CONTROLLABILITY OF 2D EULER AND NAVIER-STOKES EQUATIONS BY DEGENERATE FORCING

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Abstract. We study controllability issues for the 2D Euler and Navier-Stokes (NS) systems under periodic boundary conditions. These systems describe motion of homogeneous ideal or viscous incompressible fluid on a two-dimensional torus $\mathbb{T}^2$. We assume the system to be controlled by a degenerate forcing applied to fixed number of modes.

In our previous work [3, 5, 4] we studied global controllability by means of degenerate forcing for Navier-Stokes (NS) systems with non-vanishing viscosity ($\nu > 0$). Methods of differential geometric/Lie algebraic control theory have been used for that study. In [3] criteria for global controllability of finite-dimensional Galerkin approximations of 2D and 3D NS systems have been established. It is almost immediate to see that these criteria are also valid for the Galerkin approximations of the Euler systems. In [5, 4] we established a much more intricate sufficient criteria for global controllability in finite-dimensional observed component and for L_2-approximate controllability for 2D NS system. The justification of these criteria was based on a Lyapunov-Schmidt reduction to a finite-dimensional system. Possibility of such a reduction rested upon the dissipativity of NS system, and hence the previous approach can not be adapted for Euler system.

In the present contribution we improve and extend the controllability results in several aspects: 1) we obtain a stronger sufficient condition for controllability of 2D NS system in an observed component and for L_2-approximate controllability; 2) we prove that these criteria are valid for the case of ideal incompressible fluid ($\nu = 0$); 3) we study solid controllability in projection on any finite-dimensional subspace and establish a sufficient criterion for such controllability.

Keywords: incompressible fluid, 2D Euler system, 2D Navier-Stokes system, controllability

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1. Introduction

The present paper extends our work started in [3, 5, 4] on studying controllability of 2- and 3- dimensional Navier-Stokes equations (2D and 3D NS systems) under periodic boundary conditions. The characteristic feature of our problem setting is a choice of control functions; we are going to control the 2D NS/Euler system by means of *degenerate* forcing. The corresponding equations are

$$\partial u/\partial t + (u \cdot \nabla)u + \nabla p = \nu\Delta u + F(t, x), \quad (1)$$

$$\nabla \cdot u = 0. \quad (2)$$

The words "degenerate forcing" mean that $F(t,x)$ is a "low-order" trigonometric polynomial with respect to x, i.e. a sum of a "small number" of harmonics:

$$F(t, x) = \sum_{k \in \mathcal{K}^1} v_k(t)e^{ik\cdot x}, \ \mathcal{K}^1 \text{ is finite.}$$

The word "control" means that the components $v_k(t),\ t \in [0, T]$ of the forcing can be chosen freely among measurable essentially bounded functions. In fact to achieve controllability piecewise-constant controls suffice.

In [3, 5, 4] we derived sufficient controllability criteria for Galerkin approximations of 2D and 3D NS systems. For the 2D NS system we established sufficient criteria for so called, controllability in finite-dimensional observed component and for L_2-approximate controllability. The corresponding definitions can be found in the Section 3.

Now we consider both cases of viscous ($\nu > 0$) and *ideal* ($\nu = 0$) incompressible fluid simultaneously. To establish a possibility to propagate the action of small dimensional control to (a finite number of) higher modes we use the technique of Lie extensions developed in the scope of geometric control theory (see [2, 16]). For finite-dimensional Galerkin approximations of 2D and 3D Euler systems ($\nu = 0$) the controllability criteria turn out to be the same as for 2D and 3D NS systems ($\nu > 0$) (see [3, 5, 4]). This is due to the fact that these controllability criteria are of "purely nonlinear" nature; they are completely determined by the nonlinear term of the Euler system.

Tools of Geometric Control Theory are not yet adapted too much to infinite-dimensional case. For dealing with inifinite-dimensional dynamics we used in [3, 5, 4] a Lyapunov-Schmidt reduction to a finite-dimensional system. The possibility of such a reduction rested upon dissipativity of the NS system, which is not anymore present when one deals with Euler system.

In the present paper we abandon the Lyapunov-Schmidt reduction and instead refine the tools of geometric control in order to deal with viscous and nonviscous case at the same time. This refinement also allow us to improve the sufficient criterion of controllability in observed component for 2D NS/Euler system. The criterion, formulated in terms of so-called 'saturating property' of the set of controlled forcing modes, is stronger then the one established in [3, 5, 4]. Analysis of the saturation property in [15] showed that a generic symmetric set of 4 controlled modes suffices for achieving controllability.

For a saturating set we manage to prove L_2-approximate controllability for 2D NS/Euler system. We also study controllability in finite-dimensional projections. The latter property means that the attainable set of 2D NS/Euler system is projected surjectively onto any finite-dimensional subspace of H_2.

There was an extensive study of controllability of the Navier-Stokes and Euler equations in particular by means of boundary control. There are various results on exact local controllability of 2D and 3D Navier-Stokes equations obtained by A.Fursikov, O.Imanuilov, global exact controllability for 2D Euler equation obtained by J.-M. Coron, global exact controllability for 2D Navier-Stokes equation by A.Fursikov and J.-M. Coron. The readers may turn to the book [10] and to the surveys [11] and [8] for further references.

Our problem setting differs from the above results by the class of *degenerate distributed controls* which is involved. In closer relation to our work is a publication of M.Romito ([21]) who provided a criterion for controllability of Galerkin approximations of 3D NS systems. J.C.Mattingly and E.Pardoux adapted ([19])) the controllability result from [5] for studying properties of the solutions of stochastically forced 2D NS systems. M.Hairer and J.C.Mattingly have applied ([15]) the controllability results to studying ergodicity of 2D Navier-Stokes equation under degenerate stochastic forcing.

The structure of our paper is as follows. Section 2 contains a necessary minimum of standard preliminary material on 2D Euler and NS systems. The problem setting in the Section 3 is succeeded by the formulation of the main results in the Section 4. These results include sufficient criteria for controllability in observed component, for solid controllability in finite-dimensional projection, and for L_2-approximate controllability for both 2D NS and 2D Euler systems. The controllability criteria rest upon so-called 'saturating property' - a arithmetic property of the set $\mathcal{K}^1 \subset \mathbb{Z}^2$ of controlled modes. In Subsection 4.1 we provide necessary and sufficient conditions for this property to hold.

The rest of the paper is devoted to the proofs of these results. Among the tools involved are some results on equiboundedness and continuous dependence of solutions of 2D NS/Euler systems on relaxed forcings. Being interesting for their own sake these results are formulated in Section 5 and are proved in Appendix. In Section 6 we accomplish the proof of (solid) controllability in observed component. The construction, introduced in this proof, is crucial for the proof of controllability in a finite-dimensional projection accomplished in Section 7. Proof of L_2-approximate controllability is similar; the readers can either complete it by themselves or consult [5].

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2. Preliminaries on 2D NS/Euler System: vorticity, spectral method, Galerkin approximations

We consider the 2D NS/Euler system (1)-(2). The boundary conditions are assumed to be periodic, i.e. one may assume the velocity field u to be defined on the 2-dimensional torus $\mathbb{T}^2$. Besides we assume

$$\int_{\mathbb{T}^2} u dx = 0. \tag{3}$$

Let us introduce the vorticity $w = \nabla^{\perp} \cdot u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ of u. Applying the operator $\nabla^{\perp}$ to the equation (1) we arrive to the equation:

$$\partial w/\partial t + (u \cdot \nabla)w - \nu \Delta w = v(t, x), \tag{4}$$

where $v(t, x) = \nabla^{\perp} \cdot F(t, x)$.

Notice that: i) $\nabla^{\perp} \cdot \nabla p = 0$, ii) $\nabla^{\perp}$ and Δ commute as linear differential operators in x with constant coefficients;

$$\text{iii) } \nabla^{\perp} \cdot (u \cdot \nabla)u = (u \cdot \nabla)(\nabla^{\perp} \cdot u) + (\nabla^{\perp} \cdot u)(\nabla \cdot u) = (u \cdot \nabla)w,$$

for all u satisfying (2).

It is known that u, which satisfies the relations (2) and (3), can be recovered in a unique way from w. From now on we will deal with the equation (4).

A natural and standard (see [6, 7]) way to view the NS systems is to represent them as evolution equations in Hilbert spaces.

Consider Sobolev spaces $H^{\ell}(\mathbb{T}^s)$ with the scalar product defined as

$$\langle u, u' \rangle_{\ell} = \sum_{\alpha \leq \ell} \int_{\mathbb{T}^s} (\partial^{\alpha} u/\partial x^{\alpha})(\partial^{\alpha} u'/\partial x^{\alpha}) dx;$$

the norm $\| \cdot \|_{\ell}$ is defined by virtue of this scalar product. Denote by H_{ℓ} the closures of $\{u \in C^{\infty}(\mathbb{T}^s), \nabla \cdot u = 0\}$ in the norms $\| \cdot \|_{\ell}$ in the respective spaces $H^{\ell}(\mathbb{T}^s)$, $\ell \geq 0$. The norms in H_{ℓ} will be denoted again by $\| \cdot \|_{\ell}$. It will be convenient for us to redefine the norm of H_1 by putting $\|u\|_1^2 = \langle -\Delta u, u \rangle$, and the norm of H_2 by putting $\|u\|_2^2 = \langle -\Delta u, -\Delta u \rangle$.

Results on global existence and uniqueness of weak and classical solutions of NS systems in bounded domains can be found in [7, 6, 18]. Similar results for the inviscid (Euler) case - W.Wolibner's existence and uniqueness theorem - are presented in [17]. Formulation in [9] allows for asserting global existence and uniqueness of trajectories $t \mapsto u_t$ (respectively $t \mapsto w_t$ for the vorticity) of the 2D Euler systems in any Sobolev space H_s with $s > 2$ (respectively with $s > 1$ for the vorticity), provided that the initial data belongs to these spaces.

Let us consider now the basis of eigenfunctions $\{e^{ik \cdot x}\}$ of the Laplacian on $\mathbb{T}^2$ and take the Fourier expansion of the vorticity $w(t, x) = \sum_k q_k(t) e^{ik \cdot x}$ and control $v(t, x) = \sum_k v_k(t) e^{ik \cdot x}$; here $k \in \mathbb{Z}^2$. As far as w and f are real-valued, we have $\bar{q}_n = q_{-n}$, $\bar{v}_n = v_{-n}$. We assume $v_0 = 0$; by (3) $q_0 = 0$.

Evidently $\partial w/\partial t = \sum_k \dot{q}_k(t) e^{ik \cdot x}$. To compute $(u \cdot \nabla)w$ we write the equalities

$$\nabla^{\perp} \cdot u = w, \ \nabla \cdot u = 0 \Leftrightarrow -\partial_2 u_1 + \partial_1 u_2 = w, \ \partial_1 u_1 + \partial_2 u_2 = 0.$$

From these latter we conclude by a standard reasoning that

$$u_1 = \sum_{k \in \mathbb{Z}^2 \setminus 0} q_k(t)(ik_2/|k|^2) e^{ik \cdot x}, \ u_2 = -\sum_{k \in \mathbb{Z}^2 \setminus 0} q_k(t)(ik_1/|k|^2) e^{ik \cdot x},$$

and

$$(5) \qquad (u \cdot \nabla)w = \sum_{k \in \mathbb{Z}^2 \setminus 0} (\sum_{m+n=k} (m \wedge n)|m|^{-2} q_m(t) q_n(t)) e^{ik \cdot x},$$

where $m \wedge n = m_1 n_2 - m_2 n_1$ is the external product of $m = (m_1, m_2), n = (n_1, n_2)$.

