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YANG-MILLS-HIGGS CONNECTIONS ON CALABI-YAU MANIFOLDS

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Abstract. Let X be a compact connected Kähler–Einstein manifold with $c_1(TX) \geq 0$. If there is a semistable Higgs vector bundle (E, θ) on X with $\theta \neq 0$, then we show that $c_1(TX) = 0$; any X satisfying this condition is called a Calabi–Yau manifold, and it admits a Ricci–flat Kähler form [Ya]. Let (E, θ) be a polystable Higgs vector bundle on a compact Ricci–flat Kähler manifold X. Let h be an Hermitian structure on E satisfying the Yang–Mills–Higgs equation for (E, θ). We prove that h also satisfies the Yang–Mills–Higgs equation for $(E, 0)$. A similar result is proved for Hermitian structures on principal Higgs bundles on X satisfying the Yang–Mills–Higgs equation.

1. Introduction

Let X be a compact connected Kähler–Einstein manifold with $c_1(TX) \geq 0$. A Higgs vector bundle on X is a holomorphic vector bundle E on X equipped with a holomorphic section θ of $\text{End}(E) \bigotimes \Omega_X$ such that $\theta \bigwedge \theta = 0$. The definition of semistable and polystable Higgs vector bundles is recalled in Section 2. We prove that if there is a semistable Higgs vector bundle (E, θ) on X with $\theta \neq 0$, then $c_1(TX) = 0$ (see Proposition 2.1).

Let X be a compact connected Calabi–Yau manifold, which means that X is a Kähler manifold with $c_1(TX) = 0$. Fix a Ricci–flat Kähler form on X [Ya]. Let (E, θ) be a polystable Higgs vector bundle on X. Then there is a Hermitian structure on E that satisfies the Yang–Mills–Higgs equation for (E, θ) (this equation is recalled in Section 2). Fix a Hermitian structure h on E satisfying the Yang–Mills–Higgs equation for (E, θ).

Our main theorem (Theorem 3.3) says that h also satisfies the Yang–Mills–Higgs equation for $(E, 0)$.

We give an example to show that if a Hermitian structure h_0 on E satisfies the Yang–Mills–Higgs equation for $(E, 0)$, then h_0 does not satisfy the Yang–Mills–Higgs equation for a general polystable Higgs vector bundle of the form (E, θ) (see Remark 3.4). In

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Remark 3.5 we describe how a Yang–Mills–Higgs Hermitian structure for $(E\,,\theta)$ can be constructed from a Yang–Mills–Higgs Hermitian structure for $(E\,,0)$.

Theorem 3.3 extends to the more general context of principal G–bundles on X with a Higgs structure, where G is a connected reductive affine algebraic group defined over $\mathbb{C}$; this is carried out in Section 4.

2. Higgs field on a Kähler-Einstein manifold

We recall that a Kähler metric is called *Kähler–Einstein* if its Ricci curvature is a constant real multiple of the Kähler form. Let X be a compact connected Kähler manifold admitting a Kähler–Einstein metric. We assume that $c_1(TX) \geq 0$; this is equivalent to the condition that the above mentioned scalar factor is nonnegative. Fix a Kähler–Einstein form ω on X. The cohomology class in $H^2(X, \mathbb{R})$ given by ω will be denoted by $\widetilde{\omega}$.

Define the *degree* of a torsionfree coherent analytic sheaf F on X to be

$$\mathrm{degree}(F) \,:=\, (c_1(F) \cup \widetilde{\omega}^{d-1}) \cap [X] \,\in\, \mathbb{R}\,,$$

where d is the complex dimension of X. Throughout this paper, stability will be with respect to this definition of degree.

The holomorphic cotangent bundle of X will be denoted by Ω_X. A *Higgs field* on a holomorphic vector bundle E on X is a holomorphic section θ of $\mathrm{End}(E)\bigotimes\Omega_X = (E\bigotimes\Omega_X)\bigotimes E^*$ such that

$$\theta\bigwedge\theta \,=\, 0\,. \tag{2.1}$$

A *Higgs vector bundle* on X is a pair of the form $(E\,,\theta)$, where E is a holomorphic vector bundle on X and θ is a Higgs field on E.

A Higgs vector bundle $(E\,,\theta)$ is called *stable* (respectively, *semistable*) if for all nonzero coherent analytic subsheaves $F \,\subset\, E$ with $0 \,<\, \mathrm{rank}(F) \,<\, \mathrm{rank}(E)$ and $\theta(F) \,\subseteq\, F\bigotimes\Omega_X$, we have

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} < \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)} \;\; (\text{respectively, } \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \leq \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)})\,.$$

A semistable Higgs vector bundle $(E\,,\theta)$ is called *polystable* if it is a direct sum of stable Higgs vector bundles.

