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The Liouville side of the vortex

Giulio Bonelli,[a,b] **Alessandro Tanzini**[a] **and Jian Zhao**[a]

[a] *International School of Advanced Studies (SISSA), and INFN, Sezione di Trieste, via Bonomea 265, 34136 Trieste, Italy*

[b] *Abdus Salam International Centre for Theoretical Physics (ICTP), Strada Costiera 11, 34014 Trieste, Italy*

E-mail: bonelli@sissa.it, tanzini@sissa.it, zhaojian@sissa.it

Abstract: We analyze conformal blocks with multiple (semi-)degenerate field insertions in Liouville/Toda conformal field theories an show that their vector space is fully reproduced by the four-dimensional limit of open topological string amplitudes on the strip with generic boundary conditions associated to a suitable quiver gauge theory. As a byproduct we identify the non-abelian vortex partition function with a specific fusion channel of degenerate conformal blocks.

Keywords: Supersymmetric gauge theory, Conformal and W Symmetry, Topological Field Theories, Topological Strings

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1 Introduction

The connection between refined BPS counting in four dimensional $\mathcal{N} = 2$ quiver gauge theories — namely Nekrasov partition functions [31] — and Logarithmic Conformal Field Theories on Riemann Surfaces [6], which was originally noticed in [3], opened a renewed perspective on both the areas. This correspondence can be studied via geometric engineering [20] where topological strings [34] can be used to exactly describe the BPS protected sectors of the gauge theory both at perturbative and non perturbative level and realizes the above program in M-theory [17, 35].

In this context, in [13, 14] the role of vortex counting was noticed and proposed to encode in a two dimensional field theoretic perspective the insertion of surface operators of the type discussed in [4, 21–30]. The role of non-abelian vortices was explored in [13] by relating their partition function [33] with instanton counting and topological string amplitudes.

The Liouville/Toda descriptions of some of these amplitudes with suitable boundary conditions were provided in terms of insertions of multiple degenerate fields. In presence of more than one insertion, the conformal blocks span a vector space whose dimension is fixed by the fusion rules of a generic primary with degenerate fields. One of the main results of this paper is to provide a full realization of the above in terms of topological string

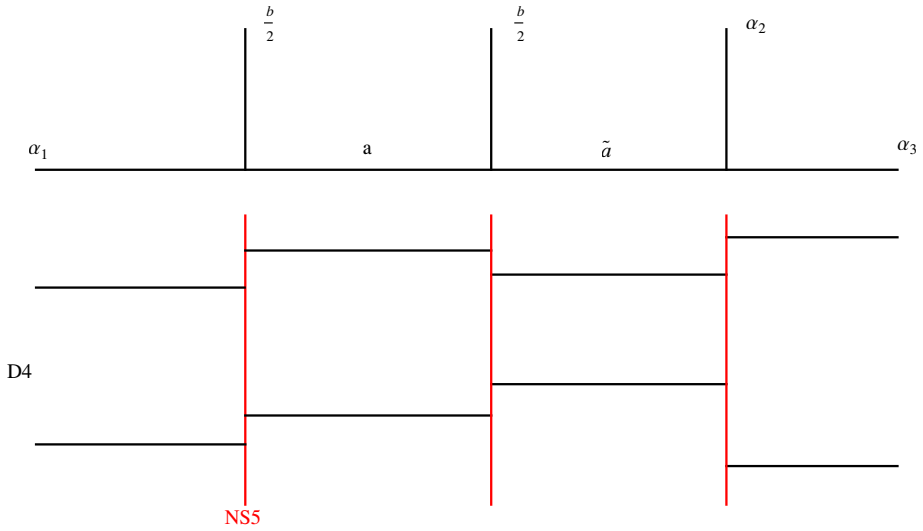

Figure 1. AGT relation between SU(2) quiver gauge theory and CFT.

amplitudes with general boundary conditions. As a byproduct we identify the non-abelian vortex partition function with a specific fusion channel of the degenerate conformal block, different to the one considered so far in the literature.

Moreover, we realize the vortex counting problem as sub-counting instantons by showing how to relate the Nekrasov partition function and its vortex counterpart by a particular choice of mass parameters in an appropriately engineered gauge theory in four dimensions. On the gauge theory side, this boils down to consider surface operator insertions in a theory with a simpler quiver structure. On the AGT dual side, we notice that the above mass parameters assignments produces the insertion of degenerate fields in the Liouville/Toda CFT amplitudes. Similar considerations were presented in [14]. We study this correspondence in depth, reproduce some known results and embed them in a wider framework. In particular we show the correspondence between the fusion channel choice in the Liouville/Toda field theory side and the choice of possible surface operator insertions. The relation with topological strings, in the form of related strip amplitudes [2, 19], is also considered in full generality for the SU(2) case and in some particular exemplificative ones for SU(N).

We organize our paper as follows. In section 2 we calculate the CFT dual of SU(2) vortex partition functions. In section 3 we extend the CFT dual for SU(N) vortices and argue its validity for general strip amplitudes. Section 4 contains our conclusion, while some technical details are left for the appendices.

2 SU(2) vortices and degenerate states

2.1 General setup

We start from two node $SU(2)$ theory with specific parameters. Its Liouville conformal block dual and brane construction is illustrated in figure 1. Following the results of our

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previous paper [13], we will focus on the free field limit, $\epsilon_+ := \epsilon_1 + \epsilon_2 = 0$. The parameters of this two node quiver are: μ_1, μ_2 are masses of antifundamental hypermultiples; μ_3, μ_4 are masses of fundamental hypermultiples; m is the mass of bifundamental hypermultiplet and $(a_1, a_2) = (a, -a); (\tilde{a}_1, \tilde{a}_2) = (\tilde{a}, -\tilde{a})$ are Coulomb branch parameters of the first and second gauge factor. On the conformal field theory side $\alpha_1, \alpha_2, \alpha_3$ are the external momenta in Liouville theory. When all parameters are generic, what we get is just the standard AGT correspondence between instanton partition functions of quiver gauge theories and conformal blocks with five operator insertions. When there are degenerate states, different fusion channels will give different results which also have different gauge theory interpretation as we will show in the following. For two node SU(2) quiver theories there are two channels, one corresponding to SU(2) vortex partition functions while the other to a simple surface operator as discussed in [21]. The general situation with the insertion of more degenerate fields is discussed in subsequent sections.

The standard AGT-relation [3] gives the following map between parameters:

$$\begin{aligned} \mu_1 &= \alpha_1 - \frac{\epsilon_2}{2} \\ \mu_2 &= -\alpha_1 - \frac{\epsilon_2}{2} \\ \mu_3 &= \alpha_2 + \alpha_3 \\ \mu_4 &= \alpha_2 - \alpha_3 \\ m &= -\frac{\epsilon_2}{2} \end{aligned} \tag{2.1}$$

The fusion rules of Liouville field theory imply that

$$\begin{aligned} \alpha_1 &= a - s_1 \frac{\epsilon_2}{2} \\ \tilde{a} &= a - s_2 \frac{\epsilon_2}{2} \end{aligned} \tag{2.2}$$

where s_1 and s_2 are ± 1. This fixes the masses to

$$\begin{aligned} \mu_1 &= a - (s_1 + 1)\, \frac{\epsilon_2}{2} \\ \mu_2 &= -a + (s_1 - 1)\, \frac{\epsilon_2}{2} \end{aligned}$$

Let us remark that when the differences between Coulomb branch parameters and fundamental/bifundamental masses are linear in ϵ_1 and ϵ_2 the instanton partition function is largely simplified. To see this let us recall the contribution from antifundamental fields

$$Z_{\text{antifund}}(m, a, Y) = \prod_{\alpha=1}^{2} \prod_{(i,j)\in Y_\alpha} (a_\alpha + m + \epsilon_2(j - i)) \tag{2.3}$$

Where $a_1 = a; a_2 = -a$, and (i, j) are the box location in the Young tableaux. If we choose $s_1 = -1$, then $\mu_1 = a; \mu_2 = -a - \epsilon_2$. The above formula then implies that $Y_2 = \emptyset$ and Y_1 to be a row. The other choice $s_1 = 1$ just exchanges the roles of Y_1 and Y_2. So the choice of fusion channel here is just a convention. What is really relevant is the choice of s_2.

