is

$$\overline{q}(\boldsymbol{x}) = q_{\mathfrak{g}}(\boldsymbol{y}), \tag{2.56}$$

where $q_{\mathfrak{g}}(\boldsymbol{y}) = \frac{1}{2}\boldsymbol{y}^t C_{\mathfrak{g}}\boldsymbol{y}$ is the Tits form of the finite-dimensional Lie algebra $\mathfrak{g}$. The integers $(\boldsymbol{y}) = (y_\star, y_{i,j})$ are attached to the vertices of the Dynkin graph $\mathfrak{g}$ as in figure 1. In particular, $\overline{q}$ and $q_{\mathfrak{g}}$ are $\mathbb{Z}$-equivalent; the isometry between the two Tits forms is just multiplication by E^{-1}. Under this isometry

$$\widehat{\Gamma}_{\text{matter}} \cong \Gamma_{\mathfrak{g}}. \tag{2.57}$$

Remark. Here we defined the matter charges as equivalence classes in $\Gamma/\operatorname{rad} q$. Of course, the physical matter charges are specific representatives of these classes. In section 3.1 we shall use the physical definition. However, the action of the S-duality group is independent of the choice of representatives, and often a different choice simplifies the computations.

JHEP09(2015)035

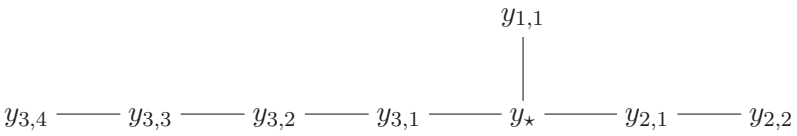


Figure 1. The assignments of integral variables ($\boldsymbol{y}$) to the nodes of the E_8 graph. For D_4, E_6, E_7 just restrict to the corresponding Dynkin subgraph.

Structure of the derived category $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}(\boldsymbol{p})$. The Abelian category $\mathcal{H} = \mathsf{coh}\,\mathbb{X}(\boldsymbol{p})$ is *hereditary*. The derived category $\mathsf{D}^b\mathcal{H}$ of a hereditary Abelian category $\mathcal{H}$ coincides with its repetitive category $\mathsf{rep}\,\mathcal{H}$ [13, 14].

Definition. Let $\mathcal{A}$ be an Abelian category. Its *repetitive category* $\mathsf{rep}\,\mathcal{A}$ is

$$\mathsf{rep}\,\mathcal{A} = \bigvee_{n\in\mathbb{Z}} \mathcal{A}[n], \tag{2.58}$$

whose objects are of the form

$$A = \bigoplus_{n\in\mathbb{Z}} A_n[n] \tag{2.59}$$

with $A_n \in \mathcal{A}$ and only finitely many A_n's non-zero; the morphisms are

$$\mathrm{Hom}(A[m], B[n]) = \mathrm{Ext}^{n-m}(A, B) \tag{2.60}$$

with Yoneda compositions. The notation in eq. (2.58) stands for eq. (2.59) together with the fact that $\mathrm{Hom}(A[m], B[n]) = 0$ for $m > n$. The translation functor is $A[m] \mapsto A[m+1]$.

Thus, to describe $\mathsf{D}^b\mathcal{H}$, it is enough to study the Abelian category $\mathcal{H}$.

The *slope* $\mu(E)$ of a coherent sheaf E is the ratio of its degree and rank[16]

$$\mu(E) = \mathsf{deg}\,E/\mathsf{rank}\,E. \tag{2.61}$$

A coherent sheaf E is *semi-stable* (resp. *stable*[17]) if for all non-zero subsheaf F one has $\mu(F) \le \mu(E)$ (respectively $\mu(F) < \mu(E)$).

The $\boldsymbol{\chi(p) = 0}$ case. We restrict ourselves to the four weighted projective lines with $\chi = 0$, i.e. to $\mathbb{X}_p$ with $p = 2, 3, 4, 6$. In this case all indecomposable coherent sheaf is semi-stable [12, 13]. Let $\mathcal{H}^{(q)}$ be the full (hereditary) Abelian subcategory of semi-stable objects of slope $q \in \mathbb{Q}\cup\infty$. For all q one has $\mathcal{H}^{(q)} \simeq \mathcal{H}_0$ the full subcategory of finite-length objects. Then

$$\mathcal{H} = \bigvee_{q\in\mathbb{Q}\cup\infty} \mathcal{H}^{(q)} \tag{2.62}$$

[16] By convention, the zero object has all slopes.

[17] This notion of stability is related but *distinct* from the one relevant for the existence of BPS states we mentioned at the beginning of this section. That notion is based on a stability function (the $\mathcal{N} = 2$ central charge Z) which depends on the couplings, masses and Coulomb branch point.

JHEP09(2015)035

and hence

$$\mathsf{D}^b\,\mathcal{H} = \bigvee_{n\in\mathbb{Z}} \bigvee_{q\in\mathbb{Q}\cup\infty} \mathcal{H}^{(q)}[n], \tag{2.63}$$

where the notation implies that $\mathrm{Hom}_{\mathcal{H}}(\mathcal{H}^{(q)}, \mathcal{H}^{(q')}) = 0$ for $q > q'$. $\mathcal{H}_0$ (and then $\mathcal{H}^{(q)}$ for all q) is a $\mathbb{P}^1$-family of stable tubes all of which are homogenous but the ones over the three (or four) special points $(\lambda_i : \mu_i)$ which have periods $p_i > 1$.

In conclusion, all indecomposable object X of the derived category $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_p$ belongs to a $\mathcal{H}^{(q)}[n]$ for some $q \in \overline{\mathbb{Q}}$ and $n \in \mathbb{Z}$; all these subcategories $\mathcal{H}^{(q)}[n]$ are equivalent to the 'perturbative' category $\mathcal{H}_0$ and hence physically well understood [3].

2.3 Telescopic functors and $\mathcal{B}_3$ braid group action on $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_p$

If our $\mathcal{N} = 2$ theory has S-duality, the duality should be, in particular, a property of its BPS sector. Hence the duality should act by automorphisms of the relevant derived category or, more precisely, of its orbit category (2.1). For the five SU(2) SCFTs we expect the S-duality group $\mathbb{S}$ to contain a $\mathrm{PSL}(2,\mathbb{Z})$ from the internal torus of its 6d construction. Before entering in the technical details, let us see why this fact is rather natural in view of the peculiar form of the derived category as described in eq. (2.63). We may identify $\overline{\mathbb{Q}} \equiv \mathbb{Q} \cup \infty$ with the projective line over the field $\mathbb{Q}$. The group $\mathrm{PSL}(2,\mathbb{Z})$ naturally acts on $\mathbb{P}^1(\mathbb{Q})$. Then the structure in the RHS of (2.63) suggests the existence of an action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_p$ which sends an object X of slope $q \in \mathbb{P}^1(\mathbb{Q})$ into an object of slope $q' = \gamma \cdot q \in \mathbb{P}^1(\mathbb{Q})$ for $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$. Since the slope is essentially the ratio of the Yang-Mills electric and magnetic charges, such a natural $\mathrm{PSL}(2,\mathbb{Z})$ action will have the physical interpretation of the electro-magnetic S-duality.

In order to implement this idea, we first need to normalize correctly the Yang-Mills charges so that they are integral, while the degree is quantized in units of $1/p$. So, for all object X of the derived category, we set

$$YM(X) \equiv \begin{pmatrix} p\,\deg X \\ \mathsf{rank}\, X \end{pmatrix} \in \mathbb{Z}^2, \tag{2.64}$$

and call the integral 2-vector $YM(X)$ the *Yang-Mills charges of* X. Then we have to construct auto-equivalences of the derived category $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_p$ which induce on the Yang-Mills charges $YM(\cdot)$ an action of $\mathrm{SL}(2,\mathbb{Z})$. To be concrete, consider the two matrices

$$\mathsf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{L} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{2.65}$$

T and L generate $\mathrm{SL}(2,\mathbb{Z})$. Indeed, the braid group on three strands, $\mathcal{B}_3$, is generated by two elements T, L subject to the single relation

$$\mathsf{TLT} = \mathsf{LTL}, \tag{2.66}$$

while we have

$$1 \to Z(\mathcal{B}_3) \to \mathcal{B}_3 \to \mathrm{PSL}(2,\mathbb{Z}) \to 1, \tag{2.67}$$

JHEP09(2015)035

where the center of the braid group, $Z(\mathcal{B}_3)$, is the infinite cyclic group generated by $(\mathsf{LT})^3$. The two 2×2 matrices in eq. (2.65) satisfy the braid relation (2.66) as well as $(\mathsf{LT})^3 = -1$, and hence generate the full $\mathrm{SL}(2,Z)$.

To prove that the electro-magnetic duality group $\mathrm{SL}(2,\mathbb{Z})$ is part of $\mathrm{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_p)$ we need to construct two functors T and L, which are auto-equivalences of the triangulated category $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_p$, and have the property that

$$YM(T(X)) = \mathsf{T}\,YM(X), \qquad YM(L(X)) = \mathsf{L}\,YM(X), \tag{2.68}$$

so that the subgroup of $\mathrm{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_p)$ generated by the two functors T, L will induce an $\mathrm{SL}(2,\mathbb{Z})$ action on the Yang-Mills charges $YM(\cdot)$. Such auto-equivalences T, L do exist: they are called *telescopic functors* [15, 16, 33].

T is simply the functor which shifts the $L(\boldsymbol{p})$ degree of the sheaf by $\vec{x}_3$ [14, 16, 33]

$$X \longmapsto X(\vec{x}_3) \equiv T(X), \tag{2.69}$$

where we ordered the weights so that $p_3 \equiv p$ is the largest one. One can see the map $X \longmapsto T(X)$ as the completion of a canonical map to a triangle of $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_p$ [15, 16, 33]

$$\bigoplus_{j=0}^{p-1} \mathrm{Hom}^\bullet(\tau^j\mathcal{S}_{3,0}, X)\otimes\tau^j\mathcal{S}_{3,0} \xrightarrow{\mathrm{can_X}} X \longrightarrow T(X), \tag{2.70}$$

so that T induces the following action on the Grothendieck group Γ [33]

$$[T(X)] = [X] - \sum_{j=0}^{p-1}\langle\mathcal{S}_{3,j}, X\rangle_E\,[\mathcal{S}_{3,j}]. \tag{2.71}$$

Explicitly, the action on the generating set $\mathcal{O}$, $\mathcal{S}_{i,j}$ is given by

$$T(\mathcal{O}) = \mathcal{O}(\vec{x}_3), \quad T(\mathcal{S}_{3,j}) = \mathcal{S}_{3,j+1}, \quad T(\mathcal{S}_{i,j}) = \mathcal{S}_{i,j} \text{ for } i\neq 3. \tag{2.72}$$

Thus T preserves the rank, while increases the degree by $1/p$ times the rank; therefore

$$YM(T(X)) = \mathsf{T}\,YM(X), \tag{2.73}$$

as required. The definition of the second functor L is similar; one introduces the triangle

$$\bigoplus_{j=0}^{p-1} \mathrm{Hom}^\bullet(\tau^j\mathcal{O}, X)\otimes\tau^j\mathcal{O} \xrightarrow{\mathrm{can_X}} X \longrightarrow L(X), \tag{2.74}$$

and proves that $X \longmapsto L(X)$ is an auto-equivalence of the derived category, see [15, 16]. The action of L on the Grothendieck group is then

$$[L(X)] = [X] - \sum_{j=0}^{p-1}\langle\tau^j\mathcal{O}, X\rangle_E\,[\tau^j\mathcal{O}]. \tag{2.75}$$

JHEP09(2015)035

This formula shows that $\mathsf{deg}\, L(X) = \mathsf{deg}\, X$ while

$$\mathsf{rank}\, L(X) = \mathsf{rank}\, X - \sum_{j=0}^{p-1} \langle \tau^j \mathcal{O}, X\rangle_E = \mathsf{rank}\, X - p\, \mathsf{deg}\, X, \tag{2.76}$$

where we used the Riemann-Roch theorem (2.41). Thus we get the desired property

$$YM(L(X)) = \mathsf{L}\, YM(X). \tag{2.77}$$

The explicit action on the generating set $\mathcal{O}$, $\mathcal{S}_{i,j}$ $(j = 0, 1, \ldots, p_i - 1)$ is

$$L(\mathcal{O}) = \tau^{-1}\mathcal{O} \equiv \mathcal{O}(-\vec{\omega}), \tag{2.78}$$

$$L(\mathcal{S}_{i,j}) = \ker\left[\bigoplus_{k=1}^{p/p_i} \mathcal{O}\Big((kp_i - 1 - j)\vec{\omega}\Big) \xrightarrow{\text{can}} \mathcal{S}_{i,j}\right][1], \tag{2.79}$$

which, in the particular case $p_i = p$, reduces to

$$L(\mathcal{S}_{i,j}) = \mathcal{O}\big(-\vec{x}_i + (p - 1 - j)\vec{\omega}\big)[1]. \tag{2.80}$$