Let Λ_ω denote the adjoint of multiplication of differential forms on X by ω. In particular, Λ_ω sends a $(p\,,q)$–form on X to a $(p-1\,,q-1)$–form. Given a Higgs vector bundle $(E\,,\theta)$ on X, the *Yang–Mills–Higgs* equation for the Hermitian structures h on E states that

$$\Lambda_\omega(\mathcal{K}_h + \theta\wedge\theta^*) \,=\, c\sqrt{-1}\cdot\mathrm{Id}_E\,, \tag{2.2}$$

where $\mathcal{K}_h \in C^\infty(X, \text{End}(E) \bigotimes \Omega^{1,1}_X)$ is the curvature of the Chern connection on E for h, the adjoint θ^* of θ is with respect to h, and c is a constant scalar (it lies in $\mathbb{R}$). A Hermitian structure on E is called Yang–Mills–Higgs for $(E\,,\theta)$ if it satisfies the equation in (2.2).

Proposition 2.1. *If there is a semistable Higgs bundle $(E\,,\theta)$ on X such that $\theta \neq 0$, then $c_1(TX) = 0$.*

Proof. The Higgs field θ on E induces a Higgs field on $\text{End}(E)$, which we will denote by $\widehat{\theta}$. We recall that for any locally defined holomorphic sections s of $\text{End}(E)$,

$$\widehat{\theta}(s) = [\theta\,, s]\,.$$

Let

$$\theta' = \widehat{\theta} \otimes \text{Id}_{\Omega_X}\,. \tag{2.3}$$

This is a Higgs field for $\text{End}(E) \bigotimes \Omega_X$. We note that the integrability condition in (2.1) implies that $\theta'(\theta) = 0$.

Assume that $(E\,,\theta)$ is semistable with $\theta \neq 0$, and also assume that $c_1(TX) \neq 0$. Since $(X\,,\omega)$ is Kähler–Einstein with $c_1(TX) \geq 0$, the condition $c_1(TX) \neq 0$ implies that the anti-canonical line bundle $\bigwedge^d TX$ is positive, so X is a complex projective manifold. Also, the cohomology class of ω is a positive multiple of the ample class $c_1(TX)$.

We shall use the fact that the tensor product of semistable Higgs bundles on a polarized complex projective manifold, with the induced Higgs field, is semistable [Si2, Cor. 3.8]. Thus, $(\text{End}(E), \widehat{\theta})$ is semistable. Moreover, since ω is Kähler–Einstein, Ω_X is a polystable vector bundle, in particular it is semistable. Then $(\Omega_X, 0)$ is a semistable Higgs bundle. As a result, the Higgs bundle $(\text{End}(E) \bigotimes \Omega_X, \theta')$ is semistable.

The homomorphism

$$\mathcal{O}_X \longrightarrow \text{End}(E) \otimes \Omega_X\,, \quad f \longmapsto f\theta$$

defines a homomorphism of Higgs vector bundles

$$\varphi\,:\, (\mathcal{O}_X\,, 0) \longrightarrow (\text{End}(E) \otimes \Omega_X\,, \theta')\,. \tag{2.4}$$

As $\theta \neq 0$, the homomorphism φ in (2.4) is nonzero. Since $(\text{End}(E) \otimes \Omega_X\,, \theta')$ is semistable, we have

$$0 = \frac{\text{degree}(\mathcal{O}_X)}{\text{rank}(\mathcal{O}_X)} = \frac{\text{degree}(\varphi(\mathcal{O}_X))}{\text{rank}(\varphi(\mathcal{O}_X))} \leq \frac{\text{degree}(\text{End}(E) \otimes \Omega_X)}{\text{rank}(\text{End}(E) \otimes \Omega_X)} = \frac{\text{degree}(\Omega_X)}{\text{rank}(\Omega_X)}\,; \tag{2.5}$$

the last equality follows from the fact that $c_1(\text{End}(E)) = 0$. Therefore,

$$\text{degree}(\Omega_X) \geq 0\,. \tag{2.6}$$

Recall that $c_1(TX) \geq 0$ and X admits a Kähler–Einstein metric. So, (2.6) contradicts the assumption that $c_1(TX) \neq 0$. Therefore, we conclude that

$$c_1(TX) = 0\,. \tag{2.7}$$

Consequently, ω is Ricci–flat, in particular, X is a Calabi–Yau manifold. □

A well-known theorem due to Simpson says that E admits an Hermitian structure that satisfies the Yang–Mills–Higgs equation for $(E\,,\theta)$ if and only if $(E\,,\theta)$ is polystable [Si1, Thm. 1] (see also [Si2]); when X is a compact Riemann surface and $\text{rank}(E) = 2$, this was first proved in [Hi].

The Chern connection on E for h will be denoted by ∇^h. Let $\widehat{\nabla}^h$ denote the connection on $\text{End}(E) = E \bigotimes E^*$ induced by ∇^h. The Levi–Civita connection on Ω_X associated to ω and the connection $\widehat{\nabla}^h$ on $\text{End}(E)$ together produce a connection on $\text{End}(E) \bigotimes \Omega_X$. This connection on $\text{End}(E) \bigotimes \Omega_X$ will be denoted by $\nabla^{\omega,h}$.

