

SOBOLEV REGULARITY FOR MONGE-AMPÈRE TYPE
EQUATIONS*GUIDO DE PHILIPPIS[†] AND ALESSIO FIGALLI[‡]

Abstract. In this note we prove that if the cost function satisfies some necessary structural conditions and the densities are bounded away from zero and infinity, then strictly c -convex potentials arising in optimal transportation belong to $W_{\text{loc}}^{2,1+\kappa}$ for some $\kappa > 0$. This generalizes some recent results concerning the regularity of strictly convex Alexandrov solutions of the Monge–Ampère equation with right-hand side bounded away from zero and infinity [G. De Philippis and A. Figalli, *Invent. Math.*, 192 (2013), pp. 55–69; G. De Philippis, A. Figalli, and O. Savin, *Math. Ann.*, to appear; T. Schmidt, *Adv. Math.*, to appear].

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1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We want to investigate the regularity of solutions to Monge–Ampère type equations of the form

$$(1.1) \quad \det(D^2u - \mathcal{A}(x, Du)) = f \quad \text{in } \Omega,$$

where $f \geq 0$ and $\mathcal{A}(x, p)$ is a $n \times n$ symmetric matrix.

This class of equations naturally arises in optimal transportation (see, for instance, [23]). There, the matrix \mathcal{A} and the right-hand-side f are given by

$$\mathcal{A}(x, Du(x)) = -D_{xx}c(x, T_u(x)), \quad f(x) = |\det(D_{xy}c(x, T_u(x)))| \frac{\rho_0(x)}{\rho_1(T_u(x))},$$

where $c(x, y)$ represents the cost function, ρ_0 and ρ_1 are probability densities, and T_u is the optimal transport map sending ρ_0 onto ρ_1 . Under a twist assumption on the cost (see (C2) below), the map T_u is uniquely determined through the relation

$$-D_x c(x, T_u(x)) = Du(x).$$

Moreover, when $\mathcal{A} \equiv 0$ the above equation reduces to the classical Monge–Ampère equation.

Regularity for the above class of equations has received a lot of attention in recent years [12, 13, 18, 19, 21, 23, 25, 26]. In particular, under some necessary structural conditions on \mathcal{A} (see (C1) below), one can show that if f is smooth, then u is smooth as well [19, 23, 25, 26]. In addition, it is proved in [12] that solutions are locally $C^{1,\alpha}$ when f is merely bounded away from zero and infinity (see also [13, 18]). Furthermore, if f is close to a constant, then $W_{\text{loc}}^{2,p}$ estimates are proved in [4] for the classical

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Monge–Ampère equation (i.e., when $\mathcal{A} \equiv 0$) and in [20] under some stronger conditions on \mathcal{A} (namely, the inequality in (1.6) should be strict when $\xi, \eta \neq 0$).

Recently, the authors introduced new techniques to address the Sobolev regularity of u when $\mathcal{A} \equiv 0$ and f is only bounded away from zero and infinity: more precisely, it is proved in [10] that $D^2u \in L \log L_{\text{loc}}(\Omega)$, and with a variant of the same techniques this result has been improved in [11] to $u \in W_{\text{loc}}^{2,1+\kappa}(\Omega)$ for some $\kappa > 0$ (see also [24]). Let us mention that these results played a crucial role in [1, 2] to show the existence of distributional solutions to the semigeostrophic system.

The aim of this paper is to extend the $W_{\text{loc}}^{2,1+\kappa}$ regularity to the general class of Monge–Ampère equations in (1.1). Apart from its own interest, because of several regularity results for the squared distance function on the sphere and its perturbations [8, 9, 14, 15, 22], it looks plausible to us that, at least in some particular regimes, this result may have applications in the study of generalized semigeostrophic system on the sphere [7].

In order to describe our result, we need to introduce some more notation and the main assumptions on the cost functions.

Let $X \subset \mathbb{R}^n$ be an open set and $u : X \rightarrow \mathbb{R}$ be a c -convex function, i.e., u can be written as

$$(1.2) \quad u(x) = \max_{y \in \bar{Y}} \{-c(x, y) + \lambda_y\}$$

for some open set $Y \subset \mathbb{R}^n$ and $\lambda_y \in \mathbb{R}$ for all $y \in \bar{Y}$. We are going to assume that u is an Alexandrov solution of (1.1) inside some open set $\Omega \subset X$, i.e.,

$$|\partial^c u(E)| = \int_E f \quad \forall E \subset \Omega \text{ Borel,}$$

where

$$\partial^c u(E) := \bigcup_{x \in E} \partial^c u(x), \quad \partial^c u(x) := \{y \in \bar{Y} : u(x) = -c(x, y) + \lambda_y\},$$

and $|F|$ denotes the Lebesgue measure of a set F . It is wellknown that in order to prove some regularity results, (1.1) needs to be coupled with some boundary conditions: for instance, when $\mathcal{A} \equiv 0$ and $f \equiv 1$, solutions are smooth whenever they are strictly convex, and to obtain strict convexity some suitable boundary conditions are needed [3, 6].

For the general case in (1.1), let u be a c -convex function associated to an optimal transport problem, and for any $y \in \bar{Y}$ define the contact set

$$\Lambda_y := \{x \in X : u(x) = -c(x, y) + \lambda_y\}.$$

Under some structural assumptions on the cost functions (which we shall describe below) and some convexity hypotheses on the supports of the source and target measure, it has been proved in [12] that u is an Alexandrov solution of (1.1) inside X , and it is strictly c -convex (i.e., for any $y \in \partial^c u(X)$ the contact set Λ_y reduces to one point) provided f is bounded away from zero and infinity.