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Let us notice that bifundamental masses can transfer degeneration between adjacent gauge factors of a quiver theory. Indeed the contribution of bifundamental hypermultiples is:

$$Z_{\text{bifund}}\left(m,a,\tilde{a},Y,W\right)=\prod_{\alpha=1}^{2}\prod_{\beta=1}^{2}Z_{\text{bifund}}^{(\alpha,\beta)} \tag{2.4}$$

$$Z_{\text{bifund}}^{(\alpha,\beta)}=\prod_{s\in Y_\alpha}\prod_{t\in W_\beta}\left(m_{\alpha,\beta}+\epsilon_2\left(A_{Y_\alpha}(s)+L_{W_\beta}(s)+1\right)\right)\left(m_{\alpha,\beta}-\epsilon_2\left(A_{W_\beta}(t)+L_{Y_\alpha}(t)+1\right)\right)$$

$$m_{\alpha,\beta}:=a_\alpha-\tilde{a}_\beta-m$$

From the second fusion relation in the diagram one gets

$$\begin{aligned}m_{1,1}&=(s_2+1)\,\frac{\epsilon_2}{2}\\ m_{2,2}&=(1-s_2)\,\frac{\epsilon_2}{2}\\ m_{1,2}&=2a+(1-s_2)\,\frac{\epsilon_2}{2}\\ m_{2,1}&=-2a+(1+s_2)\,\frac{\epsilon_2}{2}\end{aligned} \tag{2.5}$$

Moreover, AGT-correspondence implies that, up to a U(1) factor which doesn't play any role here,

$$\begin{aligned}&Z_{\text{Quiver}}\left(a,\tilde{a}=a-s_2\frac{\epsilon_2}{2};\mu_1=a,\mu_2=-a-\epsilon_2;\alpha_2+\alpha_3,\alpha_2-\alpha_3\right)\\ &\quad=\mathcal{F}\left(a+\frac{\epsilon_2}{2},\frac{-\epsilon_2}{2},a,\frac{-\epsilon_2}{2},a-s_2\frac{\epsilon_2}{2},\alpha_2,\alpha_3\right)\end{aligned} \tag{2.6}$$

where the l.h.s. is the instanton partition function of SU(2) quiver gauge theory and the r.h.s. is the conformal block of Liouville field theory.

In the following we will show that when $s_2=-1$ the quiver partition function in the above formula reduces to the SU(2) vortex partition function, while when $s_2=1$, it corresponds to the SU(2) simple surface operator.

2.2 SU(2) vortices

Let us start investigating the case $s_2=-1$ where

$$\begin{aligned}m_{1,1}&=0\\ m_{2,2}&=\epsilon_2\\ m_{1,2}&=2a+\epsilon_2\\ m_{2,1}&=-2a\end{aligned} \tag{2.7}$$

To start with, let's focus on $Z_{\text{bifund}}^{(1,1)}$:

$$\begin{aligned}&Z_{\text{bifund}}^{(1,1)}=\prod_{s\in Y_1}\left(\epsilon_2\left(A_{Y_1}(s)+L_{W_1}(s)+1\right)\right)\prod_{t\in W_1}\left(-\epsilon_2\left(A_{W_1}(t)+L_{Y_1}(t)+1\right)\right)\\ &\text{where:}\\ &\prod_{t\in W_1}\left(-\epsilon_2\left(A_{W_1}(t)+L_{Y_1}(t)+1\right)\right)=\prod_{(i,j)\in W_1}-\epsilon_2\left(A_{W_1}(i,j)+L_{Y_1}(i,j)+1\right)\end{aligned} \tag{2.8}$$

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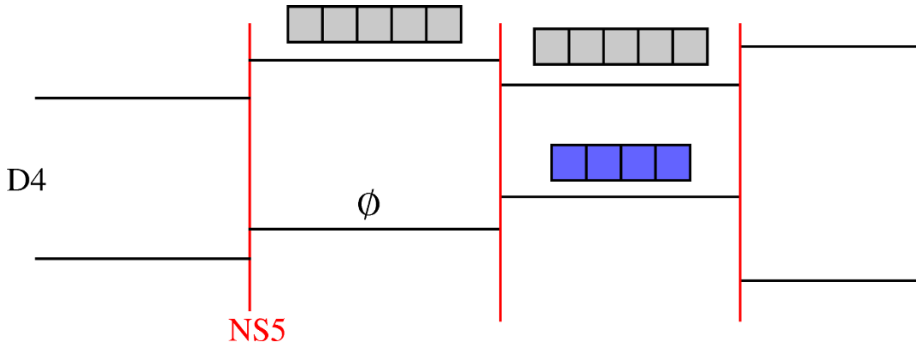

Figure 2. SU(2) vortex from quiver theory.

From the discussion of the previous section we know that the choice of the fundamental mass parameter in (2.7) implies that Y_1 is a row diagram. Moreover, from the results in the appendix, one gets that the bifundamental masses in (2.7) set also W_1 to be a row of the same length which we call k_1, see the figure 2.

To simplify the formulæ, let's define some notations:

$$(x)_{Y,W} := \prod_{s\in Y}\prod_{t\in W} \left(x+\epsilon_2\left(A_Y(s)+L_W(s)+1\right)\right)\left(x-\epsilon_2\left(A_W(t)+L_Y(t)+1\right)\right) \tag{2.9}$$

and

$$\begin{aligned}(x)_Y &:= (x)_{Y,\emptyset}\\ H_Y &:= (0)_{Y,\emptyset}\\ (x)_k &:= (x)_{\emptyset,(1^k)}\\ (x)_{k_1,k_2} &:= (x)_{(1^{k_1}),(1^{k_2})}\end{aligned}$$

Let's calculate $Z^{(1,1)}_{\text{bifund}}$ explicitly

$$Z^{(1,1)}_{\text{bifund}} = \prod_{i=1}^{k_1}\epsilon_2 i \prod_{j=1}^{k_1} -\epsilon_2 j = (\epsilon_2)^2_{k_1}\,(-1)^{k_1} \tag{2.10}$$

The contribution form $Z^{(2,2)}_{\text{bifund}}$ is instead

$$Z^{(2,2)}_{\text{bifund}} = \prod_{t\in W_2} -\epsilon_2\left(A_{W_2}(t)+L_\emptyset(t)\right) = \prod_{(i,j)\in W_2} -\epsilon_2(j-i-1) \tag{2.11}$$

which is non zero only if W_2 is a row. Let's denote its length by k_2. Then

$$Z^{(2,2)}_{\text{bifund}} = (\epsilon_2)_{k_2} \tag{2.12}$$

By including the contributions from $Z^{(1,2)}_{\text{bifund}}$ and $Z^{(2,1)}_{\text{bifund}}$ we get the final formula

$$Z_{\text{bifund}} = (\epsilon_2)^2_{k_1}\,(-1)^{k_1}\,(\epsilon_2)_{k_2}\,(-2a)_{k_1}\,(2a+\epsilon_2)_{k_1,k_2} \tag{2.13}$$

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The contribution from the anti-fundamental matter can be computed with analogous methods giving

$$Z_{\text{antifund}} = (\epsilon_2)_{k_1} (-2a)_{k_1} \tag{2.14}$$

The generic contribution from the vector multiplets is

$$\begin{aligned} Z_{\text{vect}}(a,Y) &= \prod_{\alpha,\beta=1}^{2} Z_{\text{vect}}^{(\alpha,\beta)}(a,Y) \\ Z_{\text{vect}}^{(\alpha,\beta)}(a,Y) &= \prod_{s\in Y_\alpha}\prod_{t\in Y_\beta} \big(a_{a,\beta}+\epsilon_2\big(A_{Y_\alpha}(s)+L_{Y_\beta}(s)+1\big)\big)^{-1} \big(a_{\alpha,\beta}-\epsilon_2\big(A_{Y_\beta}(t)+L_{Y_\alpha}(t)+1\big)\big)^{-1} \\ a_{\alpha,\beta} &:= a_\alpha - a_\beta \end{aligned} \tag{2.15}$$

which reduces for the first node of our specific Young tableaux to

$$Z_{\text{vect}}(a,Y) = \left((\epsilon_2)_{k_1} (-2a)_{k_1}\right)^{-2} \tag{2.16}$$

The fundamental matter Z_{fund} is in the standard form

$$Z_{\text{fund}} = (\tilde{a}+\mu_3)_{k_1} (\tilde{a}+\mu_4)_{k_2} (-\tilde{a}+\mu_3)_{k_1} (-\tilde{a}+\mu_4)_{k_2} \tag{2.17}$$

while the contribution from the second gauge factor of the quiver is:

$$Z_{\text{vect}}(\tilde{a},W) = \frac{(-1)^{k_1+k_2}}{(\epsilon_2)_{k_1}^2 (\epsilon_2)_{k_2}^2 (2a+\epsilon_2)_{k_1,k_2} (-2\tilde{a})_{k_2,k_1}} \tag{2.18}$$

In summary, the total partition function of the quiver theory with specific choice of masses reads

$$Z_{\text{Quiver}}(k_1,k_2) = \frac{(-1)^{k_1} (\tilde{a}+\mu_3)_{k_1} (\tilde{a}+\mu_4)_{k_2} (-\tilde{a}+\mu_3)_{k_1} (-\tilde{a}+\mu_4)_{k_2}}{(\epsilon_2)_{k_1} (\epsilon_2)_{k_2} (2\tilde{a})_{k_1,k_2}} \tag{2.19}$$

This, up-to a sign factor which can be absorbed in the vortex counting parameter coincides to[1] the SU(2) vortex partition function studied in [13]:

$$Z_{\text{vortex}}^{\text{SU(2)}}(\boldsymbol{k}) = \frac{(-1)^{k_2} (a-m_1)_{k_1} (-a-m_1)_{k_2} (a-m_2)_{k_1} (-a-m_2)_{k_2}}{(\epsilon_2)_{k_1} (\epsilon_2)_{k_2} (a_{1,2})_{k_1,k_1}} \tag{2.20}$$

Notice that we should identify $m_i = -\mu_{i+2}$,and $\tilde{a}$ as a, since it is the second gauge factor that couples to hypermultiplets with generic masses.