Proposition 2.2. *Assume that the Hermitian structure h satisfies the Yang–Mills–Higgs equation in* (2.2) *for $(E\,,\theta)$. Then the section θ of* $\text{End}(E) \bigotimes \Omega_X$ *is flat (meaning covariantly constant) with respect to the connection $\nabla^{\omega,h}$ constructed above.*

Proof. The Hermitian structure h on E produces an Hermitian structure on $\text{End}(E)$, which will be denoted by $\widehat{h}$. The connection $\widehat{\nabla}^h$ on $\text{End}(E)$ defined earlier is in fact the Chern connection for $\widehat{h}$. The Kähler form ω and the Hermitian structure $\widehat{h}$ together produce an Hermitian structure on $\text{End}(E) \bigotimes \Omega_X$. This Hermitian structure on $\text{End}(E) \bigotimes \Omega_X$ will be denoted by h^ω. We note that the connection $\nabla^{\omega,h}$ in the statement of the proposition is the Chern connection for h^ω.

Since ω is Kähler–Einstein, the Hermitian structure on Ω_X induced by ω satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle $(\Omega_X\,,0)$. As h satisfies the Yang–Mills–Higgs equation for $(E\,,\theta)$, this implies that h^ω satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle $(\text{End}(E) \bigotimes \Omega_X\,,\theta')$ constructed in (2.3). In particular, the Higgs vector bundle $(\text{End}(E) \bigotimes \Omega_X\,,\theta')$ is polystable. The Proposition is obvious if $\theta = 0$. Assume that $\theta \neq 0$; then φ defined in (2.4) is nonzero.

Since $c_1(\Omega_X) = 0$, the inequality in (2.5) is an equality. Now from [Si1, Prop. 3.3] it follows immediately that

- $\varphi(\mathcal{O}_X)$ in (2.4) is a subbundle of $\text{End}(E)$,
- the orthogonal complement $\varphi(\mathcal{O}_X)^\perp \subset \text{End}(E) \bigotimes \Omega_X$ of $\varphi(\mathcal{O}_X)$ with respect to the Yang–Mills–Higgs Hermitian structure h^ω is preserved by θ', and
- $(\varphi(\mathcal{O}_X)^\perp\,,\theta'|_{\varphi(\mathcal{O}_X)^\perp})$ is polystable with

$$\frac{\text{degree}(\varphi(\mathcal{O}_X)^\perp)}{\text{rank}(\varphi(\mathcal{O}_X)^\perp)} = \frac{\text{degree}(\text{End}(E) \otimes \Omega_X)}{\text{rank}(\text{End}(E) \otimes \Omega_X)} = 0\,.$$

We note that [Si1, Prop. 3.3] also says that the Hermitian structure on the image of φ induced by h^ω satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle $(\varphi(\mathcal{O}_X)\,,0)$. Since the above orthogonal complement $\varphi(\mathcal{O}_X)^\perp \,\subset\, \text{End}(E)\bigotimes\Omega_X$ is a holomorphic subbundle,

- the connection $\nabla^{\omega,h}$ preserves $\varphi(\mathcal{O}_X)$,
- and the connection on $\varphi(\mathcal{O}_X)$ obtained by restricting $\nabla^{\omega,h}$ coincides with the Chern connection for the Hermitian structure $h^\omega|_{\varphi(\mathcal{O}_X)}$.

Also, recall that $h^\omega|_{\varphi(\mathcal{O}_X)}$ satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle $(\varphi(\mathcal{O}_X)\,,0)$. These together imply that all holomorphic sections of $\varphi(\mathcal{O}_X)$ over X are flat with respect to the Yang–Mills–Higgs connection $\nabla^{\omega,h}$ on $\text{End}(E)\bigotimes\Omega_X$. In particular, the section θ is flat with respect to $\nabla^{\omega,h}$. □

2.1. **Decomposition of a Higgs field.** In view of Proposition 2.1, henceforth we assume that $c_1(TX)\,=\,0$. Therefore, the Kähler–Einstein form ω is Ricci–flat. For any point $x\,\in\,X$, the fiber of the vector bundle Ω_X over x will be denoted by $\Omega_{X,x}$.

Let $(E\,,\theta)$ be a polystable Higgs vector bundle on X. For any point $x\,\in\,X$, we have a homomorphism

$$\eta_x\,:\,T_xX\,\longrightarrow\,\text{End}(E_x)\,,\ \ \eta_x(v)\,=\,i_v(\theta(x))\,, \tag{2.8}$$

where $i_v\,:\,\Omega_{X,x}\,\longrightarrow\,\mathbb{C}$, $z\,\longmapsto\,z(v)$, is the contraction of forms by the tangent vector v.

Lemma 2.3. *For any two points x and y of X, there are isomorphisms*

$$\alpha\,:\,T_xX\,\longrightarrow\,T_yX\quad\text{and}\quad\beta\,:\,E_x\,\longrightarrow\,E_y$$

such that $\beta(\eta_x(v)(u))\,=\,(\eta_y(\alpha(v)))(\beta(u))$ for all $v\,\in\,T_xX$ and $u\,\in\,E_x$.

Proof. Let h be an Hermitian structure on E satisfying the Yang–Mills–Higgs equation for $(E\,,\theta)$. As before, the Chern connection on E associated to h will be denoted by ∇^h.