Here, since we want to investigate the interior regularity of u , instead of assuming that u comes from an optimal transportation problem where the supports of the source and target measure enjoy some global c -convexity property, we work assuming directly that u is a strictly c -convex Alexandrov solution near some point $\bar{x} \in X$, and we prove

regularity of u in a neighborhood of \bar{x} . This has the advantage of making our result more general and flexible for possible future applications.

Hence, we assume that there exist $(\bar{x}, \bar{y}) \in X \times Y$ such that $\Lambda_{\bar{y}} = \{\bar{x}\}$, and we consider a neighborhood Ω of \bar{x} given by

$$(1.3) \quad \Omega := \{x \in X : u(x) < -c(x, \bar{y}) + \lambda_y + \delta\},$$

where $\delta > 0$ is a small constant chosen so that $\Omega \subset\subset X$ and $\partial^c u(\Omega) \subset\subset Y$. (Such a constant δ exists because $\Lambda_{\bar{y}} := \{\bar{x}\}$.) Also, we assume that u is an Alexandrov solution of

$$(1.4) \quad \begin{cases} \det(D^2u - A(x, Du)) = f & \text{in } \Omega, \\ u = -c(\cdot, \bar{y}) + \text{const} & \text{on } \partial\Omega. \end{cases}$$

Before stating our result, let us introduce the main conditions on the cost function: let Ω be as above, and let $\mathcal{O} \subset\subset Y$ be an open neighborhood of $\partial^c u(\Omega)$. We define

$$(1.5) \quad \|c\| := \|c\|_{C^3(\bar{\Omega} \times \bar{\mathcal{O}})} + \|D_{xxyy}c\|_{L^\infty(\bar{\Omega} \times \bar{\mathcal{O}})} + \|\log |\det D_{xy}c|\|_{L^\infty(\bar{\Omega} \times \bar{\mathcal{O}})}$$

and assume that the following hold:

$$(C0) \quad \|c\| < \infty.$$

(C1) For every $x \in \Omega$ and $p := -D_x c(x, y)$ with $y \in \mathcal{O}$, it holds that

$$(1.6) \quad D_{p_k p_\ell} \mathcal{A}_{ij}(x, p) \xi_i \xi_j \eta_k \eta_\ell \geq 0 \quad \forall \xi, \eta \in \mathbb{R}^n, \xi \cdot \eta = 0,$$

where \mathcal{A} is defined through c by $\mathcal{A}_{ij}(x, p) := -D_{x_i x_j} c(x, y)$, and we use the summation convention over repeated indices.

Let us point out that up to reduce the size of Ω and \mathcal{O} (this is possible because $\Omega \rightarrow \{\bar{x}\}$ and $\partial^c u(\Omega) \rightarrow \{\bar{y}\}$ as $\delta \rightarrow 0$), as a consequence of (C0) (more precisely, from the fact that $\det D_{xy}c(\bar{x}, \bar{y}) \neq 0$ and by the implicit function theorem) we can assume that the following holds:

(C2) For every $(x, y) \in \Omega \times \mathcal{O}$, the maps $x \in \Omega \mapsto -D_y c(x, y)$ and $y \in \mathcal{O} \mapsto -D_x c(x, y)$ are diffeomorphisms on their respective ranges.

We also notice that because of the boundary condition $u = -c(\cdot, \bar{y}) + \text{const}$ on $\partial\Omega$, if f is bounded away from zero and infinity inside Ω , then any c -convex Alexandrov solution of (1.4) is strictly c -convex inside Ω . (This is an immediate consequence of [12, Remark 7.2].) Our result is as follows.

THEOREM 1.1. *Let $u : \Omega \rightarrow \mathbb{R}$ be a c -convex Alexandrov solution of (1.4). Assume that c satisfies conditions (C0)–(C2) and that $0 < \lambda \leq f \leq 1/\lambda$. Then $u \in W_{\text{loc}}^{2,1+\kappa}(\Omega)$ for some $\kappa > 0$ depending only on n , λ , and $\|c\|$.*

Theorem 1.1 generalizes the corresponding result for the classical Monge–Ampère equation to the wider class of equations considered here. With respect to the arguments in [10, 11], additional complications arise from the fact that in contrast with the classical Monge–Ampère equation, in general (1.1) is not affinely invariant.

2. Notation and preliminary results. Throughout the paper, we call *universal* any constant which depends only on the data, i.e., on n , λ , and $\|c\|$. We use C to denote a universal constant larger than 1 whose value may change from line to line, and we use the notation $a \approx b$ to indicate that the ratio a/b is bounded from above and below by positive universal constants.

An immediate consequence of the definition of c -convexity (1.2) is that for any $x_0 \in X$, there exists $y_0 \in \bar{Y}$ such that

$$u(x) \geq -c(x, y_0) + u(x_0) + c(x_0, y_0) \quad \forall x \in X,$$

and in this case $y_0 \in \partial^c u(x_0)$. If in addition $u \in C^2$, then it is easily seen that $Du(x_0) = -D_x c(x_0, y_0)$ and $D^2 u(x_0) \geq -D_{xx} c(x_0, y_0) = \mathcal{A}(x_0, Du(x_0))$, where \mathcal{A} is defined in (C1) above. In particular, (1.4) is degenerate elliptic when restricted to the c -convex function.

