Exact results for $\mathcal{N} = 2$ supersymmetric gauge theories on compact toric manifolds and equivariant Donaldson invariants

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ABSTRACT: We provide a contour integral formula for the exact partition function of $\mathcal{N} = 2$ supersymmetric U(N) gauge theories on compact toric four-manifolds by means of supersymmetric localisation. We perform the explicit evaluation of the contour integral for U(2) $\mathcal{N} = 2^*$ theory on $\mathbb{P}^2$ for all instanton numbers. In the zero mass case, corresponding to the $\mathcal{N} = 4$ supersymmetric gauge theory, we obtain the generating function of the Euler characteristics of instanton moduli spaces in terms of mock-modular forms. In the decoupling limit of infinite mass we find that the generating function of local and surface observables computes equivariant Donaldson invariants, thus proving in this case a long-standing conjecture by N. Nekrasov. In the case of vanishing first Chern class the resulting equivariant Donaldson polynomials are new.

Contents

1   Introduction

$\mathcal{N} = 2$ supersymmetric gauge theories are a source of many interesting results in the theory of Integrable Systems (both classical [1–3] and quantum [4]) and more recently in Conformal Field Theory in two dimensions [5] and integrable quantum hydrodynamics [6–11].

These results are mainly due to the application of *equivariant localization* to the supersymmetric path integral which reduces its evaluation to a combinatorial problem. The results obtained so far concern few examples of four-manifolds as $\mathbb{C}^2$ [12, 13], $\mathbb{C}^2/\Gamma$ [14–20], S^4 [21, 22] and $S^2 \times S^2$ [23].

On the other hand, it is known since the seminal paper [24] that twisted $\mathcal{N} = 2$ supersymmetric gauge theories can be formulated on any Riemannian four-manifold and

their observables realise many interesting topological invariants such as Donaldson invariants [25] and knot invariants [26, 27]. A refinement of these invariants can be provided on four-manifolds admitting isometries by considering their equivariant extension, which in physical language corresponds to turning on the Ω-background [12]. However, few explicit calculations are availble in this case.

The aim of this paper is to apply the supersymmetric localization technique to a suitable class of compact four manifolds. In [23] (see also [28]) Killing spinor solutions implementing an equivariant extension of the Witten twist were found on any Riemannian four manifold admitting a U(1) action and this was used to study the case of $S^2 \times S^2$. In this paper we discuss more general toric complex surfaces and perform explicit computations in the case of $\mathbb{P}^2$ as a testing ground.

An important difference between compact and non-compact four-manifolds is obviously related to the issue of boundary conditions. For $\mathcal{N} = 2$ gauge theories on non-compact manifolds the partition function depends on the v.e.v. of the scalars a_ρ sitting in the vector multiplet. The presence of this v.e.v. is indeed crucial in order to localize to isolated fixed points in the instanton moduli space and reduces the evaluation of the partition function to a combinatorial problem. In this context, a_ρs represent the equivariant weights associated to the action of the Cartan torus of the gauge group.

On the other hand, on compact manifolds, in order to have exact smooth instanton solutions one sets $a_\rho = 0$ [24]. The supersymmetric fixed-locus in this case is given by the full instanton moduli space. However, the contribution to the evaluation of 1/2 BPS observables in $\mathcal{N} = 2$ theories is fully captured by singular gauge field configurations sitting at the boundary of the instanton moduli space [29, 30]. A suitable (partial) compactification and desingularization of this space is provided by considering the moduli space of torsion free sheaves on the four-manifold, which locally corresponds to turning on a non-commutative deformation [31]. The boundary is in this case provided by ideal sheaves, which correspond to copies of point-like U(1) non-commutative instantons.

The strategy we follow is then to use the equivariant twisted supersymmetry of [23] to directly localize the path integral to point-like instantons sitting at the zeroes of the vector field generating the U(1) action. The contribution of each of these points is given by a Nekrasov partition function on the corresponding affine patch $\sim \mathbb{C}^2$. In this context, the equivariant parameters a_ρ are intended as classical solutions to the fixed point equations and as such have to be integrated over. This result is in agreement with a proposal made by Nekrasov [32] for the calculation of the $\mathcal{N} = 2$ partition function on compact toric manifolds.[1]

Let us notice that another important issue arising in the study of $\mathcal{N} = 2$ supersymmetric gauge theories on compact manifolds is the appearance of extra gaugino zero modes. As we will show in the following, a proper treatment of these modes provides the prescription for the contour integration on the Coulomb branch parameters a_ρ.

On the mathematical side, the difference between the non-compact and compact cases is that in the former one has to consider the moduli space of *framed* instantons and corre-

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[1]$\mathcal{N} = 2$ theories on toric Kähler manifolds have been recently analyzed also in [33].

spondingly of *framed* torsion-free sheaves for its compactification, while in the latter there is no framing. We recall that the framing correspond to a trivialization of the fiber at a point, which implies that the moduli space includes global gauge transformations acting on the framing. Framed instanton moduli spaces are hyperkähler and have deep links to representation theory of infinite dimensional Lie algebrae and Geometric Invariant Theory [34]. They are much more amenable to equivariant localization than the corresponding *unframed* moduli spaces. On the other hand, the latter bring important information, as for example Donaldson invariants are formulated via intersection theory on them. In [32] Nekrasov conjectured that the integration over the Coulomb branch parameters in the $\mathcal{N} = 2$ partition function over compact toric surfaces produces precisely the corresponding Donaldson invariants. In this paper we will prove this conjecture for U(2) gauge theories on $\mathbb{P}^2$ by specifying the integration contour and by spelling out the conditions imposed on the fixed point data by the stability conditions on the equivariant sheaves. For U(2) gauge theory the contour integral evaluation corresponds to taking the residue at $a_\rho = a_1 - a_2 = 0$, in line with Witten's arguments [24]. We will find that for odd first Chern class the $\mathcal{N} = 2$ generating function of local and surface observables indeed calculate the equivariant Donaldson invariants obtained in [35]. This follows by comparing our formula (3.43) with the results of theorem 6.15 in [35] as explained in detail in section 3.5. Let us underline that our approach holds also in presence of reducible connections, which contribute for even first Chern class, where the method of [35] does not apply. We calculate the equivariant Donaldson polynomials in this case too and we match their non-equivariant limit with the SU(2) Donaldson polynomials computed in [36]. Let us remark that the pure partition functions are expected to count the zero dimensional components of the instanton moduli space [24]. Our findings are in full agreement with this expectation implying non trivial cubic identities on the Nekrasov partition functions.

We also consider $\mathcal{N} = 2^*$ gauge theory, that is Super-Yang-Mills theory in presence of a hypermultiplet of mass M. This theory interpolates between pure $\mathcal{N} = 2$ in the decoupling limit $M \to \infty$ and $\mathcal{N} = 4$ for $M \to 0$. In the latter case the partition function is expected to be the generating function of the Euler characteristics of the moduli space of *unframed* sheaves. We provide a check of this for U(2) gauge theories on $\mathbb{P}^2$. For odd first Chern class we get results in agreement with [37], and for even first Chern class we compare with the results obtained by Yoshioka using finite field methods [38, 39].

The paper is organised as follows. In section 2 we discuss the general features of $\mathcal{N} = 2$ gauge theories on complex four-manifolds and discuss equivariant observables. We then specialise to compact toric surfaces discussing the supersymmetric fixed points and the contour integral formula obtained by properly treating the fermionic zero-modes. The master formula for the generating function of local and surface observables is presented in equation (3.9), specialising to U(2) gauge theories on $\mathbb{P}^2$. In section 3 we focus on U(2) Super Yang-Mills on $\mathbb{P}^2$. We study in detail the analytic structure of the integrand by making use of Zamolodchikov's recursion relations for Virasoro conformal blocks. We then evaluate explicitly the contour integral. Our main results are equation (3.43) and (3.70) for odd and even first Chern class respectively. We then proceed to the non-equivariant limit $\epsilon_1, \epsilon_2 \to 0$ and compare with the results in the mathematical literature. In subsection 3.8

we discuss the calculation of the pure partition function on $\mathbb{P}^2$ which implies remarkable cubic identities for the Nekrasov partition function. In section 4 we study the $\mathcal{N} = 2^*$ theory and discuss the zero mass limit which we find to calculate the generating function of Euler characteristics of moduli spaces of rank-two sheaves. Our main result is (4.20) which includes also the contribution of strictly semi-stable sheaves. We finally discuss the (mock-)modular properties of the $\mathcal{N} = 4$ partition function. Section 5 contains a discussion on open problems and appendix A describes the relation between the supersymmetric fixed point data and Klyachko's classification of semi-stable equivariant sheaves.

2 $\mathcal{N} = 2$ gauge theories on complex surfaces and Hermitian Yang Mills bundles

In this section we discuss U(N) $\mathcal{N} = 2$ gauge theories on complex surfaces and specify the results of [23] to toric surfaces.

Four dimensional $\mathcal{N} = 2$ gauge theories can be considered on any orientable four manifold M upon a proper choice of the $\mathcal{R}$-symmetry bundle [24]. The sum over the physical vacua contributing to the supersymmetric path-integral depends of course on the specific gauge group at hand. In the case of SU(N) gauge theories, these are completely described in terms of anti-selfdual connections $F^+ = 0$, once the orientation on M is chosen. In the U(N) case extra contributions arise from gauge bundles with non trivial first Chern class. Indeed, beyond anti-instantons, one has to consider gauge bundles with first Chern class aligned along $H^+(X, \mathbb{Z})$. This led in [23] to consider the gauge fixing of the supersymmetric path-integral in a split form, where the U(1) sector is treated separately. If M is an hermitian manifold, an equivalent procedure is given by gauge fixing the path-integral to Hermitian-Yang-Mills (HYM) connections

$$
\begin{aligned}
F^{(2,0)} &= 0 \\
g^{i\bar{j}}F_{i\bar{j}} &= \lambda \mathbb{1}
\end{aligned}
\tag{2.1}
$$

where $F^{(2,0)}$ is the $(2,0)$ component of the gauge curvature in a given complex structure, g is the hermitian metric on M and λ is a real parameter.

If the manifold M is Kähler, then (2.1) reads

$$
\begin{aligned}
F^{(2,0)} &= 0 \\
\omega \wedge F &= \lambda \, \omega \wedge \omega \mathbb{1}
\end{aligned}
\tag{2.2}
$$

where $\lambda = \frac{2\pi \int_M c_1(E) \wedge \omega}{r(E) \int_M \omega \wedge \omega} = \frac{2\pi \mu(E)}{\int_M \omega \wedge \omega}$ and $\mu(E)$ is the *slope* of the vector bundle. Here $r(E) = N$ is the rank of E and $c_1(E) = \frac{1}{2\pi}\mathrm{Tr}F_E$ its first Chern class.

In the rest of the paper we consider Kähler four manifolds admitting a U(1) action with isolated fixed points. In this case, as shown in [23], one can improve the supersymmetric localization technique by making it equivariant with respect to such a U(1) action and localize on point-like instantons. The resulting partition function is obtained by a suitable gluing of Nekrasov partition functions which includes the sum over fluxes and the integration over the Coulomb parameters.

In the twisted variables, the supersymmetry reads as

$$\begin{aligned}
\mathcal{Q}A &= \Psi, & \mathcal{Q}\Psi &= i\iota_V F + D\Phi, & \mathcal{Q}\Phi &= i\iota_V \Psi, \\
\mathcal{Q}\bar{\Phi} &= \eta, & \mathcal{Q}\eta &= i\,\iota_V D\bar{\Phi} + i[\Phi, \bar{\Phi}], & & \\
\mathcal{Q}\chi^+ &= B^+, & \mathcal{Q}B^+ &= i\mathcal{L}_V \chi^+ + i[\Phi, \chi^+].
\end{aligned} \tag{2.3}$$

In (2.3) ι_V is the contraction with the vector field V and $\mathcal{L}_V = D\iota_V + \iota_V D$ is the covariant Lie derivative. On a Kähler four manifold self-dual forms split as

$$\chi^+ = \chi^{(2,0)} \oplus \chi^{(0,2)} \oplus \chi\,\omega \ \ \text{and} \ \ B^+ = B^{(2,0)} \oplus B^{(0,2)} \oplus b\,\omega. \tag{2.4}$$

Let us notice that the supercharge (2.3) manifestly satisfies $\mathcal{Q}^2 = i\mathcal{L}_V + \delta_\Phi^{\text{gauge}}$. Consistency of the last line implies that the V-action preserves the self-duality of B^+ and χ^+, that is $L_V \star = \star L_V$, where $\star$ is the Hodge-$\star$ and $L_V = d\iota_V + \iota_V d$ is the Lie derivative. This condition coincides with the requirement that V generates an isometry of the four manifold.

The supersymmetric Lagrangian we consider is

$$L = \frac{i\tau}{4\pi} \Big(\text{Tr}\, F \wedge F - c\, \text{Tr}\, F \wedge \text{Tr}\, F \Big) + \gamma \wedge \text{Tr}\, F + \mathcal{Q}\mathcal{V} \tag{2.5}$$

where c is a constant,[2] τ is the complexified coupling constant, $\gamma \in H^2(M)$ is the source for the c_1 of the vector bundle and $\mathcal{V}$ is a gauge invariant localizing term, chosen in order to implement the Hermitean-Yang-Mills equations, namely

$$\mathcal{V} = -\text{Tr}\big[i\chi^{(0,2)} \wedge F^{(2,0)} + i\chi\,(\omega \wedge F - \lambda\,\omega \wedge \omega \mathbb{1}) + \Psi \wedge \star(\mathcal{Q}\Psi)^\dagger + \eta \wedge \star(\mathcal{Q}\eta)^\dagger\big]. \tag{2.6}$$

The integration over $B^{(0,2)}$ and b in (2.5) implies the Hermitean Yang-Mills equations (2.2) as delta-gauge conditions. In particular, the path integral over the field b ensures the semi-stability of the bundle.[3] Recall that [43] a bundle E is said to be (slope) semistable if for every proper sub-bundle $G \subset E$, the slope of the bundle $\mu(E)$, defined below (2.2), is greater or equal than the slope of the sub-bundle $\mu(G)$. If it is strictly greater E is said to be stable. If the bundle E admits a sub-bundle G, then the b field has an integration mode proportional to the projector onto G, namely $ib_0\Pi_G$. The connection splits as

$$A_E = \begin{pmatrix} A_G & n \\ n^\dagger & \star \end{pmatrix} \tag{2.7}$$

and the curvature accordingly as

$$F_E = \begin{pmatrix} F_G + n \wedge n^\dagger & \star \\ \star & \star \end{pmatrix}. \tag{2.8}$$

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[2]Different values of c in (2.5) produce different expansion in the final formula. The usual choice is $c = 0$, which produces an expansion in the instaton number, or equivalently in the second Chern character $ch_2 = c_2^2 - \frac{1}{2}c_1^2$ of the bundle. The choice $c = 1$ produces an expansion in the second Chern class c_2 and the choice $c = \frac{1}{2}$ produces an expansion on the discriminant D of the bundle. In comparing the result of the paper with the literature we will use the last two choices.

[3]The semi-stability of the bundle and HYM condition are actually equivalent. This is the so called Hitchin-Kobayashi correspondence, that was proven in [40–42].

Let us focus on the integral along the above integration mode. The corresponding term in the action comes from

$$\int_M \text{Tr}\left[b\left(\omega \wedge F_E - \lambda \, \omega \wedge \omega \mathbb{1}_E\right)\right] \tag{2.9}$$

and reads

$$ib_0 \int_M \text{Tr}\left[\Pi_G\left(\omega \wedge F_E - \lambda \, \omega \wedge \omega \mathbb{1}_E\right)\right] = ib_0\left[2\pi r(G)\left(\mu(G) - \mu(E)\right) + \int_M |n|^2\right] \tag{2.10}$$

Therefore the path integral includes the term

$$\int db_0 \, e^{ib_0\left[2\pi r(G)(\mu(G)-\mu(E))+\int_M |n|^2\right]} \sim \delta\left(2\pi r(G)\left(\mu(G) - \mu(E)\right) + \int_M |n|^2\right) \tag{2.11}$$

which, because of $\int_M |n|^2 \geq 0$, implies that the partition function is supported on vector bundles E such that

$$\mu(E) \geq \mu(G) \tag{2.12}$$

for any sub-bundle G, that is on semi-stable vector bundles. Notice that this condition depends on the point in the Kähler cone defining the polarization ω.