Now the 2D NS/Euler system can be written (cf. [12]) as an (infinite-dimensional) system of ODE for q_k:

$$(6) \qquad \dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ \ k, m, n \in \mathbb{Z}^2.$$

Observe that the product $q_m q_n$ enters the sum $\sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n$ twice with (a priori) different coefficients. Therefore this sum can be rearranged

$$(7) \quad \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n = \sum_{m+n=k, |m|<|n|} (m \wedge n)(|m|^{-2} - |n|^{-2}) q_m q_n.$$

From the last representation we conclude that $q_m q_n$ does not appear in the equations whenever $|m| = |n|$.

Consider any finite subset $\mathcal{G} \subset \mathbb{Z}^2$ and introduce the Galerkin $\mathcal{G}$-approximation of the system (4) or of the system (6) by projecting this system onto the linear space spanned by the harmonics $e^{ik \cdot x}$ with $k \in \mathcal{G}$. The result is a finite-dimensional system of ODE or a control system

$$(8) \qquad \dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ \ k, m, n \in \mathcal{G}.$$

3. 2D NS/Euler system controlled by degenerate forcing. Problem setting

We study the case where the 2D NS/Euler system is forced by a trigonometric polynomial: $v(t, x) = \sum_{k \in \mathcal{K}^1} v_k e^{ik \cdot x}$, where $\mathcal{K}^1$ is a finite set. Such forcing is called *degenerate*. As we said $v_k(\cdot)$ with $k \in \mathcal{K}^1$ are controls at our disposal; they are measurable essentially bounded functions and can be chosen arbitrary besides standard agreement of complex conjugation: $v_{-k} = \bar{v_k}$. From now on we assume the set $\mathcal{K}^1$ of controlled modes to be *symmetric*: $k \in \mathcal{K}^1 \Leftrightarrow -k \in \mathcal{K}^1$.

Let us introduce a finite symmetric set of *observed* modes indexed by $k \in \mathcal{K}^{obs} \subset \mathbb{Z}^2$. The observed modes are reunited in so-called observed component.

We assume $\mathcal{K}^{obs} \supseteq \mathcal{K}^1$. As we will see, nontrivial controllability issues arise only if $\mathcal{K}^1$ is a proper subset of $\mathcal{K}^{obs}$. We identify the space of observed modes with $\mathbb{R}^N$ and denote by Π^{obs} the operator of projection of solutions onto the space of observed modes.

We will represent the controlled 2D NS/Euler equation in the following splitted (controlled -observed -unobserved components) form:

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ k \in \mathcal{K}^1, \tag{9}$$

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k, \ k \in \mathcal{K}^{obs} \setminus \mathcal{K}^1, \tag{10}$$

$$\dot{Q} = \nu \Delta Q + B(q, Q). \tag{11}$$

In the latter equation $B(q, Q)$ stays for the projection of the nonlinear term of the Euler system onto the space of unobserved modes.

Galerkin $\mathcal{K}^{obs}$-approximation of the 2D NS/Euler system consists of the equations (9)-(10) under an additional condition $m, n \in \mathcal{K}^{obs}$ for the summation indices.

Definition 3.1. *Galerkin $\mathcal{K}^{obs}$-approximation of 2D NS/Euler systems is time-T globally controllable if for any two points $\tilde{q}, \hat{q}$ in $\mathbb{R}^N$, there exists a control which steers in time T this Galerkin approximation from $\tilde{q}$ to $\hat{q}$.* □

Definition 3.2. *(controllability in observed component) 2D NS/Euler system is time-T globally controllable in observed N-dimensional component if for any $\tilde{\varphi} \in H_2$ and any $\hat{q} \in \mathbb{R}^N$ there exists a control which steers the system in time T from $\tilde{\varphi}$ to some $\hat{\varphi} \in (\Pi^{obs})^{-1}(\hat{q})$.* □

In other words the 2D NS/Euler system is globally controllable in observed component if its time-T attainable set (from each point) is projected by Π^{obs} onto the whole coordinate subspace spanned by the observed modes. Notice that the observed dynamics (9)-(10) is affected by infinite-dimensional dynamics (11).

We generalize the previous definition.

Definition 3.3. *(controllability in finite-dimensional projection) Let $\mathcal{L}$ be a finite-dimensional subspace of $H_2(\mathbb{T}^2)$ and $\Pi^{\mathcal{L}}$ be L_2-orthogonal projection of $H_2(\mathbb{T}^2)$ onto $\mathcal{L}$. The 2D NS/Euler system is time-T globally controllable in projection on $\mathcal{L}$ if for any $\tilde{\varphi} \in H_2(\mathbb{T}^2)$ and for any $\hat{q} \in \mathbb{R}^N$ there exists a control which steers the system in time T from $\tilde{\varphi}$ to some $\hat{\varphi} \in (\Pi^{\mathcal{L}})^{-1}(\hat{q})$.* □

Remark 3.1. *Controllability in observed component amounts to controllability in finite-dimensional projection on a coordinate subspace $\mathcal{L}$.* □.

Definition 3.4. *(L_2-approximate controllability) The 2D NS/Euler system is time-T L_2-approximately controllable, if for any two points $\tilde{\varphi}, \hat{\varphi} \in H_2$ and for any $\varepsilon > 0$ there exists a control which steers the system in time T from $\tilde{\varphi}$ to the ε-neighborhood of $\hat{\varphi}$ in L_2-norm.* □

Let us introduce some useful terminology.

Definition 3.5. *Fix initial condition $\tilde{\varphi} \in H_2(\mathbb{T}^2)$ for trajectories of the controlled 2D NS/Euler system.*

The correspondence between the controlled forcing $v(\cdot) \in L_\infty\left([0,T];\mathbb{R}^d\right)$ *and the corresponding trajectory (solution)* w_t *of the system is established by forcing/trajectory map (*$\mathcal{F}/\mathcal{T}$*-map).*

The correspondence between the controlled forcing $v(\cdot)$ *and the observed component* $q(t) = \Pi^{obs} w_t$ *(an* $\mathbb{R}^N$*-valued function) of the corresponding trajectory is established by forcing/observation map (*$\mathcal{F}/\mathcal{O}$*-map).*

If NS/Euler system is considered on an interval $[0,T]$ $(T < +\infty)$*, then the map* $\mathcal{F}/\mathcal{T}_T : v(\cdot) \mapsto w_T$ *is called end-point map; the map* $\Pi^{obs} \circ \mathcal{F}/\mathcal{T}_T$ *is called end-point component map, the composition* $\Pi^{\mathcal{L}} \circ \mathcal{F}/\mathcal{T}_T$ *is called* $\mathcal{L}$*-projected end-point map.* □

Remark 3.2. *In the terminology of control theory the first two maps would be called input/trajectory and input/output maps correspondingly.* □

Remark 3.3. *Evidently time-T controllability of the NS/Euler system in observed component or in finite-dimensional projection is the same as surjectiveness of the corresponding end-point maps.* □

Invoking these maps we will introduce a stronger notion of *solid controllability.*

Definition 3.6. *Let* $\Phi : M^1 \mapsto M^2$ *be a continuous map between two metric spaces, and* $S \subseteq M^2$ *be any subset. We say that* Φ *covers* S *solidly, if* $S \subseteq \Phi(M^1)$ *and this inclusion is stable with respect to* C^0*-small perturbations of* Φ*, i.e. for some* C^0*-neighborhood* Ω *of* Φ *and for each map* $\Psi \in \Omega$*, there holds:* $S \subseteq \Psi(M^1)$*.* □

In what follows M^2 will be finite-dimensional vector space.

Definition 3.7. *(solid controllability in finite-dimensional projection) The 2D NS/Euler system is time-T solidly globally controllable in projection on finite-dimensional subspace* $\mathcal{L} \subset H_2$*, if for any bounded set* S *in* $\mathcal{L}$ *there exists a set of controls* B_S *such that* $\left(\Pi^{\mathcal{L}} \circ \mathcal{F}/\mathcal{T}_T\right)(B_S)$ *covers* S *solidly.* □

3.1. **Problem setting.** We address the following questions.

Question 1. Under what conditions the 2D NS/Euler system (9)-(10)-(11) is globally controllable in observed component? □

Question 2. Under what conditions the 2D NS/Euler system (9)-(10)-(11) is solidly controllable in any finite-dimensional projection? □

Question 3. Under what conditions the 2D NS/Euler system is L_2-approximately controllable? □

In [5, 4] we have answered the Questions 1,3 for the 2D NS system. In the present contribution we improve the previous results (provide sufficient controllability conditions under weaker hypothesi), extend them onto the case of ideal fluid (2D Euler equation) and answer the Question 2 for 2D NS and Euler systems.

4. Main results for $2D$ NS/Euler system

Let $\mathcal{K}^1 \subset \mathbb{Z}^2 \setminus \{0\}$ be a symmetric finite set of controlled forcing modes. Define the sequence of sets $\mathcal{K}^j \subset \mathbb{Z}^2$ iteratively as:

$$\mathcal{K}^j = \mathcal{K}^{j-1} \bigcup \{m+n| \ m,n \in \mathcal{K}^{j-1} \bigwedge \|m\| \neq \|n\| \bigwedge m \wedge n \neq 0\}. \tag{12}$$

Theorem 4.1. *(controllability in observed component) Let $\mathcal{K}^1$ be the set of controlled forcing modes, $\mathcal{K}^{obs}$ a symmetric finite set of observed modes. Define iteratively by (12) the sequence of sets $\mathcal{K}^j$, $j = 2, \ldots,$ and assume that $\mathcal{K}^M \supseteq \mathcal{K}^{obs}$ for some $M \geq 1$. Then for any $T > 0$ the 2D NS/Euler system (9)-(10)-(11) is time-T globally controllable in the observed component.* □

Remark 4.1. *The present sufficient criterion differs from the one, we obtained in* [5, 4]*, by the presence of the 'term' $\mathcal{K}^{j-1}$ in the right-hand side of the formula (12). With the new augmented $\mathcal{K}^j$'s and with new 'saturation property' (see Definition 4.2) controllability can be established under weaker hypothesi.* □

Theorem 4.1 characterizes controllability in projection on a finite-dimensional *coordinate* subspaces. A natural question is whether the system is controllable in projection on *any* finite-dimensional subspace (a relevance of this question for regularity of solutions for stochastically forced 2D NS system has been cleared to us by J.C.Mattingly and E.Pardoux).

Definition 4.2. *A finite set $\mathcal{K}^1 \subset \mathbb{Z}^2 \setminus \{0\}$ of forcing modes is called saturating if $\bigcup_{j=1}^{\infty} \mathcal{K}^j = \mathbb{Z}^2 \setminus \{0\}$, where $\mathcal{K}^j$ are defined by (12).* □

Theorem 4.3. *(controllability in finite-dimensional projection) Let $\mathcal{K}^1$ be a saturating set of controlled forcing modes and $\mathcal{L}$ be any finite-dimensional subspace of H_2. Then for any $T > 0$ the 2D NS/Euler system (9)-(10)-(11) is time-T solidly controllable in any finite-dimensional projection.* □

Another controllability result holds under similar assumptions.

Theorem 4.4. *(L_2-approximate controllability) Consider the 2D NS/Euler system controlled by degenerate forcing. Let $\mathcal{K}^1$ be a saturating set of controlled forcing modes. Then for any $T > 0$ the system (9)-(10)-(11) is time-T L_2-approximately controllable.* □

4.1. Saturating sets of forcing modes.

As we see the saturating property is crucial for controllability. In [15] the following characterization of this property has been established.

