$\mathcal{N}$ = 1 superpotentials from multi-instanton calculus

$\mathcal{N} = 1$ superpotentials from multi-instanton calculus

Francesco Fucito

Dipartimento di Fisica, Universitá di Roma "Tor Vergata"
I.N.F.N. Sezione di Roma II,
Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail: `francesco.fucito@cern.ch`

José F. Morales

CERN, TH Division
1211 Geneva 23, Switzerland
E-mail: `morales@lnf.infn.it`

Rubik Poghossian

Yerevan Physics Institute
Alikhanian Br. st. 2, 375036 Yerevan, Armenia
E-mail: `poghos@yerphi.am`

Alessandro Tanzini

Scuola Internazionale Superiore di Studi Avanzati
I.N.F.N. Sez. di Trieste
Via Beirut 4, 34014 Trieste, Italy
E-mail: `tanzini@sissa.it`

ABSTRACT: In this paper we compute gaugino and scalar condensates in $\mathcal{N} = 1$ super-symmetric gauge theories with and without massive adjoint matter, using localization formulae over the multi-instanton moduli space. Furthermore we compute the chiral ring relations among the correlators of the $\mathcal{N} = 1^*$ theory and check this result against the multi-instanton computation finding agreement.

Contents

1. Introduction

The vacua of a non-abelian supersymmetric gauge theory (SYM) look very different according to the number of supersymmetries (SUSY) [1]. While in the $\mathcal{N} = 1$ case a theory with gauge group $SU(N)$ has N vacuum states, for extended supersymmetry it happens that the theory has flat directions. If SUSY is unbroken, the classical degeneracy is not lifted by quantum corrections. The vacua of the quantum theory then form a moduli space. Computations of non-perturbative effects must cope with these facts.

The strategy developed in [3] for the computation of correlators in $\mathcal{N} = 1$ SYM with massive matter, m, in the fundamental representation is simple: first perform the computation in the presence of a generic vacuum expectation value (v.e.v.), a, for the scalar fields, then determine the v.e.v. by minimizing the quantum potential. In the lectures in

ref. [2] a nice description of this procedure is given. Let us focus on the computation of the quantum potential. In this example since all the gauge degrees of freedom are massive due to the Higgs mechanism, the potential can only be a function of the matter fields. By using R-symmetry arguments, in the SU(2) case one finds a quantum F-term $F = ma - 2\frac{\Lambda^5}{a^3}$. Minimizing this potential one finds $a^2 = \pm\sqrt{2}\Lambda^{5/2}/\sqrt{m}$ in agreement with the previous statements concerning the number of vacua of a $\mathcal{N} = 1$ SUSY theory. To generalize this procedure to other types of massive matter is non-trivial: for matter in the adjoint there are massless gauge degrees of freedom and R-symmetry arguments alone do not determine the quantum potential. But the findings of ref. [4] can help in this respect: the structure of the quantum effective theory studied in this reference is coded in the geometry of the Seiberg-Witten curve. Even more interestingly a picture of confinement is obtained when the curve degenerates at some special points where the monopoles are massless. These degenerate points lie in the moduli space of SYM and a connection with the v.e.v.'s of the previous discussion is not straightforward.

In this paper we will show how the dramatic progress in the understanding of multi-instanton corrections to SYM with extended SUSY, recently achieved [5 – 13], can be used to provide the missing link. In turn, this will lead to the application of these techniques to $\mathcal{N} = 1$ SYM. In particular we will focus on $\mathcal{N} = 1$ SYM obtained by mass deformations of $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SYM theories. The deformations lift the moduli space degeneracy and the $\mathcal{N} = 1$ correlators can be read from the $\mathcal{N} = 2$ multi-instanton results once they are evaluated at the minima of the quantum potential.

This is the plan of the paper: in the first part of section 2 we discuss localization formulae in presence of the chiral observables associated to the superpotential. We show how to express the chiral condensates of the $\mathcal{N} = 1$ theory in terms of correlators of the parent $\mathcal{N} = 2$ theory. The next task is to find the value of the v.e.v. of the scalar fields in the $\mathcal{N} = 1$ vacuum.

In section 3 we address this problem for $\mathcal{N} = 1$ SYM with an adjoint chiral multiplet and a non-trivial superpotential or mass. We complement the discussion with the results of [4], where a vacuum showing the signatures of confinement is exhibited. To do so one must first break $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ SYM by adding a mass. In this vacuum the monopoles are massless and this condition takes a nice geometrical form: it corresponds to the degeneration of a Riemann surface in which the B-cycles are zero. The variable $u(a) = \langle \text{tr } \phi^2 \rangle$ takes a definite value at this point. Inverting the series $u(a)$ gives the value of the v.e.v.'s in the minimum of the quantum potential. The corresponding condensates that we find via multi-instanton calculus are then easily checked against chiral ring relations along the same lines of [10].

In section 4 we discuss the $\mathcal{N} = 1^*$ case. The situation here is more involved. The results of [14, 15] are not enough to carry our program out. So we supplement them by computing chiral ring type equations and using the connection between the $\mathcal{N} = 1^*$ SYM and integrable models [16, 17]. Also in this case we check the results of multi-instanton calculus against chiral ring type relations.

Finally in section 5 we summarize our results.

2. Localization formulae for $\mathcal{N} = 1$ supersymmetric theories

2.1 Equivariant forms

Let $\mathfrak{g} = \mathrm{U}(1)^r$ be a group action on a smooth manifold M of complex dimension ℓ specified by the vector field

$$\xi = \xi^i(x)\frac{\partial}{\partial x^i}$$

Let $\Omega(M)$ be the space of differential forms on M. The equivariant extension is given by the polynomial map $\alpha : \mathfrak{g} \to \Omega(M)$ from $\mathfrak{g}$ to the algebraof differential forms on M. α may be regarded as an element of $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ with $\mathbb{C}[\mathfrak{g}]$ the algebra of complex-valued polynomials on $\mathfrak{g}$. We define a grading in $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ by letting, for homogeneous $P \in \mathbb{C}[\mathfrak{g}]$ and $\beta \in \Omega(M)$,

$$\deg(P \otimes \beta) = 2\deg(P) + \deg(\beta). \tag{2.1}$$

Let us then introduce the equivariant derivative, d_ξ, and the Lie derivative, $\mathcal{L}_\xi$,

$$d_\xi \equiv d - i_\xi, \qquad d_\xi^2 = -di_\xi - i_\xi d = -\mathcal{L}_\xi$$

where d is the exterior derivative and $i_\xi dx^i \equiv \delta_\xi x^i$ the contraction with ξ. If α satisfies

$$d_\xi \alpha = 0 \tag{2.2}$$

it is said to be equivariantly closed. Let x_0^s be the isolated critical points of the group action ξ i.e. points where $\xi^i(x_0^s) = 0 \, , \forall i$. The integral of an equivariantly closed form is given by the localization formula

$$\int_M \alpha = (-2\pi)^\ell \sum_s \frac{\alpha_0(x_0^s)}{\det^{\frac{1}{2}} \mathcal{L}_\xi(x_0^s)} \tag{2.3}$$

with $\mathcal{L}_{\xi\,i}^{\,j} = \partial_i \xi^j : T_{x_0} M \to T_{x_0} M$ the tangent space map induced by the vector field ξ. α_0 is the zero form part with respect to the grading in $\Omega(M)$.

The localization formula (2.3) is an extremely powerful tool to compute integrals as for example volumes and Chern classes of complicate riemannian manifolds (in our case the ADHM moduli space). As an example let us compute the integral of the form $e^{-a(x^2+y^2)}\, dx \wedge dy$ via $\mathrm{U}(1)$ localization on $\mathbb{R}^2$. We choose ξ to be the generator of $\mathrm{SO}(2)$ rotations in the plane: $\xi = \hbar\, (y\, \partial/\partial x - x\, \partial/\partial y)$ with critical point $x_0 = y_0 = 0$. The equivariant extension of the form given above is

$$\alpha = e^{-a(x^2+y^2)}\, dx \wedge dy - \frac{\hbar}{2a} e^{-a(x^2+y^2)} = \alpha_2 + \alpha_0 \tag{2.4}$$

With respect to the grading (2.1), α is a two-form which is made of two parts having respectively degree two and zero in $\Omega(M)$. Sending $\hbar \to 0$ we recover the starting form. Using the localization formula we get

$$\int_{\mathbb{R}^2} e^{-a(x^2+y^2)}\, dx \wedge dy = \int_{\mathbb{R}^2} \alpha = 2\pi\, \frac{\hbar\, e^{-a(x_0^2+y_0^2)}}{2a\hbar} = \frac{\pi}{a} \tag{2.5}$$

which is the result we would have obtained by performing the integral on the l.h.s. Notice that the right hand side does not depend on $\hbar$. After this quick exercise we are ready to write down the equivariant forms in $\mathbb{C}^2 = \mathbb{R}^4$ under a U(1) subgroup of SO(4) generated by $\xi = i\,\hbar\,(z^1 dz^1 - z^2 dz^2 - \text{h.c.})$

$$
\begin{aligned}
\alpha_0 &= 1 \\
\alpha_{(2,0)} &= dz^1 \wedge dz^2 + i\,\hbar\,z^1 z^2 \\
\bar{\alpha}_{(0,2)} &= d\bar{z}^1 \wedge d\bar{z}^2 - i\,\hbar\,\bar{z}^1 \bar{z}^2 \\
\alpha_{(2,2)} &= dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2 + i\,\hbar\,(z^1 z^2 d\bar{z}^1 \wedge d\bar{z}^2 - \text{h.c.}) - \hbar^2\,z^1 z^2 \bar{z}^1 \bar{z}^2
\end{aligned} \tag{2.6}
$$

The subscripts are the gradings with respect to the complex structure. Notice that all the forms in (2.6) are invariant under the U(1) action

$$
z^1 \to e^{i\hbar}\,z^1, \quad z^2 \to e^{-i\hbar}\,z^2 \tag{2.7}
$$

generated by ξ.