2.1 Equivariant observables

In this subsection we discuss equivariant observables in the topologically twisted gauge theory. These are obtained by the equivariant version of the usual descent equations.

The scalar supercharge action can be written as the equivariant Bianchi identity for the curvature $\mathbf{F}$ of the universal bundle as [44]

$$\mathbf{DF} \equiv \left(-Q + D + i\iota_V\right)\left(F + \psi + \Phi\right) = 0, \tag{2.13}$$

where D is the covariant derivative. Therefore, for any given ad-invariant polynomial $\mathcal{P}$ on the Lie algebra of the gauge group, we have

$$Q\mathcal{P}(\mathbf{F}) = (d + i\iota_V)\,\mathcal{P}(\mathbf{F}) \tag{2.14}$$

and the observables are obtained by intersection of the above with elements of the equivariant cohomology of the manifold, $\boldsymbol{\Omega} \in H_V^\bullet(M)$ as

$$\mathcal{O}\left(\boldsymbol{\Omega}, \mathcal{P}\right) \equiv \int \boldsymbol{\Omega} \wedge \mathcal{P}(\mathbf{F}). \tag{2.15}$$

As far as the $\text{U}(N)$ gauge theory is concerned, we can consider the basis of single trace observables $\mathcal{P}_n(x) = \frac{1}{n}\text{Tr}\,x^n$ with $n = 1, \dots N$.

The equivariant cohomology splits in even and odd parts which can be discussed separately. We focus on the relevant observables corresponding to the even cohomology. The two cases to discuss in the $\text{U}(2)$ theory are $n = 1, 2$. The first $\int_M \text{Tr}\,\mathbf{F} \wedge \Omega$ is the source term for the first Chern class and for the local observable $\text{Tr}\,\Phi(P)$, where P is a fixed point of the vector field V. The second is

$$\frac{1}{2}\int_M \Omega^{[\text{even}]} \wedge \text{Tr}\,\mathbf{F}^2 \tag{2.16}$$

This generates

- the second Chern character of the gauge bundle $\int_M \text{Tr}(F \wedge F)$ for $\mathbf{\Omega} = 1$ (the Poincaré dual of M),

- surface observables for $\mathbf{\Omega} = \omega + H$, where ω is a V-equivariant element in $H^2(M)$ and H a linear polynomial in the weights of the V-action satisfying $dH = \iota_V \omega$. Namely

$$\int_M \omega \wedge \text{Tr}\left(\Phi F + \Psi^2\right) + H\,\text{Tr}(F \wedge F) \tag{2.17}$$

- for $\mathbf{\Omega} = (\omega + H) \wedge (\omega' + H') + K$, with $\omega + H$ and $\omega' + H'$ as in the previous item and K a quadratic, coordinate independent, polynomial in the weights of the V-action, we get

$$\int_M \omega \wedge \omega'\,\text{Tr}\,\Phi^2 + (\omega H' + H'\omega) \wedge \text{Tr}\left(\Phi F + \frac{1}{2}\Psi^2\right) + (HH' + K)\,\text{Tr}(F \wedge F) \tag{2.18}$$

- local observables at the fixed points $\text{Tr}\,\Phi^2(P)$, for $\mathbf{\Omega} = \delta_P$ the Poincaré dual of any fixed point P under the V-action.

Let us remark that local observables in the equivariant case depend on the insertion point via the equivariant weights of the fixed point. This is due to the fact that the equivariant classes of different fixed points are distinct. From the gauge theory viewpoint one has

$$\text{Tr}\,\Phi^2(P) - \text{Tr}\,\Phi^2(P') = \int_{P'}^{P} \iota_V\,\text{Tr}\left(\Phi F + \frac{1}{2}\Psi^2\right) + Q[\ldots] \tag{2.19}$$

so that the standard argument of point location independence is flawed by the first term in the r.h.s.

Indeed the set of equivariant observables is richer than the set of non-equivariant ones. Also the observables in (2.18) reduce in the non equivariant limit to local observables up to a volume factor.

The mathematical meaning of these facts is that the equivariant Donaldson polynomials give a finer characterization of differentiable manifolds. The physical one is that the Ω-background probes the gauge theory via a finer BPS structure.

2.2 Gluino zero modes and contour integral prescription

An issue that we have not analyzed till now is the existence of gluino zero modes and its consequences in the evaluation of the path integral.

The fermionic fields are the scalar η, the 1-form Ψ and the selfdual 2-form χ^+. The number of zero modes is given by the respective Betti numbers $b_0 = 1$, $b_1 = 0$ and $b_2^+ = 1$ times the rank of the gauge group.[4] Specifically, the χ^+ zero mode is proportional to the Kähler form ω.

The discussion on the integration on the zero-modes for the complete U(N) theory is naturally split in the U(1) sector and the SU(N) sector. Actually, the two sectors are different in nature. The first is related to a global symmetry of the theory while the second to the structure of the moduli space at the fixed points of the supercharge of the microscopic theory.

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[4]We remind the reader that $b_2^+ = 1$ for all toric surfaces.

2.2.1 The zero modes in the U(1) sector

The zero modes in the U(1) sector come as a quartet of symmetry parameters of the whole twisted super-algebra. The c-number BRST charge implementing this shift symmetry is given by

$$\mathsf{q}A = 0, \qquad \mathsf{q}\Psi = 0, \quad \mathsf{q}\Phi = \kappa_\Phi \mathbb{1}, \quad \mathsf{q}\kappa_\Phi = 0,$$
$$\mathsf{q}\bar\Phi = \kappa_{\bar\Phi} \mathbb{1}, \quad \mathsf{q}\kappa_{\bar\Phi} = 0, \quad \mathsf{q}\eta = \kappa_\eta \mathbb{1}, \quad \mathsf{q}\kappa_\eta = 0, \qquad (2.20)$$
$$\mathsf{q}\chi = \kappa_\chi \omega \mathbb{1}, \quad \mathsf{q}\kappa_\chi = 0, \quad \mathsf{q}B = 0,$$

and the action of $\mathcal{Q}$ on the c-number parameters above is given by

$$\mathcal{Q}\kappa_\Phi = 0, \quad \mathcal{Q}\kappa_{\bar\Phi} = -\kappa_\eta, \quad \mathcal{Q}\kappa_\eta = 0, \quad \mathcal{Q}\kappa_\chi = 0, \qquad (2.21)$$

so that $\{\mathcal{Q}, \mathsf{q}\} = 0$. The κ-ghosts have to be supplemented by their corresponding anti-ghosts $\bar\kappa_I$ and Lagrange multipliers λ_I, with $I \in \{\Phi, \bar\Phi, \eta, \chi\}$ and $\mathsf{q}\bar\kappa_I = \lambda_I$ and $\mathsf{q}\lambda_I = 0$. It is needless to say that $\mathcal{Q}\bar\kappa_I = 0$ and $\mathcal{Q}\lambda_I = 0$.

Notice that $\mathsf{q}\mathcal{V} = 0$. The gauge fixing fermion for the U(1) zero modes then reads

$$\nu = \sum_I \bar\kappa_I \int_M \mathrm{Tr}(I) e^\omega \qquad (2.22)$$

so that the gauge fixing action $(\mathcal{Q} + \mathsf{q})\nu$ gives a suitable measure to integrate out these modes as a perfect quartet.

The only U(1) zero mode who survives is that of the B field which is still playing as a Lagrange multiplier for the HYM equations.

2.2.2 Zero modes in the SU(N) sector and integration contour prescription

In this subsection we show that by correctly treating the issue of gaugino zero modes in the SU(N) sector we get precise instructions about the integration on the leftover $N-1$ Cartan parameters $a_\rho = a_\alpha - a_\beta$.

The presence of gaugino zero modes implies a ghost number anomaly that has to be compensated by the insertion of appropriate supersymmetric terms which cancel the ghost number excess and soak-up the fermionic zero modes. The path integral as it stands is indeed undefined and its measure has to be improved. In order to do this we add to the localizing action the further term

$$S_{\mathrm{gauginos}} = s\mathcal{Q} \int_M \mathrm{Tr}\,\bar\Phi_0 \chi_0 \omega = s \int_M \mathrm{Tr}\left\{\eta_0 \chi_0 \omega + \bar\Phi_0 b_0 \omega\right\}. \qquad (2.23)$$

where s is a complex parameter and only the zero modes of the fields enter. The final result does not depend on the actual value of s as long as $s \neq 0$. The first term in the r.h.s. of (2.23) contributes to the ghost number anomaly by one insertion per element in the Cartan subalgebra of $su(N)$. Once the integral over the $N-1$ couples of gluino zero modes (η_0, χ_0) is taken, we stay with an insertion of b-field zero mode per $su(N)$ Cartan element as

$$\prod_\rho \left(\int da\, d\bar a\, db_0\, (s\omega)\, e^{s\bar a b_0 \omega}\right)_\rho e^{\mathcal{Q}\mathcal{V}} \qquad (2.24)$$

where ρ spans the $su(N)$ Cartan subalgebra. By renaming $\bar{a} \to \bar{a}/s$ and letting $s \to \infty$ we then get

$$\prod_\rho \left(\int da\, d\bar{a}\, \frac{\partial}{\partial \bar{a}} \int \frac{db_0}{b_0}\, e^{\bar{a} b_0 \omega} \right)_\rho e^{\mathcal{QV}|_{\bar{a}=0}}\,. \tag{2.25}$$

Similar arguments appeared in the evaluation of the low-energy effective Seiberg-Witten theory [45]. The integrals over the $N-1$ zero modes of b are taken by evaluating at $b = 0$ by Cauchy theorem. This implies that the leftover integral over the Cartan parameters is a total differential in the $\bar{\Phi}$ zero-mode variables, namely in $\bar{a}_\rho$, so that it gets reduced to a contour integral along the boundary of the moduli space of solutions of the fixed points equations that will be discussed in the next subsection.

Let us notice that the way in which we have soaked up the (η, χ) fermionic zero modes in (2.23) implies that the path integral localizes on configurations satisfying a more general condition than the Hermitian Yang-Mills equation. This is due to the fact that the b-field zero modes along the Cartan of $su(N)$ are not playing the role of Lagrange multipliers anymore. Therefore the gauge fixing condition results to be $F^+ = \omega \mathtt{t}$, where $\mathtt{t}$ is a constant Cartan element in $u(N)$, instead of (2.2). The former is indeed the condition satisfied by the supersymmetric fixed points that we will discuss in the next subsection.

2.3 Localization onto the fixed points

The localization proceeds as follows: by setting the fermions to zero, the fixed points of the supercharge read

$$\begin{aligned} \iota_V D\bar{\Phi} + [\Phi, \bar{\Phi}] &= 0, \\ i\iota_V F + D\Phi &= 0, \end{aligned} \tag{2.26}$$

and their integrability conditions

$$\begin{aligned} \iota_V D\Phi &= 0, \\ \mathcal{L}_V F &= [F, \Phi]. \end{aligned} \tag{2.27}$$

By using the reality condition for the scalar fields $\bar{\Phi} = -\Phi^\dagger$ and the first of (2.27), the first of (2.26) splits in two, that is

$$\iota_V D\bar{\Phi} = 0 \quad \text{and} \quad [\Phi, \bar{\Phi}] = 0 \tag{2.28}$$

which imply that Φ and $\bar{\Phi}$ lie in the same Cartan subalgebra. By reasoning in an analogous way on the second equation in (2.27), we get that the gauge curvature too is aligned along the Cartan subalgebra.

We now describe the solution in detail for compact toric manifolds. These latter are described by their toric fan [46]. The supersymmetry algebra is equivariant with respect to the maximal torus $\mathrm{U}(1)^{N+2}$, where the first factor is the Cartan torus of the gauge group and the second is the isometry V of the four manifold.[5] In components, labeled by

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[5]We remind the reader that for toric surfaces V generates a $(\mathbb{C}^*)^2$-action, which correspond to a complexification of the Ω-background parameters.
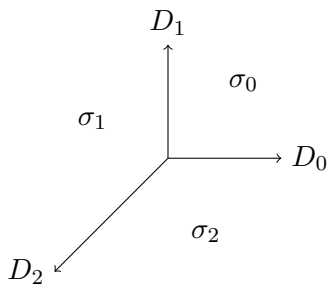
Figure 1. Toric fan of $\mathbb{P}^2$. σ_ℓ labels the cone of dimension two relative to the ℓ-th $\mathbb{C}^2$ coordinates patch.

$\alpha = 1, \ldots, N$, we have

$$(F + \Phi)_\alpha = F_\alpha^{\mathrm{point}} + a_\alpha + \sum_\ell k_\alpha^{(\ell)} \omega^{(\ell)} \tag{2.29}$$

that is, $F + \Phi$ is the $\mathrm{U}(1)^{N+2}$ equivariant curvature of the bundle. The a_α parameters generate the $\mathrm{U}(1)^N$-action. Moreover $\omega^{(\ell)} \in H_V^2(M)$ is the V-equivariant two-form Poincaré dual of the equivariant divisor D_ℓ corresponding to the ℓ-th vector of the fan (see figure 1).

Let us denote by $H^{(\ell)}$ the zero-form part of $\omega^{(\ell)}$. We get

$$\Phi_\alpha = a_\alpha + \sum_\ell k_\alpha^{(\ell)} H^{(\ell)} \,. \tag{2.30}$$

The values of Φ_α at each fixed point $P_{(\kappa)}$ will be denoted by

$$a_\alpha^{(\kappa)} \equiv \Phi_\alpha \left(P_{(\kappa)} \right) \,. \tag{2.31}$$

In (2.29), F^{point} is the contribution of point-like instantons located at the fixed points of the $\mathrm{U}(1)^2$-action. For each of these fixed points we have then an independent contribution given by the Nekrasov partition function associated to the affine patch where the fixed point is sitting. In this framework, the contribution of point-like instantons correspond to the one of ideal sheaves on $\mathbb{C}^2$ supported at the fixed points of the $\mathrm{U}(1)^2$-action, labeled by Young diagrams $\{Y_\alpha^{(\ell)}\}$.[6] We remind the reader that the Chern classes of the point-like instantons are given by

$$
\begin{aligned}
c_1^{(\ell)} &= \sum_{\alpha=1}^N k_\alpha^{(\ell)}, \\
ch_2^{(\ell)} &= \sum_{\alpha=1}^N \left| Y_\alpha^{(\ell)} \right|.
\end{aligned}
\tag{2.32}
$$

Summarizing, we find that the localization procedure implies that the partition function is written as a product of copies of the Nekrasov partition function in the appropriate shifted variables glued by the integration over the Cartan parameters $\{a_{\alpha\beta}\}$.

---

[6]Locally this compactification can be regarded as a non-commutative deformation in the affine patch of M.

The integration contour is specified according to the discussion in the previous subsection as follows. Solving the fixed point equations we bounded the field theory phase to the deep Coulomb branch by declaring Φ and $\bar{\Phi}$ to lie at a generic point in the Cartan subalgebra where the gauge symmetry is maximally broken as $\mathrm{U}(N) \to \mathrm{U}(1)^N$. This implies the integral over $(a, \bar{a})$ to be in $\mathbb{C}^{N-1} \setminus \mathcal{T}$ where $\mathcal{T}$ is a tubular neighborhood of the hyperplanes set $\Delta = \{a_\alpha - a_\beta = 0\}$. This choice guarantees maximal gauge symmetry breaking. Henceforth, by using Stokes theorem in formula (2.25), we find that the complete partition function is given by a contour integral around the above regions of the leftover terms in the path integral evaluation. In particular, for $N = 2$ we find a single contour integral around the origin in $\mathbb{C}$.