Proposition 4.5 ([15])**.** *Let $\mathcal{K}^1 \subset \mathbb{Z}^2$ be a symmetric finite set, $\mathcal{K}^j$ be defined by (12) for $j = 2, \ldots,$ and $\mathcal{K}^\infty = \bigcup_{j=1}^{\infty} \mathcal{K}^j$. The union $\mathcal{K}^\infty \bigcup \{0\}$ is an additive subgroup of $\mathbb{Z}^2$, if and only if $\mathcal{K}^1 \subset \mathbb{Z}^2$ contains two vectors which are not collinear and have different lengths.* □

Remark 4.2. *As it is well known any additive subgroup of $\mathbb{Z}^r$ is a lattice, i.e. a set of integer linear combinations $\sum_{j=1}^{s} a_j v^j$ of s $(s \leq r)$ linearly independent generators $v^1, \ldots, v^s \in \mathbb{Z}^r$ (see, for example,* [20] *or* [2]*).* $\square$

Corollary 4.6 ([15])**.** *If the additive subgroup, which appears in the Proposition 4.5, coincides with $\mathbb{Z}^2$, then the set $\mathcal{K}^1$ is saturating.* $\square$

Before providing a short proof of the Proposition 4.5 we elaborate more on the saturating property. The following elementary Lemma is involved.

Lemma 4.7. *Let $\mathcal{H}$ be an additive subgroup, generated by a set $\{v^1, \ldots, v^s\} \subset \mathbb{Z}^2$. Then $\mathcal{H}$ coincides with $\mathbb{Z}^2$ if and only if the greatest common divisor (g.c.d.) of the numbers $d_{ij} = v^i \wedge v^j,\ i, j \in \{1, \ldots, s\}$ equals* 1. $\square$

As before $x \wedge y = x_1 y_2 - x_2 y_1$ stays for the external product of $x, y \in \mathbb{Z}^2$ and it is assumed that 0 is divisible by any integer.

Proof of the Lemma. $(\Rightarrow)$ Let the g.c.d. of the numbers d_{ij} be μ. If $\mathcal{H} = \mathbb{Z}^2$, then the vectors $x = (1, 0), y = (0, 1)$ can be represented as integer linear combinations of $v^1, \ldots, v^s$ and therefore the external product $x \wedge y$ is an integer combination of d_{ij}. Hence $1 = x \wedge y$ is divisible by μ.

$(\Leftarrow)$ Let the g.c.d. of the numbers d_{ij} equal 1. Then at least one of d_{ij} does not vanish and the lattice (see Remark 4.2) $\mathcal{H}$ is 2-dimensional.

Take one of the pairs ξ, η of generators of $\mathcal{H}$ for which $\xi \wedge \eta$ admits minimal positive (integer) value ω. We claim that: i) $\omega = 1$, and ii) $\mathcal{H} = \mathbb{Z}^2$.

To conclude i) we reason as before. Indeed all v^r are linear integer combinations of ξ and η and therefore all d_{ij} have to be divisible by ω. To conclude ii) it is enough to observe that *for each* $\zeta \in \mathbb{Z}^2$ the linear equation $A\xi + B\eta = \zeta$ possesses *integer* solution

$$A = (\zeta \wedge \eta)/\omega = (\zeta \wedge \eta),\ B = (\xi \wedge \zeta)/\omega = (\xi \wedge \zeta),$$

and hence ξ, η generate $\mathbb{Z}^2 = \mathcal{H}$. $\square$

Theorem 4.8. *For a symmetric finite set $\mathcal{K}^1 = \{v^1, \ldots, v^s\} \subset \mathbb{Z}^2$ the following properties are equivalent:*

i) $\mathcal{K}^1$ is saturating;

ii) the greatest common divisor of the numbers $d_{ij} = v^i \wedge v^j,\ i, j \in \{1, \ldots, s\}$ equals 1 *and there exist $v^\alpha, v^\beta \in \mathcal{K}^1$, which are not collinear and have different lengths.* $\square$

This result is an immediate corollary of the Proposition 4.5 and the Lemma 4.7. Just note that $\mathcal{K}^\infty \bigcup \{0\}$ is obviously contained in the additive group generated by $\mathcal{K}^1$.

Corollary 4.9. *The set $\mathcal{K}^1 = \{(1, 0), (-1, 0), (1, 1), (-1, -1)\} \subset \mathbb{Z}^2$ is saturating.* $\square$

This leads to

Corollary 4.10. *Solid controllability in any finite-dimensional projection and L_2-approximately controllability can be achieved by forcing* 4 *modes.* $\square$

Now we turn to to the Proposition 4.5.

Proof of the Proposition 4.5. If any two vectors from $\mathcal{K}^1$ are either collinear or have the same length, then $\mathcal{K}^1=\mathcal{K}^2 = \cdots = \mathcal{K}^\infty$ and therefore the set $\mathcal{K}^\infty \bigcup\{0\}$ is finite, i.e. fails to be an additive subgroup of $\mathbb{Z}^2$.

Assume now that $m \wedge n \neq 0,\ |m| \neq |n|$. Without lack of generality we may assume: $m \cdot n \geq 0,\ |n| > |m|$.

By construction $\mathcal{K}^j$ and $\mathcal{K}^\infty$ are symmetric, provided $\mathcal{K}^1$ is. Therefore it suffices to prove that

$$j, k \in \mathcal{K}^\infty \bigcup\{0\} \Rightarrow\ j + k \in \mathcal{K}^\infty \bigcup\{0\}. \tag{13}$$

The case of either $j = 0$ or $k = 0$ is trivial. By virtue of (12) the implication (13) holds if

$$|j| \neq |k| \text{ and } j \wedge k \neq 0. \tag{14}$$

We have to study the resting cases.

From $m \cdot n > 0$ it follows that $|m + n| > |n| > |m|$. Besides $(m + n) \in \mathcal{K}^2 \subseteq \mathcal{K}^\infty$. Repeating the argument one obtains $\forall \sigma = 2, 3, \dots$:

$$|m + \sigma n| > \cdots > |m + n| > |n| > |m|$$

and $m + \sigma n \in \mathcal{K}^\infty$. Denote

$$m + n = p,\ m + 2n = q,\ m + 3n = r,\ m + 4n = s.$$

Each pair of vectors from the set $\{m, n, p, q, r, s\}$ is linearly independent.

1. Assume $j \wedge k = 0$, but $|j| \neq |k|$. Then $j + k \neq 0$. Exclude from the set $\{m, n, p, q, r, s\}$ the vector (at most one, if any) which is collinear to j (and then to k, and $j + k$). Exclude also the vectors (at most two, if any) whose lengths equal either $|j|$ or $|k|$.

Pick one of the remaining vectors, say m. Then $(j - m) \wedge (k + m) = (j + k) \wedge m \neq 0$. Similarly $(j + m) \wedge (k - m) \neq 0$. We claim that either $|j - m| \neq |k + m|$ or $|j + m| \neq |k - m|$. Indeed if both equalities hold, then

$$0 = (j - m) \cdot (j - m) - (k + m) \cdot (k + m) +$$
$$+(j + m) \cdot (j + m) - (k - m) \cdot (k - m) = 2(j \cdot j - k \cdot k),$$

and therefore $|j| = |k|$, which is a contradiction. Hence if , say $|j - m| \neq |k + m|$, then $j + k = (j + (-m)) + (k + m) \in \mathcal{K}^\infty$.

2. Assume $|j| = |k| \neq 0$. Exclude from the set $\{m, n, p, q, r, s\}$ the vector (at most one, if any) whose length equals $|j| = |k|$. Exclude also the vectors (at most three, if any), which are collinear to either j, or k or $j + k$. Finally exclude the vector (at most one, if any), which is orthogonal to $j + k$. There is at least one vector remained, say m.

Then

$$(j \pm m) \cdot (j \pm m) - (k \mp m) \cdot (k \mp m) = \pm 2(j + k) \cdot m \neq 0.$$

Therefore $|j + m| \neq |k - m|$ and $|j - m| \neq |k + m|$.

Calculating

$$(j + m) \wedge (k - m) - (j - m) \wedge (k + m) = 2m \wedge (j + k) \neq 0,$$

we see that at least one of two pairs $\{(j+m),(k-m)\}$, $\{(j-m),(k+m)\}$ is linearly independent. Hence we conclude, as in **1**, that $j+k \in \mathcal{K}^{\infty}$. □

5. Relaxation of forcing for 2D NS/Euler system: approximation results and uniform bounds for trajectories

In this Section we formulate some results on boundedness and continuity of solutions of 2D NS/Euler system with respect to the forcing. We assume the space of degenerate forcings to be endowed with a weak topology determined by so-called relaxation metric. These results are used in Section 6 for proving controllability in observed projection. Besides they are interesting for their own sake as an example of application of relaxed controls to NS/Euler and other classes of PDE systems. The proofs are rather technical; they are to be found in Appendix.

5.1. **Relaxation metric.**

Definition 5.1. *(see e.g.* [13, 14]*) The relaxation pseudometric in the space* $L^1\left([0,T],\mathbb{R}^d\right)$ *is defined by the seminorm*

$$\|u(\cdot)\|_{rx} = \max_{t\in[0,T]} \left\{ \left\| \int_0^t u(\tau)d\tau \right\|_{R^d} \right\}.$$

The relaxation metric is obtained by identification of the functions which coincide for almost all $\tau \in [0,T]$. □

The relaxation metric is weaker than the natural metric of $L^1\left([0,T],\mathbb{R}^d\right)$. The relaxation norms of fast oscillating functions are small, while their L_1-norms can be large. For example

$$\|\omega^{1/2}\cos\omega t\|_{rx} = \max_{t\in[0,T]} \left| \int_0^t \omega^{1/2}\cos\omega\tau d\tau \right| \leq \omega^{-1/2},$$

and $\|\omega^{1/2}\cos\omega t\|_{rx} \to 0,$ as $\omega \to +\infty$, while $\|\omega^{1/2}\cos\omega t\|_{L_1} \to +\infty,$ as $\omega \to +\infty$.

Lemma 5.2. *Let for integrable functions* $\phi_n(\cdot)$, $n=1,2,\ldots,$ *their relaxation norms* $\|\phi_n(\cdot)\|_{rx} \stackrel{n\to\infty}{\longrightarrow} 0$. *Let* $\{r_\beta(t)|\ \beta\in\mathcal{B}\}$ *be a family of absolutely-continuous functions with their* $W_{1,2}$*-norms equibounded:*

$$\exists C:\ \|r_\beta(0)\|^2 + \int_0^T (\dot{r}_\beta(\tau))^2 d\tau \leq C^2,\ \forall\beta\in\mathcal{B}.$$

Then $\|r_\beta(\cdot)\phi_n(\cdot)\|_{rx} \stackrel{n\to\infty}{\longrightarrow} 0$, *uniformly with respect to* $\beta \in B$. □

Proof.

$$\left|\int_0^\tau r_\beta(t)\phi_n(t)dt\right| = \left| r_\beta(\tau)\int_0^\tau \phi_n(t)dt - \int_0^\tau \dot{r}_\beta(t)\int_0^t \phi_n(\theta)d\theta dt\right| \leq$$
$$\leq C\left(1+2\sqrt{\tau}\right)\|\phi_n(t)\|_{rx}. \ \square$$

5.2. **Boundedness of solutions of forced 2D NS/Euler system.** Consider a set $\mathbf{F}$ of degenerate forcings $v(t,x)=\sum_{k\in\mathcal{K}^1} v_k(t)e^{ik\cdot x}$; $\sharp\mathcal{K}^1=d$. We identify these forcings with vector-functions $v(t)=(v_k(t))\in L_\infty([0,T];\mathbb{R}^d)$. Forced 2D NS/Euler system is treated as an evolution equation in $H_s,\ s\geq 2$. An example of boundedness result, we are interested in, would be the following

Lemma 5.3. *Assume the set* $\mathbf{F}$ *of degenerate forcings to be bounded in the relaxation metric. Fix the time interval* $[0,T]$ *and the initial condition* $w(0)=w_0\in H_s,\ s\geq 2$ *for the 2D NS/Euler system. Then the trajectories* w_t *of the system (4) forced by* $v(t,x)\in\mathbf{F}$ *are equibounded in* H_0 *norm:*

$$\exists b:\ vrai\sup_{t\in[0,T]}\|w_t\|_0\leq b.\ \square$$

This result is not covered by classical results on boundedness of solutions of 2D NS/Euler system because the set of forcings can be bounded in the relaxation metric while being unbounded in L_∞ and L_2 metric.

