Superlocalization formulas and supersymmetric Yang-Mills theories

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Abstract

By using supermanifold techniques we prove a generalization of the localization formula in equivariant cohomology which is suitable for studying supersymmetric Yang-Mills theories in terms of ADHM data. With these techniques one can compute the reduced partition functions of topological super Yang-Mills theory with 4, 8 or 16 supercharges. More generally, the superlocalization formula can be applied to any topological field theory in any number of dimensions.

1 Introduction

The study of nonperturbative effects in nonabelian supersymmetric gauge theories (SYM) has been the focus of much research activity in theoretical physics. In recent years, after the work of Seiberg and Witten [32, 33], many new results have been obtained and the techniques discussed in [32, 33] have also been used to shed light on analogous effects in string theory (see [25] for a review). Even if these techniques are very powerful, yet the derivation of the above mentioned results relies on a certain number of assumptions. This has triggered the interest of some authors to check these results by independent methods. A direct evaluation of nonperturbative effects is in fact possible by computing the partition function or the relevant correlators of the theory of interest. It has been a pleasant surprise of the last year that such a computation can indeed be carried out in some cases for arbitrary values of the instanton winding number.

This is the end point of a long journey that we will very briefly summarize: following the pioneering works [1, 34] in which condensates for SYM theories were computed, in [15, 16, 19] a first check for the prepotential of the SYM with eight fermionic charges was performed for instantons of winding number two. This computations were performed at the semiclassical level and agreed with the prepotential of [32, 33]. This opened the way to a reformulation in the framework of topological theories [7, 6]. The full exploitation of the power of this observation was hindered by the presence of constrained quantities in the functional integral (see [14] for a detailed account on this point). After this point was dealt with, it was possible to show that the measure of the functional integral could be written as a total derivative [12]. The derivative operator is found to be the BRST operator of the SYM. Even if the computation was then reduced to the evaluation of a boundary term there were still some issues to be settled concerning the compactness of the domain of integration. This was done for the first time in the case of instantons of winding number two [21, 22], by localization on certain manifolds of given dimensions. The computation for arbitrary winding number in the case of eight supersymmetries with matter in the fundamental and adjoint representation was presented in [31]. In turn this work stemmed from a line of research focused on computing topological invariants with the aid of topological field theories [26, 29, 28].

An interesting alternative approach to the computation in [31] appeared in [17], where use was made of the localization formula in equivariant cohomology (for this formula see e.g. [10]). In order to use that formula, the action of the SYM under study must be reinterpreted as a form over the instanton moduli space. This is quite natural for SYM theories with eight charges [36], where the fermionic moduli can be interpreted as differential forms on the instanton moduli space. In the case of instantons of arbitrary winding number, this approach can be conveniently implemented by using the ADHM contruction [2], as it was shown in [7, 6].

In this paper we want to show how by using supermanifold techniques one can write a (mild) generalization of the localization formula which is suitable for implementing this approach to the computation of the partition function to SYM theories with any number of charges (supermanifold techniques were already used in [12] to retrieve the measure of the partition function of SYM). The structure of the paper is as follow. In Section 2 we briefly recall the basics of equivariant cohomology and the localization formula. In Section 3 we spell out what are the bundles on the instanton moduli space that are relevant to different numbers of supersymmetry charges. In Section 4 we review the basics of supermanifold theory, paying special attention to the supermanifolds defined in terms of the cotangent bundle to a differentiable manifolds (the "tautological supermanifolds"). In Section 5 we describe our generalization of the superlocalization formula; this stems from the identification of the BRST operator with a suitable equivariant cohomology operator, which in the case of tautological supermanifolds (i.e., in the $\mathcal{N}=2$ case) is the usual equivariant exterior differential. Finally in Section 6 we treat in some detail the $\mathcal{N}=2$ case, also explaining how to use the superlocalization formula in the presence of constraints.

One can notice that these results can in fact be applied to any supersymmetric theory, whose Lagrangian can be written as the BRST variation of some function, in any number of dimensions.

2 Equivariant cohomology

We recall the basic formulas in equivariant cohomology. Let M an n-dimensional differentiable manifold, and G a Lie group with Lie algebra $\mathfrak{g}$ and an action ρ on M. To any $\xi \in \mathfrak{g}$ one associates the fundamental vector field (on M)

$$\xi^* = \left[\frac{d}{dt}\rho_{(-t\exp\xi)}\right]_{t=0}$$

which we shall write in local coordinates as

$$\xi^* = \xi^\alpha\, T^i_\alpha \frac{\partial}{\partial x^i}\,.$$

If $\mathbb{C}[\mathfrak{g}]$ is the algebra of polynomial $\mathbb{C}$-valued functions on $\mathfrak{g}$, and $\Omega(M)$ is the algebra of differential forms on M, the product algebra $\Omega(M,\mathfrak{g}) = \mathbb{C}[\mathfrak{g}]\otimes\Omega(M)$ has a natural grading, defined by

$$\deg(P \otimes \beta) = 2\deg(P) + \deg(\beta)$$

if P and β are homogeneous. The group G acts on $\Omega(M, \mathfrak{g})$ as

$$(g \cdot \alpha)(\xi) = {\rho_g}^*(\alpha(Ad_{g^{-1}}\xi)). \tag{1}$$

Moreover one defines the *equivariant differential*

$$d_{\mathfrak{g}} \colon \Omega(M, \mathfrak{g})^\bullet \to \Omega(M, \mathfrak{g})^{\bullet+1}, \qquad (d_{\mathfrak{g}}\alpha)(\xi) = d(\alpha(\xi)) - i_{\xi^*}\alpha(\xi);$$

one has

$$({d_{\mathfrak{g}}}^2\alpha)(\xi) = -\mathcal{L}_{\xi^*}(\alpha(\xi)) \tag{2}$$

where $\mathcal{L}$ denotes the Lie derivative. The elements of the invariant subalgebra $\Omega_G(M) = (\Omega(M, \mathfrak{g}))^G$ are called *equivariant forms*; in view of eq. (2), on them the equivariant differential squares to zero, allowing the definition of the *equivariant cohomology* of M as $H^\bullet(M, G) = H^\bullet(\Omega_G(M), d_{\mathfrak{g}})$.