Moreover, the stability condition on the equivariant *unframed* sheaves induces constraints on the allowed values of the fixed points data $\{k_{\alpha\beta}^{(\ell)} := k_\alpha^{(\ell)} - k_\beta^{(\ell)}\}$. We will describe in section 3 the details of all this for $\mathrm{U}(2)$ gauge theories on $\mathbb{P}^2$.

3 Exact partition function on $\mathbb{P}^2$ and equivariant Donaldson invariants

Let us denote the homogeneous coordinates of $\mathbb{P}^2$ by $[z_0 : z_1 : z_2]$. The $(\mathbb{C}^*)^2$ torus action, generated by the vector,[7] acts on homogeneus coordinates as $[z_0 : e^{\epsilon_1} z_1 : e^{\epsilon_2} z_2]$. In local coordinates $(x^{(\ell)}, y^{(\ell)})$ in the three coordinates patches $(z_\ell \neq 0)$ the action is $(e^{\epsilon_1^{(\ell)}} x^{(\ell)}, e^{\epsilon_2^{(\ell)}} y^{(\ell)})$ with weights

$$
\begin{array}{c|c|c}
\ell & \epsilon_1^{(\ell)} & \epsilon_2^{(\ell)} \\
\hline
0 & \epsilon_1 & \epsilon_2 \\
1 & \epsilon_2 - \epsilon_1 & -\epsilon_1 \\
2 & -\epsilon_2 & \epsilon_1 - \epsilon_2
\end{array}
\tag{3.1}
$$

ordered so that $\epsilon_1^{(\ell)} = -\epsilon_2^{(\ell+1)}$. The fixed points under the V-action are denoted by

$$
P_{(0)} = [1 : 0 : 0], \qquad P_{(1)} = [0 : 1 : 0], \qquad P_{(2)} = [0 : 0 : 1]. \tag{3.2}
$$

The generators of the global gauge transformation $(\mathbb{C}^*)^N$ are denoted by $\vec{a} = \{a_\alpha\}$, $\alpha = 1, \ldots, N$. The v.e.v. of the scalar field Φ is given by specifying (2.30) and (2.31) to $\mathbb{P}^2$. The equivariant extensions of the Fubini-Study two-form $\omega = i\partial\bar{\partial}\log(|z_0|^2 + |z_1|^2 + |z_2|^2)$ are

$$
\begin{aligned}
\omega^{(0)} &= \omega + \frac{\epsilon_1|z_0|^2 + (\epsilon_1 - \epsilon_2)|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \\
\omega^{(1)} &= \omega + \frac{\epsilon_2|z_0|^2 + (\epsilon_2 - \epsilon_1)|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \\
\omega^{(2)} &= \omega + \frac{-\epsilon_1|z_1|^2 - \epsilon_2|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}
\end{aligned}
\tag{3.3}
$$

and satisfy $(\iota_V - d)\omega^{(\ell)} = 0$. So that

$$
a_\alpha^{(\ell)} = a_\alpha + k_\alpha^{(\ell)}\epsilon_1^{(\ell)} + k_\alpha^{(\ell+1)}\epsilon_2^{(\ell)} \tag{3.4}
$$

---

[7]In local coordinates $x^{(0)} = z_1/z_0, y^{(0)} = z_2/z_0$ in the patch $z_0 \neq 0$ the vector has the following expression $V = i\epsilon_1(x^{(0)}\partial_{x^{(0)}} - \bar{x}^{(0)}\bar{\partial}_{\bar{x}^{(0)}}) + i\epsilon_2(y^{(0)}\partial_{y^{(0)}} - \bar{y}^{(0)}\bar{\partial}_{\bar{y}^{(0)}})$.

and, setting $k_\alpha^{(0)} \equiv k_\alpha^{(3)} = p_\alpha$, $k_\alpha^{(1)} = q_\alpha$ and $k_\alpha^{(2)} = r_\alpha$, we have explicitly, by (3.4) and (3.1)

$$
\begin{aligned}
\vec{a}^{(0)} &= \vec{a} + \vec{p}\epsilon_1 + \vec{q}\epsilon_2 \\
\vec{a}^{(1)} &= \vec{a} + \vec{q}(\epsilon_2 - \epsilon_1) + \vec{r}(-\epsilon_1) \\
\vec{a}^{(2)} &= \vec{a} + \vec{p}(\epsilon_1 - \epsilon_2) + \vec{r}(-\epsilon_2).
\end{aligned}
\tag{3.5}
$$

The fixed point data on $\mathbb{P}^2$ are described in terms of a collection of Young diagrams $\{\vec{Y}_\ell\}$, and of integer numbers $\{\vec{k}^{(\ell)}\}$ $\ell = 0, 1, 2$ describing respectively the $(\mathbb{C}^*)^{N+2}$-invariant point-like instantons in each patch and the magnetic fluxes of the gauge field, which correspond to the first Chern class c_1 as prescribed by (2.32).

The explicit expression at the three fixed points $P_{(\ell)}$ of the V-equivariant local and surface observables introduced in section 2.1 is given as follows. By calling for brevity

$$
\alpha = \omega + H, \qquad p = \alpha' \wedge \alpha'' + K
\tag{3.6}
$$

where H was defined in formula (2.17), we can write the most general equivariant extension α as

$$
\alpha = \omega + \frac{h|z_0|^2 + (h - \epsilon_1)|z_1|^2 + (h - \epsilon_2)|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2},
\tag{3.7}
$$

where ω is the Fubini-Study form of $\mathbb{P}^2$ and h a linear, coordinate independent, polynomial in the weights of the V-action. The evaluation at the fixed points of the observables α, p, with fugacities z, x is[8]

$$
\begin{aligned}
\imath^*_{P_{(0)}}(z\alpha + xp) &= zh + x\tilde{K} \\
\imath^*_{P_{(1)}}(z\alpha + xp) &= z(h - \epsilon_1) + x(\tilde{K} - \tilde{h}\epsilon_1 + \epsilon_1^2) \\
\imath^*_{P_{(2)}}(z\alpha + xp) &= z(h - \epsilon_2) + x(\tilde{K} - \tilde{h}\epsilon_2 + \epsilon_2^2).
\end{aligned}
\tag{3.8}
$$

The full U(2) partition function on $\mathbb{P}^2$ is given by

$$
Z_{\text{full}}^{\mathbb{P}^2}(\mathbf{q}, x, z, y; \epsilon_1, \epsilon_2) = \sum_{\{k_\alpha^{(\ell)}\}|\text{semi-stable}} \oint_\Delta da \prod_{\ell=0}^{2} Z_{\text{full}}^{\mathbb{C}^2}(\mathbf{q}^{(\ell)}; a^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) \, y^{c_1^{(\ell)}}
\tag{3.9}
$$

where $\mathbf{q} = \exp(2\pi i \tau)$ is the exponential of the gauge coupling and $\mathbf{q}^{(\ell)} = \mathbf{q} \, e^{\imath^*_{P_{(\ell)}}(\alpha z + px)}$ is the one shifted by the observable (3.8) evaluated at the fixed points $P_{(\ell)}$ of $\mathbb{P}^2$. Finally y is the source term corresponding to the Kähler form $t\omega$ with t the complexified Kälher parameter, so that $y = e^{2\pi t}$.

The integration in (3.9) realizes an isomorphism between the fixed points of the *un-framed* moduli space of equivariant rank two sheaves on $\mathbb{P}^2$ and copies of the fixed points of the *framed* moduli space on $\mathbb{P}^2$. Details of this isomorphism are presented in the explicit computation below and, in the case of odd c_1, reproduce exactly the results of [35].

---

[8]We defined $\tilde{h} = h' + h''$, $\tilde{K} = K + h'h''$ some new, coordinate independent, polynomial in ϵ_1, ϵ_2 of degree one and two respectively.

The stability conditions constraining the fixed point data $\{k_\alpha^{(\ell)}\}$'s are obtained by mapping these latter to the data describing unframed equivariant sheaves in terms of filtrations as in [47]. More details are provided in appendix A.

The factors appearing in (3.9) are the Nekrasov full partition functions

$$Z_{\text{full}}^{\mathbb{C}^2}(\mathbf{q}\,; a, \epsilon_1, \epsilon_2) = Z_{\text{class}}^{\mathbb{C}^2}(\mathbf{q}\,; a, \epsilon_1, \epsilon_2) Z_{\text{1-loop}}^{\mathbb{C}^2}(a, \epsilon_1, \epsilon_2) Z_{\text{inst}}^{\mathbb{C}^2}(\mathbf{q}\,; a, \epsilon_1, \epsilon_2) \tag{3.10}$$

whose explicit expressions we report below.

In the following we will compute the integral (3.9) with $x = z = 0$ (so $\mathbf{q}^{(\ell)} = \mathbf{q}$) and $y = 1$. The case with $x, z \neq 0, y \neq 1$ is a straightforward modification of the calculations below. In particular if one keeps $x, z \neq 0$ the result of the integration will give the generating function for equivariant Donaldson invariants for $\mathbb{P}^2$.

3.1 Classical action

The classical part of the partition function coming from (3.10) is given by evaluating (2.5) on the supersymmetric minima (2.29)

$$Z_{\text{class}}^{\mathbb{P}^2}(\mathbf{q}\,; \vec{a}, \epsilon_1, \epsilon_2) = \prod_{\ell=0}^{2} Z_{\text{class}}^{\mathbb{C}^2}(\mathbf{q}\,; \vec{a}^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) = \prod_{\ell=0}^{2} \exp\left[-\pi i \tau \frac{\sum_{\alpha=1}^{2} \left(a_\alpha^{(\ell)}\right)^2 - c\left(\sum_{\alpha=1}^{2} a_\alpha^{(\ell)}\right)^2}{\epsilon_1^{(\ell)} \epsilon_2^{(\ell)}} \right]. \tag{3.11}$$

Inserting the values of the equivariant weights (3.1) and (3.5) we obtain

$$Z_{\text{class}}^{\mathbb{P}^2}(\mathbf{q}\,; \vec{a}, \epsilon_1, \epsilon_2) = \exp\left[-\pi i \tau \left(\sum_{\alpha=1}^{2} (p_\alpha + q_\alpha + r_\alpha)^2 - c\left(\sum_{\alpha=1}^{2} p_\alpha + q_\alpha + r_\alpha \right)^2 \right) \right]. \tag{3.12}$$

Since $\mathbf{q} = \exp[2\pi i \tau]$ we have

$$Z_{\text{class}}^{\mathbb{P}^2}(\mathbf{q}\,; \vec{a}, \epsilon_1, \epsilon_2) = \mathbf{q}^{-\frac{1}{2}\left(\sum_{\alpha=1}^{2}(p_\alpha + q_\alpha + r_\alpha)^2 - c\left(\sum_{\alpha=1}^{2} p_\alpha + q_\alpha + r_\alpha\right)^2 \right)} = \mathbf{q}^{-\frac{1}{4}\left((1-2c)c_1^2 + (p+q+r)^2 \right)} \tag{3.13}$$

where we defined

$$p = p_1 - p_2, \quad q = q_1 - q_2, \quad r = r_1 - r_2, \tag{3.14}$$

and $c_1 = \sum_{(\ell)} c_1^{(\ell)}$ with $c_1^{(\ell)}$ defined in (2.32).

The sum in front of the full partition function can be rewritten as

$$\sum_{\{\vec{p}, \vec{q}, \vec{r}\} \in (\mathbb{Z}^2)^3} = \sum_{c_1 \in \mathbb{Z}} \sum_{\substack{\{p, q, r\} \in \mathbb{Z}^3 \\ p+q+r+c_1 = \text{even}}} \tag{3.15}$$

where we have performed a zeta function regularization of the sum over two integers, since the full partition function will depend only on p, q, r, c_1. Moreover is enough to consider only the cases $c_1 = \{0, 1\}$, because we are considering a rank two bundle, therefore the moduli spaces of two bundles with both $c_1 = 0$ (or 1) mod 2 are isomorphic after the twist by a line bundle.[9]

---

[9]The case $c_1 = 0$ or equivalently $c_1 = $ even hides some subtleties since the bundle can be reducible and the moduli space becomes singular [48]. We will in fact treat this case separately.

As discussed in section 2 the Hermitian-Yang-Mills equation implies semi-stability of the bundle. This in turn consists in some restrictions on the integers $\{k\}$ in the summation of (3.9) which will be discussed in subsections 3.5, 3.6 and in the appendix A.

3.2 One-loop contribution

The one-loop contribution in (3.9) is given by

$$Z_{\text{1-loop}}^{\mathbb{P}^2}(\vec{a}, \epsilon_1, \epsilon_2) = \prod_{\ell=0}^{2} Z_{\text{1-loop}}^{\mathbb{C}^2}(\vec{a}^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) = \prod_{\ell=0}^{2} \exp\left[-\sum_{\alpha\neq\beta} \gamma_{\epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}}(a_{\alpha\beta}^{(\ell)}) \right] \qquad (3.16)$$

where $a_{\alpha\beta} := a_\alpha - a_\beta$ and the double gamma-function is defined as

$$\gamma_{\epsilon_1, \epsilon_2}(x) = \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{-tx}}{(1 - e^{\epsilon_1 t})(1 - e^{\epsilon_2 t})}\, , \qquad (3.17)$$

with $\text{Re}(\epsilon_1)$ and $\text{Re}(\epsilon_2)$ positive. We have $a_{\alpha\beta} = \{a_{12}, a_{21}\} =: \{a, -a\}$ and similarly $p_{\alpha\beta} =: \{p, -p\}$ etc.[10] Inserting the values of the equivariant weights (3.1), (3.5) and using the definition of $\gamma_{\epsilon_1, \epsilon_2}$ (3.17) we can write

$$Z_{\text{1-loop}}^{\mathbb{P}^2} = \prod_{\pm} \exp\left[-\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} e^{-t(\pm a)} \frac{x^{\pm(q+r)} y^{\pm(p+r)}}{(1-x)(1-y)(x-y)} P_\pm(x, y) \right], \quad (3.18)$$

where we defined[11] $x := e^{\epsilon_1 t}$ and $y := e^{\epsilon_2 t}$, and $P_\pm(x, y)$ is a rational function in x and y

$$P_\pm(x, y) = x^{\mp N} y^{\mp N}(x - y) + x^{\mp N} y^2(1 - x) - x^2 y^{\mp N}(1 - y) \qquad (3.19)$$

with $N := p + q + r$ an integer with the same parity of c_1 (3.15). The values of $P_\pm(x, y)$ on $x = 1$, $y = 1$ and $x = y$ are zero, this means that in those points $P_\pm(x, y)$ has zeros which cancel the denominators $(1-x)^{-1}, (1-y)^{-1}, (x-y)^{-1}$ in (3.18). Making use of the identity

$$x^N - y^N = (x - y) \sum_{i=0}^{N-1} x^i y^{N-1-j} \qquad (3.20)$$

we arrive at the following expression for $P_\pm(x, y)$:

- $N \geq 0$.

$$P_+(x, y) = x^{-N} y^{-N}(1 - x)(1 - y)(x - y) \sum_{i=0}^{N} y^i \sum_{j=0}^{N-i} x^j,$$

$$P_-(x, y) = \begin{cases} (1 - x)(1 - y)(x - y) & N = 0 \\ 0 & N = 1, 2 \\ x^{N-1} y^{N-1}(1-x)(1-y)(x-y) \displaystyle\sum_{i=0}^{N-3} y^{-i} \sum_{j=0}^{N-3-i} x^{-j} & N > 2 \end{cases} \qquad (3.21)$$

---

[10]Note that this differs from the usual convention $a_{\alpha\beta} =: \{2a, -2a\}$.