We will derive the previous result from a stronger assertion (Theorem 5.4). To formulate the assertion consider the primitives $V(\cdot)=\int_0^{\cdot} v(\tau)d\tau$ of $v(\cdot)\in\mathbf{F}$. By assumptions of the Lemma 5.3 $V(\cdot)$ are equibounded in the metric of $C^0([0,T],\mathbb{R}^d)$.

Denote the trigonometric polynomial $\sum_{k\in\mathcal{K}^1} V_k(t)e^{ik\cdot x}$ by $V_t(x)$. The forced (controlled) 2D Euler system can be written as

$$\partial w_t/\partial t=(u_t\cdot\nabla)w_t+\nu\Delta w_t+\partial V_t/\partial t. \tag{15}$$

Put $y_t=w_t-V_t$. The equation (15) can be rewritten as:

$$\partial y_t/\partial t=(u_t\cdot\nabla)(y_t+V_t)+\nu\Delta y_t+\nu\Delta V_t. \tag{16}$$

Recall that u_t is the divergence-free solution of the equation $\nabla^\perp\cdot u_t=w_t=y_t+V_t$. It can be represented as a sum $\mathcal{Y}_t+\mathcal{V}_t$, where $\mathcal{V}_t,\mathcal{Y}_t$ are the divergence-free solutions of the equations $\nabla^\perp\cdot\mathcal{V}_t=V_t,\ \nabla^\perp\cdot\mathcal{Y}_t=y_t$, with periodic boundary conditions.

Hence the equation (16) allows for the representation

$$\begin{aligned}\partial y_t/\partial t&=\left((\mathcal{Y}_t+\mathcal{V}_t)\cdot\nabla\right)(y_t+V_t)\\ &=(\mathcal{Y}_t\cdot\nabla)\,y_t+(\mathcal{V}_t\cdot\nabla)\,y_t+(\mathcal{Y}_t\cdot\nabla)\,V_t+\nu\Delta y_t+\nu\Delta V_t+(\mathcal{V}_t\cdot\nabla)\,V_t.\end{aligned} \tag{17}$$

This equation can be seen as 2D NS/Euler equation forced by y-linear forcing term $(\mathcal{V}_t\cdot\nabla)\,y_t+(\mathcal{Y}_t\cdot\nabla)\,V_t$ together with y-independent forcing term $\nu\Delta V_t+(\mathcal{V}_t\cdot\nabla)\,V_t$.

Consider instead of (17) a more general equation

$$\partial y_t/\partial t=(\mathcal{Y}_t\cdot\nabla)\,y_t+\left(\mathcal{V}_t^1\cdot\nabla\right)y_t+(\mathcal{Y}_t\cdot\nabla)\,V_t^2+\nu\Delta y_t+V_t^0, \tag{18}$$

where all the forcing terms V_t^0,V_t^1,V_t^2 are now decoupled and $\mathcal{V}_t^1$ is the divergence-free solution of the equation $\nabla^\perp\cdot\mathcal{V}_t^1=V_t^1$.

Consider the set $\mathbf{F}_B$ of triples $\{(V_t^0,V_t^1,V_t^2)\}$ satisfying the condition:

$$\sup_{t\in[0,T]}\max\{\|V_t^0\|,\|V_t^1\|,\|V_t^2\|\}\leq B,\ B>0. \tag{19}$$

As far as V_t^i are trigonometric polynomials (of fixed order) in x the norms $\|\cdot\|_{H_s}$ are equivalent for all s, so one just uses the notation $\|\cdot\|$.

All the results of this Section will be proven for the equation (18) and for the forcings from the set $\mathbf{F}_B$ defined by (19).

Remark 5.1. *The equation (18) is "nonclasically forced" 2D NS/Euler equation. Remarks on existence and uniqueness results for its solutions can be found in Appendix.* $\square$

Theorem 5.4. *Let* $\mathbf{F}_B = \{(V_t^0, V_t^1, V_t^2)\}$ *be the set defined by (19). Fix the time interval* $[0,T]$ *and the initial condition* $y(0) = y_0 \in H_s,\ s \geq 2$ *for the system (18) forced by the elements of* $\mathbf{F}_B$. *Then* $\exists b > 0$ *such that for all* $(V_t^0, V_t^1, V_t^2) \in \mathbf{F}_B$ *and for the corresponding trajectories* y_t *of the equation (18) there holds:*

$$i)\ vrai \sup_{t \in [0,T]} \|y_t\|_{L_\infty} \leq b; \tag{20}$$

$$ii)\ vrai \sup_{t \in [0,T]} \|y_t\|_{H_2} \leq b; \tag{21}$$

$$iii)\ \int_0^T \left\| \frac{\partial}{\partial t} y_t \right\|_1^2 dt \leq b. \ \square \tag{22}$$

Remark 5.2. *Obviously the conclusion of the Lemma 5.3 can be derived from this theorem.*

The proof of the Theorem 5.4 is to be found in Appendix.

5.3. **Continuous dependence of trajectories on relaxed forcings.** In this subsection we establish continuous dependence of trajectories of the equation (18) on the forcing terms V_t^1, V_t^2, V_t^0, as these latter vary continuously in the relaxation metric.

Theorem 5.5. *Consider the set* $\mathbf{F}_B = \{(V_t^0, V_t^1, V_t^2)\}$ *defined by (19). Fix the time interval* $[0,T]$ *and the initial condition* $y(0) = y_0 \in H_s,\ s \geq 2$ *for the system (18) forced by the elements of* $\mathbf{F}_B$. *Endow* $\mathbf{F}_B$ *with the relaxation metric and endow the space of trajectories of the 2D NS/Euler equation with* $L_\infty((0,T); H_0)$*-metric. Then the restriction of the forcing/trajectory map onto* $\mathbf{F}_B$ *is uniformly continuous.* $\square$

The proof of the Theorem 5.5 is to be found in Appendix.

6. Proof of controllability in observed component for 2D NS/Euler system

We prove first the result on solid controllability in observed component (Theorem 4.1). The construction introduced in proof is used to establish controllability in finite-dimensional projection and L_2-approximate controllability.

First we slightly particularize the assertion of the Theorem 4.1.

Theorem 6.1. *Let $\mathcal{K}^1$ be a set of controlled forcing modes. Define according to (12) the sequence of sets $\mathcal{K}^j,\ j = 2, \ldots,$ and assume that, for some M, $\mathcal{K}^M \supset \mathcal{K}^{obs}$.*

Then for all sufficiently small $T > 0$ the 2D NS/Euler system is solidly controllable in projection on the observed component. Besides one can choose the corresponding family of controls $v(\cdot, b)$ (cf. Definition 3.7) which is parameterized continuously in L_1-metric by a compact subset B_R of a finite-dimensional linear space and is uniformly (with respect to t, b) bounded: $\forall t, b: \ \|v(t, b)\| \leq A(T, R)$. $\Box$.

The only additional restriction in the claim of the latter result is smallness of time. To deal with large T we can apply zero control on the interval $[0, T - \theta]$ with θ small and then apply the result of the Theorem 6.1.

6.1. **Sketch of the proof.** By assumption the set $\mathcal{K}^{obs}$ of observed modes is contained in some $\mathcal{K}^M,\ M \geq 1$, from the sequence defined by (12). We will proceed by induction on M.

If $M = 1$ then $\mathcal{K}^1 \supset \mathcal{K}^{obs}$, i.e. all the equations for the observed modes contain controls. Then it is easy to establish small time controllability in observed component, given the fact that there are no a priori bounds on controls (this is done in Subsection 6.2).

Let $M > 1,\ \mathcal{K}^M \supset \mathcal{K}^{obs} \supset \mathcal{K}^1$. We start acting as if independent control parameters enter all the equations indexed by $k \in \mathcal{K}^M$. Then we are under previous assumption and hence can construct a needed family of controls. Though the control parameters indexed by $k \in \mathcal{K}^M \setminus \mathcal{K}^1$ are fictitious and our next step would be approximating the actuation of (some of) these fictitious controls by actuation of controls of *smaller* dimension. Now we employ the controls, which only enter the equations indexed by $k \in \mathcal{K}^{M-1} \subset \mathcal{K}^M$. The possibility of such approximation for 2D NS/Euler system (provided that the relation between $\mathcal{K}^{M-1}$ and $\mathcal{K}^M$ is established by (12)) is the main element of our construction.

If $M - 1 > 1$, then the approximating controls are also fictitious, but we can repeat the reasoning in order to arrive after $M - 1$ steps to *true* controls indexed by $\mathcal{K}^1$.

We can look at the process the other way around. Starting with a (specially chosen) family of degenerate controls in low modes, indexed by $\mathcal{K}^1$ we transfer their actuation to the higher modes via the *nonlinear term* of 2D NS/Euler system.

6.2. **Proof of the Theorem 6.1: first induction step.** The first induction step ($M = 1$) follows from the following Lemma.

Lemma 6.2. *Let $M = 1$, and $\mathcal{K}^1 = \mathcal{K}^{obs}$. The 2D NS/Euler system is split in the subsystems (9) and (11), which can be written in a concise form as*

$$dq^1/dt = f_1(q^1, Q) + v,\ dQ/dt = F(q^1, Q) \qquad (23)$$
$$q^1(0) = q^1_0,\ Q(0) = Q_0,$$

$\dim q^1 = N$. *Then for sufficiently small* $\tau > 0$*: there exists a family of controls* $v(t;b)$ *which satisfies the conclusion of the Theorem 6.1.* □

Proof. Without lack of generality we may assume the initial condition for the observed component to be $q^1(0) = 0_{R^{\kappa_1}}$. We do not diminish generality either by assuming $\mathcal{K}^1 = \mathcal{K}^{obs}$ instead of $\mathcal{K}^1 \supseteq \mathcal{K}^{obs}$. Recall that $\Pi_1 : (q^1, Q) \to q^1$.

Define for $y \in \mathbb{R}^N$, $\|y\|_{l_1} = \sum_{j=1}^N |y_i|$. Let $\mathcal{C}_R = \{y \in \mathbb{R}^N | \ \|y\|_{l_1} \leq R\}$.

Fix $\gamma > 1$. Take the interval $[0, \tau]$; the value of *small* $\tau > 0$ will be specified later on. For each $b \in \gamma\mathcal{C}_R$ take $v(t;p,\tau) = \tau^{-1}p$ - a constant control. Obviously $\gamma\mathcal{C}_R \supset \mathcal{C}_R$ and $\int_0^\tau v(t;p,\tau)dt = p$. For fixed $\tau > 0$ the map $p \mapsto v(t;p,\tau)$ is continuous in L_1-metric.

We claim that $\exists \tau_0 > 0$ such that for $\tau \in (0, \tau_0)$ the family of controls $v(t;p,\tau)$, $p \in \gamma\mathcal{C}_R$ satisfies the conclusion of the Lemma, so one may take $b = p$, $B_R = \gamma\mathcal{C}_R$.