It has been discovered independently in [12] and [18] that because of (C1), for any $x_0 \in \Omega$ and $y_0 \in \partial^c u(x_0)$, through the change of variables $x \mapsto q(x) := -D_y c(x, y_0)$ the function

$$(2.1) \quad \bar{u}(q) := u(x(q)) + c(x(q), y_0) - u(x_0) - c(x_0, y_0)$$

has convex level sets inside Ω . (Here and in what follows $x(q)$ denotes the inverse of $q(x)$, which is well defined because of (C2).) Moreover, \bar{u} is \bar{c} -convex, where

$$(2.2) \quad \bar{c}(q, y) := c(x(q), y) - c(x(q), y_0);$$

see [12, Theorem 4.3].

Since u solves (1.4) one can check by a direct computation that \bar{u} solves

$$(2.3) \quad \det(D^2 \bar{u} - \mathcal{B}(q, D\bar{u})) = g$$

with

$$(2.4) \quad \mathcal{B}_{ij}(q, D\bar{u}(q)) = -D_{q_i q_j} \bar{c}(q, T_{\bar{u}}(q)) \quad \text{and} \quad g(q) = f(x(q)) [\det D_{xy} c(x(q), T_{\bar{u}}(q))]^{-2},$$

where $T_{\bar{u}}$ is the map uniquely identified by the relation $D\bar{u}(q) = -D_q \bar{c}(q, T_{\bar{u}}(q))$. Moreover it holds that

$$(2.5) \quad \mathcal{B}_{ij}(\cdot, 0) \equiv 0, \quad D_p \mathcal{B}_{ij}(\cdot, 0) \equiv 0,$$

so using Taylor's formula we can write

$$(2.6) \quad \mathcal{B}_{ij}(q, D\bar{u}) = \mathcal{B}_{ij, k\ell}(q, D\bar{u}) \partial_k \bar{u} \partial_l \bar{u},$$

where

$$(2.7) \quad \mathcal{B}_{ij, k\ell}(q, D\bar{u}(q)) := \int_0^1 D_{p_k p_\ell} \mathcal{B}_{ij}(q, \tau D\bar{u}(q)) d\tau.$$

In addition, since condition (C1) is tensorial [23, 21, 17] and $\|c\|$ involves only mixed fourth derivative, it is easily seen that $\|\bar{c}\| \approx \|c\|$ and \mathcal{B} satisfies the same assumptions as \mathcal{A} . In particular (C1) and (2.7) imply that

$$(2.8) \quad \mathcal{B}_{ij, k\ell} \xi_i \xi_j \eta_k \eta_\ell \geq 0 \quad \forall \xi \cdot \eta = 0.$$

Given a C^1 c -convex function as above, for any $x_0 \in \Omega$, $y_0 = T_u(x_0)$, and $h \in \mathbb{R}^+$, we define the *section* centered at x_0 of height h as

$$S_h^u(x_0) := \{x \in \Omega : u(x) \leq -c(x, y_0) + u(x_0) + c(x_0, y_0) + h\}.$$

Assuming that $S_h^u(x_0) \subset \Omega$, through the change of variables $x \mapsto q(x) := -D_y c(x, y_0)$ this section is transformed into the *convex* set

$$Q_h^{\bar{u}}(q_0) := -D_y c(S_h^u(x_0), y_0) = \{q : \bar{u}(q) \leq h\}.$$

When no confusion arises, we will often abbreviate $S_h(x_0)$ and $Q_h(q_0)$ for $S_h^u(x_0)$ and $Q_h^{\bar{u}}(q_0)$.

We also recall [16] that given an open bounded convex set Q , there exists an ellipsoid E such that

$$(2.9) \quad E \subset Q \subset nE,$$

where the dilation is done with respect to the center of E . We refer to it as the John ellipsoid of Q , and we say that Q is normalized if $E = B(0, 1)$. An immediate consequence of (2.9) is that any open bounded convex set Q admits an affine transformation L such that $L(Q)$ is normalized. Hence, given u and $S_h^u(x_0)$ as above, we can consider \bar{u} , its section $Q_h^{\bar{u}}(q_0)$, and the normalizing affine transformation L . Then we define $\bar{w} : L(Q_h) \rightarrow \mathbb{R}$ as

$$(2.10) \quad \bar{w}(q') := (\det L)^{2/n} \bar{u}(q), \quad q' := Lq.$$

It is easy to check that \bar{w} solves

$$(2.11) \quad \det(D^2 \bar{w}(q') - \mathcal{C}(q', D\bar{w}(q'))) = g(L^{-1}q'),$$

where

$$\mathcal{C}(q', D\bar{w}(q')) := (\det L)^{2/n} (L^*)^{-1} \mathcal{B}(L^{-1}q', (\det L)^{-2/n} L^* D\bar{w}(q')) L^{-1}$$

Up to an isometry, we can assume that

$$E = \left\{ q : \sum_{i=1}^n \frac{q_i^2}{r_i^2} \leq 1 \right\},$$

with $r_1 \leq \dots \leq r_n$. Then $L^{-1} = \text{diag}(r_1, \dots, r_n)$, and

$$(2.12) \quad \mathcal{C}_{ij}(q', D\bar{w}(q')) = \mathcal{C}_{ij,kl}(q', D\bar{w}(q')) \partial_k \bar{w} \partial_\ell \bar{w}$$

with

$$(2.13) \quad \mathcal{C}_{ij,kl}(q', D\bar{w}(q')) = (r_1 \dots r_n)^{2/n} \frac{r_i r_j}{r_k r_\ell} \mathcal{B}_{ij,kl}(q, D\bar{u}(q));$$

see (2.7). Moreover, by (2.8) (or again because of the tensorial nature of condition (C1))

$$(2.14) \quad \mathcal{C}_{ij,kl} \xi_i \xi_j \eta_k \eta_\ell \geq 0 \quad \forall \xi \cdot \eta = 0.$$