The $\mathcal{B}_3$ braid group relation. It is easy to see that [15, 16]

$$LTL = TLT. \tag{2.81}$$

As an illustration (and to establish a few useful equalities), we check that the two sides of the equality act in the same way on the structure sheaf $\mathcal{O}$ and on the simple sheaves $\mathcal{S}_{3,j}$ (assuming $p_3 = p$). By PROPOSITION 5.3.4 of [16] this suffices to conclude that the two automorphisms of $\mathsf{D}^b\, \mathsf{coh}\, \mathbb{X}$ are at least *isomorphic*. Since τ commutes with T, L,

$$LTL(\mathcal{O}) = LT(\tau^{-1}\mathcal{O}) = \tau^{-1}L(\mathcal{O}(\vec{x}_3)), \tag{2.82}$$

while the triangle

$$L(\mathcal{S}_{3,0})[-1] \to L(\mathcal{O}) \to L(\mathcal{O}(\vec{x}_3)) \tag{2.83}$$

gives

$$L(\mathcal{O}(\vec{x}_3)) = \mathcal{S}_{3,0} \quad \Longrightarrow \quad LTL(\mathcal{O}) = \mathcal{S}_{3,1}. \tag{2.84}$$

On the other hand,

$$TLT(\mathcal{O}) = TL(\mathcal{O}(\vec{x}_3)) = T(\mathcal{S}_{3,0}) = \mathcal{S}_{3,1} \equiv LTL(\mathcal{O}). \tag{2.85}$$

By (2.80) we have

$$TLT(\mathcal{S}_{3,j}) = \tau^{-(j+2)}\mathcal{O}[1]. \tag{2.86}$$

Eq. (2.80) gives the triangle

$$\begin{aligned} &\tau^{j+1}L(\mathcal{S}_{3,j})[-1] \to \mathcal{O} \to \mathcal{S}_{3,p-1} \\ &\xRightarrow[\text{and rotate}]{\text{apply } \tau LT} \quad \tau L(\mathcal{S}_{3,0})[-1] \to \tau^{j+2}\, LTL(\mathcal{S}_{3,j})[-1] \to \mathcal{S}_{3,p-1}. \end{aligned} \tag{2.87}$$

JHEP09(2015)035

Since $\tau L(\mathcal{S}_{3,0})[-1] \equiv \mathcal{O}(-\vec{x}_3)$ (cf. (2.80)), the above triangle yields

$$LTL(\mathcal{S}_{3,j}) = \tau^{-(j+2)}\mathcal{O}[1] \tag{2.88}$$

in agreement with eqs. (2.81), (2.86).

Eq. (2.81) says that the two functors L, T generate a subgroup of $\mathrm{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X})$ which is isomorphic to the braid group $\mathcal{B}_3$. Note that the functor TLT acts on $YM(X)$ by the matrix S

$$\mathsf{S} := \mathsf{TLT} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.89}$$

Arguing as in ref. [3] one concludes that the functor TLT has the physical effect of interchanging weak and strong Yang-Mills coupling $g_{\mathrm{YM}} \longleftrightarrow 1/g_{\mathrm{YM}}$.

The center $Z(\mathcal{B}_3)$ of the braid group. In view of eq. (2.67), we have $\mathrm{PSL}(2,\mathbb{Z}) = \mathcal{B}_3/Z(\mathcal{B}_3)$. The center $Z(\mathcal{B}_3)$ of $\mathcal{B}_3$ is the infinite cyclic group generated by $(TL)^3$ which acts on the Yang-Mills charges as the non-trivial element of the center of $\mathrm{SL}(2,\mathbb{Z})$

$$(TL)^3 \colon YM(X) \longmapsto -YM(X). \tag{2.90}$$

Using eqs. (2.85), (2.86), we get

$$(TL)^3(\mathcal{O}) = TLT \cdot LTL(\mathcal{O}) = TLT(\mathcal{S}_{3,1}) = \tau^{-3}\mathcal{O}[1], \tag{2.91}$$

$$(TL)^3(\mathcal{S}_{3,j}) = TLT \cdot LTL(\mathcal{S}_{3,j}) = \tau^{-(j+2)}TLT(\mathcal{O})[1] = \tau^{-(j+2)}\mathcal{S}_{3,1}[1] = \tau^{-3}\mathcal{S}_{3,j}[1]. \tag{2.92}$$

So (again by PROPOSITION 5.3.4 of [16]) we have the isomorphism of triangle functors

$$(TL)^3 \simeq \tau^{-3}\Sigma, \tag{2.93}$$

where Σ stands for the shift functor, $\Sigma(X) = X[1]$. Hence $(TL)^3$ induces an automorphism $(\boldsymbol{TL})^3$ of the Grothendieck lattice Γ of order $2, 2, 4, 2$ for $p = 2, 3, 4, 6$, respectively.

Consider the automorphism $\tau^3(TL)^3\Sigma^{-1}$: *(i)* it fixes the structure sheaf and the simples $\mathcal{S}_{3,j}$, *(ii)* it preserves the slope $\mu(X) = \mathsf{deg}(X)/\mathsf{rank}(X)$, and *(iii)* its action on Γ has order dividing $2, 2, 4, 2$, respectively. The group of slope preserving automorphisms of $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}$ which fix $\mathcal{O}$ is precisely $\mathrm{Aut}(\mathbb{X})$ i.e. (essentially) the permutations of equal weight special points in $\mathbb{P}^1$. Thus $\tau^3(TL)^3\Sigma^{-1}$ is a permutation of the special points which respects their weights and leaves the third one fixed (by convention $p_3 = p$). Hence $\tau^3(TL)^3\Sigma^{-1} = 1$ for $p = 4, 6$. For $p = 2$, $\tau^3(TL)^3\Sigma^{-1}$ is either the identity or an order 2 permutation of three objects (x_1, x_2, x_4) which treats the three objects on the same footing; hence $\tau^3(TL)^3\Sigma^{-1} = 1$. For $p = 3$

$$\tau^3(TL)^3\Sigma^{-1} \equiv (TL)^3\Sigma^{-1}, \tag{2.94}$$

is either the identity or the permutation π_{12} of the first two special points. From their explicit action on the Grothendieck group, see section A.1, we conclude for the second possibility. Hence

$$(TL)^3 = \begin{cases} \tau^{-3}\Sigma & p \neq 3 \\ \pi_{12}\,\Sigma & p = 3, \end{cases} \tag{2.95}$$

where π_{12} is the automorphism which interchanges the first two special points i.e., in terms of the canonical generating set (2.42), $\mathcal{O}(\ell\vec{x}_1) \leftrightarrow \mathcal{O}(\ell\vec{x}_2)$ for $\ell = 1, 2$.

We note that $(TL)^3$ acts as -1 on the Yang-Mills charges and as $+1$ on the flavor charges.

2.4 The groups $\mathrm{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X})$ and $\mathbb{S} \equiv \mathrm{Aut}\,\mathcal{C}(\mathbb{X})$

The subgroup $\mathcal{B}_3$ generated by L, T is not the full automorphism group $\mathrm{Aut}(\mathsf{D}^b\mathsf{coh}\,\mathbb{X})$. We have a surjection of the automorphism group on $\mathcal{B}_3$ whose kernel is the group of the automorphisms of the Abelian category $\mathsf{coh}\,\mathbb{X}$ which fix degree and rank. This is the group $\mathsf{Pic}(\mathbb{X})^0 \ltimes \mathrm{Aut}(\mathbb{X})$, where $\mathsf{Pic}(\mathbb{X})^0$ is the group of degree zero line bundles and $\mathrm{Aut}(\mathbb{X})$ is the group of geometric automorphisms of $\mathbb{X}$, essentially[18] the group of permutations of the special points having the same weight p_i. Thus, for a weighted projective line $\mathbb{X}$ having zero Euler characteristic, $\chi(\mathbb{X}) = 0$, we have [15, 16]

$$1 \to \mathsf{Pic}(\mathbb{X})^0 \ltimes \mathrm{Aut}(\mathbb{X}) \to \mathrm{Aut}(\mathsf{D}^b\mathsf{coh}\,\mathbb{X}) \to \mathcal{B}_3 \to 1. \tag{2.96}$$

In particular, the derived auto-equivalence

$$(TL)^3(\tau^{-1}\Sigma)^{-1} = \begin{cases} \mathcal{O}(-2\vec{\omega}) \otimes - & p \neq 3 \\ \pi_{12}\,\mathcal{O}(\vec{\omega}) \otimes - & p = 3 \end{cases} \in \mathsf{Pic}(\mathbb{X})^0 \ltimes \mathrm{Aut}(\mathbb{X}), \tag{2.97}$$

belongs to the kernel of $\mathrm{Aut}(\mathsf{D}^b\mathsf{coh}\,\mathbb{X}) \to \mathcal{B}_3$.

The cluster category of the weighted projective line $\mathbb{X}$ is defined to be the orbit category of $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}$ with respect to the cyclic subgroup generated by $\tau^{-1}\Sigma$

$$\mathcal{C}(\mathbb{X}) = \mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}/\langle\tau^{-1}\Sigma\rangle. \tag{2.98}$$

Then comparing eqs. (2.96), (2.67), and (2.97) we get:

Let $\mathbb{X}$ be a weighted projective line with $\chi(\mathbb{X}) = 0$ and $\mathcal{C}(\mathbb{X}) = \mathsf{D}^b(\mathsf{coh}\,\mathbb{X}_2)/\langle\tau^{-1}\Sigma\rangle$ its cluster category. Then (cf. **Proposition 7.4** of [17])

$$1 \to \mathsf{Pic}(\mathbb{X})^0 \ltimes \mathrm{Aut}(\mathbb{X}) \to \mathrm{Aut}(\mathcal{C}(\mathbb{X})) \to \mathrm{PSL}(2,\mathbb{Z}) \to 1. \tag{2.99}$$

We call the group $\mathrm{Aut}\,\mathcal{C}(\mathbb{X})$ the *full S-duality group* written $\mathbb{S}$

$$\mathbb{S} = \big(\mathsf{Pic}(\mathbb{X})^0 \ltimes \mathrm{Aut}(\mathbb{X})\big) \ltimes \mathrm{PSL}(2,\mathbb{Z}). \tag{2.100}$$

$\mathbb{S}$ is the S-duality group of the four SCFT $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. Its quotient group acting effectively on the Yang-Mills charges (modulo the Weyl action of the gauge group) is $\mathrm{PSL}(2,\mathbb{Z})$. The physical interpretation of the kernel subgroup

$$\mathscr{W} \equiv \mathsf{Pic}(\mathbb{X})^0 \ltimes \mathrm{Aut}(\mathbb{X}) \tag{2.101}$$

will be discussed in section 3.

[18] For $D_4^{(1,1)}$ the situation is slightly subtler [15, 16]. We shall ignore this aspect.

JHEP09(2015)035

2.5 Some useful formulae

Note that both our generating functors $L, T \in \mathrm{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X})$ are of the form $X \longmapsto \mathcal{L}_Y(X)$ where, for a fixed object Y, the functor $\mathcal{L}_Y$ is defined by the triangle

$$\bigoplus_{k=0}^{p-1} \mathrm{Hom}^\bullet(\tau^k Y, X) \otimes \tau^k Y \xrightarrow{\ \mathrm{can}\ } X \longrightarrow \mathcal{L}_Y(X). \tag{2.102}$$

Indeed one has

$$L = \mathcal{L}_{\mathcal{O}}, \qquad T = \mathcal{L}_{\mathcal{S}_{3,0}}. \tag{2.103}$$

A basic theorem (see THEOREM 5.1.3 of [16]) states that, for a weighted projective line with $\chi(\mathbb{X}) = 0$, a functor $X \longmapsto \mathcal{L}_Y(X)$ is an auto-equivalence of the derived category iff *(i)* p is length of the τ-orbit of the object Y, and *(ii)* the object Y is *quasi-simple*. These two conditions imply, in particular,[19]

$$\langle \tau^j Y, \tau^k Y\rangle_E = \dim \mathrm{Hom}(\tau^j Y, \tau^k Y) - \dim \mathrm{Hom}(\tau^k Y, \tau^{j+1} Y) = \delta^{(p)}_{j,k} - \delta^{(p)}_{j,k-1}, \tag{2.104}$$

from which we get

$$\frac{1}{p}\sum_{j,k=0}^{p-1} \exp\big[2\pi i(j-k)s/p\big]\ \langle \tau^j Y, \tau^k Y\rangle_E = 1 - \exp\big[-2\pi i s/p\big]. \tag{2.105}$$

We shall see in section 4.2 that this last equation guarantees that the induced action in the Grothendieck group

$$[X] \longmapsto [\mathcal{L}_Y, X] = [X] - \sum_{p=0}^{p-1} \big\langle [\tau^k Y], [X]\big\rangle_E\ [\tau^k Y], \tag{2.106}$$

is an isometry of the Euler form i.e.

$$\big\langle \mathcal{L}_Y X, \mathcal{L}_Y Z\big\rangle_E = \big\langle X, Z\big\rangle_E, \tag{2.107}$$

which is (obviously) a necessary condition in order $\mathcal{L}_Y$ to be an auto-equivalence.