2.2 Equivariant forms for supersymmetric theories

Here we consider $\mathcal{N} = 1$ SYM with gauge group U(N) coupled to an adjoint chiral superfield Φ with superpotential $W(\Phi)$

$$
S_{\mathcal{N}=1} = S_{\mathcal{N}=2} + \int d^4 x d^2\theta\, W(\Phi) \tag{2.8}
$$

In the multi-instanton calculus, we treat the chiral tree-level superpotential $W(\Phi)$ as the insertion of a set of non-trivial observables. In this way, the correlators of the chiral ring of $\mathcal{N} = 1$ will be related to the correlators in the underlying $\mathcal{N} = 2$ SYM. As it is well known, in a saddle point evaluation of the functional integral with action $S_{\mathcal{N}=1}$ the quantum fluctuation of the fields cancel among each other and one is left with the zero modes only. Localization will then be performed in the $\mathcal{N} = 2$ universal moduli space $\mathcal{M} \times \mathbb{C}^2$ where $\mathcal{M}$ is the moduli space of gauge connections. Our first task is to introduce equivariantly closed forms suitable for being integrated with the localization formula. These closed forms will generate the set of non-trivial observables. The equivariantly closed forms are given by a combination of the zero modes of the theory suitably deformed by the presence of the torus action generated by ξ. Let us introduce the equivariant derivative

$$
d_\xi = d_{\mathbb{C}^2} + d_{\mathcal{M}} - i_\xi \tag{2.9}
$$

$d_{\mathbb{C}^2}$ is the usual total differential in $\mathbb{C}^2$, $d_{\mathcal{M}}$ is the total differential on the moduli space of gauge connections[1] with winding number k and ξ is the fundamental vector field generating the action of the group $\text{U}(1)_\hbar \times \text{U}(1)_{a_\alpha}^N \times \text{U}(1)_\phi^k$. This group parameterizes the Cartan of the group of symmetries of the ADHM manifold: the SO(4) Lorentz group, the U(k)$_\phi$

---

[1]It is well known that this space is obtained by starting from a flat space of real dimension $4k^2 + 4kN$, subtracting $3k^2$ ADHM constraints and finally modding by the symmetry U(k). For our purposes it is enough to take $d_{\mathcal{M}}$ as the total differential of the space of real dimension $4k^2 + 4kN$.

symmetry of the ADHM constraints and the $U(N)_{a_\alpha}$ gauge symmetry. In an $\mathcal{N} = 2$ theory the a_α's are free parameters describing the expectation values of the scalar fields while in the $\mathcal{N} = 1$ gauge theory under study here they will describe the couplings in the superpotential and the $\mathcal{N} = 1$ vacuum.

Following [18], let us introduce the universal equivariant connection $\mathcal{A}$ and the universal equivariant field strength $\mathcal{F}$ (see section 7 of [19] for a detailed description of the geometrical framework). Given the kernel U of the ADHM matrix,[2] they can be written as

$$\mathcal{A} = \bar{U} \, d_\xi U = A + C$$
$$\mathcal{F} = d_\xi \left(\bar{U} \, d_\xi U \right) = F + \Psi + \varphi \tag{2.10}$$

in terms of the zero modes

$$A = \bar{U} d_{\mathbb{C}^2} U \qquad C = \bar{U} d_{\mathcal{M}} U$$
$$F = F_{\mu\nu} dx^\mu dx^\nu = d_{\mathbb{C}^2} \left(\bar{U} \, d_{\mathbb{C}^2} U \right)$$
$$\Psi = \lambda_m dz^m + \psi_{\bar{m}} d\bar{z}^{\bar{m}} = d_{\mathbb{C}^2} \left(\bar{U} \, d_{\mathcal{M}} U \right) + d_{\mathcal{M}} (\bar{U} \, d_{\mathbb{C}^2} U)$$
$$\varphi = (d_{\mathcal{M}} \bar{U})(d_{\mathcal{M}} U) - \bar{U} \, \mathcal{L}_\xi U \tag{2.11}$$

With respect to the grading of forms in the space $\mathbb{C}^2 \times \mathcal{M}$, A is a $(1,0)$ form, C a $(0,1)$ form, F is a $(2,0)$ form and Ψ a $(1,1)$ form. U is a $(0,0)$ form and therefore $i_\xi U = 0$. Notice that only φ gets deformed by ξ as expected [10]. In the limit $\hbar \to 0$ we recover the gauge connection, field strength, gauginos and scalar field solutions in the instanton background.

Notice that in (2.11) we decomposed the real one-form Ψ with respect to the complex structure of $\mathbb{C}^2$ as a complex $(1,0)$-form $\lambda_m dz^m$ plus a complex $(0,1)$-form $\psi_{\bar{m}} d\bar{z}^{\bar{m}}$. These two components can be identified respectively with the gaugino and the chiral matter field of the $\mathcal{N} = 1$ theory in the topologically twisted formulation [20, 21]. This identification allows us to write the basic operators of the $\mathcal{N} = 1$ theory in an equivariant form in terms of $\mathcal{F}$ and the "volume" forms (2.6) described in the last section[3]

$$\int_{\mathbb{R}^4} d^4x \, \mathrm{tr} \, \varphi^k = \int_{\mathbb{C}^2} \alpha_{(2,2)} \wedge \mathrm{tr} \, \mathcal{F}^k$$
$$\int_{\mathbb{R}^4} d^4x \, \mathrm{tr} \, \lambda^\alpha \lambda_\alpha \varphi^{k-2} = -\frac{1}{k(k-1)} \int_{\mathbb{C}^2} \alpha_{(0,2)} \wedge \mathrm{tr} \, \mathcal{F}^k$$
$$\int_{\mathbb{R}^4} d^4x \, d^2\theta \, W(\Phi) = \int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge \mathrm{tr} \, W(\mathcal{F}) \tag{2.12}$$

---

[2]The idea is that a non-trivial bundle of a certain dimension can always be mapped in a trivial one of greater dimension. U is the map between the two bundles and $P = UU^\dagger$ the projection operator. It is then easier to find the covariant derivative on the non-trivial bundle as the projection of the flat derivative of the trivial bundle i.e. $\nabla = P\partial$.

[3]The authors of [10] have given arguments for the closure of the form φ. Their reasonings should be replaced by the present analysis.

2.3 Correlation functions via localization

Here we apply localization techniques to the computation of condensates in $\mathcal{N}=1$ SYM with various matter content. Correlators are defined as

$$\langle\, \mathcal{O}\,\rangle_{\mathcal{N}=1} = \frac{1}{V\mathcal{Z}} \int_{\mathcal{M}\times\mathbb{C}^2} \mathcal{O} \exp\left[-S_{\mathcal{N}=2} - \int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge \operatorname{tr} W(\mathcal{F})\right] \tag{2.13}$$

where $\mathcal{O}$ is any equivariant operator, $\mathcal{Z} = \langle\, \mathbb{1}\,\rangle$ the partition function and V the volume. As we said earlier, to apply the localization theorem $\mathcal{M}$ must be smooth and, as a consequence, $\mathbb{C}^2$ must be taken non-commutative. Let V denotes the volume of the non-commutative space-time

$$V = \int_{\mathbb{C}^2} \alpha_{(2,2)} = \bar{z}_0^1 \bar{z}_0^2 z_0^1 z_0^2 = \zeta^2 \tag{2.14}$$

This integral is performed using the localization formula which instructs us to compute at the critical points the zero form part of the equivariant form to be integrated. ζ is the non-commutative parameter[4]

$$[\bar{z}^j, z^i] = \zeta\, \delta^{ij} \tag{2.15}$$

The integrals over $\mathcal{M}$ can be written as a sum over N Young tableaux Y labelling the fixed points of ξ on the instanton moduli space [8 – 10]

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}} \sum_Y \mathcal{O}_Y \frac{e^{S_Y}}{\det^{\frac{1}{2}}\mathcal{L}_{\xi_Y}} q^{|Y|}$$
$$\mathcal{Z} = \sum_Y \frac{e^{S_Y}}{\det^{\frac{1}{2}}\mathcal{L}_{\xi_Y}} q^{|Y|} \tag{2.16}$$

where S_Y and $\mathcal{O}_Y$ are the zero form parts of the action S and of the operator $\mathcal{O}$.

If $\mathcal{O}$ is taken to be $\alpha_{(2,2)} \wedge \operatorname{tr}\mathcal{F}^J$, since $S_Y = -\frac{i}{\hbar} z_0^1 z_0^2 W(\varphi) \to 0$ at the fixed points, we see that the superpotential term can be always discarded and therefore the $\mathcal{N}=1$ amplitude reduces to a scalar correlator in $\mathcal{N}=2$

$$\langle \operatorname{tr}\varphi^J\,\rangle_{\mathcal{N}=1} = \langle \operatorname{tr}\varphi^J e^{-\int_{\mathbb{C}^2}\alpha_{(2,0)}\wedge\operatorname{tr}W(\mathcal{F})}\,\rangle_{\mathcal{N}=2} = \langle\operatorname{tr}\varphi^J\,\rangle_{\mathcal{N}=2} \tag{2.17}$$

To obtain a *bona fide* $\mathcal{N}=1$ correlator we then have to fix the value of the v.e.v.'s by minimizing the quantum potential as we will discuss in the next section.

If $\mathcal{O}$ is taken to be $\alpha_{(0,2)} \wedge \operatorname{tr}\mathcal{F}^{J+2}$, a power of S_Y is needed in order to compensate for the unbalanced $\alpha_{(2,0)}$ form in $\mathcal{O}$. After this power is extracted the superpotential term can be discarded leaving again an $\mathcal{N}=2$ amplitude to compute

$$\langle \operatorname{tr}\lambda^\alpha\lambda_\alpha\,\varphi^J\,\rangle_{\mathcal{N}=1} = -\frac{1}{c_J V}\langle \int \alpha_{(0,2)}\wedge\operatorname{tr}\mathcal{F}^{J+2}\,\rangle_{\mathcal{N}=1} = \frac{1}{c_J V}\frac{\partial^2}{\partial\tau_w\partial\tau_J}\ln\mathcal{Z}(\tau_w,\tau_J)|_{\tau_w=\tau_J=0} \tag{2.18}$$

---

[4]Given our choice below we have $\bar{z}_{1,2} = \zeta\, \partial/\partial z_{1,2}$. While the notion of critical point for a commutative space is clearly well defined, when a coordinate becomes an infinite dimensional operator this notion becomes slippery. In all of this work we never need to deal with these type of problems though, and (2.14) is just taken as a suitable regularization.

with $c_J = (J + 2)(J + 1)$ and

$$\mathcal{Z}(\tau_w, \tau_J) \equiv \int_\mathcal{M} \exp\left[-S_{\mathcal{N}=2} - \tau_w \int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge W(\mathcal{F}) - \tau_J \int_{\mathbb{C}^2} \alpha_{(0,2)} \wedge \operatorname{tr} \mathcal{F}^{J+2}\right] \quad (2.19)$$

As for (2.17), also this amplitude has to be evaluated at a minimum of the quantum superpotential. In writing the gaugino correlator in the form (2.18) we use the fact that in the limit $\hbar \to 0$ only the term proportional to $\alpha_{(2,0)} \wedge \alpha_{(0,2)}$ contributes since for unbalanced powers of $\alpha_{(2,0)}$ there is no way to properly contract the fermionic zero modes. The equations (2.17), (2.18) are the generalisation of the result of [21], where the mass deformation of $\mathcal{N} = 2$ theory has been studied.

