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DONAGI–MARKMAN CUBIC FOR THE GENERALISED HITCHIN SYSTEM

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Abstract. Donagi and Markman (1993) have shown that the infinitesimal period map for an algebraic completely integrable Hamiltonian system (ACIHS) is encoded in a section of the third symmetric power of the cotangent bundle to the base of the system. For the ordinary Hitchin system the cubic is given by a formula of Balduzzi and Pantev. We show that the Balduzzi–Pantev formula holds on maximal rank symplectic leaves of the G-generalised Hitchin system.

Contents

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1. Introduction

Algebraic completely integrable Hamiltonian systems (ACIHS) form a very special class of quasi-projective algebraic varieties. They carry both a holomorphic symplectic (or Poisson) structure and the structure of a Lagrangian fibration, whose generic fibres are principal homogeneous spaces for abelian varieties (algebraic tori). The varying Hodge structure on the Lagrangian tori allows one to write, after some choices, a holomorphic period map from the base of the fibration to the Siegel upper-half space. Such structures often arise after complexification of real integrable systems appearing in classical physics.

In the C^∞-category any smooth family of tori admits locally (on the base) the structure of a Lagrangian fibration, provided the dimension of the base is half the dimension of the total space. In the holomorphic category, as discovered in [11], for a Lagrangian structure to exist, the differential of the period map must be a section of the third symmetric power of the cotangent bundle to the base, *the Donagi–Markman cubic*. Locally on the base, the cubic is given (in special coordinates) by the third derivatives of a holomorphic function $\mathbf{F}$, called *a holomorphic pre-potential*, while the (marked) period map is given by its Hessian $\mathrm{Hess}(\mathbf{F})$. For example, if the ACIHS is the family of intermediate Jacobians over the moduli space of gauged Calabi–Yau threefolds, the cubic is the Yukawa cubic ([11]).

In [20], [21] N.Hitchin introduced a beautiful ACIHS, supported on a partial compactification of the cotangent bundle to the coarse moduli space of holomorphic G-bundles on a Riemann surface X of genus at least two. Its points correspond to (semi-stable) Higgs bundles on X. These are pairs (E, θ) of a G-bundle E and a holomorphic section of $\mathrm{ad}\, E \otimes K_X$, where G is a semi-simple or reductive complex Lie group. The "conserved momenta" are the spectral invariants of θ, and the abelian varieties are Jacobians or Pryms of certain covers of X.

In [30], [5] the authors considered natural generalisations of the Hitchin system, supported on moduli spaces of meromorphic, i.e., $K_X(D)$-valued, Higgs bundles, for a divisor $D \geq 0$. Such moduli spaces exist in any genus, provided D is sufficiently positive, and Markman and Bottacin showed that they carry a natural holomorphic Poisson structure.

Many known examples of ACIHS appear as special cases of this *generalised Hitchin system*, see the survey [12]. In genus zero such examples are the finite-dimensional coadjoint orbits in loop algebras and Beauville's ([3]) polynomial matrix systems, including geodesic flows on ellipsoids and Euler–Arnold systems. In genus one we have elliptic Gaudin, Sklyanin ([26]) and Calogero–Moser ([25]) systems.

For any ACIHS, $\mathrm{Im}\,\mathrm{Hess}(\mathbf{F})$, being symmetric and positive-definite, provides a Kähler metric on the base. Together with the family of lattices, this data can be packaged into the so-called *(integral) special Kähler structure* ([17]), used by physicists to describe massive vacua in global $N = 2$ super-symmetry in 4 dimensions, e.g., in Seiberg–Witten theory.

More specifically, Donagi and Witten ([14], [15], [31]) have proposed certain special cases of the generalised Hitchin system as candidates for the Coulomb branch of the moduli of vacua for 4-dimensional $N = 2$ Yang–Mills theory with structure group G_c and massive vector multiplets.

A surprising relation between **B**-model large N duality and ADE Hitchin systems has been discovered in [8], [7]. There the authors construct a family of quasi-projective Calabi–Yau 3-folds, whose family of intermediate Jacobians is isogenous to the family of Hitchin Pryms. In that setup the Donagi–Markman cubic plays the role of the Yukawa cubic.

Identifying the cubic is thus a natural and interesting question. For the ordinary Hitchin system ($D = 0$) and $G = SL_2$ such a formula appears in [8], (47). Building on unpublished notes of T.Pantev, Balduzzi ([2]) has derived a formula for arbitrary semi-simple G and $D = 0$, and another variant of the proof is sketched in [19]. Finally, [22] treats the case of the Neumann system, i.e., $X = \mathbb{P}^1$, $G = SL_2$ and $D = 2 \cdot \infty + \sum_i n_i q_i$, using special properties of plane curves. In all of these examples the cubic is given by a "logarithmic derivative" of a discriminant.

Our goal in this note is to show that the Balduzzi–Pantev formula holds along the (good) symplectic leaves of the generalised Hitchin system.

Theorem A. *Let X be a smooth curve, G a semi-simple complex algebraic Lie group, and $D \geq 0$ a divisor on X with $L^2 = K_X(D)^2$ very ample. Let $o \in \mathcal{B} \simeq H^0(X, \bigoplus L^{d_i})$ be a generic point, corresponding to a smooth cameral cover $\pi_o : \widetilde{X}_o \to X$ with simple ramification. Denote by $\mathfrak{D} : \mathcal{B} \times X \to tot\ L^{|\mathcal{R}|}$ the discriminant, and by $\mathbf{B}$ the set*

$$\mathbf{B} = \left(\{o\} + H^0(X, \bigoplus_i L^{d_i}(-D)) \right) \cap \mathscr{B} \subset \mathcal{B},$$

where $\mathscr{B} \subset \mathcal{B}$ is the locus of generic cameral covers. Then the Donagi–Markman cubic for the (maximal rank) symplectic leaf $\mathcal{S}|_{\mathbf{B}} \to \mathbf{B}$ of the G-generalised Hitchin system is

$$c_o : H^0(\widetilde{X}_o, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_o})^W \longrightarrow Sym^2 \left(H^0(\widetilde{X}_o, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_o})^W \right)^{\vee}$$

$$c_o(\xi)(\eta, \zeta) = \frac{1}{2} \sum_{p \in Ram\ (\pi_o)} Res_p^2 \left(\pi_o^* \frac{\mathcal{L}_{Y_\xi}(\mathfrak{D})}{\mathfrak{D}} \bigg|_{\{o\} \times X} \eta \cup \zeta \right).$$

Here Y_ξ is the preimage of ξ under the isomorphism $T_{\mathbf{B},o} \simeq H^0(\widetilde{X}_o, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_o})^W$ induced by the canonical meromorphic 2-form on tot $\mathfrak{t} \otimes_{\mathbb{C}} L$, and $\mathcal{L}_{Y_\xi}$ denotes Lie derivative.

Theorem B. *With the same notation as above,*

$$c_o(\xi, \eta, \zeta) = \sum_{p \in Ram\ (\pi_o)} Res_p^2 \sum_{\alpha \in \mathcal{R}} \frac{\alpha(\xi)\alpha(\eta)\alpha(\zeta)}{\alpha(\lambda_o)},$$

where $\lambda_o \in H^0(\widetilde{X}_o, \mathfrak{t} \otimes_{\mathbb{C}} L)$ *is the tautological section.*

The proof is a local calculation for the universal family of cameral covers. The main ingredient, making this possible, is *abelianisation* ([10]), which has a long history, see [21, 4, 16, 34, 9, 10]. The family of Albanese varieties, associated to the symplectic leaves of the Hitchin system is a family of generalised Prym varieties for a family of cameral curves. By a theorem of Griffiths, the differential of the period map of the latter is given by cup product with the Kodaira–Spencer class. By results of [24] and [28], the symplectic structure on the cameral Pryms agrees with the one on the leaves of the Hitchin system, so the Kodaira–Spencer class is the only missing ingredient.

The content of the paper is as follows. In Section 2 we review the generalised Hitchin system and Markman's construction of the Poisson structure via symplectic reduction. In Section 3 we review period maps and the cubic condition. In Sections 4 and 5 we recall basics about cameral covers and abelianisation. We prove Theorems A and B in Section 6.

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Notation and conventions. We fix a smooth, compact, connected Riemann surface X of genus g, $D = \sum_{i=1}^{s} n_i q_i \geq 0$ a divisor. Set $L = K_X(D)$ and assume L^2 is very ample. We fix $G = \mathbf{G}(\mathbb{C}) \supset B \supset T$: a semi-simple complex algebraic Lie group, a Borel subgroup, and a maximal torus. Then $W = N_G(T)/T$ is the Weyl group, $\mathcal{R} \subset \mathfrak{t}^\vee$ the root system, and $\mathcal{R}^+$ the positive roots. The lattices $\mathsf{root}_{\mathfrak{g}} \subset \mathsf{char}_{\mathsf{G}} \subset \mathsf{weight}_{\mathfrak{g}}$ and $\mathsf{cochar}_{\mathfrak{g}} \subset \mathsf{coweight}_{\mathsf{G}} \subset \mathsf{coweight}_{\mathfrak{g}}$ correspond to their names. By $\mathbf{Gr} = \mathbf{G}(n, V)$ we denote Grassmannian of n-dimensional subspaces of a vector space V and by $Z(s)$ the zero locus of a section s. The Hitchin base is $\mathcal{B} = H^0(X, \mathfrak{t} \otimes_{\mathbb{C}} L/W) \simeq H^0(X, \oplus_i L^{d_i})$, and $\mathscr{B} = \mathcal{B} \backslash \Delta$ is the complement of the discriminant locus. We denote ramification and branch loci by Ram and Bra , respectively.