3 Warm-up: SU(2) SQCD with $N_f = 4$ again

As a warm-up, we consider again $\mathcal{N} = 2$ SQCD with $G_{\mathrm{gauge}} = \mathrm{SU}(2)$ and four flavors of quarks (the $D_4^{(1,1)}$ model). The eight quark states with electric charge $+1$ transform in the vector representation of SO(8). Seen as coherent sheaves on the weighted projective line $\mathbb{X}_2$ of weights $(2,2,2,2)$ these quark states correspond to the eight exceptional simple sheaves

$$\mathcal{S}_{i,j} \qquad i = 1,2,3,4, \quad j = 0,1. \tag{3.1}$$

In the $\mathbb{X}_2$ case $\tau^2 = \mathrm{Id}$, so $\mathbf{\Phi}^2 = 1$. The Yang-Mills charges correspond to the $(+1)$-eigenvectors of $\mathbf{\Phi}$ and the flavor charges to the (-1)-eigenvectors.

To make everything very transparent, in the next subsection we illustrate how flavor Spin(8) triality arise in concrete terms i.e. writing down explicit expressions in the basis of Γ_{matter} which is standard in physics. Then in section 3.2 we present a more elegant abstract viewpoint. The reader may prefer to skip section 3.1.

[19] Here $\delta^{(p)}_{j,k}$ is the mod p Kronecker delta, i.e. $\delta^{(p)}_{j,k} = 1$ if $j = k \mod p$ and zero otherwise.

JHEP09(2015)035

3.1 Flavor charges and flavor weigths

The four linear independent flavor charges are

$$\alpha_1 = [\mathcal{O}(\vec{x}_3)] - [\mathcal{O}(\vec{x}_4)] = [\mathcal{S}_{3,0}] - [\mathcal{S}_{4,0}] \tag{3.2}$$
$$\alpha_2 = [\mathcal{O}(\vec{x}_2)] - [\mathcal{O}(\vec{x}_3)] = [\mathcal{S}_{2,0}] - [\mathcal{S}_{3,0}] \tag{3.3}$$
$$\alpha_3 = [\mathcal{O}] + [\mathcal{O}(\vec{c})] - [\mathcal{O}(\vec{x}_1)] - [\mathcal{O}(\vec{x}_2)] = [\mathcal{S}_{1,1}] - [\mathcal{S}_{2,0}] \tag{3.4}$$
$$\alpha_4 = [\mathcal{O}(\vec{x}_1)] - [\mathcal{O}(\vec{x}_2)] = [\mathcal{S}_{1,0}] - [\mathcal{S}_{2,0}] \tag{3.5}$$

i.e. the Grothendieck classes which have zero rank and zero degree. Note that in the Grothendieck group we have the relation $[\mathcal{S}_{i,0}] - [\mathcal{S}_{i+1,0}] = [\mathcal{S}_{i+1,1}] - [\mathcal{S}_{i,1}]$ for all i. The Euler pairing restricted to the sublattice $\Gamma_{\text{flavor}} \subset \Gamma$ generated by the charges $\{\alpha_a\}_{a=1}^4$ is

$$\langle \alpha_a, \alpha_b \rangle_E \equiv \alpha_a^t E \alpha_b = \text{the Cartan matrix of the } D_4 \text{ graph} \qquad \begin{array}{ccccc} & & \alpha_1 & & \\ & & | & & \\ \alpha_3 & - & \alpha_2 & - & \alpha_4 \end{array} \tag{3.6}$$

Γ_{flavor} is thus identified with the root lattice of Spin(8). An object X of $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_2$ then carries a weight $(w_1(X), w_2(X), w_3(X), w_4(X))$ of the flavor Spin(8). Explicitly,

$$w_a(X) = \langle \alpha_a, [X] \rangle_E \equiv \langle [X], \alpha_a \rangle_E. \tag{3.7}$$

(Note that the Euler form is symmetric if one of its arguments is a flavor charge).

Using (3.7) we easily compute the flavor weights of *(i)* the exceptional simples (3.1), *(ii)* the degree zero line bundles (which correspond to monopoles with magnetic charge +1 and zero electric charge), and *(iii)* the degree 1 line bundles (dyons with unit electric and magnetic charges), see table 2. We see that the quarks have the weights of the vector representation $\boldsymbol{v}$ of Spin(8), the monopoles of the spinorial representation $\boldsymbol{s}$, and the dyons of the spinorial representation $\boldsymbol{c}$. Of course, this is the physically correct result [6].

PSL(2, $\mathbb{Z}$) and triality. Let us check how the two derived auto-equivalences T and L of section 2.3 act on the Yang-Mills and flavor charges. T acts on the generating set $\mathcal{O}$, $\mathcal{S}_{i,j}$ as[20]

$$\begin{aligned} T(\mathcal{O}) &= \mathcal{O}(\vec{x}_1), \\ T(\mathcal{S}_{1,0}) &= \mathcal{S}_{1,1}, \qquad T(\mathcal{S}_{1,1}) = \mathcal{S}_{1,0}, \\ T(\mathcal{S}_{i,j}) &= \mathcal{S}_{i,j} \qquad i = 2,3,4, \quad j = 0,1 \end{aligned} \tag{3.8}$$

which is consistent with the expected action on the Yang-Mills charges[21]

$$\begin{pmatrix} 2\,\mathsf{deg}\,T(X) \\ \mathsf{rank}\,T(X) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\,\mathsf{deg}\,X \\ \mathsf{rank}\,X \end{pmatrix}, \tag{3.9}$$

[20] In this subsection we change our conventions from $T\colon X \to X(\vec{x}_3)$ to $T\colon X \to X(\vec{x}_1)$ to facilitate comparison with standard conventions in SO(8) representation theory.

[21] Note that the factor 2 in the upper entry of the Yang-Mills charge vector for $N_f = 4$ as compared to $\mathcal{N} = 2^*$ precisely corresponds to the discussion in section 16 of [6]. In the general case 2 gets replaced by p.

JHEP09(2015)035

sheaf	(w_1, w_2, w_3, w_4)	sheaf	(w_1, w_2, w_3, w_4)
$\mathcal{S}_{1,0}$	$(0, 0, -1, 1)$	$\mathcal{S}_{1,1}$	$(0, 0, 1, -1)$
$\mathcal{S}_{2,0}$	$(0, 1, -1, -1)$	$\mathcal{S}_{2,1}$	$(0, -1, 1, 1)$
$\mathcal{S}_{3,0}$	$(1, -1, 0, 0)$	$\mathcal{S}_{3,1}$	$(-1, 1, 0, 0)$
$\mathcal{S}_{4,0}$	$(-1, 0, 0, 0)$	$\mathcal{S}_{4,1}$	$(1, 0, 0, 0)$

Spin(8) weights of exceptional simple sheaves

sheaf	(w_1, w_2, w_3, w_4)	sheaf	(w_1, w_2, w_3, w_4)
$\mathcal{O}$	$(0, 0, 1, 0)$	$\mathcal{O}(-\vec{\omega})$	$(0, 0, -1, 0)$
$\mathcal{O}(\vec{x}_1 + \vec{x}_2 - \vec{c})$	$(0, 1, -1, 0)$	$\mathcal{O}(\vec{x}_3 + \vec{x}_4 - \vec{c})$	$(0, -1, 1, 0)$
$\mathcal{O}(\vec{x}_1 + \vec{x}_3 - \vec{c})$	$(1, -1, 0, 1)$	$\mathcal{O}(\vec{x}_2 + \vec{x}_4 - \vec{c})$	$(-1, 1, 0, -1)$
$\mathcal{O}(\vec{x}_1 + \vec{x}_4 - \vec{c})$	$(-1, 0, 0, 1)$	$\mathcal{O}(\vec{x}_2 + \vec{x}_3 - \vec{c})$	$(1, 0, 0, -1)$

Spin(8) weights of degree zero line bundles

sheaf	(w_1, w_2, w_3, w_4)	sheaf	(w_1, w_2, w_3, w_4)
$\mathcal{O}(\vec{x}_1)$	$(0, 0, 0, 1)$	$\tau\mathcal{O}(\vec{x}_1)$	$(0, 0, 0, -1)$
$\mathcal{O}(\vec{x}_2)$	$(0, 1, 0, -1)$	$\tau\mathcal{O}(\vec{x}_2)$	$(0, -1, 0, 1)$
$\mathcal{O}(\vec{x}_3)$	$(1, -1, 1, 0)$	$\tau\mathcal{O}(\vec{x}_3)$	$(-1, 1, -1, 0)$
$\mathcal{O}(\vec{x}_4)$	$(-1, 0, 1, 0)$	$\tau\mathcal{O}(\vec{x}_4)$	$(1, 0, -1, 0)$

Spin(8) weights of degree 1 line bundles

Table 2. Spin(8) weight of coherent sheaves on $\mathbb{X}_2$.

while the induced transformation on the Grothendieck group $\boldsymbol{T}\colon \Gamma \to \Gamma$ acts on the flavor charges α_a as (cf. eqs. (3.2)–(3.5))

$$\boldsymbol{T}(\alpha_1) = \alpha_1, \quad \boldsymbol{T}(\alpha_2) = \alpha_2, \quad \boldsymbol{T}(\alpha_3) = \alpha_4, \quad \boldsymbol{T}(\alpha_4) = \alpha_3, \tag{3.10}$$

in other words by the automorphism $\alpha_3 \longleftrightarrow \alpha_4$ of the Dynkin graph (3.6). This is precisely the non-trivial element of the Spin(8) triality group $\mathfrak{S}_3$ which maps the weights of $\boldsymbol{v}$ into themselves, consistently with the fact that the T-duality makes sense at weak coupling where the quark fields keep their identity.

The functor L acts as

$$\begin{aligned} L(\mathcal{O}) &= \mathcal{O}(-\vec{\omega}), \\ L(\mathcal{S}_{i,j}) &= \mathcal{O}(-\vec{x}_i + (1-j)\vec{\omega})[1], \qquad i = 1, \dots, 4,\ j = 0, 1. \end{aligned} \tag{3.11}$$

Thus

$$\begin{pmatrix} 2\,\mathsf{deg}\, L(X) \\ \mathsf{rank}\, L(X) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2\,\mathsf{deg}\, X \\ \mathsf{rank}\, X \end{pmatrix}, \tag{3.12}$$

JHEP09(2015)035

and (cf. eq. (2.75))

$$\boldsymbol{L}[\mathcal{S}_{i,j}] = (-1)^{j-1}[\mathcal{O}(\vec{x}_i)] \mod \mathrm{rad}\, q, \qquad j = 0, 1, \tag{3.13}$$

so that the Spin(8) weights of the eight objects $L(\mathcal{S}_{ij})$ are equal to those of the eight sheaves $\{\mathcal{O}(\vec{x}_i), \tau\mathcal{O}(\vec{x}_i)\}_{i=1}^4$ which are the weights of the spinorial representation $\boldsymbol{c}$. Thus L maps the Spin(8) vector representation $\boldsymbol{v}$ into the spinorial representation $\boldsymbol{c}$. However, the action of L on the weights is not simply the automorphism $\alpha_1 \longleftrightarrow \alpha_4$ of the Dynkin graph (3.6) but rather the composition of this graph automorphism with a Spin(8) Weyl transformation. In order to see this, we need to look at the full automorphism group $\mathrm{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_2)$.