Let us now compute the scalar correlators

$$\langle \operatorname{tr} \varphi^J \rangle = \frac{1}{V\, \mathcal{Z}} \int_{\mathcal{M} \times \mathbb{C}^2} \alpha_{(2,2)} \wedge \operatorname{tr} \mathcal{F}^J = \frac{\hbar^2}{\mathcal{Z}} \sum_Y \operatorname{tr} \varphi_Y^J \frac{e^{S_Y}}{\det^{\frac{1}{2}} \mathcal{L}_{\xi_Y}} q^{|Y|}$$

$$= \frac{1}{\mathcal{Z}} \sum_Y Z_Y \operatorname{tr} \varphi_Y^J q^{|Y|} \quad (2.20)$$

with $q^{|Y|} = e^{-\frac{8\pi k}{g^2}}$ and

$$\mathcal{Z}_Y = \langle\, \mathbb{1}\, \rangle = \frac{\hbar^2}{\det^{\frac{1}{2}} \mathcal{L}_{\xi_Y}}$$

$$\operatorname{tr} \varphi_Y^J = \operatorname{tr} \varphi_{\mathrm{cl}}^J + \sum_{\alpha=1}^N \sum_{j_\alpha=1}^{k_{1\alpha}} \left[(a_\alpha + \hbar(k'_{j_\alpha} - j_\alpha + 1))^J - (a_\alpha - \hbar(j_\alpha - 1))^J\right.$$
$$\left. - (a_\alpha + \hbar(k'_{j_\alpha} - j_\alpha))^J + (a_\alpha - \hbar j_\alpha)^J\right] \quad (2.21)$$

Here a_α are the v.e.v's, j_α runs over the rows of the Y_α tableau, k'_{j_α} is the number of boxes in the column j_α and $k_{1\alpha}$ are the number of rows in Y_α.

The case $J = 2$ is particularly simple: (2.21) depends only on the number of boxes, k, (which is the instanton winding number) in the Young tableaux

$$\operatorname{tr} \varphi_Y^2 = \operatorname{tr} \varphi_{\mathrm{cl}}^2 + 2\, k\, \hbar^2 \quad (2.22)$$

Plugging into (2.20) one finds the Matone's relation [10]

$$\left\langle \frac{1}{2} \operatorname{tr} \varphi^2 \right\rangle = \frac{1}{\mathcal{Z}} \sum_k Z_k\, q^k \left[\frac{1}{2} \operatorname{tr} \varphi_{\mathrm{cl}}^2 + k\, \hbar^2\right] = \frac{1}{2} \operatorname{tr} \varphi_{\mathrm{cl}}^2 + \frac{\hbar^2}{\mathcal{Z}} \sum_k k\, Z_k\, q^k$$

$$= \frac{1}{2} \operatorname{tr} \varphi_{\mathrm{cl}}^2 - q\, \partial_q \mathcal{F}(q) \quad (2.23)$$

with

$$\mathcal{Z}(q) \equiv \sum_k Z_k\, q^k; \qquad \mathcal{F}(q) \equiv -\hbar^2 \ln \mathcal{Z}(q) \quad (2.24)$$

In a similar way one can compute higher point functions. Again correlators involving $\operatorname{tr}\varphi^2$ are particularly simple. For example

$$
\begin{aligned}
\langle \operatorname{tr}\varphi^2 \operatorname{tr}\varphi^J \rangle &= \frac{1}{\mathcal{Z}} \sum_Y (\operatorname{tr}\varphi_{\rm cl}^2 + 2\,k\,\hbar^2)\,\varphi_Y^J\,Z_Y\,q^k = \operatorname{tr}\varphi_{\rm cl}^2\,\langle \operatorname{tr}\varphi^J \rangle + \frac{2\,\hbar^2}{\mathcal{Z}}\,q\,\partial_q \sum_Y \varphi_Y^J\,Z_Y\,q^k \\
&= \operatorname{tr}\varphi_{\rm cl}^2\,\langle \operatorname{tr}\varphi^J \rangle + 2\,\hbar^2\,q\,\partial_q\,\langle \operatorname{tr}\varphi^J \rangle + 2\,\hbar^2\,\frac{\mathcal{Z}'(q)}{\mathcal{Z}}\langle \operatorname{tr}\varphi^J \rangle \\
&= \langle \operatorname{tr}\varphi^2 \rangle\,\langle \operatorname{tr}\varphi^J \rangle + 2\,\hbar^2\,q\,\partial_q\,\langle \operatorname{tr}\varphi^J \rangle
\end{aligned}
\tag{2.25}
$$

Notice that in the limit $\hbar \to 0$ the correlator factorizes as expected. Curiously it is the second term which is relevant to the computations of the gaugino condensate below.

Now let us consider correlators involving the gaugino condensate. For concreteness we choose the superpotential $W(\Phi) = \frac{1}{2}\,m\,\Phi^2$. Applying the localization formula to (2.19) one finds

$$
\begin{aligned}
\frac{1}{V}\frac{\partial^2}{\partial\tau_w \partial\tau_J}\ln\mathcal{Z}(\tau_w,\tau_J)|_{\tau_w=\tau_J=0} &= \frac{m}{\hbar^2}\left[\langle \operatorname{tr}\varphi^2 \operatorname{tr}\varphi^{J+2} \rangle - \langle \operatorname{tr}\varphi^2 \rangle\,\langle \operatorname{tr}\varphi^{J+2} \rangle\right] \\
&= 2m\,q\,\partial_q\,\langle \operatorname{tr}\varphi^{J+2} \rangle
\end{aligned}
\tag{2.26}
$$

Now comparing (2.18) with (2.26) we get

$$
\langle \operatorname{tr}\lambda\lambda\varphi^J \rangle_{\mathcal{N}=1} = -\frac{2m}{(J+2)(J+1)}\,q\,\partial_q\,\langle \operatorname{tr}\varphi^{J+2} \rangle
\tag{2.27}
$$

For $J=0$ one finds

$$
\langle\,\operatorname{tr}\lambda\lambda\,\rangle = -m\,q\,\partial_q\,\langle \operatorname{tr}\varphi^2 \rangle
\tag{2.28}
$$

3. Pure $\mathcal{N}=2$ SYM down to $\mathcal{N}=1$

Very recently the effective superpotential of $\mathcal{N}=1$ gauge theories obtained via deformations of theories with extended supersymmetry has been computed with matrix model type arguments [22]. Following this work a general strategy, based on the Konishi anomaly, has been devised to carry out these computations in various phases of the $\mathcal{N}=1$ theory [23]. See ref. [24] for a nice review. The results are summarize by the so called "chiral ring relations". The elements of the chiral ring are those gauge invariant operators that are annihilated by the $\bar{\mathcal{Q}}_{\dot\alpha}$ supersymmetric charges. The notable properties of this set of operators are at the heart of supersymmetric instanton calculus and are responsible for this class of operators to be independent from space-time variables [25]. In [10], it was already shown by a direct multi-instanton computation that $\mathcal{N}=2$ correlators of the type $\langle \operatorname{tr}\varphi^J \rangle$ satisfy the chiral ring relations predicted by Konishi anomaly arguments. In this section we extend this result to scalar and gaugino condensate correlators in the confining phase of $\mathcal{N}=1$ SYM. In these theories gaugino condensates are evaluated via equations of the type of (2.27). Following [23] we then introduce generating functions (resolvent) $\mathcal{T}(z)$, $\mathcal{R}(z)$ for the correlators $\langle \operatorname{tr}\varphi^J \rangle$ and $\langle \operatorname{tr}\lambda\lambda\varphi^J \rangle$ respectively

$$
\mathcal{T}(z) = \int d^4x\,\operatorname{tr}\frac{1}{z-\varphi} = \frac{1}{z}\sum_k \int \alpha_{(2,2)} \wedge \operatorname{tr}\left(\frac{\mathcal{F}}{z}\right)^k
\tag{3.1}
$$

$$\mathcal{R}(z) = \int d^4x \, \mathrm{tr} \, \lambda\lambda \frac{1}{z-\varphi} = -\frac{1}{z} \sum_k \frac{1}{(k+2)(k+1)} \int \alpha_{(0,2)} \wedge \left(\frac{\mathcal{F}}{z}\right)^{k+2} \tag{3.2}$$

The form of $\langle \mathcal{R}(z) \rangle$, $\langle \mathcal{F}(z) \rangle$ is highly constrained by the cancellation of Konishi anomaly. Indeed they are determined up to two arbitrary polynomials of order $n-1$, $n+1$, via

$$\langle \mathcal{R}(z) \rangle = W'(z) - \sqrt{W'(z)^2 + f(z)}$$
$$\langle \mathcal{T}(z) \rangle = -\frac{1}{4} \frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} \tag{3.3}$$

The relation (3.3) is understood in the $\hbar \to 0$ limit. $W(z)$ is the superpotential and $f(z), c(z)$ are arbitrary polynomials of order $n-1$, $n+1$ given by

$$W(z) = \sum_{k=1}^{n+1} \frac{1}{k} g_k z^k \quad f(z) = \sum_{k=0}^{n-1} f_k z^k \quad c(z) = \sum_{k=0}^{n-1} c_k z^k \tag{3.4}$$

Different phases of the theory correspond to different choices of the constants c_k, f_k and account for partial or total breaking of the gauge group. More precisely for a generic choice of the $2n$ parameters c_k, f_k, the gauge group is broken down to $\prod_{i=1}^{n} U(N_i)$ with c_k, f_k parameterizing the n ranks N_i and the n gaugino condensates S_i. Here we consider the two extreme cases where the gauge group is completely broken to $U(1)^N$ or completely unbroken.

3.1 $U(1)^N$ completely broken phase

To break completely the gauge group the superpotential should have N different minima and therefore $W(z)$ should be a polynomial of order $N+1$. We write

$$W'(z) = P_N(z) = \prod_{\alpha=1}^{N} (z - a_\alpha) \tag{3.5}$$

with a_α parameterizing the couplings in $W(z)$. The functions $f(z), c(z)$ appearing in (3.10) are given in terms of an associated Seiberg-Witten like curve

$$y^2 = W'(z)^2 + f(z) = P_N(z)^2 - 4\Lambda^{2N}$$
$$c(z) = P'_N(z) \tag{3.6}$$

i.e. $f(z) = -4\Lambda^{2N}$. Plugging in (3.6) into the chiral ring relations one finds

$$\langle \mathcal{T}(z) \rangle = \frac{P'(z)}{\sqrt{P(z)^2 - 4\Lambda^{2N}}} \tag{3.7}$$

This is precisely the form of the chiral ring relations for a pure $\mathcal{N} = 2$ theory with gauge group $U(N)$ and vev's a_α. In [10] these relations have been shown to hold for $\mathcal{N} = 2$ SYM with gauge group $SU(2)$ by explicit evaluation of the multi-instanton determinants. According to (2.17) these results imply that the same chiral ring relations hold for the $\mathcal{N} = 1$ as well.