Fix a C^∞ path $\gamma\,:\,[0\,,1]\,\longrightarrow\,X$ such that $\gamma(0)\,=\,x$ and $\gamma(1)\,=\,y$. Take α to be the parallel transport of T_xX along γ for the Levi–Civita connection associated to ω. Take β to be the parallel transport of E_x along γ for the above connection ∇^h. Using Proposition 2.2 it is straightforward to deduce that

$$\beta(\eta_x(v)(u))\,=\,(\eta_y(\alpha(v)))(\beta(u))$$

for all $v\,\in\,T_xX$ and $u\,\in\,E_x$. □

From (2.1) it follows immediately that for any $v_1\,,v_2\,\in\,T_xX$, we have

$$\eta_x(v_1)\circ\eta_x(v_2)\,=\,\eta_x(v_2)\circ\eta_x(v_1)\,,$$

where η_x is constructed in (2.8). In view of this commutativity, there is a generalized eigenspace decomposition of E_x for $\{\eta_x(v)\}_{v\in T_xX}$. More precisely, we have distinct elements $u_1^x\,,\cdots\,,u_m^x\,\in\,\Omega_{X,x}$ and a decomposition

$$E_x\,=\,\bigoplus_{i=1}^m E_x^i \tag{2.9}$$

such that

- for all $v\,\in\,T_x$ and $1\,\le\,i\,\le\,m$,
$$\eta_x(v)(E_x^i)\,\subseteq\,E_x^i\,, \tag{2.10}$$
- the endomorphism of E_x^i
$$\eta_x(v)|_{E_x^i}-u_i^x(v)\cdot \mathrm{Id}_{E_x^i} \tag{2.11}$$
is nilpotent.

Therefore, these elements $\{u_i^x\}_{i=1}^m$ are the joint generalized eigenvalues of $\{\eta_x(v)\}_{v\in T_xX}$. Note however that there is no ordering of the elements $\{u_i^x\}_{i=1}^m$. From Lemma 2.3 it follows immediately that the integer m is independent of x.

Let Y' denote the space of all pairs of the form $(x\,,\epsilon)$, where $x\,\in\,X$ and

$$\epsilon\,:\,\{1\,,\cdots\,,m\}\,\longrightarrow\,\{u_i^x\}_{i=1}^m$$

is a bijection. Clearly, Y' is an étale Galois cover of X with the permutations of $\{1\,,\cdots\,,m\}$ as the Galois group. We note that Y' need not be connected. Fix a connected component $Y\,\subset\,Y'$. Let

$$\varpi\,:\,Y\,\longrightarrow\,X\,,\quad (x\,,\epsilon)\,\longmapsto\,x \tag{2.12}$$

be the projection. So ϖ is an étale Galois covering map.

For any $y\,=\,(x\,,\epsilon)\,\in\,Y$, and any $i\,\in\,\{1\,,\cdots\,,m\}$, the element $\epsilon(i)\,\in\,\{u_i^x\}_{i=1}^m$ will be denoted by $\widehat{u}_i^{\varpi(y)}$.

Therefore, from (2.9) we have a decomposition

$$\varpi^*E\,=\,\bigoplus_{i=1}^m F_i\,, \tag{2.13}$$

where the subspace $(F_i)_y\,\subset\,(\varpi^*E)_y\,=\,E_{\varpi(y)}$, $y\,\in\,Y$, is the subspace of $E_{\varpi(y)}$ which is the generalized simultaneous eigenspace of $\{\eta_x(v)\}_{v\in T_{\varpi(y)}X}$ for the eigenvalue $\widehat{u}_i^{\varpi(y)}(v)$ (the element $\widehat{u}_i^{\varpi(y)}$ is defined above).

Clearly, (2.13) is a holomorphic decomposition of the holomorphic vector bundle ϖ^*E. Consider the Higgs field $\varpi^*\theta\,\in\,H^0(Y,\,\mathrm{End}(\varpi^*E)\bigotimes\Omega_Y)$ on ϖ^*E, where $\Omega_Y\,=\,\varpi^*\Omega_X$ is the holomorphic cotangent bundle of Y. From (2.10) it follows immediately that

$$(\varpi^*\theta)(F_i)\,\subseteq\,F_i\otimes\Omega_Y\,. \tag{2.14}$$

Let

$$\theta_i := (\varpi^*\theta)|_{F_i} \tag{2.15}$$

be the Higgs field on F_i obtained by restricting $\varpi^*\theta$.