Still with the same notation as above, we also define the *normalized size* of a section $S_h(x_0)$ as

$$(2.15) \quad \alpha(S_h(x_0)) = \alpha(Q_h(q_0)) := \frac{|L|^2}{(\det L)^{2/n}}.$$

Notice that even if L may not be unique, α is well defined up to universal constants. In case u is C^2 in a neighborhood of x_0 , by a simple Taylor expansion of \bar{u} around q_0 it is easy to see that there exists $h(x_0) > 0$ small such that

$$(2.16) \quad \alpha(S_h(x_0)) = \alpha(Q_h(q_0)) \approx |D^2\bar{u}(q_0)| \quad \forall h \leq h(x_0),$$

where $q_0 := q(x_0)$. Since u and \bar{u} are related by a diffeomorphism, the following lemma holds.

LEMMA 2.1. *Let $\Omega' \subset \Omega$ and let $u \in C^2(\Omega')$ be a strictly c -convex function such that $\|Du\|_{L^\infty(\Omega')}$ is universally bounded. Then there exists a universal constant M_1 such that the following holds: For every $x_0 \in \Omega'$ there exists a height $\bar{h}(x_0) > 0$ such that if $|D^2u(x_0)| \geq M_1$, then*

$$(2.17) \quad |D^2u(x_0)| \approx \alpha(S_{\bar{h}}(x_0)) \quad \forall h \leq \bar{h}(x_0).$$

Proof. Differentiating twice the relation (2.1) we obtain

$$\partial_{q_i q_j} \bar{u} = \partial_{q_i} x^k \partial_{q_j} x^l \partial_{x_k x_l} u + \partial_{q_i q_j} x^k \partial_{x_k} u + \partial_{q_i} x^k \partial_{q_j} x^l \partial_{x_k x_l} c + \partial_{q_i q_j} x^k \partial_{x_k} c,$$

where we have used the summation convention over repeated indices. The above equation implies that

$$(2.18) \quad \nu |D^2\bar{u}(q_0)| - C(1 + |Du(x_0)|) \leq |D^2u(x_0)| \leq \frac{1}{\nu} |D^2\bar{u}(q_0)| + C(1 + |Du(x_0)|)$$

for some universal constants $\nu, C > 0$. Since by assumption Du is universally bounded inside Ω' , (2.17) follows by (2.18) and (2.16), provided M_1 is sufficiently large. \square

We now show some geometric properties of sections and some estimates for solutions of (1.4) which will play a major role in the paper. Here, the dilation of a section $S_h(x)$ is intended with respect to x .

PROPOSITION 2.1 (properties of section). *Let u be a c -convex Alexandrov solution of (1.4) with $0 < \lambda \leq f \leq 1/\lambda$. Then, for any $\Omega' \subset \subset \Omega'' \subset \subset \Omega$, there exists a positive constant $\rho = \rho(\Omega', \Omega'')$ such that the following properties hold:*

- (i) $S_h^u(x) \subset \Omega''$ for any $x \in \Omega'$, $0 \leq h \leq 4\rho$.
- (ii) There exist $0 < \alpha_1 < \alpha_2$ universal such that for all $\mu \in (0, 1)$

$$\mu^{\alpha_2} S_h^u(x) \subset S_{\mu h}^u(x) \subset \mu^{\alpha_1} S_h^u(x)$$

for any $x \in \Omega'$, $0 \leq 2h \leq \rho$.

- (iii) There exists a universal constant $\sigma < 1$ such that if $S_h^u(x) \cap S_h^u(y) \neq \emptyset$, then $S_h^u(y) \subset S_{h/\sigma}^u(x)$ for any $x, y \in \Omega'$, $0 \leq h \leq \sigma\rho$.
- (iv) $\text{diam}(S_\rho^u(x)) \leq 1$ and $\bigcap_{0 < h \leq \rho} S_h^u(x) = \{x\}$.

Proof. Points (i) and (iv) follow from the strict c -convexity of u shown in [12, section 7] and the fact that the modulus of strict c -convexity is universal. (This last fact follows by a simple compactness argument in the spirit of [5, Theorem 1'.])

Point (iii) corresponds to the engulfing property of sections proved in [12, Theorem 9.3].

The second inclusion in point (ii) follows from [12, Lemma 9.2].¹ For the first one, it is enough to show that there exists a universal constant $\bar{s} \in (0, 1)$ such that

$$(2.19) \quad \bar{s}Q_{\bar{h}}^{\bar{u}}(\bar{q}) \subset Q_{\bar{h}/2}^{\bar{u}}(\bar{q})$$

¹To be precise, in [12] the dilation is done with respect to the center of the John ellipsoid and not with respect to the “center” x of the section. However, it is easy to see that the same statement holds also in this case.

and then iterate this estimate (here \bar{u} is defined as in (2.1), and $\bar{q} := q(x)$). To prove (2.19), let E_{2h} be the John ellipsoid associated to $Q_{2h}^{\bar{u}}(\bar{q})$ and assume without loss of generality that E_{2h} is centered at the origin. By convexity of the sections in these new variables,

$$\bar{s}(Q_h^{\bar{u}}(\bar{q}) - \bar{q}) + \bar{q} \subset Q_h^{\bar{u}}(\bar{q}) \subset Q_{2h}^{\bar{u}}(\bar{q}) \subset nE_{2h} \quad \forall \bar{s} \in (0, 1).$$