From eq. (2.96) we know that, besides the braid group generated by by T, L we have the automorphism group $\mathsf{Pic}(\mathbb{X}_2)^0 \ltimes \mathrm{Aut}(\mathbb{X}_2)$. Now

$$\mathsf{Pic}(\mathbb{X}_2)^0 = (\mathbb{Z}/2\mathbb{Z})^4/(\mathbb{Z}/2\mathbb{Z})_{\mathrm{diag}}, \tag{3.14}$$

while, neglecting the subtlety already mentioned,[22] we may effectively take $\mathrm{Aut}(\mathbb{X}_2) \simeq \mathfrak{S}_4$, the permutation of the four special points in $\mathbb{P}^1$. Then

$$\mathsf{Pic}(\mathbb{X}_2)^0 \ltimes \mathrm{Aut}(\mathbb{X}_2) \simeq (\mathbb{Z}/2\mathbb{Z})^4/(\mathbb{Z}/2\mathbb{Z})_{\mathrm{diag}} \ltimes \mathfrak{S}_4 \simeq \mathrm{Weyl}(\mathrm{Spin}(8)). \tag{3.15}$$

Let Π be the autoequivalence of the derived category associated with the permutation $\pi = (14)(23)$ of the four special points in $\mathbb{P}^1$. We define a new auto-equivalence of $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_2$

$$\widehat{L} = \Pi L. \tag{3.16}$$

$\widehat{L}$ has the same action on the Yang-Mills charges as L. It follows from eq. (3.13) that the Spin(8) weights of $\widehat{L}(\mathcal{S}_{i,1})$ are equal to those of $\mathcal{O}(\vec{x}_{\pi(i)})$; comparing with table 2 we see that the effect of $\widehat{L}$ on the flavor weights is simply

$$\big(w_1(\widehat{L}(X)), w_2(\widehat{L}(X)), w_3(\widehat{L}(X)), w_4(\widehat{L}(X))\big) = \big(w_4(X), w_2(X), w_3(X), w_1(X)\big), \tag{3.17}$$

i.e. $\widehat{L}$ induces the graph automorphism $\alpha_1 \leftrightarrow \alpha_4$ which interchanges the Spin(8) spinorial representations $\boldsymbol{v} \leftrightarrow \boldsymbol{c}$. However, $\widehat{L}$ is less convenient than L since $\widehat{L}T\widehat{L} \neq T\widehat{L}T$.

3.2 Relation with the F_4 root system

Let us look to the action of $\mathcal{B}_3$ on the flavor charges from a more conceptual standpoint. We start from the formulae proven in section 2.5 which we specialize to $N_f = 4$ SQCD. In this case both $\mathcal{O}$ and $\mathcal{S}_{3,0}$ have τ-period $p = 2$. For $p = 2$ the action of $\mathcal{L}_Y$ on the Grothendieck group, eq. (2.106), reduces to

$$\begin{aligned} \mathcal{L}_Y[X] &= [X] - \langle [Y], [X] \rangle_E\, [Y] - \langle [\tau Y], [X] \rangle_E\, [\tau Y] = \\ &= [X] - \left\langle \frac{[Y] + [\tau Y]}{2}, [X] \right\rangle_E \big([Y] + [\tau Y]\big) - \left\langle \frac{[Y] - [\tau Y]}{2}, [X] \right\rangle_E \big([Y] - [\tau Y]\big) \end{aligned} \tag{3.18}$$

[22] The points is as follows: a permutation of the four special points does not give back the same $\mathbb{X}_2$ but rather a new one with a different cross-ratio of the four special points. Since this cross-ratio does not enter anywhere in the BPS sector, as far as we are interested to BPS physics, we may identify all $\mathbb{X}_2$ and consider $\mathfrak{S}_4$ to be a symmetry of the physics. For a generic configuration of the four points the actual $\mathrm{Aut}(\mathbb{X}_2)$ is the Klein group. Hence the operation Π defined in the text is an *actual automorphism* of the derived category.

JHEP09(2015)035

with $Y = \mathcal{O}$, $\mathcal{S}_{3,0}$ ($\mathcal{S}_{1,0}$) respectively. Since $[\tau Y] = \mathbf{\Phi} \cdot [Y]$, the charges $[Y] \pm [\tau Y]$ are ± 1 eigenvectors of $\mathbf{\Phi}$ i.e. they are a Yang Mills and a matter charge, respectively. Hence, the induced action on the matter lattice is $\Gamma_{\text{matter}} \equiv \Gamma/\text{rad}\, q$ is simply

$$\overline{\mathcal{L}}_Y[X] = [X] - \frac{1}{2}\Big\langle [Y] - [\tau Y], [X] \Big\rangle_{\text{sym}} \big([Y] - [\tau Y]\big) \tag{3.19}$$

where $\langle \cdot, \cdot \rangle_{\text{sym}}$ is the inner product on $\Gamma_{\text{matter}} \simeq \Gamma_{D_4}$ given by the Cartan matrix. The action of an automorphism is an isometry of the positive definite symmetric pairing; this requires

$$\big\| [Y] - [\tau Y] \big\|^2 \equiv \big\langle [Y] - [\tau Y], [Y] - [\tau Y] \big\rangle_E = 4 \tag{3.20}$$

which is automatic in view of eq. (2.105). From eqs. (2.96), (3.15) we conclude that

$\text{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_2)$ *acts*[23] *on the flavor root lattice* $\Gamma_{\mathfrak{so}(8)}$ *as the group of reflections generated by its elements of square-length* 2 *and* 4. *This group is isomorphic to* $\text{Weyl}(F_4)$ [35]. *One has*

$$1 \to \text{Weyl}(\text{SO}(8)) \to \text{Weyl}(F_4) \to \mathfrak{S}_3 \to 1. \tag{3.21}$$

The triality group is simply $\text{Weyl}(F_4)/\text{Weyl}(\text{SO}(8))$.

3.3 The cluster category

Around eq. (2.1) we claimed that the triangulated category which describes the physics of the BPS sector is not $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}$ itself but rather an orbit category of the form $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}/\mathscr{C}$ where $\mathscr{C}$ is a certain subgroup of $\text{Aut}(\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X})$. The subgroup $\mathscr{C}$ is determined by non-perturbative physical considerations, in particular by the study of the quantum monodromy $\mathbb{M}(q)$ and its fractional powers [21, 24]. In the case of a Lagrangian field theory, however, the group $\mathscr{C}$ should also have an elementary interpretation in terms of conventional weak-coupling physics. Then we use $N_f = 4$ SU(2) SQCD as a convenient example to shed light on the physical meaning of the cluster category.

We claim that passing from $\mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}_2$ to $\mathcal{C}(\mathbb{X}_2)$ in this example corresponds to implementing the Gauss' law on the physical states. Indeed, the center $Z(\mathcal{B}_3)$ of the braid group $\mathcal{B}_3$ is the infinite cyclic group generated by $(TL)^3 = (LT)^3$. For $p = 2$ one has the equality

$$(TL)^3 \equiv \tau^{-3}\Sigma = \tau^{-1}\Sigma \tag{3.22}$$

and hence $(TL)^3$ acts on the Yang-Mills charges $YM(X)$ as multiplication by -1, while fixing all flavor charges. Thus $(TL)^3$ acts on Γ as the Weyl group of the gauge group SU(2), which is a gauge transformation belonging to the connected component of the identity which leaves invariant all physical states satisfying the Gauss' law. Hence $(TL)^3 \equiv \tau^{-1}\Sigma$ should act trivially on the physical states, i.e. two objects in the same $\tau^{-1}\Sigma$-orbit should be considered the same 'physical' object. Then the natural triangulated category which is associated to the physical BPS states is

$$\mathcal{C}(\mathbb{X}_2) = \mathsf{D}^b(\mathsf{coh}\,\mathbb{X}_2)/\langle \tau^{-1}\Sigma \rangle, \tag{3.23}$$

[23] Neglecting the subtlety in footnote 22.

JHEP09(2015)035

which is precisely the definition of the cluster category for the coherent sheaves on the weighted projective line of type $(2,2,2,2)$ (equivalently of the canonical algebra of the same type) [17, 36].

It is convenient to see the orbit category $\mathcal{C}_2 \equiv \mathsf{D}^b(\mathsf{coh}\,\mathbb{X}_2)/\langle\tau^{-1}\Sigma\rangle$ as a (triangulated) category with the same objects as the triangulated category $\mathsf{D}^b(\mathsf{coh}\,\mathbb{X}_2)$ and morphism spaces

$$\mathrm{Hom}_{\mathcal{C}_2}(X,Y) = \bigoplus_{g\in\langle\tau^{-1}\Sigma\rangle} \mathrm{Hom}_{\mathsf{D}^b(\mathsf{coh}\,\mathbb{X}_2)}(X,gY), \tag{3.24}$$

see appendix B for further details. Then the isoclasses of $\mathcal{C}_2$ objects are the orbits of isoclasses of objects of the derived category, and the physical BPS states — satisfying Gauss' law — are in one-to-one correspondence with the isoclasses of stable $\mathcal{C}_2$ objects. The physical meaning of equation (3.24) is that defining the "physical" morphism space to be the direct sum of morphisms in the original category from X to gY for all g, makes the morphism to account correctly for all the possible "relative gauge orientations" of the two objects when they form a BPS bound state.

4 *S*-duality for the $E_r^{(1,1)}$ models

4.1 Generalities

We have seen in section 2.1 that the matter charges of the five $\mathfrak{g}_r^{(1,1)}$ models ($\mathfrak{g} = A_1, D_4, E_6, E_7, E_8$) take values in the weight lattice $\Gamma_{\mathfrak{g}}^{\mathrm{w}}$ of $\mathfrak{g}$. By this we mean that they are valued on a lattice endowed with the Cartan (integral) symmetric form $\langle\cdot,\cdot\rangle_{\mathrm{sym}}$ which is preserved by all symmetries and dualities of the quantum field theory. On the root lattice $\Gamma_{\mathfrak{g}}$ there is, in addition, an integral skew-symmetric form $\langle\cdot,\cdot\rangle_{\mathrm{Dirac}}$ which is also preserved by all dualities. Indeed, the reduced Coxeter element $\overline{\mathbf{\Phi}}$ centralizes the action of all dualities on $\Gamma_{\mathfrak{g}}$, cf. eq. (2.25). However there are two major differences between the Lagrangian models $A_1^{(1,1)}$, $D_4^{(1,1)}$ and the non-Lagrangian ones $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$:

A) In the Lagrangian case the skew-symmetric form $\langle\cdot,\cdot\rangle_{\mathrm{Dirac}}$ is identically zero on $\Gamma_{\mathfrak{g}}$, i.e. the matter charges are pure flavor ($\equiv$ the matter system are free hypermultiplets);

B) in the Lagrangian case the Lie group $\exp(\mathfrak{g})$ is a *symmetry* of the theory. This is <u>not</u> true in the non-Lagrangian case.