If M is compact, and α is an equivariantly closed element in $\Omega_G(M)$, and $\xi \in \mathfrak{g}$, one denotes by $\int_M \alpha(\xi)$ the integral of the piece of $\alpha(\xi)$ which is an n-form ($n = \dim M$), in the usual gradation of $\Omega(M)$. This integral can be nicely evaluated by using a *localization formula*. For every $\xi \in \mathfrak{g}$ let $M_\xi \subset M$ be the zero-set of ξ^*. If G is compact, and the zeroes of ξ^* are isolated (so that M_ξ is finite), one has

$$\int_M \alpha(\xi) = (-2\pi)^{n/2} \sum_{p \in M_\xi} \frac{\alpha(\xi)_0(p)}{\det^{1/2} L_{p,\xi}} \tag{3}$$

where $\alpha(\xi)_0$ is the 0-form part of $\alpha(\xi)$, and $L_{p,\xi} \colon T_pM \to T_pM$ is the linear operator defined as

$$L_{p,\xi}(v) = [\xi^*, v]$$

(this is well defined since $\xi^*(p) = 0$).

3 Supersymmetry and vector bundles over the instanton moduli space

If M is a Kähler (or hyperkähler) surface, and $\mathcal{M}$ is the moduli space of instantons on M, via index-theoretic constructions one can define some vector bundles on $\mathcal{M}$. In this section we briefly review the construction of these bundles and recall how they are involved in the study of SYM with $\mathcal{N} = 1, 2$ or 4 supersymmetries.

We start by considering a Kähler surface M. If E is an $SU(N)$ vector bundle on M, with $c_2(E) = k$, there is a smooth space $\mathcal{M}$ parametrizing equivalence classes of irreducible anti-self-dual connections on E (instantons) modulo gauge equivalence (some subtleties are involved here, e.g. the fact that the Kähler metric in M should be generic in a suitable sense, but these issues will be ignored here. A good reference on this topic is [13].)

The choice of a spin structure in M amounts to a choice of a square root L of the canonical bundle K, i.e., a complex line bundle L such that $L^2 \simeq K$ (the canonical bundle K is the bundle of holomorphic 2-forms on M). We shall consider L as a smooth bundle (i.e., we also allow for smooth sections rather than just holomorphic). The spin bundles S_+, S_- (i.e., the rank-two complex bundles of undotted and dotted spinors, respectively) may be taken as

$$S_- = [\Omega^{0,0} \oplus \Omega^{0,2}] \otimes L\,, \qquad S_+ = \Omega^{0,1} \otimes L\,.$$

It this not difficult to show that $\Omega^{0,2} \otimes L \simeq L^*$, so that S_- splits as a direct sum of orthogonal (in the Kähler metric) subbundles of rank 1, $S_- \simeq L \oplus L^*$.

One trivially has $S_+ \otimes L^* \simeq \Omega^{0,1}$; moreover, since $K \simeq \det \Omega^{1,0}$ if we consider smooth bundles, one has $\Omega^{0,1} \otimes K \simeq \Omega^{1,0}$, hence $S_+ \otimes L \simeq \Omega^{1,0}$. Let us describe these isomorphisms in component notation. Let $(e^0, \dots, e^3)$ be a local orthornomal frame of 1-forms, and consider the associated local bases of sections of the spin bundles S_+, S_-. One has an isomorphism $S_+ \otimes S_- \simeq T^*M \otimes \mathbb{C}$, which is given by

$$\psi \otimes \chi \mapsto \sigma_{m\alpha\dot\beta}\, \psi^\alpha\, \chi^{\dot\beta}\, e^m\,,$$

where $\sigma_{m\alpha\dot\beta}$ are the Pauli matrices. Thus the isomorphisms $S_+ \otimes L \simeq \Omega^{1,0}$, $S_+ \otimes L^* \simeq \Omega^{0,1}$ express the fact that the forms

$$\sigma_{m\alpha\dot 1}\, e^m\,, \qquad \sigma_{m\alpha\dot 2}\, e^m$$

are of type (1,0) and (0,1), respectively. All this is explained in more "physical" terms in [37, 23].

$\mathcal{N} = 2$ **supersymmetry.** Let E be an $SU(N)$ bundle on M, and let $\mathcal{M}$ be the moduli space of instantons on E. For every $m \in \mathcal{M}$ we have a twisted Dirac operator

$$D_m \colon S_+ \otimes S_- \otimes \mathrm{ad}(E) \to S_- \otimes S_- \otimes \mathrm{ad}(E)$$

(where $\mathrm{ad}(E)$ is the adjoint bundle of E, i.e., the bundle of trace-free endomorphisms of E and the global symmetry group of the $\mathcal{N} = 2$ theory is identified with the $SU(2)$ structure group S_-), together with the adjoint operator D_m^*. The assignment

$$m \in \mathcal{M} \quad \rightsquigarrow \quad \ker D_m - \operatorname{coker} D_m = \ker D_m - \ker D_m^*$$

defines the index $\operatorname{ind}(D)$ as a class in the topological K-theory of $\mathcal{M}$. Under our assumptions (in particular, we consider only irreducible instantons) we have $\ker D_m^* = 0$ for all $m \in \mathcal{M}$, so that $\operatorname{ind}(D)$ is a vector bundle, that we denote by $\mathcal{S}_2$. The fibre of $\mathcal{S}_2$ at the point $m \in \mathcal{M}$ is the vector space $\ker D_m$. This (complex) vector bundle may be naturally identified with the complexified tangent bundle to $\mathcal{M}$ [3], and one has $\operatorname{rk} \mathcal{S}_2 = \dim_{\mathbb{R}} \mathcal{M}$ (which equals $4kn$ in the case of framed $SU(N)$ instantons on $\mathbb{R}^4$). Since we have considered the full spin bundle S_+, this is the case relevant to $\mathcal{N} = 2$ supersymmetry.

$\mathcal{N} = 1$ **supersymmetry.** In the case $\mathcal{N} = 1$ one considers the subbundle L instead of the full bundle S_+. In this way we get a vector bundle $\mathcal{S}_1$ on $\mathcal{M}$ with $\operatorname{rk} \mathcal{S}_1 = \frac{1}{2} \dim \mathcal{M}$. One can notice that the orthogonal splitting $S_- = L \oplus L^*$ implies that the complexified tangent bundle to the instanton moduli space $T\mathcal{M} \otimes \mathbb{C} \simeq \mathcal{S}_2$ splits as a direct sum of two vector bundles, $T\mathcal{M} \otimes \mathbb{C} \simeq \mathcal{S}_1 \oplus \mathcal{S}_1'$, which are dual to the bundles of differential forms of type (1,0) and (0,1) on $\mathcal{M}$. One has $\operatorname{rk} \mathcal{S}_1 = 2kN$ for the framed instantons.