Denote for fixed $\tau > 0$ the map

$$p \mapsto v(\cdot;p,\tau) \mapsto \left(\Pi_1 \circ \mathcal{F}/\mathcal{O}_\tau\right)\left(v(\cdot;p,\tau)\right)$$

by $\Phi(p;\tau)$. Recall that $\Pi_1 \circ \mathcal{F}/\mathcal{O}_\tau$ is the end-point component map (cf. Definition 3.5). The map $p \mapsto v(\cdot;p,\tau)$ is continuous in L_1-metric of controls and hence also in the relaxation metric. Therefore by Theorem 5.5 the map $p \mapsto \Phi(p;\tau)$ is continuous.

Restrict the equations (23) to the interval $[0, \tau]$ and proceed with time substitution $t = \tau\xi$, $\xi \in [0,1]$. The equations take form:

$$dq^1/d\xi = \tau f_1(q^1, Q) + p, \ dQ/d\xi = \tau F(q^1, Q), \ \xi \in [0,1]. \tag{24}$$

For $\tau = 0$ the 'limit system' of (24) is

$$dq_0^1/d\xi = p, \ dQ_0/d\xi = 0, \ \xi \in [0,1]. \tag{25}$$

The end-point component map $p \mapsto q_0^1(1)$ for the limit system is the identity.

From classical results on boundedness of solutions of 2D NS/Euler system we conclude that q^1-components of the solutions of the systems (24) and (25) (with the same initial condition) deviate by a quantity $\leq C\tau$, where the constant C can be chosen independent of p, τ for sufficiently small $\tau > 0$. Then $\|\Phi(\cdot;\tau) - Id\| \leq C\tau$. By degree theory argument there exists τ_0 such that $\forall \tau \in (0, \tau_0)$ the image of $p \mapsto \Phi(p;\tau)$ covers $\mathcal{C}_R$ solidly.

To complete the proof note that $\|v(t;p,\tau)\|$ are uniformly bounded by $\gamma R \tau^{-1}$. □

In what follows we will need a modification of the previous Lemma.

Lemma 6.3. *Consider the system (23) and impose the boundary conditions*

$$q^1(0) = \phi(p), q^1(\tau) = \psi(p), \ p \in P, \ P \text{ - compact}, \ \phi, \psi \text{ - continuous},$$

on its q^1*-component.*

Then for all sufficiently small $\tau > 0$, there exists a family of controls $v(t;p,\tau)$ defined on $[0,\tau]$, such that the corresponding trajectories, which meet the initial condition, meet the end-point condition approximately:

$$\|q^1(\tau;p) - \psi(p)\| \leq C\tau.$$

Besides $\|Q(t) - Q_0\|_0 \leq \gamma C\tau,\ \forall t \in [0,\tau]$. Here C can be chosen independent on p, τ. □

The proof is similar to the previous one. One can choose the family of controls $v(t,p) = \tau^{-1}(\psi(p) - \phi(p)),\ t \in [0,\tau]$.

6.3. **Generic induction step: solid controllability by extended controls.** Let us proceed further with the induction. Assume that the statement of the Theorem 6.1 has been proven for all $M \leq (N-1)$; we are going to prove it for $M = N$.

Consider now the system (9)-(10)-(11) with the *extended* set $\mathcal{K}_e^1 = \mathcal{K}^2$ of controlled forcing modes; $\mathcal{K}_e^1 \supseteq \mathcal{K}^1$.

Obviously this new system satisfies the conditions of the Theorem 6.1; indeed

$$\mathcal{K}_e^1 = \mathcal{K}^2 \Rightarrow \mathcal{K}_e^j = \mathcal{K}^{j+1},\ j \geq 1,$$

for the sets $\mathcal{K}_e^j, \mathcal{K}^j$, defined by (12). Hence $\mathcal{K}_e^{M-1} = \mathcal{K}^M \supseteq \mathcal{K}^{obs}$.

By induction hypothesis the system with extended controls is solidly controllable in observed projection: there exists a continuous in L_1-metric family of extended controls $v(t;b)$ which satisfies the conclusion of the Theorem 6.1.

This family of controls is uniformly bounded; assume that $\|v(t;b)\|_{l_1} \leq A,\ \forall b \in B,\ \forall t \in [0,T]$. The values of $v(t;b)$ belong to $\mathbb{R}^{\kappa_2}$, where $\kappa_2 = \#\mathcal{K}^2 = \mathcal{K}_e^1$.

Evidently these extended controls are unavailable for the original problem. We are going to approximate their action by the action controls from a more restricted set.

To this end let us first take the vectors $e_1, \ldots, e_{k_2}$ from the standard basis in $\mathbb{R}^{k_2}$ together with their opposites $-e_1, \ldots, -e_{k_2}$. Multiply each of these vectors by A and denote the set of these $2\kappa_2$ vectors by E_2^A. The convex hull $\text{conv}E_2^A$ of E_2^A contains all the values of $v(t;b)$.

First we will approximate the family of functions $v(t;b)$ which take their values in $\text{conv}E_2^A$ by E_2^A-valued functions. Such a possibility is a central result of relaxation theory.

We will apply a modification of R.V.Gamkrelidze's Approximation Lemma (see [14, Ch.3],[13, p.119]). According to it, given $\varepsilon > 0$ and a parameterized family of conv E_2-valued functions, which varies continuously in $L-1$ metric with parameter, one can construct a continuously parameterized family of E_2-valued functions which ε-approximates the first family in the relaxation metric uniformly with respect to the parameter. Moreover the functions of the approximating family can be chosen piecewise-constant and the number L of the intervals of constancy can be chosen the same for all $b \in B$.

Actually the Approximation Lemma in [14, Ch.3] regards weak approximation of strongly continuous families of relaxed controls (Young measures).

Proposition 6.4. *(cf. Approximation Lemma;* [14, Ch.3]*). Let B be a compact and $\{v(t;b)|b \in B\}$ be a family of (conv E_2^A)-valued functions, which depends on $b \in B$ continuously in L_1 metric. Then for each $\varepsilon > 0$ one can construct a L_1-continuous equibounded family $\{z(t;b)|\ b \in B\}$ of E_2^A-valued functions which ε-approximates the family $\{v(t;b)|b \in B\}$ in the relaxation metric uniformly with respect to $b \in B$. Moreover the functions $z(t;b)$ can be chosen piecewise-constant and the number L of the intervals of constancy can be chosen the same for all $b \in B$. The intervals of constancy of these controls vary continuously with $b \in B$.* □

We omit the proof, which is a variation of the proof in [14, Ch.3].

Applying this result to our case we construct a L_1-continuous family of E_2^A-valued functions $\{z(t;b)|\ b \in B\}$ which approximates the family $\{v(t;b)|\ b \in B\}$ uniformly in the relaxation metric. According to the Theorem 5.5 the end-point map $\mathcal{F}/\mathcal{O}_T$ is continuous in the relaxation metric. Therefore we conclude with the following result.

Proposition 6.5. *For some $L \geq 1$ there exists a family of piecewise-constant E_2^A-valued controls $\{z(t;b)|\ b \in B\}$ (with at most L intervals of constancy) such that $b \mapsto z(t;b)$ is continuous with respect to L_1 metric and the reduced system is solidly controllable by means of the family.* □

6.4. **Generic induction step: solid controllability of the original system.** Let us compare the original system (9)-(10)-(11) with the system driven by the E_2^A-valued controls $\{z(t;b)|\ b \in B\}$ from the Proposition 6.5.

In both systems the equations for the coordinates q_k, indexed by $k \in \mathcal{K}^1$ coincide:

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k,\ \ k \in \mathcal{K}^1. \tag{26}$$

We collect these coordinates into the vector denoted by q^1.

In the original system the equations for the variables $q_k,\ k \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right)$ are 'uncontrolled':

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k,\ \ k \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right). \tag{27}$$

They differ from the corresponding equations of the system with extended controls, which are:

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + z_k,\ \ k \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right). \tag{28}$$

We collect $q_k,\ k \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right)$ into the vector denoted by q^2 and denote $q = (q^1, q^2)$.

Finally the equation for the infinite-dimensional component Q_t, which collects the higher modes $e^{ik \cdot x},\ k \notin \mathcal{K}^2$, does not contain controls and is the same in both systems. It suffices for our goals to write this equation in a concise form as:

$$\dot{Q} = h(q, Q). \tag{29}$$

According to the Proposition 6.5 we manage to control our system solidly by means of extended E_2^A-valued piecewise-constant controls $z(t;b)$. By Proposition 6.4 the intervals of constancy vary continuously with $b \in B$. Our task now is to design a family of "small-dimensional" controls $x(t;b)$ for the equations (26)-(27)-(29), such that the maps

$$b \mapsto z(\cdot;b) \mapsto (\Pi^{obs} \circ \mathcal{F}/\mathcal{O}_\tau)\,(z(\cdot;b)) \text{ and } b \mapsto x(\cdot;b) \mapsto (\Pi^{obs} \circ \mathcal{F}/\mathcal{O}_\tau)\,(x(\cdot;b))$$

are C^0-close.

If on some interval of constancy $T \in [\underline{t}, \bar{t}]$ the value of $z(t;b)$ equals $\pm Ae_k \pm \bar{A}e_{-k}$ with $k \in \mathcal{K}^1$, then we just take the control $x(t;b)$ in (26) coinciding with $z(\cdot;b)$ on this interval.

The real problem arises when on some interval of constancy $z(t;b)$ takes value $\pm Ae_{\bar{k}}$ with $\bar{k} \in \left(\mathcal{K}^2 \setminus \mathcal{K}^1\right)$. There are no controls available in the corresponding equation (27) for $q_{\bar{k}}$ and we will "affect" the evolution of $q_{\bar{k}}$ via the variables $q_m,\ m \in \mathcal{K}^1$ which enter this equation.

More exactly the construction of the controls $x(t;b,\omega)$ on the intervals of constancy of $z(t;b)$ goes as follows:

1) on an interval $[\underline{t}, \bar{t}]$ of the first kind, where $z(t;b) = A(b)e_k + \overline{A(b)}e_{-k}$ with $k \in \mathcal{K}^1$ we take $x(t;b,\omega) = z(t;b)$;

2) on an interval $[\underline{t}, \bar{t}]$ of the second kind, where $z(t;b) = A(b)e_k + \overline{A(b)}e_{-k}$ with $k \in \mathcal{K}^2 \setminus \mathcal{K}^1$, pick a pair $m, n \in \mathcal{K}^1$ such that

$$m \wedge n \neq 0,\ |m| \neq |n|,\ m + n = k,$$

such pair exists by the definition of $\mathcal{K}^2$ (see (12)). Choose $A_m(b), A_n(b) \in \mathbb{R}$ satisfying

$$|A_m(b)| = |A_n(b)| \bigwedge A_m(b)A_n(b)(m \wedge n)(|m|^{-2} - |n|^{-2}) = A(b). \tag{30}$$

and put

$$x_m(t;b,\omega) = \overline{x_{-m}(t;b,\omega)} = A_m(b)i\omega e^{i\omega t},\ A_{-m} = \overline{A_m},$$
$$x_n(t;b,\omega) = \overline{x_{-n}(t;b,\omega)} = A_n(b)(-i\omega)e^{-i\omega t},\ A_{-n} = \overline{A_n},$$

taking other components $x_j(t;b,\omega)$ equal to 0.

It is easy to see that the primitives $X(t;b,\omega) = \int_0^t x(s;b,\omega)ds$ are bounded by a constant which can be chosen independent of b and ω. Besides $X(T;b,\omega)$ varies continuously with b (for fixed ω).