To conclude the matching, notice that in the two nodes quiver theory, we have two parameters q_1, q_2 which are the exponential of the gauge couplings of the quiver theory. These are related to the vortex counting parameters z_1, z_2 of vortex partition functions as

$$q_1^{k_1} (q_2)^{k_1+k_2} = (q_1 q_2)^{k_1} q_2^{k_2} = z_1^{k_1} z_2^{k_2} \tag{2.21}$$

From the CFT viewpoint z_i are the insertion points of the degenerate fields.

[1] With respect to [13] we set $\hbar = -\epsilon_2$. These sign factors will be disregarded in the following without further notice.

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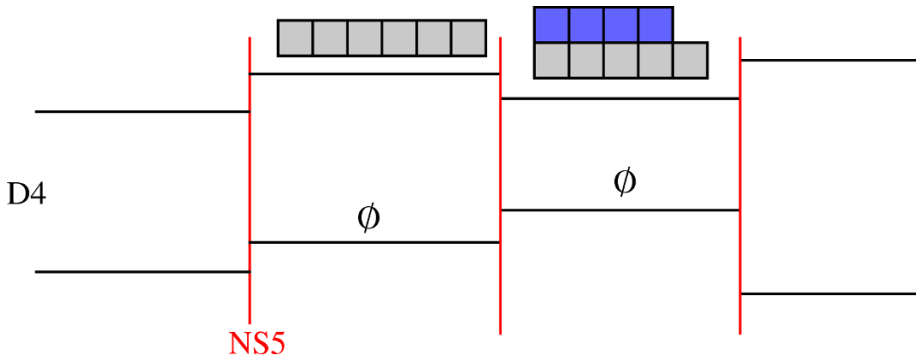

Figure 3. SU(2) simple surface operators from quiver theory.

2.3 SU(2) simple surface operators

A natural question is to find what's the result in the other channel. As expected we find it is the result of [21]. So, let's now choose $s_2 = 1$, then

$$\begin{aligned} \tilde{a} &= a - \frac{\epsilon_2}{2} \\ m_{1,1} &= \epsilon_2 \\ m_{2,2} &= 0 \\ m_{1,2} &= 2a \\ m_{2,1} &= -2a + \epsilon_2 \end{aligned} \tag{2.22}$$

In this case, the contribution of the bifundamentals reads

$$Z^{(1,1)}_{\text{bifund}} = \prod_{s\in Y_1} \left(\epsilon_2 \left(A_{Y_1}(s) + L_{W_1}(s) + 2\right)\right) \prod_{t\in W_1} \left(-\epsilon_2 \left(A_{W_1}(t) + L_{Y_1}(t)\right)\right) \tag{2.23}$$

where: (2.24)

$$\prod_{t\in W_1} \left(-\epsilon_2 \left(A_{W_1}(t) + L_{Y_1}(t)\right)\right) = \prod_{(i,j)\in W_1} -\epsilon_2 \left(A_{W_1}(i,j) + L_{Y_1}(i,j)\right)$$

Using once again the results in the appendix, the bifundamental contribution

$$Z^{(2,2)}_{\text{bifund}} = \prod_{t\in W_2} \left(-\epsilon_2 \left(A_{W_2}(t) + L_{\emptyset}(t) + 1\right)\right) = \prod_{t\in W_2} \left(-\epsilon_2 (j-i)\right) \tag{2.25}$$

is non vanishing only if $W_2 = \emptyset$, see figure 3.

Therefore, the bifundamental contributions are given by

$$\begin{aligned} Z^{(1,1)}_{\text{bifund}} &= H_{Y_1} H_{W_1} (-1)^{k_1} \\ Z^{(2,2)}_{\text{bifund}} &= 1 \\ Z^{(1,2)}_{\text{bifund}} &= (-1)^{k_1+1} (-2a)_{k_1+1} \\ Z^{(2,1)}_{\text{bifund}} &= (-1)^{k_1+k_2} \left(\tilde{a}_{1,2}\right)_{W_1} \end{aligned}$$

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The contribution from the other factors can be analogously derived to be

$$Z_{\text{antifund}} Z_{\text{vect}}(a, Y) = \frac{1}{(\epsilon_2)_{k_1+1} (-2a)_{k_1+1}} \tag{2.26}$$

$$Z_{\text{vect}}(\tilde{a}, W) = \frac{1}{\left(H_{W_1}(\tilde{a}_{1,2})_{W_1}\right)^2} \tag{2.27}$$

$$Z_{\text{fund}} = (\tilde{a} - \mu_3)_{W_1} (\tilde{a} - \mu_4)_{W_1} \tag{2.28}$$

and finally we get

$$Z_{\text{Quiver}}(W_1) = \frac{(-1)^{k_1+k_2+1} (\tilde{a} - \mu_3)_{W_1} (\tilde{a} - \mu_4)_{W_1}}{H_{W_1}(\tilde{a}_{1,2})_{W_1}} \tag{2.29}$$

which is the partition function of SU(2) simple surface operator [21]

$$Z_{\text{simple surface}} = \frac{(a + m_1)_{W_1} (a + m_2)_{W_1}}{H_{W_1}(a_{1,2})_{W_1}} \tag{2.30}$$

Now the identification of parameters goes as

$$q_1^{k_1+1} (q_2)^{k_1+k_2} = \frac{z_1}{z_2} z_1^{k_1} z_2^{k_2} \tag{2.31}$$

As already noticed, z_i are the insertion points of the degenerate fields.

2.4 Relation to open topological string amplitudes

The amplitudes discussed in the previous sections can be derived as four dimensional limits of Open Topological String amplitudes on the strip with suitable boundary conditions [13, 21]. The discussion of the previous section then provides the CFT interpretation of this class of strip amplitudes, as summarized in figure 4 and figure 5. Actually, this is the simplest situation. For example we can have more than two degenerate states, then does this story still holds? The answer is yes. From our previous calculations, we can deduce three general laws: (1) the number of nodes of the quiver equals the number of degenerate states. (2) the total number of rows of Young-tableaux increase by one when counting from left to right along the quiver of gauge theory nodes. (3) different fusion channels just tell us on which gauge factor of the quiver to associate an extra row in the partition. So if we have n degenerate states, the corresponding quiver has n nodes, and on each node there are two choices to add a new row. For convenience let's define a fusion vector $\mathfrak{V} \in \mathbb{Z}_2^n$, whose i-th component is 1 if we add a new row onto the partition attached to the first $D4$ brane and 2 if to the second. For example, the non-abelian vortex partition function is associated to $\mathfrak{V} = (1, 2)$, while the simple surface operator partition function is associated to $\mathfrak{V} = (1, 1)$.

When we have n degenerate states, the Young-tableaux on the final node are a couple (Y, W) satisfying the constraint $n_1 + n_2 = n$, where n_1, n_2 are respectively the number of rows of Y and of W. Hence we conclude that the four dimensional limit of the strip amplitudes of the form $A^{\{Y,W\}}_{\{\emptyset,\emptyset\}}$, that is with boundary conditions labeled by Y and W, reproduces the full conformal block vector space including all the possible fusion channels. For example we can choose $\mathfrak{V} = (1, \ldots, 1, 2, \ldots, 2)$, where there are n_1 1's and n_2 2's and can prove explicitly that for this choice of fusion vector our claim is correct, see figure 6.

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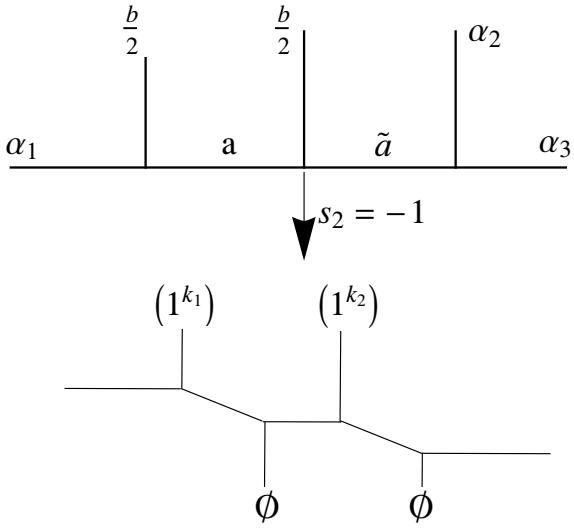

Figure 4. Relation between SU(2) vortex and CFT.