Observe now that for any $q \in Q_h^{\bar{u}}(\bar{q})$, we have (recall that $\bar{u}(\bar{q}) = 0$)

$$(2.20) \quad \bar{u}(\bar{s}(q - \bar{q}) + \bar{q}) = \bar{s} \int_0^1 D\bar{u}((1 - t\bar{s})\bar{q} + t\bar{s}q) \cdot (q - \bar{q}) dt.$$

Since $q, \bar{q} \in nE_{2h}$ we have $q - \bar{q} \in 2nE_{2h}$, hence

$$(2.21) \quad (q - \bar{q})/2n \in E_{2h} \subset Q_{2h}(\bar{q}).$$

Moreover, by convexity of $Q_{2h}(\bar{q})$, $(1 - t\bar{s})\bar{q} + t\bar{s}q \in Q_h(\bar{q}) \subset \tau_0 Q_{2h}(\bar{q})$ for some universal $\tau_0 < 1$ (see [12, Lemma 9.2]). Defining the “dual norm” $\|\cdot\|_{\mathcal{K}}^*$ associated to a convex set \mathcal{K} as

$$\|a\|_{\mathcal{K}}^* := \sup_{\xi \in \mathcal{K}} a \cdot \xi,$$

it follows from [12, Lemma 6.3] that

$$(2.22) \quad \|D\bar{u}(q)\|_{Q_{2h}(\bar{q})}^* = \|-D_q \bar{c}(q, T_{\bar{u}}(q))\|_{Q_{2h}(\bar{q})}^* \leq Ch \quad \forall q \in Q_h^{\bar{u}}(\bar{q}).$$

Thus, thanks to (2.20) and (2.21) we get

$$\begin{aligned} \bar{u}(\bar{s}(q - \bar{q}) + \bar{q}) &= 2n\bar{s} \int_0^1 D\bar{u}((1 - t\bar{s})\bar{q} + t\bar{s}q) \cdot \frac{(q - \bar{q})}{2n} dt \\ &\leq 2n\bar{s} \int_0^1 \|D\bar{u}((1 - t\bar{s})\bar{q} + t\bar{s}q)\|_{Q_{2h}(\bar{q})}^* dt \leq 2n\bar{s}Ch \leq h/2, \end{aligned}$$

provided \bar{s} is small enough. This proves the desired inclusion. \square

As shown, for instance, in [11], an easy consequence of property (iii) is the following Vitali-type covering theorem.

PROPOSITION 2.2 (Vitali covering theorem). *Let $u, f, \Omega', \Omega'', \rho, \sigma$ be as in Proposition 2.1, let D be a compact subset of Ω' , and let $\{S_{h_x}(x)\}_{x \in D}$ be a family of sections with $h_x \leq \rho$. Then we can find a finite number of these sections $\{S_{h_{x_i}}(x_i)\}_{i=1, \dots, m}$ such that*

$$D \subset \bigcup_{i=1}^m S_{h_{x_i}}(x_i) \quad \text{with } \{S_{\sigma h_{x_i}}(x_i)\}_{i=1, \dots, m} \text{ disjoint.}$$

We now want to show that sections at the same height have a comparable shape. For this, we first recall the following estimate from [12].

PROPOSITION 2.3 (Alexandrov estimates). *Let $u, f, \Omega', \Omega'', \rho$ be as in Proposition (2.1), and let $S_h(x)$ be a section of u for some $x \in \Omega'$ and $h \leq \rho$. Then*

$$(2.23) \quad |S_h(x_0)| \approx h^{n/2}.$$

Remark 2.1. Estimates (2.22) and (2.23) have the following important consequence: consider the function \bar{u} defined in (2.1), fix one of its sections Q_h such that

$Q_{2h} \subset \Omega''$ with Ω'' as above, normalize Q_h using its corresponding John’s transformation L , and define \bar{w} as in (2.10). Since $(\det L)^{-2/n} \approx |E_h|^{2/n} \approx \text{osc}_{Q_h} \bar{u} \approx \text{osc}_{Q_{2h}} \bar{u}$ (by (2.23)) and $E_h \subset Q_{2h}$, we deduce the universal gradient bound

$$\begin{aligned}
 \sup_{L(Q_h)} |D\bar{w}| &= (\det L)^{2/n} \sup_{Q_h} |(L^*)^{-1} D\bar{u}| \\
 &\leq C(\det L)^{2/n} \sup_{Q_h} \|D\bar{u}\|_{E_h}^* \\
 (2.24) \quad &\leq C(\det L)^{2/n} \sup_{Q_h} \|D\bar{u}\|_{Q_{2h}}^* \\
 &\leq C(\det L)^{2/n} \text{osc}_{Q_{2h}} \bar{u} \leq C.
 \end{aligned}$$

LEMMA 2.2. *Let $u, f, \Omega', \Omega'', \rho$ be as in Proposition 2.1, and let \bar{u} as in (2.1). Then for any $0 \leq h \leq \rho$ there exist two radii $r = r(h)$ and $R = R(h)$ such that the following holds: for every $x_0 \in \Omega'$, if E is the John ellipsoid associated to the section $Q_h^{\bar{u}}(q_0)$, $q_0 := q(x_0)$, then, up to a translation,*

$$B_r(0) \subset E \subset B_R(0).$$

Proof. Let $r_1 \leq \dots \leq r_n$ be the axes of E . Since $r_n \leq \text{diam}(E) \leq C$ (see Proposition 2.1(iv)) and by (2.23)

$$h^{n/2} \approx |E| \approx r_1 \dots r_n \leq \text{diam}(E)^{n-1} r_1,$$

we obtain the desired lower bound on r_1 . \square

Obviously analogous properties hold for the section $Q_h^{\bar{u}}(q_0)$.