$\mathcal{N} = 4$ **supersymmetry.** In this case we have a family of Dirac operators

$$D \colon S_+ \otimes \Sigma \otimes \operatorname{ad}(E) \to S_- \otimes \Sigma \otimes \operatorname{ad}(E)$$

where Σ is a bundle with structure group $SU(4)$.[1] For framed instantons the resulting bundle on $\mathcal{M}$ has rank $8kN$.

4 Supermanifolds

We want to introduce some supermanifolds whose "bosonic" part is the instanton moduli space $\mathcal{M}$; this will be done by using the vector bundles on $\mathcal{M}$ that we have introduced in the previous section. We start by briefly recalling the basic definitions about supermanifolds. (We consider supermanifolds in the sense of Berezin-Leĭtes and Konstant [9, 24]).

An (m, n) dimensional supermanifold $\mathfrak{M}$ is a pair $(M, \mathcal{A})$, where M is an m-dimensional manifold, and $\mathcal{A}$ is a sheaf of $\mathbb{Z}_2$-graded commutative algebras on M, such that:

1. there is morphism of sheaves of $\mathbb{R}$-algebras $\varepsilon \colon \mathcal{A} \to \mathcal{C}_M^\infty$;

[1] However as a consequence of the twisting of the theory [35, 27, 38] the structure group is reduced to a little group. Of the three possibilities listed in [35, 27, 38] we discuss in [11] the so-called $SU(2) \times SU(2) \times U(1)$ twisting, where the structure group is actually reduced to $S(U(2) \times U(2)) \subset SU(4)$.

2. if $\mathcal{N}$ is the nilpotent subsheaf of $\mathcal{A}$, the quotient $\mathcal{N}/\mathcal{N}^2$ is a locally free sheaf (vector bundle) E of rank n;

3. $\mathcal{A}$ is locally isomorphic to the exterior algebra sheaf $\Lambda^\bullet\mathcal{E}$, and this isomorphism is compatible with the morphism ε.

These conditions imply $\ker\varepsilon = \mathcal{N}$. Also, condition 3 implies that ε is surjective (as a sheaf morphism).

If we start from a rank n vector bundle E on M we may construct an (m,n)-dimensional supermanifold $(M,\mathcal{A})$ by letting $\mathcal{A} = \Lambda^\bullet\mathcal{E}$, with ε the natural projection $\Lambda^\bullet E \to \mathcal{E}$. Conversely, standard arguments in deformation theory allow one to prove that any supermanifold is *globally* isomorphic to a supermanifold of this kind [5].

If $(x^1,\dots,x^m)$ are local coordinates in M, and $(\theta^1,\dots,\theta^n)$ is a local basis of sections of $\mathcal{E}$, then the collection $(x^1,\dots,x^m,\theta^1,\dots,\theta^n)$ is a local coordinate chart for $\mathfrak{M}$. According to the requirements above, a section of $\mathcal{A}$ (i.e., a superfunction on $\mathfrak{M}$) has a local expression

$$f = \sum_{k=1}^{n} f_{[k]} \tag{4}$$

where

$$f_{[k]} = \sum_{\alpha_1\dots\alpha_k=1\dots n} f_{\alpha_1\dots\alpha_k}(x)\,\theta^{\alpha_1}\cdots\theta^{\alpha_k}$$

with $f_0 = \varepsilon(f)$. Thus we get the well-known superfield expansion.

If $\mathcal{A}_p$ is the stalk of $\mathcal{A}$ at $p\in M$ (i.e., the algebra of germs of sections of $\mathcal{A}$ at p), then

$$\mathcal{I}_p = \ker\varepsilon\colon \mathcal{A}_p \to (\mathcal{C}^\infty_M)_p$$

is a maximal graded ideal of $\mathcal{A}_p$. The *tangent superbundle* is by definition the sheaf $\mathcal{D}er_{\mathbb{R}}\mathcal{A}$ of graded derivations of $\mathcal{A}$. The tangent superspace $T_p\mathfrak{M}$ at a point $p\in M$ is by definition the (m,n)-dimensional graded vector space

$$T_p\mathfrak{M} = \mathcal{D}er_{\mathbb{R}}\mathcal{A}/\mathcal{I}_p\cdot\mathcal{D}er_{\mathbb{R}}\mathcal{A}$$

and one has a canonical isomorphism $T_p\mathfrak{M}\simeq T_pM\oplus\mathcal{E}^*_p$ where $\mathcal{E}^*_p$ is the fibre at p of the dual vector bundle $\mathcal{E}^*$, or, in terms of superbundles,

$$T\mathfrak{M}\simeq\mathcal{A}\otimes[TM\oplus\mathcal{E}^*]\,.$$

Tautological supermanifolds. Our strategy for the computation of the partition function of the topological SYM theory will involve considering supermanifolds based on the instanton moduli space $\mathcal{M}$, associated with the vector bundles $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_4$ we have introduced in the previous section. Since we shall study in detail the case of $\mathcal{N} = 2$ supersymmetry, and since $\mathcal{S}_2$ may be identified with the tangent bundle to $\mathcal{M}$, we shall develop in some detail the case where the vector bundle to which a supermanifold is associated with is the cotangent bundle (we choose the cotangent rather than the tangent bundle for mere reasons of convenience).

So, if given a supermanifold $\mathfrak{M} = (M, \mathcal{A})$, if the associated vector bundle $\mathcal{E}$ on M is isomorphic to the cotangent bundle T^*M, we say that $\mathfrak{M}$ is *tautologically associated with* M. If the isomorphism $\mathcal{A} \simeq \Lambda^\bullet T^*M$ has been fixed, one has a canonical isomorphism

$$T\mathfrak{M} \simeq \mathcal{A} \otimes [TM \oplus TM] \tag{5}$$

(cf. [12]), and there is a naturally defined involution Π on $T\mathfrak{M}$, exchanging the two summands.

Superfunctions on $\mathfrak{M}$ are just differential forms on M; one has an isomorphism (of sheaves of $\mathcal{C}^\infty_M$-modules) $\tau\colon \Omega^\bullet \to \mathcal{A}$. Note that if $(x^1, \ldots, x^m)$ are local coordinates in M, and we set $\theta^i = \tau(dx^i)$, then $(x^1, \ldots, x^m, \theta^1, \ldots, \theta^m)$ is a local coordinate system for $\mathfrak{M}$ (and of course, $d\theta^i \neq 0$!).