2. The generalised Hitchin system

2.1. **Meromorphic Higgs bundles.** A *holomorphic L-valued G-Higgs bundle* on X is a pair (E, θ), consisting of a (holomorphic) principal G-bundle $E \to X$ and $\theta \in H^0(X, \mathrm{ad}\, E \otimes L)$. The pair is *(semi-) stable*, if for any *θ-invariant* reduction $\sigma : X \to E/P$ of E to a maximal parabolic subgroup $P \subset G$, $\deg \sigma^* T_{G/P} > 0$ (resp. ≥ 0). Here $T_{G/P}$ is the vertical bundle of $E/P \to X$. This is Ramanathan's (semi-)stability [33], but tested on θ-invariant reductions only.

By the argument in Theorem 4.3 of [33], there exists an analytic coarse moduli space $\mathbf{Higgs}_{G,D} = \coprod_{c \in \pi_1(G)} \mathbf{Higgs}_{G,D,c}$ of semi-stable pairs (E, θ), whose connected components are labelled by the topological type of E. It is expected to be algebraic and separated by suitable modifications of the constructions in [35] and [32], see Remark 2.5 in [26]. For *everywhere regular* θ one can construct the moduli space using the spectral correspondence, see Section 5.4 of [9].

2.2. **Poisson structure.** If nonempty, $\mathbf{Higgs}_{G,D,c}$ are Poisson ([30], [5]), and in fact are ACIHS via the *Hitchin map* h_c. This is a proper morphism $\mathbf{Higgs}_{G,D,c} \to \mathcal{B}$, mapping $[(E,\theta)]$ to the spectral invariants of θ. The *Hitchin base* $\mathcal{B}$ (Section 4.2) is (non-canonically) isomorphic to $\bigoplus_i H^0(X, L^{d_i})$, where d_i are the degrees of the basic G-invariant polynomials on $\mathfrak{g}$. We recall next some definitions from [12], and the structure of the symplectic leaves from [30] and [31]. A Poisson structure on a smooth (analytic or quasi-projective algebraic) variety M is a bivector $\Pi \in H^0(M, \Lambda^2 T_M)$, endowing $\mathcal{O}_M$ with the structure of a sheaf of Lie algebras via the Poisson bracket $\{f,g\} := (df \wedge dg)(\Pi)$. The Lie subalgebra of *Casimir functions* is $\ker\left(H^0(\mathcal{O}_M) \to H^0(T_M)\right)$, $f \mapsto X_f = \{f, \ \}$.

Such a Π determines a sheaf morphism $\Psi : T_M^\vee \to T_M$, and, consequently, a stratification of M by *submanifolds* M_k (k-even) on which $\operatorname{rk} \Psi = k$. The strata are (local-analytically) foliated by k-dimensional leaves S on which $\Pi|_S$ is symplectic. These leaves are the integral leaves of $\Psi(T_M^\vee) \subset T_M$ and the level sets of the Casimirs. A subvariety $L \subset (M,\Pi)$ is called *Lagrangian*, if $L \subset \overline{S}$ for some symplectic leaf S, and $L \cap S$ is a Lagrangian subvariety of S. We say that (M,Π) is an *algebraically completely integrable Hamiltonian system (ACIHS)*, if there exist a smooth variety $\mathcal{B}$, a closed subvariety $\Delta \subsetneq \mathcal{B}$, and a proper flat morphism $h : M \to \mathcal{B}$, such that $h|_{\mathcal{B}\setminus\Delta}$ has Lagrangian fibres, which are abelian torsors.

Suppose a connected algebraic group $\mathcal{G}$ acts on (M,Π). The dual of $\mathbf{g} = \operatorname{Lie} \mathcal{G}$ carries the Kostant–Kirillov Poisson structure, whose symplectic leaves are the coadjoint orbits $\mathbf{O} \subset \mathbf{g}^\vee$. The action is said to be *Poisson*, if the Lie-algebra homomorphism $\mathbf{g} \to H^0(T_M)$ factors through Lie-algebra homomorphisms $\mathbf{g} \to H^0(\mathcal{O}_M) \to H^0(T_M)$, the second arrow being $f \mapsto X_f$. The first arrow corresponds to a *moment map* $\mu : M \to \mathbf{g}^\vee$, which is Poisson. If μ is submersive with connected fibres, and if $M/\mathcal{G}$ exists, then it carries a canonical (Marsden–Weinstein) Poisson structure, whose symplectic leaf through $m \in M$ is

$$\mu^{-1}(\mathbf{O}_{\mu(m)})/\mathcal{G} \simeq \mu^{-1}(\mu(m))/\mathcal{G}_{\mu(m)}.$$

Let E be a G-bundle, and $\eta : E|_D \simeq D \times G = \operatorname{Spec}\left(H^0(\mathcal{O}_D) \otimes H^0(\mathcal{O}_G)\right)$ a framing. Let $\widetilde{G}_D = \widetilde{\mathbf{G}}_D(\mathbb{C})$ be the group of maps from D to G. We recall that $\widetilde{\mathbf{G}}_D$ is defined in terms of its functor of points, by requiring that $\operatorname{Hom}(\operatorname{Spec} A, \widetilde{\mathbf{G}}_D) = \operatorname{Hom}(\operatorname{Spec} A \otimes H^0(\mathcal{O}_G), \mathbf{G})$, for any $\mathbb{C}$-algebra A. It is the product of the groups of (n_i-1)-jets of maps $D \to G$, i.e., $\widetilde{G}_D = \prod_{i=1}^s G_{n_i-1}$, where $\operatorname{Hom}(\operatorname{Spec} A, \widetilde{\mathbf{G}}_n) \simeq \operatorname{Hom}(\operatorname{Spec} A \otimes \mathbb{C}[t]/t^{n+1}, \mathbf{G})$. For example, an embedding $G \subset GL_N$ would identify G_n with the set of $\mathbb{C}[t]/t^{n+1}$-valued matrices, satisfying the equations of $G \mod t^{n+1}$. The *level group* is $G_D = \widetilde{G}_D/Z(G)$, with $Z(G)$ embedded diagonally. We identify $\mathfrak{g}_D^\vee = \widetilde{\mathfrak{g}}_D^\vee \simeq \mathfrak{g}^\vee \otimes H^0(K(D)|_D)$ by the duality pairing

$$H^0(\mathcal{O}_D) \otimes H^0(K(D)|_D) \longrightarrow H^0(K(D)|_D) \xrightarrow{Res} H^1(K) \simeq \mathbb{C}.$$

The moduli space $\mathcal{P}^{st} = \mathcal{P}_{G,D,c}$ of (stable) framed G-bundles of topological type c with level-D structure parametrises isomorphism classes of stable pairs (E,η), ([30]). Its cotangent bundle $T^\vee\mathcal{P}^{st}$ parametrises classes of triples (E,η,θ), $\theta \in H^0(X, \mathrm{ad}\, E \otimes K_X(D))$. The natural action of G_D on $\mathcal{P}^{st}$ lifts to $T^\vee\mathcal{P}^{st}$, and the canonical moment map for a lifted action is $\mu : [(E,\eta,\theta)] \mapsto Ad_\eta(\theta|_D)$, see [30] 6.12, [31] (10). Markman identified $T^\vee\mathcal{P}^{st}/G_D$ with a dense open $\mathbf{Higgs}^{sm}_{G,D,c}$ and extended the Poisson structure to the partial compactification.

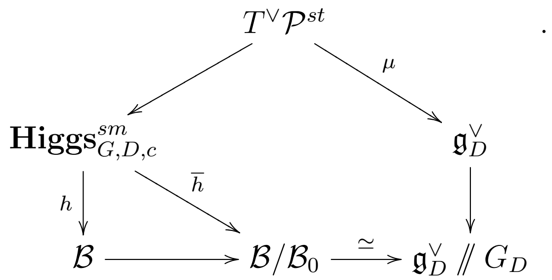
Let $\mathcal{B}_0 := H^0\left(\oplus_i L^{d_i}(-D)\right) \subset \mathcal{B}$. By Proposition 8.8 of [30], there is a canonical isomorphism $\mathcal{B}/\mathcal{B}_0 \simeq \mathfrak{g}_D^\vee /\!\!/ G_D \simeq \mathbb{C}^{r \deg D}$,

$$\bigoplus_{k=1}^{r} \frac{H^0(L^{d_k})}{H^0(L^{d_k}(-D))} \simeq \bigoplus_{i=1}^{s} \mathfrak{g}^\vee \otimes H^0(K(D_i)|_{D_i}) /\!\!/ G_{n_i-1} \simeq \bigoplus_{i=1}^{s} \mathbb{C}^{rn_i} = \mathbb{C}^{r \deg D}$$

sending $[f] \in H^0(L^{d_k})/H^0(L^{d_k}(-D))$ to the coefficients of its (n_i-1)-jet in a holomorphic trivialisation at $q_i \in \mathrm{supp}(D)$, $\forall i = 1..s$:

$$\left(a_0 + \ldots + a_{n_i-1}z^{n_i-1} + \ldots + a_{n_i d_k -1} z^{n_i d_k -1}\right) \frac{dz^{\otimes d_k}}{z^{n_i d_k}} + \ldots \longmapsto (a_0, \ldots, a_{n_i-1}) \in \mathbb{C}^{n_i}.$$

The following diagram is a variant of [30], Proposition 8.8:



The symplectic leaves $S = \mu^{-1}(\mathbf{O})/G_D$ are labelled by coadjoint orbits $\mathbf{O} = \prod_i \mathbf{O}_i \subset \mathfrak{g}_D^\vee$. The symplectic foliation is a refinement of the foliation by fibres of $\overline{h}$, every $\overline{h}$-fibre containing a unique leaf of maximal ($= \dim M_{\mathbf{Higgs}} - r \deg D$) rank.