Remark 2.2. Notice that Proposition 2.1(ii) applied to the (convex) sections of \bar{u} implies the following: given $x \in \Omega''$ and $h \leq \rho$, let $r_1 \leq \dots \leq r_n$ denote the axes of the John ellipsoid associated to $Q_h(x)$. Then

$$(2.25) \quad r_n \leq Cr_1^{\alpha_3}$$

for some universal exponent $\alpha_3 < 1$ and a constant $C(\Omega', \Omega'')$.

To see this just normalize $Q_\rho(x)$ using L and notice that by [12, Theorem 6.11], $\text{dist}(x, \partial(L(Q_\rho(x)))) \geq 1/C$ for some universal constant C . Thus, up to enlarging C ,

$$\left(\frac{h}{C\rho}\right)^{\alpha_2} B_1(x) \subset L(Q_h) \subset \left(\frac{Ch}{\rho}\right)^{\alpha_1} B_1(x).$$

Since by Lemma 2.2 sections of height ρ have bounded eccentricity (i.e., $|L| \approx C(\Omega', \Omega'')$), this implies the claim with $\alpha_3 := \alpha_1/\alpha_2$.

We now observe that $\alpha(Q_h) \approx r_n^2/(r_1 \dots r_n)^{2/n}$, from which we deduce that

$$r_1^2 \leq C \frac{r_n^2}{\alpha(Q_h)}.$$

In particular, this and (2.25) imply

$$r_n \leq Cr_1^{\alpha_3} \leq C \frac{r_n^{\alpha_3}}{\alpha(Q_h)^{\alpha_3/2}},$$

that is,

$$r_n \leq \frac{C}{\alpha(S_h)^\beta} \quad \text{with} \quad \beta := \frac{\alpha_3}{2 - 2\alpha_3}.$$

Hence, since S_h is linked to Q_h by a diffeomorphism with universal C^1 norm, and $\text{diam}(S_h) \leq \text{diam}(nE_h) = 2nr_n$, we get

$$(2.26) \quad \text{diam}(S_h) \leq \frac{\bar{C}}{\alpha(S_h)^\beta}, \quad \beta, \bar{C} > 0 \text{ universal.}$$

3. $W^{2,1+\kappa}$ estimates. Applying first a large dilation to Ω we can assume that $B(0, 1) \subset \Omega$, and by a standard covering argument (see, for instance, [10, section 3]) it suffices to prove the $W^{2,1+\kappa}$ regularity of u inside $B(0, 1/2)$. Also, by an approximation argument,² it is enough to prove the result when $u \in C^2$. Hence Theorem 1.1 is a consequence of the following.

THEOREM 3.1. *Let $u \in C^2$ be a c -convex solution of (1.4) with $\Omega \supset B(0, 1)$. Then there exist universal constants κ and C such that*

$$(3.1) \quad \int_{B(0,1/2)} |D^2 u|^{1+\kappa} \leq C.$$

We start with the following lemma.

LEMMA 3.1. *Let u be as above, $x_0 \in B(0, 3/4)$, and $h > 0$ such that $S_{2h}(x_0) \subset B(0, 5/6)$. Consider the function \bar{u} as in (2.1), its section $Q_h = Q_h(q_0)$ with $q_0 := q(x_0)$, and (up to a rotation) let $E_h = \{\sum x_i^2/r_i^2 \leq 1\}$ the John ellipsoid associated to $Q_h(q_0)$. Denote by L the affine transformation that normalizes Q_h , and define \bar{w} and $\mathcal{C}_{ij,k\ell}$ as in (2.10) and (2.13), respectively. Then*

$$(3.2) \quad \int_{L(Q_h)} |\partial_{ij}\bar{w} - \mathcal{C}_{ij,k\ell}\partial_k\bar{w}\partial_\ell\bar{w}| \leq C$$

for some universal constant C .

Proof. Since by the c -convexity of u (which is preserved under change of variables) the matrix $(\partial_{ij}\bar{w} - \mathcal{C}_{ij,k\ell}\partial_k\bar{w}\partial_\ell\bar{w})_{i,j=1,\dots,n}$ is nonnegative definite, it is enough to estimate

$$\int_{L(Q_h)} \sum_{i=1}^n (\partial_{ii}\bar{w} - \mathcal{C}_{ii,k\ell}\partial_k\bar{w}\partial_\ell\bar{w})$$

from above.

Using the bounds $\mathcal{H}^{n-1}(\partial(L(Q_h))) \leq C(n)$ (since $L(Q_h)$ is a normalized convex set) and $|D\bar{w}| \leq C$ (see (2.24)), we see that first term is controlled from above by

$$(3.3) \quad \int_{L(Q_h)} \Delta\bar{w} = \int_{\partial(L(Q_h))} D\bar{w} \cdot \nu \leq \mathcal{H}^{n-1}(\partial(L(Q_h))) \sup_{L(Q_h)} |D\bar{w}| \leq C.$$

²To approximate our solution with smooth ones, it suffices to regularize the data and then either apply [19, Remark 4.1] (notice that by Proposition 2.1(iv) and [12, Theorem 8.2], u is strictly c -convex and of class C^1 inside Ω), or approximate our cost c with cost functions satisfying the strong version of (C1) and apply [19, Theorem 1.1].