Berezin integration.[2] Let $\mathfrak{M} = (M, \mathcal{A})$ be an (m, n) dimensional supermanifold, with M an oriented manifold, and denote by $\Omega^m_{\mathfrak{M}}$ the sheaf of super differential m-forms on $\mathfrak{M}$, and by $\mathcal{P}_n$ the sheaf of graded differential operators of order n on $\mathcal{A}$. The sheaf $\Omega^m_{\mathfrak{M}}$ has its natural structure of graded left $\mathcal{A}$-module given by multiplication of forms by functions; the sheaf $\mathcal{P}_n$ has an an analogous graded left $\mathcal{A}$-module structure, but also has a (inequivalent) right $\mathcal{A}$-module structure, given by

$$(D \cdot f)(g) = D(fg)$$

where f, g are superfunctions. We consider on $\mathcal{P}_n$ this module structure, and take the graded tensor product $\Omega^m_{\mathfrak{M}} \otimes_{\mathcal{A}} \mathcal{P}_n$.

The structural morphism $\varepsilon\colon \mathcal{A} \to \mathcal{C}^\infty_M$ extends to a morphism $\Omega^m_{\mathfrak{M}} \to \Omega^m_M$, whose action we denote by a tilde. The sheaf $\Omega^m_{\mathfrak{M}} \otimes_{\mathcal{A}} \mathcal{P}_n$ has a subsheaf $\mathcal{K}$ whose sections ω are such that the differential m-form $\widetilde{\omega(f)}$ on M is exact for every choice of a superfunction f with compact support (more precisely, $\widetilde{\omega(f)} = d\eta$ for a compactly supported $(m-1)$-form η on M). The quotient $\Omega^m_{\mathfrak{M}} \otimes_{\mathcal{A}} \mathcal{P}_n/\mathcal{K}$ is denoted by $\mathcal{B}er(\mathfrak{M})$ and is called the *Berezinian sheaf* of $\mathfrak{M}$. It is a locally free graded $\mathcal{A}$-module of rank (1,0) if n is even, rank (0,1) is n is odd.

[2]This approach to the Berezin integral is taken from [20].

On its compactly supported sections one can define an integral (the Berezin integral) by letting

$$\int_{\mathfrak{M}} \omega = \int_M \widetilde{\lambda(1)}$$

where λ is any section of $\Omega^m_{\mathfrak{M}} \otimes_{\mathcal{A}} \mathcal{P}_n$ whose class in the quotient $\Omega^m_{\mathfrak{M}} \otimes_{\mathcal{A}} \mathcal{P}_n/\mathcal{K}$ is $[\lambda] = \omega$. This integral performs the usual procedure of "integrating over the fermions": indeed, given a local coordinate system $(x^1, \ldots, x^m, \theta^1, \ldots, \theta^n)$ defined in a patch U, one has

$$\omega_{|U} = \left[dx^1 \wedge \ldots \wedge dx^m \otimes \frac{\partial}{\partial\theta^1} \cdots \frac{\partial}{\partial\theta^n} \right] f$$

for a superfunction $f \in \mathcal{A}(U)$, and if ω is supported in U, one has

$$\int_{\mathfrak{M}} \omega = \int_M f_{[n]}\, dx^1 \ldots dx^m$$

i.e. the Berezin integral is the usual integral over M of the last term in the superfield expansion eq. (4) of the component superfunction f, and corresponds to the usual operation of "integrating over the fermions" in quantum field theory.

The Berezinian bundle of a 'tautological' supermanifold $\mathfrak{M}$ has a canonical global section Θ. If Θ_0 is a global nowhere vanishing differential n-form on M, and Δ is a dual derivation of order n (i.e., $\Delta(\Theta_0) = 1$), the class $\Theta = [\Theta_0 \otimes \Delta]$ is a well-defined global section of $\mathcal{B}er(\mathfrak{M})$, independent of the choices of Θ_0 and Δ. In local coordinates (x, θ), where $\theta^i = dx^i$, one has

$$\Theta = \left[dx^1 \wedge \ldots \wedge dx^n \otimes \frac{\partial}{\partial\theta^1} \cdots \frac{\partial}{\partial\theta^n} \right] .$$

If η is an n-form on M, one has

$$\int_{\mathfrak{M}} \Theta\, \tau(\eta) = \int_M \eta\, .$$

5 BRST transformations and superlocalization formulas

Let $\mathfrak{M} = (M, \mathcal{A})$ be an (m, n)-dimensional supermanifold, with $\mathcal{A}$ the sheaf of sections of the exterior algebra bundle of a rank n vector bundle $\mathcal{E}$ on M. Assume that there is an action ρ of a Lie group G on M, and that G also acts on $\mathcal{E}$ by a linear action $\hat{\rho}$ in such a way that the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\hat\rho_g} & \mathcal{E} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\rho_g} & M \end{array}$$

commutes for all $g \in G$. The action $\hat\rho$ induces a vector field $\hat\xi^*$ on $\mathcal{E}$:

$$\hat\xi^* = \left[\frac{d}{dt}\hat\rho_{\exp(-t\xi)}\right]_{t=0} .$$

If $(x^1, \dots, x^m)$ are local coordinates on M, and $(\theta^1, \dots, \theta^n)$ are a local basis of sections of $\mathcal{E}$ (so that they can be regarded as local fibre coordinates on $\mathcal{E}^*$), $\hat\xi^*$ is locally written as

$$\hat\xi^* = \xi^\alpha\, T^i_\alpha \,\frac{\partial}{\partial x^i} + \xi^\alpha\, \theta^B\, U^A_{\alpha B}\, \frac{\partial}{\partial \theta^A} \qquad (6)$$

where the functions $\xi^\alpha\, T^i_\alpha$ are the local components of the generator ξ^* of the action of G on M. The vector field (6) can be regarded as an even super vector field $\hat\xi^*$ on $\mathfrak{M}$ representing the induced infinitesimal action of G on $\mathfrak{M}$.

If $p \in M$ is a zero of ξ^*, then an endomorphism $\tilde L_{\xi,p}$ of the fibre $\mathcal{E}_p$ rests defined. We may regard this as an even endomorphism $\mathbf{L}_{\xi,p}$ of the cotangent superspace $T^*_p\mathfrak{M}$, defined as $(1, L)$ after identifying $T^*_p\mathfrak{M} \simeq T^*_pM \oplus \mathcal{E}_p$.