[11]This choice of analytic continuation implies that $\gamma_{\epsilon_1 \epsilon_2}(x)$ has a branch cut for $x > 0$.

- $N < 0$.

$$P_+(x, y) = \begin{cases} 0 & N = -1, -2 \\ x^{|N|-1}y^{|N|-1}(1-x)(1-y)(x-y) \displaystyle\sum_{i=0}^{|N|-3} y^{-i} \sum_{j=0}^{|N|-3-i} x^{-j} & N < -2 \end{cases}$$

$$P_-(x, y) = x^{-|N|}y^{-|N|}(1-x)(1-y)(x-y) \sum_{i=0}^{|N|} y^i \sum_{j=0}^{|N|-i} x^j.$$

$$(3.22)$$

Inserting this result back in (3.18) and using the definition of the Gamma function:

$$\Gamma(s) = \int_0^\infty dt\, t^{s-1}e^{-t} \tag{3.23}$$

we obtain for $Z_{\text{1-loop}}^{\mathbb{P}^2}$ of (3.16) the following results

- $N = 0$

$$Z_{\text{1-loop}}^{\mathbb{P}^2} = -\left(a + p\epsilon_1 + q\epsilon_2\right)^2 \tag{3.24}$$

- $N > 0$

$$Z_{\text{1-loop}}^{\mathbb{P}^2} = \prod_{i=0}^{N} \prod_{j=0}^{N-i} \left(a + (p-j)\epsilon_1 + (q-i)\epsilon_2\right) \cdot$$
$$\prod_{i=0}^{N-3} \prod_{j=0}^{N-3-i^\diamond} -\left(a + (p-1-j)\epsilon_1 + (q-1-i)\epsilon_2\right)$$

$$(3.25)$$

- $N < 0$

$$Z_{\text{1-loop}}^{\mathbb{P}^2} = \prod_{i=0}^{|N|} \prod_{j=0}^{|N|-i} -\left(a + (p+j)\epsilon_1 + (q+i)\epsilon_2\right) \cdot$$
$$\prod_{i=0}^{|N|-3} \prod_{j=0}^{|N|-3-i^\diamond} \left(a + (p+1+j)\epsilon_1 + (q+1+i)\epsilon_2\right)$$

$$(3.26)$$

where the symbols $\diamond$ over the products in the second lines of formulas (3.25), (3.26) mean that those products are equal to 1 if $|N| < 3$. The only relevant case is actually that with $p, q, r \in \mathbb{Z}_{\geq 0}$. This can be seen by a direct computation which shows that the final result does depend on the absolute values of p, q, r only. Therefore from now on we assume $N \geq 0$.

3.3 Instanton contribution

The instanton contribution in (3.9) is given by

$$\prod_{\ell=0}^{2} Z_{\text{inst}}^{\mathbb{C}^2}(\mathbf{q}\,;\vec{a}^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) \tag{3.27}$$

where $Z_{\text{inst}}^{\mathbb{C}^2}$ is the Nekrasov partition function defined as follows. Let $Y = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ be a Young diagram, and $Y' = \{\lambda_1' \geq \lambda_2' \geq \dots\}$ its transposed. λ_i is the length of the i-column and λ_j' the length of the j-row of Y. For a given box $s = \{i, j\}$ we define respectively the arm and leg length functions

$$A_Y(s) = \lambda_i - j, \qquad L_Y(s) = \lambda_j' - i. \tag{3.28}$$

Note that these quantities can also be negative when s does not belong to the diagram Y. The fixed points data for each patch are given by a collection of Young diagrams $\vec{Y}^{(\ell)} = \{Y_\alpha^{(\ell)}\}$, and the instanton contribution is [12, 13, 49]

$$Z_{\text{inst}}^{\mathbb{C}^2}(\mathbf{q}\,; \vec{a}, \epsilon_1, \epsilon_2) = \sum_{\{Y_\alpha\}} \mathbf{q}^{|\vec{Y}|} z_{\text{vec}}(\vec{a}, \vec{Y}, \epsilon_1, \epsilon_2) \tag{3.29}$$

where $\mathbf{q} = \exp(2i\pi\tau)$ and

$$\begin{aligned}
z_{\text{vec}}(\vec{a}, \vec{Y}, \epsilon_1, \epsilon_2) = \prod_{\alpha,\beta=1}^{N} \prod_{s \in Y_\alpha} &\big(a_{\beta\alpha} - L_{Y_\beta}(s)\epsilon_1 + (A_{Y_\alpha}(s) + 1)\epsilon_2\big)^{-1} \\
\times &\big(a_{\alpha\beta} + (L_{Y_\beta}(s) + 1)\epsilon_1 - A_{Y_\alpha}(s)\epsilon_2\big)^{-1}.
\end{aligned} \tag{3.30}$$

3.4 Analytic structure of the integrand

In order to integrate the full partition function (3.9) along a we need to study the analytic structure of the integrand.

The instanton partition function (3.29) has simple poles at

$$a \equiv a_{12} = m\epsilon_1 + n\epsilon_2, \quad m, n \in \mathbb{Z}, \quad m \cdot n > 0. \tag{3.31}$$

This behavior can be displayed explicitly by the Zamolodchikov's recursion relation [50] which was analyzed for gauge theories in [51]. In the evaluation of the integral it will be very useful to write it as

$$Z_{\text{inst}}\big(\mathbf{q}; a, \epsilon_1, \epsilon_2\big) = 1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R_{m,n} Z_{\text{inst}}\left(\mathbf{q}; m\epsilon_1 - n\epsilon_2, \epsilon_1, \epsilon_2\right)}{\big(a - m\epsilon_1 - n\epsilon_2\big)\big(a + m\epsilon_1 + n\epsilon_2\big)} \tag{3.32}$$

where

$$R_{m,n} = 2 \underbrace{\prod_{i=-m+1}^{m} \prod_{j=-n+1}^{n}}_{(i,j)\neq\{(0,0),(m,n)\}} \frac{1}{\big(i\epsilon_1 + j\epsilon_2\big)}. \tag{3.33}$$

Therefore the product of the three instanton partition functions coming from the three patches

$$Z_{\text{inst}}\big(\mathbf{q}; a^{(0)}, \epsilon_1, \epsilon_2\big) Z_{\text{inst}}\big(\mathbf{q}; a^{(1)}, -\epsilon_2, \epsilon_1 - \epsilon_2\big) Z_{\text{inst}}\big(\mathbf{q}; a^{(2)}, \epsilon_2 - \epsilon_1, -\epsilon_1\big) \tag{3.34}$$

displays a polar structure as depicted in figure 2. The lattice[12] $(x, y) = (i\epsilon_1, j\epsilon_2)$ $i, j \in \mathbb{Z}$
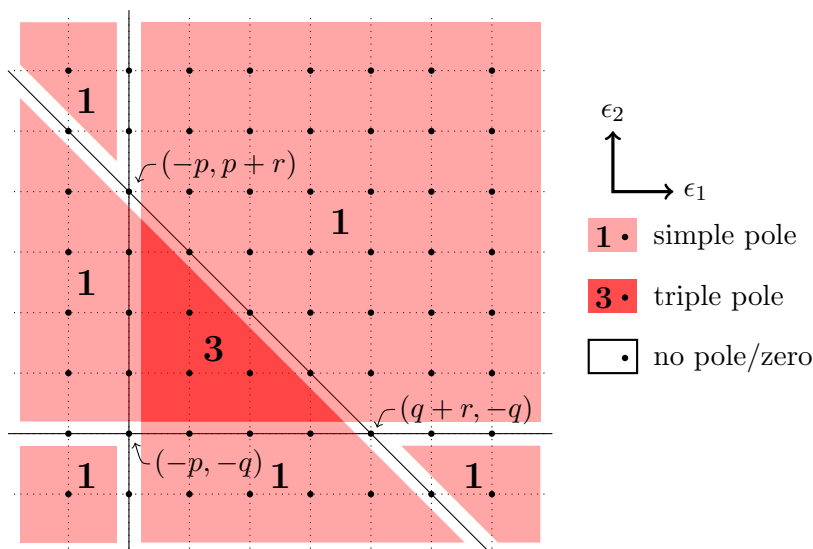
Figure 2. Poles of instanton partition function.

is separated in seven regions by three straight lines

$$x = -p, \quad y = -q, \quad y = -x + r. \tag{3.35}$$

In the interior of the triangle $T_I = \{(-p, -q), (q+r, -q), (-p, p+r)\}$ formed by these three lines there are triple poles. Along the three lines there are simple poles *only* in the segment strictly contained between two vertices of the triangle. In all the other points of the lattice there are simple poles.

In the analysis of the one-loop contribution one can see[13] that the only relevant case is $N > 0$. Looking at (3.25) one can see that this contributes with double zeros in the interior of the triangle T_I (which cancel the multiplicity of the poles of the instanton part) and simple zeros along the perimeter of T_I (which cancel the simple poles of the instanton part on the edges of the triangle).[14] The positions of the zeroes of the one-loop part is described in figure 3. The overall polar structure of the full partition function is drawn in figure 4: there are simple poles in all the points of the lattice that are not along the three straight lines (3.35). This implies that the integration of Z_{full} will be given by the sum of the residues of simple poles inside the contour of integration $\Delta = \partial C$ given in (3.9)

$$\oint_{\partial C} Z_{\text{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2) da \propto \sum_{(i,j) \in C} \text{Res}\big(Z_{\text{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2) \big| a = i\epsilon_1 + j\epsilon_2 \big)$$
$$= \sum_{(i,j) \in C} \lim_{a \to i\epsilon_1 + j\epsilon_2} (a - i\epsilon_1 - j\epsilon_2) Z_{\text{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2), \tag{3.36}$$

and from the discussion in section 2.2.2 the only residue to evaluate is the one relative to the pole at the origin.

---

[12]We consider ϵ_1, ϵ_2 to be incommensurable.

[13]Indeed in the case $N = 0$ the integrand in (3.9) does not display any pole at the origin.

[14]Of course if $N < 3$ there is none interior of the triangle, so only simple poles.
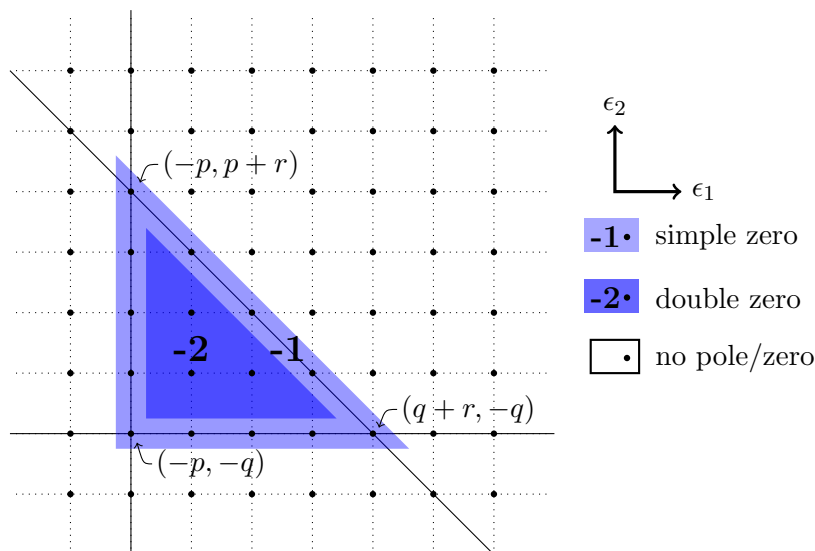
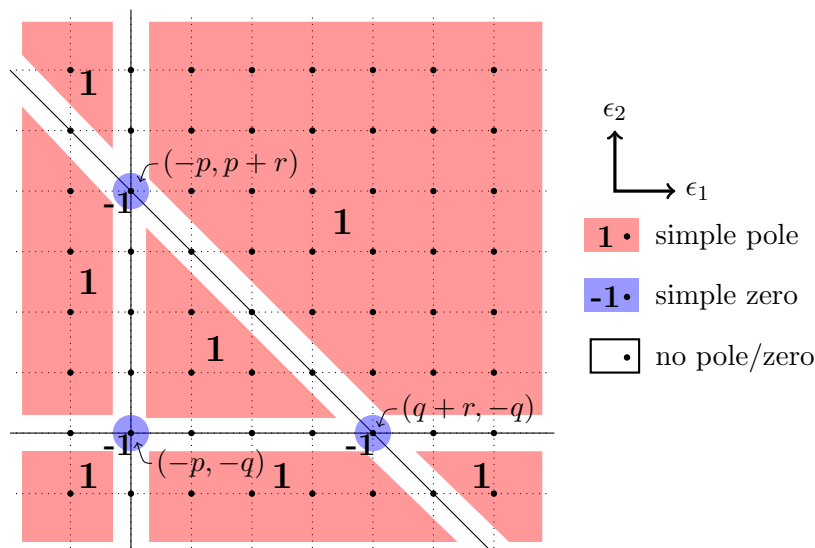**Figure 3**. Poles of one-loop partition function.

Figure 4. Poles of the full partition function.

3.5 Exact results for odd c_1

Now we can perform the integration by residues evaluation as anticipated in (3.36). We are focusing on the case with $c_1 = 1$, the other case $c_1 = 0$ is more subtle and will be studied in a separate section.

From the analysis of the previous section we know that the full partition function has a pole at the origin only if the integers $p = p_{12}$, $q = q_{12}$, $r = r_{12}$ are strictly positive. Moreover we have to impose the stability conditions, which are discussed in the appendix, see (A.13). These, together with $p + q + r + c_1 = even$ imply that the integers p, q, r have

to satisfy strict triangle inequalities, namely

$$p + q > r > 0, \quad p + r > q > 0, \quad q + r > p > 0. \tag{3.37}$$

Using the expressions for the classical (3.13), one-loop (3.25) and instanton (3.32) partition functions, we can put all together (details are given in section 3.5.1) obtaining as the final result of the integration

$$
\begin{aligned}
Z_{\mathcal{N}=2}^{\mathbb{P}^2}(\mathbf{q}\,;\epsilon_1,\epsilon_2)\big|_{c_1=1} = \\
= \mathbf{q}^{-\frac{1}{4}(1-2c)} \sum_{\{p,q,r\}} \mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \prod_{\{(i,j)\}} \frac{1}{i\epsilon_1 + j\epsilon_2} \\
Z_{\mathrm{inst}}\big(\mathbf{q}; a_{\mathrm{res}}^{(0)}, \epsilon_1, \epsilon_2\big) Z_{\mathrm{inst}}\big(\mathbf{q}; a_{\mathrm{res}}^{(1)}, \epsilon_2 - \epsilon_1, -\epsilon_1\big) Z_{\mathrm{inst}}\big(\mathbf{q}; a_{\mathrm{res}}^{(2)}, -\epsilon_2, \epsilon_1 - \epsilon_2\big)
\end{aligned}
\tag{3.38}
$$

where

- the sum is over positive integers p, q, r satisfying the triangle inequality (3.37) and also $p + q + r = odd$,

- the product is over the points of the lattice $(i,j) \in (D^{(p,q,r)} \cap \mathbb{Z}^2) \setminus (0,0)$; where the regions $D^{(p,q,r)}$ are the intersections of two triangles T_1 and T_2, one of side $p + q + r$ and the other of side $p + q + r - 3$:

$$
\begin{aligned}
T_1 &= \{(-p, -q), (q+r, -q), (-p, p+r)\}, \\
T_2 &= \{(p-1, q-1), (-q-r+2, q-1), (p-1, -p-r+2)\}.
\end{aligned}
\tag{3.39}
$$

T_1 is delimited by the three straight lines

$$x = -p, \quad y = -q, \quad y = -x + r. \tag{3.40}$$

T_2 is delimited by the three straight lines

$$x = p - 1, \quad y = q - 1, \quad y = -x - r + 1. \tag{3.41}$$

- we used the following notation

$$
\begin{aligned}
a_{\mathrm{res}}^{(0)} &= p\epsilon_1 - q\epsilon_2, \\
a_{\mathrm{res}}^{(1)} &= q(\epsilon_2 - \epsilon_1) - r(-\epsilon_1), \\
a_{\mathrm{res}}^{(2)} &= r(-\epsilon_2) - p(\epsilon_1 - \epsilon_2).
\end{aligned}
\tag{3.42}
$$

We can compare the expression (3.38) with theorem 6.15 in [35]. Indeed, (3.38) coincide with the formula in [35] with x, z set to zero. Indeed the region $D^{(p,q,r)}$ defined above coincides with the one in Lemma 6.12 of [35].