3. VHS and the cubic condition

3.1. **Variation of Hodge structures.** Let $h : \mathcal{H} \to \mathcal{B}$ be a holomorphic family of polarised compact connected Kähler manifolds over a complex manifold $\mathcal{B}$. The varying Hodge filtration on the first de Rham cohomology of the fibres $h^{-1}(b) =: \mathcal{H}_b$ is the prototypical example of an integral, weight-1 variation of polarised Hodge structures (VHS), encoded in the quadruple $(\mathcal{F}^\bullet, \mathcal{F}_\mathbb{Z}, \nabla^{GM}, S)$. Here $\mathcal{F} = R^1h_*\mathbb{C}\otimes\mathcal{O}_B$, a holomorphic vector bundle with fibres $\mathcal{F}_b = H^1(\mathcal{H}_b, \mathbb{C})$. It carries a flat holomorphic connection $\nabla^{GM}: \mathcal{F}\to\mathcal{F}\otimes\Omega^1_B$, a ∇^{GM}-flat subbundle of lattices $\mathcal{F}_\mathbb{Z}$, a holomorphic flag $\mathcal{F}^1\subset\mathcal{F}^0 = \mathcal{F}$ and a polarisation S.

The Gauss–Manin connection ∇^{GM} can be defined topologically, using Ehresmann's theorem: $\mathcal{H}$ is a fibre bundle, so homotopy invariance of de Rham cohomology implies that $R^1h_*\mathbb{C}$ is a locally constant sheaf, and the connection is then expressed by a Cartan–Lie formula ([37]). A holomorphic description, due to Katz and Oda ([27]), is as follows. The complex $\Omega^\bullet_\mathcal{H}$ can be equipped with the Koszul–Leray filtration $L^i\Omega^\bullet_\mathcal{H} = h^*\Omega^i_\mathcal{B}\wedge\Omega^\bullet_\mathcal{H}[-i]$. Then in the associated spectral sequence $(E_1^{\bullet,0}, d_1)$ is identified with $(\Omega^\bullet_\mathcal{B}(\mathcal{F}), \nabla^{GM})$, via $h^{-1}\mathcal{O}_\mathcal{B}\simeq_{qis}\Omega^\bullet_{\mathcal{H}/\mathcal{B}}[-1]$.

The Hodge filtration $\mathcal{F}^1 = h_*\Omega^1_{\mathcal{H}/\mathcal{B}}\subset\mathcal{F}^0 = \mathcal{F}$ is induced by the stupid filtration $\Omega^{\geq 1}_{\mathcal{H}/\mathcal{B}}[-1]\subset\Omega^\bullet_{\mathcal{H}/\mathcal{B}}$. We denote by $\mathrm{gr}\nabla^{GM}$ the $\Omega^1_\mathcal{B}$-valued homomorphism $\mathcal{F}^1\to\Omega^1_\mathcal{B}\otimes\mathcal{F}^0/\mathcal{F}^1$, which is Simpson's original Higgs field on $\mathcal{F}^1\oplus\mathcal{F}^0/\mathcal{F}^1$.

Integral cohomology provides a subbundle of lattices $\mathcal{F}_\mathbb{Z} = R^1h_*\mathbb{Z}$ and the induced real structure splits the Hodge filtration, giving $\mathcal{F}\simeq_{C^\infty}\mathcal{F}^1\oplus\overline{\mathcal{F}^1}$.

The fibrewise Kähler structure provides a ∇^{GM}-flat, non-degenerate, sesqui-linear, *skew-symmetric* pairing $S:\mathcal{F}^0\otimes\overline{\mathcal{F}^0}\longrightarrow\mathcal{C}^\infty_\mathcal{B}$, $S_b([\eta],[\psi]) = i\int_{\mathcal{H}_b}\eta\wedge\overline{\psi}\wedge\omega_b^{\dim\mathcal{H}/\mathcal{B}-1}$, which induces a C^∞-isomorphism $\overline{\mathcal{F}^1}\simeq_{C^\infty}\mathcal{F}^{1\,\vee}$. The Hodge–Riemann bilinear relations state that 1) $H^1(\mathcal{H}_b,\mathcal{O})$ and $H^0(\mathcal{H}_b,\Omega^1)$ are Lagrangian for $S_b(\bullet,\bar{\bullet})$ and 2) $S_b>0$ on $H^0(\mathcal{H}_b,\Omega^1)$.

3.2. **The period map.** The VHS data are equivalent to a period map from $\mathcal{B}$ to a classifying space (period domain), which for weight one is Siegel's upper half space $\mathbb{H}$. Let $o\in\mathcal{B}$ be a base point and $\check{\mathbf{D}}\subset\mathbf{Gr} = \mathbf{Gr}(h^{1,0}, H^1(\mathcal{H}_o,\mathbb{C}))$ the subvariety of *Lagrangian* (for S_o) subspaces, which is a homogeneous space $\mathrm{Sp}(H^1(\mathcal{H}_o,\mathbb{C}))/\mathrm{Stab}\left[H^0(\Omega^1_{\mathcal{H}_o})\subset H^1(\mathbb{C})\right]$. The second bilinear relation determines an open subvariety $\mathbf{D}\subset\check{\mathbf{D}}$, isomorphic to $\mathrm{Sp}(2h^{1,0},\mathbb{R})/U(h^{1,0})$. Parallel transport then gives rise to a (set-theoretic) period map $\Phi:\mathcal{B}\to\mathbf{D}/\Gamma$, where Γ is the monodromy group of the VHS.

Theorem 3.1 ([18])**.** *The period map Φ is locally liftable and holomorphic.*

Hence on a contractible neighbourhood $\mathcal{U}\subset(\mathcal{B},o)$ there is a holomorphic lift

$$\widetilde{\Phi}:\qquad \mathcal{U}\longrightarrow\mathbf{D}\subset\mathbf{Gr}(h^{0,1}, H^1(\mathcal{H}_o,\mathbb{C}))$$

$$b\longmapsto\left[H^0(\mathcal{H}_b,\Omega^1)\subset H^1(\mathcal{H}_b,\mathbb{C})\simeq H^1(\mathcal{H}_o,\mathbb{C})\right].$$

We can make $\widetilde{\Phi}$ more explicit as follows. The Hodge decomposition on $H^1(\mathcal{H}_o,\mathbb{C})$ gives a standard coordinate chart $H^0(\Omega^1)^\vee\otimes H^1(\mathcal{O})\simeq_S H^0(\Omega^1)^{\vee\,\otimes 2}\subset\mathbf{Gr}$. Then $\check{\mathbf{D}}\cap H^0(\Omega^1)^{\vee\,\otimes 2} = \mathrm{Sym}^2\left(H^0(\Omega^1)^\vee\right)$, and $\mathbf{D}$ consists of the elements with $\mathrm{Im}\ >0$.

The classical way of explicating this map is by choosing a *marking*. A harmonic basis of $H^0(\mathcal{H}_o, \Omega^1)$ identifies $\mathbf{D}$ with Siegel's upper-half space $\mathbb{H}_{h^{1,0}}$. Since $\mathcal{U} \subset \mathcal{B}$ is contractible, we have $H_1(\mathcal{U}, \mathbb{Z}) \simeq H_1(\mathcal{H}_o, \mathbb{Z})$. Choosing a basis of H_1 mod torsion, with normalised A-periods, gives a period matrix of the form $[\Delta_\delta, Z]$, and $\widetilde{\Phi}(b) = Z(b) \in \mathbb{H}_{h^{1,0}}$. Here $\Delta_\delta = \mathrm{diag}(\delta_1, \ldots)$, where δ_i are the polarisation divisors.

3.3. **A theorem of Griffiths.** We can associate to the family h various linear objects: the vertical bundle $\mathcal{V} = h_* T_{\mathcal{H}/\mathcal{B}}$, the relative Jacobian $(\overline{\mathcal{F}^1}/\mathcal{F}_{\mathbb{Z}})$ and Albanese $(\mathcal{F}^{1\,\vee}/\mathcal{F}_{\mathbb{Z}}^\vee)$ varieties. The latter is related to $d\Phi$, as follows. The infinitesimal deformations of $\mathcal{H}_o$ are controlled by the Kodaira–Spencer map $\kappa \in T^\vee_{B,o} \otimes H^1(\mathcal{H}_o, T)$. On the other hand, $T_{\mathbf{D},\widetilde{\Phi}(o)} \subset H^0(\mathcal{H}_o, \Omega^1)^\vee \otimes H^1(\mathcal{H}_o, \mathcal{O})$, so cup product and contraction $T \otimes \Omega^1 \to \mathcal{O}$ give a map $m^\vee : H^1(\mathcal{H}_o, T) \to T_{\mathbf{D},\Phi(o)}$, $m^\vee(\xi)(\alpha) := \xi \cup \alpha \in H^1(\mathcal{H}_o, \mathcal{O})$.

Theorem 3.2 ([18])**.** *The infinitesimal period map satisfies*

$$\begin{array}{ccc} T_{\mathcal{B},o} & \xrightarrow{\;d\widetilde{\Phi}_o\;} & T_{\mathbf{D},\widetilde{\Phi}(o)} \subset H^0(\mathcal{H}_o, \Omega^1)^\vee \otimes H^1(\mathcal{H}_o, \mathcal{O}) \; , \\ & {\scriptstyle \kappa} \searrow \quad \nearrow {\scriptstyle m^\vee} & \\ & H^1(\mathcal{H}_o, T) & \end{array}$$

that is, $\forall Y \in T_{\mathcal{B},o}$*, one has*

$$d\widetilde{\Phi}_o(Y) = gr\ \nabla^{GM}_Y = \kappa(Y) \cup \quad \in T_{\mathbf{D},\widetilde{\Phi}(o)} \subset Sym^2 H^0(\Omega^1)^\vee.$$

3.4. **Lagrangian structures.** The spaces $\mathbf{Higgs}_{G,D,c}$ are Lagrangian fibrations, and the existence of such structure on a family of complex tori is not automatic, as shown in [11]. Indeed, if $h : \mathcal{H} \to (\mathcal{B}, o)$ carries a holomorphic symplectic form and has Lagrangian fibres (*Lagrangian structure*), then the symplectic form induces an isomorphism $i : \mathcal{V}^\vee \to T_{\mathcal{B}}$. For a family of abelian varieties, a choice of marking gives a trivialisation $\mathcal{V} \simeq V \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{B}}$, $V := H^0(\mathcal{H}_o, \Omega^1)^\vee$, which, combined with i, gives an "affine structure" $\alpha : V^\vee \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{B}} \simeq T_{\mathcal{B}}$. Having Lagrangian fibres requires a compatibility between the affine structure and the period map: the *global cubic condition.*

Theorem 3.3 ([11],[12])**.** *For a family $h : \mathcal{H} \to (\mathcal{B}, o)$ of marked abelian varieties, with a period map Φ and an affine structure $\alpha : V^\vee \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{B}} \simeq T_{\mathcal{B}}$ the following are equivalent:*

(1) *h admits a Lagrangian structure inducing α*
(2) *$\widetilde{\Phi} : (\mathcal{B}, o) \to Sym^2 V$ is locally the Hessian of a function* $\mathbf{F}$
(3) *the differential*

$$d\Phi \circ \alpha \in H^0(V \otimes Sym^2 V \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{B}}) \simeq Hom(\mathcal{V}^\vee, Sym^2 \mathcal{V})$$

is the image under $Sym^3 V \hookrightarrow V \otimes Sym^2 V$ of a cubic $c \in Sym^3 V \otimes_{\mathbb{C}} H^0(\mathcal{B}, \mathcal{O}_{\mathcal{B}})$.