We shall consider the algebra $\mathbb{C}[\mathfrak{g}]\otimes\mathcal{A}(M)$ (where $\mathcal{A}(M)$ is the algebra of global sections of $\mathcal{A}$), which carries the action of G given by

$$(g \cdot \alpha)(\xi) = \hat\rho_g(\alpha(\mathrm{Ad}_{g^{-1}}\xi))$$

where by abuse of notation we denote by $\hat\rho$ the induced action of G on $\mathfrak{M}$. On $\mathbb{C}[\mathfrak{g}]\otimes\mathcal{A}(M)$ one considers the $\mathbb{Z}$-grading

$$\deg(P \otimes f) = 2\deg(P) + \deg(f)$$

(where $\deg(P)$ is the degree of the polynomial $P \in \mathbb{C}[\mathfrak{g}]$ and $\deg(f) = k$ if $f = f_{[k]}$) and the $\mathbb{Z}_2$-grading given by the grading of $\mathcal{A}$ (the terms "even" and "odd") will refer to this grading). We shall denote by $\mathfrak{A}_G$ the subalgebra of $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ formed by G-invariant elements.

Definition 5.1 *A BRST operator is an odd derivation Q of $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ of $\mathbb{Z}$-degree 1 such that*

1. $(Q^2F)(\xi) = \hat\xi^*(F)$ *for all* $F \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ *and* $\xi \in \mathfrak{g}$;

2. Q is equivariant, i.e., $Q \circ g = g \circ Q$ for all $g \in G$;

3. The equivariant morphism $\sigma_Q \colon \mathcal{E}^* \to TM$ defined by

$$\sigma_Q(v)(f) = i_v(Q(f)) \qquad \text{for all functions } f \text{ on } M$$

is injective.

Remark 5.2 The third condition fails in the case relevant to $\mathcal{N} = 4$ supersymmetry, and should rather be replaced by the condition that $\mathcal{E}$ is a direct sum in such a way that σ_Q is injective after restriction to any of the summands of $\mathcal{E}^*$. However for the sake of simplicity we shall stick to this condition in its present form.

In particular, this implies that $Q^2_{|\mathfrak{A}_G} = 0$, so that an equivariant cohomology $H^\bullet(\mathfrak{A}_G, Q)$ is defined. Moreover, for every $\xi \in \mathfrak{g}$ one can define an odd supervector field Q_ξ by letting $Q_\xi(F(\xi)) = Q(F)(\xi)$ for all $F \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$. In terms of this supervector field the first requirement in Definition 5.1 reads

$$[Q_\xi, Q_\xi] = 2\hat{\xi}^* \tag{7}$$

where $[\,,\,]$ is the graded commutator of supervector fields (in this case an anticommutator in fact). Let us, for future use, write this equation in local components. After writing

$$Q_\xi = a^i \frac{\partial}{\partial x^i} + b^A \frac{\partial}{\partial \theta^A},$$

eq. (7) is equivalent to the conditions

$$a^i = \sigma^i_A \, \theta^A, \qquad b^A \, \sigma^i_A = \xi^\alpha \, T^i_\alpha \tag{8}$$

where σ^i_A is the matrix representing the morphism $\sigma^*_Q \colon T^*M \to \mathcal{E}$, i.e., $\sigma^*_Q(dx^i) = \sigma^i_A \, \theta^A$.

Lemma 5.3 *If $p \in M$ is a zero of ξ^* for an element $\xi \in \mathfrak{g}$, the diagram*

$$\begin{array}{ccc} T^*_p M & \xrightarrow{\sigma^*_Q} & \mathcal{E}_p \\ \downarrow{\scriptstyle \mathcal{L}_{p,\xi}} & & \downarrow{\scriptstyle \tilde{L}_{\xi,p}} \\ T^*_p M & \xrightarrow{\sigma^*_Q} & \mathcal{E}_p \end{array} \tag{9}$$

commutes.

Proof. The commutativity of the diagram is equivalent to the infinitesimal equivariance of the morphism σ_Q. □

Finally, we require the existence of a G-invariant Riemannian metric h on M. Since the morphism σ_Q is injective this also defines a G-invariant fibre metric H on $\mathcal{E}$ by letting $H(u,v) = h(\sigma_Q(u), \sigma_Q(v))$ for all $u, v \in \mathcal{E}^*$. Using the metrics h and H one can construct a G-invariant global section of the Berezinian sheaf $\mathrm{Ber}(\mathfrak{M})$, whose local coordinate expression is

$$\Theta = \left[dx^1 \wedge \ldots \wedge dx^m \otimes \frac{\partial}{\partial \theta^1} \cdots \frac{\partial}{\partial \theta^n} \right] \frac{\det^{1/2}(h)}{\det^{1/2}(H)} .$$

Actually, this Berezinian measure does not depend on the metrics but only on the BRST operator Q via the morphism σ_Q.

Assumption 5.4 *The morphism σ_Q, regarded as a section of the bundle $\mathcal{E} \otimes TM$, is parallel with respect to the connection induced by the metrics h and H.*

Lemma 5.5 *Let ν be a superfunction which is homogeneous of degree $n-1$, i.e., $\nu = \nu_{[n-1]}$, and let $*_H \nu$ be the section of $\mathcal{E}^*$ which is Hodge dual to ν via the metric H, i.e., in local coordinates,*

$$(*_H \nu)^A = \frac{1}{(n-1)!} (\det(H))^{-1/2} \, \varepsilon^{A A_2 \ldots A_n} \, \nu_{A_2 \ldots A_n} \qquad \text{if} \qquad \nu = \nu_{A_1 \ldots A_{n-1}} \, \theta^{A_1} \ldots \theta^{A_{n-1}} ,$$

where $\varepsilon^{A_1 \ldots A_n}$ is the completely antisymmetric symbol. Then, for every regular domain $U \subset M$ with compact closure,

$$\int_{\mathfrak{M}|U} \Theta \, Q_\xi(\nu) = \int_{\partial U} *_h \sigma_Q(*_H \nu)) \, ;$$

here ∂U is equipped with the induced orientation, and $*_h$ is Hodge duality in M.

Proof. The equality is proved by direct computation. It is necessary to use the Assumption 5.4. □

We can now state the localization formula. Let Q be a BRST operator.

Theorem 5.6 *Let M and G be compact, let $F \in \mathfrak{A}_G$ be such that $Q(F) = 0$, and assume that $\xi \in \mathfrak{g}$ is such that ξ^* only has isolated zeroes. Then,*

$$\int_{\mathfrak{M}} \Theta \, F(\xi) = \frac{(-2)^{n/2} (n/2)! \pi^{m/2}}{(m/2)!} \sum_{p \in M_\xi} \mathrm{Sdet}^{1/2}(\mathbf{L}_{p,\xi}) \, F(\xi)_0(p) \qquad (10)$$

where $\mathrm{Sdet}(\mathbf{L}_{p,\xi})$ is the superdeterminant (Berezinian determinant) of the even endomorphism $\mathbf{L}_{p,\xi}$ (cf. [8, 24, 4]).