Indeed, in section 3 we saw that, in the two Lagrangian models, the class of coherent sheaves X with a fixed Yang-Mills charge $YM(X) \equiv (p\,\mathsf{deg}\,X, \mathsf{rank}\,X)^t$ have the correct $\mathfrak{g}$ weights to form a complete representation of $\mathfrak{g}$. This is certainly <u>not</u> true for $p = 3,4,6$. For instance, the number of zero-degree line bundles is equal to the order of the restricted Picard groups

$$\mathsf{Pic}(\mathbb{X}_3)^0 = \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}, \quad \mathsf{Pic}(\mathbb{X}_4)^0 = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}, \quad \mathsf{Pic}(\mathbb{X}_6)^0 = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}, \tag{4.1}$$

and no non-trivial representation of E_6, E_7 and E_8 have dimension 9, 8, and 6. Alternatively we may note that $\mathrm{Weyl}(E_r)$ is <u>not</u> a triangle auto-equivalence for $E_r^{(1,1)}$ models. The

JHEP09(2015)035

subgroups of $\mathrm{Aut}(\mathcal{C}(\mathbb{X}_p))$ which fix the rank and the degree are listed in the second column of the following table:

$$\begin{array}{l|l|l} p=3 & (\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z})\rtimes\mathfrak{S}_3 & \mathrm{SU}(3) \\ p=4 & (\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z})\rtimes\mathfrak{S}_2 & \mathrm{Sp}(2) \\ p=6 & \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z} & \mathrm{Sp}(1) \end{array} \tag{4.2}$$

They certainly do not contain the Weyl group of any Lie group but those listed in the third column and their subgroups. The lattice of flavor charges

$$\Gamma_{\mathrm{fl}} = \Big\{ x\in\Gamma \;\Big|\; \mathbf{\Phi}\cdot x = -x \Big\}, \tag{4.3}$$

has rank $0,3,2$ for $p=3,4,6$. For $p=4$ Γ_{fl} is generated by the three classes

$$\begin{aligned} \alpha_1 &= [\mathcal{S}_{2,0}] + [\mathcal{S}_{2,2}] - [\mathcal{S}_{1,1}], \\ \alpha_2 &= [\mathcal{S}_{3,0}] + [\mathcal{S}_{3,2}] - [\mathcal{S}_{1,0}], \\ \alpha_3 &= [\mathcal{S}_{1,0}] - [\mathcal{S}_{2,0}] - [\mathcal{S}_{2,2}] \end{aligned} \tag{4.4}$$

The symmetric form on Γ_{fl}

$$\langle \alpha_a, \alpha_b\rangle_E = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \tag{4.5}$$

is not related in any obvious way to a Cartan matrix, which is consistent with the fact that (contrary to the $p=2$ case studied in section 3) physically we do not expect any non-Abelian enhancement of the flavor symmetry for the $p=4$ model. $\mathrm{Aut}(\mathbb{X}_4)$ just permutes the two identical matter AD subsystems of type D_4, and acts on the flavor charges as

$$\alpha_1 \leftrightarrow \alpha_1+\alpha_2+\alpha_3, \qquad \alpha_2 \leftrightarrow -\alpha_3. \tag{4.6}$$

The two generators of $\mathsf{Pic}(\mathbb{X}_4)^0$ act as[24]

$$\mathcal{O}(\vec{x}_2-\vec{x}_3)\colon (\alpha_1,\alpha_2,\alpha_3)\mapsto(\alpha_3,-\alpha_1-\alpha_2-\alpha_3,\alpha_1) \tag{4.7}$$

$$\mathcal{O}(\vec{x}_1-\vec{2}x_2)\colon (\alpha_1,\alpha_2,\alpha_3)\mapsto(\alpha_3,\alpha_1+\alpha_2+\alpha_3,\alpha_1). \tag{4.8}$$

For $p=6$ the generators of Γ_{fl} are

$$\alpha_1 = [\mathcal{S}_{1,1}] - [\mathcal{S}_{3,0}] - [\mathcal{S}_{3,2}] - [\mathcal{S}_{3,4}], \quad \alpha_2 = -[\mathcal{S}_{1,0}] + [\mathcal{S}_{3,0}] + [\mathcal{S}_{3,2}] + [\mathcal{S}_{3,4}] \tag{4.9}$$

with

$$\langle \alpha_a, \alpha_b\rangle_E = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} = 2\,C_{A_2}. \tag{4.10}$$

Tensoring with $\mathcal{O}(\vec{x}_1-3\vec{x}_3)$ changes the sign of both flavor charges, while the flavor charges are inert under $\mathcal{O}(\vec{x}_2-2\vec{x}_3)$.

[24] A perhaps more transparent description of the action of the Picard group on the flavor charges is the following: $\mathcal{O}(\vec{x}_2-\vec{x}_3)$ inverst the signs of the flavor charges of the two D_4 matter systems, while $\mathcal{O}(\vec{x}_1-2\vec{x}_2)$ invert the sign of the flavor charge of the D_2 hypermultiplet.

JHEP09(2015)035

4.2 Emergence of Shephard-Todd complex reflection groups

From section 2.5 we know that the action of the two functors L and T on the Grothendieck group Γ is given by

$$[LX] = [X] - \sum_{k=0}^{p-1} \big\langle \mathbf{\Phi}^k \cdot [\mathcal{O}], [X] \big\rangle_E \, \mathbf{\Phi}^k \cdot [\mathcal{O}] \tag{4.11}$$

$$[TX] = [X] - \sum_{k=0}^{p-1} \big\langle \mathbf{\Phi}^k \cdot ([\mathcal{O}(\vec{c})] - [\mathcal{O}((p-1)\vec{x}_3)]), [X] \big\rangle_E \, \mathbf{\Phi}^k \cdot ([\mathcal{O}(\vec{c})] - [\mathcal{O}((p-1)\vec{x}_3)]) \tag{4.12}$$

where we used

$$[\mathcal{S}_{3,p-1}] = [\mathcal{S}(\vec{c})] - [\mathcal{O}((p-1)\vec{x}_3)]. \tag{4.13}$$

Writing $[X] = \sum_a [X]_a \phi_a$, where $\{\phi_a\}_{a=1}^n$ is the canonical basis of Γ (cf. section 2.2) and $[X]_a \in \mathbb{Z}$, we have[25]

$$[X]_b \, \boldsymbol{L}_{ba} \equiv [LX]_a = \delta_{ab} \, [X]_b - \sum_{k=0}^{p-1} (\mathbf{\Phi}^k)_{1a} \, (\mathbf{\Phi}^k)_{1c} E_{cb} [X]_b \tag{4.14}$$

$$[X]_b \, \boldsymbol{T}_{ba} \equiv [TX]_a = \delta_{ab} \, [X]_b - \sum_{k=0}^{p-1} \Big((\mathbf{\Phi}^k)_{r\,a} - (\mathbf{\Phi}^k)_{r-1\,a} \Big) \Big((\mathbf{\Phi}^k)_{r\,c} - (\mathbf{\Phi}^k)_{r-1\,c} \Big) E_{cb} [X]_b. \tag{4.15}$$

Since $\mathbf{\Phi} = -E(E^t)^{-1}$, the matrices $\boldsymbol{T}$ and $\boldsymbol{L}$ are expressed in terms of the Euler matrix E only; plugging in the explicit matrix E in eq. (2.48), we get integral $n \times n$ matrices $\boldsymbol{T}$ and $\boldsymbol{L}$ giving the concrete action of T, L on the lattice Γ. See appendix A.

For $p = 3, 4, 6$, the matrices $\boldsymbol{T}$ and $\boldsymbol{L}$ give a (reducible) $n = 8, 9, 10$ dimensional representation of $\mathcal{B}_3$ on the vector space $V = \Gamma \otimes \mathbb{C}$. Indeed they satisfy the braid relation

$$\boldsymbol{T}\boldsymbol{L}\boldsymbol{T} = \boldsymbol{L}\boldsymbol{T}\boldsymbol{L}. \tag{4.16}$$

$\boldsymbol{T}$ and $\boldsymbol{L}$ commute with $\mathbf{\Phi}$ and, for $p = 3$, also with the permutation π_{12} in eq. (2.95). We decompose V in eigenspaces of the semisimple linear map $\mathbf{\Phi}$,

$$V = \bigoplus_{s=0}^{p-1} V_s, \qquad \mathbf{\Phi}\Big|_{V_s} = e^{2\pi i s/p}. \tag{4.17}$$

From eqs. (4.14), (4.15) the restrictions of the braid generators $\boldsymbol{T}$, $\boldsymbol{L}$ to each $\mathbf{\Phi}$-eigenspace,

$$\boldsymbol{L}_s, \boldsymbol{T}_s \colon V_s \longrightarrow V_s \tag{4.18}$$

[25] The matrices $\boldsymbol{T}$ and $\boldsymbol{L}$ are defined to act on the left of the basis elements $\phi_a \xrightarrow{L,T} \boldsymbol{L}_{ab}\,\phi_b$, $\boldsymbol{T}_{ab}\,\phi_b$, and hence they act on the *right* of the coefficients $[X]_a$.

JHEP09(2015)035

are

$$\boldsymbol{L}_s = \mathbf{1} - \frac{1}{p}\left(\sum_{k=0}^{p-1} e^{-2\pi i k s/p}\,\boldsymbol{\Phi}^k v\right) v^t E \left(\sum_{\ell=0}^{p-1} e^{-2\pi i \ell s/p}\,\boldsymbol{\Phi}^\ell\right) \tag{4.19}$$

$$\boldsymbol{T}_s = \mathbf{1} - \frac{1}{p}\left(\sum_{k=0}^{p-1} e^{-2\pi i k s/p}\,\boldsymbol{\Phi}^k w\right) w^t E \left(\sum_{\ell=0}^{p-1} e^{-2\pi i \ell s/p}\,\boldsymbol{\Phi}^\ell\right) \tag{4.20}$$

$$\text{where} \qquad v = (1,0,\cdots,0)^t, \qquad w = (0,\cdots,0,-1,1)^t. \tag{4.21}$$

Eqs. (4.19), (4.20) together with eq. (2.105) imply that the *minimal* equations satisfied by the restricted generators $\boldsymbol{T}_s, \boldsymbol{L}_s$ are

$$\begin{cases} (\boldsymbol{L}_s - 1)(\boldsymbol{L}_s - e^{-2\pi i s/p}) = (\boldsymbol{T}_s - 1)(\boldsymbol{T}_s - e^{-2\pi i s/p}) = 0 & \text{if } \dim V_s > 1 \\ \boldsymbol{L}_s - e^{-2\pi i s/p} = \boldsymbol{T}_s - e^{-2\pi i s/p} = 0 & \text{if } \dim V_s = 1. \end{cases} \tag{4.22}$$

In particular, acting on V_0 — which corresponds to the Yang-Mills charges $YM(\cdot)$ — $\boldsymbol{T}_0$ and $\boldsymbol{L}_0$ are 2×2 irreducible unipotent matrices in agreement with eqs. (2.68), (2.65). For p even, $\boldsymbol{T}_{p/2}$ and $\boldsymbol{L}_{p/2}$ acting on $V_{p/2}$ — i.e. on the flavor charges — are involutions, in fact reflections on square-length $2p$ lattice vectors

$$y \longmapsto y - \frac{1}{p}\langle u, y\rangle_{\rm sym}\, u, \quad \text{for some } \; u \in \Gamma, \; \|u\|^2 = 2p, \tag{4.23}$$

just as found in section 3.2 for the $D_4^{(1,1)}$ model.

The minimal polynomial for the $n\times n$ matrices $\boldsymbol{T}$, $\boldsymbol{L}$ which give the action of the telescopic functors T, L on the Grothendieck group Γ is then

$$\big(\boldsymbol{T}-1\big)\big(\boldsymbol{T}^p-1\big) = \big(\boldsymbol{L}-1\big)\big(\boldsymbol{L}^p-1\big) = 0. \tag{4.24}$$

Indeed, more generally, the following identities are true for *all* analytic function $f(z)$

$$\Big(f(\boldsymbol{T}) - f(\boldsymbol{\Phi}^{-1})\Big)\Big(\boldsymbol{T}-1\Big) = \Big(f(\boldsymbol{L}) - f(\boldsymbol{\Phi}^{-1})\Big)\Big(\boldsymbol{L}-1\Big) = 0, \tag{4.25}$$

eq. (4.24) being the special case $f(z) = z^p$. In addition, the following identities hold (see appendix A)

$$(\boldsymbol{T}^t)^p = 1 + R_1 \otimes M^t \qquad (\boldsymbol{L}^t)^p = 1 + R_2 \otimes Q^t, \tag{4.26}$$

where M, Q, R_1, R_2 are the magnetic/electric charge and radical vectors defined in eqs. (2.44), (2.45), (2.50) and (2.51).