In case any of these conditions holds, there exists a unique symplectic structure, for which the 0-section is Lagrangian.

The key observation is that if, for a contractible open $\mathcal{U} \subset \mathcal{B}$, the canonical symplectic form on $T^\vee_{\mathcal{U}}$ descends to $\mathcal{H}|_{\mathcal{U}} \simeq \mathcal{V}/\mathcal{F}^\vee_{\mathbb{Z}}$, translations by the family of lattices (Δ_δ, Z) should be symplectomorphisms, which forces Z to be a Hessian. In terms of local coordinates and a basis of V, the cubic c is given by $c = \sum \frac{\partial^3 \mathbf{F}}{\partial a_i \partial a_j \partial a_k} da_i \cdot da_j \cdot da_k$ and the cubic condition is the equality of mixed partials. The (non-unique) function $\mathbf{F}$ is called *holomorphic pre-potential* and plays a key rôle in Seiberg–Witten theory and special Kähler geometry.

4. Cameral Covers

We describe now a family of ramified Galois covers of X, with Galois group $W = N_G(T)/T$. We refer the reader to [21, 9, 16, 34, 10] for more details and insight. What we call a "cameral cover" is an L-valued cameral cover in the sense of [9] and [10].

4.1. **The adjoint quotient.** The group W is finite, acting on $\mathfrak{t} = \mathrm{Lie}(T)$ by reflections with respect to root hyperplanes, so by [6], Theorem A the subalgebra of invariants is isomorphic to a polynomial algebra on $l = \mathrm{rk}\, \mathfrak{g}$ generators:

$$\mathbb{C}[\mathfrak{t}] = \mathrm{Sym}\, \mathfrak{t}^\vee \supset \mathbb{C}[\mathfrak{t}]^W \simeq \mathbb{C}[I_1, \ldots, I_l],$$

and so there exists a quotient morphism $\chi : \mathfrak{t} \to \mathfrak{t}/W$. The quotient $\mathfrak{t}/W$ is a cone: a variety with a $\mathbb{C}^\times$-action, induced by the homothety $\mathbb{C}^\times$-action on $\mathfrak{t}$, and a unique fixed point, the orbit of the origin. The morphism χ is $\mathbb{C}^\times$-equivariant. There is an isomorphism $\mathfrak{t}/W \simeq \mathbb{C}^l$, but it is non-canonical, since there is no preferred linearisation of the action. If we *fix* generators $\{I_i\}$ (and positive roots $\mathcal{R}^+$), then we can identify χ with a map $\mathfrak{t} \to \mathbb{C}^l$ (or respectively, $\mathbb{C}^l \to \mathbb{C}^l$), $v \mapsto (I_1(v), \ldots, I_l(v))$. For classical Lie algebras χ maps a diagonal matrix to the coefficients of its characteristic polynomial.

By construction, χ is a branched $|W| : 1$ cover, ramified along the root hyperplanes. It is a classical result ([36]) that $\det(d\chi) = c \prod_{\alpha \in \mathcal{R}} \alpha$, for some $c \neq 0$. We call $\mathfrak{D}_\chi = \prod_{\alpha \in \mathcal{R}} \alpha$ *the discriminant* of $\mathfrak{g}$. Since $\mathfrak{D}_\chi \in \mathbb{C}[\mathfrak{t}]^W$, we can write (non-canonically) $\mathfrak{D}_\chi = P(I_1, \ldots, I_l)$, for some (weighted-homogeneous) polynomial P. Then $\{P = 0\} \subset \mathbb{C}^l$ is the equation of the branch locus $\mathrm{Bra}\,(\chi) \subset \mathfrak{t}/W$, i.e., the (singular) discriminant hypersurface of χ.

The inclusion $\mathfrak{t} \subset \mathfrak{g}$ induces $\mathbb{C}[\mathfrak{t}]^W \simeq \mathbb{C}[\mathfrak{g}]^G$, where G acts on $\mathfrak{g}$ by the adjoint action. Consequently, $\mathfrak{t}/W \simeq \mathfrak{g} /\!\!/ G$ (*the adjoint quotient*), and we can choose the W-invariant polynomials on $\mathfrak{t}$ to be restrictions of G-invariant polynomials on $\mathfrak{g}$. In particular, $\mathfrak{D}_\chi$ is the restriction of a G-invariant polynomial, whose locus of non-vanishing is the set of regular elements in $\mathfrak{g}$.

The degrees $\deg I_i = d_i$, or sometimes their shifts $m_i = d_i - 1$, are called the *exponents* of $\mathfrak{g}$, and are important invariants. In particular, $\prod_i d_i = |W|$, $\sum_i m_i = |\mathcal{R}^+|$, $\sum_i (2m_i + 1) = \dim \mathfrak{g}$. We note that $|\mathcal{R}| = 2|\mathcal{R}^+| = \dim \mathfrak{g} - \operatorname{rk} \mathfrak{g}$ and that $d_i \geq 2$, as G is semi-simple and there is no linear invariant.

4.2. **Cameral covers.** The $\mathbb{C}^\times$-equivariance of $\chi : \mathfrak{t} \to \mathfrak{t}/W$ allows us to twist it by any principal $\mathbb{C}^\times$-bundle, obtaining a quotient morphism between the associated fibre bundles. Applied to $L = K_X(D)$, this gives

$$p : \qquad \text{tot } \ \mathfrak{t} \otimes_{\mathbb{C}} L \longrightarrow \text{tot } \mathfrak{t} \otimes_{\mathbb{C}} L/W \,,$$

or, non-canonically, tot $L^{\oplus l} \to$ tot $\bigoplus_i L^{d_i}$, $s = (s_1, \ldots, s_l) \mapsto (I_1(s), \ldots, I_l(s))$. We denote by $\mathcal{B}$ the Hitchin base, i.e., the space of global sections of the quotient (cone) bundle

$$\mathcal{B} = H^0(X, \mathfrak{t} \otimes_{\mathbb{C}} L/W) \simeq \bigoplus_i H^0(X, L^{d_i}).$$

Varying the evaluation morphisms $\operatorname{ev}_b : X \to \text{tot } \mathfrak{t} \otimes_{\mathbb{C}} L/W$ with $b \in \mathcal{B}$, we get a morphism

$$\mathcal{B} \times X \xrightarrow{\text{ev}} \text{tot } \mathfrak{t} \otimes_{\mathbb{C}} L/W \,,$$

which is the evaluation morphism of the tautological section over $\mathcal{B} \times X$. We can then pull back the W-cover p to $\mathcal{B} \times X$:

$$\begin{array}{ccccc}
\widetilde{X}_b & \hookrightarrow & \mathcal{X} & \xrightarrow{\quad\iota\quad} & \text{tot } \mathfrak{t} \otimes_{\mathbb{C}} L \\
\downarrow{\scriptstyle \pi_b} & & \downarrow{\scriptstyle \pi} & & {\scriptstyle p}\downarrow \\
\{b\} \times X & \hookrightarrow & \mathcal{B} \times X & \xrightarrow{\quad\text{ev}\quad} & \text{tot } \mathfrak{t} \otimes_{\mathbb{C}} L/W \\
 & & \searrow & & \swarrow{\scriptstyle q} \\
 & & & X &
\end{array} \,.$$

The result is a family $f = \text{pr}_1 \circ \pi : \mathcal{X} \to \mathcal{B}$ of (not necessarily smooth, integral or reduced) covers of X, with smooth total space $\mathcal{X}$, which is generically a $|W| : 1$ cover of $\mathcal{B} \times X$. We call the fibres $f^{-1}(b) = \widetilde{X}_b \subset \text{tot } \mathfrak{t} \otimes_{\mathbb{C}} L$ *cameral covers* of X, and $\pi : \mathcal{X} \to \mathcal{B} \times X$ *the universal cameral cover.*

We can show that $\mathcal{X} = Z(s)$, for s a section of the pullback of $\mathfrak{t} \otimes_{\mathbb{C}} L/W$ to the triple product $M = \mathcal{B} \times X \times \text{tot } \mathfrak{t} \otimes_{\mathbb{C}} L$. Indeed, let $r = p \circ q : \mathfrak{t} \otimes_{\mathbb{C}} L \to X$ be the bundle projection. The tautological section τ of $q^* \left(\mathfrak{t} \otimes_{\mathbb{C}} L/W\right)$ can be pulled to tot $\mathfrak{t} \otimes_{\mathbb{C}} L$, giving a section $p^*\tau$ of $r^* \ \mathfrak{t} \otimes_{\mathbb{C}} L/W$. Then $\mathcal{X} = Z(s) \subset M$, for $s = \text{pr}_{12}^* \text{ ev} - \text{pr}_3^* \ p^*\tau \in H^0\left(M, r^* \ \mathfrak{t} \otimes_{\mathbb{C}} L/W\right)$.