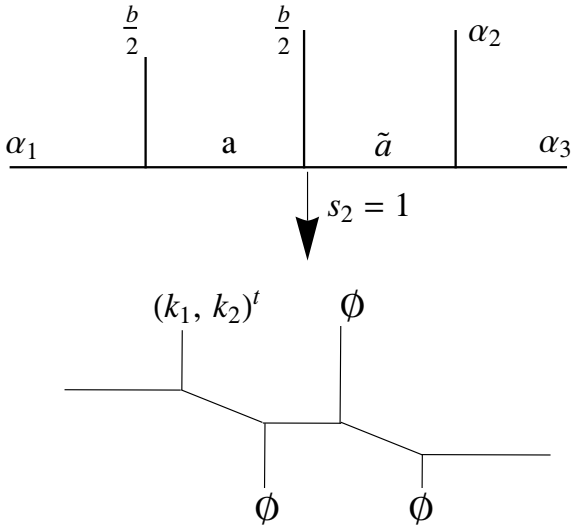

Figure 5. Relation between SU(2) simple surface operator and CFT.

3 SU(N) generalization

In the following we will give the natural generalization to SU(N) theories. We know that the SU(N) vortex partition function should have N independent counting parameters, thus from the previous section's discussion we know that the associated SU(N) quiver theory will have N nodes. The quiver configuration reads as the brane construction illustrated in figure 7.

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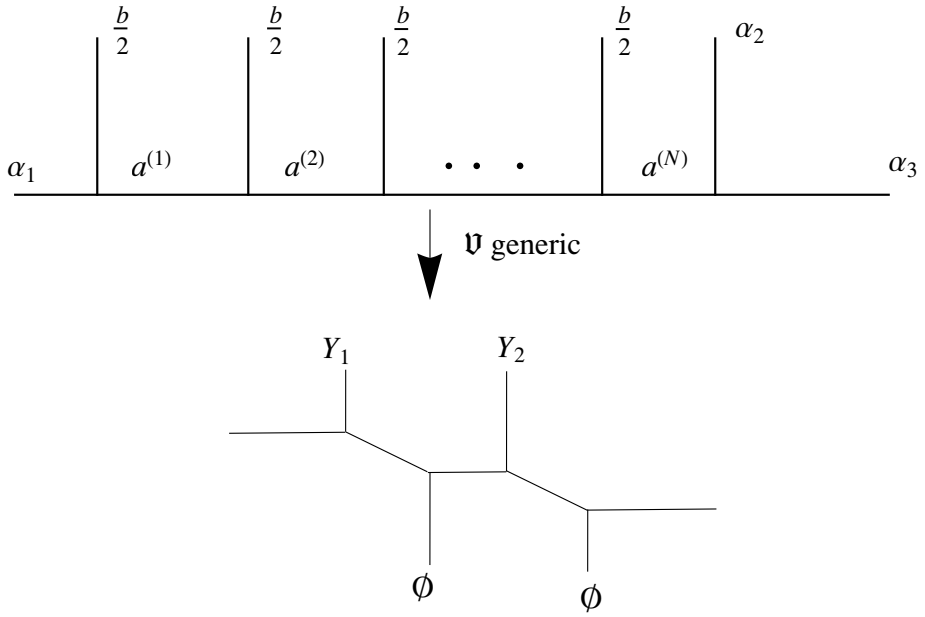

Figure 6. Relation between SU(2) strip amplitudes and CFT.

3.1 SU(N) vortices

The Young-tableaux configuration of quiver gauge theory corresponding to vortex partition function is such that at the L-th node the arrows of Young-tableaux are $Y^{(L)} = \left(1^{k_1}, \ldots, 1^{k_L}, \emptyset, \ldots, \emptyset\right)$. This configuration can be obtained from a given bifundamental mass assignments as displayed in the following. We will see that this choice of masses correctly reproduces the fusion rules for Toda field theory.

Let us consider the L-th node of the quiver and calculate $Z_L Z_{L,L+1}$, where Z_L is the vector contribution of the L-th node while $Z_{L,L+1}$ the corresponding bifundamental. Following the arguments in the appendix, we can read out the L-th bifundamental mass to be

$$m^{(L)}_{\alpha,\alpha} := a^{(L)}_{\alpha} - a^{(L+1)}_{\alpha} - m_L = \delta_{\alpha,L+1}\epsilon_2 \tag{3.1}$$

Then the matrix of masses is given by

$$m^{(L)}_{\alpha,\beta} = \begin{cases} a^{(L)}_{\alpha,\beta} = a^{(L+1)}_{\alpha,\beta} & \alpha \in [1,L]; \quad \beta = [1,L] \\ a^{(L+1)}_{\alpha,\beta} & \alpha \in [1,L]; \quad \beta \in [L+1,N] \\ a^{(L)}_{\alpha,\beta} & \alpha \in [L+1,N]; \ \beta = [1,L] \end{cases}$$

We find it better to write Z_L in three parts according to above mass matrix formula:

$$Z_L^{-1} = \prod_{\alpha,\beta=1}^{L} \left(a^{(L)}_{\alpha,\beta}\right)_{k_\alpha,k_\beta} \prod_{\alpha=1}^{L}\prod_{\beta=L+1}^{N} (-1)^{k_\alpha} \left(a^{(L)}_{\beta,\alpha}\right)_{k_\alpha} \prod_{\beta=1}^{L}\prod_{\alpha=L+1}^{N} \left(a^{(L)}_{\alpha,\beta}\right)_{k_\beta}$$

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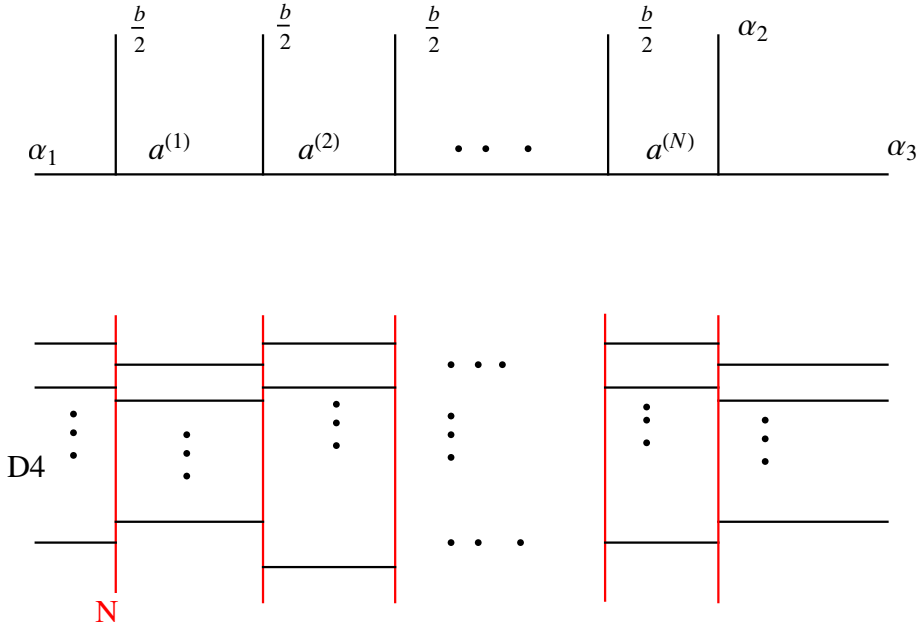

Figure 7. AGT relation between SU(N) quiver gauge theory and CFT.

Correspondingly, $Z_{L,L+1}$ read

$$Z_{L,L+1} = \left\{ \prod_{\alpha=1}^{L} \prod_{\beta=1}^{L} \left(m_{\alpha,\beta}^{(L)} \right)_{k_\alpha,k_\beta} \right\} \times \left\{ \prod_{\alpha=1}^{L} \prod_{\beta=L+2}^{N} (-1)^{k_\alpha} \left(-m_{\alpha,\beta}^{(L)} \right)_{k_\alpha} \right\} \left\{ \prod_{\alpha=1}^{L} \left(m_{\alpha,L+1}^{(L)} \right)_{k_\alpha,k_{L+1}} \right\} \times \left\{ \prod_{\alpha=L+1}^{N} \prod_{\beta=1}^{L} \left(m_{\alpha,\beta}^{(L)} \right)_{k_\beta} \right\} \left\{ \prod_{\alpha=L+1}^{N} \left(m_{\alpha,L+1}^{(L)} \right)_{k_{L+1}} \right\}$$

Then we get:

$$Z_L Z_{L,L+1} = \frac{\left\{ \prod_{\alpha=1}^{L} \left(a_{\alpha,L+1}^{(L+1)} \right)_{k_\alpha,k_{L+1}} \right\}}{\left\{ \prod_{\alpha=1}^{L} (-1)^{k_\alpha} \left(a_{L+1,\alpha}^{(L)} \right)_{k_\alpha} \right\}} \left\{ \prod_{\alpha=L+2}^{N} \left(a_{\alpha,L+1}^{(L+1)} \right)_{k_{L+1}} \right\} (\epsilon_2)_{k_{L+1}} \tag{3.2}$$