3.2 U(N) unbroken phase

Now we consider the case where the gauge group is completely unbroken. For concreteness we choose $W(z) = \frac{1}{2} m z^2$, i.e. a mass deformation of pure $\mathcal{N} = 2$ SYM

$$W(z) = \frac{1}{2}mz^2 \qquad f(z) = f_0 \qquad c(z) = c_0 \tag{3.8}$$

At the leading order $\frac{1}{z}$ one finds

$$S = \frac{f_0}{2\,m}$$
$$N = -\frac{c_0}{4m} \tag{3.9}$$

i.e. the coefficients $f_0 = 2mS$ and $c_0 = -4mN$ parameterize the gaugino condensate and the rank N of the gauge group respectively (N should not to be confused with n, the order of $W(z)$). Plugging into (3.3) one finds

$$\langle \mathcal{R}(z) \rangle = mz \left(1 - \sqrt{1 + \frac{2S}{mz^2}} \right)$$
$$\langle \mathcal{T}(z) \rangle = \frac{N}{z \sqrt{1 + \frac{2S}{mz^2}}} \tag{3.10}$$

Expanding in $\frac{1}{z}$, equations (3.10) determine all the correlators in the chiral ring in terms of the gaugino condensate S and the rank N of the gauge group.

Notice that for $N = 1$, (3.10) is of the form (3.7) with $P(z) = z$, $\Lambda^2 = -\frac{S}{2m}$. In general, (3.10) can be always put in the "$\mathcal{N} = 2$ form" (3.7) by choosing

$$P(z) = \prod_{\alpha=1}^{N} (z - a_\alpha); \qquad a_\alpha = 2\,\Lambda\,\cos \pi \frac{(j - \frac{1}{2})}{N} \quad j = 1, ..N \tag{3.11}$$

This follows from the identity

$$\frac{P'(z)}{\sqrt{P(z)^2 - 4\,\Lambda^{2N}}} = \frac{N}{z\sqrt{1 - \frac{4\Lambda^2}{z^2}}} \tag{3.12}$$

satisfied by (3.11). Therefore the chiral ring relations for $\mathcal{N} = 1$ SYM theory in the unbroken phase coincide with those of a pure $\mathcal{N} = 2$ gauge theory with vev's a_α given by (3.11). The values a_α are at the minima of the quantum potential [26]. They represent those points of the moduli space of vacua of the mass perturbed $\mathcal{N} = 2$ theory at which the monopoles are massless. From the geometrical point of view they are those degenerate points at which the elliptical Seiberg-Witten curve has two zeroes of the first order and all the other ones are of order two.

Actually there are N different possible choices for the v.e.v.'s in (3.11), which are obtained sending

$$\Lambda \to \Lambda e^{i\pi(\tau + n)/N} \quad n = 0, \dots, N - 1 \tag{3.13}$$

each choice corresponding to a different vacuum of the $\mathcal{N} = 1$ theory. All of this is very much in line with the computational strategy in [2].

Finally it is interesting to notice that the chiral ring observables $\langle \operatorname{tr} \varphi^J \rangle$ for the mass deformed $\mathcal{N} = 2$ SYM with gauge group U(N) are N times the result for the U(1) case.

3.3 Multi-instanton tests

U(1) **case.** We start by considering $\mathcal{N} = 1$ SYM with gauge group U(1) and an adjoint chiral multiplet with mass m. By U(1) instantons we refer to instanton in the non-commutative theory, but the non-commutative parameter will never enter in our formulae. This non-commutative extension is also needed in order to compactify the moduli space of U(N) instantons and will be always understood. The result for the commutative theory follow from the limit where the commutative parameter is formally turned off.

According to (2.17) and (2.18) scalar correlators and gaugino condensates in this theory are related to those in pure $\mathcal{N} = 2$ SYM. Scalar correlators in pure $\mathcal{N} = 2$ SYM are given by (2.20) and (2.21). The instanton partition function $\mathcal{Z}_Y$ for pure $\mathcal{N} = 2$ is given by

$$Z_Y(\hbar) = \prod_{s \in Y_k} \frac{1}{\hbar^2 L(s)^2} \tag{3.14}$$

Evaluating (3.14) one finds

$$\langle \operatorname{tr} \varphi^2 \rangle = 2\Lambda^2$$
$$\langle \operatorname{tr} \varphi^4 \rangle = 6\Lambda^4$$
$$\langle \operatorname{tr} \varphi^6 \rangle = 20\Lambda^6$$
$$\langle \operatorname{tr} \varphi^8 \rangle = 70\Lambda^8 \tag{3.15}$$

in agreement with the chiral ring predictions:

$$\left\langle \operatorname{tr} \left(\frac{1}{z - \varphi} \right) \right\rangle = \frac{1}{z} \frac{1}{\sqrt{1 - \frac{4\Lambda^2}{z^2}}} = \sum_{k=0}^{\infty} \frac{(2k)!}{k!^2} \frac{\Lambda^{2k}}{z^{2k+1}} \tag{3.16}$$

i.e.

$$\langle \operatorname{tr} \varphi^{2k} \rangle = \Lambda^{2k} \frac{(2k)!}{k!^2} \tag{3.17}$$

SU(2) **case.** The chiral ring relation for scalar correlators in pure $\mathcal{N} = 2$ SYM with gauge group SU(2) were tested in [6] against multi-instanton calculus. Here we recall these results and extract from them the scalar correlators for the mass deformed $\mathcal{N} = 1$ theory. The Seiberg-Witten SU(2) curve is given by the equation

$$y^2 = P(z)^2 - 4\Lambda^4 = \left(z^2 - \frac{u}{2} \right)^2 - 4\Lambda^4 \tag{3.18}$$

so that expanding the resolvent

$$\left\langle \operatorname{tr} \frac{1}{z - \varphi} \right\rangle = \frac{P'(z)}{\sqrt{P(z)^2 - 4\Lambda^4}} \tag{3.19}$$

we find $u = < \mathrm{tr}\, \varphi^2 >$. The minima of the quantum potential associated to the SU(2) unbroken phases, i.e. the points where two zeros of the Seiberg-Witten curve are colliding, are given by $u_{1,2} = \pm 4\Lambda^2$ that correspond to $u = \pm(a_1^2 + a_2^2)$ with a_α given by (3.11) for $N = 2$.

By a direct computation in the $\mathcal{N} = 2$ SYM one finds [6]

$$u = \langle \mathrm{tr}\, \varphi^2 \rangle = 2\, a^2 + \frac{q}{a^2} + \frac{5}{16}\frac{q^2}{a^6} + \frac{9}{32}\frac{q^3}{a^{10}} + \dots$$

$$\langle \mathrm{tr}\, \varphi^4 \rangle = 2\, a^4 + 6\, q + \frac{9}{8}\frac{q^2}{a^4} + \frac{7}{8}\frac{q^3}{a^8} + \dots$$

$$\langle \mathrm{tr}\, \varphi^6 \rangle = 2\, a^6 + 15\, q\, a^2 + \frac{135}{16}\frac{q^2}{a^2} + \frac{125}{32}\frac{q^3}{a^6} + \dots \tag{3.20}$$

The a here should not be confused with its quantum analog a_α entering in the Seiberg-Witten curve and given by (3.11). The right variable for a comparison is not the classical v.e.v. a but the "quantum" coordinate u. Therefore, inverting the first of these equations, we get

$$a = \sqrt{\frac{u}{2}} - \frac{1}{\sqrt{2}\, u^{\frac{3}{2}}} q - \frac{15}{4\sqrt{2}\, u^{\frac{7}{2}}} q^2 - \frac{105}{4\sqrt{2}\, u^{\frac{11}{2}}} q^3 + \dots \tag{3.21}$$

Substituting back in (3.20) we get[5]

$$\langle \mathrm{tr}\, \varphi^4 \rangle = 4q + \frac{u^2}{2}\,,$$

$$\langle \mathrm{tr}\, \varphi^6 \rangle = 6qu + \frac{u^3}{4}\,,$$

$$\langle \mathrm{tr}\, \varphi^8 \rangle = 12q^2 + 6qu^2 + \frac{u^4}{8} \tag{3.22}$$

which are exactly the chiral ring relations. We draw the reader's attention on the fact that even if the expansion parameter in (3.20) is close to 1, since we are in the strong coupling region, still all our manipulations are safe being the relations (3.22) exact. Plugging $u_{1,2} = \pm 4\Lambda^2$ in (3.22) and comparing with (3.17) one easily see that the SU(2) scalar condensates are twice the U(1) ones, in agreement with the chiral prediction (3.10). We stress that this procedure is just the extension of the strategy of ref. [2].

U(N) case. According to (3.10) U(N) scalar correlators are given by N times the result for U(1) i.e.

$$\langle \mathrm{tr}\, \varphi^{2k} \rangle = N\Lambda^{2k}\frac{(2k)!}{k!^2} \tag{3.23}$$

These results follows from those of $\mathcal{N} = 2$ choosing vev given by (3.11). For this choice of vev, the chiral ring relation (3.3) for $\mathcal{R}(z)$ becomes

$$\mathcal{R}(z) = \left\langle \mathrm{tr}\left(\frac{\lambda\lambda}{z - \varphi}\right) \right\rangle = mz\left(1 - \sqrt{1 - \frac{4\Lambda^2}{z^2}}\right) \tag{3.24}$$

---

[5]For the sake of conciseness in (3.20) we did not write the gravitational corrections to the results. They can be found in [6] together with the corrections to the formula below.

By comparing the expansion of the above formula with (3.23) we get the relation

$$\langle \mathrm{tr}\lambda\lambda\varphi^{2k}\rangle = -\frac{m}{(2k+1)N}\langle \mathrm{tr}\varphi^{2k+2}\rangle = -\frac{m}{(2k+1)(2k+2)}\Lambda_N\partial_{\Lambda_N}\langle \mathrm{tr}\varphi^{2k+2}\rangle \qquad (3.25)$$

with $\Lambda_N \equiv \Lambda^{1/N}$. This result matches with the computation via localization presented in (2.27) with $J = 2k$ and $q = \Lambda_N^2$.

In the limit $m \to \infty$, in which the massive particles decouple, from (3.25) we can recover the results for pure $\mathcal{N} = 1$ SYM.

4. Chiral ring relations for $\mathcal{N} = 1^*$ SYM

In this section we consider the $\mathcal{N} = 1^*$ gauge theory, defined as a mass deformation of $\mathcal{N} = 4$ where all three chiral multiplets have mass m. Alternatively the $\mathcal{N} = 1^*$ theory can be defined as a mass deformation of $\mathcal{N} = 2^*$. According to the discussion in section 2 scalar correlators and gaugino condensates in $\mathcal{N} = 1^*$ are related to scalar correlators in $\mathcal{N} = 2^*$ via (2.17), (2.18). As before the $\mathcal{N} = 2^*$ amplitudes should be evaluated at the v.e.v. specifying the $\mathcal{N} = 1$ vacuum, i.e. the points in the moduli space where the Seiberg-Witten curve degenerates and the $\mathrm{SU}(N)$ symmetry is unbroken.