4.3. **Genericity and Discriminants.** If L^2 is very ample, by Bertini's theorem ([34], Section 1), there exists a Zariski open subset $\mathscr{B} \subset \mathcal{B}$ of cameral covers which are *generic*, i.e., a) are smooth and b) have simple ramification. A more concrete description of $\mathscr{B}$ can be given in terms of discriminant loci. A root $\alpha \in \mathfrak{t}^\vee$ induces a bundle map $r^*\mathfrak{t} \otimes_{\mathbb{C}} L \to r^*L$, so after pullback by the tautological section of $r^*\mathfrak{t} \otimes_{\mathbb{C}} L$, can be identified with a section $\alpha \circ \iota \in H^0(\mathcal{X}, r^*L)$. Its restriction to $\widetilde{X}_b$ is $\alpha \circ \iota_b \in H^0(\widetilde{X}_b, \pi_b^* L)$. Consequently, $\mathfrak{D}_\chi$ gives a W-invariant section $\widetilde{\mathfrak{D}} \in H^0(r^*L^{|\mathcal{R}|})^W$, and $2\mathrm{Ram}\ (\pi) = Z\left(\widetilde{\mathfrak{D}}\right)$. This section descends to a section $\mathfrak{D} : \mathfrak{t} \otimes_{\mathbb{C}} L/W \to q^*L^{|\mathcal{R}|}$, whose zero locus $Z(\mathfrak{D})$ is a singular hypersurface:

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\iota} & \mathrm{tot}\ \mathfrak{t} \otimes_{\mathbb{C}} L & & \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} & \searrow^{\widetilde{\mathfrak{D}}} & \\ \mathcal{B} \times X & \xrightarrow{\mathrm{ev}} & \mathrm{tot}\ \mathfrak{t} \otimes_{\mathbb{C}} L/W & \xrightarrow{\mathfrak{D}} & \mathrm{tot}\ q^*L^{|\mathcal{R}|} \end{array} .$$

We shall occasionally write $\mathfrak{D}$ and $\widetilde{\mathfrak{D}}$ for $\mathrm{ev}^*\mathfrak{D}$ and $\iota^*\widetilde{\mathfrak{D}}$. A local calculation shows that the only possible singularities of $\widetilde{X}_b$ occur at ramification points, a) $\Rightarrow$ b), and "genericity" is equivalent to $\mathrm{ev}_b(X) \cap Z(\mathfrak{D})^{sing} = \varnothing$ and $\mathrm{ev}_b(X) \pitchfork Z(\mathfrak{D})^{sm}$. We have that $\mathrm{Bra}\ (\pi) = Z\left(\mathrm{ev}^*\mathfrak{D}\right) \subset \mathcal{B} \times X$, and away from $0 \in \mathcal{B}$, $\mathrm{pr}_1 : \mathcal{B} \times X \supset \mathrm{Bra}\ (\pi) \to \mathcal{B}$ is a ramified cover, of degree $|\mathcal{R}| \deg L$, with $\mathrm{pr}_1^{-1}(b) \simeq \mathrm{Bra}\ (\pi_b) \subset X$. *Its* ramification locus is $\mathrm{ev}^{-1}\left(Z(\mathfrak{D})^{sing}\right)$, and we have

$$\mathscr{B} \subsetneq \mathcal{B} \setminus \left\{\mathrm{pr}_1\left(\mathrm{ev}^{-1}\left(Z(\mathfrak{D})^{sing}\right)\right)\right\} \subsetneq \mathcal{B}.$$

The locus $\Delta = \mathcal{B} \setminus \mathscr{B}$ is often termed *the discriminant locus*. For later reference, we note that over simply-connected opens $\mathcal{U} \subset \mathscr{B}$, the connected components of $\mathrm{Bra}\ (\pi)$ are graphs of holomorphic maps $c : \mathcal{U} \to X$.

5. Abelianisation

The essence of the abelianisation ideology is that one can translate the non-abelian data of the Higgs field into abelian (spectral) data on a cover of X. While spectral covers suffice for classical G, cameral covers seems to be better suited for the case of arbitrary structure groups. We discuss briefly a weak version of the abelianisation theorem, and refer to [21], [4], [16], [9], [34], [10]), [7] for the details.

Let $\pi_o : \widetilde{X}_o \to X$ be a *fixed* generic cameral cover. By Corollary 17.8 of [10], the Hitchin fibre $h^{-1}(o) \subset \mathbf{Higgs}_{G,D}$ is a torsor over a *generalised Prym variety* $\mathbf{Prym}_{\widetilde{X}_o/X} := H^1(X, \mathcal{T})$, for a certain sheaf $\mathcal{T}$.

We identify $\mathrm{Hom}(\mathbb{C}^\times, T) \otimes_{\mathbb{Z}} \mathbf{Pic}_{\widetilde{X}_o}$ with the (iso classes of) T-bundles on $\widetilde{X}_o$ via $(\mu, \mathscr{L}) \mapsto (\mathscr{L} \setminus \{0\}) \times_\mu T$. As $W = N_G(T)/T \subset \mathrm{Aut}(\widetilde{X}_o)$, there is natural W-action on both factors of this product. Next, we identify the cocharacter lattice $\mathsf{cochar}_{\mathsf{G}} = \mathsf{char}_{\mathsf{G}}^{\vee}$ with $\mathrm{Hom}(\mathbb{C}^{\times}, T)$, and thus $\mathsf{cochar} \otimes_{\mathbb{Z}} \mathbb{C}^{\times} \simeq \mathsf{T}$.

Every $\check{\alpha} \in \mathsf{coroot}_{\mathfrak{g}}$ determines an ideal $\mathsf{char}_{\mathsf{G}}(\check{\alpha}) = \epsilon\mathbb{Z}$, $\epsilon \geq 0$. By inspection, $\epsilon = 1$, unless $\mathfrak{g} = B_l = SO_{2l+1}$ (or has it as a factor), and $\check{\alpha}$ is a long coroot. In that case $\epsilon = 2$ and $\check{\alpha}/2$ is a primitive element of $\mathsf{cochar}_{\mathsf{G}}$ (Lemma 3.3, [13]).

In [10] the authors define two abelian sheaves $\overline{\mathcal{T}} \supset \mathcal{T}$ on X. The first one is $\overline{\mathcal{T}} = \pi_* \left(\mathsf{cochar} \otimes \mathcal{O}_{\widetilde{\mathsf{X}}_o}^{\times}\right)^{W}$, while the second is

$$X \supset U \longmapsto \mathcal{T}(U) := \left\{ t \in \Gamma\left(\pi^{-1}(U), \mathsf{cochar} \otimes \mathcal{O}_{\widetilde{\mathsf{X}}_o}^{\times}\right)^{W} \,\middle|\, \widetilde{\alpha}(t)|_{\mathrm{Ram}\ \alpha} = 1, \forall \alpha \in \mathcal{R} \right\}.$$

Here $\mathrm{Ram}\ _{\alpha} = \mathrm{Ram}\ (\pi_o) \cap \{\alpha = 0\}$, and $\widetilde{\alpha}$ is the 1-psg corresponding to α. Notice that $\widetilde{\alpha}(t) = \pm 1$ and $\overline{\mathcal{T}}/\mathcal{T}$ is a torsion sheaf. In the absence of B_l-factors one has $\mathcal{T} = \overline{\mathcal{T}}$. The neutral components $H^1(X, \overline{\mathcal{T}})^0$ and $H^1(X, \mathcal{T})^0$ are (isogenous) abelian varieties. The connected components $\mathbf{Higgs}_{G,D,c}$, whenever non-empty, are torsors over $H^1(X, \mathcal{T})^0$.

5.1. Theorem 6.4 in [10] describes the above torsors. A (generic) point $[(E, \theta)] \in h_c^{-1}(o)$ determines a morphism $h_c^{-1}(o) \to \mathsf{cochar} \otimes_{\mathbb{Z}} \mathbf{Pic}_{\widetilde{\mathsf{X}}_o}$ with finite fibres ([34], [23]). Since $H^1(X, \overline{\mathcal{T}})$ is isogenous to

$$\left(\mathsf{cochar} \otimes_{\mathbb{Z}} \mathbf{Pic}_{\widetilde{\mathsf{X}}_o}\right)^{W} \subset \mathsf{cochar} \otimes_{\mathbb{Z}} \mathbf{Pic}_{\widetilde{\mathsf{X}}_o},$$

the question of specifying a $\mathbf{Prym}^0_{\widetilde{X}_o/X}$-torsor breaks, up to isogeny, into: 1) determining a coset for the W-invariant T-bundles in $\mathsf{cochar} \otimes_{\mathbb{Z}} \mathbf{Pic}_{\widetilde{\mathsf{X}}_o}$, and 2) determining a torsor for the neutral connected component. This is quite subtle and will not be needed later, so we refer to [10] for details.

Over the generic locus $\mathscr{B} = \mathcal{B} \backslash \Delta$, $\mathbf{Higgs}_{G,D}$ is a torsor over the relative Prym fibration $\mathbf{Prym}_{\mathcal{X}/\mathscr{B}} \to \mathscr{B}$. As $\mathbf{Higgs}_{G,D}$ admits local sections over simply-connected opens $\mathscr{U} \subset \mathscr{B}$, one can identify $\mathbf{Higgs}_{G,D,c}\big|_{\mathscr{U}} \simeq \mathbf{Prym}^0_{\mathcal{X}/\mathscr{U}}$, whenever $\mathbf{Higgs}_{G,D,c} \neq \varnothing$.