Equip Y with the pulled back Kähler form $\varpi^*\omega$. Consider the Hermitian structure ϖ^*h on ϖ^*E, where h, as before, is a Hermitian structure on E satisfying the Yang–Mills–Higgs equation for $(E\,,\theta)$. It is straightforward to check that ϖ^*h satisfies the Yang–Mills–Higgs equation for $(\varpi^*E\,,\varpi^*\theta)$. In particular, $(\varpi^*E\,,\varpi^*\theta)$ is polystable. The restriction of ϖ^*h to the subbundle F_i in (2.13) will be denoted by h_i. Since

$$(\varpi^*E\,,\varpi^*\theta) = \bigoplus_{i=1}^{m}(F_i\,,\theta_i)\,,$$

where θ_i is constructed in (2.15), and ϖ^*h satisfies the Yang–Mills–Higgs equation for $(\varpi^*E\,,\varpi^*\theta)$, it follows that h_i satisfies the Yang–Mills–Higgs equation for $(F_i\,,\theta_i)$ [Si1, p. 878, Theorem 1]. Consequently, $(F_i\,,\theta_i)$ is polystable. We note that the polystability of $(F_i\,,\theta_i)$ also follows form the fact that $(F_i\,,\theta_i)$ is a direct summand of the polystable Higgs vector bundle $(\varpi^*E\,,\varpi^*\theta)$.

Let

$$\mathrm{tr}(\theta_i) \in H^0(Y,\,\Omega_Y) \tag{2.16}$$

be the trace of θ_i. Let r_i be the rank of the vector bundle F_i. Define

$$\widetilde{\theta}_i := \theta_i - \frac{1}{r_i}\mathrm{Id}_{F_i}\otimes\mathrm{tr}(\theta_i) \in H^0(Y,\,\mathrm{End}(F_i)\otimes\Omega_Y)\,. \tag{2.17}$$

We note that $\widetilde{\theta}_i$ is also a Higgs field on F_i.

Corollary 2.4. *The section* $\theta_i \in H^0(Y,\,\mathrm{End}(F_i)\bigotimes\Omega_Y)$ *in* (2.15) *is flat with respect to the connection on* $\mathrm{End}(F_i)\bigotimes\Omega_Y$ *constructed from* h_i *and* $\varpi^*\omega$*. Similarly,* $\widetilde{\theta}_i$ *in* (2.17) *is flat with respect to this connection on* $\mathrm{End}(F_i)\bigotimes\Omega_Y$*.*

Proof. We noted earlier that h_i satisfies the Yang–Mills–Higgs equation for $(F_i\,,\theta_i)$. From this it follows that h_i also satisfies the Yang–Mills–Higgs equation for $(F_i\,,\widetilde{\theta}_i)$. Therefore, substitutions of $(F_i\,,\theta_i\,,h_i)$ and $(F_i\,,\widetilde{\theta}_i\,,h_i)$ in place of $(E\,,\theta\,,h)$ in Proposition 2.2 yield the result. $\square$

Proposition 2.5. *The Higgs field* $\widetilde{\theta}_i$ *on* F_i *in* (2.17) *vanishes identically.*

Proof. Since the endomorphism in (2.11) is nilpotent, it follows that

$$\widetilde{\theta}_i(y)(v) \in \mathrm{End}(\varpi^*E_y) = \varpi^*\mathrm{End}(E_y) = \mathrm{End}(E_{\varpi(y)})$$

is nilpotent for all $y \in Y$ and $v \in T_yY$. Consider the homomorphism

$$\widetilde{\widetilde{\theta}}_i : F_i \longrightarrow F_i\otimes\Omega_Y\,,\quad z \longmapsto \widetilde{\theta}_i(y)(z)\ \ \forall\, z \in (F_i)_y\,. \tag{2.18}$$

Let

$$\mathcal{V}_i := \text{kernel}(\widetilde{\widetilde{\theta}}_i) \subset F_i \tag{2.19}$$

be the kernel of it. From Corollary 2.4 it follows that the subsheaf $\mathcal{V}_i \subset F_i$ is a subbundle. We also note that $\mathcal{V}_i$ is of positive rank.

Let

$$\widetilde{\theta}_i^f = \widetilde{\theta}_i \otimes \text{Id}_{\Omega_Y}$$

be the Higgs field on $F_i \bigotimes \Omega_Y$. Since $\varpi^*\omega$ is Kähler–Einstein, and h_i satisfies the Yang–Mills–Higgs equation for $(F_i, \widetilde{\theta}_i)$, the Hermitian structure on $F_i \bigotimes \Omega_Y$ induced by the combination of h_i and $\varpi^*\omega$ satisfies the Yang–Mills–Higgs equation for $(F_i \bigotimes \Omega_Y, \widetilde{\theta}_i^f)$. In particular, $(F_i \bigotimes \Omega_Y, \widetilde{\theta}_i^f)$ is polystable.

Note that

$$\frac{\text{degree}(F_i \otimes \Omega_Y)}{\text{rank}(F_i \otimes \Omega_Y)} = \frac{\text{degree}(F_i)}{\text{rank}(F_i)} + \frac{\text{degree}(\Omega_Y)}{\text{rank}(\Omega_Y)} = \frac{\text{degree}(F_i)}{\text{rank}(F_i)}; \tag{2.20}$$