The mass spectrum of the antifundamental hypermultiplets is assigned as

$$(\mu_1, \mu_2, \ldots, \mu_N) = \left(-a_1^{(1)} - \epsilon_2, -a_2^{(1)}, \ldots, -a_N^{(1)} \right) \tag{3.3}$$

and the correspondent contribution to the instanton partition function is

$$Z_{\text{antifund}} = \prod_{f=1}^{N} \prod_{i=1}^{k_1} \left(a_1^{(1)} + \mu_f + \epsilon_2(1-i) \right) = (-1)^{Nk_1} (\epsilon_2)_{k_1} \prod_{i=2}^{N} \left(a_{i,1}^{(1)} \right)_{k_1} \tag{3.4}$$

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Finally, the vector contribution of the last N-th node is

$$Z_N^{-1} = \prod_{\alpha=1}^{N} (\epsilon_2)_{k_\alpha}^2 \, (-1)^{k_\alpha} \prod_{\alpha<\beta}^{N} (-1)^{k_\alpha+k_\beta} \left(a_{\alpha,\beta}^{(N)}\right)_{k_\alpha,k_\beta}^2 \tag{3.5}$$

Then the instanton partition function of this quiver is:

$$Z_{\text{Quiver}} = \frac{(-1)^{Nk_1+\sum_\alpha k_\alpha} \prod_{\alpha,f=1}^{N} (-1)^{k_\alpha} \left(-a_\alpha^{(N)} + \mu_{f+N}\right)_{k_\alpha}}{\prod_{\alpha=1}^{N} (\epsilon_2)_{k_\alpha} \prod_{\alpha<\beta}^{N} (-1)^{k_\beta} \left(a_{\alpha,\beta}^{(N)}\right)_{k_\alpha,k_\beta}} \tag{3.6}$$

Following the result of [13], and identifing $\hbar = -\epsilon_2$, the SU(N) vortex partition function can be written as:

$$\begin{aligned} Z_{\text{vortex}}^{\text{SU}(N)} &= \sum_{\boldsymbol{k}} Z_{\text{vortex}}^{\text{SU}(N)}(\boldsymbol{k}) \prod_{i=1}^{N} z_i^{k_i} \\ Z_{\text{vortex}}^{\text{SU}(N)}(\boldsymbol{k}) &= \frac{\prod_{\alpha,f=1}^{N} (-1)^{k_\alpha} \left(-a_\alpha - m_f\right)_{k_\alpha}}{\prod_{\alpha=1}^{N} (\epsilon_2)_{k_\alpha} \prod_{\alpha<\beta}^{N} (-1)^{k_\beta} \left(a_{\alpha,\beta}\right)_{k_\alpha,k_\beta}} \end{aligned} \tag{3.7}$$

This can be identified with the quiver instanton partition function by setting $a_\alpha^{(N)} = a_\alpha$ and $m_f = -\mu_{N+f}$. The counting parameters z_i are identified as

$$\begin{aligned} \prod_{i=1}^{N} q_i^{\sum_{j=1}^{i} k_j} &= \prod_{i=1}^{N} z_i^{k_i} \\ z_i &:= \prod_{j=i}^{N+1-i} q_i \end{aligned} \tag{3.8}$$

In conclusion, the instanton partition function of quiver gauge theory with $Y^{(L)} = \left(1^{k_1}, .., 1^{k_L}, \emptyset, \ldots, \emptyset\right)$ with parameters in formula (3.1) and (3.3) gives the SU(N) vortex partition function.

3.2 SU(N) simple surface operators

From the previous arguments we can argue that the four dimensional limit of the strip amplitude $A^{W,\emptyset,\ldots,\emptyset}_{\{\emptyset,\ldots,\emptyset\}}$, with $W = (k_1, k_2, \ldots, k_N)$, corresponds to the quiver gauge theory with the following Young-tableaux assignment

$$\begin{aligned} Y^{(L)} &= (Y_L, \emptyset, \ldots, \emptyset) \\ Y_L^t &= (k_1 + (N-L), k_2 + (N-L), \ldots, k_L + (N-L)) \end{aligned} \tag{3.9}$$

The corresponding bifundamental masses can be obtained by following the arguments displayed in the appendix to be

$$m_{\alpha,\alpha}^{(L)} = a_\alpha^{(L)} - a_\alpha^{(L+1)} - m_L = \delta_{\alpha,1}\epsilon_2 \tag{3.10}$$

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for the L-th node. The corresponding vector contribution for the L-th node is

$$Z_L^{-1} = (-1)^{|Y_L|} H_{Y_L}^2 \prod_{\beta=2}^{N} \left(a_{1,\beta}^{(L)}\right)_{Y_L} \prod_{\alpha=2}^{N} (-1)^{|Y_L|} \left(a_{1,\alpha}^{(L)}\right)_{Y_L} \tag{3.11}$$

while the bifundamental is

$$Z_{L,L+1} = (\epsilon_2)_{Y_L,Y_{L+1}} \prod_{\beta=2}^{N} \left(a_{1,\beta}^{(L)}\right)_{Y_L} \prod_{\alpha=2}^{N} (-1)^{|Y_{L+1}|} \left(a_{1,\alpha}^{(L+1)}\right)_{Y_{L+1}} \tag{3.12}$$

so that

$$\prod_{L=1}^{N-1} Z_L Z_{L,L+1} = \left\{ \prod_{L=1}^{N-1} \frac{(\epsilon_2)_{Y_L,Y_{L+1}}}{(-1)^{|Y_L|} H_{Y_L}^2} \right\} \frac{\prod_{\alpha=2}^{N} (-1)^{|Y_N|} \left(a_{1,\alpha}^{(N)}\right)_{Y_N}}{\prod_{\alpha=2}^{N} (-1)^{|Y_1|} \left(a_{1,\alpha}^{(1)}\right)_{Y_1}} \tag{3.13}$$

Using the result of the last appendix, we can rewrite

$$(\epsilon_2)_{Y_L,Y_{L+1}} = (-1)^{|Y_L|+L} H_{Y_L} H_{Y_{L+1}} \tag{3.14}$$

and finally get

$$\prod_{L=1}^{N-1} Z_L Z_{L,L+1} = (-1)^{\sum_{L=1}^{N-1} L} \frac{H_{Y_N} \prod_{\alpha=2}^{N} (-1)^{|Y_N|} \left(a_{1,\alpha}^{(N)}\right)_{Y_N}}{H_{Y_1} \prod_{\alpha=2}^{N} (-1)^{|Y_1|} \left(a_{1,\alpha}^{(1)}\right)_{Y_1}} \tag{3.15}$$

Notice that, as in SU(2) case, the spectrum of antifundamental hypermultiplets is fixed to be the same both for simple surface operator and nonabelian vortices. What distinguishes the different cases are the different fusion rules channels. The corresponding factors are then

$$Z_{\text{fund}} = \prod_{f=1}^{N} \left(a_1^{(N)} - \mu_{f+N}\right)_{Y_N} \tag{3.16}$$

$$Z_{\text{antifund}} = (-1)^{|Y_1|} H_{Y_1} \prod_{\alpha=2}^{N} \left(a_{1,\alpha}^{(1)}\right)_{Y_1} \tag{3.17}$$

$$Z_N^{-1} = (-1)^{N|Y_N|} H_{Y_N}^2 \prod_{\alpha=2}^{N} (a_{1,\alpha})_{Y_N}^2 \tag{3.18}$$

which finally give

$$Z_{\text{Quiver}} = (-1)^{\sum_{L=1}^{N} L+N|Y_1|+|Y_N|} \frac{\prod_{f=1}^{N} \left(a_1^{(N)} - \mu_{f+N}\right)_{Y_N}}{H_{Y_N} \prod_{\alpha=2}^{N} \left(a_{1,\alpha}^{(N)}\right)_{Y_N}}. \tag{3.19}$$

This, after the identifications $\hbar = -\epsilon_2$, $a_1^{(N)} = a_1$, $m_f = -\mu_{f+N}$ and $\lambda = Y_N$, is the simple surface operator partition function discussed in [21] under the same counting parameters identification that we used in the last section.