6. Proof of Theorems A and B

6.1. **The Symplectic Leaf.** Suppose $o \in \mathscr{B} \subset \mathcal{B}$ is generic (in the sense of Section 4.2), and consider the affine subspace

$$\{o\} + \mathcal{B}_0 = \{o\} + \bigoplus_{i=1}^{r} H^0(L^{d_i}(-D)) \subset \mathcal{B}$$

of codimension (rk $\mathfrak{g}$ deg D). By its genericity, $\{o\}$ corresponds to a regular coadjoint orbit $\mathbf{O}_o$, with $[o] \mapsto \overline{\mathbf{O}_o}$ under $\mathcal{B}/\mathcal{B}_0 \simeq \mathfrak{g}_D^{\vee} /\!\!/ G_D$. Fixing $c \in \pi_1(G)$, we get the closure

$h_c^{-1}(\{o\} + \mathcal{B}_0) = \overline{\mathcal{S}} \subset \mathbf{Higgs}_{G,D,c}$ (if $\neq \varnothing$) of the symplectic leaf $\mathcal{S} := S(\mathbf{O}_o)$. Intersecting the $\mathcal{B}_0$-coset with the generic locus we obtain a locally closed set

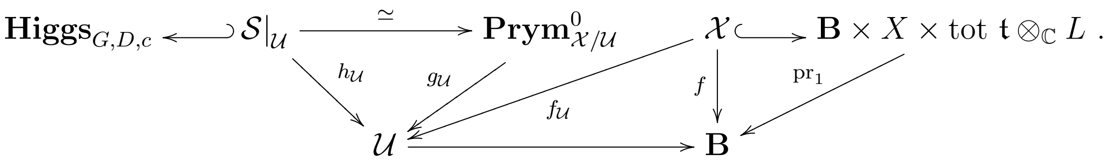
$$\mathbf{B} := (\{o\} + \mathcal{B}_0) \cap \mathscr{B} \subset \mathcal{B}$$

and, with $h_{\mathbf{B}} := h_c|_{\mathbf{B}}$, a Lagrangian fibration

$$\mathbf{Higgs}_{G,D,c} \qquad \supset \qquad \mathcal{S} = h^{-1}(\mathbf{B}) \xrightarrow{h_{\mathbf{B}}} (\mathbf{B}, o) \qquad ,$$

which is an analytic family of regularly stable Higgs bundles, having smooth cameral covers with simple Galois ramification. The corresponding family of cameral covers $f_{\mathbf{B}} : \mathcal{X}|_{\mathbf{B}} \to (\mathbf{B}, o)$ parametrises deformations of $\widetilde{X}_o$ with *fixed* intersection $\widetilde{X}_b \cap r^{-1}(D) \subset \operatorname{tot} \mathfrak{t} \otimes_{\mathbb{C}} L$.

As with any integrable system, we can associate with $h_{\mathbf{B}}$ a family of abelian varieties – the family of Albanese varieties $\operatorname{Alb}(h^{-1}(b))$, $b \in \mathbf{B}$, see Section 3 in [17]. By the abelianisation theorem (Section 5), the latter is the family of (neutral connected components of the) generalised Pryms associated with the family of cameral curves. We then have, for a contractible open set $\mathcal{U} \subset (\mathbf{B}, o)$, the following diagram:



The family of relative Pryms carries a holomorphic symplectic structure, which can be described in terms of cameral curves, as we now explain. Since $\mathbf{B}$ is an affine space over $\mathcal{B}_0$, we have

$$T_{\mathbf{B},o} = \mathcal{B}_0 \simeq H^0\left(X, \bigoplus_i L^{d_i}(-D)\right).$$

Let N_i, $i = 1, 2, 3$, denote the normal bundles for the inclusions of $\widetilde{X}_o$ in $\mathcal{X}|_{\mathbf{B}}$, $\mathcal{X}$ and $\operatorname{tot} \mathfrak{t} \otimes_{\mathbb{C}} L$, respectively. We have

$$T_{\mathbf{B},o} \simeq H^0(\widetilde{X}_o, N_1) \simeq H^0(\widetilde{X}_o, N_2(-r^*D)) \simeq H^0(\widetilde{X}_o, N_3(-r^*D))^W,$$

and all of these isomorphisms are induced by restricting the differential of the respective projection ($f_{\mathbf{B}}$, f or r) to $\widetilde{X}_o$.

On $\operatorname{tot} \mathfrak{t} \otimes_{\mathbb{C}} L$ there is a canonical meromorphic 2-form $\omega_{\mathfrak{t}} \in H^0(\mathfrak{t} \otimes_{\mathbb{C}} \Omega^2_{\mathfrak{t} \otimes_{\mathbb{C}} L}(r^*D))^W$, induced by the canonical symplectic form on $\operatorname{tot} K_X$, see e.g., [24]. To construct it, observe that for a vector space V, there is a natural $\mathfrak{t}$-valued skew-form on $V \oplus (V^\vee \otimes \mathfrak{t})$, given by $((x, \alpha \otimes s), (y, \beta \otimes t)) = \alpha(y)s - \beta(x)t$. This gives rise to a $\mathfrak{t}$-valued 2-form on $\operatorname{tot} \mathfrak{t} \otimes_{\mathbb{C}} K_X$, given locally by $dz \wedge \left(dp \otimes \sum_i dt_i \otimes \frac{\partial}{\partial t_i}\right)$. The local $\mathcal{O}_X$-module isomorphisms $L|_U \simeq K|_U$ twist this to give the meromorphic 2-form $\omega_{\mathfrak{t}}$. Alternatively, $\omega_{\mathfrak{t}}$ can be constructed from a

Liouville form: if λ is the tautological section of $r^*(\mathfrak{t}\otimes_{\mathbb{C}} L) \to \operatorname{tot}\, \mathfrak{t}\otimes_{\mathbb{C}} L$, then $\omega_{\mathfrak{t}} = -d\lambda_o$, $\lambda_o = \lambda|_{\widetilde{X}_o}$.

From the tangent sequence

$$0 \longrightarrow T_{\widetilde{X}_o} \longrightarrow T_{\mathfrak{t}\otimes_{\mathbb{C}} L}|_{\widetilde{X}_o} \longrightarrow N_3 \longrightarrow 0$$

one obtains, by contraction with $\omega_{\mathfrak{t}}|_{\widetilde{X}_o}$, a sheaf homomorphism $N_3 \to \mathfrak{t}\otimes_{\mathbb{C}} K_{\widetilde{X}}(r^*D)$. In general, this map is not an isomorphism, but induces one on invariant global sections:

$$H^0(\widetilde{X}_o, N_3(-r^*D))^W \simeq_{\omega_{\mathfrak{t}}} H^0(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} K_{\widetilde{X}_o})^W,$$

see [28], [23]. In fact, this is a special case of a much more general phenomenon: as shown in [28], the generalised Hitchin system satisfies the rank 2 condition of Hurtubise and Markman, and this isomorphism becomes a special case of Proposition 2.11 in [24].

The tangent sequence of $g_{\mathbf{B}} : \mathbf{Prym}^0_{\mathcal{X}/\mathbf{B}} \to \mathbf{B}$ exhibits $T_{\mathbf{Prym}^0_{\mathcal{X}/\mathbf{B}}}$ as an extension of $g^*_{\mathbf{B}} T_{\mathbf{B}}$ by $T_{\mathbf{Prym}^0/\mathbf{B}}$, which, upon restriction to $P_o = \mathbf{Prym}^0_{\widetilde{X}_o/X}$, gives an extension of $T_{\mathbf{B},o} \otimes \mathcal{O}_{P_o}$ by $T_{P_o} = H^1(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} \mathcal{O})^W \otimes \mathcal{O}_{P_o}$. As $T_{\mathbf{B},o} \simeq H^0(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} K)^W$, at any $\mathscr{L} \in P_o$ the tangent space $T_{\mathbf{Prym}^0,\mathscr{L}}$ is an extension of a pair of Serre-dual spaces, $H^0(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} K)^W$ and $H^1(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} \mathcal{O})^W$. This extension can be split, and the canonical symplectic form transported to $T_{\mathbf{Prym}^0,\mathscr{L}}$. Theorem 4.1 of [28] implies that under the identification $\mathcal{S}|_{\mathcal{U}} \simeq \mathbf{Prym}^0_{\mathcal{X}/\mathcal{U}}$ the symplectic form obtained by moment-map reduction coincides with the canonical one, independent of the splitting. See [23], Theorem 1.10 for the case $X = \mathbb{P}^1$.

6.2. **The Donagi-Markman cubic.** We now give the first version of our result.

Proposition 6.1. *Let $\pi_o : \widetilde{X}_o \to X$ be the cameral cover, corresponding to a generic point $o \in \mathbf{B}$. Then the Donagi–Markman cubic at o for $h_{\mathbf{B}} : \mathcal{S}|_{\mathbf{B}} \to \mathbf{B}$ is given by*

$$c_o : H^0(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} K_{\widetilde{X}_o})^W \longrightarrow \mathit{Sym}^2\left(H^0(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} K_{\widetilde{X}_o})^W\right)^\vee,$$

$$c_o(\xi)(\eta,\zeta) = \frac{1}{2\pi i}\int_{\widetilde{X}_o} \kappa(Y_\xi)\cup\eta\cup\zeta,$$

where $\kappa : T_{B,o} \to H^1(\widetilde{X}_o, T_{\widetilde{X}_o})$ is the Kodaira–Spencer map of the family $f_{\mathcal{U}} : \mathcal{X}|_{\mathcal{U}} \to \mathcal{U}$, and Y_ξ is the preimage of ξ under the isomorphism $T_{B,o} \simeq_{\omega_{\mathfrak{t}}} H^0(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} K_{\widetilde{X}_o})^W$.