To reproduce the full generating function of equivariant Donaldson invariant in [35] one should repeat the computation and the integration of $Z_{\mathrm{full}}^{\mathbb{P}^2}$ with $x, z \neq 0$ in (3.9). This implies a light modification in the calculations, namely one should replace $\mathbf{q}$ with $\mathbf{q}^{(\ell)}$ in every copy of $Z_{\mathrm{full}}^{\mathbb{C}^2}$, with $\mathbf{q}^{(\ell)}$ defined below (3.9). Moreover we need to expand in the

discriminant of the bundle (see (A.9) in appendix A), that is choosing $c = \frac{1}{2}$ in (2.5). The result in this case is

$$Z^{\mathbb{P}^2}_{\mathcal{N}=2}(\mathbf{q}, x, z, \epsilon_1, \epsilon_2)\big|_{c_1=1} =$$

$$= \sum_{\{p,q,r\}} \mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \exp\left(-\frac{1}{4}\sum_{\ell=0}^{2} \frac{(a^{(\ell)}_{\mathrm{res}})^2\, \imath^*_{P_{(\ell)}}(\alpha z + px)}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\right) \prod_{\{(i,j)\}} \frac{1}{i\epsilon_1 + j\epsilon_2}$$

$$Z_{\mathrm{inst}}\big(\mathbf{q}^{(0)}; a^{(0)}_{\mathrm{res}}, \epsilon_1, \epsilon_2\big) Z_{\mathrm{inst}}\big(\mathbf{q}^{(1)}; a^{(1)}_{\mathrm{res}}, \epsilon_2-\epsilon_1, -\epsilon_1\big) Z_{\mathrm{inst}}\big(\mathbf{q}^{(2)}; a^{(2)}_{\mathrm{res}}, -\epsilon_2, \epsilon_1-\epsilon_2\big)$$

(3.43)

where sum and product are the same of (3.38). Since $q = \Lambda^4$, formula (3.43) matches completely with the theorem 6.15 of [35].[15]

3.5.1 Proof of (3.38)

We evaluate the residue of Z_{full} at $a = 0$, namely

$$a^{(0)} = p\epsilon_1 + q\epsilon_2$$
$$a^{(1)} = q(\epsilon_2 - \epsilon_1) + r(-\epsilon_1)$$
$$a^{(2)} = p(\epsilon_1 - \epsilon_2) + r(-\epsilon_2).$$

(3.44)

We know from section 3.4 that p, q, r are strictly positive. Therefore we see from (3.31) and (3.34) that the three instanton partition functions have a simple pole each, which identifies the region with triple poles in figure 2. Moreover

$$p, q, r \geq 1 \quad \Rightarrow \quad N = p + q + r \geq 3$$

(3.45)

so we get a double zero from the one-loop part. Using (3.32) the instanton part is

$$Z^{\mathbb{P}^2}_{\mathrm{inst}} = \left(1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R^{(0)}_{m,n} Z_{\mathrm{inst}}\left(\mathbf{q}; m\epsilon_1 - n\epsilon_2, \epsilon_1, \epsilon_2\right)}{\left(a^{(0)} - m\epsilon_1 - n\epsilon_2\right)\left(a^{(0)} + m\epsilon_1 + n\epsilon_2\right)}\right)$$

$$\cdot \left(1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R^{(1)}_{m,n} Z_{\mathrm{inst}}\left(\mathbf{q}; m(\epsilon_2 - \epsilon_1) - n(-\epsilon_1), \epsilon_2 - \epsilon_1, -\epsilon_1\right)}{\left(a^{(1)} - m(\epsilon_2 - \epsilon_1) - n(-\epsilon_1)\right)\left(a^{(1)} + m(\epsilon_2 - \epsilon_1) + n(-\epsilon_1)\right)}\right)$$

(3.46)

$$\cdot \left(1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R^{(2)}_{m,n} Z_{\mathrm{inst}}\left(\mathbf{q}; m(-\epsilon_2) - n(\epsilon_1 - \epsilon_2), -\epsilon_2, \epsilon_1 - \epsilon_2\right)}{\left(a^{(2)} - m(-\epsilon_2) - n(\epsilon_1 - \epsilon_2)\right)\left(a^{(2)} + m(-\epsilon_2) + n(\epsilon_1 - \epsilon_2)\right)}\right)$$

where similarly to (3.33)

$$R^{(\ell)}_{m,n} = 2 \underbrace{\prod_{i=-m+1}^{m} \prod_{j=-n+1}^{n}}_{(i,j)\neq\{(0,0),(m,n)\}} \frac{1}{\left(i\epsilon_1^{(\ell)} + j\epsilon_2^{(\ell)}\right)}.$$
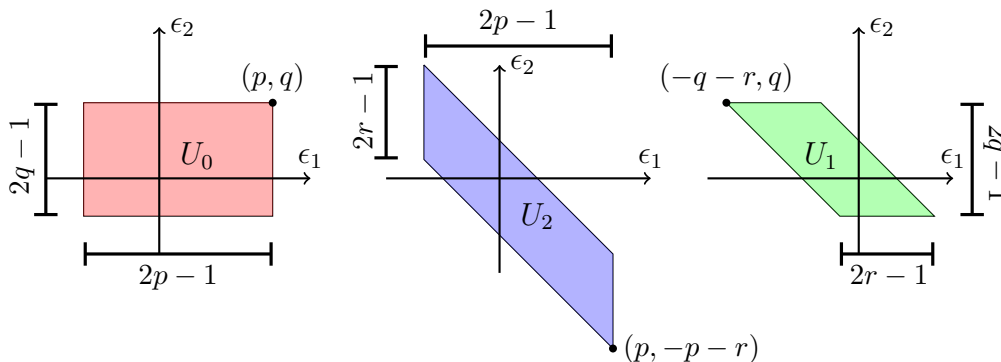
(3.47)

Figure 5. Regions U_ℓ.

The triple pole is obtained by picking respectively from the three sums the terms $(m = p, n = q)$, $(m = q, n = r)$, $(m = r, n = p)$ giving

$$Z_{\text{inst}}^{\mathbb{P}^2} = -\frac{1}{a^3}\, \mathbf{q}^{pq+pr+qr}\, \tilde{R}_{p,q}^{(0)}\, \tilde{R}_{q,r}^{(1)}\, \tilde{R}_{r,p}^{(2)}\, Z_{\text{Res}} + O\left(\frac{1}{a^2}\right) \tag{3.48}$$

where

$$\tilde{R}_{m,n}^{(\ell)} = \frac{1}{a^{(\ell)} + m\epsilon_1^{(\ell)} + n\epsilon_2^{(\ell)}} R_{m,n}^{(\ell)} \tag{3.49}$$

and we defined

$$\begin{aligned} Z_{\text{Res}} =\,& Z_{\text{inst}}\big(\mathbf{q}; p\epsilon_1 - q\epsilon_2, \epsilon_1, \epsilon_2\big) Z_{\text{inst}}\big(\mathbf{q}; q(\epsilon_2 - \epsilon_1) - r(-\epsilon_1), \epsilon_2 - \epsilon_1, -\epsilon_1\big) \\ & Z_{\text{inst}}\big(\mathbf{q}; r(-\epsilon_2) - p(\epsilon_1 - \epsilon_2), -\epsilon_2, \epsilon_1 - \epsilon_2\big). \end{aligned} \tag{3.50}$$

Note that Z_{Res} is equal to the last line of (3.38).

When calculated at the point $a = 0$ the three factors $\tilde{R}^{(\ell)}$ can be rewritten as

$$\tilde{R}^{(\ell)} = \prod_{(i,j) \in U_\ell \backslash (0,0)} \frac{1}{\big(i\epsilon_1 + j\epsilon_2\big)}, \tag{3.51}$$

where the three regions U_ℓ are depicted in figure 5 and are defined as:

- U_0 is a rectangle $2p - 1 \times 2q - 1$ delimited by the four straight lines

$$x = -p + 1, \quad x = p, \quad y = -q + 1, \quad y = q. \tag{3.52}$$

- U_1 is a parallelogram delimited by the four straight lines

$$y = -q + 1, \quad y = q, \quad y = -x - r, \quad y = -x + r - 1. \tag{3.53}$$

- U_2 is a parallelogram delimited by the four straight lines

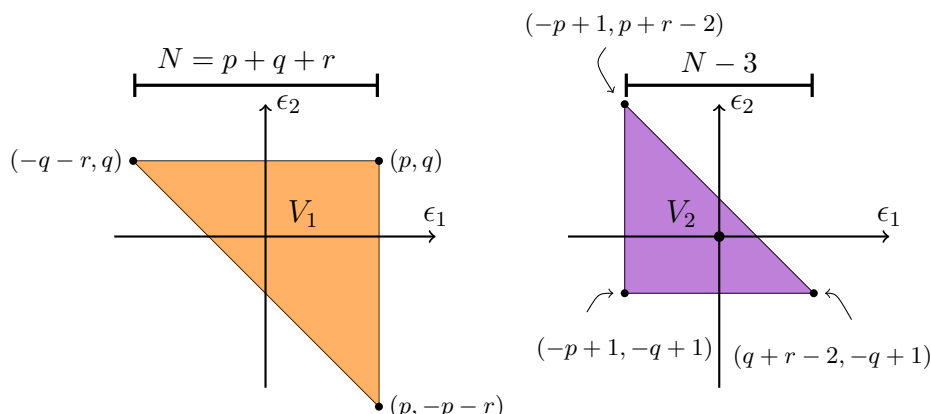$$x = -p + 1, \quad x = p, \quad y = -x - r, \quad y = -x + r - 1. \tag{3.54}$$

Figure 6. Regions V_1, V_2.

Since $N \geq 3$ (3.45), from (3.25) we get for the one-loop part

$$Z_{\text{1-loop}}^{\mathbb{P}^2} = \prod_{i=0}^{N} \prod_{j=0}^{N-i} \big(a + (p-j)\epsilon_1 + (q-i)\epsilon_2\big) \prod_{i=0}^{N-3} \prod_{j=0}^{N-3-i} -\big(a + (p-1-j)\epsilon_1 + (q-1-i)\epsilon_2\big). \quad (3.55)$$

The double zero in $a = 0$ is hidden in the products

$$Z_{\text{1-loop}}^{\mathbb{P}^2} = -a^2 \underbrace{\prod_{i=0}^{N} \prod_{j=0}^{N-i}}_{(i,j) \neq (q,p)} \big(a + (p-j)\epsilon_1 + (q-i)\epsilon_2\big) \underbrace{\prod_{i=0}^{N-3} \prod_{j=0}^{N-3-i}}_{(i,j) \neq (q-1,p-1)} -\big(a + (p-1-j)\epsilon_1 + (q-1-i)\epsilon_2\big).$$

$$(3.56)$$

When evaluated in $a = 0$ the two products in (3.56) can be rewritten as

$$\prod_{(i,j) \in V_1 \backslash (0,0)} \big(i\epsilon_1 + j\epsilon_2\big) \prod_{(i,j) \in V_2 \backslash (0,0)} \big(i\epsilon_1 + j\epsilon_2\big) \quad (3.57)$$

where V_1, V_2 are two triangles depicted in figure 6 and defined as:

- V_1 is the triangle with vertices $\{(p,q), (-q-r,q), (p,-p-r)\}$. It is delimited by the three straight lines

$$x = p, \quad y = q, \quad y = -x - r. \quad (3.58)$$

- V_2 is the triangle with vertices $\{(-p+1,-q+1), (q+r-2,-q+1), (-p+1,p+r-2)\}$. It is delimited by the three straight lines

$$x = -p+1, \quad y = -q+1, \quad y = -x + r - 1. \quad (3.59)$$

The residue evaluation is therefore

$$\text{Res}\big(Z_{\text{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2)\big| a = 0\big) = \lim_{a \to 0} a\, Z_{\text{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2) \quad (3.60)$$

$$= \mathbf{q}^{-\frac{1}{4}(1-2c)} \mathbf{q}^{-\frac{1}{4}(p+q+r)^2} \prod_{(i,j) \in V_1 \backslash (0,0)} \big(i\epsilon_1 + j\epsilon_2\big) \prod_{(i,j) \in V_2 \backslash (0,0)} \big(i\epsilon_1 + j\epsilon_2\big)$$

$$\cdot \mathbf{q}^{pq+pr+qr} \prod_{(i,j) \in U_0 \backslash (0,0)} \frac{1}{\big(i\epsilon_1 + j\epsilon_2\big)} \prod_{(i,j) \in U_1 \backslash (0,0)} \frac{1}{\big(i\epsilon_1 + j\epsilon_2\big)} \prod_{(i,j) \in U_2 \backslash (0,0)} \frac{1}{\big(i\epsilon_1 + j\epsilon_2\big)} Z_{\text{Res}}(\mathbf{q}).$$

Comment: it is simple to verify that the number of points different from $(0,0)$ in the regions $U_\ell \cap \mathbb{Z}^2$ and $V_{1,2} \cap \mathbb{Z}^2$ sum together to an *even* number. This means that the total product over these regions in (3.60) is invariant under the reflection $(i,j) \to (-i,-j)$.

The final result (3.38) is recovered by imposing the stability conditions (3.37) on (3.60). The detailed derivation of these conditions is performed in appendix A. Due to the strict triangle inequality we have

$$U_0 \cap U_1 \cap U_2 = U_0 \cap U_1 = U_0 \cap U_2 = U_1 \cap U_2 = V_1 \cap V_2; \qquad (3.61)$$

and

$$(U_0 \cup U_1 \cup U_2) \cap \mathbb{Z}^2 = (V_1 \cup V_2) \cap \mathbb{Z}^2. \qquad (3.62)$$

This means that (3.60) reduces to

$$
\begin{aligned}
&\mathrm{Res}\big(Z_{\text{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2)\big|a = 0\big) \\
&= \mathbf{q}^{-\frac{1}{4}(1-2c)-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \prod_{(i,j)\in[(V_1\cap V_2)\cap\mathbb{Z}^2]\backslash(0,0)} \frac{1}{(i\epsilon_1 + j\epsilon_2)} Z_{\text{Res}}(\mathbf{q}) \qquad (3.63)
\end{aligned}
$$

Moreover we see from (3.39), (3.40), (3.41) and (3.58), (3.59) that $V_1 = \overline{T}_1$, $V_2 = \overline{T}_2$ where the bar indicates the reflection of the two axis highlighted above. Therefore the intersection $V_1 \cap V_2$ is precisely the region $D^{(p,q,r)}$ mirrored through the origin, and from the above comment this means that (3.63) is equal to (3.38) once summed over all the (proper) integers p, q, r.