For the second term, we claim the following: there exists a universal constant C such that

$$(3.4) \quad \inf_{L(Q_h)} \sum_{i=1}^n \mathcal{C}_{ii,kl} \partial_k \bar{w} \partial_\ell \bar{w} \geq -C.$$

To see this we write

$$\sum_{i=1}^n \mathcal{C}_{ii,kl} \partial_k \bar{w} \partial_\ell \bar{w} = \sum_{i=1}^n \sum_{k, \ell \neq i} \mathcal{C}_{ii,kl} \partial_k \bar{w} \partial_\ell \bar{w} + 2 \sum_{i=1}^n \sum_{k \neq i} \mathcal{C}_{ii,ik} \partial_i \bar{w} \partial_k \bar{w} + \sum_{i=1}^n \mathcal{C}_{ii,ii} \partial_i \bar{w} \partial_i \bar{w}.$$

We first observe that since for any $i = 1, \dots, n$ the vector $(\partial_1 \bar{w}, \dots, \partial_{i-1} \bar{w}, 0, \partial_{i+1} \bar{w}, \dots, \partial_n \bar{w})$ is orthogonal to coordinate vector e_i , the first term in the right-hand side is nonnegative by condition (C1).

Concerning the second and third terms, taking into account the definition of $\mathcal{C}_{ij,kl}$ in (2.13) we can rewrite them as

$$(3.5) \quad 2 \sum_{i=1}^n \sum_{k \neq i} (r_1 \dots r_n)^{2/n} \frac{r_i}{r_k} \mathcal{B}_{ii,ik} \partial_i \bar{w} \partial_k \bar{w} + \sum_{i=1}^n (r_1 \dots r_n)^{2/n} \mathcal{B}_{ii,ii} \partial_i \bar{w} \partial_i \bar{w}.$$

Observe that by (2.23),

$$(3.6) \quad (r_1 \dots r_n)^{2/n} \approx h.$$

In addition, by the Lipschitz regularity of \bar{u} (which is simply a consequence of the fact that u is locally Lipschitz inside Ω),

$$(3.7) \quad h/r_k \leq C \quad \forall k = 1, \dots, n.$$

Since $\|D\bar{w}\|_\infty \leq C$ (see (2.24)) and the size of \mathcal{B} is controlled by $\|\bar{c}\| \approx \|c\|$, by (3.6) and (3.7) we see that the expression in (3.5) is universally bounded.

This proves (3.4), which combined with (3.3) concludes the proof. \square

LEMMA 3.2. *With the same notation and hypotheses as in Lemma 3.2, let σ be as in Proposition 2.2. Then there exists a universal constant C such that*

$$(3.8) \quad |\{\tilde{q} \in L(Q_{\sigma h}) : \text{Id}/C \leq \partial_{ij} \bar{w} - \mathcal{C}_{ij,kl} \partial_k \bar{w} \partial_\ell \bar{w} \leq C \text{Id}\}| \geq \frac{1}{C}.$$

Proof. Since σ is universal and $L(Q_h)$ is normalized, by Proposition 2.1(ii) we get

$$|L(Q_{\sigma h})| \approx |L(Q_h)| \approx 1.$$

So, using Lemma 3.1 and the Chebychev inequality, we deduce the existence of a universal constant C such that

$$|\{\tilde{q} \in L(Q_{\sigma h}) : \partial_{ij} \bar{w} - \mathcal{C}_{ij,kl} \partial_k \bar{w} \partial_\ell \bar{w} \leq C \text{Id}\}| \geq \frac{1}{C}.$$

Since by (2.11) the product of the eigenvalues of the matrix $(\partial_{ij} \bar{w} - \mathcal{C}_{ij,kl} \partial_k \bar{w} \partial_\ell \bar{w})_{i,j=1, \dots, n}$ is of order one, whenever the eigenvalues are universally bounded from above, they also have to be universally bounded from below. Hence, up to enlarging the value of C , this proves (3.8). \square

Remark 3.1. Recalling the definition (2.15) of $\alpha(Q_h) = \alpha(S_h)$, we can rewrite both (3.2) and (3.8) in terms of \bar{u} and $Q_h = Q_h(q_0)$, obtaining that

$$\int_{Q_h} |\partial_{ij}\bar{u} - \mathcal{B}_{ij,k\ell}\partial_k\bar{u}\partial_l\bar{u}| \leq C\alpha(Q_h) \left| \{x \in Q_{\sigma h} : \alpha(Q_h)/C \leq |\partial_{ij}\bar{u} - \mathcal{B}_{ij,k\ell}\partial_k\bar{u}\partial_l\bar{u}| \leq C\alpha(Q_h)\} \right|.$$

(See, for instance, the proof of [11, Lemma 3.2].) In terms of u , this estimate becomes

$$(3.9) \quad \int_{S_h} |D^2u - \mathcal{A}(x, Du)| \leq C_0\alpha(S_h) \left| \{x \in S_{\sigma h} : \alpha(S_h)/C_0 \leq |D^2u - \mathcal{A}(x, Du)| \leq C_0\alpha(S_h)\} \right|,$$

where $S_h = S_h(x_0)$ with x_0 an arbitrary point inside $B(0, 3/4)$, and C_0 is universal.