6.2.1. *Proof:* Either of the maps $h_{\mathcal{U}}$ or $g_{\mathcal{U}}$ determines the classifying map $\Phi : \mathcal{U} \to \mathbb{H}_{\dim \mathbf{B}}/\Gamma$. The first-order deformations of $P_o = \mathbf{Prym}^0_{\widetilde{X}_o/X}$ are controlled by $H^1(P_o, T_{P_o}) = H^1(P_o, \mathcal{O}_{P_o}) \otimes H^1(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} \mathcal{O})^W$. The polarisation on the Prym is determined by the one on $\widetilde{X}_o$, so $H^1(P_o, T_{P_o}) \simeq H^1(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} \mathcal{O})^{W\otimes 2}$ and polarisation-preserving deformations are contained in $\mathrm{Sym}^2 H^1(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} \mathcal{O})^W \simeq \mathrm{Sym}^2 H^0(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} K_{\widetilde{X}_o})^{W\vee}$, where the last identification uses Serre duality *on* $\widetilde{X}_o$. Hence the derivative of the period map for $h_{\mathcal{U}}$ can be identified with the derivative of the period map for the family of cameral curves $f_{\mathbf{B}}$, and we can apply Theorem 3.2 to the latter. Finally, we notice now the map $m^\vee$ from Griffiths' theorem is dual to the multiplication map $m : H^0(\widetilde{X}_o, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_o})^{W\otimes 2} \to H^0(\widetilde{X}_o, \mathfrak{t} \otimes \mathfrak{t} \otimes K^2) \to_{\mathrm{tr}} H^0(\widetilde{X}_0, K^2)$. This is quite clear, as one defines Serre duality as a $H^1(\widetilde{X}_0, K_{\widetilde{X}_0})$-values pairing, followed by integration (trace) map to $\mathbb{C}$. For a proof, see [37], Lemma 10.22 or [1], Chapter XI, section 8. □We make a linear-algebraic comment on the double-dualisation used here. If V is a finite-dimensional vector space, the natural isomorphism $\mathrm{Hom}(V^\vee, V) = \mathrm{Hom}(V^{\vee\otimes 2}, \mathbb{C})$ is given by

$$f \longmapsto (\alpha \otimes \beta \mapsto \beta(f(\alpha)))\,.$$

Consequently, $\mathrm{Hom}(V^\vee, \mathrm{Hom}(V^\vee, V)) = \mathrm{Hom}(V^{\vee\otimes 3}, \mathbb{C})$ is given by

$$F \longmapsto (Y \otimes \alpha \otimes \beta \mapsto \beta(F(Y)(\alpha)))\,.$$

We apply these to $V^\vee = H^0(\widetilde{X}_o, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_o})^W$. Our genericity assumptions imply that $\mathrm{Bra}\ (\pi_b) \cap \mathrm{Supp}\ D = \varnothing$ for generic $b \in \mathbf{B}$. Indeed, cameral curves from the symplectic leaf have fixed intersection with r^*D. On the other hand (Section 4.3), $\mathrm{Bra}\ (\pi) \subset \mathcal{B} \times X$ is an unbranched covering of $\mathscr{B} \subset \mathcal{B}$, with local sections by graphs of holomorphic maps $\mathscr{B} \to X$. Keeping some of the ramification points fixed corresponds to holding some of these local sections constant, i.e., to a fixed part of the linear system of ramification divisors, and these form a (proper) closed subset of $\{o\} + \mathcal{B}_0$. Points from $\mathrm{Ram}\ (\pi_b) \cap r^*D$, even if non-singular, do not contribute to the Kodaira–Spencer class: in Section 6.4 we are going to see that κ is determined by the *derivatives* of the positions of the branch points. Hence from now on we assume that $\mathrm{Bra}\ (\pi_b) \cap \mathrm{Supp}\ D = \varnothing$.

6.3. **Kodaira–Spencer calculation.** We turn now to calculating the Kodaira–Spencer map of the family $f_{\mathcal{U}} : \mathcal{X}_{\mathcal{U}} \to \mathcal{U}$, $o \in \mathcal{U} \subset \mathbf{B}$. Recall that the (global) Kodaira–Spencer map is the connecting homomorphism $\varkappa : \Gamma(T_{\mathcal{U}}) \to R^1 f_* T_{\mathcal{X}/\mathcal{U}}$, and $\kappa = \varkappa|_o : T_{\mathcal{U},o} \to H^1(\widetilde{X}_o, T_{\widetilde{X}_o})$. We have the following cocycle description of $\varkappa$ from [29], Chapter 4. Let $\{\mathscr{U}_\delta\}$ be an (acyclic) open cover of $\mathcal{X}_{\mathcal{U}}$, $\widetilde{X}_o = \bigcup_\delta \widetilde{X}_o \cap \mathscr{U}_\delta$, on which the tangent sequence of $f_{\mathcal{U}}$ splits. Fix such splittings, i.e., trivialisations $\Phi_\delta : \mathscr{U}_\delta \simeq \Phi_\delta(\mathscr{U}_\delta) \subset \mathcal{U} \times \mathbb{C}$, with coordinates $\{\underline{\beta} = (\beta_k)_k, z_\delta\}$, where the vertical coordinates are related by $z_\delta = \varphi_{\delta\sigma}(\underline{\beta}, z_\sigma)$. Let $Y = \frac{\partial}{\partial \beta} \in \Gamma(T_{\mathcal{U}})$ be a vector field. Then a Čech representative for $\varkappa(Y)$ is given by the differences of the lifts of Y, i.e., by the 1-cocycle

$$\varkappa(Y)_{\delta\sigma} = \left.\frac{\partial \varphi_{\delta\sigma}}{\partial \beta}\right|_{z_\sigma = \varphi_{\sigma\delta}(\beta, z_\delta)} \frac{\partial}{\partial z_\delta}.$$

Here $\frac{\partial \varphi_{\delta\sigma}}{\partial \beta} = \mathcal{L}_Y \varphi_{\delta\sigma} = Y(\varphi_{\delta\sigma})$ denotes the Lie derivative.

What we need next is a convenient open cover of $\mathcal{X}_{\mathbf{B}}$. As we are considering deformations of a map $\pi_o : \widetilde{X}_o \to X$ with a fixed target, this cover should be adapted to the dynamics of the branch points. Let Bra $(\pi_o) = \{p_1, \dots, p_N\}$, with $N = |\mathcal{R}| \deg L$. We consider again a contractibe open $\mathcal{U} \subset (\mathbf{B}, o)$, and describe a 2-set open cover $\mathbf{U} \cup \mathbf{V} = \mathcal{X}_{\mathcal{U}}$. The first open is simply $\mathbf{U} := \mathcal{X}_{\mathcal{U}} \backslash \text{Ram } (\pi)$, the complement of the ramification locus. The set $\mathbf{V}$ is a neighbourhood of Ram (π), constructed as follows. We fix an atlas $\mathfrak{U} = \{(U_j, z_j)\}$ of X, where $U_0 = X \backslash \{p_1, \dots, p_N\}$, and $\{U_j\}$ are small non-intersecting open disks, centered at $\{p_j\}$, and by assumption, $\text{supp}(D) \subset U_0$. Being unbranched, $\mathcal{U} \times X \supset \text{Bra } (\pi)|_{\mathcal{U}} \to \mathcal{U}$ admits holomorphic sections over $\mathcal{U}$, given by $c_j : \mathcal{U} \to X$, $j = 1 \dots N$. We assume that $c_j(o) = p_j$, and $c_j(\mathcal{U}) \subset U_j$, i.e., the branch points of π_b are contained in the open disks U_j for all $b \in \mathcal{U}$. This is always possible by the open mapping theorem and by the simply-connectedness of $\mathcal{U}$. We thus obtain a (trivial) disk bundle $\mathcal{U} \times X \supset \coprod_{j \neq 0} \text{graph } c_j \to \mathcal{U}$ and take as our second open set $\mathbf{V} := \pi^{-1} \left(\coprod_{j \neq 0} \text{graph } c_j \right) \subset \mathcal{X}|_{\mathcal{U}}$. The set $\mathbf{V}$ has $\frac{1}{2}|W||\mathcal{R}| \deg L$ connected components. Recall that, for simple $\mathfrak{g}$, the Weyl group acts transitively on the set of roots of fixed length. Thus, $\text{Ram } (\pi_o)_{p_j} = \coprod_{\alpha \in \mathcal{R}_j^+} \text{Ram }_{\alpha}$, and by genericity $|\text{Ram }_{\alpha}| = |W|/2$. Here $\mathcal{R}_j^+ \in \{\mathcal{R}^+, \mathcal{R}^+_{shrt}, \mathcal{R}^+_{long}\}$, depending on whether $\mathfrak{g}$ is simply laced or not, and on the point p_j, if it is not. If $\mathfrak{g}$ is semi-simple and not simple, there can be different types of fibres. Correspondingly, $\mathbf{V} = \coprod \mathbf{V}_{j\alpha}$, $\mathbf{V}_{j\alpha} = \pi^{-1} (\text{graph } c_j) \cap \{Z(\alpha)\} = \coprod_k \mathbf{V}^k_{j\alpha}$. The connected components $\mathbf{V}^k_{j\alpha}$, $1 \leq k \leq |W|/2$, are tubular neighbourhoods of the connected components of Ram (π). Finally, we have $\mathbf{U} \cap \mathbf{V} \subset \pi^{-1} \left(\mathcal{U} \times \left(\coprod_{j \neq 0} U_j \right) \right)$. This is indeed a good cover, since $\widetilde{X}_o \cap \mathbf{U}$ is affine, and $\widetilde{X}_o \cap \mathbf{V}$ is Stein.