Finally we show (3.61) (3.62). Eq.(3.61) comes directly from the construction of the five regions. Indeed each U_i shares a couple of "delimitation" parallel straight lines with another U_j and the other parallel couple with the remaining U_k. Moreover each U_i shares a couple of consecutive non-parallel lines with one V_i and the other couple with the other V_j. See figure 7. In formulae, we define the region $\langle r_i, r_j, r_k \ldots \rangle$ as the convex hull of the intersection points of all the straight lines $r_i, r_j, r_k \ldots$ and call

$$
\begin{aligned}
r_1 &= \{x = -p+1\}, & r_2 &= \{x = p\}, \\
r_3 &= \{y = -q+1\}, & r_4 &= \{y = q\}, \\
r_5 &= \{y = -x+r-1\}, & r_6 &= \{y = -x-r\}.
\end{aligned} \qquad (3.64)
$$

Then we have

$$
\begin{aligned}
U_0 &= \langle r_1, r_2, r_3, r_4 \rangle, \quad U_1 = \langle r_3, r_4, r_5, r_6 \rangle, \quad U_2 = \langle r_1, r_2, r_5, r_6 \rangle, \\
V_1 &= \langle r_2, r_4, r_6 \rangle, \qquad V_2 = \langle r_1, r_3, r_5 \rangle,
\end{aligned} \qquad (3.65)
$$

from which (3.61) directly follows.

We will now show that (3.62) is equivalent to the triangle inequality. Indeed in general $(V_1 \cup V_2) \cap \mathbb{Z}^2$ can exceed $(U_0 \cup U_1 \cup U_2) \cap \mathbb{Z}^2$, (causing the appearance of terms $(i\epsilon_1 + j\epsilon_2)^{+1}$ in (3.63)). This does not happen if the following three conditions are satisfied:

1. the segment between the vertex $(p, -q+1)$ of U_0 and the vertex $(p, r-p-1)$ of U_2 has distance strictly less than 2 (so that it cannot contain points of the lattice), so

$$-q + 1 - (r-p-1) < 2 \iff -q-r+p+2 < 2 \iff q+r > p; \qquad (3.66)$$
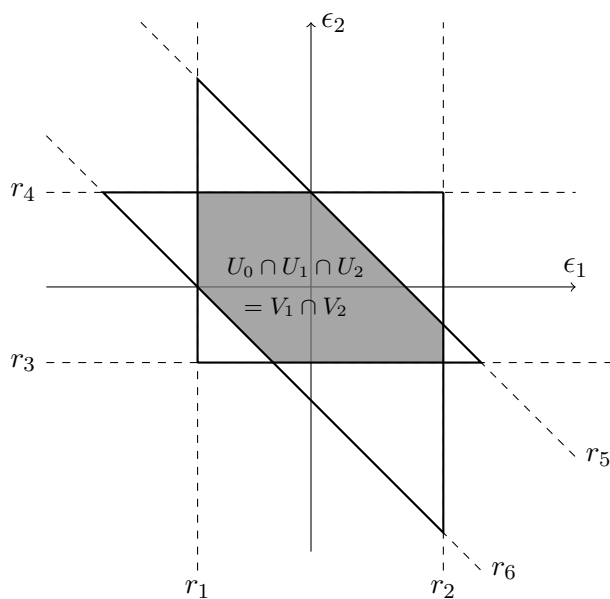
see figure 8.

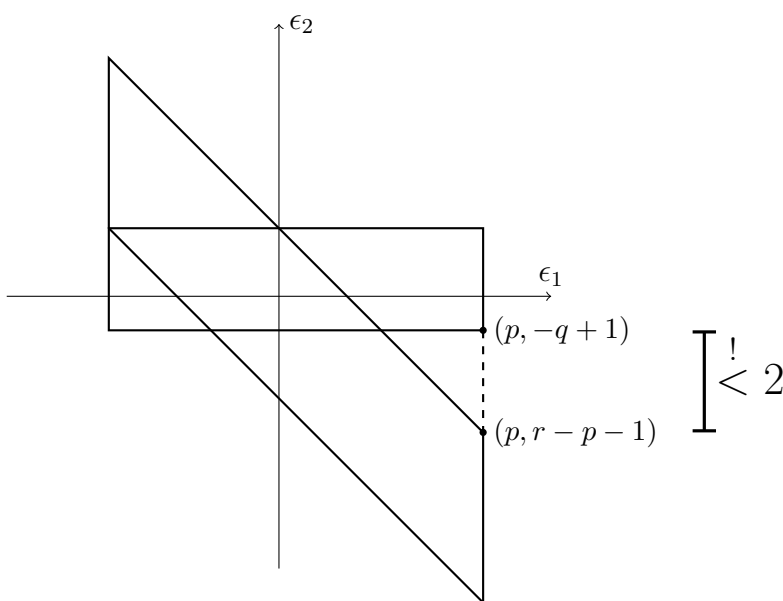**Figure 7**. Intersections of the regions U_ℓ, V_1, V_2.

Figure 8. The union $V_1 \cup V_2$ exceed the union $U_0 \cup U_1 \cup U_2$ iff the strict triangle inequality is not satisfied.

2. the distance between the vertex $(-p+1, q)$ of U_0 and the vertex $(r-q-1, q)$ of U_1 must be strictly less than 2

$$-p + 1 - (r - q - 1) < 2 \iff -p - r + q + 2 < 2 \iff p + r > q; \qquad (3.67)$$

3. the distance between the vertex $(-p+1, -r+p-1)$ of U_2 and the vertex $(-r+q-1, -q+1)$ of U_1 must be strictly less than $2\sqrt{2}$

$$-p+1-(-r+q-1) < 2 \iff -p-q+r+2 < 2 \iff p+q > r. \quad (3.68)$$

3.6 Exact results for even c_1

The case with even first Chern class is subtle because it allows for reducible connections. Namely the bundle can be written as a direct sum of line bundles, and the presence of this kind of connections makes the moduli space singular ([48] section 4.2).

Indeed one can saturate one of the three inequalities, and so define a strict *semi*-stable bundle, only if the sum of the three integers p, q, r is even

$$p+q \geq r, \qquad p+r \geq q, \qquad q+r \geq p, \quad (3.69)$$

e.g. $p+q=r$. From the discussion about the supersymmetric fixed point locus of section 2 we know that we should consider also this kind of configurations in the construction of the partition function.

Technically nothing changes in the calculation since we already noticed that the full partition function $Z_{\text{full}}^{\mathbb{P}^2}$ has a pole at the origin only if $p, q, r > 0$. We have only to add the contribution saturating (3.69). These kind of configurations have non trivial automorphism group, that is the action of a $\mathbb{Z}_2$-group.[16] Therefore in counting gauge invariant configurations one has to divide by the order of the automorphism group, namely $\sharp\mathbb{Z}_2 = 2$. This appears as a coefficient $1/2$ on the sum over strictly semi-stable configurations in the final result. Henceforth the gauge theoretical conjecture for the generating function of equivariant Donaldson invariants reads,[17]

$$Z_{\mathcal{N}=2}^{\mathbb{P}^2}(\mathbf{q}, x, z, \epsilon_1, \epsilon_2)\big|_{c_1=0} = \left(\sum_{\substack{\{p,q,r\} \\ \text{strictly stable}}} + \frac{1}{2} \sum_{\substack{\{p,q,r\} \\ \text{strictly semi-stable}}} \right) \mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)}$$

$$\exp\left(-\frac{1}{4} \sum_{\ell=0}^{2} \frac{(a_{\text{res}}^{(\ell)})^2 \, i_{P_{(\ell)}}^*(\alpha z + px)}{\epsilon_1^{(\ell)} \epsilon_2^{(\ell)}} \right) \prod_{(i,j) \in V_1 \backslash (0,0)} \left(i\epsilon_1 + j\epsilon_2 \right) \prod_{(i,j) \in V_2 \backslash (0,0)} \left(i\epsilon_1 + j\epsilon_2 \right)$$

$$\prod_{(i,j) \in U_0 \backslash (0,0)} \left(i\epsilon_1 + j\epsilon_2 \right)^{-1} \prod_{(i,j) \in U_1 \backslash (0,0)} \left(i\epsilon_1 + j\epsilon_2 \right)^{-1} \prod_{(i,j) \in U_2 \backslash (0,0)} \left(i\epsilon_1 + j\epsilon_2 \right)^{-1}$$

$$Z_{\text{inst}}\left(\mathbf{q}^{(0)}; a_{\text{res}}^{(0)}, \epsilon_1, \epsilon_2\right) Z_{\text{inst}}\left(\mathbf{q}^{(1)}; a_{\text{res}}^{(1)}, \epsilon_2 - \epsilon_1, -\epsilon_1\right) Z_{\text{inst}}\left(\mathbf{q}^{(2)}; a_{\text{res}}^{(2)}, -\epsilon_2, \epsilon_1 - \epsilon_2\right)$$
$$(3.70)$$

where $p+q+r = even$, $a_{\text{res}}^{(\ell)}$ are defined in (3.42), $(i,j) \in \mathbb{Z}^2$ and the regions U, V are defined in (3.52)–(3.53) and (3.58), (3.59). As (3.43), expression (3.70) is obtained taking $c = \frac{1}{2}$ in (2.5). For the stricly stable configurations the products in (3.70) can be rewritten

---

[16] A reducible U(2)-bundle splits in the sum of two line bundles as $E = L_1 \oplus L_2$. There is a $\mathbb{Z}_2$ gauge symmetry exchanging the two line bundles as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} L_2 & 0 \\ 0 & L_1 \end{pmatrix}$.

[17] To obtain the partition function on $\mathbb{P}^2$ is enough to put to zero x and z in (3.70) so that also $\mathbf{q}^{(\ell)} \to \mathbf{q}$.

as the product over the regions $D^{(p,q,r)}$ described below (3.43), but this is no more true for the strictly semi-stable ones (see the discussion at the end of subsection 3.5.1).

The result (3.70) provides a conjecture for equivariant SU(2) Donaldson invariants. These are not known in the mathematical literature. In the next section we show that in the limit $\epsilon_1, \epsilon_2 \to 0$ the formula (3.70) reproduces the SU(2) Donaldson invariants for $\mathbb{P}^2$.

Let us underline that imposing the stability condition is crucial in order to get a finite $\epsilon_1, \epsilon_2 \to 0$ limit for the gauge theory partition function. Indeed we checked that removing the stability condition from (3.43) and (3.70) would produce partition functions which are diverging in that limit.

3.7 Non equivariant limit

In this section we will compare our results in the limit $\epsilon_1, \epsilon_2 \to 0$ with Donaldson invariants.

We start with the example of formula (3.43), that is known [35] to be the generating function of equivariant Donaldson invariants in the case of U(2)-bundle with $c_1 = 1$. This bundle can be reduced to a projective unitary group bundle PU(2) = SU(2)/$\mathbb{Z}_2$ = SO(3). Therefore, in the limit $\epsilon_1, \epsilon_2 \to 0$ (3.43) should produce SO(3)-Donaldson invariants on $\mathbb{P}^2$. Indeed expanding (3.43) in series, before in $\mathbf{q}$ and then in x, z, and performing the limit[18] $\epsilon_1, \epsilon_2 \to 0$, we obtain

$$\lim_{\epsilon_1,\epsilon_2 \to 0} Z^{\mathbb{P}^2}_{\text{full}}(\mathbf{q}, x, z, \epsilon_1, \epsilon_2)\big|_{c_1=1} =$$
$$= 1 + \mathbf{q}\frac{1}{16}\left(19\frac{x^2}{2!} + 5\frac{xz^2}{2!} + 3\frac{z^4}{4!}\right) + \mathbf{q}^2\frac{1}{32}\left(85\frac{x^4}{4!} + 23\frac{x^3z^2}{2!\,3!} + 17\frac{x^2z^4}{2!\,4!} + 19\frac{xz^6}{6!} + 29\frac{z^8}{8!}\right)$$
$$+ \mathbf{q}^3\frac{1}{4096}\left(29557\frac{x^6}{6!} + 8155\frac{x^5z^2}{2!\,5!} + 6357\frac{x^4z^4}{4!\,4!} + 7803\frac{x^3z^6}{3!\,6!} + 12853\frac{x^2z^8}{2!\,8!} + \right.$$
$$\left. + 26907\frac{xz^{10}}{10!} + 69525\frac{z^{12}}{12!}\right) + O(\mathbf{q}^4) \tag{3.71}$$

this result is in perfect agreement with the literature [36] Theorem 4.4.

In the case $c_1 = 0$ we obtained expression (3.70), in this case the U(2)-bundle can be reduced to the SU(2)-bundle. With the same procedure as before we can check that the limit $\epsilon_1, \epsilon_2 \to 0$ produces SU(2)-Donaldson invariants on $\mathbb{P}^2$. Indeed we get

$$\lim_{\epsilon_1,\epsilon_2 \to 0} Z^{\mathbb{P}^2}_{\text{full}}(\mathbf{q}, x, z, \epsilon_1, \epsilon_2)\big|_{c_1=0} =$$
$$= \mathbf{q}\left(-\frac{3}{2}z\right) + \mathbf{q}^2\left(-\frac{13}{8}\frac{x^2z}{2!} - \frac{xz^3}{3!} + \frac{z^5}{5!}\right)$$
$$+ \mathbf{q}^3\left(-\frac{879}{256}\frac{x^4z}{4!} - \frac{141}{64}\frac{x^3z^3}{3!\,3!} - \frac{11}{16}\frac{x^2z^5}{2!\,5!} + \frac{15}{4}\frac{xz^7}{7!} + 3\frac{z^9}{9!}\right)$$
$$+ \mathbf{q}^4\left(-\frac{36675}{4096}\frac{x^6z}{6!} - \frac{1515}{256}\frac{x^5z^3}{5!\,3!} - \frac{459}{128}\frac{x^4z^5}{4!\,5!} + \frac{51}{16}\frac{x^3z^7}{3!\,7!} + \frac{159}{8}\frac{x^2z^9}{2!\,9!} + 24\frac{xz^{11}}{11!} + 54\frac{z^{13}}{13!}\right)$$
$$+ \mathbf{q}^5\left(-\frac{850265}{32768}\frac{x^8z}{8!} - \frac{143725}{8192}\frac{x^7z^3}{7!\,3!} - \frac{3355}{256}\frac{x^6z^5}{6!\,5!} - \frac{5}{16}\frac{x^5z^7}{5!\,7!} + \frac{2711}{64}\frac{x^4z^9}{4!9!} + \right.$$
$$\left. + \frac{2251}{16}\frac{x^3z^{11}}{3!\,11!} + \frac{487}{2}\frac{x^2z^{13}}{2!\,13!} + 694\frac{xz^{15}}{15!} + 2540\frac{z^{17}}{17!}\right) + O(\mathbf{q}^6) \tag{3.72}$$

---

[18]The limit sets to zero also $h, \tilde{h}, \tilde{K}$ in (3.8), being these polynomials in ϵ_1, ϵ_2.

and we again have agreement with the literature [36] Theorem 4.2. This show that formula (3.70) is indeed a good candidate for the generating function of equivariant Donaldson invariants for an SU(2)-bundle, even in the cases where reducible connections are present.

3.8 Remarkable identities from the evaluation of the partition function

In this subsection we specify our computation to the partition functions without any inserion of observables.

It was noticed in [24] that the partition function of twisted $\mathcal{N} = 2$ Super Yang-Mills theory on a differentiable oriented four manifold is vanishing, due to the presence of ψ-zero modes. These span the tangent space of the instanton moduli space. Therfore the only case in which the partition function is non vanishing correspondes to zero-dimensional components of the moduli space. The partition function is a topological invariant counting, with signs dictated by their relative orientation, the number of the above connected components.

By inspecting our results on the pure partition functions, we obtain results in agreement with the above observation. This in turn implies some remarkable cubic identities on the Nekrasov partition function that we display below.