Consider two trajectories $w_t^{b,\omega}, \bar{w}_t,\ t \in [0,T]$, which are driven by the controls $x(t;b,\omega)$ and $z(t;b)$ correspondingly. We will prove that $w_t^{b,\omega}$ and $\bar{w}_t$ match asymptotically (as $\omega \to \infty$) in all the components but q^1. Let Π_1 be the projection onto the space of modes $\{e^{im\cdot x} |\ m \in \mathcal{K}^1\}$, while $\Pi_1^\perp$ be the projection onto its orthogonal complement.

Proposition 6.6. *i) The trajectories $w_t^{b,\omega}$ are equibounded:*

$$\exists C: \ \|w_t^{b,\omega}\|_0 \leq C, \ \forall t \in [0,T], \ b \in B, \ \omega > 0;$$

ii) For fixed $\omega > 0$ the dependence $b \mapsto w_t^{b,\omega}$ on b is continuous in $C^0[0,T]$-metric of the controls;

iii) For any $\varepsilon > 0$ there exists $\delta > 0$ and ω_0 such that if $\omega > \omega_0$ and $\left\|w^{b,\omega}|_{t=0} - \bar{w}|_{t=0}\right\|_0 \leq \delta$, then $\forall t \in [0,T]: \ \|\Pi_1^{\perp}\left(w_t^{b,\omega} - \bar{w}_t\right)\|_0 \leq \varepsilon$. $\square$

Assuming the claim of this Proposition (which is proven in the next subsection) to hold true let us complete the induction.

By assumption the system (26)-(28)-(29) is solidly controllable in observed component by means of the family of extended controls $z(t;b)$, i.e. the map $b \mapsto \left(\Pi^{obs} \circ \mathcal{F}/\mathcal{T}_T\right)(z(t;b))$ covers solidly the cube $\mathcal{C}_R$ in $\Pi^{obs}(H_2)$. According to the Proposition 6.6 all the components, but q^1, of the trajectories driven by $x(t;b,\omega_0)$ match up to arbitrarily small ε, provided ω is sufficiently large. For fixed ω the controls $x(t;b,\omega)$ depend continuously (in L_1-metric) on $b \in B$.

By the degree theory argument we may conclude that for large ω the map

$$b \mapsto \left(\Pi_1^{\perp} \circ \Pi^{obs} \circ \mathcal{F}/\mathcal{T}_T\right)(x(t;b,\omega))$$

covers solidly the set $\Pi_1^{\perp}(\mathcal{C}_R)$.

Still the map $b \mapsto (\Pi_1 \circ \mathcal{F}/\mathcal{T}_T)(x(t;b,\omega))$ does not necessarily match with $b \mapsto (\Pi_1 \circ \mathcal{F}/\mathcal{T}_T)(z(t;b))$. We have to settle the q^1-component.

Considering $q^1(t;\omega,b)$ and evaluating it at T we observe that according to the Proposition 6.6 the values $q^1(T;\omega,b)$ are equibounded for all b,ω and for fixed $\omega > 0$ the dependence $b \to q^1(T;\omega,b)$ is continuous. Put

$$\phi(b) = q^1(T;\omega,b), \psi(b) = (\Pi_1 \circ \mathcal{F}/\mathcal{T}_T)(z(t;b)).$$

We can apply Lemma 6.3 for constructing controls $x(t;b,\omega)$ defined on an arbitrarily small interval $[T, T+\tau]$ such that

$$\|q^1(T+\tau;b) - \psi(b)\| = O(\tau),$$
$$\|(\Pi_1^{\perp} \circ \mathcal{F}/\mathcal{T}_{T+\tau})(x(t;\omega)) - (\Pi_1^{\perp} \circ \mathcal{F}/\mathcal{T}_T)(z(t;b))\| = O(\tau), \text{ as } \tau \to 0.$$

Then choosing $\tau > 0$ sufficiently small we prove that the maps

$$b \mapsto (\Pi^{obs} \circ \mathcal{F}/\mathcal{T}_T)(z(t;b)) \text{ and } b \mapsto (\Pi^{obs} \circ \mathcal{F}/\mathcal{T}_{T+\tau})(x(t;b,\omega))$$

are close in C^0-metric and therefore by the degree theory argument the last map covers solidly the cube $\mathcal{C}_R$. It means that the system is time-$(T+\tau)$ solidly controllable. $\square$

6.5. **Proof of the Proposition 6.6.** We proceed by induction on a uniformly (with respect to $b \in B$) bounded number of the intervals of constancy of controls $z(\cdot;b)$. Since on the intervals of the first kind the controls $z(\cdot;b)$ and $x(\cdot;b)$ coincide it suffices to consider one interval $[\underline{t}, \bar{t}]$ of the second kind. We may think that $[\underline{t}, \bar{t}] = [0,T]$.

Recall that on an interval of the second kind the equation for $\bar{w}_t$, driven by the constant control $z(t;b)$, is:

$$\partial_t \bar{w}_t = \left(\bar{\mathcal{W}}_t \cdot \nabla\right) \bar{w}_t + \nu\Delta\bar{w}_t + Ae_{m+n} + \bar{A}e_{-(m+n)}, \ m,n \in \mathcal{K}^1,$$

where $\bar{\mathcal{W}}_t$ is the divergence-free solution of the equation $\nabla^{\perp} \cdot \bar{\mathcal{W}}_t = \bar{w}_t$ under periodic boundary conditions.

Now take the control $x(\tau;\omega,b)$, constructed in the precedent Subsection and consider its primitive $V_t^{\omega}(b) = \int_0^t x(\tau;\omega,b)d\tau$. The 2D NS/Euler system can be written on $[0,T]$ as

$$\partial w_t^{\omega}/\partial t = (u_t^{\omega} \cdot \nabla)w_t^{\omega} + \nu\Delta w_t^{\omega} + \partial V_t^{\omega}/\partial t.$$

Put $y_t^{\omega} = w_t^{\omega} - V_t^{\omega}$. Notice that y_t^{ω} and w_t^{ω} differ only in $\mathcal{K}^1$-indexed modes, i.e. $\Pi_1^{\perp}y_t^{\omega} = \Pi_1^{\perp}w_t^{\omega}$. The equation for y_t^{ω} is:

$$\partial y_t^{\omega}/\partial t = (u_t^{\omega} \cdot \nabla)(y_t^{\omega} + V_t^{\omega}) + \nu\Delta\left(y_t^{\omega} + V_t^{\omega}\right). \tag{31}$$

The function u_t^{ω} can be represented as a sum $\mathcal{Y}_t^{\omega} + \mathcal{V}_t^{\omega}$, where $\mathcal{V}_t^{\omega}, \mathcal{Y}_t^{\omega}$ are the divergence-free solutions of the equations: $\nabla^{\perp} \cdot \mathcal{V}_t^{\omega} = V_t^{\omega}, \ \nabla^{\perp} \cdot \mathcal{Y}_t^{\omega} = y_t^{\omega}$, under periodic boundary conditions.

Hence the equation (31) allows for the representation

$$\begin{aligned} \partial y_t^{\omega}/\partial t = &\left(\left(\mathcal{Y}_t^{\omega} + \mathcal{V}_t^{\omega}\right) \cdot \nabla\right)\left(y_t^{\omega} + V_t^{\omega}\right) + \nu\Delta\left(y_t^{\omega} + V_t^{\omega}\right) = \left(\mathcal{Y}_t^{\omega} \cdot \nabla\right) y_t^{\omega} + \\ &+ \left(\mathcal{V}_t^{\omega} \cdot \nabla\right) y_t^{\omega} + \left(\mathcal{Y}_t^{\omega} \cdot \nabla\right) V_t^{\omega} + \nu\Delta y_t^{\omega} + \nu\Delta V_t^{\omega} + \left(\mathcal{V}_t^{\omega} \cdot \nabla\right) V_t^{\omega}. \end{aligned} \tag{32}$$

Denote $e^{i\ell\cdot x}$ by e_{ℓ}; then

$$V_t^{\omega} = (A_m e_m + A_{-n}e_{-n})e^{i\omega t} + (A_{-m}e_{-m} + A_n e_n)e^{-i\omega t}.$$

Proceeding as in Section 2 we compute

$$\begin{aligned} (\mathcal{V}_t^{\omega} \cdot \nabla)V_t^{\omega} = &A_m(b)A_n(b)(m \wedge n)(|m|^{-2} - |n|^{-2})e_{m+n} + \\ +&\overline{A_m(b)A_n(b)}(m \wedge n)(|m|^{-2} - |n|^{-2})e_{-(m+n)} + (\cdots)e^{i2\omega t} + (\cdots)e^{-i2\omega t}, \end{aligned}$$

or by virtue of (30),

$$(\mathcal{V}_t^{\omega} \cdot \nabla)V_t^{\omega} = A(b)e_{m+n} + \overline{A(b)}e_{-(m+n)} + (\cdots)e^{i2\omega t} + (\cdots)e^{-i2\omega t}. \tag{33}$$

Here $(\cdots)$ stay for (unspecified) factors, which do not depend on t.

The control-quadratic term $\left(\mathcal{V}_t^{\omega} \cdot \nabla\right) V_t^{\omega}$ at the righ-hand side of (32) will act as our new control. For sufficiently large ω the summands $(\cdots)e^{i2\omega t} + (\cdots)e^{-i2\omega t}$ in the control-quadratic term (cf. (33)) together with control-linear terms are small in relaxation metric (see Lemma 5.2). Hence for large ω the controls $x(\tau;\omega,b)$ approximate properly the actuation of the control $A(b)e_{m+n} + \overline{A(b)}e_{-(m+n)}$.

To formalize this introduce the notation $\eta_t^{\omega} = y_t^{\omega} - \bar{w}_t$, we obtain for η_t^{ω} the equations:

$$\begin{aligned} \partial_t \eta_t^{\omega} = &\left(\left(\mathcal{W}_t^{\omega} + \mathcal{V}_t^{\omega}\right) \cdot \nabla\right) y_t^{\omega} - \left(\bar{\mathcal{W}}_t \cdot \nabla\right) \bar{w}_t + \\ +&\nu\Delta\eta_t^{\omega} + \left(\left(\mathcal{V}_t^{\omega} \cdot \nabla\right) V_t^{\omega} - \left(Ae_{m+n} + \bar{A}e_{-(m+n)}\right)\right). \end{aligned}$$

Subtracting and adding $\left(\left(\mathcal{W}_t^\omega + \mathcal{V}_t^\omega\right) \cdot \nabla\right) \bar{w}_t$ to the right-hand side of the latter equation we transform it into

$$\partial_t \eta_t^\omega = \left(\mathcal{H}_t^\omega \cdot \nabla\right) \eta_t^\omega + \left(\left(\bar{\mathcal{W}}_t + \mathcal{V}_t^\omega\right) \cdot \nabla\right) \eta_t^\omega + \left(\left(\mathcal{H}_t^\omega + \mathcal{V}_t^\omega\right) \cdot \nabla\right) \bar{w}_t + \\ + \nu \Delta \eta_t^\omega + \left(\left(\mathcal{V}_t^\omega \cdot \nabla\right) V_t^\omega - A e_{m+n}\right), \tag{34}$$

where $\mathcal{H}_t^\omega$ is the divergence-free solution of the equation $\nabla^\perp \cdot \mathcal{H}_t^\omega = \eta_t^\omega$.