$\mathcal{B}_3$ action on the matter charges. The braid group action on the *matter* vector space

$$V_{\rm matter} = \bigoplus_{s\neq 0} V_s \equiv \Gamma_{\rm matter} \otimes \mathbb{C} \tag{4.27}$$

preserves the positive-definite Hermitian product induced by the Euler form as well as the lattice $\Gamma_{\rm matter} \subset V_{\rm matter}$. Hence the quotient of $\mathcal{B}_3$ which acts effectively on $V_{\rm matter}$ is a *finite* group $G_{\rm matter}$. This group is generated by the restriction of the matrices $\boldsymbol{T}$ and $\boldsymbol{L}$

JHEP09(2015)035

to the $(n-2)$-dimensional space V_{matter}. By abuse of notation, we denote these 'matter' restrictions by the same symbols $\boldsymbol{T}$, $\boldsymbol{L}$. Then the finite group G_{matter} is generated by the two elements $\boldsymbol{T}$, $\boldsymbol{L}$ subjected at least to the obvious relations

$$\boldsymbol{L}^p = \boldsymbol{T}^p = 1, \quad \boldsymbol{TLT} = \boldsymbol{LTL}. \tag{4.28}$$

Each eigenspace V_s with $s \neq 0$ carries a unitary representation of G_{matter} (with respect to the Hermitian product induced by the Euler form). From eqs. (4.19), (4.20) we see that

The multiplicity of $e^{-2\pi i s/p}$ as an eigenvalue of $\boldsymbol{T}_s$ and $\boldsymbol{L}_s$ in V_s is precisely one. Their other eigenvalues are all equal 1 (cf. eq. (4.22)). (4.29)

Given the underling real structure on the complex vector space V_{matter} induced by the lattice Γ_{matter} we know that the G_{matter}-modules V_s and V_{p-s} are conjugate

$$V_{p-s} = \overline{V_s}, \qquad p = 3, 4, 6, \quad s = 1, 2, \ldots, p-1. \tag{4.30}$$

The case $\boldsymbol{V_s}$ is one-dimensional. If $\dim V_s = 1$ the action of both generators $\boldsymbol{T}_s$, $\boldsymbol{L}_s$ is just multiplication by $e^{-2\pi i s/p}$, and the quotient group of G_{matter} which acts effectively on V_s is the cyclic group

$$\mathbb{Z} \Big/ \tfrac{p}{\gcd(s,p)}\mathbb{Z}. \tag{4.31}$$

In this case the action of the generator of $Z(\mathcal{B}_3)$ on V_s is

$$(\boldsymbol{T}_s\boldsymbol{L}_s)^3 = e^{-12\pi i s/p} \equiv \begin{cases} 1 & \text{for } p = 3, 6 \\ (-1)^s & \text{for } p = 4. \end{cases} \tag{4.32}$$

From section 2.3 we know that

$$(\boldsymbol{TL})^3 = \begin{cases} -\pi & p = 3 \\ -\boldsymbol{\Phi}^{-3} & p = 2, 4, 6. \end{cases} \tag{4.33}$$

Thus for $p \neq 3$ we have

$$(\boldsymbol{TL})^3\big|_{V_s} = -e^{-6\pi i s/p}, \tag{4.34}$$

which is consistent with eq. (4.32) only for $p = 6$, $s = 1, 3, 5$ and $p = 4$, $s = 2$. In particular,

in all one-dimensional representations of G_{matter}: $(\boldsymbol{T}_s\boldsymbol{L}_s)^3 = 1$. (4.35)

$\boldsymbol{V_s}$ of dimension larger than 1. If $\dim V_s > 1$, both $\boldsymbol{T}$, $\boldsymbol{L}$ are *complex reflections* i.e. unitary matrices with all but one eigenvalues equal 1 [9], cf. eq. (4.29). A group generated by complex reflections is called a *complex reflection group*. We conclude that the quotient group G_s of G_{matter} which acts effectively on V_s is a *finite* complex reflection group.

JHEP09(2015)035

4.2.1 Finite complex reflection groups

Finite complex reflection groups have been classified by Shephard and Todd [9]: there are three infinite families and 34 exceptional groups denoted $G_4, \cdots, G_{37}$. Real reflection groups (Coxeter groups) and rational reflection groups (Weyl groups) are encoded in Dynkin graphs; complex reflection groups are also encoded in certain graphs of a more general kind [10]. We look for Shephard-Todd groups generated by two elements $\boldsymbol{T}$, $\boldsymbol{L}$ satisfying relations which imply (4.28) with $p = 3, 4, 6$. A part for the Weyl group of SU(2) (which corresponds to the already discussed $D_4^{(1,1)}$ model), there are two other candidates in the classification list, namely the exceptional Shephard-Todd groups G_4 and G_8, see the following table:

G_4 is a subgroup of U(2) which as an abstract group (i.e. forgetting its realization as a complex reflection group) is

$$G_4 \simeq \text{binary tetrahedral group } \overline{T} \simeq \mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z}). \tag{4.36}$$

Consider the standard Coxeter presentation of $\overline{T}$ (which is the subgroup of SU(2) corresponding to the affine graph $E_6^{(1)}$ in the McKay correspondence)

$$\overline{T} = \left\langle A, B \;\middle|\; A^2 = B^3 = (-AB)^3 = -1 \right\rangle \tag{4.37}$$

The identification

$$\boldsymbol{T} = -B, \qquad \boldsymbol{L} = AB, \tag{4.38}$$

maps (4.37) into the standard presentation of G_4. However its explicit realization as a subgroup of SU(2) does <u>not</u> realize it as a complex reflection group. To get a complex reflection group we must twist the standard quaternionic degree 2 Klein realization Q of $\overline{T}$ by the character χ of one of its two non-trivial one-dimensional representations (the two choices producing equivalent results[26]). Thus, G_4 is the subgroup of U(2) generated by the two matrices

$$\boldsymbol{T} = -\chi(B)\, B_Q, \qquad \boldsymbol{L} = \chi(AB)\, A_Q B_Q \tag{4.39}$$

or, explicitly,

$$\boldsymbol{T} = -\frac{1}{\sqrt{2}}\,\omega \begin{pmatrix} \epsilon & \epsilon^3 \\ \epsilon & \epsilon^7 \end{pmatrix}, \qquad \boldsymbol{L} = \frac{1}{\sqrt{2}}\,\omega \begin{pmatrix} \epsilon^3 & \epsilon^5 \\ \epsilon^7 & \epsilon^5 \end{pmatrix} \tag{4.40}$$

where ω, ϵ are primitive roots of unity of order, respectively, 3 and 8. The irreducible representations of G_4 may be then read directly from the character table of the binary tetrahedral group $\overline{T}$.

In the same fashion, G_8 is a subgroup of U(2) which is a central extension by $\mathbb{Z}/2\mathbb{Z}$ of the binary octahedral group $\overline{O}$, i.e. of the subgroup of SU(2) associated to the affine graph $E_7^{(1)}$. Explicitly [9]

$$\boldsymbol{T} = -\frac{1}{\sqrt{2}}\,\epsilon^3 \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \boldsymbol{L} = \epsilon^3 \begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^5 \end{pmatrix} \tag{4.41}$$

[26] The two choices are related by interchanging the defining representation and its conjugate $R \leftrightarrow \overline{R}$.

JHEP09(2015)035

with ϵ as before. Again we may obtain the irreducible representations of G_8 by twisting the character table of $\overline{O}$. We claim that G_8 is a central extension by $Z(G_8) \equiv \mathbb{Z}/4\mathbb{Z}$ of the group $\mathrm{PSL}(2, \mathbb{Z}/4\mathbb{Z})$

$$1 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow G_8 \xrightarrow{P} \mathrm{PSL}(2, \mathbb{Z}/4\mathbb{Z}) \longrightarrow 1. \tag{4.42}$$

To show our claim, the only thing we have to prove is that the quotient of G_8 by its center $Z(G_8)$ is the group $\mathrm{PSL}(2, \mathbb{Z}/4\mathbb{Z})$. Now

$$G_8/Z(G_8) = \left\{ \boldsymbol{T}, \boldsymbol{L} \;\middle|\; \boldsymbol{TLT} = \boldsymbol{LTL},\ (\boldsymbol{TL})^3 = \boldsymbol{T}^4 = \boldsymbol{L}^4 = 1 \right\} \equiv \mathrm{PSL}(2, \mathbb{Z})/N(4), \tag{4.43}$$

where $N(n)$ (for $n \in \mathbb{N}$) stands for the normal closure[27] of T^n in $\mathrm{PSL}(2, \mathbb{Z})$ i.e. the intersection of all normal subgroups of $\mathrm{PSL}(2, \mathbb{Z})$ containing T^n. Comparing presentations, for all $n \in \mathbb{N}$, one has

$$\begin{aligned} \mathrm{PSL}(2, \mathbb{Z})/N(n) &= \text{the } (2, 3, n) \text{ triangle group} \equiv \\ &\equiv \left\langle a, b, c \mid a^2 = b^3 = c^n = abc = 1 \right\rangle. \end{aligned} \tag{4.44}$$

Klein proved that $N(n) = \Gamma(n)$ for $n \leq 5$ [8] so[28]

$$\begin{aligned} \left\langle \boldsymbol{T}, \boldsymbol{L} \in \mathrm{PSL}(2, \mathbb{Z}) \;\middle|\; \boldsymbol{T}^n = \boldsymbol{L}^n = 1 \right\rangle &\equiv \mathrm{PSL}(2, \mathbb{Z})/N(n) = \\ &= \mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \text{dihedral} & D_5^{(1)} \quad n = 2 \\ \text{tetrahedral} & E_6^{(1)} \quad n = 3 \\ \text{octahedral} & E_7^{(1)} \quad n = 4 \\ \text{icosahedral} & E_8^{(1)} \quad n = 5. \end{cases} \end{aligned} \tag{4.45}$$

The affine diagrams $\mathfrak{g}^{(1)}$ in the second column are the McKay graphs of the *double cover* of the $\mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z})$ group which is a finite subgroup of $\mathrm{SU}(2)$. It is amusing that the three Lie algebras $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ appear again in the game, now as quotients of the modular group by principal congruence subgroups $\Gamma(n)$. As abstract groups, the dihedral group of order 6 is isomorphic to $\mathfrak{S}_3$, while the tetrahedral one is isomorphic to $\mathfrak{A}_4$.

In view of eq. (4.43), the case $n = 4$ in (4.45) proves our claim (4.42).

4.2.2 The $E_6^{(1,1)}$ model

Let us look case by case. For $p = 3$ we have $\dim V_1 = \dim V_2 = 3$. Comparing the presentation of G_4 in table 3 with eq. (4.28) we conclude that V_1, V_2 carry a representation of $G_4 \simeq \mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z})$. We decompose V_s $(s = 1, 2)$ into irreducible representations of

[27] T is the 2×2 matrix defined in (2.65). In eq. (4.43) we use that L^{-1} is conjugate to T in $\mathrm{PSL}(2, \mathbb{Z})$.

[28] The case $n = 5$ in eq. (4.45) is never used in our present analysis: we listed it merely to give the complete statement of Klein's fundamental result. The case $n = 6$ is radically different, since the triangle group $(2, 3, 6)$ yields a tessellation of the Euclidean plane, and hence is an *infinite* group, while all groups of interest here must be *finite* by the argument around eq. (2.25). We have already pointed out that the case of the $E_8^{(1,1)}$ model, corresponding to $p = 6$, is slightly different from the other ones: using the Chinese remainder theorem, $\mathrm{PSL}(2, \mathbb{Z}/6\mathbb{Z})$ is written as the product of *two* triangle groups instead of a single triangle group as for $p = 2, 3, 4$, cf. eq. (1.6).

JHEP09(2015)035

name	graph	order	defining relations	center Z	order of Z
G_4		24	$\boldsymbol{L}^3 = \boldsymbol{T}^3 = 1$, $\boldsymbol{TLT} = \boldsymbol{LTL}$	$(\boldsymbol{LT})^{3k}$	2
G_8		96	$\boldsymbol{L}^4 = \boldsymbol{T}^4 = 1$, $\boldsymbol{TLT} = \boldsymbol{LTL}$	$(\boldsymbol{LT})^{3k}$	4

Table 3. Shephard-Todd groups G_4 and G_8: graphs, orders, presentations and centers.

G_4. First we rule out that V_s decomposes into the direct sum of three one-dimensional representations since, in that case we would get $(\boldsymbol{TL})^3 = 1$ (by eq. (4.35)), while we know that $(\boldsymbol{TL})^3$ is the non-trivial involution $-\pi_{12}$; its action on V_s has eigenvalues $(-1, -1, 1)$. We also rule out the irreducible three dimensional representation of G_4 — which coincides with the 3-dimensional irrepresentation of $\overline{T}$ — which, being real, cannot be a complex reflection subgroup of U(3). We remain with the defining two-dimensional representation R in eq. (4.40) and its conjugate $\overline{R}$. For the defining representation R (with the standard choice $\omega = e^{2\pi i/3}$) we have

$$\text{eigenvalues of } \boldsymbol{T},\ \boldsymbol{L} \text{ in } R = (e^{-2\pi i/3}, 1), \tag{4.46}$$

and we conclude that

$$V_1 = R \oplus \mathbf{1}, \qquad V_2 = \overline{R} \oplus \mathbf{1}, \tag{4.47}$$

where $\mathbf{1}$ stands for the trivial representation. As a further check, note that in the representation R, $(\boldsymbol{TL})^3 = -1$ so that the eigenvalues of $(\boldsymbol{TL})^3$ acting on V_s are $(-1, -1, +1)$ as they should.