More explicitly, by computing the coefficients of the power series in $\mathbf{q}$ of the partition function (i.e. formula (3.38) for $c_1 = 1$ and formula (3.70) in the limit $x, z \to 0$ for $c_1 = 0$), one can see that they are almost all equal to zero! Actually only one term survives, namely $p = q = r = 1$ that contributes to the $c_1 = 1$ case. So we can rewrite the partition function for the pure $\mathcal{N} = 2$ theory as

$$Z_{\mathcal{N}=2}^{\mathbb{P}^2}(\mathbf{q})\big|_{c_1=1} = \mathbf{q}^{(1+c)/2}, \qquad Z_{\mathcal{N}=2}^{\mathbb{P}^2}(\mathbf{q})\big|_{c_1=0} = 0. \tag{3.73}$$

This result is in full agreement with the expected behavior of the equivariant partition function in the limit $\epsilon_1, \epsilon_2 \to 0$. In this limit the partition function is expected to be a finite function of the gauge coupling. Indeed, looking at (3.38) at fixed power in the expansion in $\mathbf{q}$, all the dependence on ϵ_1, ϵ_2 appears in the product and in the $Z_{\text{inst}}^{(\ell)}$, the latter depending on ϵ_1, ϵ_2 in the denominators only. So, to obtain a finite limit for $\epsilon_1, \epsilon_2 \to 0$, these terms should sum up to zero but for the term $p = q = r = 1$ in which case both the product and the instanton partition functions contribute as 1. A similar argument holds for the case with $c_1 = 0$. As expected, the non zero term is the contribution of the zero dimensional moduli space components, since $\dim \mathcal{M} = D - 3$ (where the discriminant D is given in (A.9)).

These results imply the following cubic identities for the Nekrasov partition function

$$\mathbf{q}^{-\frac{3}{4}} \sum_{\substack{\{p,q,r\} \\ \text{strictly stable}}} \left[\mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \prod_{\{(i,j)\}} \frac{1}{i\epsilon_1 + j\epsilon_2} \times Z_{\text{inst}}\big(\mathbf{q}; p\epsilon_1 - q\epsilon_2, \epsilon_1, \epsilon_2\big) \right.$$

$$\left. Z_{\text{inst}}\big(\mathbf{q}; q(\epsilon_2 - \epsilon_1) + r\epsilon_1, \epsilon_2 - \epsilon_1, -\epsilon_1\big) Z_{\text{inst}}\big(\mathbf{q}; -r\epsilon_2 - p(\epsilon_1 - \epsilon_2), -\epsilon_2, \epsilon_1 - \epsilon_2\big) \right] = 1 \tag{3.74}$$

and

$$
\left(\sum_{\substack{\{p,q,r\} \\ \text{strictly stable}}} + \frac{1}{2} \sum_{\substack{\{p,q,r\} \\ \text{strictly semi-stable}}} \right) \left[\mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \prod_{\substack{\{(i,j)\} \\ \{(k,l)\}}} \frac{i\epsilon_1 + j\epsilon_2}{k\epsilon_1 + l\epsilon_2} \right.
$$

$$
\times \, Z_{\text{inst}}\big(\mathbf{q}; p\epsilon_1 - q\epsilon_2, \epsilon_1, \epsilon_2\big) \tag{3.75}
$$

$$
\left. \times \, Z_{\text{inst}}\big(\mathbf{q}; q(\epsilon_2 - \epsilon_1) + r\epsilon_1, \epsilon_2 - \epsilon_1, -\epsilon_1\big) Z_{\text{inst}}\big(\mathbf{q}; -r\epsilon_2 - p(\epsilon_1 - \epsilon_2), -\epsilon_2, \epsilon_1 - \epsilon_2\big) \right] = 0
$$

where the product on $\{i,j\}$ and $\{k,l\}$ in (3.74) and (3.75) can be read from (3.38) and (3.70) respectively.

4 $\mathcal{N} = 2^\star$ theory and Euler characteristics

In this section we extend our results to the presence of a hypermultiplet in the adjoint representation with mass M, namely to the so-called $\mathcal{N} = 2^\star$ theory. In the limit $M \to 0$, one gets $\mathcal{N} = 4$ gauge theory whose partition function is the generating function of the Euler characteristics of the moduli spaces of unframed semi-stable equivariant torsion free sheaves [39].

In the following we will compute the full U(2) partition function of the $\mathcal{N} = 2^\star$ theory on $\mathbb{P}^2$ and, after an integration over the v.e.v. of the scalar field, analogous to the one performed in the previous section, we will take the massless limit checking the relation with the Euler characteristics computed in [37–39]. The insertion of the hypermultiplet modifies both the one-loop and the instanton part of the partition function. The one-loop partition function has the extra factor

$$
Z_{\text{1-loop,hyp}}^{\mathbb{P}^2}(\vec{a}, M, \epsilon_1, \epsilon_2) = \prod_{\ell=0}^{2} \exp\left[\sum_{\alpha \neq \beta} \gamma_{\epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}} (a_{\alpha\beta}^{(\ell)} + M) \right]. \tag{4.1}
$$

Following the same steps as in section 3.2, and assuming again $N > 2$ as in (3.45), we obtain similarly to (3.55)

$$
Z_{\text{1-loop,hyp}}^{\mathbb{P}^2}(\vec{a}, M, \epsilon_1, \epsilon_2) = \prod_{i=0}^{N} \prod_{j=0}^{N-i} \big(a + M + (p-j)\epsilon_1 + (q-i)\epsilon_2 \big)^{-1} \times
$$

$$
\prod_{i=0}^{N-3} \prod_{j=0}^{N-3-i} - \big(a - M + (p-1-j)\epsilon_1 + (q-1-i)\epsilon_2 \big)^{-1}, \tag{4.2}
$$

where $N = p + q + r$ with p, q, r defined in (3.14). For the instanton part we should consider the appropriate recursion relation in the presence of an adjoint hypermultiplet that generalizes (3.32). The instanton partition function on $\mathbb{C}^2$ (3.29) in the presence of an adjoint hypermultiplet becomes

$$
Z_{\text{inst,adj}}^{\mathbb{C}^2}(\mathbf{q}; a, M, \epsilon_1, \epsilon_2) = \sum_{\{Y_\alpha\}} \mathbf{q}^{|\vec{Y}|} z_{\text{adj}}(a, M, \vec{Y}, \epsilon_1, \epsilon_2) \tag{4.3}
$$

where $\mathbf{q} = \exp(2i\pi\tau)$ and

$$z_{\text{adj}} = \prod_{\alpha,\beta=1}^{2} \frac{\prod_{s \in Y_\alpha} \left(a_{\beta\alpha} - M - L_{Y_\beta}(s)\epsilon_1 + (A_{Y_\alpha}(s)+1)\epsilon_2\right)\left(a_{\alpha\beta} - M + (L_{Y_\beta}(t)+1)\epsilon_1 - A_{Y_\alpha}(t)\epsilon_2\right)}{\prod_{s \in Y_\alpha} \left(a_{\beta\alpha} - L_{Y_\beta}(s)\epsilon_1 + (A_{Y_\alpha}(s)+1)\epsilon_2\right)\left(a_{\alpha\beta} + (L_{Y_\beta}(t)+1)\epsilon_1 - A_{Y_\alpha}(t)\epsilon_2\right)}.$$
(4.4)

A recursion relation for (4.4) similar to (3.32) is also reported in [51], and has the form

$$Z_{\text{inst,adj}}^{\mathbb{C}^2}(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2) = \left(\hat\eta(\mathbf{q})\right)^{-2\frac{(M-\epsilon_1)(M-\epsilon_2)}{\epsilon_1\epsilon_2}} H(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2),$$
(4.5)

where $\hat\eta(q) = \prod_{n=1}^{\infty}(1-q^n)$ and

$$H(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2) = 1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R_{m,n}^{\text{adj}} H\left(\mathbf{q};\, m\epsilon_1 - n\epsilon_2, M, \epsilon_1, \epsilon_2\right)}{\left(a - m\epsilon_1 - n\epsilon_2\right)\left(a + m\epsilon_1 + n\epsilon_2\right)}$$
(4.6)

with

$$R_{m,n}^{\text{adj}} = 2 \left(\prod_{i=-m+1}^{m} \prod_{j=-n+1}^{n} \left(M - i\epsilon_1 - j\epsilon_2\right)\right) \Bigg/ \left(\underbrace{\prod_{i=-m+1}^{m} \prod_{j=-n+1}^{n}}_{(i,j)\neq\{(0,0),(m,n)\}} \left(i\epsilon_1 + j\epsilon_2\right)\right).$$
(4.7)

The instanton partition function for $\mathbb{P}^2$ is obtained by multiplying (4.5) over the three patches

$$
\begin{aligned}
Z_{\text{inst,adj}}^{\mathbb{P}^2}(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2) &= \prod_{\ell=0}^{2} Z_{\text{inst,adj}}^{\mathbb{C}^2}(\mathbf{q};\, a^{(\ell)}, M, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) \\
&= \left(\hat\eta(\mathbf{q})\right)^{-6} \prod_{\ell=0}^{2} \left(1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R_{m,n}^{\text{adj},(\ell)} H\left(\mathbf{q};\, m\epsilon_1^{(\ell)} - n\epsilon_2^{(\ell)}, M, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}\right)}{\left(a^{(\ell)} - m\epsilon_1^{(\ell)} - n\epsilon_2^{(\ell)}\right)\left(a^{(\ell)} + m\epsilon_1^{(\ell)} + n\epsilon_2^{(\ell)}\right)}\right).
\end{aligned}
$$
(4.8)

Before discussing the limit $M \to 0$ let us make a preliminary comment. First of all notice that, where z_{adj} (4.4) is regular, we have

$$\lim_{M \to 0} z_{\text{adj}}(a, M, \vec{Y}, \epsilon_1, \epsilon_2) = 1.$$
(4.9)

Since

$$\sum_{\{Y_\alpha\}} \mathbf{q}^{|\vec{Y}|} = \left(\hat\eta(\mathbf{q})\right)^{-2}$$
(4.10)

we get from (4.3), (4.9) and (4.5) that

$$\lim_{M \to 0} H\left(\mathbf{q};\, m\epsilon_1 - n\epsilon_2, M, \epsilon_1, \epsilon_2\right) = 1,$$
(4.11)

because in $a = m\epsilon_1 - n\epsilon_2$ we are away from the poles of H.

We will now compute the residue of Z_{full} in the origin as we did in section 3.5. We assume $M > 0$ and, since we want to take eventually the massless limit, M small enough not to meet poles of $Z_{\text{1loop,hyp}}$. We recall that

$$Z_{\text{full}}^{\mathcal{N}=2^\star} = Z_{\text{class}}\, Z_{\text{1loop}}\, Z_{\text{1loop,hyp}}\, Z_{\text{inst,adj}}$$
(4.12)

with components reported in (3.13), (3.55), (4.2) and (4.8) respectively. At the origin:

- Z_{class} and $Z_{\text{1loop,hyp}}$ have neither poles nor zeros,

- Z_{1loop} has a double zero,

- $Z_{\text{inst,adj}}$ has a triple pole.

Indeed we can write

$$Z_{\text{1-loop}}^{\mathbb{P}^2}(a,\epsilon_1,\epsilon_2) = a^2 \prod_{(i,j)\in V_1\backslash(0,0)} (a + i\epsilon_1 + j\epsilon_2) \prod_{(i,j)\in V_2\backslash(0,0)} (-a + i\epsilon_1 + j\epsilon_2).$$

$$Z_{\text{1-loop,hyp}}^{\mathbb{P}^2}(a,M,\epsilon_1,\epsilon_2) = \prod_{(i,j)\in V_1} (a + M + i\epsilon_1 + j\epsilon_2)^{-1} \prod_{(i,j)\in V_2} (-a + M + i\epsilon_1 + j\epsilon_2)^{-1}. \tag{4.13}$$

where the region V_1 and V_2 are described in (3.58) and (3.59) respectively. Similarly to (3.48)

$$Z_{\text{inst,adj}}^{\mathbb{P}^2} = \big(\hat{\eta}(\mathbf{q})\big)^{-6} \frac{1}{a^3}\, \mathbf{q}^{pq+pr+qr}\, \tilde{R}_{p,q}^{\text{adj},(0)}\, \tilde{R}_{q,r}^{\text{adj},(1)}\, \tilde{R}_{r,p}^{\text{adj},(2)}\, H_{\text{Res}}(\mathbf{q};M) + O\!\left(\frac{1}{a^2}\right) \tag{4.14}$$

where

$$\tilde{R}_{m,n}^{\text{adj},(\ell)} = \frac{1}{a^{(\ell)} + m\epsilon_1^{(\ell)} + n\epsilon_2^{(\ell)}} R_{m,n}^{\text{adj},(\ell)} \tag{4.15}$$

and

$$\begin{aligned} H_{\text{Res}}(\mathbf{q};M) =& H\big(\mathbf{q};\, p\epsilon_1 - q\epsilon_2, M, \epsilon_1, \epsilon_2\big) H\big(\mathbf{q};\, q(\epsilon_2 - \epsilon_1) - r(-\epsilon_1), M, \epsilon_2 - \epsilon_1, -\epsilon_1\big) \\ & \times H\big(\mathbf{q};\, r(-\epsilon_2) - p(\epsilon_1 - \epsilon_2), -\epsilon_2, M, \epsilon_1 - \epsilon_2\big). \end{aligned} \tag{4.16}$$

By calculating the factors $R^{\text{adj},(\ell)}$ in $a = 0$ we get

$$\tilde{R}^{(\ell)} = \frac{\prod_{(i,j)\in U_\ell}(M - i\epsilon_1 - j\epsilon_2)}{\prod_{(i,j)\in U_\ell\backslash(0,0)}(i\epsilon_1 + j\epsilon_2)}, \tag{4.17}$$

with U_ℓ defined in (3.52), (3.54), (3.53).

All in all, $Z_{\text{full}}^{\mathcal{N}=2^\star}$ has a simple pole located at the origin whose residue is[19]

$$\begin{aligned} & M^{-1}\text{Res}\big(Z_{\text{full}}^{\mathcal{N}=2^\star}(\mathbf{q};a,M,\epsilon_1,\epsilon_2)\big|a=0\big) = M^{-1} \lim_{a\to 0} a\, Z_{\text{full}}^{\mathcal{N}=2^\star}(\mathbf{q};a,M,\epsilon_1,\epsilon_2) \\ &= M^{-1}\mathbf{q}^{-\frac{1}{4}(1-2c)c_1^2}\mathbf{q}^{-\frac{1}{4}(p+q+r)^2} \\ & \times \prod_{(i,j)\in V_1\backslash(0,0)}\big(i\epsilon_1 + j\epsilon_2\big) \prod_{(i,j)\in V_2\backslash(0,0)}\big(i\epsilon_1 + j\epsilon_2\big) \prod_{(i,j)\in V_1}\big(M + i\epsilon_1 + j\epsilon_2\big)^{-1} \prod_{(i,j)\in V_2}\big(M + i\epsilon_1 + j\epsilon_2\big)^{-1} \\ & \times M^3 \prod_{(i,j)\in U_0\backslash(0,0)} \frac{\big(M - i\epsilon_1 - j\epsilon_2\big)}{\big(i\epsilon_1 + j\epsilon_2\big)} \prod_{(i,j)\in U_1\backslash(0,0)} \frac{\big(M - i\epsilon_1 - j\epsilon_2\big)}{\big(i\epsilon_1 + j\epsilon_2\big)} \prod_{(i,j)\in U_2\backslash(0,0)} \frac{\big(M - i\epsilon_1 - j\epsilon_2\big)}{\big(i\epsilon_1 + j\epsilon_2\big)} \\ & \times \big(\hat{\eta}(\mathbf{q})\big)^{-6}\mathbf{q}^{pq+pr+qr}\, H_{\text{Res}}(\mathbf{q};M). \end{aligned} \tag{4.18}$$

_____

[19]We normalize the integrated partition function with M^{-1} to get dimensionless quantities.

Taking the limit $M \to 0$, and using the fact that from (4.11) $H_{\mathrm{Res}}(\mathbf{q}; M) \to 1$, we obtain

$$\lim_{M \to 0} \frac{1}{M} \mathrm{Res}\left(Z_{\mathrm{full}}^{\mathcal{N}=2^\star}(\mathbf{q}; a, M, \epsilon_1, \epsilon_2) \big| a = 0\right) = \left(\hat{\eta}(\mathbf{q})\right)^{-6} \mathbf{q}^{-\frac{1}{4}c_1^2} \mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)},$$
(4.19)

where $6 = \chi(\mathbb{P}^2) \cdot \mathrm{rank}\,(\mathrm{U}(2))$.