The proof of this localization formula follows the pattern of the proof of the usual formula, cf. [10]. So we need the following preliminary results. We assume that M and G are both compact, and an element $\xi \in \mathfrak{g}$ such that ξ^* has isolated zeroes has been fixed.

Lemma 5.7 *There exists a superfunction β (actually, a section of $\mathcal{E}$) such that*

1. $\hat{\xi}^*(\beta) = 0$*;*
2. $Q_\xi(\beta)$ *is invertible outside* M_ξ*;*
3. *Every $p \in M_\xi$ has a neighbourhood on which the function $H(\beta, \beta)$ equals the square distance from the point p.*

Proof. One can construct a differential 1-form λ on M such that $\mathcal{L}_{\xi^*}(\lambda) = 0$ and $\lambda(\xi^*) = d_p^2$, where $d_p(x)$ is the distance of x from p in the metric h [10]. The section $\beta = \sigma_Q^*(\lambda)$ of $\mathcal{E}$ satisfies the required conditions. □

Lemma 5.8 *The superfunction $F(\xi)_{[n]}$ is Q-exact outside M_ξ, i.e., there is a section ν of $\mathcal{A}$ on $M \setminus M_\xi$ such that*

$$F(\xi)_{[n]|M\setminus M_\xi} = Q_\xi(\nu)\,.$$

Proof. In view of the previous Lemma, outside M_ξ we may set

$$\nu = \left(\beta\, F(\xi)\, Q_\xi(\beta)^{-1}\right)_{[n-1]}\,,$$

and again using the previous Lemma, one gets the desired equality. □

Proof of Theorem 5.6. For every $p \in M_\xi$ let $B_\epsilon(p)$ be the ball of radius ϵ (measured with the metric h) around p, and denote by $\mathfrak{M}_\epsilon$ the supermanifold $\mathfrak{M}$ restricted to the complement of the union of the closures of the balls $B_\epsilon(p)$. Then, using Lemma 5.5,

$$\int_{\mathfrak{M}} \Theta\, F(\xi) = \lim_{\epsilon\to 0} \int_{\mathfrak{M}_\epsilon} \Theta\, Q_\xi(\nu) = -\lim_{\epsilon\to 0} \sum_{p\in M_\xi} \int_{S_\epsilon(p)} *_h\sigma_Q(*_H\nu)$$

where $S_\epsilon(p)$ is the boundary of $B_\epsilon(p)$. Under the rescaling

$$x \mapsto \epsilon^{1/2}\, x, \qquad \theta \mapsto \epsilon^{1/2}\, \theta$$

the term $\mu = \beta\, Q_\xi(\beta)^{-1}$ is homogeneous of degree zero, so that we get

$$\int_{\mathfrak{M}} \Theta\, F(\xi) = -\sum_{p\in M_\xi} F(\xi)_0(p) \int_{S_1(p)} *_h\sigma_Q(*_H\mu_{[n-1]})\,.$$

The integrals in the r.h.s., again using Lemma 5.5, may be recast as Berezin integrals over the supermanifolds $\mathfrak{M}|B_1(p)$. These may be evaluated by writing their integrands in local coordinates, obtaining

$$(-1)^{n/2} \int_{\mathfrak{M}|B_1(p)} \Theta \left[\sum_{A,B=1}^{n} a_{AB}(p)\, \theta^A\, \theta^B \right]^{n/2}$$

where $a_{AB}(p)$ is a skew-symmetric matrix of constants which represents the morphism $\tilde{L}_{p,\xi}$ (with an index lowered with the metric H). From this we get (cf. e.g. [39])

$$\int_{\mathfrak{M}} \Theta\, F(\xi) = (-2)^{n/2}(n/2)! \sum_{p\in M_\xi} F(\xi)_0(p)\, \mathrm{Pf}(a(p)) \int_{B_1(p)} \mathrm{vol}(h)$$

where $\mathrm{vol}(h)$ is the Riemannian volume form, and $\mathrm{Pf}(a(p))$ is the Pfaffian of the matrix $a(p)$. Since

$$\mathrm{Pf}(a(p)) = \mathrm{Sdet}^{1/2}(\mathbf{L}_{p,\xi}) \qquad \text{and} \qquad \int_{B_1(p)} \mathrm{vol}(h) = \frac{\pi^{m/2}}{(m/2)!}$$

we eventually obtain the superlocalization formula.

Example 5.9 The simplest example is provided by the tautological supermanifolds; then $\mathcal{E} = T^*M$ and for every $\xi \in \mathfrak{g}$ the vector field Q_ξ is

$$Q_\xi = d + \Pi(\xi^*, 0)$$

where the exterior differential d is regarded as an odd supervector field on $\mathfrak{A}$, and Π is the morphism which interchanges the two summands in eq. (5). σ turns out to be the identity morphism. The superlocalization formula reduces to the usual localization formula (eq. (3)); note that $\mathrm{Sdet}^{1/2}(\mathbf{L}_{p,\xi}) = \det^{-1/2}(L_{p,\xi})$. In this case the isomorphism $\tau\colon \Omega^\bullet \to \mathcal{A}$ intertwines the equivariant differential with the BRST operator Q. This is the suitable framework for $\mathcal{N} = 2$ supersymmetry.

Example 5.10 Let M be a Kähler manifold of even complex dimension m, with Kähler form ω, and assume that G acts on M by Kähler isometries (so every ρ_g is a holomorphic isometry for the Kähler metric h). After complexifying the tangent bundle TM, we take $\mathcal{E} = \Omega^{1,0}$, and

$$Q_\xi = \partial + \Pi(\xi^*, 0)\,.$$

The morphism σ_Q^* is the projection $T^*M \otimes \mathbb{C} \to \Omega^{1,0}$. We obtain the superlocalization formula eq. (10) with numerical factor $(-2)^{m/2}\pi^m(m/2)!/m!$. This is the picture relevant to $\mathcal{N} = 1$ supersymmetry.

6 Application to topological $\mathcal{N} = 2$ SYM

We want to apply the superlocalization formula to the computation of the partition function for topological Yang-Mills theory. We consider explicitely the case of $\mathcal{N} = 2$ supersymmetry but the cases $\mathcal{N} = 1, 4$ may be dealt with along the same lines after choosing the relevant supermanifolds on the instanton moduli space.