The chosen root data and atlas allow us to split the tangent sequence of $f_{\mathcal{U}}$ on $\mathbf{U}$ and $\mathbf{V}$. To be explicit, a choice of basis of $\mathcal{B}_0 \simeq H^0 \left(X, \bigoplus_i L^{d_i}(-D) \right)$ identifies $\mathcal{U} \subset \mathbf{B}$ with nested open subsets of $\mathbb{C}^{\dim \mathcal{B}_0}$, and we denote by $\underline{\beta} = (\beta_k)_k$ the corresponding coordinates. Then $\{\underline{\beta}, z_j\}$ are (étale!) coordinates on $\mathbf{U} \cap \pi^{-1} (\mathcal{U} \times U_j)$. The above choices also give (candidates for) étale coordinates on the $\mathbf{V}_{j\alpha}$. Indeed, each root $\alpha \in \mathfrak{t}^{\vee}$ gives $\mathcal{X}_{\mathcal{U}} \subset \mathcal{U} \times \text{tot } (\mathfrak{t} \otimes_{\mathbb{C}} L) \to \mathcal{U} \times \text{tot } L$. Since $\text{supp } D \cap U_j = \varnothing$, $K(D)|_{U_j} = K|_{U_j}$. The local coordinate z_j induces a trivialisation of K and we denote by φ_{z_j} the composition $K(D)|_{U_j} = K|_{U_j} \simeq U_j \times \mathbb{C} \to \mathbb{C}$. Then

$$\alpha_j = \varphi_{z_j} \circ \alpha : \qquad \text{tot } \mathfrak{t} \otimes_{\mathbb{C}} L|_{U_j} \xrightarrow{\alpha} L|_{U_j} \xrightarrow{\varphi_{z_j}} \mathbb{C}$$

and $\{\underline{\beta}, \alpha_j\}$ are étale coordinates on $\mathbf{V}_{j\alpha}$. Since the cover is Galois, it is enough to choose, for each j, a simple root, say α_{1j}, and then all other α_j are obtained as $s \cdot \alpha_{1j}$, $s \in W$.

We make some comments and clarifications about these coordinates. They are only étale coordinates, and one could further refine the cover to turn them into analytic ones. As étale maps induce isomorphisms on tangent bundles, this is sufficient for our purposes. One can understand better the local picture as follows. The choice of simple roots determines a basis (of fundamental coweights) in $\mathfrak{t}$, and by the discussion in Sections 4.1 and 4.2 the local equations of $\mathcal{X}|_{\mathcal{U}\times U_j}$ are given by a system of equations

$$\left| \begin{array}{l} I_1(\alpha_1,\ldots,\alpha_r) = b_1(\underline{\beta}, z) \\ \cdots \\ I_r(\alpha_1,\ldots,\alpha_r) = b_r(\underline{\beta}, z) \end{array} \right. ,$$

where b_i are holomorphic functions and $z := z_j$. Over (a contractible subset of) $\mathcal{U} \times U_0$, we have, on a chosen component of $\mathbf{U}$, a parametrisation $\alpha_i = g_i(\underline{\beta}, z)$, and all other components are obtained using the W-action, $g_i \mapsto g_i - n_{ij} g_j$. Over the ramification locus, say, on $\mathbf{V}_{j\alpha}$, $\alpha = \alpha_1$, Weierstrass preparation theorem tells us that

$$\alpha^2 = (z - c(\underline{\beta}))v(\underline{\beta}, z) \qquad \text{or } z = \alpha^2 u(\underline{\beta}, \alpha) + c(\underline{\beta}),$$

and $\alpha_i = g_i(\alpha, z)$, $i \geq 2$. Here u and v are holomorphic functions, satisfying $v(\underline{\beta}, c(\underline{\beta})) \neq 0$, $u(\underline{\beta}, 0) \neq 0$ and $v(\underline{\beta}, \alpha^2 u(\underline{\beta}, \alpha) + c(\underline{\beta})) = u(\underline{\beta}, \alpha)^{-1}$.

Next, the cameral curves may well have "horizontal tangents", so $\varphi_{z_j} \circ \alpha$ is an étale coordinate only on a (Zariski) open subset of $\mathbf{V}_{j\alpha}$. But for us is sufficient that this open set contains Ram (π), which follows from the genericity and the Lie-algebraic fact that on $\ker \alpha \cap Z(\mathfrak{D})^{sm}$, $\ker d\chi|_{\ker \alpha} = \mathbb{C}\check{\alpha}$, see [36].

Finally, we observe that if indeed $p_j = q_i \in \operatorname{supp} D$ for some i and j, $D = n_i q_i + \ldots$, then the frame $\frac{dz_j}{z_j^{n_i}}$ gives a canonical trivialisation of $K(D)|_{U_j}$, induced by the local coordinate z_j.

6.4. **Proof of Theorem A.** We substitute the Kodaira–Spencer class κ_{Y_γ} in Proposition 6.1. Applying the cocycle description of κ to the cover $\{\mathbf{U}, \mathbf{V}\}$, we see that the only contributions arise from $z_j = \varphi_{j\alpha}(\underline{\beta}, \alpha_j)$, i.e., from the intersections $\mathbf{U} \cap \mathbf{V}_{j\alpha}$. The discriminant is $\widetilde{\mathfrak{D}} = \prod_{\alpha\in\mathcal{R}} \alpha = \pm \prod_{\alpha\in\mathcal{R}^+} \alpha^2$, and so

$$\frac{\partial_\beta \widetilde{\mathfrak{D}}}{\widetilde{\mathfrak{D}}} = \sum_{\alpha\in\mathcal{R}^+} \pi^* \frac{\partial_\beta \alpha^2}{\alpha^2},$$

where the α-th summand has second order pole along Ram $(\pi) \cap \mathbf{V}_{j\alpha}$, and is regular elsewhere. Fix $j_0 \neq 0$, $\alpha_{j_0} \in \mathcal{R}^+_{j_0}$, and denote z_{j_0}, α_{j_0} and c_{j_0} by z, α and c, respectively. Using the local equation for $\mathcal{X}_{\mathcal{U}} \subset \mathcal{U} \times U_j \times \mathbb{C}$ from the previous subsection, we have that the contribution to the discrimant ratio $\pi^* \frac{\partial_\beta \mathfrak{D}}{\mathfrak{D}}$ is

$$-\pi^* \frac{\partial_\beta c}{z - c(\underline{\beta})} + \pi^* \frac{\partial_\beta v}{v},$$

where the first term has a second order pole along $\alpha = 0$, while the second term is regular there. On the other hand, we have by the implicit function theorem,

$$\varkappa_{\alpha z}(Y) = \frac{\alpha}{2}\,\pi^*\frac{\partial_\beta \alpha^2}{\alpha^2}\bigg|_{z=\varphi(\beta,\alpha)}\frac{\partial}{\partial\alpha} = -\frac{\partial_\beta c}{2\alpha u(\beta,\alpha)}\frac{\partial}{\partial\alpha} + \dots,$$

where the first summand has a first-order pole at $\alpha = 0$ and $\dots$ denotes terms which are regular at $\alpha = 0$. Setting $\underline{\beta} = 0$ gives the cocycle representatives of $\kappa_{\alpha z}(Y)$. Finally, observe that for a meromorphic function g the last formula implies

$$\mathrm{Res}_{\alpha=0}\left(\kappa_{\alpha z} \lrcorner g(\alpha)d\alpha^2\right) = \frac{1}{2}\mathrm{Res}^2_{\alpha=0}\left(\pi^*\frac{\partial_\beta \mathfrak{D}}{\mathfrak{D}}g(\alpha)d\alpha^2\right).$$

Summing over $j = 1\dots N$, $\alpha \in \mathcal{R}_j^+$ and the $\mathbb{Z}/2\mathbb{Z}$-cosets in W completes the proof. $\square$

6.5. **Proof of Theorem B.** Recall that the cup product on $H^0(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} K_{\widetilde{X}_o})$ is obtained by combining the Killing form $\mathrm{tr} = \sum_{\alpha\in\mathcal{R}}\alpha^2 \in \mathrm{Sym}^2\mathfrak{t}^\vee$ with the cup product on $H^0(\widetilde{X}_o, K_{\widetilde{X}_o})$. With the setup and notation from the previous subsection, we have $\alpha dz = \alpha^2(2u + \alpha\partial_\alpha u)d\alpha + \sum_k \alpha\left(\alpha^2\partial_{\beta_k}u + \partial_{\beta_k}c\right)d\beta_k$, and hence

$$\frac{1}{2}\mathrm{Res}^2_{\alpha=0}\left(-\pi_o^*\,\frac{\partial_\beta c}{z - c(\underline{\beta})}\bigg|_{\underline{\beta}=0} g(\alpha)d\alpha^2\right) = \mathrm{Res}^2_{\alpha=0}\left(\frac{\alpha(\partial_\beta\lrcorner(-d\lambda))}{\alpha(\lambda)}\bigg|_{\underline{\beta}=0} g(\alpha)d\alpha^2\right),$$

since $\lambda = \sum_\alpha \alpha dz \otimes \frac{\partial}{\partial\alpha}$. This is the contribution to the residue from each of the connected components of $\mathrm{Ram}_{\,\alpha}$. We complete the proof by arguing that

$$\mathrm{Res}^2_p \sum_{\alpha\in\mathcal{R}^+}\left(\frac{\alpha(\xi)}{\alpha(\lambda_o)}\left(\sum_{\alpha'\in\mathcal{R}}\alpha'(\eta)\alpha'(\zeta)\right)\right) = \mathrm{Res}^2_p\sum_{\alpha\in\mathcal{R}}\frac{\alpha(\xi)\alpha(\eta)\alpha(\zeta)}{\alpha(\lambda_o)}.$$

Indeed, the terms with $\alpha \neq \alpha'$ vanish due to the W-invariance of ξ, η, ζ, as follows. The W-action on $H^0(\widetilde{X}_o, \mathfrak{t}\otimes K)$ is a combination of the actions on $\mathfrak{t}$ and $\widetilde{X}_o$. The choice of simple roots gives, dually, a basis of $\mathfrak{t}$, so we have $H^0(\widetilde{X}_o, \mathfrak{t}\otimes_{\mathbb{C}} K) \simeq H^0(\widetilde{X}_o, K)^{\oplus l}$. Then $\eta = (\eta_1, \dots, \eta_l)$ and invariance with respect to the symmetry s_α means $s_\alpha^*\eta = s_\alpha^{-1}(\eta_1,\dots,\eta_l)$. The Weyl group acts by $s_\alpha(\beta) = \beta - n\alpha$, hence locally $s_\alpha^*\eta_i|_{\alpha=0} = \eta_i$, and $\eta_i = f d\alpha$, for some *odd* function of α, which does not contribute to the residue at $\alpha = 0$. $\square$

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