By construction $V_t^\omega, \mathcal{V}_t^\omega, \left(\left(\mathcal{V}_t^\omega \cdot \nabla\right) V_t^\omega - \left(A e_{m+n} + \bar{A} e_{-(m+n)}\right)\right)$ converge to 0 in relaxation metric, as $\omega \to +\infty$. By the continuity result (Theorem 5.5) trajectories of (34) converge in C^0 to the trajectories of the equation

$$\partial_t \eta_t = \left(\left(\mathcal{H}_t + \bar{\mathcal{W}}_t\right) \cdot \nabla\right) \eta_t + \left(\mathcal{H}_t \cdot \nabla\right) \bar{w}_t + \nu \Delta \eta_t, \tag{35}$$

as $\omega \to +\infty$,

To estimate the evolution of $\|\eta_t\|_0$ by virtue of (35) we multiply both parts of (35) by η_t in H_0. Integrating the resulting equality on $[0, \tau]$ and observing that $\langle \left(\left(\bar{\mathcal{W}}_t + \mathcal{H}_t\right) \cdot \nabla\right) \eta_t, \eta_t \rangle = 0$ we conclude

$$\frac{1}{2}\|\eta_\tau\|_0^2 + \nu \int_0^\tau \|\eta_t\|_1^2 dt = \frac{1}{2}\|\eta_0\|_0^2 + \int_0^\tau \langle \left(\mathcal{H}_t \cdot \nabla\right) \bar{w}_t, \eta_t \rangle dt.$$

The summand $\langle \left(\mathcal{H}_t \cdot \nabla\right) \bar{w}_t, \eta_t \rangle$ can be estimated as in the Section 5.3:

$$|\langle \left(\mathcal{H}_t \cdot \nabla\right) \bar{w}_t, \eta_t \rangle| \leq C\|\mathcal{H}_t\|_1 \|\nabla \bar{w}_t\|_1 \|\eta_t\|_0 \leq C\|\nabla \bar{w}_t\|_1 \|\eta_t\|_0^2.$$

One concludes $\|\nabla \bar{w}_t\|_1 \leq c'\|\bar{w}_t\|_2$, while $\|\bar{w}_t\|_2$ are equibounded according to the Proposition 5.4.

Thus we get

$$\frac{1}{2}\|\eta_\tau\|_0^2 \leq \frac{1}{2}\|\eta_0\|_0^2 + c\int_0^\tau \|\eta_t\|_0^2 dt,$$

and by application of Gronwall inequality we conclude

$$\frac{1}{2}\|\eta_\tau\|_0^2 \leq \frac{1}{2}\|\eta_0\|_0^2 e^{cT}. \ \square$$

7. Proof of the Theorem 4.3

The proof of the result regarding L_2-approximate controllability (Theorem 4.4) for 2D NS system can be found in [5]; it holds also for 2D Euler system. Here we provide a proof of the Theorem 4.3, which regards controllability in finite-dimensional projection. Following the steps of our proof the readers can recover the proof of Theorem 4.4.

Let $\mathcal{L}$ be a ℓ-dimensional subspace of H_2 and $\Pi^{\mathcal{L}}$ be L_2-orthogonal projection of H_2 onto $\mathcal{L}$. We start with constructing a finite-dimensional coordinate subspace which is projected by $\Pi^{\mathcal{L}}$ onto $\mathcal{L}$.

To find one it suffices to pick a $(\dim \mathcal{L}) \times (\dim \mathcal{L})$-sub-matrix, from the $(\dim \mathcal{L}) \times \infty$ matrix which is a coordinate representation of $\Pi^{\mathcal{L}}$. We look for more: for each $\varepsilon > 0$ we would like to find a finite-dimensional coordinate subspace, which contains an ℓ-dimensional (non-ccordinate) subsubspace $\mathcal{L}_\varepsilon$, which is ε-close to $\mathcal{L}$. The latter means that not only $\Pi^{\mathcal{L}} \mathcal{L}_\varepsilon = \mathcal{L}$ but also $\Pi^{\mathcal{L}}|_{\mathcal{L}_\varepsilon}$ is ε-close to the identity operator.

To achieve this we choose an orthonormal basis $e_1, \ldots, e_\ell$ in $\mathcal{L}$ and take for each e_i its finite-dimensional component (truncation) $\bar{e}_i$, which is ε-close to e_i. All $\bar{e}_i$ belong to some finite-dimensional coordinate subspace $\mathcal{S}$ of H_2; which reunites modes indexed by some symmetric set $S \subset \mathbb{Z}^2$. Let Π_S be L_2-orthogonal projection of H_2 onto $\mathcal{S}$. The subspace $\mathcal{S}$ together with the subsubspace $\mathcal{L}_\varepsilon$ spanned by $\bar{e}_1, \ldots, \bar{e}_\ell$ are the ones we looked for. Indeed

$$\left\|\Pi^{\mathcal{L}}\bar{e}_i - \bar{e}_i\right\| = \left\|\sum_{j=1}^{\ell}\langle \bar{e}_i, e_j\rangle e_j - \bar{e}_i\right\| \leq \sum_{j=1}^{\ell}|\langle \bar{e}_i - e_i, e_j\rangle| + \|e_i - \bar{e}_i\| \leq (\ell+1)\varepsilon.$$

Without lack of generality we may assume that $\|\Pi_S(\tilde{\varphi}) - \tilde{\varphi}\|_0 \leq \varepsilon$.

The set $\mathcal{K}^1$ of controlled modes is saturating, i.e. for $\mathcal{K}^j$ defined by (12), $\mathcal{K}^M \supseteq S$ for some M. This means that the system is solidly controllable in the observed component q^S.

In the proof of the Theorem 4.1 (Section 6) we started with a "full-dimensional" set of controlled modes indexed by $\mathcal{K}^M$ and then constructed successively controls which only enter the equations for the modes indexed by $\mathcal{K}^{M-1}, \ldots, \mathcal{K}^1$.

Assume that we are at the first induction step under the conditions of the Lemma 6.2, i.e. that all the coordinates of the component q^S are controlled. Following Lemma 6.2 let us construct a family of controls which steers the q^S-components of the corresponding trajectories w_t from $\Pi_S(\tilde{\varphi})$ to the points of the "ball" $\mathcal{C}_R$ in $\mathcal{S}$. We can construct these controls to actuate on an interval of arbitrarily small length $\tau > 0$. Denoting by $Q_.$ the component of w_t's which is orthogonal to q^s we can conclude from (24) and (25):

$$\|Q_t\|_0 \leq \|Q_0\|_0 + C\tau, \ t \in [0, \tau],$$

for some constant $C > 0$.

Recall that $\|Q_0\|_0 = \|\Pi_S(\tilde{\varphi}) - \tilde{\varphi}\|_0 \leq \varepsilon$. Choosing $\tau \leq \varepsilon/C$, we conclude $\|Q_t\|_0 \leq 2\varepsilon, \ \forall t \in [0, \tau]$.

Let us check what happens with the component $Q.$ at generic induction step of the proof of the Theorem 6.1. At the first stage of each step (Subsection 6.3) we apply the Approximation Lemma (Proposition 6.4). At this stage the trajectories are approximated up to arbitrary small (uniformly for $t \in [0, \tau]$) error $\delta > 0$. We can choose $\delta \leq \varepsilon/(2M)$.

At the second stage of each induction step (Subsection 6.4) the component $Q.$ (which belongs to the image of the projection Π_2) suffers arbitrarily small alteration. We can make it (uniformly for $t \in [0, \tau]$) smaller than $\varepsilon/(2M)$.

Therefore at each induction step the component $Q.$ suffers alteration by value $\leq \varepsilon/M$; total alteration is $\leq \varepsilon$. Hence after the induction procedure $\|Q_\tau\|_0 \leq 2\varepsilon + \varepsilon = 3\varepsilon$.

At the end we arrive to a family of controls $x(\cdot; b)$ such that the map $b \mapsto \Pi_S \circ \mathcal{F}/\mathcal{T}_T)(x(t; b))$ covers solidly the ball $\mathcal{C}_R$ in $\mathcal{S}$. Besides

$$\|(\Pi_S^{\perp} \circ \mathcal{F}/\mathcal{T}_T)(x(t; b))\| \leq 3\varepsilon.$$

Then by the choice of S the map $b \mapsto \Pi_{\mathcal{L}} \circ \Pi_S \circ \mathcal{F}/\mathcal{T}_T)(x(t;b))$ covers the set $\mathcal{C}_{R/2} \cap \mathcal{L}$, if $\varepsilon > 0$ is sufficiently small. □

8. Appendix

8.1. **Forced 2D Euler equation: existence and uniqueness of solutions.** We outline the proof which is a modification of the proof of the existence and uniqueness theorem for 2D Euler equation to be found in [17]. Recall that the original proof of existence and uniqueness of classical solutions has been accomplished by W.Wolibner in [22].

Consider the 'nonclassically forced' equation (18) with $\nu = 0$:

$$\partial y_t/\partial t = (\mathcal{Y}_t \cdot \nabla)\, y_t + \left(\mathcal{V}_t^1 \cdot \nabla\right) y_t + (\mathcal{Y}_t \cdot \nabla)\, V_t^2 + V_t^0,$$

where V_t^j $(j = 0,1,2)$ are trigonometric polynomials and $\mathcal{V}_t^1$ and $\mathcal{Y}_t$ are divergence-free solutions of the equations

$$\nabla^{\perp} \cdot \mathcal{V}_t^1 = V_t^1,\ \nabla^{\perp} \cdot \mathcal{Y}_t = y_t,$$

under periodic boundary conditions.

Following the approach of [17] let us introduce a map $\xi. \mapsto \Phi(\xi) = \eta.$ which is defined by means of the *linear* differential equations

$$\begin{aligned} \nabla^{\perp} \cdot \zeta_t &= \xi_t, \\ \partial \eta_t/\partial t = (\zeta_t \cdot \nabla)\, \eta_t + \left(\mathcal{V}_t^1 \cdot \nabla\right) \eta_t &+ (\zeta_t \cdot \nabla)\, V_t^2 + V_t^0. \end{aligned}$$

It is easy to see that fixed points of the map Φ correspond to classical solutions of the equation (18).

Choosing an appropriate set Ω of Hölderian (of exponent $\delta \in (0,1)$) with respect to time and space variables) functions with $L_\infty^{(x)}$-norms bounded by a constant, one is able to establish, as in [17], that Φ maps Ω in itself. Besides S is compact convex subset of C^0 and existence of fixed point is derived from Schauder theorem.

Analysis of the proof shows that the equiboundedness of the $L_\infty^{(x)}$-norms of V_t^j guarantee equiboundedness of the $L_\infty^{(x)}$-norms of the corresponding solutions of (18). This will prove the statement i) of the Proposition 5.4.

8.2. **Forced 2D NS equation: existence, uniqueness and boundedness of solutions.** The existence of solutions from $L_\infty([0,t];H_2)$ for the nonclassically forced NS equation (18) can be established in the same way as for classically forced NS equation, for example by energy estimates for Galerkin approximations.

In the same classical way we prove the boundedness of $\|y_t\|_2$, and of $\int_0^T \|\frac{d}{dt}y_t\|_1^2 dt$ i.e. the estimates (21) and (22). The boundedness of $L_\infty^{(x)}$-norms (the estimate (20)) follows then from Sobolev inequality (see [1]).

8.3. **Proof of the Theorem 5.4: equiboundedness of solutions for 2D Euler equation.** For the nonclassically forced 2D Euler equation the $L_\infty^{(x)}$-equiboundedness of solutions (Theorem 5.4; item i)) comes with the proof of existence (see Subsection 8.1).