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3.3 Toda fusion rules from quiver gauge theory

In this subsection we show how to *derive* fusion rules of semidegenerate states of Toda field theory from our construction. Let's concentrate on the L-th node of the quiver and recall the diagonal part of the mass assignment

$$m^{(L)}_{\alpha,\alpha}{:=}a^{(L)}_{\alpha} - a^{(L+1)}_{\alpha} - m_L \tag{3.20}$$

By denoting $\mathtt{m}_L = m_L(1,1,\ldots,1)$, being a vector of N entries all equal to m_L, we can write the above formula as

$$\mathtt{a}^{(L)} - \mathtt{a}^{(L+1)} = \mathtt{m}^{(L)} - \mathtt{m}_L \tag{3.21}$$

where $\mathtt{a}^{(L)}$ denotes the vector of internal momenta at the L-th node and $\mathtt{m}^{(L)}$ the vector of diagonal entries of the mass matrix at the L-node. Actually, for this assignment of external momenta, Toda fusion rules have N channels. For the i-th channel $\mathtt{m}^{(L)} = \epsilon_2 u_i = \epsilon_2 \left(u_1 - \sum_{j=1}^{i-1} e_j\right)$. Where u_i is the unit vector in the i-th direction in $\mathbb{R}^N$ and $e_j{:=}u_j - u_{j+1}$ are the simple roots of the $\mathfrak{sl}_N$ algebra. Then we have

$$\mathtt{a}^{(L)} - \mathtt{a}^{(L+1)} = \epsilon_2 \left(u_1 - \sum_{j=1}^{i-1} e_j\right) - \mathtt{m}_L = \epsilon_2 \left(u_1 - \frac{\mathtt{m}_L}{\epsilon_2}\right) - \epsilon_2 \sum_{j=1}^{i-1} e_j \tag{3.22}$$

If we set $\mathtt{m}_L = \epsilon_2 \frac{1}{N}(1,1,\ldots,1)$, then

$$\mathtt{a}^{(L)} - \mathtt{a}^{(L+1)} = \epsilon_2 \left(-\omega_1\right) - \epsilon_2 \sum_{j=1}^{i-1} e_j \tag{3.23}$$

where ω_1 is the highest weight of the fundamental representation of $\mathfrak{sl}_N$. The above formula can be recognized as the fusion rule calculated in [16].

For SU(N) N nodes quiver, we can have N semidegenerate states, for each one of them we have N channels. We can use a N-dimensional vector of integer entries $\mathfrak{V}$ to denote the choice of the fusion channels. The fusion vector $\mathfrak{V}$ is built as follows: if on the L-th node we choose k-th channel, namely $m^{(L)}_{\alpha,\alpha} = \epsilon_2 \delta_{\alpha,k}$, then the corresponding L-th component of $\mathfrak{V}$ is set equal to k. For example for the SU(N) vortex $\mathfrak{V}_{\text{vortex}} = (1,2,\ldots,N)$, while for SU($N$) simple surface operator, $\mathfrak{V}_{\text{simplesurface}} = (1,1,\ldots,1)$.

The relation with the four dimensional limit of strip amplitudes goes as in the SU(2) as depicted in figures 8, 9 and figure 10.

Notice that the four dimensional limit of strip amplitudes correspond to conformal blocks with only two independent external momenta, and one independent internal momentum. The number of degenerate states inserted in the conformal block corresponds to the total number of rows of the Young tableaux parametrizing the open string boundary conditions. This suggests that in order to have arbitrary boundary conditions one should consider conformal blocks with an arbitrary number of degenerate field insertions. Since we know that the full instanton partition function can be obtained by gluing two strip amplitudes with generic boundary conditions, this would provide a conformal field theory picture of this operation. From the CFT viewpoint, the infinite number of degenerate insertions could condense in a line operator [15] which could be used to glue the two CFT amplitudes to obtain the full result.

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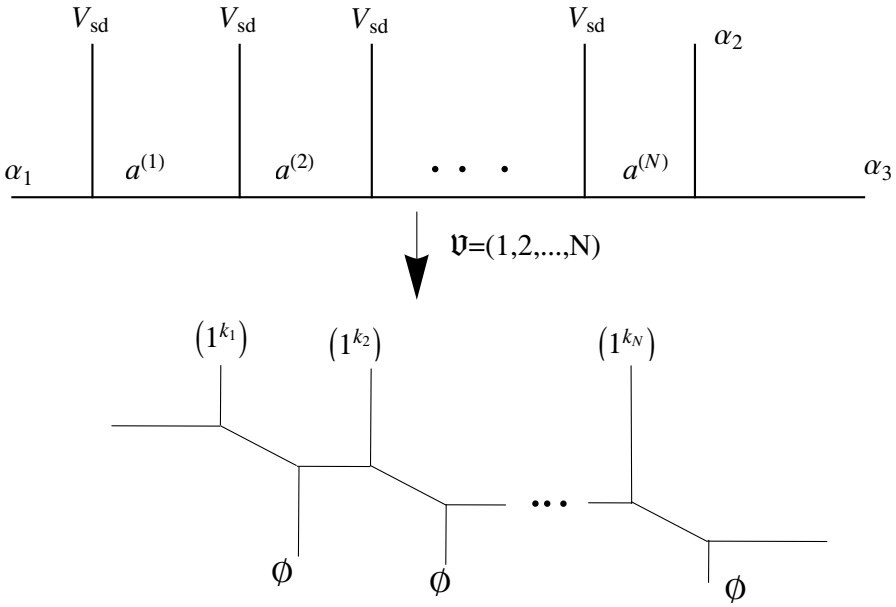

Figure 8. Relation between SU(N) vortex and CFT.

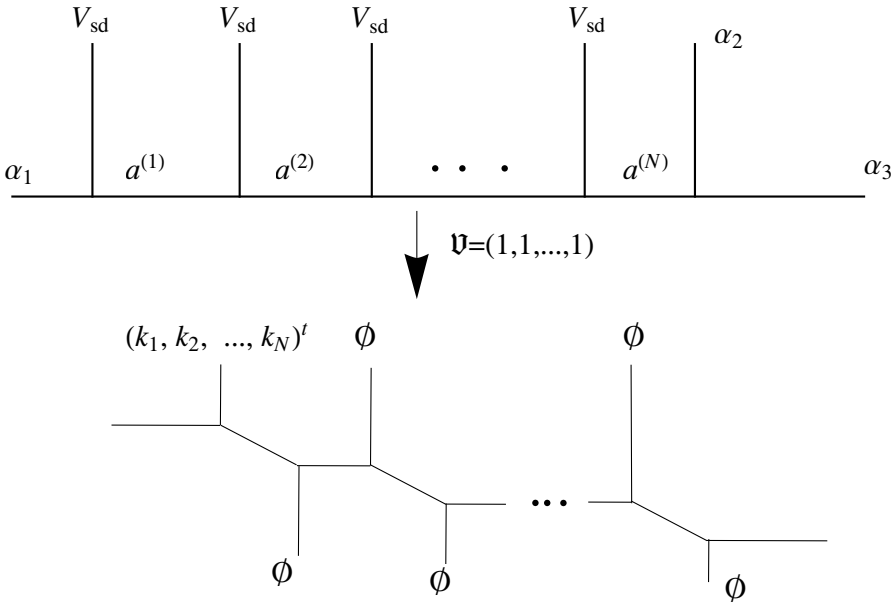

Figure 9. Relation between SU(N) simple surface operators and CFT.

4 Conclusions

In this paper we studied the relation between non-abelian vortex partition functions and Liouville/Toda conformal field theories, by showing how to reproduce these partition functions from conformal blocks with degenerate field insertions. Moreover, we performed a general analysis using geometric engineering for open topological strings and found that

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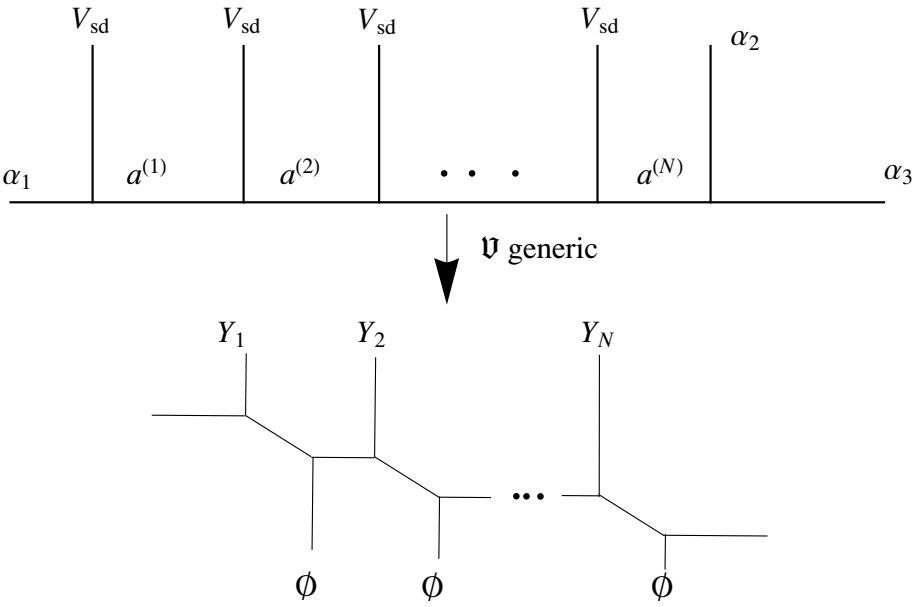

Figure 10. Relation between SU(N) strip amplitudes and CFT.

there is a much richer structure in this correspondence which arises by identifying the full vector space of degenerate conformal blocks with the four dimensional limit of open topological strings amplitudes on a strip with general boundary conditions. A natural generalisation of this approach would be to analyse the full refined topological string amplitudes from the CFT viewpoint possibly along the lines of [5]. Another interesting venue is the investigation of the correspondence with integrable systems and their quantization [1, 10–12] and in particular their relevance in vortex counting and more in general for open topological string amplitudes on the strip.