4.2.3 The $E_7^{(1,1)}$ model

By comparing presentations, we see that for $p = 4$ the space V_s $(s = 1, 2, 3)$ carry a representation of the Shephard-Todd group G_8. We have

$$\dim V_1 = \dim V_3 = 2, \quad \dim V_2 = 3 \tag{4.48}$$

and $(\boldsymbol{TL})^3$ acts on V_s as $\tau^{-3}\Sigma$, i.e. as multiplication by $-e^{-3\pi i s/2}$. From eq. (4.35) it follows that V_1, V_3 are irreducible representations of G_8 of degree 2. There are four such representations F_χ which are obtained by twisting the defining representation F in eq. (4.41) by one of the four one-dimensional characters χ of G_8 (with $\chi^4 = 1$). One has (choosing $\epsilon = e^{\pi i/4}$)

$$\text{eigenvalues of } \boldsymbol{T},\ \boldsymbol{L} \text{ in } F_\chi = (\chi\, e^{3\pi i/2}, \chi) \tag{4.49}$$

so the only realizations as complex reflection groups in $\mathbb{C}^2$ are F and its conjugate $\overline{F}$. On F the central element $(\boldsymbol{TL})^3$ acts as $-e^{\pi i/2}$. Comparing with eq. (4.34) we see that, as representations of G_8,

$$V_1 = F, \qquad V_3 = \overline{F}. \tag{4.50}$$

Next we consider the space V_2 which is spanned over $\mathbb{C}$ by the three strict-sense flavor charges of the QFT model. The eigenvalues of $\boldsymbol{T}$, $\boldsymbol{L}$ in V_2 are $(+1, +1, -1)$ so G_{matter} acts

on V_2 as a *real* reflection group, i.e. a Coxeter group. The central element $(\boldsymbol{TL})^3$ acts as 1. Since $\boldsymbol{L}^2$ and $\boldsymbol{T}^2$ act trivially, the group which acts effectively on V_2 is

$$\mathrm{PSL}(2,\mathbb{Z})/N(2) \equiv \mathrm{PSL}(2,\mathbb{Z})/\Gamma(2) \equiv \mathrm{PSL}(2,\mathbb{Z}/2\mathbb{Z}) \simeq \mathfrak{S}_3. \tag{4.51}$$

We have two possibilities: either V_2 is two copies of the trivial representation of the symmetric group $\mathfrak{S}_3$ plus the sign one-dimensional representation σ, or it is the trivial representation plus the irreducible degree 2 representation of $\mathfrak{S}_3$, W. To distinguish the two possibilities, note that in the first case $\boldsymbol{TL}$ acts as the identity, and then (in particular)

$$V_2 = \mathbf{1} \oplus \mathbf{1} \oplus \sigma \quad \Longrightarrow \quad \sum_{k=0}^{3} (-1)^k \Big\{ [TLT(\tau^{-k}\mathcal{O})] - [T(\tau^{-k}\mathcal{O})] \Big\} = 0. \tag{4.52}$$

Using eq. (2.85), the above sum becomes

$$\begin{aligned} &\sum_{k=0}^{3} (-1)^k \Big\{ [\mathcal{S}_{3,k+1}] - [\mathcal{O}(\vec{x}_3 - k\vec{\omega})] \Big\} = \\ &= \sum_{k=0}^{3} (-1)^k \Big\{ [\mathcal{S}_{3,k+1}] - \sum_{\ell=0}^{k} [\mathcal{S}_{3,\ell}] - \sum_{i=1}^{2} \sum_{\ell=0}^{k-1} [\mathcal{S}_{i,\ell}] \Big\} \mod \mathrm{rad}\, q \end{aligned} \tag{4.53}$$

collecting the terms proportional to the classes localized, say, at the first special point, $[\mathcal{S}_{1,\ell}]$ we see that the sum does not vanish mod rad q. We conclude that

$$V_2 = W \oplus \mathbf{1}, \tag{4.54}$$

that is: (in a suitable basis) *the* 3 *flavor charges of the* $E_7^{(1,1)}$ *theory are permuted by the action of the S-duality group* which acts through its $\mathfrak{S}_3 \equiv \mathrm{PSL}(2,\mathbb{Z}/2\mathbb{Z})$ factor group, cf. eq. (1.13). This fact was already noted in refs. [39, 40].

4.2.4 The $E_8^{(1,1)}$ model

For $p = 6$ we have

$$\dim V_1 = \dim V_5 = 1, \quad \dim V_2 = \dim V_3 = \dim V_4 = 2. \tag{4.55}$$

The central element $(\boldsymbol{TL})^3$ acts as multiplication by $-e^{\pi i s}$ on V_s.

On the one-dimensional representations V_1, V_5, the generators $\boldsymbol{L}$, $\boldsymbol{T}$ act as multiplication by $e^{-\pi i/3}$ and $e^{\pi i/3}$, respectively. We denote these two characters by χ and $\overline{\chi}$.

The two conjugate representation V_2 and V_4 are irreducible by criterion (4.35). Acting on these representations we have

$$\boldsymbol{T}^3 = \boldsymbol{L}^3 = 1, \qquad \boldsymbol{TLT} = \boldsymbol{LTL}, \tag{4.56}$$

and hence the group acting effectively on V_2, V_4 is G_4. Acting on V_2

$$\text{eigenvalues of } \boldsymbol{T},\ \boldsymbol{L} = (e^{-2\pi i/3}, 1). \tag{4.57}$$

JHEP09(2015)035

Comparing with eq. (4.46) we get

$$V_2 = R, \qquad V_4 = \overline{R}. \tag{4.58}$$

We remain with the flavor charge sublattice V_3 of dimension 2. The group acting effectively on this flavor lattice has a presentation

$$\boldsymbol{T}^2 = \boldsymbol{L}^2 = 1, \qquad \boldsymbol{TLT} = \boldsymbol{LTL}, \tag{4.59}$$

and hence it is identified with the Weyl group

$$\mathrm{Weyl}(A_2) \simeq \mathrm{PSL}(2, \mathbb{Z}/2\mathbb{Z}) \simeq \mathfrak{S}_3. \tag{4.60}$$

Again, we have two possibilities: either *(i)* V_3 is the direct sum of the trivial and the sign representations, $V_3 = \mathbf{1} \oplus \sigma$, or *(ii)* V_3 is the irreducible two-dimensional representation W of $\mathfrak{S}_3$. Again, the first possibility implies $\boldsymbol{TL} = 1$. We repeat the diagnostics in eq. (4.52)

$$V_3 = \mathbf{1} \oplus \sigma \quad \Longrightarrow \quad \sum_{k=0}^{5} (-1)^k \Big\{ [TLT(\tau^{-k}\mathcal{O})] - [T(\tau^{-k}\mathcal{O})] \Big\} = 0. \tag{4.61}$$

Explicitly the sum has the form

$$\begin{aligned} &\sum_{k=0}^{5} (-1)^k \Big\{ [\mathcal{S}_{3,k+1}] - [\mathcal{O}(\vec{x}_3 - k\vec{\omega})] \Big\} = \\ &= \sum_{k=0}^{5} (-1)^k \Big\{ [\mathcal{S}_{3,k+1}] - \sum_{\ell=0}^{k} [\mathcal{S}_{3,\ell}] - \sum_{i=1}^{2} \sum_{\ell=0}^{k-1} [\mathcal{S}_{i,\ell}] \Big\} \mod \mathrm{rad}\, q \end{aligned} \tag{4.62}$$

while the last sum does not vanish mod $\mathrm{rad}\, q$. Again, we conclude that

$$V_3 = W. \tag{4.63}$$

This completes the proof of eqs. (1.9)–(1.11).

Acknowledgments

We have benefit from discussions with Bernhard Keller and Dirk Kussin. We thank Helmut Lenzing for making available to us one of his unpublished manuscripts.

A Explicit matrices in the canonical basis and additional identities

In this appendix, we write the explicit matrices which give the action of T and L on the canonical basis i.e.

$$[T\mathcal{L}_a] = \boldsymbol{T}_{ab}\, \phi_b, \qquad [L\mathcal{L}_a] = \boldsymbol{L}_{ab}\, \phi_b, \tag{A.1}$$

where $\{\mathcal{L}_a\}_{a=1}^{n}$ are the line bundles in eq. (2.42) such that $\phi_a = [\mathcal{L}_a]$. From section 4 we have

$$\boldsymbol{L} = 1 - \sum_{k=0}^{p-1} E^t (\boldsymbol{\Phi}^t)^k v \otimes v^t \boldsymbol{\Phi}^k, \qquad \boldsymbol{T} = 1 - \sum_{k=0}^{p-1} E^t (\boldsymbol{\Phi}^t)^k w \otimes w^t \boldsymbol{\Phi}^k, \tag{A.2}$$

JHEP09(2015)035

where E is the upper triangular matrix in eq. (2.48), $\mathbf{\Phi} = -E(E^t)^{-1}$ and

$$v = (1, 0, \ldots, 0)^t, \qquad w = (0, \ldots, 0, -1, 1)^t. \tag{A.3}$$

For many purposes it is more natural to use the transpose matrices $\boldsymbol{T}^t$ and $\boldsymbol{L}^t$ giving the action on the coefficient vectors $([X]_a)$, where, for all $X \in \mathsf{D}^b\,\mathsf{coh}\,\mathbb{X}$ we set $[X] = [X]_a\phi_a$.

The matrices $\boldsymbol{T}$ and $\boldsymbol{L}$ have quite remarkable properties, some of which were already discussed in section 4.

A.1 $E_6^{(1,1)}$ model

With respect to the canonical basis (2.42), one has

$$\boldsymbol{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{L} = \begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 & 2 \\ -2 & 0 & -1 & 0 & -1 & 0 & -1 & 3 \end{pmatrix} \tag{A.4}$$

It is easy to check that

$$(\boldsymbol{T} - 1)(\boldsymbol{T}^3 - 1) = (\boldsymbol{L} - 1)(\boldsymbol{L}^3 - 1) = 0, \qquad \boldsymbol{TLT} = \boldsymbol{LTL}, \tag{A.5}$$

$$\operatorname{spec} \boldsymbol{T} = \operatorname{spec} \boldsymbol{L} = (1, 1, \ldots, 1, e^{2\pi/3}, e^{-2\pi/3}). \tag{A.6}$$

Using the explicit matrices (A.4), we confirm that the central element is

$$(\boldsymbol{TL})^3 = -\pi_{12}, \tag{A.7}$$

proving the claim in eq. (2.95). Moreover

$$(\boldsymbol{T}^t)^3 - 1 = R_1 \otimes M^t \tag{A.8}$$

where M is the vector of *magnetic* charges of the ϕ_a's, eq. (2.44), and $R_1 \equiv E^{-1}M$ is the first generator of $\operatorname{rad} q$, eq. (2.50), and . Likewise

$$(\boldsymbol{L}^t)^3 - 1 = R_2 \otimes Q^t \tag{A.9}$$

where Q is the vector of *electric* charges of the ϕ_a's, eq. (2.45), and $R_2 \equiv E^{-1}Q$ is the second generator of $\operatorname{rad} q$. Note the similarity of the last two equations with the equation satisfied by the Coxeter element of an Euclidean algebra.

JHEP09(2015)035

A.2 $E_7^{(1,1)}$ model

For the $E_7^{(1,1)}$ model the matrices are

$$\boldsymbol{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{L} = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & 3 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 3 \\ -2 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 4 \end{pmatrix} \tag{A.10}$$

It is easy to check that they satisfy the identities:

$$(\boldsymbol{T}-1)(\boldsymbol{T}^4-1) = (\boldsymbol{L}-1)(\boldsymbol{L}^4-1) = 0, \tag{A.11}$$

$$\operatorname{spec}\boldsymbol{T} = \operatorname{spec}\boldsymbol{L} = (1,1,\ldots,1,i,-1,-i) \tag{A.12}$$

$$(\boldsymbol{TL})^3 = -\boldsymbol{\Phi}^{-3} \qquad \boldsymbol{TLT} = \boldsymbol{LTL}, \tag{A.13}$$

$$(\boldsymbol{T}^t)^4 - 1 = R_1 \otimes M^t, \qquad (\boldsymbol{L}^t)^4 - 1 = R_2 \otimes Q^t. \tag{A.14}$$

A.3 $E_8^{(1,1)}$ model

For the $E_8^{(1,1)}$ model the matrices are

$$\boldsymbol{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{L} = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 3 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & 4 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 2 \\ -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 3 \\ -1 & -2 & -1 & -2 & 0 & -1 & -1 & -1 & 0 & 5 \\ -2 & -2 & -1 & -2 & 0 & -1 & -1 & -1 & -1 & 6 \end{pmatrix} \tag{A.15}$$

which satisfy the identities:

$$(\boldsymbol{T}-1)(\boldsymbol{T}^6-1) = (\boldsymbol{L}-1)(\boldsymbol{L}^6-1) = 0, \tag{A.16}$$

$$\operatorname{spec}\boldsymbol{T} = \operatorname{spec}\boldsymbol{L} = (1,1,\ldots,1,e^{\pi i/3},e^{2\pi i/3},e^{\pi i},e^{4\pi i/3},e^{5\pi i/3}) \tag{A.17}$$

$$(\boldsymbol{TL})^3 = -\boldsymbol{\Phi}^{-3}, \qquad \boldsymbol{TLT} = \boldsymbol{LTL}, \tag{A.18}$$

$$(\boldsymbol{T}^t)^6 - 1 = R_1 \otimes M^t, \qquad (\boldsymbol{L}^t)^6 - 1 = R_2 \otimes Q^t. \tag{A.19}$$

B Cluster-tilting

In this appendix we give some more details on the relation between the present approach to the BPS spectra of the four SCFT $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ — which is based on the

JHEP09(2015)035

Abelian category $\mathsf{coh}\,\mathbb{X}_p$ — and the standard quiver approach [1, 26] which is based on the module category of the Jacobian algebra $\mathbb{C}Q/(\partial\mathcal{W})$ of the quiver Q with superpotential $\mathcal{W}$.