4.1 U(1) group

Here the situation is particularly simple since the v.e.v. of the scalar field can be shifted to zero by a redefinition since the theory has only one vacuum. Using the results of [7] (and setting for simplicity $\epsilon_1 = -\epsilon_2 = \hbar$) one can write

$$\langle \mathrm{tr}\,(\varphi^2)\rangle_{\mathcal{N}=2^*} = -\frac{\hbar^2}{Z(q)}\sum_k kZ_k q^k \qquad (4.1)$$

where Z_k is the k-instanton partition function given by

$$Z_k(\hbar) = \sum_{\{Y_k\}}\prod_{s \in Y_k}\left[1 - \frac{m^2}{E(s)^2}\right] \qquad (4.2)$$

Here $E(s) = -\hbar L(s)$, with $L(s)$ the length of the hook centered in the s-box of the Young tableau and stretched between the top and left end of the tableau. For the first few instanton numbers one finds:

$$\begin{aligned}
Z_{\square} &= 1 - \frac{m^2}{\hbar^2}\\
2\,Z_{\square\square} &= 2\left(1 - \frac{m^2}{4\hbar^2}\right)\left(1 - \frac{m^2}{\hbar^2}\right)\\
2\,Z_{\square\square\square} &= 2\left(1 - \frac{m^2}{9\hbar^2}\right)\left(1 - \frac{m^2}{4\hbar^2}\right)\left(1 - \frac{m^2}{\hbar^2}\right)\\
Z_{\square\square\atop\square} &= \left(1 - \frac{m^2}{9\hbar^2}\right)\left(1 - \frac{m^2}{\hbar^2}\right)^2
\end{aligned} \qquad (4.3)$$

– 13 –

The factor 2 in the second and third equations come from the identical contributions of a diagram and its transpose. Putting them together we obtain

$$
\begin{aligned}
\langle \operatorname{tr} \varphi^2 \rangle_{\mathcal{N}=1^*} &= (m^2 - \hbar^2)\,(1 + q + 3q^2 + 4q^3 + 7q^4 + \ldots) \\
&= (m^2 - \hbar^2) \sum_{d|k} d\, q^k = -\frac{1}{24}\,(m^2 - \hbar^2)\, E_2(q)
\end{aligned} \tag{4.4}
$$

This formula is exact and includes the gravitational corrections: the term multiplied by $\hbar^2$ is the only gravitational correction to this correlator. Finally, taking the $\hbar \to 0$ limit, applying (2.28) and using the relation $\partial_\tau E_2(\tau) = \frac{i\pi}{6}(E_2^2 - E_4)$ we get the gaugino condensate

$$
\langle \operatorname{tr} \lambda^\alpha \lambda_\alpha \rangle_{\mathcal{N}=1^*} = \frac{m^3}{24 \cdot 12}(E_2^2(q) - E_4(q)) \tag{4.5}
$$

4.2 Scalar correlators in $\mathrm{SU}(2)$

The situation in this case is more subtle since some extra work is required to find out the exact value of the v.e.v. in the $\mathcal{N}=1$ vacua. The Seiberg-Witten curve for the SYM with gauge group $\mathrm{SU}(2)$ is given by a genus one Riemann surface. Indeed the case in which matter gets massive was analyzed in [14] with methods similar to those in [4]. For gauge groups of higher rank we need some other input which was provided by the connection with integrable models [16, 17, 15] and will be the subject of one of the next subsections. Let us start with the so called $\mathcal{N}=2^*$ theory [14] and its Seiberg-Witten curve

$$
y^2 = \prod_{i=1}^{3}(x - \mathcal{E}_i) \qquad \mathcal{E}_i = e_i \frac{\tilde{u}}{2} + \frac{1}{4} e_i^2 m^2 \tag{4.6}
$$

Following [27], the e_i's are given by

$$
\begin{aligned}
e_1 &= \frac{1}{3}\left(\frac{\pi}{2\omega_1}\right)^2 \left[\vartheta_3^4(0|q) + \vartheta_4^4(0|q)\right] \\
e_2 &= \frac{1}{3}\left(\frac{\pi}{2\omega_1}\right)^2 \left[\vartheta_2^4(0|q) - \vartheta_4^4(0|q)\right] \\
e_3 &= -\frac{1}{3}\left(\frac{\pi}{2\omega_1}\right)^2 \left[\vartheta_3^4(0|q) + \vartheta_2^4(0|q)\right]
\end{aligned} \tag{4.7}
$$

with $q = e^{2\pi i\tau}$, $\tau = \omega_2/\omega_1$ and $\omega_{1,2}$ the semi-periods.[6] Notice that $e_1 + e_2 + e_3 = 0$. To simplify the computations in this subsection we choose the semi-period $\omega_1 = \pi/2$. Given this choice, the q-expansion of the e_i's is given in (A.8) in the appendix. The variable $\tilde{u}$ is related to the physical one, $u = \langle \operatorname{tr} \phi^2 \rangle$, by

$$
\tilde{u} = u - m^2 \left(\frac{1}{6} + \sum_{n=1}^{\infty} \alpha_n q^n \right) \tag{4.8}
$$

In [14] a guess is given for the form of the coefficients α_n which does not seem to agree with an explicit instanton computation of winding number one [28]. Our first task in this section is to compute all the α_n's.

---

[6]Apart from this choice of q we adopt the notation in [27].

The variable $\tilde{u}$ is related to the vev a in such a way that $\frac{\partial a}{\partial \tilde{u}}$ and $\frac{\partial a}{\partial \tilde{u}}$ reproduce the periods of the Seiberg-Witten curve (4.6), i.e. via the differential equations[7]

$$\frac{\partial a}{\partial \tilde{u}} = \frac{1}{2\pi} \frac{K(k^2)}{\sqrt{\mathcal{E}_1 - \mathcal{E}_2}} \qquad \frac{\partial a_D}{\partial \tilde{u}} = \frac{1}{2\pi} \frac{K(k'^2)}{\sqrt{\mathcal{E}_1 - \mathcal{E}_2}} \qquad (4.9)$$

with $k^2 = (\mathcal{E}_3 - \mathcal{E}_2)/(\mathcal{E}_1 - \mathcal{E}_2)$ and $k'^2 = (\mathcal{E}_1 - \mathcal{E}_3)/(\mathcal{E}_1 - \mathcal{E}_2)$ and $K(k^2)$ is the complete elliptic integral of first kind (A.7). Expanding in q one finds

$$
\begin{aligned}
\frac{\partial a}{\partial \tilde{u}} = {} & \frac{1}{2}z + \frac{1}{8}m^2 z^3 \left(4 + 3\,m^2\,z^2\right) q^2 + \frac{3}{128}\,m^2\,z^3 \left(64 + 128\,m^2\,z^2 + 35\,m^6\,z^6\right) q^4 \\
& + \frac{1}{512}\,m^2\,z^3 \left(1024 + 4608\,m^2\,z^2 + 1600\,m^4\,z^4 + 6720\,m^6\,z^6 \right. \\
& \qquad\qquad \left. - 1260\,m^8\,z^8 + 1155\,m^{10}\,z^{10}\right) q^6 + \dots
\end{aligned}
\qquad (4.10)
$$

with $z = (\frac{1}{2}\tilde{u} + \frac{1}{12}m^2)^{-\frac{1}{2}}$. In order to find $\tilde{u}$ we integrate (4.10) (fixing the integration constant to zero to satisfy monodromies) and invert this equation. One finds

$$
\begin{aligned}
\tilde{u} = {} & 2\,a^2 - \frac{m^2}{6} + \left(4\,m^2 + \frac{m^4}{a^2}\right) q + \left(12\,m^2 + \frac{6\,m^4}{a^2} - \frac{3\,m^6}{a^4} + \frac{5\,m^8}{16\,a^6}\right) q^2 \\
& + \left(16\,m^2 + \frac{12\,m^4}{a^2} - \frac{24\,m^6}{a^4} + \frac{15\,m^8}{a^6} - \frac{7\,m^{10}}{2\,a^8} + \frac{9\,m^{12}}{32\,a^{10}}\right) q^3 + \dots
\end{aligned}
\qquad (4.11)
$$

This can be compared with the results in [7, 10] for $u = \langle \operatorname{tr} \phi^2 \rangle$. In the limit $\hbar \to 0$

$$
\begin{aligned}
u = \langle \operatorname{tr} \varphi^2 \rangle = {} & 2a^2 + \left(-4m^2 + \frac{m^4}{a^2}\right) q + \left(-12m^2 + \frac{6m^4}{a^2} - \frac{3m^6}{a^4} + \frac{5m^8}{16a^6}\right) q^2 \\
& + \left(-16m^2 + \frac{12m^4}{a^2} - \frac{24m^6}{a^4} + \frac{15m^8}{a^6} - \frac{7m^{10}}{2a^8} + \frac{9m^{12}}{32a^{10}}\right) q^3 + \dots
\end{aligned}
\qquad (4.12)
$$

Comparing (4.11) and (4.12) one finds

$$\tilde{u} = u - \frac{1}{6}m^2 + 8m^2 \sum_{d|k} d\,q^k = u + \frac{1}{6}m^2 - \frac{1}{3}\,m^2\,E_2(q) \qquad (4.13)$$

To extract from the $\mathcal{N} = 2^*$ result (4.12) the result for the $\mathcal{N} = 1^*$ theory we need to specify the value of $\tilde{u}$ which corresponds to the points in the moduli space where two solutions of the cubic (4.6) are coincident i.e. $\tilde{u} = \frac{m^2}{2}e_i$. Plugging this back into (4.13) and using the identities (A.9) displayed in the appendix one finds

$$
\begin{aligned}
u_1 &= \frac{m^2}{6}(4\,E_2(q^2) - 1) \\
u_2 &= \frac{m^2}{6}(E_2(-\sqrt{q}) - 1) \\
u_3 &= \frac{m^2}{6}(E_2(\sqrt{q}) - 1)
\end{aligned}
\qquad (4.14)
$$

---

[7]The dictionary between the variables employed here and those in [28] is $v_{DKM} = \sqrt{2}a, u_{DKM} = u/2$. Moreover these authors follow the notations of [29] in which e_2 and e_3 are interchanged with respect to our choice. Finally their ω_1 is the period and not the semi-period.

u_1 gives the result in the Higgs phase, while in the confining phase $u = \frac{1}{2}(u_2 - u_3)$, with $u_{2,3}$ the contributions coming from the two vacua. The results for pure $\mathcal{N} = 1$ gauge theory can be read from this by sending $m \to \infty$ keeping $\Lambda^2_{\mathcal{N}=1} = m^2 q^{\frac{1}{2}}$ in the confining phase: $u_{2,3} = \pm 4\Lambda^2_{\mathcal{N}=1}$.