the last equality follows from the fact that $c_1(\Omega_Y) = 0$. The homomorphism $\widetilde{\widetilde{\theta}}_i$ in (2.18) is compatible with the Higgs fields $\widetilde{\theta}_i$ and $\widetilde{\theta}_i^f$ on F_i and $F_i \bigotimes \Omega_Y$ respectively, meaning $\widetilde{\theta}_i \circ \widetilde{\widetilde{\theta}}_i = \widetilde{\widetilde{\theta}}_i \circ \widetilde{\theta}_i$. From the definition of $\mathcal{V}_i$ in (2.19) it follows immediately that $\widetilde{\theta}_i|_{\mathcal{V}_i} = 0$. Hence $(\mathcal{V}_i, 0)$ is a Higgs subbundle of $(F_i, \widetilde{\theta}_i)$. Since both $(F_i, \widetilde{\theta}_i)$ and $(F_i \bigotimes \Omega_Y, \widetilde{\theta}_i^f)$ are semistable of same slope (see (2.20)), we conclude that $(\mathcal{V}_i, 0)$ is a Higgs subbundle of $(F_i, \widetilde{\theta}_i)$ of same slope (same as that of F_i). Now, as $(F_i, \widetilde{\theta}_i)$ is polystable, the Higgs subbundle $(\mathcal{V}_i, 0)$ of same slope has a direct summand.

Let $(W_i, \theta_i^c) \subset (F_i, \widetilde{\theta}_i)$ be a direct summand of $(\mathcal{V}_i, 0)$. If $W_i = 0$, then the proof is complete. So assume that $W_i \neq 0$.

Substituting (W_i, θ_i^c) in place of $(F_i, \widetilde{\theta}_i)$ in the above argument and iterating the argument, we conclude that $\widetilde{\theta}_i = 0$. □

Corollary 2.6. *Let X be a compact 1–connected Calabi–Yau manifold. If (E, θ) is a polystable Higgs vector bundle on X, then $\theta = 0$.*

Proof. Since X is simply connected, it follows that ϖ in (2.12) is an isomorphism. We have $H^0(X, \Omega_X) = 0$, because $b_1(X) = 0$ and $\dim H^0(X, \Omega_X) = b_1(X)/2$. Therefore, $\text{tr}(\theta_i)$ in (2.16) vanishes identically, and hence $\widetilde{\theta}_i$ in (2.17) is θ_i itself. Now Proposition 2.5 completes the proof. □

3. Independence of Yang–Mills–Higgs Hermitian structure

As before, X is a compact connected Kähler manifold with $c_1(TX) = 0$, and ω is a Ricci–flat Kähler form on X. Let $(E\,,\theta)$ be a polystable Higgs vector bundle on X. Let h be a Hermitian structure on E satisfying the Yang–Mills–Higgs equation for $(E\,,\theta)$. We will continue to use the set-up of Section 2.

Lemma 3.1. *The decomposition in* (2.13) *is orthogonal with respect to the pulled back Hermitian structure* ϖ^*h *on* ϖ^*E*.*

Proof. The decomposition in (2.13) gives a decomposition of the Higgs vector bundle $(\varpi^*E\,,\varpi^*\theta)$

$$(\varpi^*E\,,\varpi^*\theta) \,=\, \bigoplus_{i=1}^{m}(F_i\,,\theta_i)\,,$$

where θ_i are constructed in (2.15). Recall that $(\varpi^*E\,,\varpi^*\theta)$ and all $(F_i\,,\theta_i)$ are polystable. If $\widetilde{h}_i$, $1 \,\leq\, i \,\leq\, m$, is a Hermitian structure on F_i satisfying the Yang–Mills–Higgs equation for $(F_i\,,\theta_i)$, then the Hermitian structure $\bigoplus_{i=1}^m \widetilde{h}_i$ on ϖ^*E, constructed using the decomposition in (2.13), clearly satisfies the Yang–Mills–Higgs equation for $(\varpi^*E\,,\varpi^*\theta)$.

Any two Hermitian structures on ϖ^*E that satisfy the Yang–Mills–Higgs equation for $(\varpi^*E\,,\varpi^*\theta)$, differ by a holomorphic automorphism of the Higgs vector bundle $(\varpi^*E\,,\varpi^*\theta)$ [Si1, p. 878, Theorem 1]. In particular, there is a holomorphic automorphism

$$T\,:\,\varpi^*E\,\longrightarrow\,\varpi^*E$$

such that $(T\otimes \mathrm{Id}_{\Omega_Y})\circ(\varpi^*\theta) \,=\, (\varpi^*\theta)\circ T$, and

$$\bigoplus_{i=1}^{m}\widetilde{h}_i(a\,,b) \,=\, \varpi^*h(T(a)\,,T(b))\,. \tag{3.1}$$

Therefore, the lemma follows once it is shown that any holomorphic automorphism of the Higgs vector bundle $(\varpi^*E\,,\varpi^*\theta)$ preserves the decomposition in (2.13). Note that the decomposition in (2.13) is orthogonal for the above Hermitian structure $\bigoplus_{i=1}^m \widetilde{h}_i$ on ϖ^*E. If the above automorphism T preserves the decomposition in (2.13), then from (3.1) it follows immediately that the decomposition in (2.13) is orthogonal with respect to ϖ^*h.