To prove the statement ii) of the Theorem 5.4 we observe first that uniform (in t) L_∞-equiboundedness of y_t implies their uniform (in t) H_0-equiboundedness. To arrive to the conclusion of the assertion ii) let us differentiate both sides of the equation (18), say, with respect to x_i. Abbreviating $\partial/\partial x_i$ to ∂_i we get:

$$\frac{\partial}{\partial t}(\partial_i y_t) = \left(\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\cdot\nabla\right)(\partial_i y_t) + \left(\left(\partial_i\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\right)\cdot\nabla\right)y_t + \\ + \left((\partial_i\mathcal{Y}_t)\cdot\nabla\right)V_t^2 + (\mathcal{Y}_t\cdot\nabla)\left(\partial_i V_t^2\right) + \partial_i V_t^0.$$

Multiplying both sides of the latter equality by $\partial_i y_t$ in H_0 we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|\partial_i y_t\|_0^2 = \\ = \left\langle\left(\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\cdot\nabla\right)\partial_i y_t, \partial_i y_t\right\rangle + \left\langle\left(\left(\partial_i\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\right)\cdot\nabla\right)y_t, \partial_i y_t\right\rangle \\ +\langle(\partial_i\mathcal{Y}_t\cdot\nabla)V_t^2, \partial_i y_t\rangle + \langle(\mathcal{Y}_t\cdot\nabla)\partial_i V_t^2, \partial_i y_t\rangle + \langle\partial_i V_t^0, \partial_i y_t\rangle. \tag{36}$$

At the right-hand side of (36) the summand $\left\langle\left(\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\cdot\nabla\right)(\partial_i y_t), \partial_i y_t\right\rangle$ is known to vanish, while the summand $\left\langle\left(\left(\partial_i\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\right)\cdot\nabla\right)y_t, \partial_i y_t\right\rangle$ admits an upper estimate:

$$\left\langle\left(\left(\partial_i\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\right)\cdot\nabla\right)y_t, \partial_i y_t\right\rangle \leq \\ \leq C\left\|\partial_i\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\right\|_{L_\infty}\|\nabla y_t\|_{L_2}\|\partial_i y_t\|_{L_2} \leq \\ \leq C'\left\|\left(\partial_i\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\right)\right\|_{L_\infty}\|y_t\|_{H_1}^2. \tag{37}$$

Evidently $\|((\partial_i\mathcal{Y}_t)\|_{L_\infty} \leq c\|y_t\|_{L_\infty}$ and since, by virtue of i), $\|y_t\|_{L_\infty}$ are bounded, then the upper estimate (37) can be changed to

$$\left\langle\left(\partial_i\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\cdot\nabla\right)y_t, \partial_i y_t\right\rangle \leq C''\|y_t\|_{H_1}^2.$$

The summand $\langle(\partial_i\mathcal{Y}_t\cdot\nabla)V_t^2, \partial_i y_t\rangle$ can be estimated from above by

$$a\|\partial_i\mathcal{Y}_t\|_{L_2}\|\nabla V_t^2\|_{L_\infty}\|\partial_i y_t\|_{L_2}.$$

As long as V_t^2 is trigonometric polynomial in x we can change the latter estimate to $a'\|y_t\|_{H_1}^2$. A similar upper estimate is valid for the summand $\langle(\mathcal{Y}_t\cdot\nabla)(\partial_i V_t^2), \partial_i y_t\rangle$.

Finally $\langle\partial_i V_t^0, \partial_i y_t\rangle$ admits an upper estimate

$$\alpha\left(\|\partial_i V_t^0\|_{L_2}^2 + \|\partial_i y_t\|_{L_2}^2\right) \leq \alpha' + \alpha''\|y_t\|_{H_1}^2.$$

Then we come to the differential inequality for $\|y_t\|_{H_1}^2$ denoted for brevity by $\|y_t\|_1^2$:

$$\frac{\partial}{\partial t}\|y_t\|_1^2 \leq c' + c\|y_t\|_1^2,$$

wherefrom by the application of the Gronwall inequality we conclude

$$\|y_t\|_1^2 \leq \|y_0\|_1^2 e^{ct} + (c'/c)(e^{ct} - 1),$$

and consequently $\sup_{t\in[0,T]} \|y_t\|_1 \leq b$ for some $b > 0$.

To arrive to the estimate (21) for $\|y_t\|_2$ (given that the initial value y_0 belongs to H_2) we have to derivate (36) with respect to x_j, arriving to a differential equation for $\partial_j \partial_i y$. Multiplying both parts of this equation by $\partial_j \partial_i y$ we obtain

$$\frac{\partial}{\partial t}\|\partial_j \partial_i y\|_0^2 = \langle\left(\left(\mathcal{Y}_t + \mathcal{V}_t^1\right) \cdot \nabla\right) \partial_j \partial_i y_t), \partial_j \partial_i y_t\rangle + \cdots .$$

The first term at the right-hand side vanishes and then the needed estimate for $\|y_t\|_2^2$ is derived from the estimate for $\|y_t\|_1^2$ by application of Young and Gronwall inequalities.

The integral estimate iii) and even a stronger "pointwise" estimate for $\left\|\frac{\partial}{\partial t}(\partial_i y_t)\right\|$ can be concluded from (36). Indeed

$$\left\|\frac{\partial}{\partial t}(\partial_i y_t)\right\| \leq C\left(\|\mathcal{Y}_t + \mathcal{V}_t^1\|_{L_\infty}\|\nabla \partial_i y_t\| + \|\partial_i\left(\mathcal{Y}_t + \mathcal{V}_t^1\right)\|_{L_\infty}\|\nabla y_t\|\right) + \\ + C_1\|y_t\|_1 + c_2 \leq C'(1 + \|y_t\|_0)\|y_t\|_2 + c_2'.$$

8.4. **Continuity with respect to relaxation metric: proof of the Theorem 5.5.** Pick an element $(\bar{V}_t^0, \bar{V}_t^1, \bar{V}_t^2)$ from $\mathbf{F}_B$ and denote by $\bar{y}_t$ the solution of the equation

$$\partial_t \bar{y}_t = \left(\bar{\mathcal{Y}}_t \cdot \nabla\right) \bar{y}_t + \left(\bar{\mathcal{V}}_t^1 \cdot \nabla\right) \bar{y}_t + \left(\bar{\mathcal{Y}}_t \cdot \nabla\right) \bar{V}_t^2 + \nu \Delta \bar{y}_t + \bar{V}_t^0. \tag{38}$$

Let y_t be a solution of the "perturbed" equation

$$\partial_t y_t = \left(\mathcal{Y}_t \cdot \nabla\right) y_t + \left(\left(\bar{\mathcal{V}}_t^1 + \mathcal{V}_t^1\right) \cdot \nabla\right) y_t + \left(\mathcal{Y}_t \cdot \nabla\right)\left(\bar{V}_t^2 + V_t^2\right) + \\ + \nu \Delta y_t + \bar{V}_t^0 + V_t^0. \tag{39}$$

Recall that $\bar{\mathcal{V}}_t^1$ is the divergence-free solution of the equation: $\nabla^\perp \bar{\mathcal{V}}_t^1 = \bar{V}_t^1$.

Subtracting (38) from (39) and introducing the notation

$$\eta_t = y_t - \bar{y}_t, \ \mathcal{H}_t = \mathcal{Y}_t - \bar{\mathcal{Y}}_t,$$

we obtain the equation for η_t:

$$\partial_t \eta_t = \left(\mathcal{Y}_t \cdot \nabla\right) \eta_t + \left(\mathcal{H}_t \cdot \nabla\right) \bar{y}_t + \left(\left(\bar{\mathcal{V}}_t^1 + \mathcal{V}_t^1\right) \cdot \nabla\right) \eta_t + \left(\mathcal{V}_t^1 \cdot \nabla\right) \bar{y}_t + \\ + \left(\mathcal{Y}_t \cdot \nabla\right) V_t^2 + \left(\mathcal{H}_t \cdot \nabla\right) \bar{V}_t^2 + \nu \Delta \eta_t + V_t^0. \tag{40}$$

We would like to evaluate $\|\eta\|_0$; to this end we multiply in H_0 both sides of (40) by η_t. At the left-hand side we obtain $\frac{1}{2}\partial_t\|\eta_t\|_0^2$, while at the right-hand side the terms $\langle\left(\mathcal{Y}_t \cdot \nabla\right) \eta_t, \eta_t\rangle$ and $\langle\left(\left(\bar{\mathcal{V}}_t^1 + \mathcal{V}_t^1\right) \cdot \nabla\right) \eta_t, \eta_t\rangle$ both vanish. Taking into account that $\langle \Delta \eta_t, \eta_t\rangle \leq 0$ at the right-hand side, we arrive to the inequality:

$$\frac{1}{2}\partial_t\|\eta_t\|_0^2 \leq \langle\left(\mathcal{H}_t \cdot \nabla\right)\left(\bar{y}_t + \bar{V}_t^2\right), \eta_t\rangle + \\ + \langle\left(\mathcal{V}_t^1 \cdot \nabla\right) \bar{y}_t, \eta_t\rangle + \langle\left(\mathcal{Y}_t \cdot \nabla\right) V_t^2, \eta_t\rangle + \langle V_t^0, \eta_t\rangle.$$

Hence

$$\frac{1}{2}\|\eta_\tau\|_0^2 \leq \frac{1}{2}\|\eta_0\|_0^2 + \int_0^\tau \langle (\mathcal{H}_t \cdot \nabla)\left(\bar{y}_t + \bar{V}_t^2\right), \eta_t\rangle dt + \tag{41}$$

$$+ \int_0^\tau \left(\langle \left(\mathcal{V}_t^1 \cdot \nabla\right) \bar{y}_t, \eta_t\rangle + \langle (\mathcal{Y}_t \cdot \nabla) V_t^2, \eta_t\rangle + \langle V_t^0, \eta_t\rangle\right) dt.$$

What for the first integrand in the right-hand side, then

$$\left|\langle (\mathcal{H}_t \cdot \nabla)\left(\bar{y}_t + \bar{V}_t^2\right), \eta_t\rangle\right| \leq$$
$$\leq c\|\mathcal{H}_t\|_1 \|\nabla\left(\bar{y}_t + \bar{V}_t^2\right)\|_1 \|\eta_t\|_0 \leq c'\|\bar{y}_t + \bar{V}_t^2\|_2 \|\eta_t\|_0^2.$$

As long as $\bar{V}_t^2$ are trigonometric polynomials with uniformly bounded coefficients and according to Theorem 5.4 $\|\bar{y}_t\|_2$ are equibounded, then the latter estimate can be changed to $c''\|\eta_t\|_0^2$.

All terms of the second integrand at the right-hand side of (41) contain "factors" V_t^0, V_t^1, V_t^2 which are small in relaxation metric. To estimate this integral one can use Lemma 5.2. Its assumptions are verified as far as the values

$$\int_0^T \left\|\frac{d}{dt}\partial_i(\bar{y}_t)_j\right\|_0^2 dt, \int_0^T \|\dot{\eta}_t\|_0^2\, dt, \int_0^T \|\dot{y}_t\|_0^2\, dt$$

are equibounded. Say, the value of the integral $\int_0^\tau \left(\langle \left(\mathcal{V}_t^1 \cdot \nabla\right) \bar{y}_t, \eta_t\right\rangle dt$ is small, because $\mathcal{V}_t^1$ is small in relaxation norm, and $\int_0^T \left(\frac{d}{dt}\left(\partial_i(\bar{y}_t)_j(\eta_t)_j\right)\right)^2 dt$ are bounded.

For any $\delta > 0$ we can take V_t^0, V_t^1, V_t^2 sufficiently small in relaxation metric, in such a way that (41) implies

$$\frac{1}{2}\|\eta_\tau\|_0^2 \leq \frac{1}{2}\|\eta_0\|_0^2 + c''\int_0^\tau \|\eta_t\|_0^2 dt + \delta.$$

Then the smallness of $\|\eta_t\|_0$ is concluded by application of the Gronwall inequality. □

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