As discussed in [13, 14], vortices partition functions arise in the classical limit of four dimensional gauge theories with surface operator insertions. The analysis presented in this paper should then provide the classical limit of multiple surface operator insertions. In particular the approach of quiver gauge theories that we presented can be generalised to encompass also the four-dimensional instanton corrections. This should be completed with a description of the moduli space of instantons with wild ramification [18, 36].

An analogous analysis could be performed for surface operators in gauge theories on ALE spaces which have been recently related to para Liouville/Toda conformal CFTs [7–9, 32]. In this case the relevant vortex moduli space should be obtained as a lagrangian submanifold of the moduli space of instantons on ALE spaces.

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A Instanton partition functions

Let us consider the instanton partition function of a linear quiver with N nodes. The corresponding brane construction has $N+2$ sets of D4-branes and $N+1$ $NS5$ branes. We will focus on unrefined limit $\epsilon_1 = -\epsilon_2$.

$$Z_{\text{Quiver}} = Z_{\text{fund}} Z_{\text{antifund}} Z_N \prod_{i=1}^{N-1} Z_i Z_{i,i+1} \tag{A.1}$$

Z_{fund} and Z_{antifund} are the contributions from fundamental and antifundamental hypermultiplets. Z_i is the contribution from the i-th gauge factor, while $Z_{i,i+1}$ is the contribution from the i-th bifundamental hyper. These depend on the following parameters

$$\begin{aligned}
a_i^{(N)} &= \text{the i-th Coulomb branch parameter of the N-th gauge factor.} \\
m_i &= \text{the i-th mass of bifundamental hypermultiplet} \\
\mu_i &= \begin{cases} \text{masses of antifundamental hypermultiplets} & i \in [1,N] \\ \text{masses of fundamental hypermultiplets} & i \in [N+1,2N] \end{cases} \\
Y^{(i)} &: \text{the arrow of Young-tableaux on the i-th node.}
\end{aligned}$$

More explicitly:

$$Z_{\text{antifund}}\left(a^{(1)}, \mu, Y^{(1)}\right) = \prod_{f=1}^{N} \prod_{\alpha=1}^{N} \prod_{(i,j)\in Y_\alpha^{(1)}} \left(a_\alpha^{(1)} + \mu_f + \epsilon_2(j-i)\right) \tag{A.2}$$

$$Z_{\text{fund}}\left(a^{(N)}, \mu, Y^{(N)}\right) = \prod_{f=1}^{N} \prod_{\alpha=1}^{N} \prod_{(i,j)\in Y_\alpha^{(N)}} \left(a_\alpha^{(N)} - \mu_{f+N} + \epsilon_2(j-i)\right) \tag{A.3}$$

The L-th bifundamental hypermultiplet contribution is:

$$Z_{L,L+1} = \prod_{\alpha=1}^{N} \prod_{\beta=1}^{N} Z_{L,L+1}^{(\alpha,\beta)} \tag{A.4}$$

$$\begin{aligned}
Z_{L,L+1}^{(\alpha,\beta)} = & \prod_{s\in Y_\alpha^{(L)}} \left(m_{\alpha,\beta}^{(L)} + \epsilon_2 \left(A_{Y_\alpha^{(L)}}(s) + L_{Y_\beta^{(L+1)}}(s) + 1\right)\right) \\
& \prod_{t\in Y_\beta^{(L+1)}} \left(m_{\alpha,\beta}^{(L)} - \epsilon_2 \left(A_{Y_\beta^{(L+1)}}(t) + L_{Y_\alpha^{(L)}}(t) + 1\right)\right)
\end{aligned}$$

$$m_{\alpha,\beta}^{(L)} := a_\alpha^{(L)} - a_\beta^{(L+1)} - m_L \tag{A.5}$$

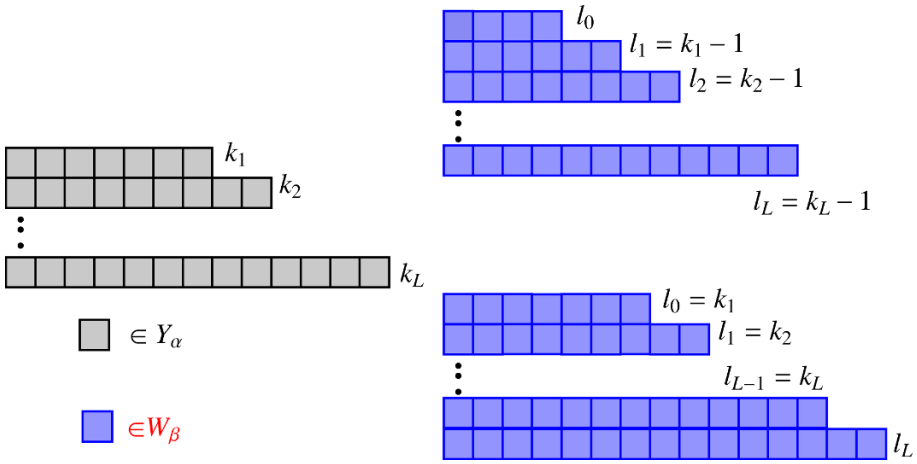

Figure 11. Relation between Y_α and W_β when $m_{\alpha,\beta} = \epsilon_2$. Then consider contribution from s.

The L-th gauge factor contribution is:

$$Z_L = \prod_{\alpha=1}^{N} \prod_{\beta=1}^{N} Z_L^{(\alpha,\beta)} \tag{A.6}$$

$$\left(Z_L^{(\alpha,\beta)}\right)^{-1} = \prod_{s \in Y_\alpha^{(L)}} \left(a_{\alpha,\beta}^{(L)} + \epsilon_2 \left(A_{Y_\alpha^{(L)}}(s) + L_{Y_\beta^{(L)}}(s) + 1\right)\right) \prod_{t \in Y_\beta^{(L)}} \left(a_{\alpha,\beta}^{(L)} - \epsilon_2 \left(A_{Y_\beta^{(L)}}(t) + L_{Y_\alpha^{(L)}}(t) + 1\right)\right) \tag{A.7}$$

For a Young-tableau Y, one box s has coordinates (i, j), where i counts the number of columns and j counts the number of rows. Then the arm and leg of s relative to another Young-tableau W,are defined as $A_W(s) := W_i - j; L_W(s) := W_j^t - i$. Where W^t is the dual partition of W. $|Y| := \sum_i Y_i$. We call a partition of the form (1^k) a row partition of length k, and a partition of the form (k) a column partition.

B Degeneration from bifundamental masses

Let us state our results and then prove them. The claim is that when $m_{\alpha,\beta} = 0$, $W_\beta = Y_\alpha$ and when $m_{\alpha,\beta} = \epsilon_2$, W_β has one row more than that of Y_α. In this situation, if we suppose Y_α has L rows with lengths $k_1 \leq k_2 \leq \ldots \leq k_L$ and W_β had $L + 1$ rows with lengths $l_0 \leq l_1 \leq \ldots \leq l_L$, then for $1 \leq i \leq L$ either $k_i = l_{i-1}$ or $k_i = l_i + 1$. Please refer to figure 11 for a pictorial illustration. Let's start from the simpler case $m_{\alpha,\beta} = 0$.

$$Z_{\text{bifund}}^{(\alpha,\beta)} = \prod_{s \in Y_\alpha} \left(\epsilon_2 \left(A_{Y_\alpha}(s) + L_{W_\beta}(s) + 1\right)\right) \prod_{t \in W_\beta} \left(-\epsilon_2 \left(A_{W_\beta}(t) + L_{Y_\alpha}(t) + 1\right)\right)$$

Let's suppose $Y_\alpha^t = (k_1, k_2, \ldots, k_L)\,; W_\beta^t = (l_1, l_2, \ldots, l_M)$. We will proceed in our proof by induction from the top row to the bottom.

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If $M > L$, then the result is non vanishing only if $L_{Y_\alpha}(t)+1 = L_\emptyset(t)+1 = -i+1 \neq 0$. The same argument applies for $L \leq M$, so that we stay with $M = L$.

The first induction step is when t is on the top row of W_β, so that $A_{W_\beta}(t) = 0$, and $L_{Y_\alpha}(t)+1 = 1+k_1-i_t$

$$\begin{cases} L_{Y_\alpha}(t)+1 = 1+k_1-i_t \neq 0 \\ i_t \in [1,l_1] \end{cases} \Longrightarrow k_1 \geq l_1$$

Similarly for the contribution from $s \in Y_\alpha$, we get $l_1 \geq k_1$, implying $k_1 = l_1$. Suppose now $k_i = l_i$ when $i \leq p-1$ and let's prove that $k_p = l_p$.