From the construction of the decomposition in (2.13) it follows that the m sections

$$\frac{1}{r_1}\mathrm{tr}(\theta_1)\,,\cdots\,,\frac{1}{r_m}\mathrm{tr}(\theta_m)\,\in\, H^0(Y,\,\Omega_Y)$$

in (2.16) and (2.17) are distinct; as mentioned just before (2.9), the elements $\{u_i^x\}_{i=1}^m$ are all distinct. Indeed, (2.13) is the generalized eigenspace decomposition for $\varpi^*\theta$, and $\frac{1}{r_1}\mathrm{tr}(\theta_1)\,,\cdots\,,\frac{1}{r_m}\mathrm{tr}(\theta_m)$ are the eigenvalues. It now follows that any automorphism of the Higgs vector bundle $(\varpi^*E\,,\varpi^*\theta)$ preserves the decomposition in (2.13). As observed earlier, this completes the proof. □

Lemma 3.2. *The section*

$$\theta \bigwedge \theta^* \in C^\infty(X, \operatorname{End}(E) \otimes \Omega_X^{1,1})$$

(see (2.2)*) vanishes identically.*

Proof. Consider θ_i defined in (2.15). From Proposition 2.5 it follows immediately that

$$\widetilde{\theta}_i \bigwedge \widetilde{\theta}_i^* = 0 . \tag{3.2}$$

Since the decomposition in (2.13) is orthogonal by Lemma 3.1, from (3.2) and (2.17) we conclude that

$$(\varpi^*\theta) \bigwedge (\varpi^*\theta^*) = 0 .$$

This implies that $\theta \wedge \theta^* = 0$. □

Theorem 3.3. *Let* (E, θ) *be a polystable Higgs vector bundle on* X *equipped with a Yang–Mills–Higgs structure* h*. Then* h *also satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle* $(E, 0)$.

Proof. In view of Lemma 3.2, this follows immediately from (2.2). □

Remark 3.4. It should be clarified that the converse of Theorem 3.3 is not valid. In other words, if h is an Hermitian structure on E satisfying the Yang–Mills–Higgs equation for $(E, 0)$, then h need not satisfy the Yang–Mills–Higgs equation for (E, θ). The reason for it is that the automorphism group of $(E, 0)$ is in general bigger than the automorphism group of (E, θ). To give an example, take X to be a complex elliptic curve equipped with a flat metric. Take E to be the trivial vector bundle $\mathcal{O}_X^{\oplus 2}$ on X of rank two. Let θ be the Higgs field on $\mathcal{O}_X^{\oplus 2}$ given by the matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} ;$$

fixing a trivialization of Ω_X, we identify the Higgs fields on $\mathcal{O}_X^{\oplus 2}$ with the 2×2 complex matrices. This Higgs vector bundle (E, θ) is polystable because the matrix A is semisimple. The Hermitian structure on $\mathcal{O}_X^{\oplus 2}$ given by the standard inner product on $\mathbb{C}^2$ satisfies the Yang–Mills–Higgs equation for $(E, 0)$, but this Hermitian structure does not satisfy Yang–Mills–Higgs equation for (E, θ) (because $AA^* \neq A^*A$).

Remark 3.5. Let (E, θ) be a polystable Higgs vector bundle on X. From Theorem 3.3 we know that the Higgs vector bundle $(E, 0)$ is polystable. Fix a Hermitian structure h_0 on E satisfying the Yang–Mills–Higgs equation for $(E, 0)$. Any other Hermitian structure on E that satisfies the Yang–Mills–Higgs equation for $(E, 0)$ differs from h_0 by a holomorphic automorphism of E. Take a holomorphic automorphism T of E such that the Hermitian structure $h := T^*h_0$ on E has the following property:

$$\theta \bigwedge \theta^{*_h} = 0 ,$$

where θ^{*_h} is the adjoint of θ constructed using h. From Lemma 3.2 and Theorem 3.3 it follows that such an automorphism T exists. The above Hermitian structure h satisfies the Yang–Mills–Higgs equation for $(E\,,\theta)$.

4. Polystable principal Higgs G–bundles

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. The Lie algebra of G will be denoted by $\mathfrak{g}$. As before, X is a compact connected Kähler manifold equipped with a Ricci–flat Kähler form ω. Let $E_G \longrightarrow X$ be a holomorphic principal G–bundle. Its adjoint vector bundle $E_G \times^G \mathfrak{g}$ will be denoted by $\text{ad}(E_G)$. A Higgs field on E_G is a holomorphic section

$$\theta \in H^0(X,\, \text{ad}(E_G) \otimes \Omega_X)$$

such that the section $\theta \bigwedge \theta$ of $\text{ad}(E_G) \bigotimes \Omega^2_X$ vanishes identically. A Higgs G–bundle on X is a pair of the form $(E_G\,,\theta)$, where E_G is a holomorphic principal G–bundle on X, and θ is a Higgs field on E_G.