We write T for the direct sum of all sheaves of the canonical basis (2.42)

$$T = \bigoplus_{\vec{a}\in C} \mathcal{O}(\vec{a}) \qquad C = \big\{0,\ \vec{c},\ \ell_i\vec{x}_i, 1 \le \ell_i \le p_i - 1\big\}. \tag{B.1}$$

We already know that the canonical algebra Λ_p having the same weight type $(p_1, \cdots, p_s)$ as the line $\mathbb{X}_p$ is given by (here $\mathcal{H}_p = \mathsf{coh}\,\mathbb{X}_p$)

$$\Lambda_p = \mathrm{End}_{\mathcal{H}_p}(T). \tag{B.2}$$

However, Λ_p does not coincide with the Jacobian algebra $\mathbb{C}Q/(\partial\mathcal{W})$ for any choice of $(Q, \mathcal{W})$ in its mutation class. What is true [3] is that there exists a Q in the class which is the completion of the quiver $Q_{\rm can}$ of Λ_p: Q contains $(s-2)$ extra arrows going from the sink of $Q_{\rm can}$ to its source and Q is endowed with a superpotential linear in the new arrows η_a so that the Jacobian relations $\partial\mathcal{W}/\partial\eta_a$ give back the original relations of Λ_p. For instance, for $E_r^{(1,1)}$ we add just one new arrow η and the superpotential becomes

$$\mathcal{W} = \eta(X_1^{p_1} + X_2^{p_2} + X_3^{p_3}). \tag{B.3}$$

The modules of Λ_p are then identified with the class of modules of the Jacobian algebra with $\eta = 0$. One may wonder whether our treatment 'forgets' the modules with $\eta \neq 0$. The answer is that these modules are already properly taken into account thanks to the properties of the cluster-category $\mathcal{C}_p$.

The category $\mathcal{C}_p$ has the same objects as $\mathsf{D}^b\,\mathcal{H}_p$ and morphism spaces

$$\mathcal{C}_p(X, Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{D}^b\,\mathcal{H}_p}(X, (\tau^{-1}\Sigma)^n Y). \tag{B.4}$$

and hence it is equivalent [17] to the category $\widetilde{\mathcal{H}}_p$ having the same objects as $\mathcal{H}_p$ and $\mathbb{Z}_2$-graded morphism spaces

$$\mathrm{Hom}_{\widetilde{\mathcal{H}}_p}(X, Y) = \mathrm{Hom}_{\mathcal{H}_p}(X, Y) \oplus \mathrm{Ext}^1_{\mathcal{H}_p}(X, \tau^{-1}Y), \tag{B.5}$$

with the appropriate composition law [17]. The tilting object $T \in \mathcal{H}_p$ in eq. (B.1) is also a *cluster-tilting object* [37] for $\mathcal{C}_p$ (for a review see [38]). With respect to $\mathcal{H}_p$, the category $\widetilde{\mathcal{H}}_p$ has additional (odd) morphisms; for instance the new morphisms $\mathcal{O}(\vec{c}) \to \mathcal{O}$ are

$$\mathrm{Hom}_{\widetilde{\mathcal{H}}_p}(\mathcal{O}(\vec{c}), \mathcal{O}) = \mathrm{Ext}^1_{\mathcal{H}_p}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega})) \simeq D\,S_{\vec{c}+2\vec{\omega}}, \tag{B.6}$$

which precisely correspond to the new $(s-2)$ arrows of the completed quiver Q with respect to old $Q_{\rm can}$. More generally, one shows [17, 37]

$$\mathbb{C}Q/(\partial\mathcal{W}) = \mathrm{End}_{\mathcal{C}_p}(T)^{\rm op}. \tag{B.7}$$

Then we have a functor

$$\mathrm{Hom}_{\mathcal{C}_p}(T, \cdot)\colon \mathcal{C}_p \longrightarrow \mathsf{mod}\,\mathbb{C}Q/(\partial\mathcal{W}), \tag{B.8}$$

JHEP09(2015)035

which is full and dense; it gives an equivalence of categories

$$\mathcal{C}_p/\mathsf{add}\,\tau T \simeq \mathsf{mod}\,\mathbb{C}Q/(\partial\mathcal{W}). \tag{B.9}$$

In particular, the indecomposable objects of the category $\mathcal{H}_p \equiv \mathsf{coh}\,\mathbb{X}_p$ are the indecomposable modules of the Jacobian algebra together with the $n \equiv r(\mathfrak{g})+2$ line bundles $\mathcal{O}(\vec{a}+\vec{\omega})$ with $\vec{a} \in C$ (cf. eq. (B.1)).

JHEP09(2015)035

References

[1] S. Cecotti and C. Vafa, *Classification of complete $\mathcal{N}=2$ supersymmetric theories in 4 dimensions*, *Surveys in differential geometry* **18** (2013) [arXiv:1103.5832] [INSPIRE].

[2] T. Eguchi and K. Hori, *$N=2$ superconformal field theories in four-dimensions and A-D-E classification*, in *Saclay 1996, The mathematical beauty of physics* [hep-th/9607125] [INSPIRE].

[3] S. Cecotti, *Categorical tinkertoys for $N=2$ gauge theories*, *Int. J. Mod. Phys.* **A 28** (2013) 1330006 [arXiv:1203.6734].

[4] S. Cecotti, M. Del Zotto and S. Giacomelli, *More on the $N=2$ superconformal systems of type $D_p(G)$*, *JHEP* **04** (2013) 153 [arXiv:1303.3149] [INSPIRE].

[5] M. Del Zotto, C. Vafa and D. Xie, *Geometric Engineering, Mirror Symmetry and 6d $(1,0) \to 4d$, $N=2$*, arXiv:1504.08348 [INSPIRE].

[6] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N=2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099] [INSPIRE].

[7] F. Diamond and J. Shurman, *Graduate Texts in Mathematics. Vol. 228: A first course in modular forms*, Springer, Berlin Germany (2005).

[8] F. Klein, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, (1884). Reprinted with an introduction and commentary by P. Slodowy, Birkhäuser Verlag, Basel Switzerland (1993).

[9] G.C. Shephard and J.A. Todd, *Finite unitary reflection groups*, *Canad. J. Math.* **6** (1954) 274.

[10] M. Broué, G. Malle and R. Rouquier, *On complex reflection groups and their associated braid groups*, *CMS Conf. Proc.* **16** (1995) 1.

[11] R. Stekolshchik, *Notes on Coxeter transformations and the McKay correspondence*, *Springer Monographs in Mathematics*, Springer, Heidelberg Germany (2008).

[12] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in the representation theory of finite dimensional algebras*, in *Lectures Notes in Mathematics. Vol. 1273: Singularities, Representation of Algebras, and Vector Bundles*, Springer, Berlin Germany (1987).

[13] H. Lenzing, *Hereditary categories*, in *London Mathematical Society Lecture Note Series. Vol. 332: Handbook of tilting theory*, L. Angeleri Hügel, D. Happel and H. Krause eds., Cambridge University Press, Cambridge U.K. (2007).

[14] H. Lenzing, *Hereditary categories*, ICTP lectures 2006. Available on line: http://webusers.imj-prg.fr/ bernhard.keller/ictp2006/lecturenotes/lenzing1.pdf, http://webusers.imj-prg.fr/ bernhard.keller/ ictp2006/lecturenotes/lenzing2.pdf.

[15] H. Lenzing and H. Meltzer, *The automorphism group of the derived category for a weighted projective line*, *Comm. Algebra* **28** (2000) 1685.

[16] H. Meltzer, *Memoirs of the American Mathematical Society. Vol. 808: Exceptional vector bundles, tilting sheaves and tilting complexes for weighted projective lines*, AMS Press, Providence U.S.A. (2004).

[17] M. Barot, D. Kussin and H. Lenzing, *The cluster category of a canonical algebra*, arXiv:0801.4540.

[18] D. Happel and C.M. Ringel, *Lecture Notes in Mathematics. Vol. 1177: The derived category of a tubular algebra*, Springer, Berlin Germany (1986), pg. 156.

[19] D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, math/0503632.

[20] D. Kussin, H. Lenzing and H. Meltzer, *Triangle singularities, ADE-chains, and weighted projective lines*, arXiv:1203.5505.

[21] S. Cecotti, A. Neitzke and C. Vafa, *R-Twisting and 4d/2d Correspondences*, arXiv:1006.3435 [INSPIRE].

[22] O. Schiffmann, *Noncommutative projective curves and quantum loop algebras*, math/0205267.

[23] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435 [INSPIRE].

[24] S. Cecotti and M. Del Zotto, *Y systems, Q systems and 4D $\mathcal{N} = 2$ supersymmetric QFT*, *J. Phys.* **A 47** (2014) 474001 [arXiv:1403.7613] [INSPIRE].

[25] T. Bridgeland, *Stability conditions on triangulated categories*, *Ann. Math.* **166** (2007) 317 [math/0212237].

[26] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi and C. Vafa, *$\mathcal{N} = 2$ quantum field theories and their BPS quivers*, *Adv. Theor. Math. Phys.* **18** (2014) 27 [arXiv:1112.3984] [INSPIRE].

[27] S. Cecotti and M. Del Zotto, *Galois covers of $N = 2$ BPS spectra and quantum monodromy*, arXiv:1503.07485 [INSPIRE].

[28] C.M. Ringel, *Lecture Notes in Mathematics. Vol. 1099: Tame Algebras and Integral Quadratic Forms*, Springer, Berlin Germany (1984).

[29] R. Moody, S. Rao and T. Yokonuma, *Toroidal Lie algebras and vertex representations*, *Geom. Dedicata* **35** (1990) 283.

[30] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras. Vol. 1: Techniques of Representation Theory*, Cambridge University Press, Cambridge U.K. (2006).

[31] M.F. Atiyah, *Vector bundles over an elliptic curve*, *Proc. London Math. Soc.* **7** (1957) 414.

JHEP09(2015)035

[32] L. Hille and M. Van den Bergh, *Fourier-Mukai transforms*, in *Handbook of tilting theory*, L. Angeleri Hügel, D. Happel and H. Krause eds., Cambridge University Press, Cambridge U.K. (2007), pg. 147.

[33] D. Kussin, *Memoirs of the American Mathematical Society. Vol. 201: Noncommutative curves of genus zero: related to finite dimensional algebras*, AMS Press, Providence U.S.A. (2009).

[34] X.-W. Chen and H. Krause, *Introduction to coherent sheaves on weighted projective lines*, arXiv:0911.4473.

[35] J. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, *Grundlehren der mathematischen Wissenschaften*, Springer, Heidelberg (1999).

[36] B. Keller, *Cluster algebras, quiver representations and triangulated categories*, in *London Mathematical Society Lecture Note Series. Vol. 375: Triangulated categories*, Cambridge University Press, Cambridge U.K. (2010), pg. 76.

[37] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, *Adv. Math.* **204** (2006) 572 [math/0402054].

[38] I. Reiten, *Tilting theory and cluster algebras*, ICTP lectures 2006. Available on line: http://webusers.imj-prg.fr/ bernhard.keller/ictp2006/lecturenotes/reiten.pdf.

[39] M. Buican, S. Giacomelli, T. Nishinaka and C. Papageorgakis, *Argyres-Douglas Theories and S-duality*, *JHEP* **02** (2015) 185 [arXiv:1411.6026] [INSPIRE].

[40] M. Buican and T. Nishinaka, *On the Superconformal Index of Argyres-Douglas Theories*, arXiv:1505.05884 [INSPIRE].

JHEP09(2015)035