1. $i_t \in [1,l_1]$, $A_{W_\beta}(t) = p-1$

$$\begin{cases} L_{Y_\alpha}(t)+p = p+k_p-i_t \neq 0 \\ i_t \in [1,l_1] \end{cases} \Longrightarrow k_p \geq l_1-(p-1)$$

2. when $i_t \in [l_1+1, l_2]$ $A_{W_\beta}(t) = p-2$

$$\begin{cases} L_{Y_\alpha}(t)+1+p-2 = p-1+k_p-i_t \neq 0 \\ i_t \in [1+l_1, l_2] \end{cases} \Longrightarrow k_p \leq l_1+1-p \text{ or } k_p \geq l_2-p+2$$

Since $k_1 = l_1$, then $k_p \neq l_1+1-p$ and $k_p \geq l_2-p+2$. By iterating this procedure we find $k_p \geq l_p$, and symmetrically $l_p \geq k_p$, namely $l_p = k_p$. This ends the proof of the first statement.

Now let us concentrate on $m_{\alpha,\beta} = \epsilon_2$

$$Z^{(\alpha,\beta)}_{\text{bifund}} = \prod_{s\in Y_\alpha} \left(\epsilon_2 \left(A_{Y_\alpha}(s) + L_{W_\beta}(s)+2\right)\right) \prod_{t\in W_\beta} \left(-\epsilon_2 \left(A_{W_\beta}(t)+L_{Y_\alpha}(t)\right)\right)$$

It is easy to show that W_β can have at most one row more than Y_α. Suppose that $Y^t_\alpha = (k_1,k_2,\ldots,k_L)\,; W^t_\beta = (l_0,l_1,l_2,\ldots,l_M)$ and apply induction again from top to bottom.

When t is on the top row of W_β there is no constraint for the length l_0.

When t is on the next to top row of W_β then:

1. for $i_t \in [1,l_0]$, in this case $A_{W_\beta}(t) = 1$

$$\begin{cases} L_{Y_\alpha}(t)+1 = 1+k_1-i_t \neq 0 \\ i_t \in [1,l_0] \end{cases} \Longrightarrow k_1 \geq l_0$$

2. for $i_t \in [l_0+1, l_1]$, in this case $A_{W_\beta}(t) = 0$

$$\begin{cases} L_{Y_\alpha}(t) = k_1-i_t \neq 0 \\ i_t \in [1+l_0, l_1] \end{cases} \Longrightarrow k_1 \leq l_0 \text{ or} k_1 \geq l_1+1$$

so we have $k_1 = l_0$ or $k_1 \geq l_1+1$.

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Let us consider now the contribution from Y_α. When s is on the top row of Y_α, $A_{Y_\alpha}(s) = 0$ and we get

$$\begin{cases} L_{W_\beta}(s) + 2 = l_1 - i_s + 2 \neq 0 \\ i_s \in [1, k_1] \end{cases} \Longrightarrow l_1 \geq k_1 - 1$$

so $k_1 = l_0$ or $k_1 \geq l_1 + 1 \cap l_1 \geq k_1 - 1 \Longrightarrow\ k_1 = l_0$ or $\ k_1 = l_1 + 1$. Now suppose that for $i \leq p - 1$, we have $l_i = k_i - 1$ or $l_{i-1} = k_i$. Then

1. for $i_t \in [1, l_0]$, $A_{W_\beta}(t) = p$ and

$$\begin{cases} L_{Y_\alpha}(t) + p = p + k_p - i_t \neq 0 \\ i_t \in [1, l_0] \end{cases} \Longrightarrow k_p \geq l_0 - p + 1$$

2. for $i_t \in [l_0 + 1, l_1]$, $A_{W_\beta}(t) = p - 1$

$$\begin{cases} L_{Y_\alpha}(t) + p - 1 = p - 1 + k_p - i_t \neq 0 \\ i_t \in [1 + l_0, l_1] \end{cases} \Longrightarrow k_p \leq l_0 + 1 - p \text{ or } k_p \geq l_1 + 2 - p$$

so we have $k_p = l_0 + 1 - p$ or $k_p \geq l_1 + 2 - p$ By iterating this procedure we get: $k_p = l_0 + 1 - p$ or $k_p = l_1 + 2 - p, \ldots,$ or $k_p = l_{p-1}$ or $k_p \geq l_p + 1$. From the induction assumption we have $k_p \geq l_p + 1$ or $k_p = l_{p-1}$.

Let us now consider the contribution from $s \in Y_\alpha$.

1. for $i_s \in [1, k_1]$, $A_{Y_\alpha}(s) = p - 1$

$$\begin{cases} L_{W_\beta}(s) + 1 + p = l_p - i_s + 1 + p \neq 0 \\ i_s \in [1, k_1] \end{cases} \Longrightarrow l_p \geq k_1 - p$$

2. for $i_s \in [k_1 + 1, k_2]$, $A_{Y_\alpha}(s) = p - 2$

$$\begin{cases} L_{W_\beta}(s) + p = l_p - i_s + p \neq 0 \\ i_s \in [k_1 + 1, k_2] \end{cases} \Longrightarrow l_p \leq k_1 - p \text{ or} l_p \geq k_2 - p + 1$$

so we find $l_p = k_1 - p$ or $l_p \geq k_2 - p + 1$. By iterating the procedure we find $l_p = k_1 - p$ or $l_p = k_2 - p + 1, \ldots,$ or $l_p = k_{p-1} - 2$ or $l_p \geq k_p - 1$. From the induction assumption it follows that $l_p \geq k_p - 1$.

Finally, combining the results from W_β and Y_α, we have: $k_p = l_p + 1$ or $k_p = l_{p-1}$, which is what we wanted to prove.

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C Factorization formulae

When $Y_L^t = (l_1+1, l_2+1, \ldots, 1+l_L)$ $Y_{L+1}^t = (l_0, l_1, l_2, \ldots, l_L)$ $(l_i \leq l_{i+1})$

$$\begin{aligned}
(\epsilon_2)_{Y_L,Y_{L+1}} &= \prod_{(i,j)\in Y_L} \left(\epsilon_2 + \epsilon_2 \left(A_{Y_L}(i,j) + L_{Y_{L+1}}(i,j) + 1\right)\right) \\
&\quad \prod_{(a,b)\in Y_{L+1}} \left(\epsilon_2 - \epsilon_2 \left(A_{Y_{L+1}}(a,b) + L_{Y_L}(a,b) + 1\right)\right) \\
&= \prod_{(i,j)\in Y_L} \left(\epsilon_2 + \epsilon_2 \left(A_{Y_L}(i,j) + L_{Y_L}(i,j)\right)\right) \\
&\quad \prod_{(a,b)\in Y_{L+1}\cap Y_L} \left(\epsilon_2 - \epsilon_2 \left(A_{Y_{L+1}}(a,b) + L_{Y_{L+1}}(a,b) + 2\right)\right) \\
&\quad \prod_{(a,b)\in Y_{L+1}\setminus Y_L} \left(\epsilon_2 - \epsilon_2 \left(A_{Y_{L+1}}(a,b) + L_{Y_L}(a,b) + 1\right)\right) \\
&= H_{Y_L} \prod_{(a,b)\in Y_{L+1}\cap Y_L} -h_{Y_{L+1}}(a,b) \prod_{(a,b)\in Y_{L+1}\setminus Y_L} h_{Y_{L+1}}(a,b) \\
&= (-1)^{|Y_L|-L} H_{Y_L} H_{Y_{L+1}}
\end{aligned}$$

Similarly ,when $Y_L^t = (l_1, l_2, \ldots, l_L)$ and $Y_{L+1}^t = (l_1, l_2, \ldots, l_{L+1})$ $(l_i \leq l_{i+1})$

$$\begin{aligned}
(\epsilon_2)_{Y_L,Y_{L+1}} &= \prod_{(i,j)\in Y_L} \left(\epsilon_2 \left((A_{Y_L}(i,j) + 1) + L_{Y_{L+1}}(i,j) + 1\right)\right) \\
&\quad \prod_{(a,b)\in Y_{L+1}} \left(-\epsilon_2 \left(\left(A_{Y_{L+1}}(a,b) - 1\right) + L_{Y_L}(a,b) + 1\right)\right) \\
&= \prod_{(i,j)\in Y_L} h_{Y_{L+1}}(i,j) \prod_{(a,b)\in Y_L} (-1)^{|Y_L|} h_{Y_L}(a,b) \\
&\quad \prod_{(a,b)\in Y_{L+1}\setminus Y_L} \left(-\epsilon_2 \left(\left(A_{Y_{L+1}}(a,b) - 1\right) + L_{Y_L}(a,b) + 1\right)\right) \\
&= (-1)^{|Y_L|} H_{Y_L} H_{Y_{L+1}}
\end{aligned}$$

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