The complete result holds with both $c_1 = 0, 1$, once the contribution of the stricly semi-stable bundles (the ones allowing for reducible connections) are weighed with the factor $1/2$ as in (3.70)

$$Z_{\mathcal{N}=4}^{\mathbb{P}^2}(\mathbf{q}) = \left(\hat{\eta}(\mathbf{q})\right)^{-6} \sum_{c_1=0,1} \left(\underbrace{\sum_{\{p,q,r\}}}_{\text{strictly stable}} + \frac{1}{2} \underbrace{\sum_{\{p,q,r\}}}_{\text{strictly semi-stable}} \right) \mathbf{q}^{-\frac{1}{4}(1-2c)c_1^2} \mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)}$$
(4.20)

where p, q, r are positive integers with $p + q + r + c_1 = even$, and they satisfy respectively strict triangle inequalities in the stable case and large triangle inequalities in the semi-stable one. In the case with only strictly stables configurations this result reduce to the one computed by Kool in [37] when we take the expansion in the second Chern class c_2 ($c = 1$).

Moreover we have checked up to high orders in the power series that for both $c_1 = 0, 1$ (4.20) is in agreement with the mock-modular form of [39]

$$
\begin{aligned}
Z_0(\mathbf{q}) &= \left(\hat{\eta}(\mathbf{q})\right)^{-6} \sum_{n=0}^{\infty} 3H(4n)\mathbf{q}^n && c_1 = 0 \\
Z_1(\mathbf{q}) &= \left(\hat{\eta}(\mathbf{q})\right)^{-6} \sum_{n=0}^{\infty} 3H(4n-1)\mathbf{q}^n && c_1 = 1
\end{aligned}
$$
(4.21)

where $H(n)$ is the Hurwitz class number [52].

5   Discussion

Let us discuss some further directions and open issues. The next natural step to take is to analyse in detail a general compact toric surface M. The conjectural master formula arising from the supersymmetric localisation discussed in section 2 reads

$$Z^M\left(\mathbf{q}, x, z, y\,; \epsilon_1, \epsilon_2\right) = \sum_{\{k_\alpha^{(\ell)}\}|\text{semi-stable}} \oint_\Delta d\vec{a} \prod_{\ell=1}^{\chi(M)} Z_{\mathrm{full}}^{\mathbb{C}^2}\left(\mathbf{q}^{(\ell)}; \vec{a}^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}\right) y^{c_1^{(\ell)}}$$
(5.1)

where $\mathbf{q}^{(\ell)} = \mathbf{q}\, e^{\imath_{P_{(\ell)}}^*(\alpha z + px)}$. Equation (5.1) has to be supplemented by suitable stability conditions constraining the sum over $k_\alpha^{(\ell)}$s. Notice that for $b_2^+ = 1$, the partition function exhibits the wall crossing phenomenon which one should evaluate from the gauge theory path integral and compare with the known results in mathematics, see [35] for the rank two case. Indeed we remind the reader that for manifolds with $b_2^+ = 1$ Donaldson invariants are only piece-wise metric independent. Their behavior is described by a chamber structure in

$H^2(M, \mathbb{R})$ with walls located at $H^2(M, \mathbb{Z}) \cap H^{2,-}(M, \mathbb{R})$. A common strategy to calculate Donaldson invariants is then given by identifying a vanishing chamber and then compute the invariants in the other chambers via wall crossing. In these cases, our formulas for rank two should reproduce the wall crossing terms as computed in [35]. Notice that for $M = \mathbb{P}^2$ there is a single chamber and the above procedure is not available. Moreover, it is neither possible to deform to $\mathcal{N} = 1$ supersymmetry with mass terms as in [25]. This makes this case particularly interesting since it has to be computed directly and we focused on it in this paper.

Let us also notice that E-strings BPS state counting in terms of elliptic genera can be realized as twisted $\mathcal{N} = 4$ partition functions [53–55]. These partition functions enjoy interesting and non-trivial modular properties [56]. It would be useful to explore if and how these properties are realized for non-vanishing mass $M \neq 0$.

The AGT correspondence relates the partition function of $\mathcal{N} = 2$ four dimensional SU(2) gauge theories on S^4 with the correlation functions of primary fields in Liouville conformal field theory [5]. In particular, the instanton contributions are realized to be conformal blocks of the Virasoro algebra with central charge[20] $c = 1 + 6\frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}$. This correspondence has been extended to other four dimensional manifolds M the central charge being computed from the reduction of the M5-brane anomaly polynomial by compactification on M [57, 58]. Explicit examples are provided by toric singularities $\mathbb{C}^2/\Gamma$ with Γ a discrete subgroup in SU(2), whose most studied case is $\Gamma = \mathbb{Z}_2$. The conformal field theory of the latter case has been shown to be $\mathcal{N} = 1$ SuperLiouville theory [59–63].

Another case which has been studied is that of $S^2 \times S^2$ whose gauge theory partition function is build out of chiral copies of Liouville gravity conformal blocks and three point functions [23]. In the same spirit one can try to find a general pattern for this correspondence in the partition function of the $\mathcal{N} = 2$ four dimensional SU(2) gauge theories on a general compact toric manifold. Our result suggests to read the gauge theory partition function in terms of a chiral CFT whose sectors are in one-to-one correspondence with the toric patches. The contribution of each sector to the correlation number is given by a copy of Virasoro conformal block with central charge $c^{(\ell)} = 1 + 6\frac{\left(\epsilon_1^{(\ell)} + \epsilon_2^{(\ell)}\right)^2}{\epsilon_1^{(\ell)} \epsilon_2^{(\ell)}}$ in the ℓ-th sector and three point functions related to the corresponding one-loop contributions of the gauge theory. The change of $(\epsilon_1^{(\ell)}, \epsilon_2^{(\ell)})$ under change of patch is related to the intersection of the corresponding divisors. Investigations in similar directions for Hirzebruch surfaces have been pioneered in [64].

Let us underline the relevance of the cubic identities we obtained in subsection 3.8. These are remarkable identities on the Nekrasov partition function and therefore, via AGT correspondence, on Virasoro conformal blocks. It would be very interesting to understand their interpretation in two dimensional Conformal Field Theory and their generalization to other toric geometries and in higher rank.

Let us notice that a crucial tool for the evaluation of the contour integral appearing in the supersymmetric partition function is Zamolodchikov's recursion relation for the

---

[20]In the round S^4 metric $\epsilon_1 = \epsilon_2 = \frac{1}{r}$, r being the S^4 radius [21]. The case of arbitrary independent real values is obtained by squashing the four sphere [22].

Virasoro conformal blocks which, via AGT correspondence, allows to locate the poles of the integrand and to compute the integral for all instanton numbers. On the other hand, an extension of the gauge theory results to higher rank would provide hints on an analogous recursion relation for $\mathcal{W}$-algebrae. Moreover, this should give a computational tool for Donaldson invariants in higher rank where wall-crossing formulas are notoriously difficult.

We finally remark that we expect that our approach can be uplifted to BPS state counting of gauge theories in higher dimensions, for example by considering supersymmetric gauge theories on five-manifolds given by circle fibrations over toric surfaces. A noticeable example is S^5, whose study is expected to provide information about the M5-brane superconformal index [65–67].

Acknowledgments

A Stability conditions for equivariant vector bundles

In this appendix we make a dictionary between Klyachko's classification of semi-stable equivariant vector bundles on $\mathbb{P}^2$ [47] (for a review see [43], section 4) and the gauge theory fixed point data we sum over in the partition function, in order to discover the constraints to be imposed because of the stability conditions. Klyachko's main result is that equivariant vector bundles on $\mathbb{P}^2$ can be completely described by sets of decreasing filtrations of vector spaces $E_\ell(i)$, one filtration for each open subset of the standard cover $\mathcal{U}_\ell$ ($\ell = 0, 1, 2$). Explicitly

$$E = E_\ell(I_\ell) \supsetneq E_\ell(I_\ell + 1) \supset \cdots \supset E_\ell(I_\ell + n_\ell) \supsetneq E_\ell(I_\ell + n_\ell + 1) = 0 \qquad (A.1)$$

where $E \simeq \mathbb{C}^N$ is the fiber of the bundle (N is the rank of the bundle) at the ℓ-th point and $E_\ell(i) = E$, $\forall i \leq I_\ell$ and $E_\ell(i) = 0$, $\forall i > I_\ell + n_\ell$. The explicit form of the vector subspaces $E_\ell(i)$ in the filtration (A.1) for a given equivariant bundle is reported in [47]. Starting from

the filtration (A.1) it is possible to compute the Chern classes of the vector bundle by the following formulae

$$c_1(E) = \sum_{\ell=0}^{2} \sum_{i} i \dim \left(E_\ell(i)/E_\ell(i+1) \right),$$

$$ch_2(E) \equiv c_2 - \frac{1}{2}c_1^2 = -\frac{1}{2} \sum_{\ell=0}^{2} \sum_{i} i^2 \dim \left(E_\ell(i)/E_\ell(i+1) \right) - \sum_{\ell<\ell'} \sum_{i,j} ij \dim E^{[\ell\ell']}(i,j),$$

$$(A.2)$$

where

$$E^{[\ell\ell']}(i,j) := E_\ell(i) \cap E_{\ell'}(j)/\left(E_\ell(i+1) \cap E_\ell(j) + E_\ell(i) \cap E_\ell(j+1) \right). \qquad (A.3)$$

Let us consider in detail the case of $N = 2$. The relevant steps of the filtration are the ones where the dimension of the subspaces jumps. In the rank two case these are two of them: $i = I_\ell$ in which the dimension jumps from 2 to 1, and $i = I_\ell + n_\ell$ when it jumps from 1 to 0. In particular $n_\ell = \sharp\{i| \dim E_\ell(i) = 1\}$. We then obtain

$$c_1(E) = \sum_{\ell=0}^{2} (2I_\ell + n_\ell),$$

$$ch_2(E) \equiv c_2 - \frac{1}{2}c_1^2 = -\frac{1}{2} \sum_{\ell=0}^{2} \left(I_\ell^2 + (I_\ell + n_\ell)^2 \right) - \sum_{\ell\neq\ell'} I_\ell(I_{\ell'} + n_{\ell'}).$$

$$(A.4)$$

To compare with the gauge theory it is more convenient to use the discriminant D, that for $N = 2$ is

$$\frac{1}{4}D(E) := c_2 - \frac{1}{4}c_1^2 \equiv ch_2 + \frac{1}{4}c_1^2 = -\frac{1}{4} \left(\sum_{\ell=0}^{2} n_\ell^2 - \sum_{\ell<\ell'} 2n_\ell n_\ell' \right). \qquad (A.5)$$

Actually this quantity D has a more fundamental geometric interpretation, indeed it completely determines the isomorphism class of the moduli space $\mathcal{M}(c_1,c_2)$ of the equivariant bundles with given Chern classes c_1 and c_2. In the gauge theory parametrization the first Chern class is

$$c_1(\mathcal{E}) = \sum_{\ell=0}^{2} \sum_{\alpha=1}^{2} k_\alpha^{(\ell)}. \qquad (A.6)$$

To extract the ch_2 for unframed sheaves $\mathcal{E}_0$ we just expand

$$Z_{\text{full}} = \mathbf{q}^{ch_2(\mathcal{E}_0)} \times \left(\cdots \right) \qquad (A.7)$$

so that $ch_2(\mathcal{E}_0)$ can be directly obtained from (3.60)

$$ch_2(\mathcal{E}_0) = \sum_{\ell=0}^{2} |\vec{Y}^{(\ell)}| - \frac{1}{4} \left[\left(\sum_{\ell=0}^{2} k_1^{(\ell)} + k_2^{(\ell)} \right)^2 + \sum_{\ell=0}^{2} (k^{(\ell)})^2 - \sum_{\ell<\ell'} 2k^{(\ell)}k^{(\ell')} \right],$$

$$= \sum_{\ell=0}^{2} |\vec{Y}^{(\ell)}| + ch_2(E) \qquad (A.8)$$

where $k^{(\ell)} := k_1^{(\ell)} - k_2^{(\ell)}$ and we isolated in the second line the vector bundle contribution from the one of the ideal sheaves. The discriminant of the vector bundle E is then

$$\frac{1}{4}D(E) := ch_2(E) + \frac{1}{4}c_1(E)^2 = -\frac{1}{4}\left(\sum_{\ell=0}^{2}(k^{(\ell)})^2 - \sum_{\ell<\ell'}2k^{(\ell)}k^{(\ell')}\right). \tag{A.9}$$

Comparing (A.2) and (A.5) with (A.6) and (A.9) is immediately clear what the dictionary between gauge theory and Klyachko's parameters is

$$I_\ell = \min(k_1^{(\ell)}, k_2^{(\ell)}), \quad I_\ell + n_\ell = \text{Max}(k_1^{(\ell)}, k_2^{(\ell)}), \quad n_\ell = k^{(\ell)} = |k_1^{(\ell)} - k_2^{(\ell)}|. \tag{A.10}$$

Namely the $k_\alpha^{(\ell)}$ are labeling the positions of the jumps in the filtration. Then by making use of Weyl symmetry one can always assume $k_1^{(\ell)} \geq k_2^{(\ell)}$, which we used in the main text.

By using the dictionary (A.10) it is possible to finally read the stability conditions for the equivariant vector bundles directly from the following

Theorem (Klyachko [47]). *The equivariant vector bundle on $\mathbb{P}^2$ defined by the filtrations (A.1) is slope-stable iff for any proper subspace $0 \subsetneq F \subsetneq E$ one has for $\tilde{\imath} \ll 0$*

$$\sum_{\ell=0}^{2}\sum_{i>\tilde{\imath}}\frac{\dim(E_\ell(i)\cap F)}{\dim F} < \sum_{\ell=0}^{2}\sum_{i>\tilde{\imath}}\frac{\dim(E_\ell(i))}{\dim E}. \tag{A.11}$$

The slope-semi-stable case has a large inequality in (A.11).

We work out explicitly the case of $N=2$. The three filtrations for $\mathbb{P}^2$ are of this form

$$E = \mathbb{C}^2 \supsetneq W_\ell \supset \cdots \supset W_\ell \supsetneq 0 \tag{A.12}$$

for each $\ell = 0, 1, 2$. Here W_ℓ is a line in $\mathbb{C}^2$, so $W_\ell \in Gr(1,2) \simeq \mathbb{P}^1$ and appears n_ℓ time in the filtration since $n_\ell = \sharp\{i|\dim E_\ell(i) = 1\}$.

We can assume that all W_ℓ ($\ell = 0, 1, 2$) are distinct[21] and also that $n_\ell > 0$, $\forall\ell$. Indeed it turns out that this is the only relevant case for stability. Either if two or more W_ℓ are equal, or if at least one $n_\ell = 0$, the bundle described by such a filtration does not admit stability, i.e. the strict inequalities (A.11) are mutually incompatible.

Finally we apply the theorem $\forall F \subsetneq E = \mathbb{C}^2$. The relevant conditions come from the choices $F = W_\ell$, $\ell = 0, 1, 2$. The only contribution in (A.11) that is not equal on the r.h.s. and l.h.s. of the inequality is the one relative to the one-dimensional n_ℓ subspaces W_ℓ of the filtrations. Eventually we obtain conditions on n_0, n_1, n_2, namely they have to satisfy strict triangle inequalities

$$n_\ell + n_{\ell'} > n_{\ell''}, \quad \text{for all the choices } \{\ell, \ell', \ell''\} = \{0, 1, 2\}. \tag{A.13}$$

The dictionary (A.10) implies that the gauge parameters $k^{(0)}, k^{(1)}, k^{(2)}$ (often called p, q, r in the main text) have to satisfy the same inequalities.

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[21]We have actually used this assumption when computing (A.4).

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