We start by briefly recapping the ADHM construction for framed $SU(N)$ instantons on $\mathbb{R}^4$.[3] Framed instantons are anti-self-dual $SU(N)$ connections on (trivial bundle on) $\mathbb{R}^4$ with a fixed framing at infinity (i.e., if we transfer the instanton to the sphere S^4 via a stereographic projection, there is a fixed isomorphism of the fibre at a given point with $\mathbb{C}^N$, and this isomorphism is part of the data specifying the instanton). The moduli space of framed instantons under gauge equivalence is a singular manifold of dimension $4kN$, where k in the second Chern class (instanton number) of the instanton. The ADHM description is obtained in terms of data consisting of $k \times k$ matrices B_1, B_2, a $N \times k$ matrix I and a $k \times N$ matrix J, all with complex entries. These are subject to the constraints

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \tag{11}$$

$$[B_1, B_2] + IJ = 0 \tag{12}$$

where † denotes hermitian conjugation. The group $U(k)$ acts on these data, by adjunction on B_1, B_2 and by multiplication from the suitable side on I and J, and this action preserves the constraints. The moduli space of framed instantons is obtained by taking all the data (B_1, B_2, I, J) satisfying the constraints and taking equivalence classes under the action of $U(k)$. The resulting moduli space is singular, its smooth points corresponding to data with trivial stabilizer under the $U(k)$ action. Singularities may be resolved, either by standard blowup techniques, or using the hyperkähler quotient costruction of the moduli space. We shall denote by $\mathcal{M}$ the smooth moduli space so obtained.

In SYM one supplements the ADHM data by fermionic moduli provided by the zero modes of the gaugino field. For $\mathcal{N} = 2$ the fermionic moduli can be identified with differential forms on the bosonic moduli space $\mathcal{M}$; this is the reason of the introduction of the "tautological supermanifolds" of Section 4. The constraints on the fermionic data are obtained by linearizing the bosonic constraints, and the multi-instanton action eventually obtained is obtained by plugging into the SYM action the bosonic and fermionic zero modes in terms of the (unconstrained) ADHM data and imposing the ADHM constraints

[3] The original source is [2]; a useful reference on this construction, and other issues that will be touched upon in this and the following sections, is [30]

via Lagrangian multipliers. The resulting action turns out to be BRST-exact, hence, given its invariance under action of the groups involved, also BRST-closed.

Moreover as we have hinted in the previous Sections, if one associates a tautological supermanifold to the bosonic moduli space, the operator Q of the previous section — the counterpart on the supermanifold side of the equivariant differential — is exactly the BRST operator. Putting all this together, this opens the way to the computation of integrals over the moduli space of quantities depending on the SYM action, such as the partition and correlation functions, by means of a superlocalization formula.

We shall at first ignore the existence of the constraints on the ADHM data. So the field content of the theory is given by the matrices B_1, B_2, I, J with their fermionic partners M_1, M_2, μ_I, μ_J. We would like to consider the action of the group $U(k) \times SU(N)$, but if we do so the fixed points of the group action will not be isolated. It is therefore convenient, following Nakajima [30], to introduce also an action of the group T^2, given by

$$(B_1, B_2, I, J) \mapsto (e^{i\epsilon_1} B_1, e^{i\epsilon_2} B_2, I,\ e^{i(\epsilon_1+\epsilon_2)} J).$$

If we denote by ϕ, a, (ϵ_1, ϵ_2) elements in the Lie algebras of $U(k)$, $SU(N)$, T^2 respectively, we obtain for the vector fields ξ^* and Q_ξ the following expressions:

$$\xi^* = (\phi I - Ia)\frac{\partial}{\partial I} + (-J\phi + aJ + \epsilon J)\frac{\partial}{\partial J} + ([\phi, B_\ell] + \epsilon_\ell)\frac{\partial}{\partial B_\ell}$$

$$\begin{aligned} Q_\xi &= \mu_I \frac{\partial}{\partial I} + \mu_J \frac{\partial}{\partial J} + M_\ell \frac{\partial}{\partial B_\ell} \\ &+ (\phi I - Ia)\frac{\partial}{\partial \mu_I} + (-J\phi + aJ + \epsilon J)\frac{\partial}{\partial \mu_J} + ([\phi, M_\ell] + \epsilon_\ell M_\ell)\frac{\partial}{\partial M_\ell} \end{aligned}$$

(here $\epsilon = \epsilon_1 + \epsilon_2$). One recognizes in Q_ξ the standard expression of the infinitesimal BRST transformations in the theory under consideration.

Introduction of constraints. If $N \subset M$ is a submanifold of M, locally given in some coordinate patch $(x^1, \ldots, x^n)$ by a set of constraints $V_1 = \ldots = V_r = 0$, we may consider the tautological supermanifold $\mathfrak{N} = (N, \mathcal{B})$ associated to the cotangent bundle T^*N.

Proposition 6.1 $\mathfrak{N}$ *is a sub-supermanifold of the tautological supermanifold* $\mathfrak{M}$*, whose equations in the local coordinate patch* $(x^1, \ldots, x^n, \theta^1 = dx^1, \ldots \theta^n = dx^n)$ *are*

$$V_1 = \ldots = V_r = 0, \qquad W_1 = \ldots = W_r = 0,$$

where the superfunctions W_i *expressing the fermionic constraints are given by*

$$W_a = \frac{\partial V_a}{\partial x^k}\,\theta^k.$$

We consider now a situation where the coordinates in M are the bosonic ADHM parameters which appear in the Lagrangian L of a $\mathcal{N}=2$ SYM [15, 18] and the functions V_a express the ADHM constraints. One also introduces fermionic partners θ, subject to the constraints $W_a=0$. The constraints are implemented by the Lagrange multipliers H^a and χ^a, so that one considers a Lagrangian

$$L' = L + H^a\, V_a(x,\theta) + \chi^a\, \frac{\partial V_a}{\partial x^k}\, \theta^k\,.$$