In a similar way one can consider higher scalar correlators. Once again following [7, 10] we find

$$\langle \mathrm{tr}\, \varphi^4 \rangle = 2\, a^4 - 6\, m^2 \left(4\, a^2 - m^2 \right) q + \left(-72\, a^2\, m^2 + 48\, m^4 - \frac{18\, m^6}{a^2} + \frac{9\, m^8}{8\, a^4} \right) q^2$$
$$+ \left(-96\, a^2\, m^2 + 152\, m^4 - \frac{156\, m^6}{a^2} + \frac{72\, m^8}{a^4} - \frac{55\, m^{10}}{4\, a^6} + \frac{7\, m^{12}}{8\, a^8} \right) q^3 + \dots \quad (4.15)$$

Inverting (4.12) we find

$$a = \sqrt{\frac{u}{2}} - q \left(\frac{m^4}{\sqrt{2}\, u^{\frac{3}{2}}} - \frac{\sqrt{2}\, m^2}{\sqrt{u}} \right) + q^2 \left(\frac{-15\, m^8}{4\sqrt{2}\, u^{\frac{7}{2}}} + \frac{6\sqrt{2}\, m^6}{u^{\frac{5}{2}}} - \frac{4\sqrt{2}\, m^4}{u^{\frac{3}{2}}} + \frac{3\sqrt{2}\, m^2}{\sqrt{u}} \right)$$
$$+ q^3 \left(\frac{-105\, m^{12}}{4\sqrt{2}\, u^{\frac{11}{2}}} + \frac{245\, m^{10}}{2\sqrt{2}\, u^{\frac{9}{2}}} - \frac{90\sqrt{2}\, m^8}{u^{\frac{7}{2}}} + \frac{53\sqrt{2}\, m^6}{u^{\frac{5}{2}}} - \frac{12\sqrt{2}\, m^4}{u^{\frac{3}{2}}} + \frac{4\sqrt{2}\, m^2}{\sqrt{u}} \right) + \dots$$
$$(4.16)$$

Finally plugging this inside (4.15) we get

$$\langle \mathrm{tr}\, \varphi^4 \rangle = \frac{u^2}{2} - 8m^2(q + 3q^2 + 4q^3 + \dots)u - 4m^4(-q + q^2 + 28q^3 + \dots) \quad (4.17)$$

In the next section we will express the terms multiplied by m^2 and m^4 in (4.17) in terms of known functions using the chiral ring relations of $\mathcal{N} = 2^*$ SYM.

4.3 Chiral ring relations

In this section we will derive the chiral ring relations for the $\mathcal{N} = 1^*$ scalar correlators and test them against the multi-instanton results in the previous section. Here we focus on the SU(2) case. Higher rank groups will be studied in next subsection exploiting the connection between SYM and integrable models. Following [9] we define the following generating function for correlators with arbitrary powers of the scalar fields[8]

$$G(z) = \left\langle \mathrm{tr}\, \frac{1}{z - \varphi - \frac{i}{2}m_c} \right\rangle - \left\langle \mathrm{tr}\, \frac{1}{z - \varphi + \frac{i}{2}m_c} \right\rangle \quad (4.18)$$

$G(z)$ is an analytic function with branch cuts in $[\alpha_i^- \pm \frac{i}{2}m_c, \alpha_i^+ \pm \frac{i}{2}m_c]$, $i = 1, \dots, N$. Let U be its domain of definition.[9] Out of $G(z)$ we can build the function [30]

$$\varpi(z) = \frac{1}{2\pi i} \int_\infty^z G(y)dy \quad (4.19)$$

---

[8]In our notations $\varphi_{\mathrm{cl}} = \mathrm{diag}\{a, -a\}$ and $\mathrm{tr}\,\mathbf{1} = 2$.

[9]To compare with [9] we should replace m by its analytic continuation im_c.

a map from U to the cover, $\mathcal{C}_N$, of an elliptic curve, $\mathcal{C}$. $\mathcal{C}$ is obtained identifying the points $(\varpi + 1, z) \sim (\varpi, z)$, $(\varpi + \tau, z) \sim (\varpi, z + im_c)$. In [35] a function $f(z, \varpi)$ with this double-periodic property was introduced

$$
\begin{aligned}
f(z, \varpi) &= \frac{1}{\vartheta_1(\varpi|\tau)} \sum_{n=0}^{N} \frac{1}{n!} \frac{\partial^n}{\partial \varpi^n} \vartheta_1(\varpi|\tau) \left(-im_c \frac{\partial}{\partial z} \right)^n H(z) \\
&= \sum_{n=0}^{N} \frac{h_n(\varpi)}{n!} \left(-im_c \frac{\partial}{\partial z} \right)^n H(z)
\end{aligned}
\tag{4.20}
$$

with

$$
h_n = \frac{1}{\vartheta_1(z|\tau)} \frac{\partial^n}{\partial z^n} \vartheta_1(z|\tau) \qquad H(z) = \prod_{i=1}^{N} (z - z_i)
\tag{4.21}
$$

and z_i free parameters specifying the vacuum. In the above formulae and in the following computations we choose $\omega_1 = \frac{1}{2}$. When comparing with the results of the previous subsection we will take care of reintroducing the suitable factors of π. It is then natural to define $\varpi(z)$ as the solution of

$$
f(-2\pi i z, \varpi) = 0
\tag{4.22}
$$

For the SU(2) case we have $H(z) = z^2 - z_1^2$ and[10]

$$
f(z, \varpi) = z^2 - 2im_c z h_1(\varpi) + (im_c)^2 h_2(\varpi) - z_1^2
\tag{4.23}
$$

The resolvent $G(z)$ then follows from (4.19)

$$
G(z) = 2\pi i \, \varpi(z)'
\tag{4.24}
$$

To find an explicit expression for (4.24) we should first compute $\varpi(z)$ from (4.22). We need

$$
\begin{aligned}
h_1(\varpi) &= \frac{\vartheta_1(\varpi|\tau)'}{\vartheta_1(\varpi|\tau)} = \pi \cot \pi \varpi + 4\pi \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin 2\pi n \varpi \\
&= \frac{1}{\varpi} + \left(\frac{-\pi^2}{3} + 8 f_1(q) \pi^2 \right) \varpi + \left(\frac{-\pi^4}{45} - \frac{16 f_3(q) \pi^4}{3} \right) \varpi^3 \\
&\quad + \left(\frac{-2\pi^6}{945} + \frac{16 f_5(q) \pi^6}{15} \right) \varpi^5 + \ldots \\
h_2(\varpi) &= \frac{\vartheta_1(\varpi|\tau)''}{\vartheta_1(\varpi|\tau)} = h_1(\varpi)' + h_1(\varpi)^2
\end{aligned}
\tag{4.25}
$$

---

[10]This form of the equation can be immediately compared with that of [15]. In this last reference the equation for $\mathcal{C}_N$ is given in the form $P_N + A_2 P_{N-2} + \ldots + A_N P_0 = 0$. The P_i are polynomials in $z, \wp(\varpi)$. Setting $2\pi i t = -2\pi i z - im_c h_1(\varpi)$ we find $0 = f(2\pi i t, \varpi) = -4\pi^2 t^2 + (im_c)^2 h_1' - z_1^2 = -4\pi^2 t^2 - (im_c)^2 \wp(\varpi) - (im_c)^2 \pi^2 E_2(q)/3 - z_1^2 = -4\pi^2 [t^2 + x + (im_c)^2 E_2(q)/12 + z_1^2/4\pi^2]$. The two last terms in this equation give an explicit expression for the A_2 introduced above. In turn, according to [15], A_2 should be identified with $-\tilde{u}/2$ thus giving (we revert to using the physical mass) $\tilde{u} = -m^2 E_2(q)/6 - z_1^2/2\pi^2$. This relation will also be found from the chiral ring relations later on.

with $f_p(q)$ defined by (A.4). Out of the two roots of (4.22) we choose that one with a pole at $\varpi = 0$. Expanding around $\varpi = 0$ one finds

$$
\begin{aligned}
z &= \frac{1}{2\pi}\left(-m_c h_1 - \sqrt{-m_c{}^2\, h_1' - z_1^2}\right) \\
&= -\frac{m_c}{\pi\,\varpi} + \left[\frac{E^2\,\pi}{m_c} + \left(\frac{1}{12} - 2\,f_1(q)\right)m_c\,\pi\right]\varpi \\
&\quad + \left[\frac{E^2\pi^2}{6}\left(-1 + 24\,f_1(q)\right) + \frac{E^4\pi^2}{m_c{}^2}\right. \\
&\qquad \left. + \frac{m_c{}^2\pi^2}{720}\left(1 - 240\,f_1(q) + 2880\,f_1(q)^2 - 960\,f_3(q)\right)\right]\frac{\pi\varpi^3}{m_c} \\
&\quad + \left[E^2\left(\frac{1}{120} - 2\,f_1(q) + 24\,f_1(q)^2 - 8\,f_3(q)\right) - \frac{E^4}{m_c{}^2}\frac{1}{2} + 12\,f_1(q)) + \frac{2E^6}{m_c{}^4} + \frac{m_c{}^2}{30240}\right. \\
&\qquad \left. + m_c{}^2\left(\frac{f_1(q)}{60} - 2f_1(q)^2 + 16f_1(q)^3 + \frac{2f_3(q)}{3} - 16f_1(q)f_3(q) + \frac{4f_5(q)}{5}\right)\right]\frac{\pi^5\varpi^5}{m_c} + \dots
\end{aligned}
\tag{4.26}
$$

where $E^2 = z_1^2/(4\pi^2)$. Inverting (4.26) we find

$$
\begin{aligned}
\varpi(z) &= -\frac{m_c}{z\,\pi} - \frac{m_c{}^3}{12\pi z^3}\left(-1 + 24\,f_1(q) - \frac{12E^2}{m_c{}^2}\right) \\
&\quad - \frac{1}{\pi z^5}\left[m_c E^4 - \frac{E^2}{2}(-1 + 24f_1(q))m_c{}^3 + (3 - 80f_1(q)\right. \\
&\qquad \left. + 960f_1(q)^2 + 320f_3(q))\frac{m_c{}^5}{240}\right] + \dots
\end{aligned}
\tag{4.27}
$$

Taking the derivative and comparing the result for $G(z) = 2\pi i \varpi'(z)$ with the expansion of (4.18)

$$
\begin{aligned}
G(z) &= \frac{2im_c}{z^2} + \frac{1}{z^4}\left(\frac{(im_c)^3}{2} + 3im_c\langle\operatorname{tr}\varphi^2\rangle\right) \\
&\quad + \frac{1}{z^6}\left(\frac{(im_c)^5}{8} + \frac{5}{2}(im_c)^3\langle\operatorname{tr}\varphi^2\rangle + 5im_c\langle\operatorname{tr}\varphi^4\rangle\right) + \dots
\end{aligned}
\tag{4.28}
$$

one finds

$$
\langle\operatorname{tr}\varphi^2\rangle = 4\,f_1(q)\,m_c{}^2 - \frac{z_1^2}{2\pi^2}
\tag{4.29}
$$

$$
\langle\operatorname{tr}\varphi^4\rangle = \frac{(\langle\operatorname{tr}\varphi^2\rangle)^2}{2} + 8m_c{}^2 f_1(q)\langle\operatorname{tr}\varphi^2\rangle + m_c{}^4\left(\frac{4}{3}f_1(q) - 32f_1(q)^2 + \frac{8}{3}f_3(q)\right)
\tag{4.30}
$$

and so on. Formula (4.30) agrees with (4.17) (once we take $im_c \to m$) and confirms the multi-instanton result. Formula (4.29) relates z_1 to u. Substituting (4.13) in (4.29) we find

$$
\frac{z_1^2}{2\pi^2} = -\tilde{u} - \frac{m_c{}^2}{6} + 4m_c{}^2 f_1(q),
\tag{4.31}
$$

which is the result obtained in the footnote of the previous page from the definition of the Seiberg-Witten curve. In the remaining of this section we compute the z_1 corresponding to the SU(2) $\mathcal{N} = 1^*$ vacua.