Fix a maximal compact subgroup

$$K_G \subset G\,.$$

A Hermitian structure on a holomorphic principal G–bundle E_G on X is a C^∞ reduction of structure group of E_G

$$E_{K_G} \subset E_G$$

to the subgroup K_G. There is a unique C^∞ connection ∇ on the principal K_G–bundle E_{K_G} such that the connection on E_G induced by ∇ is compatible with the holomorphic structure of E_G [At, p. 191–192, Proposition 5]. Using the decomposition $\mathfrak{g} = \text{Lie}(K) \oplus \mathfrak{p}$, given any Higgs field θ on E_G, we have

$$\theta^* \in C^\infty(X;\, \text{ad}(E_G) \otimes \Omega^{0,1}_X)\,.$$

Let $(E_G\,,\theta)$ be a Higgs G–bundle on X. The center of the Lie algebra $\mathfrak{g}$ will be denoted by $z(\mathfrak{g})$. Since the adjoint action of G on $z(\mathfrak{g})$ is trivial, we have an injective homomorphism

$$\psi : X \times z(\mathfrak{g}) \hookrightarrow \text{ad}(E_G) \tag{4.1}$$

from the trivial vector bundle with fiber $z(\mathfrak{g})$. This homomorphism ψ produces an injective homomorphism

$$\widehat{\psi} : z(\mathfrak{g}) \hookrightarrow H^0(X,\, \text{ad}(E_G))\,.$$

A Hermitian structure $E_{K_G} \subset E_G$ is said to satisfy the Yang–Mills–Higgs equation for $(E_G\,,\theta)$ if there is an element $c \in z(\mathfrak{g})$ such that

$$\Lambda_\omega(\mathcal{K}(\nabla) + \theta \bigwedge \theta^*) = \widehat{\psi}(c)\,,$$

where $\mathcal{K}(\nabla)$ is the curvature of the connection ∇ associated to the reduction E_{K_G}, and θ^* is defined above.

It is known that $(E_G\,,\theta)$ admits a Yang–Mills–Higgs Hermitian structure if and only if $(E_G\,,\theta)$ is polystable [Si2], [BS, p. 554, Theorem 4.6]. (See [BS] for the definition of a polystable Higgs G–bundle.)

Lemma 4.1. *Let $(E_G\,,\theta)$ be a Higgs G–bundle on X equipped with an Hermitian structure $E_{K_G}\subset E_G$ that satisfies the Yang–Mills–Higgs equation for $(E_G\,,\theta)$. Then*

$$\theta\bigwedge\theta^*\,=\,0\,.$$

Proof. This follows by applying Lemma 3.2 to the Higgs vector bundle associated to $(E_G\,,\theta)$ for the adjoint action of G on $\mathfrak{g}$. Consider the adjoint Higgs vector bundle $(\mathrm{ad}(E_G)\,,\mathrm{ad}(\theta))$. The reduction E_{K_G} produces a Hermitian structure on the vector bundle $\mathrm{ad}(E_G)$ that satisfies the Yang–Mills–Higgs equation for $(\mathrm{ad}(E_G)\,,\mathrm{ad}(\theta))$. Now Lemma 3.2 says that

$$\mathrm{ad}(\theta)\bigwedge\mathrm{ad}(\theta)^*\,=\,0\,.$$

This immediately implies that the C^∞ section $\theta\bigwedge\theta^*$ of $\mathrm{ad}(E_G)\bigotimes\Omega^{1,1}_X$ is actually a section of $\psi(z(\mathfrak{g}))\bigotimes\Omega^{1,1}_X$, where ψ is the homomorphism in (4.1).

Take any holomorphic character $\chi\,:\,G\,\longrightarrow\,\mathbb{C}^*$. Let

$$L^\chi\,:=\,E_G\times^\chi\mathbb{C}\,\longrightarrow\,X$$

be the holomorphic line bundle associated to E_G for χ. The Higgs field θ defines a Higgs field on L^χ using the homomorphism of Lie algebras

$$d\chi\,:\,\mathfrak{g}\,\longrightarrow\,\mathbb{C} \tag{4.2}$$

associated to χ; this Higgs field on L^χ will be denoted by θ^χ. Since L^χ is a line bundle, we have $\theta^\chi\bigwedge(\theta^\chi)^*\,=\,0$ (Lemma 3.2 is not needed for this). As $\theta\bigwedge\theta^*$ is a section of $\psi(z(\mathfrak{g}))\bigotimes\Omega^{1,1}_X$, from this it can be deduced that $\theta\bigwedge\theta^*\,=\,0$. Indeed, given any nonzero element $v\,\in\,z(\mathfrak{g})$, there is a holomorphic character

$$\chi\,:\,G\,\longrightarrow\,\mathbb{C}^*$$

such that $d\chi(v)\,\neq\,0$ (defined in (4.2)). □

Theorem 4.2. *Let $(E_G\,,\theta)$ be a polystable Higgs G–bundle on X, and let $E_{K_G}\subset E_G$ be an Hermitian structure that satisfies the Yang–Mills–Higgs equation for $(E_G\,,\theta)$. Then the Hermitian structure $E_{K_G}\subset E_G$ also satisfies the Yang–Mills–Higgs equation for $(E_G\,,0)$.*

Proof. In view of the Yang–Mills–Higgs equation for $(E_G\,,\theta)$, this follows immediately from Lemma 4.1. □

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