The Lagrange multipliers should be considered as additional coordinates on an enlarged supermanifold $\mathfrak{M}'$. We have a BRST vector field Q_ξ for the unconstrained theory, and we want to complete it to a new field

$$Q'_\xi = Q_\xi + \tilde{Q}_\xi = Q_\xi - R^a\, \frac{\partial}{\partial H^a} - S^a\, \frac{\partial}{\partial \chi^a}$$

which leaves the Lagragian L' invariant. To simplify the treatment we assume that the odd superfunctions R^a are linear in the coordinates χ, i.e., $R^b=\chi^a\, N_a^b$ for a matrix of ordinary functions N. Since

$$Q'_\xi(L') = Q_\xi(L) - R^a\, V_a + H^a\, \frac{\partial V_a}{\partial x^k}\, \theta^k - S^a\, \frac{\partial V_a}{\partial x^k}\, \theta^k - \chi^a\, \frac{\partial V_a}{\partial x^k}\, \xi^\alpha T^k_\alpha$$

and $Q_\xi(L)=0$ we obtain the conditions

$$S^a = H^a\,, \qquad N_a^b\, V_b = -\frac{\partial V_a}{\partial x^k}\, \xi^\alpha T^k_\alpha\,. \tag{13}$$

One assumes that the group G acts also on the "new sector" of the supermanifold $\mathfrak{M}'$, so that the vector field $\hat{\xi}^*$ acquires a new contribution

$$\tilde{\xi}^* = \xi^\alpha\, \tilde{T}^a_\alpha\, \frac{\partial}{\partial H^a} + \xi^\alpha\, \chi^b\, \frac{\partial \tilde{T}^a_\alpha}{\partial H^b}\, \frac{\partial}{\partial \chi^a};$$

one should have $[\tilde{Q}_\xi, \tilde{Q}_\xi]_+ = 2\tilde{\xi}^*$, which is equivalent to

$$R^a = \xi^\alpha\, \chi^b\, \frac{\partial \tilde{T}^a_\alpha}{\partial H^b}\,, \qquad S^a\, \frac{\partial R^b}{\partial \chi^a} = \xi^\alpha\, \tilde{T}^b_\alpha.$$

If the functions $\tilde{T}$ are linear in the H^a, these conditions are solved by

$$N_b^a = \xi^\alpha\, \frac{\partial \tilde{T}^a_\alpha}{\partial H^b}. \tag{14}$$

The second equation in eq. (13) becomes

$$\xi^\alpha\, \frac{\partial \tilde{T}^a_\alpha}{\partial H^b}\, V_b = -\frac{\partial V_a}{\partial x^k}\, \xi^\alpha T^k_\alpha\,. \tag{15}$$

Provided that this constraint is satisfied, eq. (14) yields a solution to the problem of finding the BRST transformations for the Lagrange multipliers.

Since

$$[\frac{\partial}{\partial\theta^k}, Q'_\xi] = [\frac{\partial}{\partial\theta^k}, Q_\xi] = \frac{\partial}{\partial x^k}\,, \qquad [\frac{\partial}{\partial\chi^a}, Q'_\xi] = [\frac{\partial}{\partial\chi^a}, \tilde{Q}_\xi] = N_a^b\,\frac{\partial}{\partial H^b}$$

the additional requirement for the conditions of Definition 5.1 to hold is that the matrix N is invertible.

$\mathcal{N}=2$ SYM theory follows this pattern: one adds fermionic partners to the fields in the ADHM realization of the theory, and the constraints on the fermionic fields are obtained by linearizing the constraints eq. (11) and eq. (12). At the Lagrangian level one implements the constraints via Lagrangian multipliers $H_{\mathbb{C}}$ and $H_{\mathbb{R}}$ which multiply eq. (11) and eq. (12), and by their fermionic partners $\chi_{\mathbb{R}}$ and $\chi_{\mathbb{C}}$ which are then regarded as additional fields. In [11], whose notation we follow, the reader will find a detailed analysis of this case. After regularizing the moduli space of gauge connections by minimally resolving the singularities, the BRST transformations of the theory lead to

$$\begin{aligned}
\hat{\xi}^* &= (\phi I - Ia)\frac{\partial}{\partial I} + (-J\phi + aJ + \epsilon J)\frac{\partial}{\partial J} + ([\phi, B_\ell] + \epsilon_\ell)\frac{\partial}{\partial B_\ell} \\
&+ [\phi, H_{\mathbb{R}}]\frac{\partial}{\partial H_{\mathbb{R}}} + ([\phi, H_{\mathbb{C}}] + \epsilon H_{\mathbb{C}})\frac{\partial}{\partial H_{\mathbb{R}}} + [\phi, \bar{\phi}]\frac{\partial}{\partial\bar{\phi}} \\
&+ (\phi\mu_I - \mu_I a)\frac{\partial}{\partial\mu_I} + (-\mu_J\phi + a\mu_J + \epsilon\mu_J)\frac{\partial}{\partial\mu_J} + ([\phi, M_\ell] + \epsilon_\ell M_\ell)\frac{\partial}{\partial M_l} \\
&+ [\phi, \chi_{\mathbb{R}}]\frac{\partial}{\partial\chi_{\mathbb{R}}} + ([\phi, \chi_{\mathbb{C}}] + \epsilon\chi_{\mathbb{C}})\frac{\partial}{\partial\chi_{\mathbb{R}}} + [\phi, \eta]\frac{\partial}{\partial\eta}
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
Q_\xi &= \mu_I\frac{\partial}{\partial I} + \mu_J\frac{\partial}{\partial J} + M_\ell\frac{\partial}{\partial B_\ell} + [\phi, \chi_{\mathbb{R}}]\frac{\partial}{\partial H_{\mathbb{R}}} + ([\phi, \chi_{\mathbb{C}}] + \epsilon\chi_{\mathbb{C}})\,\frac{\partial}{\partial H_{\mathbb{C}}} + \eta\frac{\partial}{\partial\bar{\phi}} \\
&+ (\phi I - Ia)\frac{\partial}{\partial\mu_I} + (-J\phi + aJ + \epsilon J)\frac{\partial}{\partial\mu_J} + ([\phi, B_\ell] + \epsilon_\ell B_\ell)\frac{\partial}{\partial M_\ell} \\
&+ H_{\mathbb{R}}\frac{\partial}{\partial\chi_{\mathbb{R}}} + H_{\mathbb{C}}\frac{\partial}{\partial\chi_{\mathbb{C}}} + [\phi, \bar{\phi}]\frac{\partial}{\partial\eta}.
\end{aligned} \tag{17}$$

(One also includes an “auxiliary” bosonic field $\bar{\phi}$ with its partner η.) This new vector field Q_ξ satisfies the conditions of Definition 5.1, so that *for the BRST vector field of the ADHM formulation of $\mathcal{N}=2$ SYM, the superlocalization formula eq. (10) holds.* The reader will find in [11] the evaluation of the superdeterminant which arises from the application of eq. (10).

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