The SU(2) unbroken vacuum is given by choosing z_1 such that the corresponding Seiberg-Witten curve degenerates. Solving for z in (4.23), we find

$$2\pi z = m_c \left(-h_1(\varpi) \pm \sqrt{-\frac{z_1^2}{m_c^2} - h_1'(\varpi)} \right)$$

$$= m_c \left(-h_1(\varpi) \pm \sqrt{-\frac{z_1^2}{m_c^2} + \frac{\pi^2}{3} E_2(q) + \wp(\varpi)} \right) \tag{4.32}$$

The positions of the end points of the cuts are defined by[11]

$$2\pi \frac{dz}{d\varpi} = m_c \left(-h_1'(\varpi) \pm \frac{\wp'(\varpi)}{2\sqrt{-\frac{z_1^2}{m_c^2} + \frac{\pi^2}{3} E_2(q) + \wp(\varpi)}} \right) = 0 \tag{4.33}$$

or

$$\sqrt{-\frac{z_1^2}{m_c^2} + \frac{\pi^2}{3} E_2(q) + \wp(\varpi)} = \mp \frac{\wp'(\varpi)}{2h_1'(\varpi)}. \tag{4.34}$$

The condition that the branch points collide should imply that the square root vanishes which according to the last equality is equivalent to

$$\wp'(\varpi) = 0 \tag{4.35}$$

$\varpi = \omega$, ω' and $\omega + \omega'$ are the zeroes of $\wp'(\varpi)$ and at these points $\wp(\varpi)$ takes the values $\pi^2 e_1$, $\pi^2 e_3$ and $\pi^2 e_2$ respectively.[12] Thus at degenerated points we have

$$\frac{z_1^2}{2\pi^2} = \frac{m_c^2}{6} E_2(q) + \frac{m_c^2}{2} e_i, \qquad i = 1, 2, 3. \tag{4.36}$$

Plugging this into (4.29) we get

$$u_i = \langle \text{tr}\, \varphi^2 \rangle = \frac{m_c^2}{6} - \frac{m_c^2}{3} E_2(q) - \frac{m_c^2}{2} e_i \tag{4.37}$$

in agreement with (4.13) after replacing $im_c \to m$ and using $\tilde{u}_i = \frac{m^2}{2} e_i$

4.4 SU(N) case and integrable models

The general SU(N) case can be studied exploiting the description of $\mathcal{N} = 1^*$ in terms of the Calogero-Moser integrable system. This system is defined by the hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2} m^2 \sum_{i<j}^{N} \wp(x_i - x_j) \tag{4.38}$$

---

[11]The cuts are the regions where there are the allowed values of the spectrum. At the end points of this regions the velocity must be zero.

[12]Since we compare with a result in the previous subsection, we must take care of the difference in our two choices of semi-period ($\omega_1 = \pi/2$ then and $\omega_1 = 1/2$ now. See the appendix for more details.

At equilibrium $p_i = 0$ and $\wp'(x_i - x_j) = 0$. This generalizes the condition (4.35) for SU(2). As in the SU(2) case we can introduce a curve $\mathcal{C}_N$, an N-fold branch cover of $\mathcal{C}$, via

$$0 = f(z, \varpi) = f(\lambda + mh_1(\varpi), \varpi)) = \det(\lambda \, \mathbb{1}_{N \times N} - L(\varpi))$$
$$= \lambda^N + \sum_{k=1}^{N} J_k(\varpi)\lambda^{N-k} = \prod_{k=1}^{N} (\lambda - \lambda_i) \tag{4.39}$$

For each value of ϖ in (4.39), which is the spectral parameter of $\mathcal{C}$, we find N values of λ which describes $\mathcal{C}_N$. The matrix $L(\varpi)$ together with a matrix $M(\varpi)$, form a Lax pair for the Calogero-Moser system with spectral parameter (ϖ) [31]. Their explicit form is [32]

$$L_{ij}(\varpi) = p_i \delta_{ij} - m(1 - \delta_{ij})\Phi(x_i - x_j, \varpi)$$
$$M_{ij}(\varpi) = m\delta_{ij} \sum_{k \neq i} \wp(x_i - x_k) - m(1 - \delta_{ij})\Phi'(x_i - x_j, \varpi) \tag{4.40}$$

where

$$\Phi(x, \varpi) = \frac{\sigma(\varpi - x)}{\sigma(\varpi)\sigma(x)} e^{\zeta(\varpi)x} \tag{4.41}$$

σ is the standard Weierstrass function [27]. The coefficients J_k's in (4.39) are connected with the integrals of motions

$$I_k = \frac{1}{k} \text{tr}\,(L^k) = \frac{1}{k} \left(\sum_{i=1}^{N} \lambda_i \right)^k \tag{4.42}$$

I_2 is the hamiltonian of the classical system and it is connected with the variable u in (4.12) [33, 34].

The relation between the parameters x_i and z_i can be found at equilibrium, where $p_i = 0$, using (4.20). In the SU(2) case, we have $N = 2$ in (4.40) and (4.41). It is therefore easy to compute

$$\det \begin{pmatrix} \lambda - p_1 & -m\Phi((x_1 - x_2, \varpi) \\ -m\Phi(x_2 - x_1, \varpi) & \lambda - p_2 \end{pmatrix}$$
$$= -m^2 \Phi(x_1 - x_2, \varpi)\Phi(x_2 - x_1, \varpi) + \lambda^2 + p_1 p_2 - \lambda(p_1 + p_2)$$
$$= m^2 h_1'(\varpi) - m^2 h_1'(x_1 - x_2) + \lambda^2 - \lambda(p_1 + p_2) + p_1 p_2 \tag{4.43}$$

In the last equality of the above equation we used the following identity for the odd function σ [27]

$$\sigma(\lambda - (x_1 - x_2))\sigma(\lambda + (x_1 - x_2)) = -\sigma^2(\lambda)\sigma^2(x_1 - x_2)[\wp(\lambda) - \wp(x_1 - x_2)] \tag{4.44}$$

and the explicit form of the Weierstrass function given in the appendix , eqs. (A.5) and (A.6).

Then comparing with $f(\lambda + mh_1(\varpi), \varpi) = m^2 h_1'(\varpi) - z_1^2 + \lambda^2$ one can see that the dependence on the spectral parameter ϖ cancels. This is a general feature which is valid for arbitrary values of N. The general relation between the parameters z_i and the momenta

and positions of the Calogero-Moser system was found in [36]. The vacua of the $\mathcal{N} = 1^*$ theory , whose quantum potential is given by the Weierstrass $\wp$-function [33], are given by the equilibrium positions of the associated Calogero-Moser system. Solving the equation $\wp'(x_1 - x_2) = 0$ [33] we then find[13]

$$z_1^2 = m^2 h_1'(x_1 - x_2)|_{x_i = x_{vac}} = -m^2 \pi^2 e_i - \frac{m^2 \pi^2}{3} E_2(q) \tag{4.45}$$

in agreement with (4.36) after the replacement $i m_c \to m$.

5. Summary of results

Here we summarize our results. We consider $\mathcal{N} = 1$ gauge theories with or without adjoint matter. The two main examples are mass deformations or pure $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ gauge theories down to $\mathcal{N} = 1$. The later one is refereed as $\mathcal{N} = 1^*$. For the chiral correlators and gaugino condensates one finds

$$\langle \operatorname{tr} \varphi^J \rangle_{\mathcal{N}=1} = \langle \operatorname{tr} \varphi^J e^{-\int_{\mathbb{C}^2} \alpha_{(2,0)} \wedge \operatorname{tr} W(\mathcal{F})} \rangle_{\mathcal{N}=2} = \langle \operatorname{tr} \varphi^J \rangle_{\mathcal{N}=2}$$
$$\langle \operatorname{tr} \lambda\lambda \varphi^J \rangle_{\mathcal{N}=1} = -\frac{2m}{(J+2)(J+1)} q\, \partial_q \langle \operatorname{tr} \varphi^{J+2} \rangle_{\mathcal{N}=2} \tag{5.1}$$

where the r.h.s. correlators are to be evaluated in the parent $\mathcal{N} = 2$ or $\mathcal{N} = 2^*$ SYM and then their v.e.v.'s have to be chosen so to minimize the quantum potential. For instance for the $\mathcal{N} = 1$ descending for pure $\mathcal{N} = 2$ gauge theory with gauge group SU(2) in the SU(2) unbroken phase one finds $u_{1,2} = \langle \operatorname{tr} \varphi^2 \rangle = \pm 4\Lambda^2$ and higher order scalar correlators are given by the (3.22).

For $\mathcal{N} = 1^*$ with gauge group SU(2) one finds

$$u_1 = \frac{m^2}{6}(4\,E_2(q^2) - 1)$$
$$u_2 = \frac{m^2}{6}(E_2(-\sqrt{q}) - 1)$$
$$u_3 = \frac{m^2}{6}(E_2(\sqrt{q}) - 1) \tag{5.2}$$

u_1 gives the result in the Higgs phase, while in the confining phase $u = \frac{1}{2}(u_2 - u_3)$, with $u_{2,3}$ the contributions coming from the two vacua. The results for pure $\mathcal{N} = 1$ gauge theory can be read from this by sending $m \to \infty$ keeping $\Lambda_{\mathcal{N}=1}^2 = m^2 q^{\frac{1}{2}}$ in the confining phase: $u_{2,3} = \pm 4\Lambda_{\mathcal{N}=1}^2$.

The scalar correlator with the next higher power is given by

$$\langle \operatorname{tr} \varphi^4 \rangle = \frac{1}{2}u^2 + \frac{1}{3}m^2(E_2(q) - 1)u + \frac{1}{18}m^4\left(\frac{1}{5} + E_2(q) - E_2(q)^2 - \frac{1}{5}E_4(q)\right) \tag{5.3}$$

---

[13]We scale by a factor of minus four the result found for the confining vacuum in [33]: $\wp(x_1 - x_2)|_{x_i = x_{conf}} = N^2/24(E_2(q) - 1/N E_2(q^{1/N}))$, due to the different choice of the semi-period $\omega_1 = i\pi$ used in that paper.