Proof of Theorem 3.1. Let $M \gg 1$ be fixed later, set $R_0 := 3/4$, and for all $m \geq 1$ define

$$(3.10) \quad R_m := R_{m-1} - \bar{C}M^{-\beta}$$

with \bar{C} and β as in (2.26). Let us denote $B(0, R_m)$ by B_{R_m} , set

$$(3.11) \quad F(x) := |D^2u(x) - \mathcal{A}(x, Du(x))|,$$

and define

$$(3.12) \quad D_m := \{x \in B_{R_m} : F(x) \geq M^m\}.$$

Thanks to Proposition 2.1, there exists $\rho > 0$ universal such that $S_h(x) \subset B(0, 5/6)$ for any $x \in B(0, 3/4)$ and $h \leq 2\rho$, and by Lemma 2.2 applied with $h = \rho$ we get $\alpha(S_\rho(x)) \approx 1$. In addition, since (C0) implies that $|\mathcal{A}(x, Du)| \leq C_1$ inside $B(0, 1)$ for some C_1 universal, we see that $||D^2u| - F| \leq C_1$. Hence, using Lemma 2.1, we deduce that if $M \gg M_1 + C_1$, then there exists a small universal constant $\nu > 0$ such that

$$\alpha(S_h(x)) \geq \nu M^m \quad \forall x \in D_m, \quad h \leq \min\{\bar{h}(x), \rho\}, \quad m \geq 1.$$

So, by choosing $M \geq \max\{1/\nu^4, M_1\}$ (so that $\nu M^{m+1} \geq M^{m+1/2}/\nu$), by continuity we obtain that for every point in D_{m+1} , there exists $h_x \in (0, \min\{\bar{h}(x), \rho\})$ such that

$$(3.13) \quad \alpha(S_{h_x}) \in (\nu M^{m+1/2}, M^{m+1/2}/\nu).$$

In particular, by (2.26) we have $\text{diam}(S_{h_x}) \leq \bar{C}M^{-\beta}$, which implies that (recall (3.10))

$$(3.14) \quad \bigcup_{x \in D_{m+1}} S_{h_x} \subset B(0, R_{m+1} + \bar{C}M^{-\beta}) = B_{R_m}.$$

According to Proposition 2.2, we can cover D_{m+1} with finitely many sections $\{S_{h_{x_j}}\}_{x_j \in D_{m+1}}$ such that $S_{\sigma h_{x_j}}$ are disjoint. Then (3.13) and (3.9) imply (recall (3.11))

$$\begin{aligned} \int_{D_{m+1}} F &\leq \sum_j \int_{S_{h_{x_j}}} F \\ &\leq \sum_j C_0 \alpha(S_{h_{x_j}}) |\{x \in S_{\sigma h_{x_j}} : \alpha(S_{h_{x_j}})/C_0 \leq F \leq C_0\alpha(S_{h_{x_j}})\}| \\ &\leq \sum_j \frac{C_0}{\nu} M^{m+1/2} |\{x \in S_{\sigma h_{x_j}} : \nu M^{m+1/2}/C_0 \leq F \leq C_0 M^{m+1/2}/\nu\}|. \end{aligned}$$

Assuming now that $\sqrt{M} \geq C_0/\nu$ and recalling (3.14), we obtain

$$\begin{aligned}
 (3.15) \quad \int_{D_{m+1}} F &\leq \sum_j \int_{S_{h_{x_j}}} F \\
 &\leq \sum_j \frac{C_0}{\nu} M^{m+1/2} |\{x \in S_{\sigma h_{x_j}} : \nu M^{m+1/2}/C_0 \leq F \leq C_0 M^{m+1/2}/\nu\}| \\
 &\leq \sum_j \frac{C_0}{\nu} M^{m+1/2} |\{x \in S_{\sigma h_{x_j}} : M^m \leq F \leq M^{m+1}\}| \\
 &= \sum_j \frac{C_0}{\nu} M^{m+1/2} |\{x \in S_{\sigma h_{x_j}} : M^m \leq F \leq M^{m+1}\} \cap B_{R_m}| \\
 &\leq \frac{C_0 \sqrt{M}}{\nu} \int_{D_m \setminus D_{m+1}} F.
 \end{aligned}$$

Adding $\frac{C_0 \sqrt{M}}{\nu} \int_{D_{m+1}} F$ to both sides of the previous inequality, we obtain

$$\left(1 + \frac{C_0 \sqrt{M}}{\nu}\right) \int_{D_{m+1}} F \leq \frac{C_0 \sqrt{M}}{\nu} \int_{D_m} F,$$

which implies

$$\int_{D_{m+1}} F \leq (1 - \tau) \int_{D_m} F$$

for some small constant $\tau = \tau(M) > 0$. We finally fix M so that it also satisfies

$$(3.16) \quad \sum_{m \geq 1} \bar{C} M^{-m\beta} \leq \frac{1}{4}.$$

In this way $R_m \geq 1/2$ for all $m \geq 1$, so that the above inequalities and the definition of R_m imply

$$\int_{\{F \geq M^m\} \cap B(0,1/2)} F \leq \int_{D_m} F \leq (1 - \tau)^m \int_{D_0} F \leq C(1 - \tau)^m.$$

(Here we used that $\int_{B(0,3/4)} F \leq C$, which can be easily proved arguing as in the proof of Lemma 3.2.) Thus, choosing $\kappa > 0$ such that $1 - \tau = M^{-2\kappa}$, we deduce that

$$|\{F \geq t\} \cap B(0,1/2)| \leq \frac{1}{t} \int_{\{F \geq t\} \cap B(0,1/2)} F \leq C t^{-1-2\kappa}$$

for some $C > 0$ universal, which implies that $F \in L^{1+\kappa}(B(0,1/2))$. Recalling the definition of F (see (3.11)) and that $|\mathcal{A}(x, Du)| \leq C$ inside $B(0,1)$ (by (C0)), this concludes the proof. \square

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