In a similar way one can study gravitational corrections. For instance for $\mathcal{N} = 1^*$ with gauge group U(1) one finds

$$\begin{aligned}
\langle \mathrm{tr}\, \varphi^2 \rangle_{\mathcal{N}=1^*} &= \left(m^2 - \hbar^2 \right) \left(1 + q + 3q^2 + 4q^3 + 7q^4 + \ldots \right) \\
&= \left(m^2 - \hbar^2 \right) \sum_{d|k} d\, q^k = -\frac{1}{24} \left(m^2 - \hbar^2 \right) E_2(q)
\end{aligned} \tag{5.4}$$

The term $\hbar^2$ represents the only gravitational correction to u! We remind the reader that even if our focus in this paper was on gauge theories and, in order to keep our formulae down to a manageable size, we have omitted gravitational corrections, they are easily accounted for and could be compared with analogous results obtained for the topological string [37–39]. The U(N) case of (5.4) can be easily extracted from [6].

Acknowledgments

A. Elliptic functions

Here we collect some useful formulae and definitions:

Theta functions

$$\vartheta_{\alpha\beta}(v|q) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}\left(n-\frac{\alpha}{2}\right)^2} e^{2\pi i \left(n-\frac{\alpha}{2}\right)\left(v-\frac{\beta}{2}\right)} \tag{A.1}$$

with $q = e^{2\pi i \tau}$. We adopt the standard shorthand notation: $\vartheta_1 = \vartheta_{11}$, $\vartheta_2 = \vartheta_{10}$, $\vartheta_3 = \vartheta_{00}$, $\vartheta_4 = \vartheta_{01}$. At $v = 0$ the theta functions satisfy the Riemann Identity

$$\vartheta_3(0|q)^4 - \vartheta_2(0|q)^4 - \vartheta_4(0|q)^4 = 0 \tag{A.2}$$

Einstein series

$$\begin{aligned}
E_2(q) &= 1 - 24 f_1(q) \\
E_4(q) &= 1 + 240 f_3(q) \\
E_6(q) &= 1 - 504 f_5(q)
\end{aligned} \tag{A.3}$$

with

$$f_p(q) = \sum_{n=1}^{\infty} \frac{n^p q^n}{1 - q^n} \tag{A.4}$$

Weierstrass Function

$$\wp(z, w_1, w_2) = -\frac{1}{3}\left(\frac{\pi}{2\omega_1}\right)^2 E_2(q) - \partial_z^2 \log \vartheta_1\left(\frac{z}{2\omega_1}\Big|q\right) \tag{A.5}$$

with $\tau = \omega_1/\omega_2$ and $\omega_{1,2}$ the half-periods. Typical choices for the periods are $(\omega_1, \omega_2) = (\frac{1}{2}, \frac{\tau}{2})$ or $(\omega_1, \omega_2) = (\frac{\pi}{2}, \frac{\pi\tau}{2})$. The two notations are related by $\wp(tz|t\omega_1, t\omega_2) = t^{-2}\wp(z|\omega_1, \omega_2)$. In particular one finds $\wp(\frac{\pi}{2}|\frac{\pi}{2}, \frac{\pi\tau}{2}) = e_1$ and $\wp(\frac{1}{2}|\frac{1}{2}, \frac{\tau}{2}) = \pi^2 e_1$. In the main text we use the shorthand notation

$$\wp(z) \equiv \wp\left(z|\frac{1}{2}, \frac{\tau}{2}\right) \tag{A.6}$$

The complete elliptic integral of first kind

$$K(k^2) \equiv \int_0^1 \frac{dt}{(1-t^2)(1-k^2\,t^2)} \tag{A.7}$$

For the reader's convenience we collect here the q-expansions of the e_i's given in (4.7)

$$e_1 = \frac{2}{3} + 16q + 16q^2 + 64q^3 + 16q^4 + 96q^5 + 64q^6 + 128q^7 + 16q^8 + \dots$$

$$e_2 = -\frac{1}{3} - 8q^{1/2} - 8q - 32q^{3/2} - 8q^2 - 48q^{5/2} - 32q^3 - 64q^{7/2} - 8q^4 - 104q^{9/2} - 48q^5$$
$$-96q^{11/2} - 32q^6 - 112q^{13/2} - 64q^7 - 192q^{15/2} - 8q^8 + \dots$$

$$e_3 = -\frac{1}{3} + 8q^{1/2} - 8q + 32\,q^{3/2} - 8q^2 + 48q^{5/2} - 32q^3 + 64q^{7/2} - 8q^4 + 104q^{9/2} - 48q^5$$
$$+96q^{11/2} - 32q^6 + 112q^{13/2} - 64q^7 + 192q^{15/2} - 8q^8 + \dots \tag{A.8}$$

and the following identities

$$e_1 = -\frac{2}{3}\left(E_2(q) - 2E_2(q^2)\right)$$
$$e_2 = -\frac{2}{3}\left(E_2(q) - \frac{1}{2}E_2(\sqrt{q})\right)$$
$$e_3 = -\frac{2}{3}\left(E_2(q) - \frac{1}{2}E_2(-\sqrt{q})\right) \tag{A.9}$$

References

[1] E. Witten, *Constraints on supersymmetry breaking*, *Nucl. Phys.* **B 202** (1982) 253.

[2] M.A. Shifman, *Nonperturbative dynamics in supersymmetric gauge theories*, *Prog. Part. Nucl. Phys.* **39** (1997) 1 [`hep-th/9704114`].

[3] M.A. Shifman, *Instantons in gauge theories*, World Scientific, Singapore 1994.

[4] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [`hep-th/9407087`], erratum *ibid.* **B 430** (1994) 485.

[5] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [`hep-th/0206161`].

[6] R. Flume and R. Poghossian, *An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential*, *Int. J. Mod. Phys.* **A 18** (2003) 2541 [`hep-th/0208176`].

[7] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [hep-th/0211108].

[8] A.S. Losev, A. Marshakov and N.A. Nekrasov, *Small instantons, little strings and free fermions*, hep-th/0302191.

[9] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238.

[10] R. Flume, F. Fucito, J.F. Morales and R. Poghossian, *Matone's relation in the presence of gravitational couplings*, *JHEP* **04** (2004) 008 [hep-th/0403057].

[11] M. Marino and N. Wyllard, *A note on instanton counting for $N = 2$ gauge theories with classical gauge groups*, *JHEP* **05** (2004) 021 [hep-th/0404125].

[12] N. Nekrasov and S. Shadchin, *ABCD of instantons*, *Commun. Math. Phys.* **252** (2004) 359 [hep-th/0404225].

[13] F. Fucito, J.F. Morales and R. Poghossian, *Instantons on quivers and orientifolds*, *JHEP* **10** (2004) 037 [hep-th/0408090].

[14] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[15] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys.* **B 460** (1996) 299 [hep-th/9510101].

[16] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, *Integrability and Seiberg-Witten exact solution*, *Phys. Lett.* **B 355** (1995) 466 [hep-th/9505035].

[17] E.J. Martinec and N.P. Warner, *Integrable systems and supersymmetric gauge theory*, *Nucl. Phys.* **B 459** (1996) 97 [hep-th/9509161].

[18] L. Baulieu and I.M. Singer, *Topological Yang-Mills symmetry*, *Nucl. Phys.* **5B** (Proc. Suppl.) (1988) 12.

[19] L. Baulieu, G. Bossard and A. Tanzini, *Topological vector symmetry of brstqft and construction of maximal supersymmetry*, *JHEP* **08** (2005) 037 [hep-th/0504224].

[20] A. Johansen, *Twisting of $N = 1$ SUSY gauge theories and heterotic topological theories*, *Int. J. Mod. Phys.* **A 10** (1995) 4325 [hep-th/9403017];
L. Baulieu and A. Tanzini, *Topological symmetry of forms, $N = 1$ supersymmetry and S-duality on special manifolds*, hep-th/0412014.

[21] E. Witten, *Supersymmetric Yang-Mills theory on a four manifold*, *J. Math. Phys.* **35** (1994) 5101 [hep-th/9403195].

[22] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[23] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *JHEP* **12** (2002) 071 [hep-th/0211170].

[24] R. Argurio, G. Ferretti and R. Heise, *An introduction to supersymmetric gauge theories and matrix models*, *Int. J. Mod. Phys.* **A 19** (2004) 2015 [hep-th/0311066].

[25] D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, *Nonperturbative aspects in supersymmetric gauge theories*, *Phys. Rept.* **162** (1988) 169.

[26] M.R. Douglas and S.H. Shenker, *Dynamics of* SU(N) *supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 271 [`hep-th/9503163`].

[27] *Higher transcendental functions*, A. Erdelyi ed., R.E. Krieger Publishing Co., Florida 1981.

[28] N. Dorey, V.V. Khoze and M.P. Mattis, *On mass-deformed $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 396** (1997) 141 [`hep-th/9612231`].

[29] K. Chandrasekharan, *Elliptic functions*, Springer-Verlag, Heidelberg 1985.

[30] V.A. Kazakov, I.K. Kostov and N.A. Nekrasov, *D-particles, matrix integrals and KP hierarchy*, *Nucl. Phys.* **B 557** (1999) 413 [`hep-th/9810035`].

[31] I. Krichever, *Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles*, *Funct. Anal. Appl.* **14** (1980) 282.

[32] M.A. Olshanetsky and A.M. Perelomov, *Classical integrable finite dimensional systems related to Lie algebras*, *Phys. Rept.* **71** (1981) 313.

[33] N. Dorey, *An elliptic superpotential for softly broken $N = 4$ supersymmetric Yang-Mills theory*, *JHEP* **07** (1999) 021 [`hep-th/9906011`].

[34] N. Dorey and A. Sinkovics, $N = 1^*$ *vacua, fuzzy spheres and integrable systems*, *JHEP* **07** (2002) 032 [`hep-th/0205151`].

[35] E. D'Hoker and D.H. Phong, *Calogero-Moser systems in* SU(N) *Seiberg-Witten theory*, *Nucl. Phys.* **B 513** (1998) 405 [`hep-th/9709053`].

[36] E. D'Hoker and D.H. Phong, *Order parameters, free fermions and conservation laws for calogero-moser systems*, `hep-th/9808156`.

[37] A. Klemm, M. Marino and S. Theisen, *Gravitational corrections in supersymmetric gauge theory and matrix models*, *JHEP* **03** (2003) 051 [`hep-th/0211216`].

[38] T.J. Hollowood, A. Iqbal and C. Vafa, *Matrix models, geometric engineering and elliptic genera*, `hep-th/0310272`.

[39] T. Eguchi and H. Kanno, *Topological strings and Nekrasov's formulas*, *JHEP* **12** (2003) 006 [`hep-th/0310235`].